Kernels

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1 Adjunction, pull-back and push-forward

1.1 Categorical framework

We need a powerful way of thinking about the relation between the functors g_* and g^{-1}. We first discuss the categorical setting, following [Wei94, Appendix A.6].

Definition 1.1. A pair (L, R) of functors $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ are called *adjoint* if there is a natural isomorphism τ between $\mathrm{Mor}_{\mathcal{B}}(L\cdot, \cdot)$ and $\mathrm{Mor}_{\mathcal{A}}(\cdot, R\cdot)$ seen as functors from $\mathcal{A}^{\mathrm{op}} \times \mathcal{B}$ to Sets.

Warning: here the letter R stands for "right adjoint", it has nothing to do with derived functors.

Unravelling the above definition a bit, we see that, for every A in $\mathcal{A}$ and B in $\mathcal{B}$, there is a bijection

$$\tau_{AB}\colon\ \mathrm{Mor}_{\mathcal{B}}(L(A), B) \to \mathrm{Mor}_{\mathcal{A}}(A, R(B)).$$

This bijection is natural in A and B in the sense that, for every $f\colon\ A \to A'$ and $g\colon\ B \to B'$, there is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Mor}_{\mathcal{B}}(L(A'), B) & \xrightarrow{\circ L(f)} & \mathrm{Mor}_{\mathcal{B}}(L(A), B) & \xrightarrow{g\circ} & \mathrm{Mor}_{\mathcal{B}}(L(A), B') \\
\downarrow{\scriptstyle \tau_{A'B}} & & \downarrow{\scriptstyle \tau_{AB}} & & \downarrow{\scriptstyle \tau_{AB'}} \\
\mathrm{Mor}_{\mathcal{A}}(A', R(B)) & \xrightarrow{\circ f} & \mathrm{Mor}_{\mathcal{A}}(A, R(B)) & \xrightarrow{R(g)\circ} & \mathrm{Mor}_{\mathcal{A}}(A, R(B'))
\end{array}$$

Proposition 1.2. *A pair (L, R) is an adjoint pair if and only if there are natural transformations $\eta\colon \mathrm{Id}_{\mathcal{A}} \implies RL$ and $\varepsilon\colon\ LR \implies \mathrm{Id}_{\mathcal{B}}$ such that, for all A and B,*

$$1_{LA} = \varepsilon_{LA} \circ L(\eta_A)$$
$$1_{RB} = R(\varepsilon_B) \circ \eta_{RB}.$$

Proof. Suppose the adjunction transformations ε and η exist. For all $f\colon\ L(A) \to B$, we set $\tau_{AB}(f) = R(f) \circ \eta_A$:

$$A = \mathrm{Id}_{\mathcal{A}}(A) \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B)$$

$$A \xrightarrow{\tau_{AB}(f)} R(B)$$

and similarly, for all $g\colon\ A \to R(B)$, $\tau_{AB}^{-1}(g) = \varepsilon_B \circ L(g)$.

Conversely, if τ exists we don't have any choice, we have to set $\eta_A = \tau_{AL(A)}(\mathrm{Id}_{L(A)})$ and $\varepsilon_B = \tau_{R(B)B}^{-1}(\mathrm{Id}_{R(B)})$. This is too big to fail. $\square$

The natural transformation η (resp. ε) is called the *unit* (resp. *co-unit*) of the adjunction. In general they are not isomorphisms of functors.

1.2 Pull back and push forward

Proposition 1.3. *Let $f\colon\ Y \to X$ be a continuous map between topological spaces. The functors $f^{-1}\colon\ \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and $f_*\colon\ \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ form an adjoint pair (f^{-1}, f_*).*

Proof. We will construct the unit and then the co-unit of the adjunction.

Let $\mathscr{G}$ be a sheaf on X. We want to construct $\eta_{\mathscr{G}}\colon\ \mathscr{G} \to f_* f^{-1}\mathscr{G}$ (which should of course be functorial in $\mathscr{G}$). For any open subset U_X in X,

$$f_* f^{-1}\mathscr{G}(U_X) = \varinjlim_{V_X \supset f(f^{-1}(U_X))} \mathscr{G}(V_X).$$

Because $f(f^{-1}(U_X)) \subset U_X$, the module $\mathscr{G}(U_X)$ is part of the direct system above hence we have our canonical map.

Moving to the co-unit, we consider any sheaf $\mathscr{F}$ on Y. We want to construct $\varepsilon_{\mathscr{F}}\colon\ f^{-1}f_*\mathscr{F} \to \mathscr{F}$. For any open subset U_Y in Y,

$$f^{-1}f_*\mathscr{F}(U_Y) = \varinjlim_{U_X \supset f(U_Y)} \mathscr{F}(f^{-1}(U_X)).$$

Because $f(U_Y) \subset U_X \iff U_Y \subset f^{-1}(U_X)$, every module in the directed system has a restriction map to $\mathscr{F}(U_Y)$. Those maps are obviously compatible with the restriction maps defining the limit so we have our canonical map.
Those constructions are so canonical that they cannot fail to give an adjunction. $\square$

The unit and co-unit of the above adjunction are not isomorphisms in general. Suppose first that X is a single point x_0 so f maps every point of Y to x_0. Then for all open subset U_Y in Y, $f^{-1}f_*\mathscr{F}(U_Y) = \mathscr{F}(Y)$ and $\varepsilon_{\mathscr{F}}(U_Y)$ restricts a global section (seen as a section of $f^{-1}f_*\mathscr{F}$) to U_Y. Dually one can consider the case where Y is a single point y and we see f as the inclusion of this point in X. Then $f_*f^{-1}\mathscr{G}$ is the sheaf which has the same stalk as $\mathscr{G}$ at y and vanishing stalk everywhere else. So $\eta_{\mathscr{G}}(U_X)$ sends a section to its germ at y if y is in U_X and zero otherwise.

2 Proper base change

2.1 Proper push-forward and c-soft sheaves

Let $f\colon Y \to X$ be a continuous map between two locally compact spaces. Recall that the *proper push-forward* functor $f_!\colon \operatorname{Sh}(Y) \to \operatorname{Sh}(X)$ is defined by:

$$f_!\mathscr{F}(U) = \left\{ s \in \mathscr{F}(f^{-1}(U));\ f \text{ is proper on } \operatorname{supp}(s) \right\}.$$

An important special case is when X is a single point. Then $f_!\mathscr{F}(U)$ is the module $\Gamma_c(U, \mathscr{F})$ of sections of $\mathscr{F}$ over U with compact support.

Lemma 2.1. *For any continuous map $f\colon Y \to X$, any sheaf $\mathscr{F}$ on Y and any x in X:*

$$(f_!\mathscr{F})_x = \Gamma_c(f^{-1}(x), \mathscr{F}).$$

The next section will generalize this formula to get the proper base change theorem.

Before that we need a class of sheaves adapted to proper push forward. Let F be a closed subset of X and $j\colon F \hookrightarrow X$ the inclusion map. By definition, for every sheaf $\mathscr{F}$ on X, $\Gamma(F, \mathscr{F}) = \Gamma(F, j^{-1}\mathscr{F}) = \varinjlim_{U \supset F} \mathscr{F}(U)$ so in particular there is a well defined restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(F, \mathscr{F})$. The sheaf $\mathscr{F}$ is *c-soft* if this map is surjective for every compact F.

Lemma 2.2. *The class of c-soft sheaves is adaptated to every functor $f_!$. In particular it is adapted to $\Gamma_c(X, \cdot)$.*

2.2 Proper base change

A *cartesian square* is a commutative diagram of maps

$$\begin{array}{ccc} Y' & \xrightarrow{\bar{g}} & Y \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$

such that there exists a homeomorphism between Y' and

$$Y \times_X X' = \{(y, x') \in Y \times X';\ f(y) = g(x')\}$$

which transforms f' and $\bar{g}$ into the obvious maps. The square in the middle of the above diagram indicates this property. An important case to keep in mind is when $f\colon Y \to X$ is a bundle and $f'\colon Y' \to X'$ as its pull-back under $g\colon X' \to X$.

Lemma 2.3 ([KS94, Proposition 2.5.11])**.** *In the cartesian square situation above, there is a canonical isomorphism of functors:*

$$g^{-1} \circ f_! \simeq f'_! \circ \bar{g}^{-1}$$

in $\mathrm{Fun}(\mathrm{Sh}(Y), \mathrm{Sh}(X'))$.

In particular, for any map $f\colon Y \to X$ and any x in X, we can consider the cartesian square:

$$\begin{array}{ccc} f^{-1}(x) & \xhookrightarrow{\ \bar{\imath}\ } & Y \\ {\scriptstyle f}\big\downarrow & \square & \big\downarrow{\scriptstyle f} \\ \{x\} & \xhookrightarrow[\ i\ ]{} & X \end{array}$$

The above lemma then gives back the stalk formula of Lemma 2.1. This formula will be used crucially in the proof.

Proof. Because a cartesian square is commutative, we know that $f_* \circ \bar{g}_* = g_* \circ f'_*$. One can check the properness condition to see that the obvious map still gives a well defined natural transformation $\delta\colon\ f_! \circ \bar{g}_* \implies g_* \circ f'_!$.

We then play with the adjunctions (g^{-1}, g_*) and $(\bar{g}^{-1}, \bar{g}_*)$. Specifically, we use the unit $\eta^{\bar{g}}\colon \mathrm{Id}_{\mathrm{Sh}(Y)} \implies \bar{g}_* \circ \bar{g}^{-1}$ and the co-unit $\varepsilon^g\colon g^{-1} \circ g_* \implies \mathrm{Id}_{\mathrm{Sh}(X')}$ to build the composition

$$g^{-1} \circ f_! \xRightarrow{\eta^{\bar{g}}} g^{-1} \circ f_! \circ \bar{g}_* \circ \bar{g}^{-1} \xRightarrow{\delta} g^{-1} \circ g_* \circ f'_! \circ \bar{g}^{-1} \xRightarrow{\varepsilon^g} f'_! \circ \bar{g}^{-1}.$$

It remains to prove that this composition is an isomorphism. We need to check that, for any sheaf $\mathscr{G}$ on Y, we get an isomorphism between $g^{-1} \circ f_!(\mathscr{G})$ and $f'_! \circ \bar{g}^{-1}(\mathscr{G})$. This can be check at the level of stalks for a point x' in X'. Recall that, for any continuous map $h\colon A \to B$, $(h^{-1}\mathscr{F})_a = \mathscr{F}_{h(a)}$ and, according to Lemma 2.1, $(h_!\mathscr{G})_b = \Gamma_c(h^{-1}(b), \mathscr{G})$. So here we get

$$(g^{-1} f_! \mathscr{G})_{x'} = \Gamma_c\big(f^{-1}(g(x')), \mathscr{G}\big).$$

Because we consider a cartesian square, the map $\bar{g}$ induces a homeomorphism between $(f')^{-1}(x')$ and $f^{-1}(g(x'))$ and an isomorphism

$$\Gamma_c\big(f^{-1}(g(x')), \mathscr{G}\big) \simeq \Gamma_c\big((f')^{-1}(x'), \bar{g}^{-1}\mathscr{G}\big).$$

The later module is $(f'_!\bar{g}^{-1}\mathscr{G})_{x'}$. ☐

We can now consider the derived version of the above lemma.

Proposition 2.4 (Proper base change formula [KS94, Proposition 2.6.7])**.** *In the cartesian square situation above, there is a canonical isomorphism of derived functors:*

$$g^{-1} \circ Rf_! \simeq Rf'_! \circ \bar{g}^{-1}$$

in $\mathrm{Fun}(\mathrm{D^b}(Y), \mathrm{D^b}(X'))$.

Proof. We want to apply the above Lemma 2.3 and Grothendieck's composition theorem. Because g^{-1} is exact, $R(g^{-1} \circ f) \simeq g^{-1} \circ RF$ for any left-exact functor F. So, in view of Lemma 2.3, we only need to prove that $Rf'_! \circ \bar{g}^{-1}$ is a derived functor of $f'_! \circ \bar{g}^{-1}$. It suffices to find a class of objects which are send by $\bar{g}^{-1}$ to a class of objects adapted to $f'_!$.

Let I_Y be the class of sheaves on Y whose restriction to every fiber $f^{-1}(x)$ is c-soft. Let $I_{Y'}$ be the analogous class on Y'. The functor $\bar{g}^{-1}$ sends I_Y tp $I_{Y'}$ because $\bar{g}$ induces a homeomorphism between $(f')^{-1}(x')$ and $f^{-1}(g(x'))$ for all x'.

So we only need to check that $I_{Y'}$ is adapted to $f'_!$. Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence in $I_{Y'}$. We want to prove that $0 \to f'_!\mathscr{F} \to f'_!\mathscr{G} \to f'_!\mathscr{H} \to 0$ is exact. This is equivalent to exactness at the stalk level. By Lemma 2.1 the stalk sequence is:

$$0 \to \Gamma_c((f')^{-1}(x'), \mathscr{F}) \to \Gamma_c((f')^{-1}(x'), \mathscr{G}) \to \Gamma_c((f')^{-1}(x'), \mathscr{H}) \to 0.$$

By definition of $I_{Y'}$ all the sheaves appearing in this sequence are c-soft. And Lemma 2.2 applied to $(f')^{-1}(x')$ then guaranties exactness. ☐

3 Tensor products and the projection formula

Let X be topological space and $\mathscr{F}$ and $\mathscr{G}$ two sheaves of modules over some ring $\mathbf{k}$. The tensor product $\mathscr{F} \otimes \mathscr{G}$ is the sheafification of the presheaf $U \mapsto \mathscr{F}(U) \otimes \mathscr{G}(U)$. In particular it has the same stalks so, using that tensor products commute with direct limits:

$$(\mathscr{F} \otimes \mathscr{G})_x = \mathscr{F}_x \otimes \mathscr{G}_x. \tag{1}$$

This stalk formula can be used to prove that pull-back commute with tensor products ([KS94, Proposition 2.3.5]):

$$f^{-1}(\mathscr{F} \otimes \mathscr{G}) \simeq f^{-1}\mathscr{F} \otimes f^{-1}\mathscr{G}. \tag{2}$$

We now investigate relations between our usual functors and tensor products. In this section $\mathbf{k}$ is assumed to be a field so that $\cdot \otimes \mathscr{G}$ is an exact functor. For the next lemma, recall that, for any $\mathbf{k}$-module M, the sheaf M_X is the sheaf of locally constant functions with values in M.

Lemma 3.1 ([KS94, Lemma 2.5.12])**.** *Let* $\mathbf{k}$ *be a field and* M *a* $\mathbf{k}$*-module. For any sheaf* $\mathscr{F}$ *of* $\mathbf{k}$*-modules, there is a natural isomorphism:*

$$\Gamma_c(X, \mathscr{F} \otimes M_X) \simeq \Gamma_c(X, \mathscr{F}) \otimes M.$$

In particular, if $\mathscr{F}$ *is c-soft then* $\mathscr{F} \otimes M_X$ *is c-soft.*

Lemma 3.2 ([KS94, Proposition 2.5.13])**.** *Let* $f\colon Y \to X$ *be a continuous map,* $\mathscr{F}$ *a sheaf on* X *and* $\mathscr{G}$ *a sheaf on* Y*. If* $\mathbf{k}$ *is a field then there is a canonical isomorphism*

$$f_!\mathscr{G} \otimes \mathscr{F} \to f_!\big(\mathscr{G} \otimes f^{-1}\mathscr{F}\big).$$

Proof. The adjunction transformation $\mathrm{Id}_{\mathrm{Sh}(X)} \implies f_*f^{-1}$ gives a map $f_*\mathscr{G} \otimes \mathscr{F} \to f_*\mathscr{G} \otimes f_*f^{-1}\mathscr{F}$. In the category of presheaves there is an obvious equality $f_*\mathscr{G} \otimes f_*f^{-1}\mathscr{F} = f_*(\mathscr{G} \otimes f^{-1}\mathscr{F})$. The sheafified version gives a map from $f_*\mathscr{G} \otimes f_*f^{-1}\mathscr{F} \to f_*(\mathscr{G} \otimes f^{-1}\mathscr{F})$ which is the obvious thing on stalks since sheafification preserves the stalk level. So we have a map from $f_*\mathscr{G} \otimes \mathscr{F}$ to $f_*\big(\mathscr{G} \otimes f^{-1}\mathscr{F}\big)$. Its restriction to $f_!\mathscr{G} \otimes \mathscr{F}$ goes to $f_!\big(\mathscr{G} \otimes f^{-1}\mathscr{F}\big)$.

We need to check that it is an isomorphism. As usual we look at the stalk level. Note that the restriction of $f^{-1}\mathscr{F}$ to any fiber $f^{-1}(x)$ is obviously the constant sheaf $(\mathscr{F}_x)_{f^{-1}(x)}$.

$$\begin{aligned}
\Big(f_!\big(\mathscr{G} \otimes f^{-1}\mathscr{F}\big)\Big)_x &\simeq \Gamma_c(f^{-1}(x), G \otimes f^{-1}\mathscr{F}) && \text{by Lemma 2.1}\\
&\simeq \Gamma_c(f^{-1}(x), G \otimes (\mathscr{F}_x)_{f^{-1}(x)}) &&\\
&\simeq \Gamma_c(f^{-1}(x), G) \otimes \mathscr{F}_x && \text{by Lemma 3.1}\\
&\simeq (f_!\mathscr{G})_x \otimes \mathscr{F}_x && \text{by Lemma 2.1}\\
&\simeq (f_!\mathscr{G} \otimes \mathscr{F})_x && \text{by Equation 1}
\end{aligned}$$

The above sequence of isomorphism corresponds to the map we constructed. □

Proposition 3.3 (Projection formula [KS94, Proposition 2.6.6])**.** *Let* $f\colon Y \to X$ *be a continuous map,* $\mathscr{F}^\bullet$ *a complex of sheaves on* X *and* $\mathscr{G}^\bullet$ *on* Y*. If* $\mathbf{k}$ *is a field then there is a natural isomorphism*

$$Rf_!(\mathscr{G}^\bullet) \otimes \mathscr{F}^\bullet \simeq Rf_!(\mathscr{G}^\bullet \otimes f^{-1}\mathscr{F}^\bullet).$$

Proof. Of course we want to apply the previous lemma and Grothendieck's composition theorem. The assumption that $\mathbf{k}$ is a field could be avoided by using resolution by flat sheaves but then one would need derived tensor products in the statement. More seriously, we need a version of Grothendieck's theorem for bifunctors, see [KS94, Section I.1.10]. Here we only explain what are the facts allowing to apply this theorem. Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X and Y.

First notice that $\cdot \otimes f^{-1}\mathscr{F}$ is an exact functor so that $(\cdot \otimes f^{-1}\mathscr{F}) \circ Rf_! = Rf_!(\cdot) \otimes f^{-1}\mathscr{F}$ is a derived functor of $\cdot \otimes f^{-1}\mathscr{F} \circ f_! = f_!(\cdot) \otimes f^{-1}\mathscr{F}$.

Next the last part of Lemma 3.1 guaranties that $\cdot \otimes f^{-1}\mathscr{F}$ sends c-soft sheaves and Lemma 2.2 that this class of sheaves is adapted to $f_!$. So $Rf_!(\cdot \otimes f^{-1}\mathscr{F})$ is a derived functor of $f_!(\cdot \otimes f^{-1}\mathscr{F})$. □

4 Kernels

In this section we explain how objects in $\mathrm{D^b}(X \times Y)$ define functors between $\mathrm{D^b}(X)$ and $\mathrm{D^b}(Y)$. We assume that X and Y are two manifolds (although it would be enough to assume there exists n such that any sheaf on X or Y has a resolution by c-soft sheaves of length less than n). We denote by q_X and q_Y the projections of $X \times Y$ to X and Y. We assume that $\mathbf{k}$ is a field to avoid derived tensor products and Hom.

Definition 4.1. For any bounded complex of sheaves $\mathscr{K}^\bullet$ on $X \times Y$, we define the functors $\Phi_{\mathscr{K}^\bullet} \colon \mathrm{D}^+(Y) \to \mathrm{D}^+(X)$ by:

$$\Phi_{\mathscr{K}^\bullet}(\mathscr{G}^\bullet) = Rq_{X!}\big(\mathscr{K}^\bullet \otimes q_Y^{-1}\mathscr{G}^\bullet\big).$$

For $\mathscr{K}_1^\bullet$ in $\mathrm{D^b}(X \times Y)$ and $\mathscr{K}_2^\bullet$ in $\mathrm{D^b}(Y \times Z)$, we set

$$K_1 \circ K_2 = Rq_{XZ!}\big(q_{XY}^{-1}K_1 \otimes q_{YZ}^{-1}K_2\big).$$

Proposition 4.2. *For every $\mathscr{K}_1^\bullet$ in $\mathrm{D^b}(X \times Y)$ and $\mathscr{K}_2^\bullet$ in $\mathrm{D^b}(Y \times Z)$:*

$$\Phi_{K_1 \circ K_2} = \Phi_{K_1} \circ \Phi_{K_2}.$$

Proof. We consider all projection maps entering this situation:

In this proof we use the notation $K_i = \mathscr{K}_i^\bullet$ to save space. We denote by G a complex of sheaves on Z. In the computation below, the indication "b.c." refers the the base change formula (Proposition 2.4) in the cartesian square

$$\begin{array}{ccc} X \times Y \times Z & \xrightarrow{q_{YZ}} & Y \times Z \\ {\scriptstyle q_{XY}}\downarrow & \square & \downarrow{\scriptstyle q_Y^{YZ}} \\ X \times Y & \xrightarrow[q_Y^{XY}]{} & Y \end{array}$$

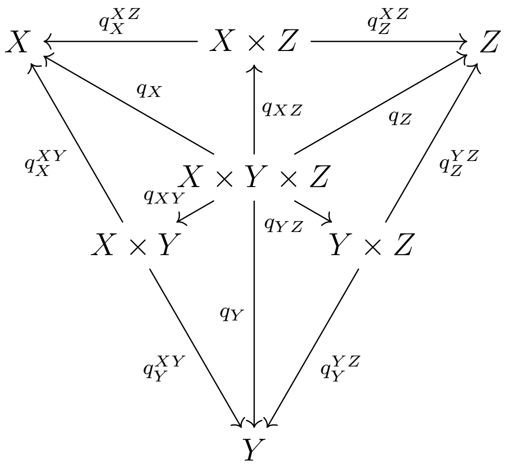

Figure 1: Some projection maps for kernel composition

$$(\Phi_{K_1} \circ \Phi_{K_2})G = (Rq_X^{XY})_! \Big[K_1 \otimes \underbrace{(q_Y^{XY})^{-1} \Big((Rq_Y^{YZ})_! \big(K_2 \otimes (q_Z^{YZ})^{-1} G \big) \Big)} \Big]$$

$$\simeq (Rq_{XY})_! \Big(\underbrace{q_{YZ}^{-1} \big(K_2 \otimes (q_{YZ}^{Z})^{-1} G \big)} \Big) \quad \text{(b. c.)}$$

$$\simeq q_{YZ}^{-1} K_2 \otimes q_{YZ}^{-1} (q_Z^{YZ})^{-1} G \text{ (by Eq.2)}$$

$$\simeq q_{YZ}^{-1} K_2 \otimes q_Z^{-1} G$$

$$\simeq (Rq_X^{XY})_! \Big[\underbrace{K_1 \otimes (Rq_{XY})_! \Big(q_{YZ}^{-1} K_2 \otimes q_Z^{-1} G \Big)} \Big]$$

$$\simeq (Rq_{XY})_! \big(q_{XY}^{-1} K_1 \otimes q_{YZ}^{-1} K_2 \otimes q_Z^{-1} G \big) \quad \text{(proj. } q_{XY}\text{)}$$

$$\simeq \underbrace{(Rq_X)_!}_{=(Rq_X^{XZ})_!(Rq_{XZ})_!} \big(q_{XY}^{-1} K_1 \otimes q_{YZ}^{-1} K_2 \otimes q_Z^{-1} G \big) \quad (q_X^{XY} \circ q_{XY} = q_X)$$

$$\simeq (Rq_X^{XZ})_! (Rq_{XZ})_! \big(q_{XY}^{-1} K_1 \otimes q_{YZ}^{-1} K_2 \otimes \underbrace{q_Z^{-1} G}_{=q_{XZ}^{-1}(q_Z^{XZ})^{-1}G} \big)$$

$$\simeq (Rq_X^{XZ})_! \Big((Rq_{XZ})_! \big(q_{XY}^{-1} K_1 \otimes q_{YZ}^{-1} K_2 \big) \otimes (q_Z^{XZ})^{-1} G \Big) \quad \text{(proj } q_{XZ}\text{)}$$

$$= \Phi_{K_1 \circ K_2} G.$$

□

References

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