Sheaf Theory

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The goals of this talk are

- to define a generalization denoted by $R\Gamma(\mathcal{F})$ of de Rham cohomology;
- to explain the notation $R\Gamma(\mathcal{F})$ (here $\mathcal{F}$ is a sheaf and $R\Gamma$ is a derived functor).

1 Presheaves and sheaves

1.1 Definitions and examples

Let X be a topological space.

Definition 1.1. A *presheaf of k-modules $\mathcal{F}$ on X* is defined by the following data:

- a k-module $\mathcal{F}(U)$ for each open set U of X;
- a map $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for each pair $V \subset U$ of open subsets such that
  - $r_{WV} \circ r_{VU} = r_{WU}$ for all open subsets $W \subset V \subset U$;
  - $r_{UU} = \mathrm{Id}$ for all open subsets U.

Therefore, a presheaf is a functor from the opposite category of open sets to the category of k-modules. If $\mathcal{F}$ is a presheaf, $\mathcal{F}(U)$ is called the *set of sections* of U and r_{VU} the *restriction* from U to V.

Definition 1.2. A presheaf $\mathcal{F}$ is a sheaf if

- for any family $(U_i)_{i\in I}$ of open subsets of X
- for any family of elements $s_i \in \mathcal{F}(U_i)$ such that

$$r_{U_i\cap U_j,U_i}(s_i) = r_{U_i\cap U_j,U_j}(s_j) \text{ for all } i, j \in I$$

there exists a unique $s \in \mathcal{F}(U)$ where $U = \cup_{i\in I} U_i$ such that $r_{U_i,U}(s) = s_i$ for all $i \in I$.

This means that we can extend a locally defined section.

Definition 1.3. A *morphism of presheaves* $f : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$: for each open set U, there exists a morphism $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the following diagram is commutative for $V \subset U$.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{r^{\mathcal{F}}_{VU}} & \mathcal{F}(V) \\ {\scriptstyle f(U)}\downarrow & & \downarrow{\scriptstyle f(V)} \\ \mathcal{G}(U) & \xrightarrow[r^{\mathcal{G}}_{VU}]{} & \mathcal{G}(V) \end{array}$$

A *morphism of sheaves* is a morphism of the associated preshaves.

We denote by $k-\mathrm{Presheaf}(X)$ and $k-\mathrm{Sheaf}(X)$ the categories of preshaves and sheaves of k-modules.

Example 1.4.

1. For any open set U, let $\mathcal{C}(U)$ be the set of continuous functions $f : U \to \mathbb{R}$. For $V \subset U$, let r_{VU} be the restriction of function in the ordinary sense. Then $\mathcal{C}$ is a sheaf of $\mathbb{R}$-modules.

2. The sheaf of holomorphic functions on $\mathbb{C}$, the sheaf of k-forms on a manifold...

3. Let $\mathcal{F}(U)$ be the set of constant functions $f : U \to k$ and r_{VU} be the restriction of function in the ordinary sense. Then $\mathcal{F}$ is a presheaf but not a sheaf.

4. Let $\mathcal{G}(U)$ be the set of *locally constant* functions $f : U \to k$ and r_{VU} be the restriction of function in the ordinary sense. Then $\mathcal{G}$ is a sheaf.

5. Let A be a closed set in X, then k_A is the sheaf where $k_A(U)$ is the set of locally constant function $A \cap U \to k$. This sheaf is called the *constant sheaf over* A.

6. Let Ω be an open set in X, then k_Ω is the sheaf where $k_\Omega(U)$ is the set of locally constant function $\Omega \cap U \to k$ with closed support. This sheaf is called the *constant sheaf over* Ω.

1.2 Localization and sheafification

In this section we define the germ of a presheaf at $x \in X$.

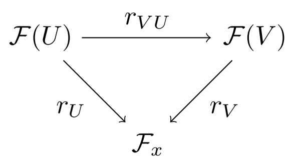
1.2.1 Stalks

Let $x \in X$. The set of open neighborhoods of x ordered by inclusion is a directed set. Let $\mathcal{F}$ be a presheaf on X. It induces a directed system of k-modules on this directed set.

Definition 1.5. The *stalk* of $\mathcal{F}$ at x is defined by

$$\mathcal{F}_x = \lim_{\rightarrow} \mathcal{F}(U)$$

By definition of the direct limit, for any open set U there exists a map $r_U : \mathcal{F}(U) \to \mathcal{F}_x$ such that the following diagram is commutative.



A morphism a preshaves $f : \mathcal{F} \to \mathcal{G}$ induces a morphism of directed systems and therefore a map

$$f_x : \mathcal{F}_x \to \mathcal{G}_x.$$

For each $x \in X$, we obtain a functor from the category of presheaves to the category of k-modules.

Example 1.6.

1. If $\mathcal{C}$ is the sheaf of continuous functions on X, then $\mathcal{C}_x$ is the set of germs of continuous functions at x.

2. Assume X is locally connected. Let A be a closed subset and k_A be the constant sheaf over A. Then

   - if $x \in A$, $(k_A)_x = k$ and the map $k_A(U) \to (k_A)_x$ is the evaluation $f \mapsto f(x)$
   - if $x \notin A$, $(k_A)_x = \{0\}$.

   Similarly, if Ω is an open subset and k_Ω the constant sheaf over Ω, then

   - if $x \in \Omega$, $(k_\Omega)_x = k$
   - if $x \notin \Omega$, $(k_\Omega)_x = \{0\}$.

Some properties of a morphism of sheaves can be read at the stalk level.

Proposition 1.7. *Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X.*

- *If $f, g : \mathcal{F} \to \mathcal{G}$ are two morphisms such that $f_x = g_x$ for all $x \in X$ then $f = g$.*

- *If $f : \mathcal{F} \to \mathcal{G}$ is a morphism such that f_x is an isomorphism for all $x \in X$ then f is an isomorphism.*

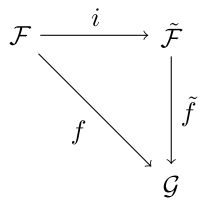
1.2.2 Sheafification

In this section we associate a sheaf $\tilde{\mathcal{F}}$ to a presheaf $\mathcal{F}$.

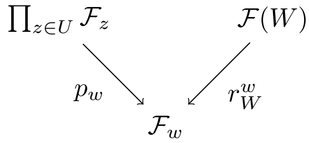
Definition 1.8. Let $\mathcal{F}$ be a presheaf. The *sheafification* $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is the sheaf characterized by the following property: there exists a morphism $i : \mathcal{F} \to \tilde{\mathcal{F}}$ such that

- $i_x : \mathcal{F}_x \to \tilde{\mathcal{F}}_x$ is an isomorphism for all $x \in X$;

- for any morphism $f : \mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf there exists a unique morphism $\tilde{f} : \tilde{\mathcal{F}} \to \mathcal{G}$ such that the following diagram is commutative.



The sheafification of $\mathcal{F}$ is constructed as follows. Let U be an open subset. A section s of $\tilde{\mathcal{F}}(U)$ is an element of $\prod_{z \in U} \mathcal{F}_z$ satisfying the following condition: for all $x \in U$ there exist an open subset $W \subset U$ and a section $t \in \mathcal{F}(W)$ such that $p_w(s) = r_W^w(t) \in \mathcal{F}_w$ for all $w \in W$.



Here $p_w : \prod_{z \in U} \mathcal{F}_z \to \mathcal{F}_w$ is the projection and $r_W^w : \mathcal{F}(W) \to \mathcal{F}_w$ is defined in the previous section. The restriction maps are given by projections.

Example 1.9. If $\mathcal{F}$ is the presheaf of constant functions then $\tilde{\mathcal{F}}$ is the sheaf of locally constant functions.

1.3 Operation on sheaves

Let $f : X \to Y$ be a continuous map.

Definition 1.10. Let $\mathcal{F}$ be a sheaf on X, then $f_*\mathcal{F}$ is the sheaf on Y given by

- $f_*\mathcal{F}(U) = \mathcal{F}\left(f^{-1}(U)\right)$;
- $r_{VU}^{f_*\mathcal{F}} = r_{f^{-1}(V)f^{-1}(U)}^{\mathcal{F}}$ if $V \subset V$.

Definition 1.11. If $\mathcal{G}$ is a sheaf on Y, the sheaf $f^{-1}\mathcal{G}$ on X is the sheafification of the presheaf $Pf^{-1}\mathcal{G}$ defined by $Pf^{-1}\mathcal{G}(U) = \lim_{V \supset f(U)} \mathcal{G}(V)$.

Example 1.12. If $A \subset X$ is a closed subset and $i : A \to X$ is the inclusion then $i_*k = k_A$ and $i^{-1}k_A = k$ (here k is the constant sheaf on A).

Definition 1.13. If $\mathcal{F}$ is a sheaf on X then $f_!\mathcal{F}$ is the sheaf on Y defined by

$$f_!\mathcal{F}(U) = \left\{ s \in \mathcal{F}\left(f^{-1}(U)\right), f : \operatorname{supp}(s) \to U \text{ is proper} \right\}.$$

Example 1.14. If $\Omega \subset X$ is an open subset and $i : \Omega \to X$ is the inclusion then $i_!k = k_\Omega$.

2 Čech cohomology

In this section, we define the cohomology of sheaves geometrically. Let $\mathcal{U} = (U_i)_i \in I$ be an open cover of X and $\mathcal{F}$ be a sheaf. The 0-*cochains* are functions which assign to each open set U_i an element of $\mathcal{F}(U_i)$. More generally a *q-cochain* $s \in C^q(\mathcal{U},\mathcal{F})$ is a function which assign to any $q+1$-tuple $(U_{i_0},\dots,U_{i_q})$ an element $s(i_0,\dots,i_q)$ in $\mathcal{F}(U_{i_0}\cap\dots\cap U_{i_q})$ such that $s(i_{\sigma(0)},\dots,i_{\sigma(q)}) = \varepsilon(\sigma)s(i_0,\dots,i_q)$ for any permutation σ.

We define a coboundary operator $\delta : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(\mathcal{U},\mathcal{F})$ by

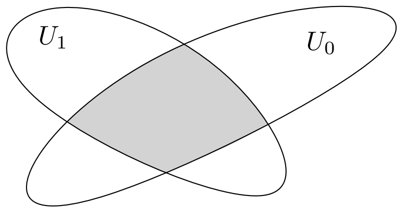
$$(\delta s)(i_0,\dots,i_{q+1}) = \sum_{j=0}^{q+1}(-1)^j s(i_0,\dots,\hat{i_j},\dots,i_{q+1})_{|U_{i_0}\cap\dots\cap U_{i_{q+1}}}.$$

Here $r_{VU}(s)$ is denoted by $s_{|V}$ if $s \in \mathcal{F}(U)$ and $V \subset U$. Note that

$$s(i_0,\dots,\hat{i_j},\dots,i_{q+1})_{|U_{i_0}\cap\dots\cap U_{i_{q+1}}}$$

is in $\mathcal{F}(U_{i_0}\cap\dots\cap U_{i_{q+1}})$.

Example 2.1. If $q = 1$, we obtain $(\delta s)(0,1) = s(1)_{|U_0\cap U_1} - s(0)_{|U_0\cap U_1}$.



Proposition 2.2. $\delta\circ\delta = 0$

The cohomology $H^*(\mathcal{U},\mathcal{F})$ of the complex $(C(\mathcal{U},\mathcal{F}),\delta)$ is called the *Čech cohomology of the cover $\mathcal{U}$ with values in $\mathcal{F}$*.

If $\mathcal{U}'$ is a refinement of $\mathcal{U}$, there exists a map

$$\rho : H^p(\mathcal{U},\mathcal{F}) \to H^p(\mathcal{U}',\mathcal{F}).$$

The map ρ is constructed as follows : pick a map $\varphi : I' \to I$ such that $U_{\varphi(i)} \subset U_i$ for all $i \in I'$ and let

$$\tilde{\rho}(s)(i_0,\dots,i_q) = s(\varphi(i_0),\dots,\varphi(i_q))_{|U_{i_0}\cap\dots\cap U_{i_q}}.$$

The map $\tilde{\rho}$ is a chain map and induces a map ρ in cohomology.

We define the p-th *Čech cohomology group $\check{H}^p(X,\mathcal{F})$ (or $R\Gamma^p(X,\mathcal{F})$) of $\mathcal{F}$* as the direct limit of the $H^p(\mathcal{U},\mathcal{F})$. Computing the direct limit is more or less impossible but there exist special covers whose Čech cohomology computes the Čech cohomology $\check{H}(X,\mathcal{F})$.

Definition 2.3. A cover is *acyclic* if $\check{H}^p(U_{i_0}\cap\dots\cap U_{i_q},\mathcal{F}) = \{0\}$ for all $p > 0$ and distinct indices $i_0,\dots,i_q$.

Theorem 2.4 (Leray)**.** *If the cover $\mathcal{U}$ is acyclic for $\mathcal{F}$ then*

$$\check{H}(\mathcal{F}) = H^*(\mathcal{U},\mathcal{F})$$

The Čech cohomology generalizes de Rham cohomology.

Proposition 2.5. *Let $\mathbb{R}_M$ be the constant sheaf on a manifold M. Then the Čech cohomology of M with values in $\mathbb{R}_M$ is isomorphic to the de Rham cohomology of M.*

Remark 2.6. To compute Čech cohomology for the constant sheaf, one can use good covers (a good cover is a cover such that $U_{i_0}\cap\dots\cap U_{i_q}$ is contractible). Good covers are acyclic for the constant sheaf.

3 Injective resolutions and cohomology of sheaves

The goal of this section is to present the cohomology of sheaves in a more algebraic and general setting.

3.1 The category of sheaves

Proposition 3.1. *The categories* $k-\mathrm{Presheaf}(X)$ *and* $k-\mathrm{Sheaf}(X)$ *are abelian.*

Proposition 3.2. *The sequence* $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ *is exact if and only if the sequences* $\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ *are exact for all* $x \in X$*.*

Definition 3.3. Γ_X is the functor from the category $k - \mathrm{Presheaf}(X)$ to the category of k-modules given by $\Gamma_X(\mathcal{F}) = \mathcal{F}(X)$.

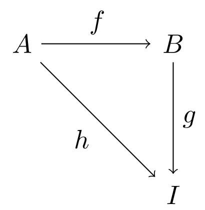
Proposition 3.4. *The functor* Γ_X *is left exact.*

3.2 Injective objects and resolutions

Let $\mathcal{C}$ be a category.

Definition 3.5. A map $f \in \mathrm{Mor}(B, C)$ is a *monomorphism* if for any $g_1, g_2 \in \mathrm{Mor}(A, B)$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

Definition 3.6. An object I of $\mathcal{C}$ is *injective* if for any map $h \in \mathrm{Mor}(A, I)$ and any monomorphism $f \in \mathrm{Mor}(A, B)$, there exists $g \in \mathrm{Mor}(B, C)$ such that the following diagram is commutative.



Definition 3.7. A category has *enough injectives* if any object has a monomorphism into an injective object.

Theorem 3.8. *The category* $k - \mathrm{Sheaf}(X)$ *has enough injectives.*

Proposition 3.9. *If* $\mathcal{C}$ *has enough injectives and* B *is an object in* $\mathcal{C}$*, there exists an exact sequence*

$$0 \to B \to J_0 \to J_1 \to \dots$$

where the J_k *are injectives. This exact sequence is called an* injective resolution *of* B*.*

Definition 3.10. Let $\mathcal{F}$ be a sheaf and let

$$0 \to \mathcal{F} \to \mathcal{J}_0 \to \mathcal{J}_1 \to \dots$$

be an injective resolution of $\mathcal{F}$. Then the *cohomology of the sheaf* $\mathcal{F}$, $R\Gamma(X, \mathcal{F})$, is the cohomology of the complex

$$0 \to \mathcal{J}_0(X) \to \mathcal{J}_1(X) \to \dots .$$

Remark 3.11. $R\Gamma^0(X, \mathcal{F}) = \mathcal{F}(X)$

More generally we have the following definition.

Definition 3.12. Let $\mathcal{C}$ be a category with enough injectives and let F be a left exact functor. If A is an object in $\mathcal{C}$, then $RF^j(A)$ is the j-th cohomology group of the complex

$$0 \to F(J_0) \to F(J_1) \to \dots$$

where

$$0 \to A \to J_0 \to J_1 \to \dots$$

is an injective resolution of A.

Remark 3.13. For $\mathcal{C} = k - \mathrm{Sheaf}(X)$ and $F = \Gamma_X$ we obtain the cohomology of sheaves.

3.3 Acyclic resolutions and examples

Injective elements are not so easy to find. To compute cohomology, we replace injective resolutions with acyclic resolutions.

3.3.1 Acyclic sheaves

Definition 3.14. Let $\mathcal{C}$ be a category with enough injectives and let F be a left exact functor. An object $A \in \mathcal{C}$ is F-*acyclic* if $RF^j(A) = \{0\}$ for all $j > 0$.

Theorem 3.15. *Let $\mathcal{C}$ be a category with enough injectives and let F be a left exact functor. Let*

$$0 \to A \to L_0 \to L_1 \to \dots$$

be a resolution such that L_j is F-acyclic for all j. Then $RF(A)$ is quasi-isomorphic to the complex

$$0 \to F(L_0) \to F(L_1) \to \dots.$$

In particular, $RF^j(A)$ is the j-th cohomology group of this chain complex.

Example 3.16. The following sheaves are Γ_X-acyclic

1. *flabby sheaves* : $\mathcal{F}(X) \to \mathcal{F}(U)$ is onto for any U (a section defined on U can be extended to X) ;
2. *soft sheaves* : $\mathcal{F}(X) \to \mathcal{F}(K) = \lim_{U \supset K} \mathcal{F}(U)$ is onto for any closed set K (a section defined in a neighborhood of K can be extended X).

Example 3.17. The sheaf of continuous functions on a manifold is soft but not flabby as a section that diverges near the boundary of an open set cannot be extended to the whole manifold.

3.3.2 De Rham cohomology

Let $\mathbb{R}_M$ be the constant sheaf on a manifold M. Let Ω^k be the sheaf of k-forms on M. It is a soft sheaf and the complex

$$0 \to \mathbb{R}_M \to \Omega^0 \to \Omega^1 \to \dots$$

is an acyclic resolution as it is exact at the stalk level by the Poincaré Lemma. Therefore, $R\Gamma(\mathbb{R}_M)$ is the cohomology of the complex

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \dots.$$

Thus $R\Gamma(\mathbb{R}_X)$ is isomorphic to the de Rham cohomology of M.

3.3.3 Singular cohomology

Let C^q be the sheaf of singular cochains on X. It is a flabby sheaf and the complex

$$0 \to k_X \to C^0 \to C^1 \to \dots$$

is an acyclic resolution. Therefore, $R\Gamma(k_X)$ is the cohomology of the complex

$$0 \to C^0(X) \to C^1(X) \to \dots$$

and thus is isomorphic to the singular cohomology.

As a corollary, we obtain that de Rham cohomology and singular cohomology are isomorphic.

3.3.4 Čech cohomology

Let $\mathcal{U}$ be a cover of X. Let $C^p(U, \mathcal{F})$ be the p-cochains of U associated to the cover $U \cap \mathcal{U}$. The complex

$$0 \to \mathcal{F} \to C^0 \to C^1 \to \dots$$

is a resolution of $\mathcal{F}$. If the cover is acyclic, the resolution is acyclic. Therefore $R\Gamma(\mathcal{F})$ is the cohomology of the complex

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to \dots$$

and our two definitions of sheaf cohomology coincide.

3.4 Derived categories and derived functors

Let $\mathcal{C}$ be an abelian category. We can define its derived category by "formally inverting" quasi-isomorphisms in the category of complexes $K(\mathcal{C})$ of $\mathcal{C}$. In general, a functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories does not induce a morphism between the associated derived categories. Yet, if F is left exact, it induces a functor

$$D(F) : D^b(\mathcal{C}) \to D^b(\mathcal{D})$$

called the derived functor. This functor is constructed in the following way:

1. select a suitable subclass of objects $\mathcal{C}$;

2. if F acts nicely on this subclass, it induces a derived functor on the derived category of this subclass;

3. if the chosen subclass is large enough, the associated derived category is equivalent to the derived category of $\mathcal{C}$ and the derived functor is completely determined.

If $\mathcal{C}$ has enough injectives, the suitable subclass is the class of injective objects. The cohomology of sheaves of $\mathcal{F}$ corresponds to the derived functor of Γ_X applied to the complex

$$0 \to \mathcal{F} \to 0.$$

This complex is quasi-isomorphic to an injective resolution of $\mathcal{F}$.