chapter 8 Conformal invariance for critical percolation

8.1 - Introduction

Recall that for 2D Bernoulli perco. on $\mathbb{Z}^2$,

- For $p < p_c$, $\mathbb{P}_p\Big($  $\Big) \sim e^{-n/L(p)}$

  ($\sim$ means $\frac{\log \mathbb{P}_p(\cdot)}{n} \to -\frac{1}{L(p)}$)

- For $p > p_c$, $\mathbb{P}_p\Big($  $\Big) \to \theta(p) > 0$

- At $p_c = \frac{1}{2}$, it is expected that $\mathbb{P}_{1/2}\Big($  $\Big) \sim n^{-\alpha}$

  (conjecture is $\alpha = \frac{5}{48}$ I think)

and more generally,

$\mathbb{P}_{p_c}\Big($  $\Big) \doteq \left(\frac{n}{N}\right)^{\alpha + o(1)}$ as $n, N \to \infty$.

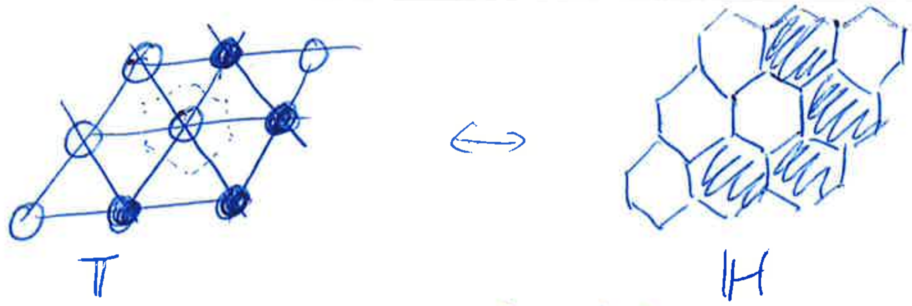
This indicates that critical percolation should "look the same at all scales", i.e. that its distribution is invariant under <u>scaling</u>.

In fact, it should be invariant also under <u>rotations</u> (we saw an example with $\mathbb{P}_{p_c}\Big($  $\Big) \xrightarrow[n\to\infty]{} \frac{1}{2}$) and even <u>conformal</u> maps (injective holomorphic maps)

... But the only percolation model where this is known is site percolation on the triangular lattice.
critical

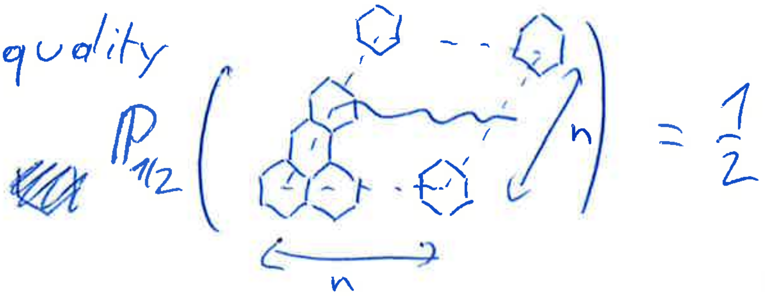
8.2 - Setting

We consider site perco. on the triangular lattice $\mathbb{T}$, which is the same as face perco. on the hexagonal lattice $\mathbb{H}$.



Each site is open (black) w. prob p
closed (white) ——— $1-p$.

One can check that we still have
- Harris inequality
- "duality":



as if there is no black crossing,
there is a white crossing.

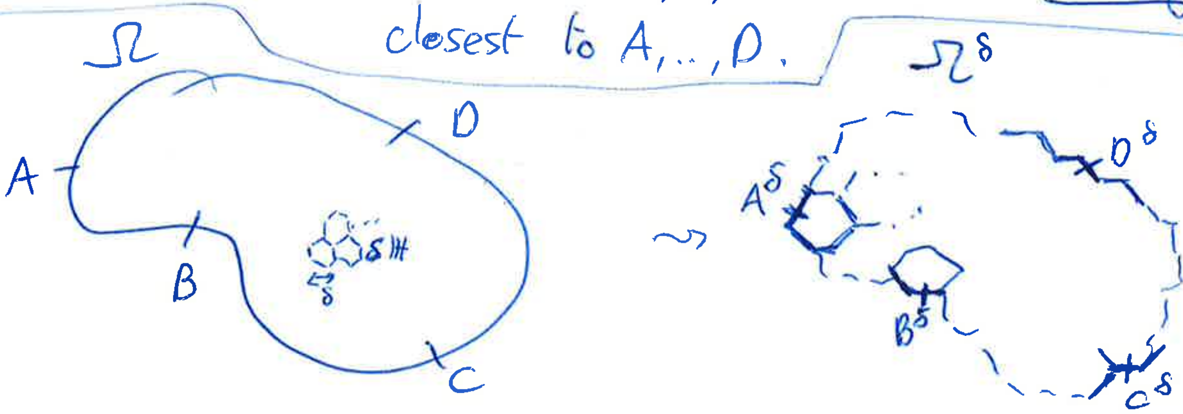
- RSW theory.

This shows that $p_c = \frac{1}{2}$ and all the results of the previous lecture (crossing estimates) apply.

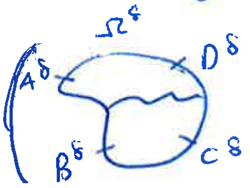
Definition 8.1 • A topological rectangle (Ω, A, B, C, D) is an open simply connected subset $\Omega \subset \mathbb{C}$ with a smooth boundary, and 4 points A, B, C, D on $\partial\Omega$ in counter-clockwise order.

Ω^c is connected, i.e. no hole

- Let $\delta > 0$, then $\Omega_\delta := \Omega \cap \delta\mathbb{H}$, and let $A^\delta, \dots, D^\delta$ be the mid-edge of $\delta\mathbb{H}$ closest to $A, \dots, D$.



We want to show that $\forall(\Omega, A, B, C, D)$ topo. rectangle,

$$\mathbb{P}_{1/2}\Big($$  $$\Big) \xrightarrow[\delta \to 0]{} f(\Omega, A, B, C, D)$$

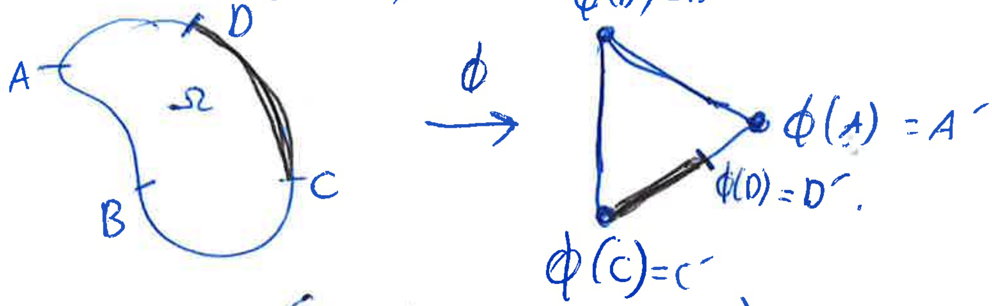
where f is conformal invariant: $\forall \phi$ conformal,

$$f(\phi(\Omega), \phi(A), \phi(B), \phi(C), \phi(D)) = f(\Omega, A, B, C, D).$$

8.3 - The Cardy-Smirnov Formula

In fact we will be able to give f "explicitely".

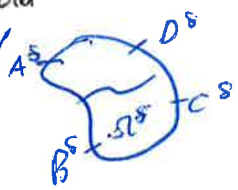
By Riemann uniformization, there is a unique conformal map ϕ that sends Ω to the equilateral triangle of edge 1, st:



Theorem 8.2 (Cardy - Smirnov)

conjectured conf. invariance & a formula

proof '01

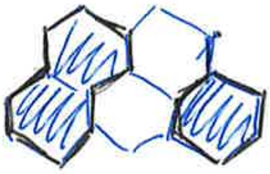
> As $\delta \to 0$, $\mathbb{P}_{1/2}\Big($  $\Big) \to f(\Omega, A, B, C, D) = |C'D'|$

This implies conf. invariance!

8.4 - The proof

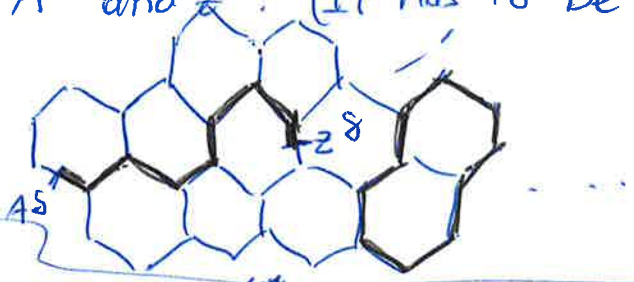
We consider (Ω, A, B, C) a topo. triangle. (D will come later)

We will see percolation configurations on Ω^δ as loop conf. on Ω^δ, by looking at the black/white interface (fixing all hexagons of $(\Omega^\delta)^c$ to be white)



If Ω^δ has N hexagons, there are 2^N loop conf.

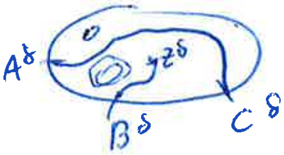
For $z \in \Omega$, we also consider loop configurations with an extra path joining A^δ and z^δ. (It has to be disjoint from the loops)



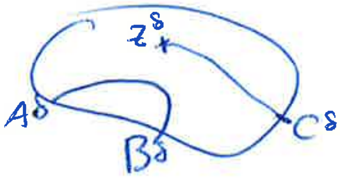
Let $F_a^\delta(z^\delta) = 2^{-N} \# \text{conf.}$

paths joining A^δ w. z^δ
B^δ w. C^δ.

$$F_b^\delta(z^\delta) = 2^{-N} \# \text{conf}$$



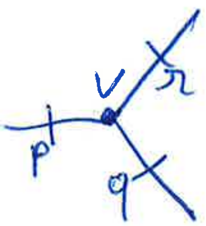
$$F_c^\delta(z^\delta) = 2^{-N} \# \text{conf}$$



Let $F^\delta(z^\delta) = F_a^\delta(z) + j\, F_b^\delta(z) + j^2 F_c^\delta(z)$

$(j = e^{2i\pi/3})$

Lemma 8.3 For any v vertex of $\delta\mathbb{H}$ inside Ω^δ, let p, q, r be the neighbouring half-edges of v in c-cw order. Then



$$(p-v)\,F^\delta(p) + (q-v)\,F^\delta(q) + (r-v)\,F^\delta(r) = 0 \quad (*)$$

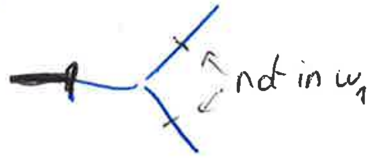
seen as a complex number

Proof Each loop config. ω with extra paths joining one of $A^\delta, B^\delta, C^\delta$ to one of p, q, r & the other together contributes to the l.h.s. of $(*)$, with a factor that looks like $\left.\begin{array}{l}(p-v)\\ \text{or } (q-v)\\ \text{or } (r-v)\end{array}\right. j^{0,1 \text{ or } 2}$.

We will group configurations by three, $(\underset{\text{ends at } p}{\omega_1}, \underset{\text{ends at } q}{\omega_2}, \underset{\text{ends at } r}{\omega_3})$ s.t. the sum of their contribution cancel.

Let ω_1 be a config ~~whose path~~ with a path that ends at p, ~~and which does not use the half-edge p...~~ s.t. the path ending at p does not go through v.

Case 1:



ω_1

We group it with

ω_3

ω_2

Contributions. if ω_1 joins A^δ with p:

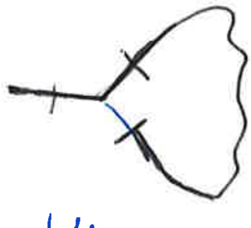
$$\left.\begin{array}{l}\omega_1 \to (p-v)\\ \omega_2 \to (q-v)\\ \omega_3 \to (r-v)\end{array}\right\} \Sigma = 0.$$

If ω_1 joins B^δ or C^δ w. p, similar.

Case 2:

ω_1

Then



ω_2

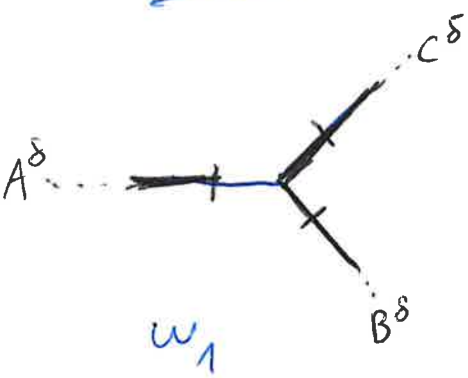
ω_3

If ω_1 joins A^δ w. p,

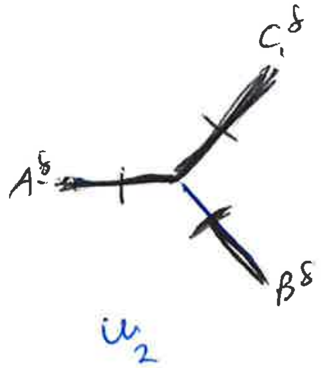
$\omega_1 \to (p-v)$ $\quad \omega_2 \to (q-v)$ $\quad \omega_3 \to (r-v)$. $\Sigma = 0$.

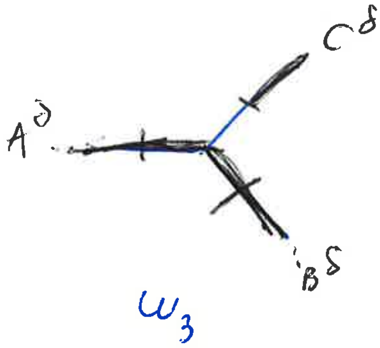
Same for B^δ, C^δ.

Case 3:



Then

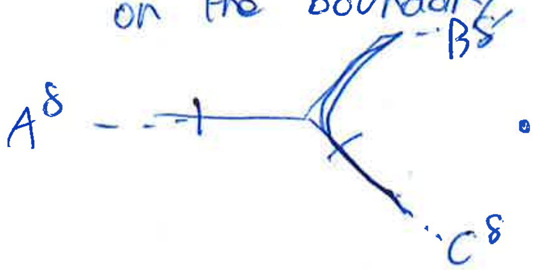




$$\left.\begin{array}{l} w_1 \to (p-v) \\ w_2 \to j(q-v) \\ w_3 \to j^2(r-v) \end{array}\right\} \Sigma = 0$$ as well! (quick check)

Works for a <u>cyclic permutation</u> of $(A^\delta, B^\delta, C^\delta)$..

↑ only possible case as A, B, C are in this order on the boundary, so we cannot have



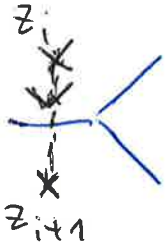
Notice all the w_2, w_3 have a path to q/r going through v. Doing the same starting from config. with the path to q/r not going through v gives all config once.

☒.

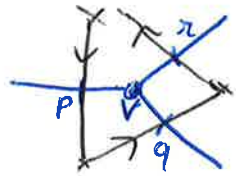
Lemma 8.3 can be seen as a form of "discrete holomorphicity" of F^δ, in the following way. If γ is a path on $\mathbb{ST}$ (dual to $\mathbb{SH}$), and F goes from mid-edges of $\mathbb{H}$ to $\mathbb{C}$, we define

$(z_0, \dots, z_n)$

$$\oint_\gamma F(z)\,dz := \sum_{i=0}^{n-1} F\left(\frac{z_i+z_{i+1}}{2}\right)(z_{i+1}-z_i).$$



Then for γ the closed path we have



$$\oint_\gamma F^\delta(z)\,dz = 2i(p-v)F^\delta(p) + 2i(q-v)F^\delta(q) + 2i(r-v)F^\delta(r)$$
$$= 0.$$

This implies that for any closed path γ on $\delta\mathbb{T}\cap\Omega$,

$$\oint_\gamma F^\delta = 0$$ ... just like for holom. functions.

On with the proof.

<u>Lemma 8.4</u> $F_a^\delta + F_b^\delta + F_c^\delta = 1$.

<u>Proof</u> Let $\mathcal{H}^\delta_{az,bc}$ be the set of loop conf in Ω^δ with a path $A^\delta - z^\delta$ and — $B^\delta - C^\delta$.

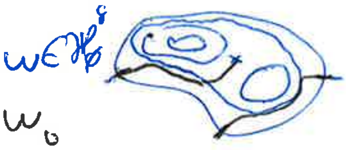
$\mathcal{H}^\delta_\emptyset$ ——— with no extra path.

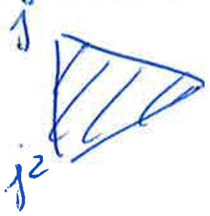
Let ω_0 be a fixed element of $\mathcal{H}^\delta_{az,bc} \cup \mathcal{H}^\delta_{bz,ac} \cup \mathcal{H}^\delta_{cz,ab}$.

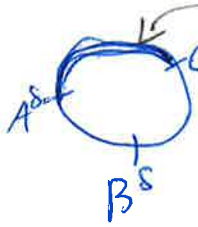
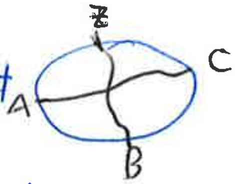
Then $\omega \mapsto \omega \Delta \omega_0$ (symmetric difference, or XOR operation)

is involutive, and is a bijection between $\mathcal{H}^\delta_\emptyset$ and ◯.

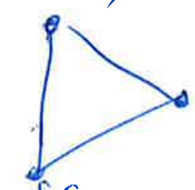
So $\#\mathcal{H}^\delta_{az,bc} + \#\mathcal{H}^\delta_{bz,ac} + \#\mathcal{H}^\delta_{cz,ab} = \#\mathcal{H}^\delta_\emptyset = 2^N$. $\boxtimes$



With this lemma, $F^\delta(z) = F_a^\delta(z) + j\, F_b^\delta(z) + j^2 F_c^\delta(z)$ is a convex combination of $1, j, j^2$, so F^δ takes values in  $1 =: T$

Moreover, for $z \in (C^\delta A^\delta)$,  $F_b^\delta(z) = 2^{-N} \#_A$ 
$= 0$ (no crossing)

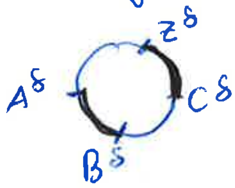
so $F^\delta(z) \in$ 

Therefore $\partial\Omega \xrightarrow{F^\delta} \partial T$, and 

Again for $z \in (C^\delta A^\delta)$,

$$F_a^\delta(z) = \mathbb{P}_{1/2}\left((a^\delta b^\delta) \xleftrightarrow{\text{path of black hexagons}} (c^\delta z^\delta) \right)$$

indeed, let ω_0 be the two paths on the boundary $\partial\Omega^\delta$;



then $\mathcal{H}_\emptyset^\delta \to \mathcal{H}_{az,bc}^\delta \cup \ldots \cup \ldots$
$\omega \mapsto \omega \,\Delta\, \omega_0$

is a bijection between perco. conf st  and $\mathcal{H}_{az,bc}^\emptyset$.

a path of black hexagons

(Ω, A, B, C, D) topo rect.

So $F_a^\delta(D)$ is the proba. we want to compute!

To find that $\lim_{\delta\to 0} F_a^\delta(D) = \ell$, it is enough to

find $\lim_{\delta \to 0} F^\delta(D)$, which would be $\ell + j^2(1-\ell)$.

$$'' \; F_a^\delta(D) + j \underbrace{\cancel{F_b^\delta(D)}}_{0} + j^2 \underbrace{F_c^\delta(D)}_{= 1 - F_a^\delta(D)}$$

So let us find $\lim_{\delta\to 0} F^\delta(D)$.

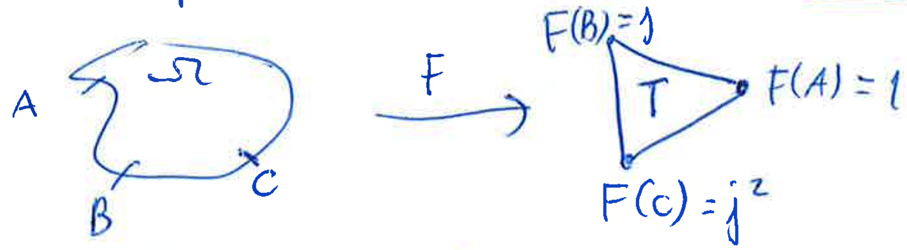
Imagine that we have $F^\delta \xrightarrow[\delta\to 0]{} F$ uniformly on compacts of Ω.

Then F is continuous.

Moreover, for any closed path γ in Ω,

$\oint_\gamma F = 0$ (by taking limits of $\oint_{\gamma^\delta} F^\delta \dots$).

Moréra's theorem implies that F is holomorphic. Moreover, F can be seen to be injective on $\partial\Omega$, which implies that F is injective (conformal) (results of complex analysis). So F is <u>the</u> conformal map



$(\partial\Omega \to \partial T)$.

and $F(D)$ is on the segment $[F(C)\ F(A)]$.

Therefore, $\lim_{\delta\to 0} F^\delta(D) = F(D)$
$= \ell + j^2(1-\ell)$

and $|F(D) - F(C)| = |\ell(1-j^2)| = \sqrt{3}\,\ell$.

Notice that T has edge length $\sqrt{3}$, so composing F with a scaling gives ϕ and the theorem.

But why do we have $F^\delta \xrightarrow[\delta\to 0]{} F$ unif. on compacts?

We already know that if a subsequence on δ's is such that $(F^{\delta_k})_{k\geq 0}$ converges unif. on compacts, the limit has to be F, which is uniquely determined. By a standard argument, it is enough to show that $(F^\delta)_{\delta>0}$ is a precompact family, i.e. has subsequential limits.

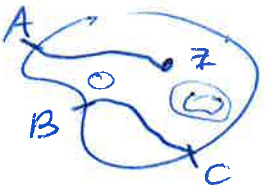
Lemma 8.5 $(F^\delta)_{\delta>0}$ is precompact for the topology of uniform convergence on compacts of Ω.

Proof By Ascoli-Arzela's theorem, it is enough to prove uniform equicontinuity (as $(F^\delta(z))_{\delta>0}$ is clearly bounded for any $z\in\Omega$ ...).

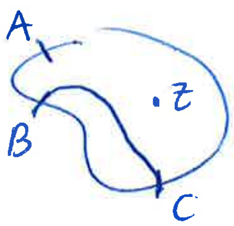
$\forall K$ compact of Ω,

$$\forall \varepsilon>0,\ \exists \eta>0 \ /\ \forall \delta>0,\ \forall z,z'\in K,$$
$$|z-z'|\leq\eta \Rightarrow |F^\delta(z)-F^\delta(z')|\leq\varepsilon.$$

We prove this for $(F_a^\delta)_{\delta>0}$, which easily gives it for $(F^\delta)_{\delta>0}$.

$F_a^\delta(z) = 2^{-N}\ \#\,\mathrm{conf}$ 

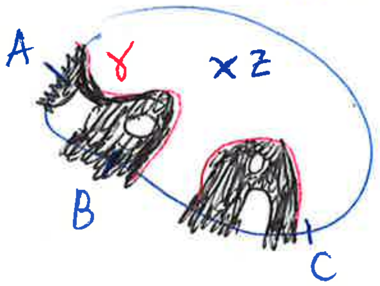
symmetric diff. with a path $A^\delta - z^\delta$ fixed

$= 2^{-N}\ \#\,\mathrm{conf}$ 

loops & a path $B^\delta - C^\delta$ that does not disconnect A^δ from z^δ.

sym. diff. with a boundary path $B^\delta - C^\delta$

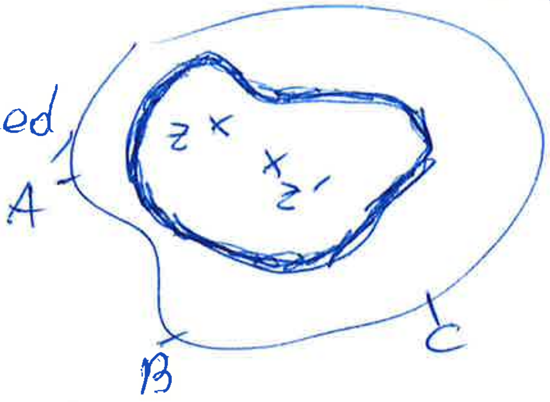
$$= \mathbb{P}_{1/2}\left(\gamma \text{ the boundary of black hex. touching } (B^\delta C^\delta) \text{ does not disconnect } A^\delta \text{ from } z^\delta\right)$$



So $|F_a^\delta(z) - F_a^\delta(z')| \lesssim \mathbb{P}_{1/2}(\gamma$ disconnects exactly one of z^δ, z'^δ from $A^\delta)$

$\lesssim \mathbb{P}_{1/2}($no circuit of black hex. around both $z, z')$

Indeed,



implies that γ has to disconnect both z, z' from A or none of them.

We can now use RSW theory to bound $|F_a^\delta(z) - F_a^\delta(z')|$. Recall $\forall n$, $\mathbb{P}_{1/2}(\,\cdot\,) \geq c > 0$. ($3^n$, n)

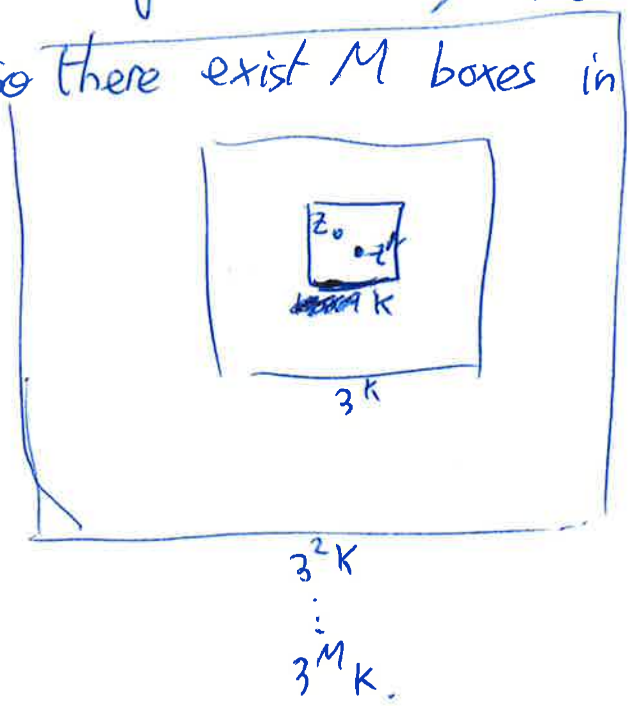
Suppose $z, z' \in K$ compact, and $\varepsilon > 0$ is fixed.

We know that $d(K, \partial\Omega) = c(K) > 0$

Let $M \geq \frac{\log \varepsilon}{\log 1-c}$, and $\delta \leq \frac{c(K)}{3^M}$.

Then if $|z'-z| \leq \delta$, we have $3^M|z-z'| \leq c(K) = d(K, \partial\Omega)$

so there exist M boxes in Ω s.t.:



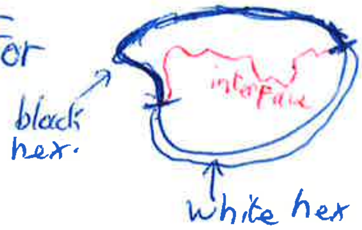
Therefore

$$\mathbb{P}_{1/2}(\text{no } z\ z') \leq (1-c)^M \leq (1-c)^{\log \varepsilon / \log 1-c} = \varepsilon .$$

$//$

$$|F_a(z) - F_a(z')|$$

Remark: Conformal invariance goes deeper than crossing estimates. For instance, it can be shown that the black white interface for



converges to a random continuous curve, whose distribution is conformal invariant, and markovian in some sense.

It is known as a Schramm-Loewner Evolution (of parameter 6).