CONTACT TAUTOLOGIES

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ABSTRACT. Arnold conjecture form the conical view point.

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Tautologie de Contact. EF 15 nov 2013

CONTACT TAUTOLOGY

~~and the "conic" version of~~ Arnold's conjecture from the conical view point

I) Tautologies
II) Conical statements
III) Idea of proof, generating hypersurfaces
IV) Persitence of GFQI
V) Generating objects

I) T^*M, CM, J^1M.

M $\mathcal{C}^\infty$ manifold $\quad$ T^*M its cotangent bundle

(q,p) canonical coordinates $\quad$ $\lambda =$ "$p\,dq$" Liouville form

Goal / Motivation: Arnold's Lagrangian intersection conjecture:

$L \hookrightarrow T^*M$ Hamiltonian-isotopic to the zero section O_M

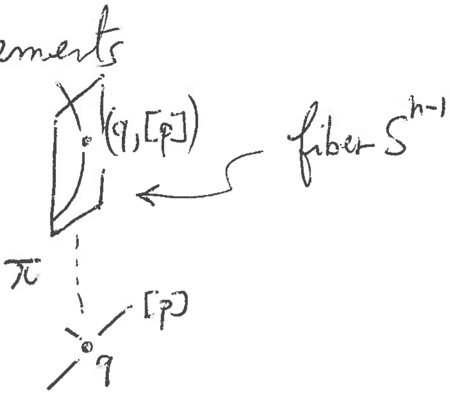
$\Rightarrow \#\{L \cap O_M\} \geqslant \text{Morse}(M)$ (with multiplicities in not $\pitchfork$)

(M is compact)

Motivation: Morse theory.

CM = mfld of cooriented contact elements
$\ni (q,[p])$

$$CM = \frac{T^*M \setminus 0_M}{\mathbb{R}^{>0}} \quad \xrightarrow{\pi} M$$



tautological contact structure:

$$\underset{\displaystyle T_{(q,[p])}(CM)}{\xi_{(q,[p])} \cap} = D\pi_{(q,[p])}^{-1} \underset{\displaystyle T_qM}{([p] \cap)}$$

Rem: $CM \cong T_1M$ (using Riemannian metric) $\cong \{\|p\|=1\} \subset T^*M$

$\xi = \ker \alpha \quad \alpha = pdq$

Rem: Contact manifold := localy like CM
(odd dimensional)

local Model: $M \cong U \subset \mathbb{R}^n \quad q = (q_1, \dots, q_n)$

choose one fixed direction: $\frac{\partial}{\partial q_n}$

$\mathcal{U} \subset CU \quad \mathcal{U} = \{ [p] \pitchfork \frac{\partial}{\partial q_n}, \text{ positively} \} \quad (p \cdot \frac{\partial}{\partial q_n} > 0)$

$[p] = [p_1, \dots, p_{n-1}, 1] \quad \alpha = dq_n + \sum_1^{n-1} p_i dq_i$

this looks like....

$$\begin{array}{l} J^1M = \underset{(q,p)}{T^*M} \times \underset{u}{\mathbb{R}} \\ \downarrow \\ T^*M \\ \vdots \\ \downarrow \\ CM \end{array} \quad , \quad \begin{array}{l} \alpha = du - p\,dq \\ \xi = \ker\alpha \end{array} \quad \subset \; C(M\times\mathbb{R})$$

open set in...

Def: legendrian submanifold : integral submfld of ξ, of maximal dimension.

Example: $f : M \to \mathbb{R}$ graph of df Lagrangian $\subset T^*M$

$$j^1 f \subset J^1M$$
$$\| $$
$$\left\{ q, p = \frac{\partial f}{\partial q}(q), \; u = f(q) \right\}$$

Example: $N \subset M$ submanifold

$\nu N =$ spherization of its conormal bundle

$\nu N = \{ (q, [p]) \;/\; q \in N, \; T_qN \subset [p] \}$

$(p_{|T_qN} = 0)$

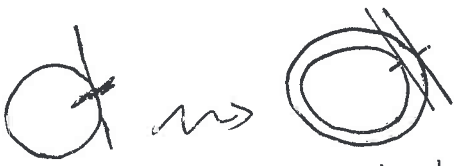
. observe $\dim \nu N = n-1$ (if $n = \dim M$)

whatever $\dim N$ is.

$N = 1$ points $\qquad \nu N$ = fiber = sphere.

N = knot $\subset \mathbb{R}^3 \qquad \nu N$ = Legendrian torus in $\mathbb{R}^3 \times S^2$

N = hypersurface $\qquad \nu N$ double cover of N



small perturb.

$\nu N = \nu^+ N \cup \nu^- N$ (if N cooriented)

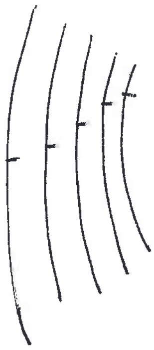
<u>Def</u> Front of L legendrian $\subset CM \xrightarrow{\pi} M$

$\hat{\pi}(L) :=$

Typical front if $\dim M = 2$: 

<u>Def:</u> "Wall": assume M open, $\varphi : M \to \mathbb{R}$ w.o. critical pts

$$W_\varphi = \bigcup_{t \in \mathbb{R}} \nu^+(\varphi^{-1}(t)) = \{(q, [d\varphi(q)]), q \in M\}$$
$$\cap$$
$$CM$$

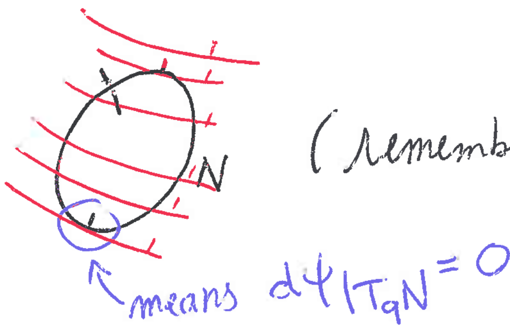


II Conical formulation version of Arnold's intersection conjecture.

Morse theory: $\overset{\text{cpct}}{N} \hookrightarrow \overset{\text{open}}{M} \xrightarrow[\psi]{} \mathbb{R}$

$$\#\{\nu N \cap W_\psi\} \geq \mathrm{Morse}(N) \geq \dots$$

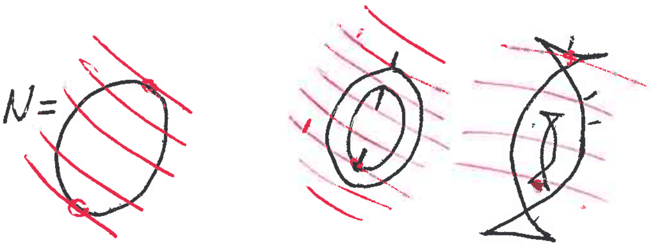
in one-one correspondance with critical pts of $\psi|N$

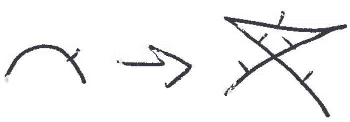


(remember Lagrange multipliers)

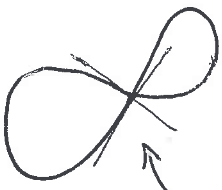
<u>THM 1</u> L_t, $t \in [0,1]$ one parameter familly of Legendrian embeddings s.t $L_0 = \nu N$ for some cpct N.

then: $\#\{L_1 \cap W_\psi\} \geq \mathrm{StabMorse}(N) \geq \dots > 0$

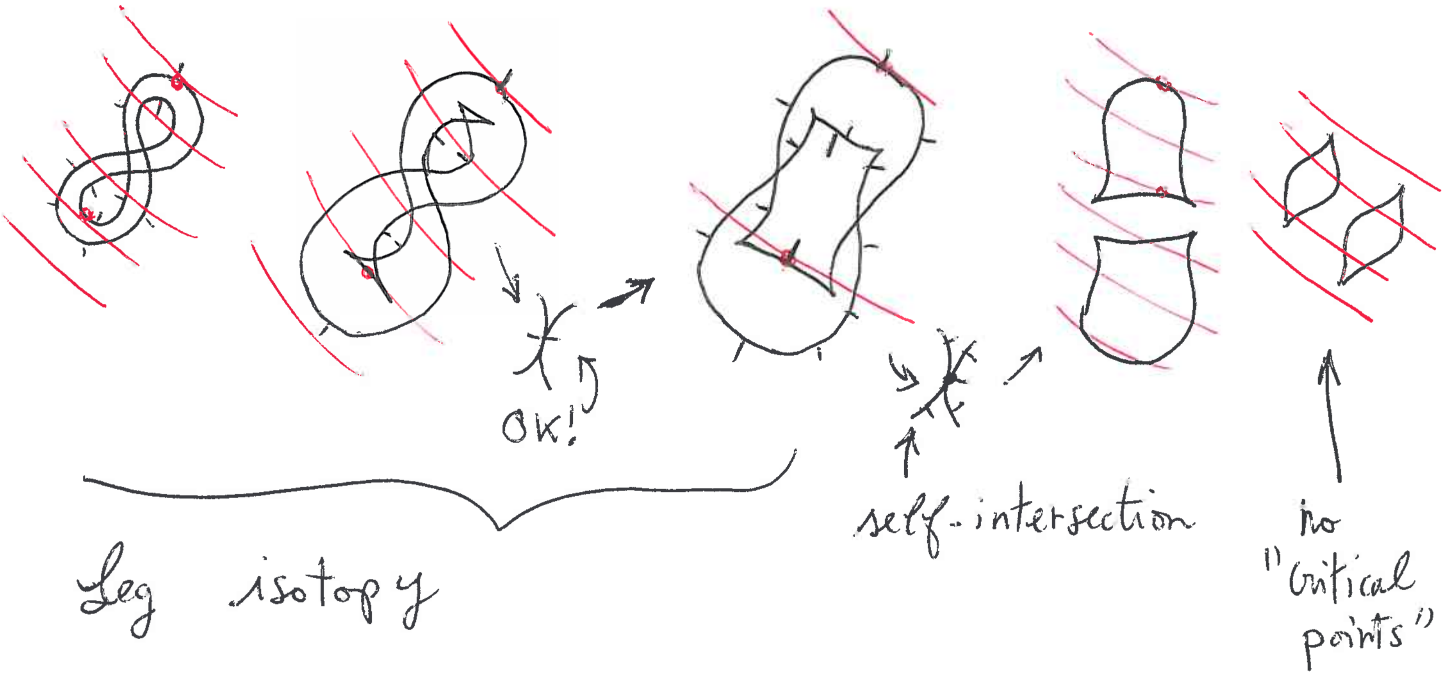
Example: $N =$ 



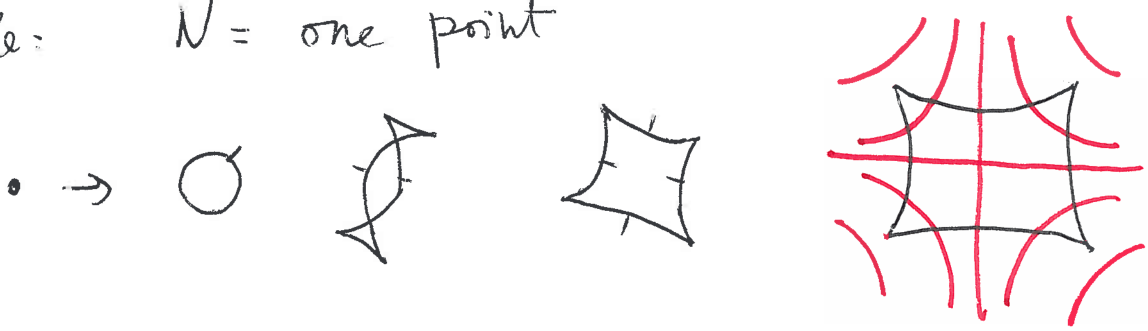
Example: Remark that νN is well defined (and generically embedded) when N is an immersed submanifold.



νN is embedded above this point.



Example: N = one point

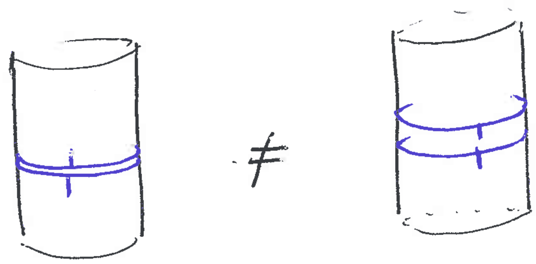


$\rightsquigarrow$ the hypothesis the ψ has no critical pt is needed!

Rem: $N = S^1$ $\nu N \subset S^1 \times S^1 \times \mathbb{R}$ is a 2-comp. link $\nu^+ N \cup \nu^- N$

this argument shows that this link is not Legendrian equivalent to $\nu^- N \cup \nu^-_\varepsilon N$

small shift of $\nu^- N$



which has no intersection with W_ϕ

but those 2 links have the same "classical invariants" (i.e. same framed knot type and same Legendrian immersion class)

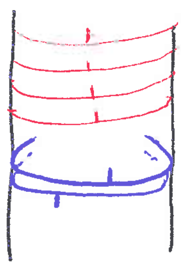
THM 2 $L_t,\ t\in[0,1]$ 1-param family of Leg embeddings in J^1N st $L_0=\{u=0,\ p=0\}=0_N$
cpct.

then $\#\{L_1\cap\{p=0\}\}\geq \mathrm{StabMorse}(M)\geq\dots>0$

Rem:
- this implies a weak form of Arnold's conjecture in T^*M because an Hamiltonian isotopy of 0_M can be lifted in J^1
- this is slightly more general: Self-intersections may appear in the Lagrangian projection of L_t

THM 1 ⇒ THM 2: $M=N\times\mathbb{R}$ $J^1N\subset CM$

$\varphi: M\to\mathbb{R}$
$(q,t)\mapsto t$

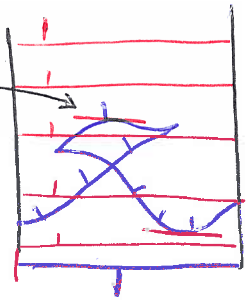


$W_\varphi=\{p=0\}$

$\nu N=\nu^+N\cup\nu^-N$

$L_0=\nu^+N$

$p=0$
$\Leftrightarrow$
horizontal tangent



$L_t\cup\nu^-N=$ deformation of νN through leg. embeddings

$\nu^-N\cap W_\varphi=\emptyset$

III Proof of THM 1

We will use the so called "Chekanov thm", which immediatly implies THM 2 as well (Chekanov's proof of it)
The only interest in explaining things in this way is to Emphasize the "conical version".
We will se ~~below~~ later that, combined with the notion of "generating hypersurface", this conical version leads to some interesting applications.

~~THM 3 ("Chekanov's thm") [See [Fe] for more references and historical remarks]~~

<u>Def</u> (FGQI) Consider $F : M \times \mathbb{R}^{2k} \to \mathbb{R}$
$$(q, w) \longmapsto F(q,w)$$

s.t. $F(q,w) = Q(w)$ Non-degenerate quadratic form of signature (k,k)

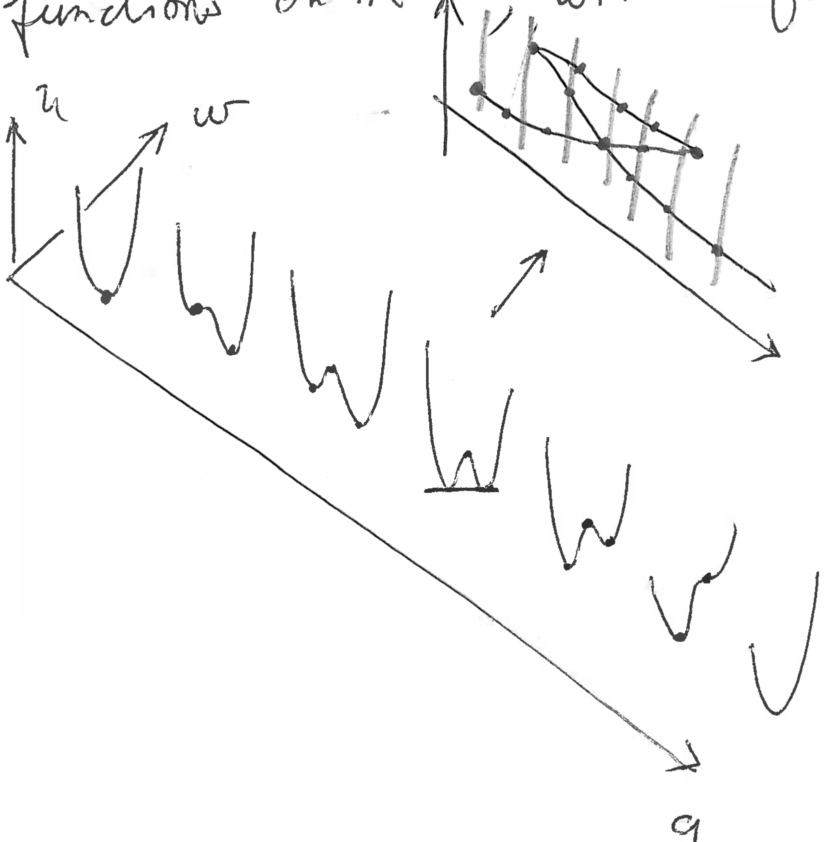
outside of a compact set.

the associated Legendrian submanifold $L_F \subset J^1M$ is

$$L_F = \left\{ (q,p,u) \mid \exists w, \frac{\partial F}{\partial w}(q,w) = 0, \; p = \frac{\partial F}{\partial q}(q,w) \right\}$$

We assume it is a $\pitchfork$ equation.

• the front of L_F ("forgetting p") is the bifurcation diagram of the critical values of a family of functions on $\mathbb{R}^{2k}$, with a fixed behavior at ∞



• L_F is obtained by contact reduction from $j^1F \subset J^1(M \times \mathbb{R}^{2k})$
if (q, w, p, r, u) are canonical coordinates on $J^1(M \times \mathbb{R}^{2k})$: $\alpha = du - p\,dq - r\,dw$
let $\mathcal{C} = \{r = 0\}$ (of codim $2k$). There is a natural projection $pr : \mathcal{C} \to J^1M$ ("forgetting w"), and $L_F = pr(j^1F \cap \mathcal{C})$

one can prove that, generically, L_F is an embedded Legendrian submanifold.

THM 3 "Chekanov's thm"

L_t, $t \in [0,1]$ 1-param family of Leg embeddings in J^1M with compact support. Assume $\exists F_0 : M \times \mathbb{R}^{2k} \to \mathbb{R}$ FGQI s.t. $L_0 = L_{F_0}$.
Then, $\exists F : M \times [0,1] \times \mathbb{R}^{2k'} \to \mathbb{R}$ 1-param family of FGQI ($k' \geq k$) t.q. $\forall t \in [0,1]$, $L_t = L_{F_t}$.

[the idea of the proof is given in the next section]

THM 3 $\Rightarrow$ THM 1 (change of notations: $L := L_1$ $L_1 := L_0$)

recall that we have $N \subset M$ $\psi : M \to \mathbb{R}$ $W_\psi \subset CM$
$L_t \subset CM$ s.t. $L_1 = \nu N$. One can assume that there exists ϕ_t, $t \in [0,1]$ 1-param family of contactomorphisms s.t. $L_t = \phi(L_1)$ ("isotopy extension Lemma", easy to prove with the notion of contact hamiltonian).
(we want ϕ_t brings L_1 to $L_1 = \nu N$): not the same notations as in the formulation of thm 1

We want to estimate $\#\{L_1 \cap W_\psi\}$. This is the same as

$$\#\{L_0 = \nu N = \phi_1(L_1) \cap \phi_1(W_\psi)\}$$

One can lift ϕ_t ~~as Hamiltonian isotopy~~ in $T^*M \setminus 0_M \cong \overset{\circ}{T}{}^*M$

graph of $d\psi$ $\rightarrow$

$$\begin{tikzcd} \Gamma_\psi \subset \dot{T}^*M \arrow[r, "\tilde{\Phi}_t"] \arrow[d] & \dot{T}^*M \arrow[d] \\ W_\psi \subset CM \arrow[r, "\Phi_t"'] & CM \end{tikzcd} \qquad \tilde{L}_t = \tilde{\Phi}_t(\Gamma_\psi)$$

$\tilde{\Phi}_t$ has not a compact support (conical!)
but $\hat{L}_t$ is a compactly supported deformation of Γ_ψ.

$\left(\hat{L}_t - \Gamma_\psi\right)_{t\in[0,1]}$ is a compactely supported hamiltonian deformation of the 0-section of T^*M.

By THM 3 (Sikorav version, for the symplectic case)

$\exists\ F: M\times\mathbb{R}^{2k} \to \mathbb{R}$, FGQI st.

$$\tilde{\Phi}_1(\Gamma_\psi) = \text{“}\Gamma_\psi + L_F\text{”}$$
$$= \left\{(q,p) \,/\, \exists w\ \frac{\partial F}{\partial w}(q,w) = 0,\ p = \frac{\partial\psi}{\partial q} + \frac{\partial F}{\partial q}(q,w)\right\}$$

$$\Phi_1(W_\psi)\cap\nu N \longleftrightarrow \tilde{\Phi}_1(\Gamma_\psi)\cap\nu^*N \text{ in } T^*M$$
$$\updownarrow$$
critical points of $\psi + F_{|N\times\mathbb{R}^{2k}}$

Using the same trick one can observe:

Proposition: M open, $\varphi : M \to \mathbb{R}$ non critical

$N = \varphi^{-1}(0)$ $\quad L_t$, $t \in [0,1]$ 1-param Legendrians

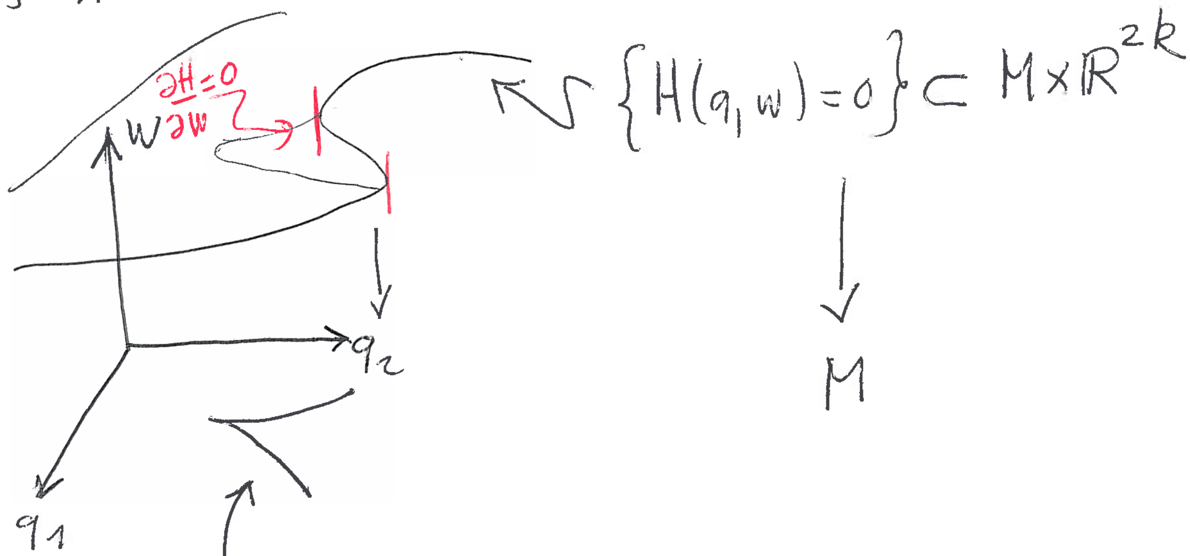
s.t. $L_0 = \nu^+ N$. then $\exists\ F : M \times \mathbb{R}^{2k}$ FGQI

s.t. $L_1 = \left\{ (q,[p]) \,/\, \exists w, \varphi(q)+F(q,w) = 0,\ \frac{\partial F}{\partial w}(q,w) = 0 \right.$
$\left. p = \frac{\partial F}{\partial q}(q,w) + \frac{\partial \varphi}{\partial q}(q) \right\}$

$$L_1 = \left\{ (q,[p]) \,/\, \exists w,\ H(q,w) = 0,\ \frac{\partial H}{\partial w}(q,w) = 0,\ p = \frac{\partial H}{\partial q}(q,w) \right\}$$

$$H(q,w) = \varphi(q) + F(q,w)$$

this means



$\pi(L_1)$ = front of L_1 = "Contour" of the hypersurface $\{H = 0\}$

IV) Persitence of GFQI (Proof of THM 3 : geometric veiwpoint)

for simplicity: $M = \mathbb{R}$ $\quad J^1M = \underset{(q,p,u)}{\mathbb{R}^3}$

Stabilization: $f : M \to \mathbb{R}$ $\quad j^1 f \subset J^1M$ its "one-graph"

Consider $F(q, w_1, w_2) = f(w_1) + w_2(q - w_1)$

- it is almost a FGQI (one can fix this easily)
- $L_F = j^1 f$: $\quad \dfrac{\partial F}{\partial w_2} = 0 \iff q = w_1$

$$\frac{\partial F}{\partial w_1} = 0 \iff w_2 = \frac{\partial f}{\partial q}(w_1) = \frac{\partial f}{\partial q}(q)$$

$$p = \frac{\partial F}{\partial q}(q, w_1, w_2) = \frac{\partial f}{\partial q}(q)$$

$$u = F(q, w_1, w_2) = f(w_1) = f(q)$$

Consider now a contactomorphism $\Phi : J^1M \to J^1M$ with compact support.

Claim: if Φ is close enough to the identity (*), then, for any affine function $g_{a,b} : q \mapsto aq + b$, $\Phi(j^1 g_{a,b})$ is still a graph, $\Phi(j^1 g_{a,b}) = j^1 G_{a,b}(q)$

Proof: there is a "compact choice" of affine functions on which Φ acts non-trivially.

(*) one can assume this: Cut Φ into small slices $\Phi = \Phi_1 \circ \Phi_2 \circ \dots \circ \Phi_K$

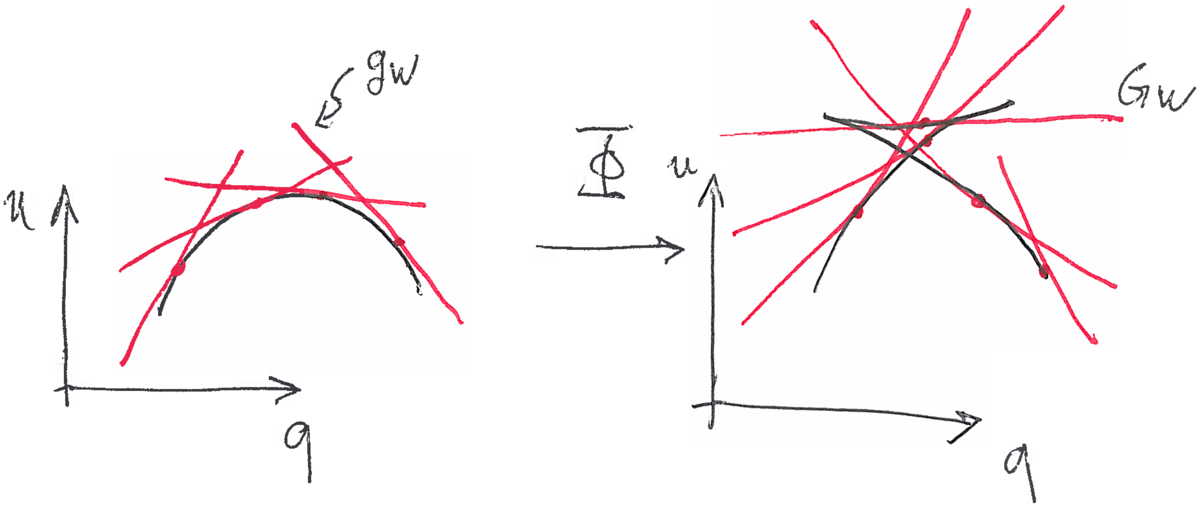
Enveloppes:

the formula $L_F = \{ u = F(q,w),\ p = \frac{\partial F}{\partial q}(q,w) \mid \exists\, w\ \frac{\partial F}{\partial w}(q,w) = 0 \}$
means that the front of $L_F = \{ (q,u),\ u = F(q,w),\ \frac{\partial F}{\partial w}(q,w) = 0 \}$
is the enveloppe of the family, parametrized by w,
of the smooth graphs of the functions

$$q \mapsto F(q,w)$$

for example, the graph of f is the enveloppe of the graphs of $g_w : q \mapsto f(w_1) + w_2(q - w_1)$

But a contact transformation "preserves the enveloppes"!



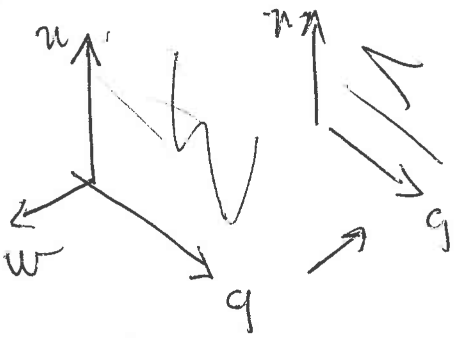
⚠ this is a schematic picture: w is 2-dimensional! Furthermore it is not true that a graph is the enveloppe of its tangents.

One can translate this idea into the language of differential calculus and prove that

$$\Phi(j^1 f) = L_G$$

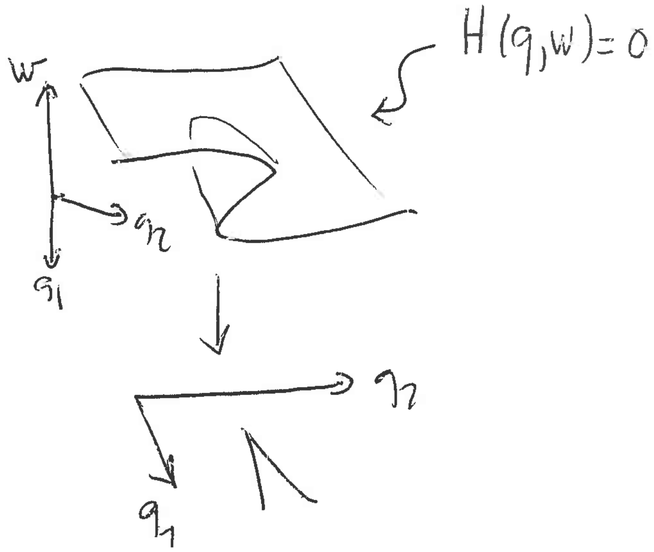
V) Generating things

Let's come back to this picture



Generating Functions

or



generating hyper-surfaces.

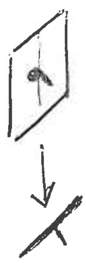
Consider a bundle

$$\begin{tikzcd} N \subset E \arrow[d, "pr"'] & CE \supset \mathcal{C} \arrow[l] \arrow[dl] \\ M & CM \arrow[l, "\pi"] \end{tikzcd}$$

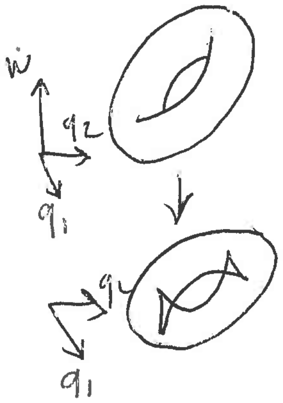
$CE \supset \nu N$

$\mathcal{C} = \{$ctct elements of E, which are tangent to the fiber $E \to M\}$

$\exists$ natural projection $pr : \mathcal{C} \to CM$



Prop $L_N = pr(\nu N \cap \mathcal{C})$ is generically an embedded Legendrian submanifold in CM

Examples: 

, Boy , Klein , etc...