Micro-support of sheaves

Vincent Humilière

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The microlocal theory of sheaves and in particular the definition of the micro-support is due to Kashiwara and Schapira (the main reference is their book "Sheaves on manifolds"). These notes follow very closely the lecture notes of Viterbo "An introduction to symplectic topology through sheaf theory".

We assume that the reader has some knowledge on the cohomology of sheaves and derived functors (but not much). We use as black boxes a few facts related to the so-called Mittag-Leffler condition and some properties of derived functors: at least in the cases we consider commuting functors induce commuting derived functors.

1 Definitions, examples

1.1 Propagation and micro-support of a sheaf

Let $\mathcal{F}$ be a sheaf of R-modules over a manifold X. We want to define what it means for a sheaf to propagate at some point and in some direction. To prepare the definition let us recall a few notations/facts.

The stalk $\mathcal{F}_x := \lim_{U \ni x} \mathcal{F}(U)$ at a point x is the set of germs of sections near x. One can also consider the stalk of the cohomology presheaf

$$H^j(\mathcal{F})_x := \lim_{U \ni x} H^j(U; \mathcal{F}).$$

An element in $H^j(\mathcal{F})_x$ is a cohomology class of a neighborhood of x modulo the equivalence relation for which two classes are equivalent when they agree on some smaller neighborhood of x. In particular, if $H^j(U; \mathcal{F})$ vanishes for some neighborhood U of x, then the stalk $H^j(\mathcal{F})_x$ vanishes. This is what happens for constant sheaves. We can define the induced sheaf to an open subset V by:

$$\Gamma_V \mathcal{F}(U) = \mathcal{F}(U \cap V).$$

Its support is included in the closure of V. Its stalk at a point of ∂V may be non trivial (for instance, this is what happens for the constant sheaf).

Definition 1. *We will say that $\mathcal{F}$* propagates *at x* in the direction of *p (or simply* at *(x,p)) if for all C^1-function ϕ defined in the neighborhood of x with $\phi(x)=0$, $d\phi(x)=p$, the natural map*

$$\varinjlim_{U\ni x} H^j(U;\mathcal{F}) \to \varinjlim_{U\ni x} H^j(U\cap\{\phi<0\};\mathcal{F}).$$

is an isomorphism for any j.[1]

In particular, if $\mathcal{F}$ propagates at x in the direction of p, every section of $\mathcal{F}$ defined on some "half-neigborhood $\{p<0\}$" of x, uniquely extends to a germ of sections at x.

Definition 2 (Micro-support I)**.** *The micro-support (or singular support) of a sheaf $\mathcal{F}$, denoted $SS(\mathcal{F})$, is the closure of all $(x,p)\in T^*X$ such that $\mathcal{F}$ does not propagate at x in the direction of p.*

Remark 3.

1. $SS(\mathcal{F})$ is conical: for all $t>0$, $(x,p)\in SS(\mathcal{F})$ iff $(x,tp)\in SS(\mathcal{F})$.

2. If $(x,p)\in SS(\mathcal{F})$, then x belongs to the support of $\mathcal{F}$. Indeed, if x is not in the support of $\mathcal{F}$, then $H^*(U;\mathcal{F})$ vanishes for some open neighborhood U of x, and $H^*(U\cap\{\phi<0\};\mathcal{F})$ vanishes as well, so that propagation holds for any possible ϕ.

3. The intersection of the micro-support with the zero section $SS(\mathcal{F})\cap 0_X$ is the support of the sheaf. Indeed, we have already proved one inclusion. Conversely, if $(x,0)$ is not in the micro-support, then for all y in some neighborhood of x, we have propagation and in particular for $\phi=0$, we get $H^*(\mathcal{F})_y\simeq 0$. Hence $H^0(\mathcal{F})=\mathcal{F}$ vanishes in some neighborhood of x.

4. The propagation condition is local: if two sheaves agree on some open set, the micro-support agree above this set.

5. The propagation condition does not depend on the choice of ϕ. It follows from Lemma 9 below.

[1] We could write equivalently $H^j(\mathcal{F})_x = (\Gamma_{\{\phi<0\}}H^j(\mathcal{F}))_x$

1.2 A few examples

EXAMPLE. (Constant sheaves). Let $\mathbb{R}_X$ be the constant sheaf on X, then $SS(\mathbb{R}_X) = 0_X$.

The support of the constant sheaf is the entire X so we only have to show that if $p \neq 0$, then $\mathbb{R}_X$ propagates in the direction of p. Let ϕ be a function s.t. $\phi(x) = 0$ and $d\phi(x) = p \neq 0$. The point x admits a fundamental system of neighborhoods U such that U and $U \cap \{\phi < 0\}$ are both contractible. Then,

$$H^j(U, \mathbb{R}_X) \simeq H^j(U \cap \{\phi < 0\}; \mathbb{R}_X) \simeq \begin{cases} \mathbb{R} & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases}$$

After taking limits other all such neighborhoods, we see that we have propagation.

EXAMPLE. (Constant sheaf on closed domains) Let V be the closure of an open set with smooth boundary in X. For $x \in \partial V$, denote by $\nu(x)$ an exterior normal covector, i.e., $\ker(\nu(x)) = T_x\partial V$ and $\langle \nu(x), u\rangle > 0$ for $u \in T_xX$ pointing outward. Denote by $\mathbb{R}_V$ the constant sheaf on V: by definition $\mathbb{R}_V(U)$ is the set of locally constant functions on $V \cap U$. Then,

$$\begin{aligned} SS(\mathbb{R}_V) = \{&(x,p) \in T^*X \mid \\ &(x \in V, p = 0) \text{ or } (x \in \partial V, \exists t \geqslant 0, p = -t\nu(x))\}. \end{aligned}$$

The cases $x \in V$ and $x \notin V \cup \partial V$ have already been treated. Let $x \in \partial V$. First note that the stalk of $\mathbb{R}_V$ at x is $\mathbb{R}$ since its set of sections is $\mathbb{R}$ for every neighborhood of x. There exists a smooth function ϕ such that $V = \{\phi \geqslant 0\}$ in the neighborhood of x. Then, for every $U \ni x$ a sufficiently small open subset,

$$H^0(U \cap \{\phi < 0\}; \mathbb{R}_V) = \mathbb{R}_V(U \cap \{\phi < 0\}) = 0,$$

But, $H^0(\mathbb{R}_V)_x = (\mathbb{R}_V)_x = \mathbb{R}$. Hence, the map $H^0(\mathbb{R})_x \to \lim_{U \ni x} H^0(U \cap \{\phi < 0\}; \mathbb{R}_V)$, is not an isomorphism. It follows that $(x, d\phi(x)) \in SS(\mathbb{R}_V)$. Since $d\phi(x)$ is a negative multiple of $\nu(x)$ it shows that for all $t > 0$, $(x, -t\nu(x)) \in SS(\mathbb{R}_V)$.

Conversely, if ϕ is any function such that $d\phi(x)$ is not a negative multiple of $\nu(x)$, then we can find a fundamental system of neighborhoods U of x such that $U \cap V$ and $U \cap V \cap \{\phi < 0\}$ are all contractible. In particular, for such U's,

$$H^j(U, \mathbb{R}_V) \simeq H^j(U \cap \{\phi < 0\}; \mathbb{R}_V),$$

and we still have an isomorphism at the level of stalks. It follows that $\mathbb{R}_V$ propagates at any such x in the direction of any such p.

We have not treated yet the particular case of points $(x, 0)$ with $x \in \partial V$. But these points belongs to the micro-support since they belong to the support.

Lemma 4. *Assume we have a short exact sequence*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0.$$

Then, for every i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$,

$$SS(\mathcal{F}_i) \subset SS(\mathcal{F}_j) \cup SS(\mathcal{F}_k),$$
$$SS(\mathcal{F}_i) \,\Delta\, SS(\mathcal{F}_j) \subset SS(\mathcal{F}_k),$$

where Δ is the symmetric difference.

Proof. We have long exact sequences in cohomology, one for U and one for $U \cap \{\phi < 0\}$, and a morphism of complex between the long exact sequences. If two of the three sheaves propagate at (x, p), then two thirds of the arrows that constitute the morphism of complex are isomorphisms. By the 5-Lemma all the remaining arrows are isomorphisms. This shows the first relation. For the same reason, if one of the three propagates at (x, p), then, either both other two propagate or both other two do not propagate. This shows the second relation. □

Example. (Constant sheaf on open domains) Let V be an open domain with smooth boundary in X, with exterior normal covector ν, and denote $\mathbb{R}_V$ the constant sheaf on V: $\mathbb{R}_V(U)$ is the set of locally constant functions on U that vanish near $U \cap \partial V$. Then,

$$\begin{aligned} SS(\mathbb{R}_V) = \{&(x, p) \in T^*X \mid \\ &(x \in V, p = 0) \text{ or } (x \in \partial V, \exists t \geqslant 0, p = t\nu(x))\}. \end{aligned}$$

Indeed, the exact sequence

$$0 \to \mathbb{R}_V \to \mathbb{R}_X \to \mathbb{R}_{X \setminus V} \to 0$$

and the above lemma give $SS(\mathbb{R}_V) \subset SS(\mathbb{R}_{X\setminus V}) \cup 0_X$ and $SS(\mathbb{R}_{X\setminus V}) \subset SS(\mathbb{R}_V) \cup 0_X$. Hence, $SS(\mathbb{R}_V) \cap 0_X = SS(\mathbb{R}_{X\setminus V}) \cap 0_X$.

Example. Let Z be a submanifold of X. Then, the micro-support of the constant sheaf on Z is the conormal to Z. Locally, $Z = \partial V$ for some open set V, and we can deduce the microsupport from the above examples and the exact sequence

$$0 \to \mathbb{R}_V \to \mathbb{R}_{\bar{V}} \to \mathbb{R}_{\partial V} \to 0.$$

1.3 Microsupport for objects of the derived category of sheaves

Lemma 5. *If we denote $\Gamma_Z\mathcal{F}$ the sheaf given by $\Gamma_Z\mathcal{F}(U) = \ker(\mathcal{F}(U) \to \mathcal{F}(U \setminus Z))$, the condition*

$$H^j(\mathcal{F})_x \simeq \lim_{U\ni x} H^j(U \cap \{\phi < 0\}; \mathcal{F})$$

is equivalent to

$$H^j(\Gamma_{\{\phi\geqslant 0\}}\mathcal{F})_x = 0.$$

Proof. The short exact sequence

$$0 \to \Gamma_{\{\phi\leqslant 0\}}\mathcal{F} \to \mathcal{F} \to \Gamma_{\{\phi<0\}}\mathcal{F} \to 0$$

gives rise to a long exact sequence for the cohomology presheaves and their stalk. The lemma follows immediately. □

The condition $H^j(\Gamma_{\{\phi\geqslant 0\}}\mathcal{F})_x = 0$ can be written $(R^j\Gamma_{\{\phi\geqslant 0\}}\mathcal{F})_x = 0$, where $R\Gamma_{\{\phi\geqslant 0\}}$ is the right derived functor associated to $\Gamma_{\{\phi\geqslant 0\}}$. We see that this condition makes sense not only for a sheaf but for an object in the derived category of sheaves.

Definition 6 (Micro-support II)**.** *The micro-support (or singular support) of an object in the derived category of sheaves $\mathcal{F}^\bullet$, denoted $SS(\mathcal{F}^\bullet)$, is the closure of all $(x,p) \in T^*X$ such that there exists a function ϕ defined in the neighborhood of x, with $\phi(x) = 0$ and $d\phi(x) = p$ and $(R\Gamma_{\{\phi\geqslant 0\}}\mathcal{F}^\bullet)_x \neq 0$.*

REMARK 7.

1. Quasi-isomorphic complexes of sheaves have the same micro-support.

2. A distinguished triangle $\mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \stackrel{+1}{\to}$ yields to the same relations as in Lemma 4, with the same proof.

The following example is at the origin of the definition of the micro-support.

EXAMPLE. Assume X is an open subset of $\mathbb{C}$, endowed with its sheaf of holomorphic functions $\mathcal{O}_X$. Let $\mathcal{P}$ be a differential operator on $\mathcal{O}_X$. Then the micro-support of the complex

$$0 \to \mathcal{O}_X \xrightarrow{\mathcal{P}} \mathcal{O}_X \to 0$$

is the characteristic variety of $\mathcal{P}$, i.e., the zero-locus of its principal symbol. One inclusion is a consequence of the Cauchy-Kowalewsky theorem. (See Kashiwara-Schapira)

1.4 Direct image of a sheaf, Lagrangian relations and micro-support

Direct image of a sheaf

Let $f : X \to Y$ be a continuous map, for any sheaf $\mathcal{F}$ on X, one defines a sheaf $f_*\mathcal{F}$ by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

The functor f_* is left exact, hence induces a derived functor Rf_*.

REMARK 8.

1. For every closed subset Z, $\Gamma_Z(f_*\mathcal{F}) = f_*(\Gamma_{f^{-1}(Z)}\mathcal{F})$. The same holds for derived functors:

   $$R\Gamma_Z(Rf_*\mathcal{F}^\bullet) = Rf_*(R\Gamma_{f^{-1}(Z)}\mathcal{F}^\bullet).$$

   Indeed, since the functors Γ_Z and f_* send injectives to injectives, it follows from Theorem 14.

2. If f is proper on the support of $\mathcal{F}$, then $(f_*\mathcal{F})_y = \lim_{f^{-1}(y)\subset U} \Gamma(U;\mathcal{F})$. Thus for derived functors:

   $$(Rf_*\mathcal{F}^\bullet)_y = \varinjlim_{f^{-1}(y)\subset U} R\Gamma(U;\mathcal{F}^\bullet).$$

   To prove it we start from $\Gamma(U; f_*\mathcal{F}) = \Gamma(f^{-1}(U);\mathcal{F})$ which holds by definition. Since f_* preserves injectives, we can apply Theorem 14 and get $R\Gamma(U; Rf_*\mathcal{F}) = R\Gamma(f^{-1}(U);\mathcal{F})$. Taking direct limits over $U \ni y$, we get $(Rf_*\mathcal{F})_y$ on the left hand side, and $\varinjlim_{f^{-1}(y)\subset U} R\Gamma(U;\mathcal{F}^\bullet)$ on the right hand side, since the family of open sets $f^{-1}(U)$ is cofinal among the neighborhoods of $f^{-1}(y)$.

Lagrangian relations

Let Λ be a Lagrangian submanifold in $\overline{T^*X} \times T^*Y$. Then, we can define the image of a subset $C \subset T^*X$ by Λ as the reduction of the subset $C \times \Lambda \subset T^*X \times \overline{T^*X} \times T^*Y$ by the coisotropic submanifold $W = \Delta \times T^*Y$, where Δ is the diagonal in $T^*X \times \overline{T^*X}$, i.e. $\pi((C \times \Lambda) \cap W)$, where π is the natural projection onto the T^*Y factor.

For instance, if ϕ is a symplectic diffeomorphism $T^*X \to T^*Y$, its graph defines a Lagrangian relation for which $\Lambda(C) = \phi(C)$.

We will use the following family of Lagrangian relations. For every $f : X \to Y$ smooth, define

$$\Lambda_f := \{(x, \xi, y, p) \in \overline{T^*X} \times T^*Y \mid f(x) = y, \xi = p \circ df(x)\}.$$

Then, for every subset $C \subset T^*X$, $(y,p) \in \Lambda_f(C)$ iff there exists $x \in f^{-1}(y)$ such that $(x, p \circ df(x)) \in C$.

Micro-support of a direct image

Lemma 9. *Assume $f : X \to Y$ is a smooth map which is proper on the support of $\mathcal{F}^\bullet$. Then,*

$$SS(Rf_*\mathcal{F}^\bullet) \subset \Lambda_f(SS(\mathcal{F}^\bullet)).$$

Moreover, this is an equality if f is a closed embedding.

Proof. Assume that $\mathcal{F}$ propagates at $(x, p \circ df(x))$ for every x in $f^{-1}(y)$. We want to show that $Rf_*\mathcal{F}$ propagates at (y,p).

Let ϕ be such that $\phi(y) = 0$ and $d\phi(y) = p$. For all $x \in f^{-1}(y)$, then, $\psi := \phi \circ f$ satisfies $\psi(x) = 0$ and $d\psi(x) = p \circ df(x)$. Our assumption implies $(R\Gamma_{\{\phi \circ f \geqslant 0\}}(\mathcal{F}))_x = 0$. Since this is the case for every $x \in f^{-1}(y)$, we get:

$$\lim_{f^{-1}(y) \subset U} R\Gamma_{\{\phi \circ f \geqslant 0\}}(U; \mathcal{F})) = 0.$$

But, according to the remarks made above,

$$\lim_{f^{-1}(y) \subset U} R\Gamma_{\{\phi \circ f \geqslant 0\}}(U; \mathcal{F})) = (Rf_*(R\Gamma_{\{\phi \circ f \geqslant 0\}}(\mathcal{F})))_y = (R\Gamma_{\phi \geqslant 0}(Rf_*\mathcal{F}))_y.$$

Thus, $Rf_*\mathcal{F}$ propagates at (y,p).

If f is a closed embedding $f^{-1}(y)$ is a point and both propagations are clearly equivalent. □

2 Some propagation theorems

2.1 Propagation for sheaves on the interval

Lemma 10. *Let $\mathcal{F}^\bullet$ be a complex of sheaves on $\mathbb{R}$. Assume that over some interval I the microsupport of $\mathcal{F}^\bullet$ is included in the zero section:*

$$SS(\mathcal{F}^\bullet) \cap T^*I \subset 0_I.$$

Then, for any $a < b \in I$, the following map is an isomorphism:

$$R\Gamma((-\infty, b); \mathcal{F}^\bullet) \to R\Gamma((-\infty, a); \mathcal{F}^\bullet).$$

Proof. Let us consider the following two families of properties:

$$(A_k): \quad \forall s \in I, \qquad \varinjlim_{\varepsilon \to 0} R\Gamma^k((-\infty, s+\varepsilon); \mathcal{F}^\bullet) \xrightarrow{\sim} R\Gamma^k((-\infty, s); \mathcal{F}^\bullet).$$

$$(B_k): \quad \forall s \in I, \qquad R\Gamma^k((-\infty, s); \mathcal{F}^\bullet) \xrightarrow{\sim} \varprojlim_{\varepsilon \to 0} R\Gamma^k((-\infty, s-\varepsilon); \mathcal{F}^\bullet).$$

Let us now assume that (A_k) and (B_k) hold for some k and show that we have the isomorphism

$$(C_k): \quad R\Gamma^k((-\infty, b); \mathcal{F}^\bullet) \simeq R\Gamma^k((-\infty, a); \mathcal{F}^\bullet).$$

Assume the map $R\Gamma^k((-\infty, b); \mathcal{F}^\bullet) \to R\Gamma^k((-\infty, a); \mathcal{F}^\bullet)$ is not injective. Then, there is $u \neq 0$ in $R\Gamma^k((-\infty, b); \mathcal{F}^\bullet)$ which restricts to 0 in $R\Gamma^k((-\infty, a); \mathcal{F}^\bullet)$. Let

$$s_0 = \sup\{s \leqslant b \,|\, u = 0 \text{ in } R\Gamma^k((-\infty, s); \mathcal{F}^\bullet)\}.$$

The injectivity of the map in (A_k) shows that if u vanishes in $R\Gamma^k((-\infty, s_0); \mathcal{F}^\bullet)$, then it vanishes in $R\Gamma^k((-\infty, s_0+\varepsilon); \mathcal{F}^\bullet)$ for some $\varepsilon > 0$, contradicting the definition of s_0. Hence, u does not vanish in $R\Gamma^k((-\infty, s_0); \mathcal{F}^\bullet)$. But this contradicts (B_k) since u vanishes in $\varprojlim_{\varepsilon \to 0} R\Gamma^k((-\infty, s-\varepsilon); \mathcal{F}^\bullet)$. Thus, such a u can not exist and our map is injective.

The surjectivity is worked out in exactly the same way as injectivity. Assume for a contradiction that there is a u in $R\Gamma^k((-\infty, a); \mathcal{F}^\bullet)$ which is not in the image of $R\Gamma^k((-\infty, b); \mathcal{F}^\bullet)$, and set

$$s_0 = \inf\{s \leqslant b \,|\, u \text{ is not in the image of } R\Gamma^k((-\infty, s); \mathcal{F}^\bullet)\}.$$

From (A_k), we deduce that u is not in the image of $R\Gamma^k((-\infty, s_0); \mathcal{F}^\bullet)$ and from (B_k) that it is. Hence a contradiction.

Let us now prove (A_k) for all k. Let $s \in I$. By definition $\Gamma_{[s,+\infty)}\mathcal{F}(U)$ is the kernel of the map $\mathcal{F}(U) \to \mathcal{F}(U \cap (-\infty, s))$. Hence we have an exact sequence

$$0 \to \Gamma_{[s,+\infty)}\mathcal{F}((-\infty, s+\varepsilon)) \to \mathcal{F}((-\infty, s+\varepsilon)) \to \mathcal{F}((-\infty, s)).$$

Now note that a section over $(-\infty, s+\varepsilon)$ that vanishes on $(-\infty, s)$ can be identified with a section over $(s-\varepsilon, s+\varepsilon)$ that vanishes on $(s-\varepsilon, s)$. It follows that we can identify $\Gamma_{[s,+\infty)}\mathcal{F}((-\infty, s+\varepsilon))$ with $\Gamma_{[s,+\infty)}\mathcal{F}((s-\varepsilon, s+\varepsilon))$ and get an exact sequence

$$0 \to \Gamma_{[s,+\infty)}\mathcal{F}((s-\varepsilon, s+\varepsilon)) \to \mathcal{F}((-\infty, s+\varepsilon)) \to \mathcal{F}((-\infty, s)).$$

If $\mathcal{F}$ was flabby, the last map of this sequence would be onto and we would have a "complete" short exact sequence. But when we compute the cohomology, we may replace $\mathcal{F}$ by a flabby resolution. We thus get a distinguished triangle

$$R\Gamma_{[s,+\infty)}\mathcal{F}((s-\varepsilon,s+\varepsilon)) \to R\Gamma((-\infty,s+\varepsilon);\mathcal{F}) \to R\Gamma((-\infty,s);\mathcal{F}) \overset{+1}{\to} .$$

Since direct limit is an exact functor and since, by assumption, $(R\Gamma_{[s,+\infty)}\mathcal{F})_s = 0$, we get the required isomorphism.

We prove property (B_k) by induction on k. First note that (B_0) is satisfied by definition of sheaves. Assume (B_{k-1}) holds. Then, (C_{k-1}) holds, hence the projective system $R\Gamma^{k-1}((-\infty,s-\frac{1}{n}),\mathcal{F})$ satisfies the Mittag-Leffler condition. It follows from Proposition 15 (see at the end of these notes) that (B_k) holds. □

2.2 The sheaf-theoretic Morse Lemma

Lemma 11. *Let $f : X \to \mathbb{R}$ proper on the support of $\mathcal{F}^\bullet$. Assume that*

$$\forall x \in f^{-1}([a,b]),\ df(x) \notin SS(\mathcal{F}^\bullet).$$

Then, the following natural map is an isomorphism:

$$R\Gamma(f^{-1}((-\infty,b));\mathcal{F}^\bullet) \to R\Gamma(f^{-1}((-\infty,a));\mathcal{F}^\bullet).$$

Remark 12. When $\mathcal{F}$ is the constant sheaf, we get the usual Morse Lemma.

Proof. This is equivalent to showing that we have an isomorphism

$$R\Gamma((-\infty,b),Rf_*\mathcal{F}) \to R\Gamma((-\infty,a),Rf_*\mathcal{F}).$$

But we know that

$$SS(Rf_*\mathcal{F}) \subset \Lambda(f)(SS(\mathcal{F})),$$

and our assumption implies that $\Lambda(f)(SS(\mathcal{F})) \cap [a,b] \subset 0_{[a,b]}$. Hence, it follows from the previous lemma. □

2.3 Sheaves whose micro-support is contained in the zero section

Lemma 13. *If $SS(\mathcal{F}^\bullet) \subset 0_X$, then $\mathcal{F}^\bullet$ is quasi-isomorphic to a locally constant sheaf.*

Proof. Since the problem is local, we can work in the neighborhood of a point x_0. Consider the fonction "square of the distance to x_0", for some Riemannian metric. It has only one critical point in x_0. Hence, from the Morse Lemma, we deduce that for some $R > 0$ and for all $\varepsilon \in (0, R)$,

$$R\Gamma(B(x_0, R); \mathcal{F}) \to R\Gamma(B(x_0, \varepsilon); \mathcal{F})$$

is an isomorphism. Taking limit $\varepsilon \to 0$, we get $R\Gamma(B(x_0, R), \mathcal{F}) \simeq R\Gamma(\mathcal{F})_{x_0}$. Hence, $R\Gamma(\mathcal{F})$ is locally constant. □

Appendix: a short list of notations and properties we used

- $\Gamma(U; \mathcal{F}) := \mathcal{F}(U)$.
- if V open, $\Gamma_V(U; \mathcal{F}) := \mathcal{F}(U \cap V) = \Gamma(U \cap V; \mathcal{F})$.
- if V closed, $\Gamma_V(U, \mathcal{F}) := \ker(\mathcal{F}(U) \to \mathcal{F}(U \setminus V))$.
- Cohomology of a sheaf: $H^j(U; \mathcal{F}) := R^j\Gamma(U; \mathcal{F})$.
- $H^0(U; \mathcal{F}) = \mathcal{F}(U)$.
- Constant sheaf with real coefficient: $\mathbb{R}_X(U)$ is the set of locally constant functions on U
- Constant sheaf on V open: $\mathbb{R}_V(U)$ is the set of locally constant functions on U that vanish in a neighborhood of $U \cap \partial V$.
- Constant sheaf on V closed: $\mathbb{R}_V(U)$ is the set of locally constant functions on $V \cap U$.
- If V is open or closed, the stalk of the constant sheaf is given by

$$(\mathbb{R}_V)_x = \begin{cases} \mathbb{R} \text{ if } x \in V \\ 0 \text{ if } x \notin V \end{cases} .$$

- The cohomology of the constant sheaf $H^j(U; \mathbb{R}_V)$ coincides with the usual (for example singular) cohomology $H^j(U \cap V; \mathbb{R})$.
- $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. The functor f_* is left exact in the category of sheaves. It is exact in the category of presheaves. It sends injective sheaves to injective sheaves.

- The functor Γ_V is left exact. It sends injective sheaves to injective sheaves.
- The functor $\varinjlim$ is exact.
- The following result seems to be fundamental:

  Theorem 14 (Grothendieck's spectral sequence)**.** *Assume we are in a category with enough injectives and that we have two left exact functors F and G, such that G transforms injectives into F-acyclic objects. Then $R(F \circ G) = RF \circ RG$.*

  In particular, the theorem holds if G sends injectives to injectives.

- (The Mittag-Leffler condition) We consider a projective system of (complexes of) abelian groups $(X_n, \rho_{n,p})$, i.e., abelian groups X_n, $n \in \mathbb{N}$ and morphisms $\rho_{n,p} : X_p \to X_n$ for $p \geqslant n$ such that $\rho_{n,n} = \mathrm{Id}$ and $\rho_{n,p} \circ \rho_{p,q} = \rho_{n,q}$. It satisfies the Mittag-Leffler condition if for any $n \in \mathbb{N}$, the sequence of ranges $(\rho_{n,p}(X_p))_{p \geqslant n}$ (subgroups of X_n) is stationary. We use the following proposition (Prop 1.12.4 in Kashiwara-Schapira):

  Proposition 15. *Assume that a projective system of complex of Abelian groups satisfies the M-L property. then,*

  1. *Then, for all k, $\phi^k : H^k(\varprojlim X_n) \to \varprojlim H^k(X_n)$ is onto.*
  2. *If for some k, the projective system $H^{k-1}(X_n)$ satisfies M-L, then, ϕ^k is bijective.*