Micro-support of sheaves in symplectic topology

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1 Some symplectic and contact tautology

1.1 Symplectic manifolds

A *symplectic form* (or symplectic structure) on a $2n$-dimensional manifold X is a 2-form ω such that $d\omega = 0$ and ω^n is a volume form.

Example 1.1 (Tautological structure on T^*N)**.** *Denote by* $\hat{\pi}_N \colon T^*N \to N$ *the cotangent bundle of* N*. On* T^*N *one has the Liouville 1-form* $\lambda_{(q,p)} = p \circ T_{(q,p)}\hat{\pi}_N$*. Its derivative* $\omega = d\lambda$ *is called the canonical symplectic structure on* N*.*

A n-dimensional submanifold L in X is called *Lagrangian* if the restriction of ω to L vanishes. If the symplectic form ω is exact and we fix a primitive λ then L is a Lagrangian submanifold if $\iota^*\lambda$ is closed. It is called *exact* if $\iota^*\lambda$ is exact.

Example 1.2. *The Liouville form has the tautological property that, for any 1-form* α *on* M*,* $\alpha^*\lambda = \alpha$*. In particular the graph of* α *in* T^*M *is Lagrangian (resp. exact Lagrangian) if and only if* α *is closed (resp. exact). Fibers of* T^*N *are also exact Lagrangian submanifolds but they are not compact.*

A *symplectomorphism* of X is a diffeomorphism preserving ω. A *symplectic isotopy* is a path $(\varphi_t)_{t\in[0,1]}$ of symplectomorphisms with $\varphi_0 = \mathrm{Id}$. It is *hamiltonian* if it "sweeps out cylinders with vanishing symplectic area": for every loop $\gamma\colon S^1 \to X$ and every $t \in [0,1]$, the cylindrical map $C_t(\gamma)\colon S^1 \times [0,t] \to X$ sending (θ, s) to $\varphi_s(\gamma(\theta))$ satisfies $\int C_t(\gamma)^*\omega = 0$. In that case there is a function $H\colon X \times I \to \mathbb{R}$ such that the vector field X_t generating the isotopy satisfies $\iota_{X_t}\omega = dH_t$. This function is unique up to addition of a function of t only. It is called a *hamiltonian function* generating φ. After choosing a base point x_0 in X one can define

$$K_t(x) = \frac{d}{dt}\int C_t(\alpha_x)^*\omega$$

where α_x is any path from x_0 to x and $C_t(\alpha_x)$ is the rectangle analogous to $C_t(\gamma)$ above. The condition on cylinders ensures that K_t is well-defined. Then $H_t = -K_t \circ \varphi_t^{-1}$ is a hamiltonian function generating φ.

Conversely, starting with a time-dependant function H_t, the condition $\omega^n \neq 0$ ensures the existence and uniqueness of a vector field X_t with $\iota_{X_t}\omega = dH_t$. The fact that ω is closed then ensures that X_t generates a hamiltonian isotopy (provided the isotopy exists, e.g. if X_t has compact support).

One of Arnold's conjectures in symplectic topology is that, for any closed manifold N and any hamiltonian isotopy φ in T^*N, $\varphi_t(0_N)$ intersects 0_N for all t. A (still open) conjecture of Arnold says that any closed exact Lagrangian submanifold of T^*N can be obtained as $\varphi_1(0_N)$ for some hamiltonian isotopy φ.

1.2 Contact manifolds

Given a cooriented hyperplane field ξ on a manifold V, one considers

$$S(V, \xi) = \{(x, p) \in T^*V;\ \ker p = \xi_x\}.$$

One says that ξ is a *contact structure* if the symplectic form of T^*V restricts to a symplectic form on $S(V, \xi)$. The symplectic manifold $S(V, \xi)$ is then called the *symplectization* of the contact manifold (V, ξ). It is a $\mathbb{R}_{>0}$-principal bundle over V: the action of a positive real number λ is $\lambda \cdot (q, p) = (q, \lambda p)$. A section of this bundle is called a contact form for ξ.

Example 1.3 (Tautological structure on $\mathcal{C}M$)**.** *Denote by $\pi_M \colon \mathcal{C}M \to M$ the bundle of cooriented contact elements of a manifold M, i.e. cooriented hyperplanes in TM. For any $(x, p) \in \dot{T}^*M := T^*M \setminus 0_M$, we will denote by $(x, [p])$ the corresponding element $\ker p$ of $\mathcal{C}M$. The tautological contact structure on $\mathcal{C}M$ is the hyperplane field ξ defined by $\xi_H := T_H\pi_M^{-1}(H)$. Its symplectization is isomorphic to $\dot{T}^*M$: the map $(x, p) \mapsto \big((x, [p]), p \circ T\pi_M\big)$ is a symplectomorphism from $\dot{T}^*M$ to $S(\mathcal{C}M, \xi)$.*

A *Legendrian* submanifold of a $(2n+1)$-dimensional contact manifold (V, ξ) is a n-dimensional submanifold L which is tangent to ξ.

Example 1.4. *(Variations on conormal bundles) We want to geometrically build Legendrian submanifolds of $\mathcal{C}M$. First one can start with a submanifold $Z \subset M$ and consider its conormal bundle*

$$\mathcal{N}Z = \{H \in \mathcal{C}M; \pi_M(H) \in Z, T_{\pi_M(H)}Z \subset H\}.$$

If Z is a cooriented hypersurface we can also consider $\mathcal{N}_+Z = \{T_zZ; z \in Z\}$ which is one of the two connected components of $\mathcal{N}Z$.

If $\psi\colon M \to \mathbb{R}$ is a function without critical point then it defines a wall in $\mathcal{C}M$ which is foliated by Legendrian submanifolds:

$$W_\psi = \{[d\psi(x)],\ x \in M\} = \bigsqcup_{t\in\mathbb{R}} \mathcal{N}_+(\psi^{-1}(t)).$$

A *contactomorphism* is a diffeomorphism of V preserving ξ. It lifts as a $\mathbb{R}_{>0}$-equivariant symplectomorphism of $S(V,\xi)$: $S\varphi(v,p) = \big(\varphi(v), p\circ(T_v\varphi)^{-1}\big)$.

A *contact isotopy* is a path $(\varphi_t)_{t\in[0,1]}$ of contactomorphisms with $\varphi_0 = \mathrm{Id}$. In contrast to the symplectic case, φ automatically has a canonical *hamiltonian function* which is defined on the symplectization $S(V,\xi)$ and is $\mathbb{R}_{>0}$-equivariant. If we denote by X_t the vector field generating φ then $H_t(v,p) = p(X_t(v))$. Conversely any $\mathbb{R}_{>0}$-equivariant function on $S(V,p)$ gives a contact isotopy (provided the relevant flow exists up to time one).

Contact Hamiltonians have two kinds of avatars. First one can simply projects X_t to a (time-dependant) section of TV/ξ which is a trivial line bundle. Less canonically one can choose a contact form α and get the function $H_t\circ\alpha$ which is defined on V itself and is often called the hamiltonian function of φ with respect to α. Of course α also gives a trivialization of TV/ξ and all three incarnations of contact hamiltonians are equivalent.

1.3 Contact lifts of exact symplectic objects

If $L \subset T^*N$ is any connected exact Lagrangian (i.e. $\iota\colon L \to T^*N$ satifies $\iota^*\lambda = df$) then it lifts as a Legendrian submanifold of $\mathcal{C}(N\times\mathbb{R})$

$$\hat{L} = \Big\{\big((q, -f(q,p_q)), [p_q, 1]\big) \in \mathcal{C}(N\times\mathbb{R});\ (q,p_q)\in L\Big\}.$$

In the above definition and elsewhere in this text, we use the canonical isomorphism $T^*\mathbb{R} \simeq \mathbb{R}\times\mathbb{R}$. Since f is well-defined up to addition of a constant, the lift $\hat{L}$ is well-defined up to "translation in the $\mathbb{R}$ direction".

Example 1.5. *If $\alpha = df$ then the graph of α lifts to $\{\big((q,-f(q)),[\alpha_q,1]\big)\}$. In particular the 0-section $0_N \subset T^*N$ (seen as the graph of the differential of the zero function) lifts to $\mathcal{N}_+(N\times\{0\})$. If we consider the projection $\psi\colon N\times\mathbb{R}\to\mathbb{R}$ then $\hat{L}\cap W_\psi$ corresponds bijectively to $L\cap 0_N$.*

Inside $\mathcal{C}(N\times\mathbb{R})$ one has the dense open set $\mathcal{C}'(N\times\mathbb{R})$ of hyperplanes not tangent to the $\mathbb{R}$ direction. Its sympectization seen inside $\dot{T}^*(N\times\mathbb{R})$ is

$$T^*N\times\dot{T}^*\mathbb{R} = \big\{((n,s),(p_n,p_s)); p_s\neq 0\big\}.$$

Hamiltonian isotopies in T^*N functorially lift to contact isotopies in $\mathcal{C}'(N\times\mathbb{R})$. If φ is generated by the time-dependant Hamiltonian H_t then, by definition, its lift is generated by $\hat{H}_t((n,s),(p_n,p_s)) = p_s H_t(n, p_n/p_s)$. If φ has compact support then $\hat{H}_t$ canonically extends to a contact hamilonian for the whole $\mathcal{C}(N\times\mathbb{R})$. Indeed there exists a positive ε such that, for any (n,p_n) and any time t, $H_t(n,p_n/p_s)$ becomes independant of p_s when $|p_s|$ is in $(0,\varepsilon)$.

We are now ready to lift the Arnold conjecture about $\varphi_t(0_N)\cap 0_N$ to a contact statement. Since 0_N is compact and hamiltonian isotopies can be cut-off, we can assume without loss of generality that φ is compactly supported.

A conjecture by Arnold Let N be a closed manifold and φ a compactly supported hamiltonian isotopy. We denote by Φ the lift of φ as a contact isotopy of $\mathcal{C}(N\times\mathbb{R})$, consider $L = \mathcal{N}_+(N\times\{0\})$ the lift of 0_N and $W_\psi = \{\big((n,s),[0,1]\big)\}$ the wall associated to $\psi\colon\ (n,s)\mapsto s$. Then, for any time t, $\Phi_t(L)$ intersects W_ψ.

2 Micro-support and persistent intersections

We fix a ring $\mathbf{k}$ which is either $\mathbb{R}$ or $\mathbb{Z}$ in this text. Ordinary cohomology will always use coefficients in $\mathbf{k}$. One can study Legendrian submanifolds of $\mathcal{C}M$ using $\mathrm{D^b}(M)$, the bounded derived category of sheaves of $\mathbf{k}$-modules on M (or its locally bounded cousin $\mathrm{D^{lb}}(M)$), seen as a quantum version of contact topology in $\mathcal{C}M$. Objects in this category generalize (among other things) submanifolds of M and local systems of coefficients. The object corresponding to a submanifold Z is denoted by $\mathbf{k}_Z$. In this text we do not need to know precisely what is $\mathrm{D^b}(M)$, we will use only the existence and functorial properties of the following constructions.

- Any object $\mathscr{F}$ has a *support* $\mathrm{supp}(\mathscr{F})$ which is a closed subset of M. The support of $\mathbf{k}_Z$ is Z.
- Any object $\mathscr{F}$ has a *micro-support* $\mathcal{N}\mathscr{F}$ which is a closed subset of $\mathcal{C}M$. If $Z\subset M$ is a closed submanifold then $\mathcal{N}\mathbf{k}_Z$ is $\mathcal{N}Z$. If U is a codimension 0 submanifold with boundary then $\mathcal{N}\mathbf{k}_U = \mathcal{N}_+\partial U$.
- For any object $\mathscr{F}$ and any subset $A\subset M$, there is a *cohomology object* $R\Gamma(A,\mathscr{F})$. If $\mathrm{supp}(\mathscr{F})$ is entirely contained in A then $R\Gamma(A,\mathscr{F})\simeq R\Gamma(M,\mathscr{F})$. If A is a nice subspace of M (say locally closed) and $\mathbf{k}=\mathbb{R}$ then the object $R\Gamma(A,\mathbf{k}_M)$ contains exactly the same information as the usual cohomology of A with coefficients in $\mathbf{k}$. As a trivial special case, $R\Gamma(\varnothing,\mathscr{F}) = 0$.

Cohomology is linked with micro-supports by the following “Morse lemma”.

Proposition 2.1. *Let $\psi : M \to \mathbb{R}$ be a function which is proper on the support of some $\mathscr{F} \in \mathrm{D^b}(M)$ and let $a < b$ be two real numbers. If, for all x in M,*

$$a \le \psi(x) < b \implies \begin{cases} d\psi(x) = 0 \textit{ and } x \notin \operatorname{supp}(\mathscr{F}) \\ \textit{or} \\ [d\psi(x)] \notin \mathcal{N}\mathscr{F} \end{cases}$$

then $R\Gamma(\{\psi < a\}, \mathscr{F}) \simeq R\Gamma(\{\psi < b\}, \mathscr{F})$.

Since $\operatorname{supp}(\mathbf{k}_M) = M$ and $\mathcal{N}\mathbf{k}_M = \mathcal{N}M = \varnothing$, the above lemma gives back that the usual cohomology of sub-level sets do not change until one crosses a critical value.

The following is a version of the main quantization result of [GKS12].

Theorem 2.2 (Quantization version I)**.** *Suppose $\mathscr{F}_0 \in \mathrm{D^b}(M)$ has compact support and Φ is a contact isotopy of $\mathcal{C}M$. Then there is a family $\mathscr{F}_t \in \mathrm{D^b}(M)$ with compact support such that $\Phi_t(\mathcal{N}\mathscr{F}_0) = \mathcal{N}\mathscr{F}_t$ and, for all t, $R\Gamma(M, \mathscr{F}_t) \simeq R\Gamma(M, \mathscr{F}_0)$.*

Corollary 2.3. *Let $\psi : M \to \mathbb{R}$ be a function without critical point so that it defines a wall W_ψ in $\mathcal{C}M$. If $\mathscr{F}_0 \in \mathrm{D^b}(M)$ has compact support and $R\Gamma(M, \mathscr{F}_0) \neq 0$ then, for any contact isotopy Φ and any time t, $\Phi_t(\mathcal{N}\mathscr{F}_0) \cap W_\psi$ is non-empty.*

Proof of corollary. Let $\mathscr{F}_t$ be the family given by Theorem 2.2. Suppose for contradiction that the intersection of W_ψ and $\Phi_t(\mathcal{N}\mathscr{F}_0) = \mathcal{N}\mathscr{F}_t$ is empty. Since $\operatorname{supp}(\mathscr{F}_t)$ is compact, $\psi(\operatorname{supp}(\mathscr{F}_t))$ is also compact. So we can apply the Morse lemma with a such that $\{\psi < a\}$ is empty and b such that $\operatorname{supp}(\mathscr{F}_t) \subset \{\psi < b\}$ to get $R\Gamma(\varnothing, \mathscr{F}_t) \simeq R\Gamma(M, \mathscr{F}_t)$. Hence the later vanishes. But it is isomorphic to $R\Gamma(M, \mathscr{F}_0)$ so we have a contradiction. □

This corollary contains (the contact reformulation of) the Arnold conjecture about persistence of intersections with the zero section in cotangent bundles. Indeed suppose N is a closed manifold and φ a compactly supported hamiltonian isotopy. The zero section lifts to $\mathcal{N}_+(N \times \{0\}) = \mathcal{N}\mathscr{F}_0$ for $\mathscr{F}_0 = \mathbf{k}_{t \le 0}$. The cohomology hypothesis is satisfied because $R\Gamma(M, \mathscr{F}_0) \simeq H^*(N)$.

3 Functoriality in contact tautology

3.1 Push forward and pull back

Let $f : N \to M$ be any map[1]. There is no hope to promote f to a honest map between $\mathcal{C}N$ and $\mathcal{C}M$ (think of the case of constant maps for instance). But we can

[1]of course a map between manifolds is always assumed to be at least of class C^1

push-forward or pull-back subsets. Let L be any subset in $\mathcal{C}N$. The *push-forward* of L under f is:

$$f_*(L) := \{(m, [p_m]) \in \mathcal{C}M \ ; \ \exists (n, [p_n]) \in L, \ m = f(n), \ p_m \circ T_n f = p_n\}.$$

Note that the above definition makes sense because it depends only on $[p_m]$ and not the specific p_m in this class.

Dually, if L' is a subset of $\mathcal{C}M$ then the *pull-back* of L' under f is:

$$f^*(L') := \{(n, [p_n]) \in \mathcal{C}N \ ; \ \exists (m, [p_m]) \in L', \ m = f(n), \ p_m \circ T_n f = p_n\}.$$

Example 3.1. *If f is a diffeomorphism from M to M and L is a single point in $\mathcal{C}M$ we get the obvious lift of f to a diffeomorphism of $\mathcal{C}M$: $f_*(m, [p_m]) = T_m f(\ker p_m)$. This lift is a contactomorphism.*

Example 3.2. *If $f\colon E \to M$ is a submersion and W is a cooriented hypersurface in E then W is called a generating hypersurface for $L := f_*(\mathcal{N}_+ W)$. If W is generic then L is an immersed Legendrian submanifold.*

3.2 Contact correspondences

In order to better understand Theorem 2.2, we need a generalization of the push-forward and pull-back operations. Let N and M be two manifolds and denote by λ_N and λ_M the Liouville forms on their tangent bundle. The 1-form $-\lambda_N + \lambda_M$ on $T^*N \times T^*M$ descends to a contact form on $\mathcal{C}(N \times M)$ which is not quite the tautological one. We will denote by $\mathcal{C}(N, M)$ the corresponding contact manifold.

A *contact correspondence* K from $\mathcal{C}N$ to $\mathcal{C}M$ is a Legendrian submanifold of $\mathcal{C}(N, M)$. It "maps" any subset L of $\mathcal{C}N$ to:

$$K(L) = \Big\{(m, [p_m]) \in \mathcal{C}M; \ \exists \big((n, m), [p_n, p_m]\big) \in K, \ (n, [p_n]) \in L\Big\}.$$

If L is a generic Legendrian submanifold of $\mathcal{C}N$ then $K(L)$ is an immersed Legendrian of $\mathcal{C}M$.

The obvious diffeomorphism from $N \times M$ to $M \times N$ induces a (coorientation reversing) contactomorphism from $\mathcal{C}(N, M)$ to $\mathcal{C}(M, N)$. The image K^t of a correspondence K under this isomorphism is a correspondence from $\mathcal{C}M$ to $\mathcal{C}N$ called the dual of K.

There is also a contactomorphism from $\mathcal{C}(N \times M)$ to $\mathcal{C}(N, M)$ which sends $((n, m), [p_n, p_m])$ to $((n, m), [-p_n, p_m])$. This will be denoted by $(\cdot)^a$.

To a map $f\colon N \to M$ we associate the correspondence

$$f_* = \Big\{\big((n, m), [p_n, p_m]\big) \in \mathcal{C}(N, M), \ m = f(n), \ p_n = p_m \circ T_n f\Big\}$$

which gives back the push-forward operation (hence the notation). The dual correspondence $(f_*)^t$ is f^*.

We denote by pr_M to projection of $N \times M$ to M. A subset L of $\mathcal{C}M$ is *non-characteristic* for f if $f_* \cap \big(\mathrm{pr}_M^*(L)\big) \subset \mathcal{C}(N \times M)$ is empty. Using less fancy notations, this means:

$$\forall n \in N, \forall p_m \in T^*_{f(n)}M, \quad (m, [p_m]) \in L \implies p_m \circ T_n f \neq 0.$$

Example 3.3. *For t in $I = [0, 1]$, let j_t be the injection of M into $M \times I$ defined by $j_t(m) = (m, t)$ and let pr_I be the projection of $M \times I$ to I. One observes that a subset L of $\mathcal{C}M$ is non-characteristic for j_t if and only if $L \cap \mathrm{pr}_I^*(\mathcal{N}\{t\}) = \varnothing$. Hence:*

L non-characteristic for all j_t, $t \in I$ $\iff L \cap \mathrm{pr}_I^*(\mathcal{C}I) = \varnothing \iff (\mathrm{pr}_I)_*(L) = \varnothing.$

Two correspondences K from $\mathcal{C}N$ to $\mathcal{C}M$ and K' from $\mathcal{C}M$ to $\mathcal{C}R$ can be *composed* to get a correspondence from $\mathcal{C}N$ to $\mathcal{C}R$:

$$K' \circ K = \Big\{ \big((n, r), [p_n, p_r]\big); \\ \exists (m, p_m) \in T^*M, \big((n, m), [p_n, p_m]\big) \in K, \big((m, r), [p_m, p_r]\big) \in K' \Big\}.$$

3.3 Legendrian graphs

Let φ be a contact transformation of $\mathcal{C}M$. As any contact transformation, it has a unique lift $S\varphi$ to the symplectization of the ambiant contact manifold. Since the symplectization of $\mathcal{C}M$ is canonically isomorphic to $\dot{T}^*M = T^*M \setminus 0_M$, the graph of $S\varphi$ is a Lagrangian submanifold of $(-\dot{T}^*M) \times \dot{T}^*M$. Its projection to $\mathcal{C}(M, M)$ is, by definition, the *Legendrian graph* of φ:

$$\Gamma_\varphi := \{((x, y), [p_x, p_y]) \in \mathcal{C}(M, M);\ (y, [p_y]) = \varphi(x, [p_x])\}.$$

This graph is a correspondence between $\mathcal{C}M$ and itself and $\varphi(L) = \Gamma_\varphi(L)$ for any subset L of $\mathcal{C}M$. If φ is the lift of $f \in \mathrm{Diff}(M)$ then $\Gamma_\varphi = f_*$.

Any contact isotopy Φ in $\mathcal{C}M$ with Hamiltonian H_t has a *Legendrian graph* $\Gamma_\Phi \subset \mathcal{C}(M, M \times I)$ defined as:

$$\Gamma_\Phi := \{((q, q', t), [p_q, p'_q, p_t]);\ \Phi_t(q, [p_q]) = (q', [p'_q]),\ p_t = H_t(q, p_q)\}.$$

Note that the condition $p_t = H_t(q, p_q)$ makes sense because of equivariance of H_t. Denoting by j_t the inclusion of M in $M \times I$ as $M \times \{t\}$, we have

$$\Gamma_{\Phi_t} = j_t^* \circ \Gamma_\Phi.$$

Remark 3.4. *By construction, any* $((q, q', t), [p_q, p'_q, p_t])$ *in* Γ_Φ *satisfies* $p_q \neq 0$ *and* $p'_q \neq 0$. *Hence, for any subset* L *in* $\mathcal{C}M$, $\Gamma_\Phi(L) \cap \mathrm{pr}_I^*(\mathcal{C}I)$ *is empty. According to Example 3.3, this implies that* $\Gamma_\Phi(L)$ *is non-characteristic for all* j_t *and* $(\mathrm{pr}_I)_*(\Gamma_\Phi(L))$ *is empty.*

4 Kernels and quantizations

Theorem 2.2 is proved by quantizing the Legendrian graph Γ_Φ. Recall that there is a contactomorphism $(\cdot)^a$ from $\mathcal{C}(M \times M \times I)$ to $\mathcal{C}(M, M \times I)$.

Theorem 4.1 (Quantization version II). *For any manifold* M *and any contact isotopy* $(\Phi_t)_{t \in I}$ *of* $\mathcal{C}M$, *there exists* $\mathscr{K}_\Phi \in \mathrm{D^{lb}}(M \times M \times I)$ *such that* $(\mathcal{N}\mathscr{K}_\Phi)^a = \Gamma_\Phi$.

In order to understand why the existence of such an object implies Theorem 2.2, we need to know about functorial properties of the micro-support.

- Any map $f\colon N \to M$ induces a functor $Rf_!\colon \mathrm{D^b}(N) \to \mathrm{D^b}(M)$. The object $Rf_!\mathscr{F}$ is strongly related to the cohomology of the restriction of $\mathscr{F}$ to fibers of f. If f is proper on the support of $\mathscr{F} \in \mathrm{D^b}(N)$ then

  $$\mathcal{N}(Rf_!\mathscr{F}) \subset f_*(\mathcal{N}\mathscr{F})$$

  and this inclusion is an equality if f is a closed embedding.

- Any map $f\colon N \to M$ induces a functor $f^{-1}\colon \mathrm{D^b}(M) \to \mathrm{D^b}(N)$. If f is non-characteristic for $\mathcal{N}\mathscr{F}$ then $\mathcal{N}(f^{-1}(\mathscr{F})) \subset f^*(\mathcal{N}\mathscr{F})$.

- Any object whose micro-support is empty come from local systems on M.

- One can see any object $\mathscr{K}$ in $\mathrm{D^b}(M \times N)$ as a quantum version of a correspondence between $\mathcal{C}M$ and $\mathcal{C}N$. In particular it can be used to send an object $\mathscr{F}$ of $\mathrm{D^b}(M)$ to an object $\mathscr{K}(\mathscr{F})$ in $\mathrm{D^b}(N)$. Using the contactomorphism $(\cdot)^a$ from $\mathcal{C}(M \times N)$ to $\mathcal{C}(M, N)$ one has, for sufficiently nice $\mathscr{K}$,

  $$\mathcal{N}(\mathscr{K}(\mathscr{F})) \subset \big(\mathcal{N}\mathscr{K}\big)^a(\mathcal{N}(\mathscr{F})).$$

  Such an object $\mathscr{K}$ is called a kernel, by analogy with integral transformations.

We now sketch why Theorem 4.1 implies Theorem 2.2. Starting with $\mathscr{F}_0$ and $\mathscr{K}_\Phi$, we set $\mathscr{F} = \mathscr{K}_\Phi(\mathscr{F}_0) \in \mathrm{D^b}(M \times I)$. One can prove that $\mathscr{K}_\Phi$ is nice enough to have $\mathcal{N}\mathscr{F} \subset \mathcal{N}(\mathscr{K}_\Phi)(\mathcal{N}\mathscr{F}_0)$. Still denoting by j_t the inclusion of M in $M \times I$ as $M \times \{t\}$, we set $\mathscr{F}_t = j_t^{-1}(\mathscr{F}) \in \mathrm{D^b}(M)$. The object $\mathscr{F}_t$ is the restriction of $\mathscr{F}$ to

$M \times \{t\}$ identified with M. Remark 3.4 ensures that j_t is non-characteristic for $\mathcal{N}\mathscr{F}$ so

$$\begin{aligned} \mathcal{N}\mathscr{F}_t &\subset j_t^*(\mathcal{N}\mathscr{F}) \\ &\subset \big(j_t^* \circ (\mathcal{N}\mathscr{K}_\Phi)^a\big)\big(\mathcal{N}\mathscr{F}_0\big) \\ &= \big(j_t^* \circ \Gamma_\Phi\big)\big(\mathcal{N}\mathscr{F}_0\big) \\ &= \Gamma_{\Phi_t}(\mathcal{N}\mathscr{F}_0) = \Phi_t(\mathcal{N}\mathscr{F}_0) \end{aligned}$$

So $\mathcal{N}\mathscr{F}_t \subset \Phi_t(\mathcal{N}\mathscr{F}_0)$. One can prove the reverse inclusion by using the isotopy $(\Phi_t)^{-1}$ and understand how its quantization $\mathscr{K}_{\Phi^{-1}}$ is related to $\mathscr{K}_\Phi$. Note however that the applications to persistent intersections use only the inclusion we proved above.

It remains to explain why $R\Gamma(M, \mathscr{F}_t) \simeq R\Gamma(M, \mathscr{F}_0)$. Let π denote the projection of $M \times I$ to I. What we want to prove is roughly that the cohomology of fibers $M \times \{t\}$ of π with respect to $\mathscr{F}_t$ is independant of t. This cohomology is described by $R\pi_!\mathscr{F}$ so we want to prove that $R\pi_!\mathscr{F}$ is (locally) constant on I. On the micro-local side, this means $\mathcal{N}(R\pi_!\mathscr{F}) = \varnothing$. This holds because $\mathcal{N}(R\pi_!\mathscr{F}) \subset \pi_*(\mathcal{N}\mathscr{F}) \subset \pi_*(\Gamma_\Phi(\mathcal{N}\mathscr{F}_0))$ and the later is empty according to Remark 3.4.

References

[GKS12] S. Guillermou, M. Kashiwara, and P. Schapira, *Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems*, Duke Math. J. **161** (2012), no. 2, 201–245, DOI 10.1215/00127094-1507367. MR2876930