Micro-support of complexes of sheaves

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January 24, 2014

1 Derived categories and functors

We saw with sheaf cohomology that a functor F can hide some "higher order", "derived" functors R^iF containing information of a cohomological nature obtained through resolutions. Indeed the global section functor sends a sheaf to a rather weak information. For instance it sends the constant sheaf $\mathbb{R}_X$ to $H^0(X;\mathbb{R})$ which only counts the number of connected components of X. But if one considers an injective resolution of $\mathbb{R}_X$ and apply the same functor then one gets the full cohomology of X with real coefficients. More generally, the cohomology of the complex obtained by applying a functor F to any injective resolution is independent of the resolution up to unique isomorphism.

This is already nice but not enough if we want to compose functors. Indeed a sheaf and its resolutions don't live in the same world so we cannot iterate the above idea. A prototypical[1] example of this situation is cohomology of a fiber bundle. Say $\pi\colon V \to \mathbb{S}^2$ is a circle bundle. It could be $\mathbb{S}^2 \times \mathbb{S}^1$ or $\mathbb{S}^3$ for instance. Those are distinguish by cohomology, hence by the sheaf $\mathbb{R}_V$. We want to reconstruct this cohomology from a sheaf on the base $\mathbb{S}^2$. Since fibers are connected, the push-forward sheaf $\pi_*\mathbb{R}_V$ is $\mathbb{R}_{\mathbb{S}^2}$. Hence this functor π_* is too crude. Even applying the derived functor of global sections to the resulting sheaf won't help. We need to apply it to a derived push-forward, hence compose derived functors.

So we need a general setup for this, hopefully understandable and flexible enough. We need objects and we need resolutions of objects and, at least for the purposes of composition, we need arbitrary complexes and not only resolutions. So let's recall some homological algebra explained e.g. in [GM03] or [Wei94]. Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Kom}(\mathcal{A})$ the category of cochain complexes of $\mathcal{A}$. The homotopy category $\mathrm{K}(\mathcal{A})$ is the category having the same objects as $\mathrm{Kom}(\mathcal{A})$ and whose morphisms are morphisms of Kom up to homotopy: $\mathrm{Mor}_{\mathrm{K}(\mathcal{A})}(X,Y) = \mathrm{Mor}_{\mathrm{Kom}(\mathcal{A})}(X,Y)/\sim$ where $f \sim g$ if there is some h such that

[1] and very much related to generating functions

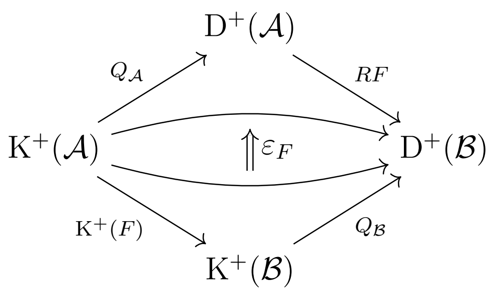
$f - g = hd + dk$. The derived category $\mathrm{D}(\mathcal{A})$ of $\mathcal{A}$ is the localization of $\mathrm{K}(\mathcal{A})$ with respect to quasi-isomorphisms (morphisms of $\mathrm{K}(\mathcal{A})$ which induce isomorphisms on cohomology). It has the same objects as $\mathrm{Kom}(\mathcal{A})$ and $\mathrm{K}(\mathcal{A})$ but there are more morphisms in a sense than in $\mathrm{K}(\mathcal{A})$. More precisely, there is a projection functor $Q_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ which does nothing on objects and such that any quasi-isomorphism f becomes an isomorphism $Q_{\mathcal{A}}(f)$ because it gained a formal inverse.

Note that the above discussion, together with an explicit construction of localization, unambiguously defines $\mathrm{D}(\mathcal{A})$. One can also characterize it up to unique isomorphism by a universal property. We will also use the variant D^{b} (or D^{+}) of bounded (or bounded below) complexes.

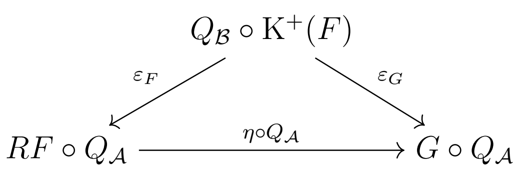
We now fix two abelian categories $\mathcal{A}$ and $\mathcal{B}$. Any (additive) functor F from $\mathcal{A}$ to $\mathcal{B}$ obviously defines a functor $\mathrm{Kom}(F)$ from $\mathrm{Kom}(\mathcal{A})$ to $\mathrm{Kom}(\mathcal{B})$ by applying F to each object of a complex. This functor descends to a functor $\mathrm{K}(F)$ between the homotopy categories $\mathrm{K}(\mathcal{A})$ and $\mathrm{K}(\mathcal{B})$ because it preserves homotopy equivalences. However it doesn't preserve quasi-isomorphisms so something smart has to be done in order to get functors between derived categories. Also the derived category is not abelian so the notion of exact functors has to be modified.

The following definition doesn't really matter. Its only purpose is to make sense of the next theorem.

Definition 1.1. Let F be a left-exact functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$. A (right) derived functor of F is a pair (RF, ε_F) where RF is an exact functor from $\mathrm{D}^{+}(\mathcal{A})$ to $\mathrm{D}^{+}(\mathcal{B})$ and ε_F is a natural transformation from $Q_{\mathcal{B}} \circ \mathrm{K}^{+}(F)$ to $RF \circ Q_{\mathcal{A}}$



satisfying the following universal property: for any other such pair (G, ε_G), there exists a unique natural transformation $\eta\colon\ FR \implies G$ making the diagram



commutative.

The universal property of derived functors guaranties that, if RF and RF' are derived functors of f then there are unique natural transformations $\eta : RF \implies RF'$ and $\eta' : RF' \implies RF$ such that $\eta \circ \eta'$ and $\eta' \circ \eta$ are the identity natural transformation of RF and RF' respectively. This is a very strong uniqueness result hence people always write *the* derived functor RF.

Of course an object A of $\mathcal{A}$ can always be seen as the complex $\cdots \to 0 \to A \to 0 \to \cdots$ concentrated in degree 0. Hence one can consider $RF(A)$ for any derived functor RF. But it is very important to understand that, in general, the cohomology of the object $RF(A)$ is not concentrated in degree zero hence there is no object B in $\mathcal{B}$ such that $RF(A)$ is isomorphic to $\cdots \to 0 \to B \to 0 \to \cdots$ (see the fiber bundle example above).

By contrast to the above definition, the following one is crucial for applications.

Definition 1.2. Let F be a left-exact functor from an abelian category $\mathcal{A}$ to another one $\mathcal{B}$. A class of objects $\mathcal{R} \subset \mathcal{A}$ is adapted to F (or F-injective) if:

- F maps any acyclic complex of $\mathrm{Kom}^+(\mathcal{R})$ to an acyclic complex
- $\mathcal{R}$ is stable by finite sums
- any object from $\mathcal{A}$ has a monomorphism to an object of $\mathcal{R}$.

In particular, if $\mathcal{A}$ has enough injectives (for instance $\mathcal{A}$ could be a category of modules or sheaves of modules) then the class of injective objects is adapted to *all* left-exact functors. However it is crucial to consider broader classes which are still adapted to some interesting functors. First it can be more convenient in concrete computation and, most of all, it's crucial in order to compose derived functors.

Proposition 1.3. *If $\mathcal{R}$ is a class of objects associated to some left-exact functor then any object $A^\bullet$ in $\mathrm{D}^+(\mathcal{A})$ is isomorphic to some complex of objects of $\mathcal{R}$.*

Note that the conclusion of the proposition means $A^\bullet \simeq Q_{\mathcal{R}}(R^\bullet)$ for some $R^\bullet$ in $\mathrm{Kom}^+(\mathcal{R})$ and the isomorphism is of course in $\mathrm{D}^+(\mathcal{A})$ so that it only means that complexes are quasi-isomorphic in Kom^+.

Example 1.4. *A sheaf is flabby is all restriction maps are surjective. The class of flabby sheaves is adapted to the functor of global sections.*

The next theorem is the result of this section. It both guaranties existence of derived functors and explains how to handle them.

Theorem 1.5. *If a left exact functor F from $\mathcal{A}$ to $\mathcal{B}$ has a class $\mathcal{R}$ of adapted objects then it has a derived functor. Moreover, for any derived functor RF and for any object $A^\bullet$ in $\mathrm{D}^+(\mathcal{A})$, $RF(A^\bullet)$ is isomorphic to $F(R^\bullet)$ for any object $R^\bullet$ in $\mathrm{Kom}^+(\mathcal{R})$ which is isomorphic to $A^\bullet$ in $\mathrm{D}^+(\mathcal{A})$.*

Note that, in the above theorem, all isomorphisms are isomorphisms in $\mathrm{D}^+(\mathcal{A})$ or $\mathrm{D}^+(\mathcal{B})$ and $F(R)$ is a very slightly sloppy notation for $Q_{\mathcal{B}}(\mathrm{K}^+(F)(R))$.

The reason why the second part of the above theorem cannot be used as a definition of derived functors is that they wouldn't be functors. The image of an object under a functor has to be an object, not an isomorphism class of objects.

The construction of derived functors involves choices but the theorem claims those choices can be made functorially to get *a* derived *functor* and not only a collection of objects $RF(A)$ for A in $\mathcal{A}$. It also claims that if ones chooses an adapted object $R^\bullet$ then $F(R^\bullet)$ (seen as a concrete computable beast) is isomorphic to $RF(A)$ for any (abstract) choice of functor RF.

Also note that, to any derived functor RF and any object A, one can associate the homology of the complex $RF(A)$. This homology is still an object of the derived category but with trivial differential. Its j graded part is denote by $R^jF(A)$. It is an object in $\mathcal{A}$. It depends on the choice of RF only up to unique isomorphism. An object A in $\mathcal{A}$ is called F-acyclic if $R^jf(A) = 0$ as soon as $j > 0$.

We can now reach our goal of composing derived functors.

Theorem 1.6 (Grothendieck's composition theorem)**.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let F and G be left-exact functors from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{B}$ to $\mathcal{C}$ respectively. If F sends injective objects to G-acyclic objects then, for any choices of derived functors RF, RG and $R(G \circ F)$, there is a natural isomorphism:*

$$R(G \circ F) \simeq RG \circ RF.$$

The hypothesis of the composition theorem cannot be dropped.

Exercise 1.7. Let $\mathcal{A}$ be the category of $\mathbb{Z}/p$-vector spaces, $\mathcal{B} = \mathcal{C}$ the category of abelian groups, F the embedding and $G = \mathrm{Hom}(\mathbb{Z}/p, \cdot)$. Prove that $RF = F$, $R(G \circ F) = G \circ F$ but $RG \circ F \neq G \circ F$.

2 Derived category of sheaves

In this section we fix a nice ring $\mathbf{k}$ and denote by Mod the category of $\mathbf{k}$-modules. For any manifold[2] X, we denote by $\mathrm{Sh}(X)$ the category of sheaves of $\mathbf{k}$-modules on X and set $\mathrm{D^b}(X) := \mathrm{D^b}(\mathrm{Sh}(X))$.

[2] A nice topological space would be good enough until we discuss micro-support.

The most important left exact functor from $\mathrm{Sh}(X)$ to Mod is the global sections functor $\Gamma(X, \cdot)$ which sends $\mathscr{F}$ to $\Gamma(X, \mathscr{F}) = \mathscr{F}(X)$. By definition, the cohomology of an object $\mathscr{F}^\bullet$ in $\mathrm{D^b}(X)$ is its image by the derived functor of $\Gamma(X, \cdot)$. So it is an object in $\mathrm{D^b}(\mathrm{Mod})$. The j-th cohomology module of $\mathscr{F}^\bullet$ is the $\mathbf{k}$-module $R^j(\Gamma(X, \cdot))(\mathscr{F}^\bullet)$. It is usually denoted by $R^j\Gamma(X, \mathscr{F}^\bullet)$. In case $\mathscr{F}^\bullet$ comes from a single sheaf $\mathscr{F}$ (in degree zero) then $R^j\Gamma(X, \mathscr{F}^\bullet) = H^j(X; \mathscr{F})$. Of course the derived functor and the cohomology modules are defined only up to unique isomorphism but tradition requires that we ignore this issue[3].

Definition 2.1. The support of a complex of sheaves $\mathscr{F}^\bullet$ is the closure of the set of x in X such that $H(\mathscr{F}^\bullet_x) \neq 0$.

Note that isomorphic objects in $\mathrm{D^b}(X)$ have the same support and that this definition is compatible with the classical definition of support of sheaves.

3 Pull-back and push-forward

Recall that any continuous map $f\colon X \to Y$ induces a pull-back functor $f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ which is the sheafification of $U \mapsto \varinjlim_V \mathscr{F}(V)$ for V open in Y and containing V. This is a complicated definition but the result has a nice stalk formula: $(f^{-1}\mathscr{F})_y = \mathscr{F}_{f(y)}$. In particular f^{-1} is an exact functor (recall exactness for sheaves is measured at the stalk level). This operation behaves well under composition of maps: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

The example of inclusion maps is important. Let ι_A^B denote the inclusion of A in B. Then, by definition, $\mathscr{F}_x = (\iota_x^X)^{-1}\mathscr{F}$. For general subsets $Z \subset X$ one uses the notation $\mathscr{F}_{|Z} = (\iota_Z^X)^{-1}\mathscr{F}$. Suppose x is a point in Z. Then one has the comforting equality $(\mathscr{F}_{|Z})_x = \mathscr{F}_x$ which follows from $\iota_x^X = \iota_Z^X \circ \iota_x^Z$.

There is also a push-forward functor $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ whose definition is straightforward: $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$. The result doesn't need to be sheafified. Obviously, we have $(f \circ g)_* = f_* \circ g_*$. This functor is left-exact but not exact in general. Indeed the stalk of $f_*\mathscr{F}$ is more complicated to understand than the pull-back version. Let y be a point in Y. By definition, $(f_*\mathscr{F})_y = \varinjlim_{U \ni y} \mathscr{F}(f^{-1}(U))$. In general we cannot say more. If we assume that f is proper on the support of $\mathscr{F}$ then this can be rewritten as

$$(f_*\mathscr{F})_y = \varinjlim_{V \supset f^{-1}(y)} \mathscr{F}(V).$$

The later limit is, again by definition, the module of global sections of the sheaf $\mathscr{F}_{|f^{-1}(y)}$. So, under this properness assumption:

$$(f_*\mathscr{F})_y = \Gamma(f^{-1}(y), \mathscr{F}_{|f^{-1}(y)}). \tag{1}$$

[3] Of course we are doing much worse when considering Floer homology.

See [KS94, Proposition 2.5.2] for a version involving $f_!$ and Γ_c.

We now turn to derived versions of those functors and their interaction. Because $(\iota_Z^X)^{-1}$ is an exact functor, we can use the notation $\mathscr{F}^\bullet_{|Z}$ without danger of confusion.

Another conveniently compact notation is:

$$R\Gamma(Z, \mathscr{F}^\bullet) := R\Gamma(Z, \mathscr{F}^\bullet_{|Z}).$$

Recall that the right hand side is already a compactified notation for $R(\Gamma(Z, \cdot))(\mathscr{F}^\bullet_{|Z})$.

The push-forward functor f_* is left-exact hence it has a derived functor Rf_*. In addition, f_* preserves injective sheaves. Since injective sheaves are F-acyclic for any left-exact functor F, we can apply the Grothendieck spectral sequence theorem to the composition $\Gamma(X, \cdot) = \Gamma(Y, \cdot) \circ f_*$ to get:

Theorem 3.1 (Cohomological Fubini for sheaves). *For any map $f\colon X \to Y$ and any $\mathscr{F}^\bullet$ in $\mathrm{D^b}(X)$, there is an isomorphism $R\Gamma(X, \mathscr{F}^\bullet) \simeq R\Gamma(Y, Rf_*\mathscr{F}^\bullet)$.*

This is called Fubini because $Rf_*\mathscr{F}^\bullet$ can be thought as the cohomology of fibers $f^{-1}(y)$, as explained by the next lemma.

Lemma 3.2. *If f is proper on the support of $\mathscr{F}^\bullet$ then*

$$(Rf_*\mathscr{F}^\bullet)_y \simeq R\Gamma(f^{-1}(y), \mathscr{F}^\bullet).$$

Proof. Let $\mathscr{F}$ be a sheaf on X and y a point in Y. We denote by ι_y and $\iota_{f^{-1}(y)}$ the inclusion of y in Y and $f^{-1}(y)$ in X. In Equation (1) we learned that, if we restrict to sheaves on whose support f is proper:

$$(\iota_y)^{-1} \circ f_* = \Gamma(f^{-1}(y), \cdot) \circ \iota_{f^{-1}(y)}^{-1}.$$

Because f_* preserves injectives and $\iota_{f^{-1}(y)}$ is even exact, we can apply Grothendieck composition theorem to get:

$$(\iota_y)^{-1} \circ Rf_* \simeq R\Gamma(f^{-1}(y), \cdot) \circ \iota_{f^{-1}(y)}^{-1}.$$

So we proved the announced formula and even proved it is functorial on the sub-category of complexes of sheaves on whose support f is proper. □

If f is a locally trivial fibration over a manifold and $\mathscr{F}^\bullet = \mathbb{R}_Y$ is the sheaf of locally constant real-valued functions then $(R^j f_*\mathscr{F}^\bullet)_y \simeq H^j(f^{-1}(y); \mathbb{R})$ and we get back the Leray-Serre spectral sequence. The discussion at the very beginning shows that, even if $\mathscr{F}^\bullet$ is concentrated in degree zero, one cannot replace $Rf_*(\mathscr{F}^\bullet)$ by $f_*(\mathscr{F}^\bullet)$.

4 Micro-support

In this section we fix a nice ring $\mathbf{k}$ and a manifold X.

In order to introduce a refinement of the support, we need a new functor. Let Z be a closed subset of X and $\mathscr{F}$ a sheaf on X. The sheaf $\Gamma_Z(\mathscr{F})$ of sections of $\mathscr{F}$ with support in Z is defined by:

$$U \mapsto \ker(\mathscr{F}(U) \to \mathscr{F}(U \setminus (U \cap Z)).$$

This defines a functor Γ_Z from $\mathrm{Sh}(X)$ to itself. One can check that it is left exact hence it has a right derived functor from $\mathrm{D^b}(X)$ to itself.

Lemma 4.1. *For any closed subset Z in X, the functor Γ_Z sends flabby sheaves to flabby sheaves.*

Proof. Let $\mathscr{F}$ be a flabby sheaf on X and U an open subset. We want to prove that any section s in $\Gamma_Z(U)$ extends to a section in $\Gamma_Z(X)$. The zero element in $\mathscr{F}(X \setminus Z)$ agrees with s on $(X \setminus Z) \cap U = U \setminus (U \cap Z)$. Hence, since $\mathscr{F}$ is a sheaf and not only a presheaf, there is a section s_1 on $(X \setminus Z) \cup U$ which restricts to 0 on $X \setminus Z$ and to s on U. Because $\mathscr{F}$ is flabby, s_1 extends to a section s_2 in $\mathscr{F}(X)$. This section is in $\Gamma_Z(X)$ by construction. □

Definition 4.2. Let $\mathscr{F}^\bullet$ be an object in $\mathrm{D^b}(X)$, x a point in X and H a (cooriented) contact element at x. The complex $\mathscr{F}^\bullet$ propagates through H if, for every relatively compact domain D whose boundary is tangent to $-H$ at x, $R\,\Gamma_{\mathrm{D}}(\mathscr{F}^\bullet)_x \simeq 0$. The micro-support $\mathcal{N}\mathscr{F}^\bullet$ is the closure in $\mathcal{C}X$ of the set of H through which $\mathscr{F}^\bullet$ does not propagate.

The meaning of the above definition will be clarified by the next proposition. First we can clarify the notations in a remark.

Remark 4.3. The stalk $R\,\Gamma_{\mathrm{D}}(\mathscr{F}^\bullet)_x$ belongs to $\mathrm{D^b}(\mathrm{Mod})$ and it vanishes[4] if and only its cohomology modules $H^j(R\,\Gamma_{\mathrm{D}}(\mathscr{F}^\bullet)_x)$ vanish[5]. Because direct limits commute with cohomology,

$$H^j(R\,\Gamma_{\mathrm{D}}(\mathscr{F}^\bullet)_x) = \varinjlim_{U} \left(R^j\,\Gamma_{\mathrm{D}}(\mathscr{F}^\bullet)(U)\right)$$

and this module can be denoted by $R^j\,\Gamma_{\mathrm{D}}(\mathscr{F}^\bullet)_x$ without ambiguity.

Given a contact element H at some point x, it will sometimes be convenient to consider a function $\varphi\colon X \to \mathbb{R}$ such that $H = \ker d\varphi(x)$. Such a function will be called associated to H. In particular, one can use $D = \{\varphi \geq 0\}$ as a domain whose boundary is tangent to $-H$ at x.

[4] A less abusive way of writing would be to say it is isomorphic to zero.

[5] Here we really mean zero.

Proposition 4.4. *Let $\mathscr{F}$ be a sheaf on X seen as a complex concentrated in degree zero. If φ is associated to H then $\mathscr{F}$ propagates through H if and only if restriction maps induce isomorphisms for all j*

$$\varinjlim_{U \ni x} H^j(U; \mathscr{F}) \xrightarrow{\sim} \varinjlim_{U \ni x} H^j(U \cap \{\varphi < 0\}; \mathscr{F}).$$

Proof. Let $0 \to \mathscr{F} \to \mathscr{I}^0 \to \cdots$ be an injective resolution of $\mathscr{F}$ so that $\mathscr{I}^\bullet$ is isomorphic to $\mathscr{F}$ in $\mathrm{D^b}(X)$. We set $D = \{\varphi \geq 0\}$ and $D^* = \{\varphi < 0\}$. By definition of Γ_D, we have for every open set U and every j, an exact sequence

$$0 \to \Gamma_\mathrm{D}\, \mathscr{I}^j(U) \to \mathscr{I}^j(U) \to \mathscr{I}^j(U \cap D^*).$$

This would work with any sheaf. But, since injective sheaves are flabby, we also get that the last arrow is surjective. So we have a short exact sequence

$$0 \to \Gamma_\mathrm{D}\, \mathscr{I}^j(U) \to \mathscr{I}^j(U) \to \mathscr{I}^j(U \cap D^*) \to 0.$$

Because Γ_D sends flabby sheaves to flabby sheaves, the sequence $0 \to \Gamma_\mathrm{D}\, \mathscr{F} \to \Gamma_\mathrm{D}\, \mathscr{I}^0 \to \cdots$ is a flabby resolution of $\Gamma_\mathrm{D}\, \mathscr{F}$ so it can be used to compute its cohomology on U and the long exact sequence in cohomology associated to the above short exact sequence is:

$$\cdots \to H^j(U; \Gamma_\mathrm{D}\, \mathscr{I}^\bullet) \to H^j(U; \mathscr{I}^\bullet) \to H^j(U \cap D^*, \mathscr{I}^\bullet) \to H^{j+1}(U; \Gamma_\mathrm{D}\, \mathscr{I}^\bullet) \to \cdots$$

The main theorem on derived functors guarantees that $R\Gamma_\mathrm{D}(\mathscr{F}) \simeq \Gamma_\mathrm{D}(\mathscr{I}^\bullet)$ so that passing to the direct limit in the long exact sequence above and using the preceding remark yields the proposition. □

References

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