Topology in mathlib

Patrick Massot

Université Paris-Saclay at Orsay

July 17, 2020

Limits

$$\lim_{n\to\infty} u_n = x \qquad \lim_{x\to x_0} f(x) = y_0 \qquad \lim_{x\to x_0^+} f(x) = y_0^-$$

$$\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y_0 \qquad \lim_{x\to x_0} f(x) = +\infty \qquad \lim_{x\to +\infty} f(x) = +\infty$$

Limits

$$\lim_{n\to\infty} u_n = x \quad \lim_{x\to x_0} f(x) = y_0 \quad \lim_{x\to x_0^+} f(x) = y_0^-$$

$$\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y_0 \quad \lim_{x\to x_0} f(x) = +\infty \quad \lim_{x\to +\infty} f(x) = +\infty$$

Stick to $f : \mathbb{R} \to \mathbb{R}$, source and target could be $\pm\infty$, x_0, $x_0^\pm$, plus variations where $x \neq x_0$. Also x could be constrained to be rational.

Limits

$$\lim_{n\to\infty} u_n = x \quad \lim_{x\to x_0} f(x) = y_0 \quad \lim_{x\to x_0^+} f(x) = y_0^-$$

$$\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y_0 \quad \lim_{x\to x_0} f(x) = +\infty \quad \lim_{x\to +\infty} f(x) = +\infty$$

Stick to $f : \mathbb{R} \to \mathbb{R}$, source and target could be $\pm\infty$, x_0, $x_0^\pm$, plus variations where $x \neq x_0$. Also x could be constrained to be rational.

That's $13 \times 13 = 169$ definitions.

Limits

$$\lim_{n\to\infty} u_n = x \qquad \lim_{x\to x_0} f(x) = y_0 \qquad \lim_{x\to x_0^+} f(x) = y_0^-$$

$$\lim_{\substack{x\to x_0\\ x\neq x_0}} f(x) = y_0 \qquad \lim_{x\to x_0} f(x) = +\infty \qquad \lim_{x\to +\infty} f(x) = +\infty$$

Stick to $f : \mathbb{R} \to \mathbb{R}$, source and target could be $\pm\infty$, x_0, $x_0^\pm$, plus variations where $x \neq x_0$. Also x could be constrained to be rational.
That's $13 \times 13 = 169$ definitions.

Limits compose. Eg. $\lim_{x\to x_0} f(x) = y_0$ and $\lim_{y\to y_0} g(x) = z_0$ implies $\lim_{x\to x_0} g \circ f(x) = z_0$.

Limits

$$\lim_{n\to\infty} u_n = x \quad \lim_{x\to x_0} f(x) = y_0 \quad \lim_{x\to x_0^+} f(x) = y_0^-$$

$$\lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = y_0 \quad \lim_{x\to x_0} f(x) = +\infty \quad \lim_{x\to +\infty} f(x) = +\infty$$

Stick to $f : \mathbb{R} \to \mathbb{R}$, source and target could be $\pm\infty$, x_0, $x_0^{\pm}$, plus variations where $x \neq x_0$. Also x could be constrained to be rational.
That's $13 \times 13 = 169$ definitions.

Limits compose. Eg. $\lim_{x\to x_0} f(x) = y_0$ and $\lim_{y\to y_0} g(x) = z_0$ implies $\lim_{x\to x_0} g \circ f(x) = z_0$.
That's $13 \times 13 \times 13 = 2197$ lemmas.

Filters

A *filter* on X is a collection F of subsets of X such that

- $X \in F$
- $U \in F$ and $U \subseteq V$ implies $V \in F$
- $U \in F$ and $V \in F$ implies $U \cap V \in F$

Examples:

- $\mathcal{N}_x =$ nhds of x
- $\mathcal{N}_{+\infty} = \{U : \text{set}\, \mathbb{R} \mid \exists A, [A, +\infty) \subseteq U\}$
- $+\infty_{\mathbb{N}} = \{U : \text{set}\, \mathbb{N} \mid \exists N_0, [N_0, +\infty) \subseteq U\}$
- given $A : \text{set}\, X$, $\mathcal{P}(A) = \{U : \text{set}\, X \mid A \subseteq U\}$

Filters and limits

Bourbaki : given a filter F on X, and a point $y \in Y$, say $f : X \to Y$ converges to y along F if:

$$\forall V \in \mathcal{N}_y,\ f^{-1}V \in F.$$

Filters and limits

Bourbaki : given a filter F on X, and a point $y \in Y$, say $f : X \to Y$ converges to y along F if:

$$\forall V \in \mathcal{N}_y,\ f^{-1}V \in F.$$

This is not general enough. Replace $\mathcal{N}_y$ by any filter on Y. Say that f converges to a filter G on Y along F if

$$\forall V \in G,\ f^{-1}V \in F.$$

Compositions

Order filters by (reverse) inclusion, and define the push-forward filter f_*F by:

$$V \in f_*F \Leftrightarrow f^{-1}V \in F.$$

So f converges to G on Y along F iff $f_*F \leq G$.

Compositions

Order filters by (reverse) inclusion, and define the push-forward filter f_*F by:

$$V \in f_*F \Leftrightarrow f^{-1}V \in F.$$

So f converges to G on Y along F iff $f_*F \leq G$.

Trivial facts: $(g \circ f)_*F = g_*f_*F$ and each f_* is monotone.

Compositions

Order filters by (reverse) inclusion, and define the push-forward filter f_*F by:

$$V \in f_*F \Leftrightarrow f^{-1}V \in F.$$

So f converges to G on Y along F iff $f_*F \leq G$.

Trivial facts: $(g \circ f)_*F = g_*f_*F$ and each f_* is monotone.

Limits compose: Assume $f_*F \leq G$ and $g_*G \leq H$, then:

$$\begin{aligned}(g \circ f)_*F &= g_*f_*F \\ &\leq g_*G \\ &\leq H\end{aligned}$$

What did we gain?

1. This was 100% mathematics, no computer science
2. This doesn't exist in the real world
3. This is everywhere in proof assistants
4. There is no going back

Properties holding eventually

- For N large enough, $P(N)$
- For x close enough to y, $P(x)$
- For almost every x, $P(x)$

Properties holding eventually

- For N large enough, $P(N)$
- For x close enough to y, $P(x)$
- For almost every x, $P(x)$

They all mean: $\{x \mid P(x)\} \in F$ for some filter F.

Notation: $\forall^f x \text{ in } F, P(x)$

Properties holding eventually

- For N large enough, $P(N)$
- For x close enough to y, $P(x)$
- For almost every x, $P(x)$

They all mean: $\{x \mid P(x)\} \in F$ for some filter F.

Notation: $\forall^f x \text{ in } F, P(x)$

Example: $\varphi : X \to Y$, F non-trivial filter on X

$\forall^f x \text{ in } F, \varphi(x) \in V$ and $\varphi_* F \leq \mathcal{N}_y$ imply $y \in \text{closure} V$.

Pulling-back filters

Given $f : X \to Y$ and a filter G on Y:

$$f^*G = \{U \mid \exists V \in G, f^{-1}V \subseteq U\}$$

Example: given top spaces $\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle i}\big\downarrow & & \\ Y & & \end{array}$ and $y_0 : Y$,

$$\lim_{\substack{x \to y_0 \\ x \in X}} f(x) = z_0 \Leftrightarrow f_* i^* \mathcal{N}_{y_0} \leq \mathcal{N}_{z_0}$$

Pulling-back filters

Given $f : X \to Y$ and a filter G on Y:

$$f^*G = \{U \mid \exists V \in G, f^{-1}V \subseteq U\}$$

Example: given top spaces $\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle i}\big\downarrow & & \\ Y & & \end{array}$ and $y_0 : Y$,

$$\lim_{\substack{x \to y_0 \\ x \in X}} f(x) = z_0 \Leftrightarrow f_* i^* \mathcal{N}_{y_0} \leq \mathcal{N}_{z_0}$$

f^* is monotone and (f_*, f^*) form a Galois connection:
$f_*F \leq G \Leftrightarrow F \leq f^*G$.

Lattice structure

Filters on X form a complete lattice. $F\ G : \mathrm{filter}\, X$, $U : \mathrm{set}\, X$

$$U \in F \sqcup G \Leftrightarrow U \in F \wedge U \in G$$

$$U \in F \sqcap G \Leftrightarrow \exists V \in F, \exists W \in G, V \cap W \subseteq U$$

$$U \in \bot$$

$$U \in \top \Leftrightarrow U = X$$

Lattice structure

Filters on X form a complete lattice. $F\ G : \text{filter}\, X$, $U : \text{set}\, X$

$$U \in F \sqcup G \Leftrightarrow U \in F \wedge U \in G$$

$$U \in F \sqcap G \Leftrightarrow \exists V \in F, \exists W \in G, V \cap W \subseteq U$$

$$U \in \bot$$

$$U \in \top \Leftrightarrow U = X$$

Example:

- in $X \times Y$, $\mathcal{N}_{(x,y)} = \text{pr}_X^* \mathcal{N}_x \sqcap \text{pr}_Y^* \mathcal{N}_y =: \mathcal{N}_x \times \mathcal{N}_y$.
- $i : A \hookrightarrow X$,

$$x \in \text{closure} A \Leftrightarrow \mathcal{N}_x \sqcap \mathcal{P}(A) \neq \bot$$

$$\Leftrightarrow i^* \mathcal{N}_x \neq \bot$$

Bases

$B : \iota \to \operatorname{set} X,\ F : \operatorname{filter} X$

F has basis B if $\quad \forall U,\ U \in F \Leftrightarrow \exists\, i,\ B_i \subseteq U$

Bases

$B : \iota \to \mathrm{set}\, X,\ F : \mathrm{filter}\, X$

F has basis B if $\quad \forall U,\ U \in F \Leftrightarrow \exists i,\ B_i \subseteq U$

```
structure has_basis (l : filter X)
  (p : ι → Prop) (B : ι → set X) :=
(mem_iff : ∀ U, t ∈ l ↔ ∃ i (hi : p i), B i ⊆ U)
```

Bases

$B : \iota \to \mathrm{set}\, X,\ F : \mathrm{filter}\, X$

F has basis B if $\quad \forall U,\ U \in F \Leftrightarrow \exists\, i,\ B_i \subseteq U$

```
structure has_basis (l : filter X)
  (p : ι → Prop) (B : ι → set X) :=
(mem_iff : ∀ U, t ∈ l ↔ ∃ i (hi : p i), B i ⊆ U)
```

If F has basis $S : I \to \mathrm{set}\, X$ and G has basis $T : J \to \mathrm{set}\, Y$ then

$$f_* F \leq G \Leftrightarrow \forall j, \exists i, S_i \subseteq f^{-1}(T_j)$$

Extension by continuity

X, Y topological spaces, Y regular, $A \subseteq X$ dense subspace

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \overset{\varphi}{\nearrow} & \\ X & & \end{array}$$

if $\forall x, \exists y, f_* i^* \mathcal{N}_x \leq \mathcal{N}_y$ then $\exists \varphi, \varphi \circ \iota = f$.

Extension by continuity

X, Y topological spaces, Y regular, $A \subseteq X$ dense subspace

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & Y \\ {\scriptstyle i}\big\downarrow & \nearrow_{\varphi} & \\ X & & \end{array}$$

if $\forall x, \exists y, f_* i^* \mathcal{N}_x \leq \mathcal{N}_y$ then $\exists \varphi, \varphi \circ \iota = f$.

Choose φ such that $\forall x, f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$.
Let's prove φ is continuous at x.

By regularity, suffices to prove $\forall V' \in \mathcal{N}_{\varphi(x)}$ *closed*, $\varphi^{-1} V' \in \mathcal{N}_x$.

Extension by continuity

X, Y topological spaces, Y regular, $A \subseteq X$ dense subspace

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \overset{\varphi}{\nearrow} & \\ X & & \end{array}$$

if $\forall x, \exists y, f_* i^* \mathcal{N}_x \leq \mathcal{N}_y$ then $\exists \varphi, \varphi \circ \iota = f$.

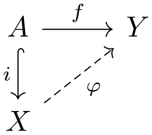
Choose φ such that $\forall x, f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$.
Let's prove φ is continuous at x.

By regularity, suffices to prove $\forall V' \in \mathcal{N}_{\varphi(x)}$ *closed*, $\varphi^{-1} V' \in \mathcal{N}_x$.

Since $f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$, $\exists V \in \mathcal{N}_x$ *open*, $i^{-1} V \subseteq f^{-1} V'$

Extension by continuity

X, Y topological spaces, Y regular, $A \subseteq X$ dense subspace



if $\forall x, \exists y, f_* i^* \mathcal{N}_x \leq \mathcal{N}_y$ then $\exists \varphi, \varphi \circ \iota = f$.

Choose φ such that $\forall x, f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$.
Let's prove φ is continuous at x.

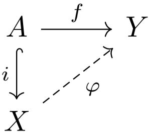
By regularity, suffices to prove $\forall V' \in \mathcal{N}_{\varphi(x)}$ *closed*, $\varphi^{-1} V' \in \mathcal{N}_x$.

Since $f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$, $\exists V \in \mathcal{N}_x$ *open*, $i^{-1} V \subseteq f^{-1} V'$

Suffices to prove $V \subseteq \varphi^{-1} V'$. Fix $y \in V$. Since V is open, $V \in \mathcal{N}_y$. In particular $i^{-1} V \in i^* \mathcal{N}_y$ hence $f^{-1} V' \in i^* \mathcal{N}_y$.

Extension by continuity

X, Y topological spaces, Y regular, $A \subseteq X$ dense subspace



if $\forall x, \exists y, f_* i^* \mathcal{N}_x \leq \mathcal{N}_y$ then $\exists \varphi, \varphi \circ \iota = f$.

Choose φ such that $\forall x, f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$.
Let's prove φ is continuous at x.

By regularity, suffices to prove $\forall V' \in \mathcal{N}_{\varphi(x)}$ *closed*, $\varphi^{-1} V' \in \mathcal{N}_x$.

Since $f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$, $\exists V \in \mathcal{N}_x$ *open*, $i^{-1} V \subseteq f^{-1} V'$

Suffices to prove $V \subseteq \varphi^{-1} V'$. Fix $y \in V$. Since V is open, $V \in \mathcal{N}_y$. In particular $i^{-1} V \in i^* \mathcal{N}_y$ hence $f^{-1} V' \in i^* \mathcal{N}_y$.

Hence $\forall^f z \text{ in } i^* \mathcal{N}_y, f(z) \in V'$, $f_* i^* \mathcal{N}_y \leq \mathcal{N}_{\varphi(y)}$ and $i^* \mathcal{N}_y \neq \bot$ by density so $\varphi\ y \in \mathrm{closure} V' = V'$.

Extension by continuity (continued)

X, Y topological spaces, Y regular, $A \subseteq X$ dense subspace

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \overset{\varphi}{\nearrow} & \\ X & & \end{array}$$

$\forall x, f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$. Let's prove $\varphi \circ i = f$.

Extension by continuity (continued)

X, Y topological spaces, Y regular, $A \subseteq X$ dense subspace

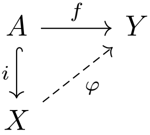
$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & \overset{\varphi}{\nearrow} & \\ X & & \end{array}$$

$\forall x, f_* i^* \mathcal{N}_x \leq \mathcal{N}_{\varphi(x)}$. Let's prove $\varphi \circ i = f$.

Fix $a \in A$. $f_* \mathcal{N}_a = f_* i^* \mathcal{N}_{i(a)} \leq \mathcal{N}_{\varphi(i(a))}$.
But we also know $f_* \mathcal{N}_a \leq \mathcal{N}_{f(a)}$, and Y is Hausdorff so $f(a) = \varphi(i(a))$.

Extension by continuity (continued)

X, Y topological spaces, Y regular, $A \subseteq X$ dense subspace



$\forall x, f_* i^* \mathcal{N}_x \le \mathcal{N}_{\varphi(x)}$. Let's prove $\varphi \circ i = f$.

Fix $a \in A$. $f_* \mathcal{N}_a = f_* i^* \mathcal{N}_{i(a)} \le \mathcal{N}_{\varphi(i(a))}$.
But we also know $f_* \mathcal{N}_a \le \mathcal{N}_{f(a)}$, and Y is Hausdorff so
$f(a) = \varphi(i(a))$.

Note how injectivity of i is used nowhere!
We use $\mathrm{image}(i)$ is dense and $\mathcal{T}_A = i^* \mathcal{T}_X$ (to get $\mathcal{N}_a = i^* \mathcal{N}_{i(a)}$).

```
dense_inducing i
```