Internal Hom for sheaves and Poincaré-Verdier duality

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1 Double complexes

In this section, $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ will be abelian categories. A double complex in $\mathcal{C}$ is a family $X^{p,q}$ of objects of $\mathcal{C}$ together with maps

$$d^h : X^{p,q} \to X^{p+1,q} \text{ and } d^v : X^{p,q} \to X^{p,q+1}$$

such that $d^h \circ d^h = d^v \circ d^v = d^v d^h - d^h d^v = 0$ It is useful to have the following picture in mind :

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & X^{p-1,q+1} & \xrightarrow{d^h} & X^{p,q+1} & \xrightarrow{d^h} & X^{p+1,q+1} & \longrightarrow & \cdots \\
 & & d^v\uparrow & & d^v\uparrow & & d^v\uparrow & & \\
\cdots & \longrightarrow & X^{p-1,q} & \xrightarrow{d^h} & X^{p,q} & \xrightarrow{d^h} & X^{p+1,q} & \longrightarrow & \cdots \\
 & & d^v\uparrow & & d^v\uparrow & & d^v\uparrow & & \\
\cdots & \longrightarrow & X^{p-1,q-1} & \xrightarrow{d^h} & X^{p,q-1} & \xrightarrow{d^h} & X^{p+1,q-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}$$

Remark 1. Another convention for double complexes is to ask the property $d^v d^h + d^h d^v = 0$ instead of $d^h d^v = d^v d^h$. We can switch from one convention to the other by replacing d^v by $(-1)^p d^v$. We can define cohomology groups of double complexes :

$$(H_I(X))^{i,j} = H^i(X^{\cdot,j}) \text{ and } (H_{II}(X))^{i,j} = H^j(X^{i,\cdot})$$

with the differentials :

$$d^h_{H_I} = 0, d^v_{H_I} \text{ induced by } d^v_X,$$

$$d^h_{H_{II}} \text{ induced by } d^h_X, d^v_{H_{II}} = 0.$$

To a double complex X with a nice finiteness property, we can associate a simple complex $s(X)$. We suppose that

$$\text{for any } n \in \mathbb{Z}, \text{ the set } \{(p,q) \in \mathbb{Z} \times \mathbb{Z}; p+q=n, X^{p,q} \neq 0\} \text{ is finite.}$$

Then we define $s(X)$ the simple complex defined by

$$s(X)^n = \bigoplus_{n=p+q} X^{p,q} \text{ with differential } \; d|_{X^{p,q}} = (-1)^p d^v + d^h.$$

We can check that $s(X)$ is indeed a complex :

$$d \circ d|_{X^{p,q}} = d \circ \left((-1)^p d^v + d^h\right) = (-1)^{2p} d^v d^v + (-1)^p d^h d^v + (-1)^{p+1} d^v d^h + d^h d^h = 0.$$

We won't prove the following theorem that ensures us that $s(X)$ is the good simple complex associated to a double complex.

Theorem 1. *If $f : X \to Y$ is a morphism of double complexes with the finiteness property such that f induces an isomorphism*

$$H_{II} H_I(X) \xrightarrow{\sim} H_{II} H_I(Y)$$

then $s(f)$ is a quasi-isomorphism.

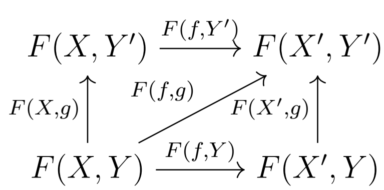
2 Bifunctor

A bifunctor $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ consists of the following data :

- a map $F : \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}') \to \mathrm{Ob}(\mathcal{C}'')$
- for all $X \in \mathrm{Ob}(\mathcal{C})$, $F(X, \cdot)$ is a functor from $\mathcal{C}'$ to $\mathcal{C}''$
- for all $X' \in \mathrm{Ob}(\mathcal{C}')$, $F(\cdot, X')$ is a functor from $\mathcal{C}$ to $\mathcal{C}''$
- If $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}'}(X',Y')$ then there exists

$$F(f,g) : F(X,X') \to F(Y,Y')$$

such that the following diagram commutes :



The only condition for this diagram to commute is that $F(Y,g) \circ F(f,X') = F(f,Y') \circ F(X,g)$.

If X is a complex in $\mathrm{Ob}(C^+(\mathcal{C}))$ and X' a complex in $\mathrm{Ob}(C^+(\mathcal{C}'))$ then $F(X,X')$

is a double complex in $\mathcal{C}''$ satisfying the finiteness condition. So by associating to $F(X, X')$ the simple complex $s(F(X, X'))$, we define a bifunctor :

$$C^+(F) : C^+(\mathcal{C}) \times C^+(\mathcal{C}') \to C^+(\mathcal{C}'').$$

The construction of s ensures that if X is homotopic to zero then $s(F(X, Y))$ is also homotopic to zero for any Y. We can pass to the quotient and get a bifunctor

$$K^+(F) : K^+(\mathcal{C}) \times K^+(\mathcal{C}') \to K^+(\mathcal{C}'').$$

Recall that if we have a functor G, a class of adapted objects $\mathcal{R}$ to G [1] is a class such that

- G is exact on $\mathcal{R}$.
- $\mathcal{R}$ is big enough (any objects injects itself in an object of $\mathcal{R}$).
- if $0 \to F' \to F \to F'' \to 0$ is an exact sequence and F' and F are in $\mathcal{R}$ then F'' is in $\mathcal{R}$.

For a bifunctor F, we will say that $(\mathcal{R}, \mathcal{R}')$ is adapted to F if

- for all $X \in \mathrm{Ob}(\mathcal{R})$, $\mathcal{R}'$ is adapted to $F(X, \cdot)$,
- for all $X' \in \mathrm{Ob}(\mathcal{R}')$, $\mathcal{R}$ is adapted to $F(\cdot, X')$.

In this case, we can derive F and we have for $X^\bullet \in D^+(\mathcal{C})$ and $Y^\bullet \in D^+(\mathcal{C}')$,

$$RF(X^\bullet, Y^\bullet) \simeq s(F(I^\bullet, J^\bullet))$$

where $X^\bullet \simeq I^\bullet$, $I^\bullet \in K^+(\mathcal{R})$, $Y^\bullet \simeq J^\bullet$ and $J^\bullet \in K^+(\mathcal{R}')$. We have the following commutative diagram :

$$\begin{array}{ccc} K^+(\mathcal{R}) \times K^+(\mathcal{R}') & \xrightarrow{F} & K^+(\mathcal{C}'') \\ \downarrow{\scriptstyle Q \times Q'} & & \downarrow{\scriptstyle Q''} \\ D^+(\mathcal{C}) \times D^+(\mathcal{C}') & \xrightarrow{RF} & D^+(\mathcal{C}''). \end{array}$$

Moreover suppose that we can derive the functor $F(\cdot, Y^\bullet)$ for all $Y^\bullet \in \mathrm{Ob}(K^+(\mathcal{C}'))$. Denote by $R_I F(\cdot, Y^\bullet)$ this right derived functor. Then by construction of the derived functor we have

$$RF(X^\bullet, Y^\bullet) \simeq R_I F(X^\bullet, Y^\bullet).$$

A suffisant condition for this to happen is that there exists a subcategory $\mathcal{R}$ of $\mathcal{C}$ such that for all $Y \in \mathrm{Ob}(\mathcal{C}'), \mathcal{R}$ is adapted to $F(\cdot, Y)$. Another way to formulate it is that $(\mathcal{R}, \mathcal{C}')$ is adapted to F. There is also an analogous theorem for composition of derived functor :

1. People usually say $\mathcal{R}$ is G-injective

Theorem 2 (Grothendieck's composition theorem)**.** *Let $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ be a left exact bifunctor of abelian categories and let $G : \mathcal{C}'' \to \mathcal{C}'''$ be a left exact functor of abelian categories. Assume that there exists full additive subcategories $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$ of $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ respectively such that $(\mathcal{R}, \mathcal{R}')$ is adapted to F, $\mathcal{R}''$ is adapted to G and :*

$$F(Ob(\mathcal{R}), Ob(\mathcal{R}')) \subset Ob(\mathcal{R}'').$$

Then the derived functor $R(G \circ F) : D^+(\mathcal{C}) \times D^+(\mathcal{C}') \to \mathcal{C}''$ exists and we have

$$R(G \circ F) \simeq RG \circ RF.$$

We finish this section by a theorem that might be useful for comprehension :

Theorem 3. *Let $\mathcal{C}$ be an abelian category with enough injective. The functor $Hom_{\mathcal{C}}$ is left exact with respect of each of its argument so we can derive it and we have the following identity :*

$$H^0(\cdot) \circ RHom_{\mathcal{C}} \simeq Hom_{D^+(\mathcal{C})}$$

and you may already know the bifunctor $H^n(\cdot) \circ RHom_{\mathcal{C}}$. It is denoted $Ext^n_{\mathcal{C}}(\cdot, \cdot)$.

Démonstration. We will do the proof as it gives us an explicit construction of a derived bifunctor. The first lemma, we want to have is :

Lemme 1. *If $\mathcal{C}$ has enough injectives, $(\mathcal{C}^{op}, \mathcal{I})$ is adapted to the bifunctor Hom where $\mathcal{I}$ denotes the injective objects of $\mathcal{C}$.*

The proof of this lemma is trivial once you have all the definitions : an object I is injective if and only if $\mathrm{Hom}(\cdot, I) : \mathcal{C}^{op} \to \mathrm{Ab}$ is an exact functor. And as $\mathrm{Hom}(F, \cdot)$ is left exact, injectives objects are adapted to $\mathrm{Hom}(F, \cdot)$. We have proved that $(\mathcal{C}^{op}, \mathcal{I})$ is adapted to Hom.

Now take $X^\bullet \in D^+(\mathcal{C}^{op})$ and $Y^\bullet \in D^+(\mathcal{C})$. We want to compute $H^0 \circ R\mathrm{Hom}(X^\bullet, Y^\bullet)$. Take $I^\bullet \in K^+(\mathcal{I})$ such that $Y^\bullet \simeq I^\bullet$. Then

$$R\mathrm{Hom}(X^\bullet, I^\bullet) \simeq s(\mathrm{Hom}(X^\bullet, I^\bullet)).$$

Let's write explicitly the double complex to understand what's happening. We have (recall that $X^\bullet$ is an element of $D^+(\mathcal{C}^{op})$ and will be seen later as an element of $D^-(\mathcal{C})$ via $X^\bullet \to X^{-\bullet}$) :

$$\cdots \leftarrow X^{-1} \leftarrow X^0 \leftarrow X^1 \leftarrow \cdots$$
$$\cdots \to I^{-1} \to I^0 \to I^1 \to \cdots$$

so $\operatorname{Hom}(X^\bullet, I^\bullet)$ is

$$\begin{array}{ccccccc}
 & \dots & & \dots & & \dots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
\dots \longrightarrow & \operatorname{Hom}(X^{-1}, I^1) & \longrightarrow & \operatorname{Hom}(X^0, I^1) & \longrightarrow & \operatorname{Hom}(X^1, I^1) & \longrightarrow \dots \\
 & \uparrow & & \uparrow & & \uparrow & \\
\dots \longrightarrow & \operatorname{Hom}(X^{-1}, I^0) & \longrightarrow & \operatorname{Hom}(X^0, I^0) & \longrightarrow & \operatorname{Hom}(X^1, I^0) & \longrightarrow \dots \\
 & \uparrow & & \uparrow & & \uparrow & \\
\dots \longrightarrow & \operatorname{Hom}(X^{-1}, I^{-1}) & \longrightarrow & \operatorname{Hom}(X^0, I^{-1}) & \longrightarrow & \operatorname{Hom}(X^1, I^{-1}) & \longrightarrow \dots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \dots & & \dots & & \dots &
\end{array}$$

We want to understand the H^0 of the simple complex S associated to this double complex. We need to understand $\ker(d_S^0)$ and $\operatorname{im}(d_S^{-1})$. First

$$S^0 = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}(X^{-i}, I^i)$$
$$S^{-1} = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}(X^{-i-1}, I^i)$$
$$S^1 \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}(X^{-i+1}, I^i)$$

and

$$d_S(f_i)_{i\in\mathbb{Z}} = (d_I \ \circ f_{i-1} \ \pm f_i \circ d_X)_{i\in\mathbb{Z}}.$$

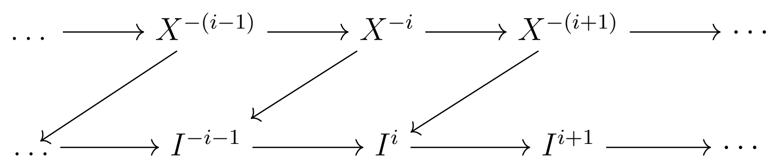
The proof is now an easy check : see $X^\bullet$ as an element of $D^-(\mathcal{C})$ by the identification $(X^i)_i \mapsto (X^{-i})_i$. Then an element of S^0 is a family of morphisms

$$\begin{array}{ccccccc}
\dots \longrightarrow & X^{-(i-1)} & \longrightarrow & X^{-i} & \longrightarrow & X^{-(i+1)} & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\dots \longrightarrow & I^{-i-1} & \longrightarrow & I^i & \longrightarrow & I^{i+1} & \longrightarrow \cdots .
\end{array}$$

An element of S^1 is a family of morphisms

$$\begin{array}{ccccccc}
\dots \longrightarrow & X^{-(i-1)} & \longrightarrow & X^{-i} & \longrightarrow & X^{-(i+1)} & \longrightarrow \cdots \\
 & & \searrow & & \searrow & & \searrow \\
\dots \longrightarrow & I^{-i-1} & \longrightarrow & I^i & \longrightarrow & I^{i+1} & \longrightarrow \cdots .
\end{array}$$

And an element of S^{-1} is a family of morphisms



The kernel of d_S^0 is composed of family of morphisms which commute with d i.e. of morphisms of chains from $X^{-\bullet}$ to $I^\bullet$. The image of d_S^{-1} are morphisms of chains homotopic to 0. So we have proved that

$$H^0 \circ R\mathrm{Hom}(X^\bullet, Y^\bullet) \simeq \mathrm{Hom}_{K(\mathcal{C})}(X^\bullet, I^\bullet) \simeq \mathrm{Hom}_{D(\mathcal{C})}(X^\bullet, Y^\bullet)$$

and this concludes the proof. $\square$

3 Operations on sheaves

3.1 The bifunctor $\mathcal{H}om$

In this section, we fix a field $\mathbf{k}$ so that the tensor product is exact. We will denote by Mod the category of $\mathbf{k}$-modules. For any manifold X, we denote by $\mathrm{Sh}(X)$ the category of sheaves of $\mathbf{k}$-modules on X and set $D^b(X) := D^b(\mathrm{Sh}(X))$. Let $\mathscr{F}$ and $\mathscr{G}$ be two elements of $\mathrm{Sh}(X)$. The sheaf $\mathcal{H}om_{\mathbf{k}}(\mathscr{F}, \mathscr{G})$ is defined by

$$\mathcal{H}om_{\mathbf{k}}(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}_{\mathrm{Sh}(U)}\left(\mathscr{F}|_U, \mathscr{G}|_U\right)$$

where $\mathrm{Hom}_{\mathrm{Sh}(U)}\left(\mathscr{F}|_U, \mathscr{G}|_U\right)$ is the $\mathbf{k}$-module of morphisms of sheaves between $\mathscr{F}|_U$ and $\mathscr{G}|_U$.

Proposition 1. *The bifunctor $\mathcal{H}om_{\mathbf{k}}(\cdot, \cdot)$ from $Sh(X)^o \times Sh(X)$ to $Sh(X)$ is left exact with respect to each of its arguments.*

The fact that $\mathcal{H}om_{\mathbf{k}}(\mathscr{F}, \mathscr{G})$ is indeed a sheaf is an easy check. We will see the left exactness later as a consequence of the $\otimes, \mathcal{H}om$ adjunction. What I really want to stress here is that

$$U \mapsto \mathrm{Hom}_{Mod}(F(U), G(U))$$

has no reason to be a presheaf except if F is flabby. There is no way to define natural restrictions maps

$$\mathrm{Hom}_{Mod}(F(U), G(U)) \to \mathrm{Hom}_{Mod}(F(V), G(V)) \text{ when } V \subset U.$$

One important is example is the following. Let Z be any open or closed set of X and $\mathscr{F} \in \mathrm{Sh}(X)$ then :

$$\mathcal{H}om_{\mathbf{k}}(k_Z, \mathscr{F}) \simeq \Gamma_Z \mathscr{F}. \tag{1}$$

Suppose first that Z is an open set. It is enough to check that for any open set U of X, we have $\Gamma(U; \mathcal{H}om(k_Z, \mathscr{F})) \simeq \Gamma(U; \Gamma_Z F)$. We are left to prove that $\mathrm{Hom}(k_{Z\cap U}, \mathscr{F}|_U) = F(Z \cap U)$. Without loss of generality, we can suppose $X = U$. Then we can also suppose $Z = X$ because :

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(k_Z, \mathscr{F}) \simeq \mathrm{Hom}_{\mathrm{Sh}(Z)}(k_Z|_Z, \mathscr{F}|_Z).$$

Indeed if we have a morphism of sheaves between $k_Z|_Z$ and $\mathscr{F}|_Z$, we can extend it to a morphism between k_Z and $\mathscr{F}$ because any section of k_Z is equal to zero near the boundary of Z.

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}(Z)}(k_Z, \mathscr{F}) &\simeq \mathrm{Hom}_{\mathrm{PSh}(Z)}(\check{k}_Z, \mathscr{F}) \\ &\simeq \{\phi_V : k \to F(V) \text{ with compatibility conditions}\}_{V\subset Z} \\ &\simeq \{\phi_V \in F(V) \text{ if } \text{ with compatibility conditions}\}_{V\subset Z} \\ &\simeq F(Z).\end{aligned}$$

where $\check{k}_Z$ designs the presheaf of constant functions with values in k. First equality is due to an adjunction formula between the forgetful functor $\mathrm{Sh}(Z) \to \mathrm{PSh}(Z)$ and the sheafification functor $\mathrm{PSH}(Z) \to \mathrm{Sh}(Z)$.

The case where Z is closed follows by applying the left exact functor $\mathcal{H}om_{\mathbf{k}}(\cdot, \mathscr{F})$ to the exact sequence $0 \to k_{X\setminus Z} \to k_X \to k_Z \to 0$ and comparing it with the exact sequence $0 \to \Gamma_Z(F) \to F \to \Gamma_{X\setminus Z}F$.

We can also prove a very useful result :

$$\mathcal{H}om_{\mathbf{k}}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \simeq \mathcal{H}om_{\mathbf{k}}(\mathscr{F}, \mathcal{H}om_{\mathbf{k}}(\mathscr{G}, \mathscr{H})). \tag{2}$$

Démonstration. We just need to prove that

$$\mathrm{Hom}_{\mathrm{Sh(X)}}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \simeq \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathscr{F}, \mathcal{H}om(\mathscr{G}, \mathscr{H})).$$

The right side is

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh(X)}}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) &\simeq \mathrm{Hom}_{\mathrm{PSh(X)}}(\mathscr{F} \check{\otimes} \mathscr{G}, \mathscr{H}) \\ &\simeq \{\phi_U : \mathscr{F}(U) \otimes \mathscr{G}(U) \to \mathscr{H}(U) \text{ with compatibility conditions}\}_{U\subset X} \\ &\simeq \{\phi_U : \mathscr{F}(U) \to \mathrm{Hom}_{\mathrm{Mod}}(\mathscr{G}(U), \mathscr{H}(U)) \text{ with compatibility conditions}\}_{U\subset X}\end{aligned}$$

and the left side is

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathscr{F}, \mathcal{H}om(\mathscr{G}, \mathscr{H})) \simeq \left\{\phi_U : \mathscr{F}(U) \to \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathscr{G}|_U, \mathscr{H}|_U) \text{ with compatibility conditions}\right\}_{U\subset X}$$

What we need to check is that giving ourselves a family of morphisms of modules from $\mathscr{F}(U)$ to $\mathrm{Hom}_{\mathrm{Mod}}(\mathscr{G}(U), \mathscr{H}(U))$ with some compatibility conditions is the same as giving ourselves a family of morphism from $\mathscr{F}(U)$ to $\mathrm{Hom}_{\mathrm{Sh}(X)}((\mathscr{G}|_U, \mathscr{H}|_U)$ with compatibility conditions. This is true even though $\mathrm{Hom}_{\mathrm{Sh}(X)}((\mathscr{G}|_U, \mathscr{H}|_U) \neq \mathrm{Hom}_{\mathrm{Mod}}(\mathscr{G}(U), \mathscr{H}(U))$. □

The adjunction formula between f^{-1} and f_* can be rewritten using $\mathcal{H}om$. Indeed, if we take $f : Y \to X$ a continuous map, $F \in D^b(X)$ and $G \in D^b(Y)$ then :

$$f_* \mathcal{H}om_{k_Y}(f^{-1}F, G) \simeq \mathcal{H}om_{k_X}(F, f_*G)$$

Démonstration.

$$\begin{aligned}
\Gamma(U; f_* \mathcal{H}om_{k_Y}(f^{-1}F, G)) &= \Gamma(f^{-1}(U); \mathcal{H}om_{k_X}(f^{-1}F, G)) \\
&= \mathrm{Hom}_{\mathrm{Sh}(f^{-1}(U))}(f^{-1}F\big|_{f^{-1}(U)}, G|_{f^{-1}(U)}) \\
&= \mathrm{Hom}_{\mathrm{Sh}(U)}((F|_U, f_*G|_U) \\
&= \Gamma(U; \mathcal{H}om_{k_X}(F, f_*G)).
\end{aligned}$$

We have just applied the adjunction formula Patrick proved for $f : f^{-1}(U) \to U$. □

3.2 Some reminders and useful formulas

A subcategory $\mathcal{R} \subset \mathcal{C}$ is adapted to a functor F (or is said to be F-injective) if :

(i) for any $X \in \mathrm{Ob}(\mathcal{C})$, there exists $X' \in \mathrm{Ob}(\mathcal{R})$ and an exact sequence $0 \to X \to X'$.

(ii) F is exact on $\mathcal{R}$.

(iii) If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{C}$ with X' and X in $\mathcal{R}$ then X'' is in $\mathcal{R}$.

Theorem 4. *Let $f : Y \to X$ be a continuous map.*

(1) Flabby sheaves are adapted to $f_\star$ and if F is flabby then $f_\star(F)$ is flabby.

(2) Flabby sheaves are adapted to Γ_Z and $\Gamma(X, \cdot)$ and if F is flabby then $\Gamma_Z f$ is flabby.

(3) c-soft sheaves are adapted to $f_!$ and if F is c-soft then $f_! F$ is c-soft.

(4) If I is an injective sheaf and F any sheaf then $\mathcal{H}om(F, I)$ is an injective sheaf. (Sh(X), $\mathcal{I}$) is $\mathcal{H}om$-injective where $\mathcal{I}$ designs injective sheaves.

This theorem could be simplified : if you don't want to compute explicitly any derived functor, you just need to say that all these functors map injective objects to injective objects and that injective objects are adapted to all these functors. As you can see, with this result, Grothendieck's composition theorem becomes easy to apply. But injective resolutions are not easy to find. This is why we use other classes of adapted objects.

Démonstration. We already saw (1), (2) and (3) during Patrick's and Vincent's lectures so there won't be any proofs of these results. Let's prove the (4). We recall the definition of injectivity : I is injective if and only if $\mathrm{Hom}_{\mathrm{Sh}(X)}(\cdot, I)$ is an exact functor. As

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\cdot, \mathcal{H}om(F, I)) = \mathrm{Hom}_{\mathrm{Sh}(X)}(\cdot \otimes F, I)$$

is the composition of two exact functor, $\mathcal{H}om(F, I)$ is an injective sheaf. As $\mathrm{Hom}_{\mathrm{Sh}(X)}(\cdot, I)$ is exact and the restriction of an injective sheaf to an open set is still an injective sheaf, we know that $\mathcal{H}om(\cdot, I)$ is an exact functor. Take now $F \in \mathrm{Sh}(X)$, as $\mathcal{H}om(F, \cdot)$ is left exact, injective sheaves are adapted to $\mathcal{H}om(F, \cdot)$. This concludes the proof. □

What we should now remember is that if we want to compute $R\mathcal{H}om(\mathscr{F}^\bullet, \mathscr{G}^\bullet)$, we just need to replace $\mathscr{G}^\bullet$ by a complex of injectives sheaves $I^\bullet$ and then

$$\text{If } \mathscr{G}^\bullet \simeq I^\bullet, R\mathcal{H}om(\mathscr{F}^\bullet, \mathscr{G}^\bullet) \simeq s\left(\mathcal{H}om(\mathscr{F}^\bullet, I^\bullet)\right)$$

We will derive now some formulas we have proved for the category $\mathrm{Sh}(X)$. As we said previously, the Grothendieck composition theorem has easy to verify hypotheses and we won't do any proofs. We take $f : Y \to X$ a continuous map, $F \in D^b(X)$, $G \in D^b(Y)$, $Z \subset X$ a closed or an open set of X and F', F'' in $D^b(X)$.

$$Rf_* R\mathcal{H}om_{D^b(Y)}(f^{-1}F, G) \simeq R\mathcal{H}om_{D^b(X)}(F, Rf_* G).$$

$$R\mathcal{H}om(k_Z, F) \simeq R\Gamma_Z F.$$

$$R\mathcal{H}om(F \otimes F', F'') \simeq R\mathcal{H}om(F, R\mathcal{H}om(F', F'')).$$

4 Poincaré-Verdier duality for closed or open subset

In this section, $Z \subset X$ will be either an open or a closed subset of X [2] and j will be the inclusion map. In this case, we can compute explicitly what $j_!$, j_* and j^{-1} are. Indeed we have :

Lemme 2. *Let $\mathscr{F}$ ∈Sh(Z). Then we have :*

$$j_! \mathscr{F}(U) = \{s \in \mathscr{F}(U \cap Z), \mathit{supp}(s) \text{ is closed in } U \subset X\}.$$

This comes from the trivial fact that $j : \mathrm{supp}(s) \to U$ is proper if and only if $\mathrm{supp}(s)$ is closed in U.

Lemme 3. *Take $\mathscr{F} \in Sh(Z)$ and j as before. Then*

$$(j_! \mathscr{F})_x = \begin{cases} \mathscr{F}_x \text{ if } x \in Z \\ 0 \text{ otherwise.} \end{cases}$$

Démonstration. Let's prove the fact that $(j_!\mathscr{F})_x = 0$ if $x \notin Z$. This is clearly true if you take $x \notin \bar{Z}$ If you take $x \in \partial Z \setminus Z$, U a neighbourhood of x and s a section of $U \cap Z$ with closed support in U, you can see that you have to vanish near x. Otherwise x would be in the support of s because the support of s is closed in U [3].

2. every proof we will give in this section works also for locally closed subset i.e. sets that can be written $U \cap F$ where U is open and F is closed.

3. you can check that for any sheaf $\mathscr{F}$ and for any section $s \in \mathscr{F}(U)$, $\mathrm{supp}(s)$ is closed in U.

But $x \notin U \cap Z$ and this is a contradiction.

Take now $x \in Z$ and we want to prove that $(j_!\mathscr{F})_x = \mathscr{F}_x$. If we take U a small neighborhood of $x \in X$ then we have $\mathscr{F}(U \cap Z) \simeq j_!\mathscr{F}(U)$. Indeed every section s of $\mathscr{F}(U \cap Z)$ will have closed support in U. If Z is closed, the support of s being closed in $U \cap Z$ will be also closed in U. If Z is open, you just have to take U small enough such that $U \cap Z = U$. □

Denote by $\mathrm{Sh}_Z(X)$ the full subcategory of sheaves on X vanishing on $X \setminus Z$: a sheaf $\mathscr{F}$ of $\mathrm{Sh}_Z(X)$ verifies $\mathscr{F}_x = 0$ for all $x \notin Z$.

Lemme 4. *We have an equivalence of category :*

$$j_! : Sh(Z) \to Sh_Z(X).$$

The inverse is functor is the one induced by j^{-1}.

That means that for any sheaf $\mathscr{G} \in \mathrm{Sh}(Z)$, $j^{-1}j_!\mathscr{G} \simeq \mathscr{G}$ and for any sheaf $\mathscr{F} \in \mathrm{Sh}_Z(X), j_!j^{-1}\mathscr{F} \simeq \mathscr{F}$. The first equality is easy to check. We have a natural morphism of $j_!\mathscr{G} \to j_*\mathscr{G}$ so with the adjunction map we get natural maps :

$$j^{-1}j_!\mathscr{G} \to j^{-1}j_*\mathscr{G} \to \mathscr{G}.$$

And, as $(j^{-1}j_!\mathscr{G})_x = \mathscr{G}_x$ and as everything is canonical, this cannot fail to give us what we want. The second equality is a little trickier to see. Indeed we have the following diagram :

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & j_*j^{-1}\mathscr{F} \\ & & \uparrow \\ & & j_!j^{-1}\mathscr{F} \end{array}$$

and we would like to factorize it by $\mathscr{F} \to j_!j^{-1}\mathscr{F}$. What we need to check is that the image of a section $s \in \mathscr{F}(U)$ seen as an element of $\Gamma(U \cap Z, j^{-1}\mathscr{F})$ has closed support in $U \cap Z \subset X$. (Recall that

$$j_!j^{-1}\mathscr{F}(U) = \{s \in (j^{-1}\mathscr{F})(U \cap Z), \mathrm{supp}(s) \text{ is closed in } U\}.)$$

But we know that s has closed support in U and that s vanishes on $X \setminus Z$ so nothing bad can happen and we get our factorization. It is a trivial check to see that we have the good stalks so we have $\mathscr{F} \simeq j_!j^{-1}\mathscr{F}$.

We have now done all the preliminaries to show that $j_!$ has a right adjoint functor :

Theorem 5. *Define $j^!\mathscr{F} := j^{-1}\Gamma_Z\mathscr{F}$. Then for all $\mathscr{G} \in Sh(Z)$ and $\mathscr{F} \in Sh(X)$, we have the following adjunction formula :*

$$Hom_{Sh(X)}(j_!\mathscr{G}, \mathscr{F}) \simeq Hom_{Sh(Z)}(\mathscr{G}, j^!\mathscr{F}).$$

Démonstration. We must first note that is $\mathscr{H} \in \mathrm{Sh}_Z(X)$ then $\mathscr{H} \otimes k_Z = \mathscr{H}$ as they have the same stalks. So we have :

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sh}(X)}(j_!\mathscr{G}, \mathscr{F}) &\simeq \mathrm{Hom}_{\mathrm{Sh}(X)}(j_!\mathscr{G} \otimes k_Z, \mathscr{F}) \\
&\simeq \mathrm{Hom}_{\mathrm{Sh}(X)}(j_!\mathscr{G}, \mathcal{H}om(k_Z, \mathscr{F})) && \text{(cf (2))} \\
&\simeq \mathrm{Hom}_{\mathrm{Sh}(X)}(j_!\mathscr{G}, \Gamma_Z\mathscr{F}) && \text{(cf (1))} \\
&\simeq \mathrm{Hom}_{\mathrm{Sh}(Z)}(j^{-1}j_!\mathscr{G}, j^{-1}\Gamma_Z\mathscr{F}) && \text{(cf lemme 4)} \\
&\simeq \mathrm{Hom}_{\mathrm{Sh}(Z)}(\mathscr{G}, j^!\mathscr{F}). && \text{(cf lemme 4)}
\end{aligned}$$

And this concludes the proof. □

5 Poincaré-Verdier duality

We have seen in the previous section a very special case of Poincaré-Verdier duality. We will now admit the general construction of $f^!$ and see what we can deduce from there.

Theorem 6. *Let $f : Y \to X$ be a continuous map of locally compact spaces such that $f_!$ has finite cohomological dimension*[4]*. Then there exists a functor $f^! : D^b(X) \to D^b(Y)$ and an isomorphism of bifunctors on $D^b(Y)^o \times D^b(X)$*

$$Hom_{D^b(X)}(Rf_!(\cdot), \cdot) \simeq Hom_{D^b(Y)}(\cdot, f^!(\cdot))$$

The preceding adjunction formula is not nice because we don't like the bifunctor $\mathrm{Hom}_{D^b(X)}$. The useful analogous of this bifunctor is the bifunctor $R\mathcal{H}om$ and we have the following formula :

$$\mathrm{Hom}_{D^b(X)}(\cdot, \cdot) = H^0 \circ R\Gamma(X, \cdot) \circ R\mathcal{H}om(\cdot, \cdot).$$

So we will quickly forget the last theorem and always use this one instead :

Theorem 7. *Let $f : Y \to X$ be a continuous map of locally compact spaces such that $f_!$ has finite cohomological dimension. Then for $F \in D^b(X)$ and $G \in D^b(Y)$, we have :*

$$R\mathcal{H}om_X(Rf_!G, F) \simeq Rf_*R\mathcal{H}om_Y(G, f^!F)$$

Démonstration. Let's give a proof of this theorem using the previous one. Let H be in $D^b(X)$. Then we have :

$$\begin{aligned}
\mathrm{Hom}_{D^b(X)}(H, Rf_\star R\mathcal{H}om_Y(G, f^!F)) &\simeq \mathrm{Hom}_{D^b(Y)}(f^{-1}H, R\mathcal{H}om_Y(G, f^!F)) \\
&\simeq \mathrm{Hom}_{D^b(Y)}(f^{-1}H \otimes G, f^!F)) \\
&\simeq \mathrm{Hom}_{D^b(X)}(Rf_!(f^{-1}H \otimes G), F) \\
&\simeq \mathrm{Hom}_{D^b(X)}(H \otimes Rf_!G, F) \\
&\simeq \mathrm{Hom}_{D^b(X)}(H, R\mathcal{H}om_Y(Rf_!G, F))
\end{aligned}$$

4. We say that $f_!$ has finite cohomological dimension if there exists an integer $r > 0$ such that $R^j f_!(X) = 0$ for any $j > r$ and any $X \in \mathrm{Sh}(X)$ seen as an element of $D^+(X)$

As this is true for all $H \in D^b(X)$, we have finished the proof. $\square$

We shall state a theorem assuring us that $f_!$ will always have finite cohomological dimension in our cases.

Theorem 8. *Let $f : Y \to X$ be a continuous function where Y is a topological manifold of dimension n. Then $f_!$ has finite cohomological dimension. We even have $R^j f_!(F) = 0$ for all $j > n+1$ and all $F \in Sh(Y)$.*

Démonstration. The proof of this statement uses the following lemma. See the propositions 3.2.2 and 3.3.11 in [3] for a proof of this lemma.

Lemme 5. *Let Y be a n-dimensional C^0-manifold and let $F \in Sh(Y)$. Then :*
(i) F admits a resolution of length at most n by c-soft sheaves.
(ii) F admits a resolution of length at most $n+1$ by flabby sheaves.
(iii) F admits a resolution of length at most $3n+1$ by injective sheaves

It is then easy to deduce our theorem : just take a resolution

$$0 \to F \to G_0 \to G_1 \to \cdots \to G_n \to 0$$

of F by c-soft sheaves. Then

$$Rf_!(F) \simeq 0 \to F(G_0) \to F(G_1) \to \cdots \to F(G_n) \to 0$$

and $R^{n+1} f_!(F) = 0$. $\square$

We will now see why this duality is called Poincaré-Verdier duality. Take $f : X \to \{pt\}$ and define

$$\omega_X = f^! \mathbb{Z}_{\{pt\}}$$

Proposition 2. *If X is a manifold of dimension n, then*

$$\omega_X \simeq or_X[n]$$

Démonstration. Let x be a point of X and U a neighborhood of x diffeomorphic to a n-dimensional ball. We have :

$$\begin{aligned} R\Gamma(U, \omega_X) &\simeq R\Gamma(X, \cdot) \circ R\Gamma_U(\omega_X) \\ &\simeq R\Gamma(\{pt\}, \cdot) \circ Rf_\star \circ R\mathcal{H}om_{\mathbb{Z}_X}(\mathbb{Z}_U, \omega_X) \\ &\simeq R\Gamma(\{pt\}, \cdot) \circ R\mathcal{H}om_{\mathbb{Z}_{\{pt\}}}(Rf_! \mathbb{Z}_U, \mathbb{Z}_{\{pt\}}) \\ &\simeq R\mathrm{Hom}(R\Gamma_c(U; \mathbb{Z}_U), \mathbb{Z}) \end{aligned}$$

But we know a flabby resolution of $\mathbb{Z}_U$ given by Cech cochains since Anne's lecture. So we are computing the cohomology with compact support of the ball. So we know that given an orientation on U, $R\Gamma_c(U, \mathbb{Z}_U) \simeq \mathbb{Z}[-n]$. This concludes the proof. [5] $\square$

5. or_X is the sheaf associated to the presheaf $U \mapsto \mathrm{Hom}(H^n_c(U; k_X), k)$

Theorem 9 (Poincaré duality). *Let X be a topological manifold of dimension n. For all i, we have :*

$$H_c^i(X, \mathbb{Q}_X)^* \simeq H^{n-i}(X; or_X)$$

Démonstration. Take as before $f : X \to \{pt\}$ the projection. We take this time ω_X as $f^!\mathbb{Q}_{\{pt\}}$. We have :

$$\begin{aligned} H^{n-i}(X; or_X) &\simeq H^{-i}(X, \omega_X) \simeq H^{-i}R\Gamma(X, \omega_X) \\ &\simeq H^{-i}R\mathrm{Hom}_{\{pt\}}(R\Gamma_c(X, \mathbb{Q}_X), \mathbb{Q}) \end{aligned}$$

Here we take the derived functor of Hom on sheaves of vector space on one point i.e. of vector spaces. But Hom is exact on vector space so we have :

$$H^{n-i}(X; or_X) \simeq H^{-i}R\Gamma_c(X, \mathbb{Q}_X)^* \simeq (H^i R\Gamma_c(X, \mathbb{Q}_X))^* \simeq H_c^i(X, \mathbb{Q}_X)^*$$

□

We can also compute $f^!$ when f is a fibration. The following proposition is an analogous of proposition 2 and will follow exactly the same proof.

Proposition 3. *Let $f : Y \to X$ be a fibration such that the fibers are l-dimensional manifolds. Then for all $X \in D^b(X)$, we have :*

$$f^!G \simeq f^{-1}G \otimes or_{X|Y}[l]$$

where $or_{X|Y}$ is the relative orientation sheaf (its restriction to each fiber is the orientation sheaf of the fiber).

Démonstration. Take an open set of Y of the form $U \times V$ such that $f|_{U\times V}$ is the second projection and such that U is a l-dimensional ball. Then

$$\begin{aligned} R\Gamma(U \times V; f^!G) &\simeq R\Gamma(Y, \cdot) \circ R\mathcal{H}om(k_{U\times V}, f^!G) \\ &\simeq R\Gamma(X, \cdot) \circ Rf_\star \circ R\mathcal{H}om(k_{U\times V}, f^!G) \\ &\simeq R\mathrm{Hom}(Rf_!k_{U\times V}, G) \end{aligned}$$

We know that on V, f is the identity and on U, f is the projection on one point. So $Rf_!k_{U\times V}$ must be $R\Gamma_c(U, k_U) \times k_V$. If you want to convince yourself of that fact, you can take the usual flabby resolution of $k_{U\times V}$ and check. So we can continue our computation :

$$\begin{aligned} R\Gamma(U \times V; f^!G) &\simeq R\mathrm{Hom}(R\Gamma_c(U; k_U) \otimes k_V, G) \\ &\simeq R\mathrm{Hom}(R\Gamma_c(U; k_U), k) \otimes R\mathrm{Hom}(k_V, G) \\ &\simeq R\Gamma(U; \omega_U) \otimes R\Gamma(V, G) \\ &\simeq R\Gamma(U \times V; or_{X|Y}[l]) \otimes R\Gamma(U \times V; f^{-1}G) \end{aligned}$$

□

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