Notes on combinatorics of permutations

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0 Introduction: definitions

These notes are mostly based on the book Combinatorics of Permutations by Miklos Bona [Bon22]; mainly a sub-collection of topics/results found in the book, picked and occasionally rewritten according to our taste. There is of course a small amount of additional material as well.

Definition 0.1. *A linear ordering of the elements of the set* $[n] = \{1, 2, 3, \dots, n\}$ *is called a permutation.*

We denote an n-permutation by $p = p_1p_2 \dots p_n$ with p_i being the ith entry in the linear order p. Well, the number on n-permutations is $n!$.

1 Breaking the order locally

1.1 Descents

Definition 1.1. *Let* $p = p_1p_2 \dots p_n$ *be a permutations and* $i < n$ *be a positive integer. We say that* i *is a descent of* p *if* $p_i > p_{i+1}$*. Similarly, we say that* i *is an ascent of* p *if* $p_i < p_{i+1}$*.*

Example 1.2. *Let* $p = 351264$*. Then* 2 *and* 5 *are descents of* p*.*

The immediate question for an enumerative combinatorist: *What is the number* $A(n, k)$ *of* n*-permutations with* $k - 1$ *descents?* Note that a permutation with $k - 1$ descents has k increasing runs, hence the convention.

On the go. Write down the 3-permutations with one descent.

The numbers $A(n, k)$ are called Eulerian numbers. We will see why so later on. Let's first prove a recurrence relation.

Theorem 1.3. *For all positive integers* k *and* n *satisfying* $k < n$*, one has*

$$A(n, k + 1) = (k + 1)A(n - 1, k + 1) + (n - k)A(n - 1, k)\,.$$

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Proof. Note that one can obtain an n-permutation p by inserting the entry n into an $(n-1)$-permutation p' in n different positions. If p' has k descents, $k+1$ out of n positions will produce an n-permutation with k descents again, in cases where we insert the entry n right after one of the descents of p' or at the end of p', i.e. in $k+1$ cases out of n. All other insertions give an n-permutation with $k+1$ descents.

Hence we can obtain an n-permutation with k descents in

1. $(k+1)$ ways from each $(n-1)$-permutation with k descents, resulting in $(k+1)A(n-1,k+1)$
2. $(n-k)$ ways from each $(n-1)$-permutation with $k-1$ descents, resulting in $(n-k)A(n-1,k)$.

□

Note that $A(n,k+1) = A(n,n-k)$, i.e. the Eulerian numbers are *symmetric*. Indeed, if $p = p_1p_2\ldots p_n$ has k descents, then its reverse $p^r = p_np_{n-1}\ldots p_1$ has $n-1-k$ descents. Actually, there exists a closed formula for $A(n,k)$ (in the form of a sum, but ok...)!

Theorem 1.4. *For all non-negative integers n and k such that $k \leq n$, the following holds:*

$$A(n,k) = \sum_{i=0}^{k}(-1)^i\binom{n+1}{i}(k-i)^n\,.$$

Proof. Let $k>0$ and let us place our n elements in k boxes (once you put them in the box, you can order them in increasing order). We will draw this using $k-1$ bars. There are k^n ways to do this, this is the first term in the sum, giving us the number of permutations with *at most* $k-1$ descents. The problem is, it also counts the ones like

$$2\ 3 \mid 1\ 4 \mid 5 \mid 6 \quad \text{or} \quad 2\ 3 \mid \mid 1\ 4\ 5 \mid 6\,.$$

Let us call a bar a **wall** if it's not followed by another bar. We call it an **extra wall** if after removing it we still get a valid configuration, i.e. one with elements in increasing order in each box. The permutations with no extra walls are exactly the ones that we want to count.

Let $S \subset [n+1]$ be a subset of $n+1$ positions between the n entries (positions might hold multiple bars). Let A_S be the set of arrangements in which there is an extra wall at each position in S and possibly elsewhere.

Note that we want the cardinality of $A \setminus \left(\bigcup_{i=1}^{n+1} A_{\{i\}}\right)$ where A is the set of all configurations i.e. $|A| = k^n$. Noting that $A_{\{i,j\}} = A_{\{i\}} \cap A_{\{j\}}$ etc, inclusion/exclusion principle implies:

$$\begin{aligned} A \setminus \left(\bigcup_{i=1}^{n+1} A_{\{i\}}\right) &= |A| - |A_{\{1\}}| - \cdots - |A_{\{n+1\}}| \\ &\quad + |A_{\{1,2\}}| + |A_{\{1,3\}}| + \cdots + |A_{\{n,n+1\}}| \\ &\quad - |A_{\{1,2,3\}}| - \ldots \\ &= \sum_{\substack{S \subset \{1,2,\ldots,n+1\} \\ |S| \leq k-1}} (-1)^{|S|}|A_S|\,. \end{aligned}$$

Now we show that $|A_S| = (k-i)^n$ where $|S| = i$. To see this, take a valid configuration with $k-i-1$ bars. There are $(k-i)^n$ such arrangements. Now insert a bar in every position in S (if the position is occupied, we can still insert a bar to the right of it, creating an empty box). Inserting a bar to each position in S now will give an element of A_S. Conversely, one can start with an element of A_S and remove the bars in the positions belonging to S, leading to a valid configuration with $k-i$ boxes. Together with the fact that the i elements of S can be chosen in $\binom{n+1}{i}$ ways, this concludes the proof. □

1.2 Generating functions & Eulerian numbers

Recall that $A(0,0) = 1$ and $A(n,0) = 0$ for $n > 0$.

Definition 1.5. *For all non-negative integers n, the n'th Eulerian polynomial is given by*

$$A_n(z) = \sum_{k=0}^{n} A(n,k) z^k \,.$$

However, Euler first defined these polynomials in the following form.

Theorem 1.6. *For all positive integers n, the n'th Eulerian polynomial can also be expressed as*

$$A_n(z) = (1-z)^{n+1} \sum_{i \geq 0} i^n z^i \,.$$

On the go. Write down $A_1(z)$ and $A_2(z)$.

Proof. By Theorem 1.4,

$$\begin{aligned}
\sum_{k=0}^{n} A(n,k) z^k &= \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n z^k \\
&= \sum_{k=0}^{n} \left(\sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} i^n \right) z^k \\
&= \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} (-1)^{k-i} \binom{n+1}{k-i} z^{k-i} i^n z^i \\
&= \sum_{i \geq 0} i^n z^i \sum_{k \geq i} \binom{n+1}{k-i} (-z)^{k-i} \\
&= (1-z)^{n+1} \sum_{i \geq 0} i^n z^i
\end{aligned}$$

where from second to third line we used that the expression in the parenthesis being equal to $A(n,k)$ vanishes for $k > n$. □

We are now ready to write down the exponential generating function of Eulerian polynomials. This has a nice, compact form.

Theorem 1.7. *Let*

$$r(t,z) = \sum_{n\geq 0}\sum_{k\geq 0} A(n,k) t^k \frac{z^n}{n!} .$$

Then the following holds:

$$r(t,z) = \frac{1-t}{1-te^{z(1-t)}} .$$

Proof. By Theorem 1.6,

$$\begin{aligned} r(t,z) &= \sum_{n\geq 0}\sum_{k\geq 0} A(n,k) t^k \frac{z^n}{n!} \\ &= \sum_{n\geq 0} (1-t)^{n+1} \sum_{i\geq 0} i^n t^i \frac{z^n}{n!} \\ &= (1-t) \sum_{i\geq 0} t^i \sum_{n\geq 0} \frac{(iz(1-t))^n}{n!} \\ &= \frac{1-t}{1-te^{z(1-t)}} . \end{aligned}$$

□

1.2.1 The other way around: Euler sums

This part is the other, historical way to discover Eulerian numbers.

Suppose you know nothing of the previous results, and you want to look at the power series $S_n(z) = \sum_{i\geq 0} i^n z^i$ for successive values of n, as a function of $z \in \mathbb{C}$ such that $|z| < 1$. When $n = 0$, this is the well-known geometric series

$$S_0(z) = \sum_{i\geq 0} z^i = \frac{1}{1-z}. \tag{1.1}$$

Then, a classical trick allows us to compute $S_1(z)$ by differentiating:

$$S_1(z) = \sum_{i\geq 0} iz^i = z \sum_{i\geq 1} iz^{i-1} = zS_0'(z) = \frac{z}{(1-z)^2}. \tag{1.2}$$

This trick can in fact be iterated, and we get similarly, after some computations,

$$S_2(z) = \sum_{i\geq 0} i^2 z^i = zS_1'(z) = \cdots = \frac{z+z^2}{(1-z)^3}. \tag{1.3}$$

In general, we have $S_{n+1}(z) = zS_n'(z)$, and after a straightforward iteration, $S_n(z)$ can be written as

$$S_n(z) = \frac{B_n(z)}{(1-z)^{n+1}} \tag{1.4}$$

where $B_n(z)$ is a polynomial. The degree of B_n is n, and it must have a root at 0, except for $n = 0$. Also, we can see in the iteration that its coefficients are integers. Therefore, let us write

$$B_n(z) = \sum_{k=0}^{n} B(n,k) z^k \tag{1.5}$$

and here, big surprise... The $B(n,k)$ are indeed $A(n,k)$, number of permutations with $k-1$ descents!

Exercise 1.8. *Show by induction (without making use of Theorem 1.6) that*

$$\sum_{i\geq 0} i^n z^i = \frac{A_n(z)}{(1-z)^{n+1}}\,.$$

Hint: Take the derivative of both sides and multiply by z, then compare the coefficients of z^k of what you get in the numerator and $A_{n+1}(z)$ to conclude the proof. You might need Theorem 1.3.

n=1						1						
n=2					1		1					
n=3				1		4		1				
n=4			1		11		11		1			
n=5		1		26		66		26		1		
n=6	1		57		302		302		57		1	

1.3 Eulerian numbers and random variables

Let P be a *random* permutation taken uniformly among the $n!$ possibilities. That is, for any fixed permutation p, we have $\mathbb{P}(P=p)=\frac{1}{n!}$.

We are interested in the random variable counting the number of descents in P, which we denote by D_n. In other words, it has the following distribution:

$$\forall k \in [n], \mathbb{P}(D_n = k-1) = \frac{A(n,k)}{n!}. \tag{1.6}$$

Theorem 1.9 (Tanny [Tan73])**.** *The random variable D_n has the same distribution as $\lfloor U_1 + \cdots + U_n \rfloor$, where $U_1, \ldots, U_n$ are iid random variables uniform in $[0,1]$ (and $\lfloor \cdot \rfloor$ is the integer part).*

Proof. Let us construct a random permutation using uniform random variables. Since $U_1, \ldots, U_n$ are iid uniform, we can use them to produce a uniform permutation: they are almost surely all different, so looking at their relative value produces a permutation (formally defined by $\sigma(i) = \mathrm{card}\{j \mid U_j \leq U_i\}$, for instance, for $n=3$, if $U_3 > U_1 > U_2$ we get $\sigma = (213)$). Clearly, all permutations are equally likely, so σ is uniformly distributed. That construction is interesting, but it doesn't give the theorem as such, we have to do this on another family of random variables instead of the U_i. The following idea seems to be due to Stanley [Sta77].

Let V_i be the *fractional* part of $U_1 + \cdots + U_i$, that is,

$$V_i = U_1 + \cdots + U_i - \lfloor U_1 + \cdots + U_i \rfloor.$$

We claim that $V_1, \ldots, V_n$ are also iid uniform random variables in $[0,1]$. Indeed, $V_1 = U_1$, and V_{i+1} is the fractional part of $V_i + U_{i+1}$; but U_{i+1} is independent from whatever happened before, and if we condition on V_i, we are just adding a uniform random variable to a constant and reducing the result modulo 1, which creates a new uniform random variable, no matter what the constant was.

Therefore, we can use the V_i to create a random permutation as we did before, and it is again a uniform permutation. But now, when does this permutation have a descent? It has a descent at i iff $V_{i+1} < V_i$, which is equivalent to the fact that $V_i + U_{i+1} \geq 1$ (we went overflow when adding the uniform U_{i+1}). But this also means that $\lfloor U_1 + \cdots + U_i \rfloor < \lfloor U_1 + \cdots + U_{i+1} \rfloor$. In other words, we have a descent whenever the integer part $\lfloor U_1 + \cdots + U_i \rfloor$ increases by one. In total, the number of descents is $\lfloor U_1 + \cdots + U_n \rfloor$. □

Thus we get the interesting formula:

$$A(n,k) = n!\mathbb{P}(U_1 + \cdots + U_n \in [k-1,k]). \tag{1.7}$$

Exercise 1.10. *Use the Central Limit Theorem and Slutsky's lemma to prove*

$$\frac{1}{\sqrt{n}}\left(\lfloor U_1 + \cdots + U_n \rfloor - \frac{n}{2}\right) \to^d \mathcal{N}\left(0, \frac{1}{12}\right).$$

Deduce an asymptotic statement about the numbers $A(n,k)$ for n large, of the form

$$\frac{1}{n!}\sum_{k=\frac{n}{2}+\alpha\sqrt{n}}^{\frac{n}{2}+\beta\sqrt{n}} A(n,k) \sim_{n\to\infty} \cdots .$$

1.4 Sequences of Eulerian numbers

Definition 1.11. *We say that the sequence $a_1, \ldots, a_n$ of positive real numbers is log-concave if for every indice k, $a_{k-1}a_{k+1} \leq a_k^2$.*

Definition 1.12. *We say that a sequence $a_1, \ldots, a_n$ of positive real numbers has real roots only if the polynomial $\sum\limits_{k=1}^{n} a_k x^k$ has real roots only.*

We want to show that for any fixed n, the Eulerian numbers $\{A(n,k)\}_k$ are log-concave. For this, we will pass by the following theorem of Newton.

Theorem 1.13. *If a sequence of positive real numbers has real roots only, then it is log-concave.*

Proof. Let $a_0, \ldots, a_n$ be our real positive sequence and $P(x) = \sum_{k=0}^n a_k x^k$ be the corresponding polynomial having real roots only. We consider its homogenized polynomial:

$$Q(x,y) = \sum_{k=0}^{n} a_k x^k y^{n-k}.$$

We say that Q *has only real roots* if for every $(x,y) \in \mathbb{C}^2 \setminus \{(0,0)\}$ that is a root of Q, we have $\frac{x}{y} \in \mathbb{R}$. Note that y cannot be zero, because $Q(x,0) = a_n x^n$ has no root, except for $x = 0$ which is excluded in the definition. In fact, we would get an equivalent definition by requiring that $\frac{y}{x} \in \mathbb{R}$: the definition is symmetric in the two variables x and y.

Note also that if the sequence $a_0, \ldots, a_n$ has real roots only, then Q has only real roots since otherwise $\frac{x}{y}$ would be a non-real root of P.

We claim that when a homogeneous polynomial of two variables has only real roots, then so does $\frac{\partial Q}{\partial x}$. Indeed, let $y_0 \in \mathbb{R}$ be fixed. Then we see $Q(x, y_0)$ as a polynomial in x. It has degree

n, so it has n complex roots with multiplicity, but by the assumption on Q they are all real (this is true as well for $y_0 = 0$, for which $x = 0$ is the only root and has multiplicity n). We can apply Rolle's theorem to this one-variable polynomial, which gives that its derivative has $n-1$ real roots (with multiplicity). Those are all the zeros of $x \mapsto \frac{\partial Q}{\partial x}(x, y_0)$ since its degree is $n-1$. Therefore,

$$\frac{\partial Q}{\partial x}(x, y_0) = 0 \implies \frac{x}{y_0} \in \mathbb{R}.$$

Now, when $y_0 \in \mathbb{C}$, we write $y_0 = \rho e^{i\theta}$. If (x, y) is a root of $\frac{\partial Q}{\partial x}$, by homogeneity, $(e^{-i\theta}x, e^{-i\theta}y_0)$ is also a root. But this is also $(e^{-i\theta}x, \rho)$, and by the first case, $\frac{e^{-i\theta}x}{\rho}$ is real, but this is also $\frac{x}{y}$.

Similarly, by the aforementioned symmetry in x/y of the definition, $\frac{\partial Q}{\partial y}$ has real roots only. We can iterate this argument and conclude that the polynomial $\partial^{a+b}Q/\partial x^a \partial y^b$ has real roots only for all a, b such that $a + b < n$. In particular this is true for $a = j-1$ and $b = n-j-1$, for some fixed $1 \le j \le n-1$. If $(x, y), y \neq 0$ is a root of $R(x, y) = \partial^{n-2}Q/\partial x^{j-1}\partial y^{n-j-1}$, so is $\frac{x}{y}$ of $\frac{R(x,y)}{y^2}$ seen as a second degree polynomial in $\frac{x}{y}$. Since $\frac{x}{y}$ is real, the latter has non-negative discriminant. We can directly compute the non-zero terms of $R(x, y)$ coming from $k = \{j-1, j, j+1\}$ terms in the summand, since all others vanish after derivation. Hence

$$R(x, y) = a_{j-1}(j-1)!\frac{(n-j+1)!}{2}y^2 + a_j j!(n-j)!xy + a_{j+1}\frac{(j+1)!}{2}(n-j-1)!x^2 .$$

Its discriminant Δ being ≥ 0 implies that

$$a_j^2 \ge \frac{j+1}{j}\frac{n-j+1}{n-j}a_{j-1}a_{j+1} \implies a_j^2 \ge a_{j-1}a_{j+1} .$$

□

We can now use this result to conclude from the following theorem that Eulerian numbers are log-concave.

Theorem 1.14. *For any fixed n, the sequence $\{A(n, k)\}_k$ of Eulerian numbers has real roots only. In other words, all roots of the polynomial*

$$A_n(z) = \sum_{k=1}^{n} A(n, k)z^k$$

are real.

Proof. You have already shown in Exercise 1.8 that for $n \ge 1$,

$$\begin{aligned} A_n(z) &= z(1-z)A'_{n-1}(z) + nzA_{n-1}(z) \\ &= z(1-z)^{n+1}\frac{d}{dz}\left(\frac{A_{n-1}}{(1-z)^n}\right) . \end{aligned} \tag{1.8}$$

The Eulerian polynomial $A_1 = z$ vanishes only at $z = 0$. Suppose that A_{n-1} has $n-1$ negative roots with one at 0 (note that the roots cannot be positive since the coefficients of our polynomials are positive). By Rolle's theorem Eq. (1.8) implies that between any two distinct roots of A_{n-1}, A_n will have a root. If A_{n-1} has some root with multiplicity $l > 1$, A_n will have the same root with multiplicity $l-1$. Together with the root at 0 which can be seen from Eq. (1.8), this accounts for $n-1$ roots of A_n. The remaining root must also be real, since the complex roots of polynomials with real coefficients come in conjugate pairs. This concludes the proof. □

1.5 Alternating permutations

In this course, we do not dive into the study of permutations through alternating runs. We refer the curious reader to the Chapter 1.2 and 1.3 of the book. We will still conclude this chapter by a nice result concerning the alternating permutations.

Definition 1.15. *Given a permutation of length n, an alternating (resp. reverse alternating) subsequence is a subsequence $p_{i_1} \dots p_{i_k}$ s.t. $p_{i_1} > p_{i_2} < p_{i_3} > \dots$ (resp. $p_{i_1} < p_{i_2} > p_{i_3} < \dots$). We say that p is (reverse) alternating if its longest (reverse) alternating subsequence is of length n.*

The number of alternating n-permutations is called an Euler zigzag number and denoted by E_n.

On the go. Find out the first 4 Euler zigzag numbers (we set $E_0 = E_1 = 1$, skip them).

Theorem 1.16. *The exponential generating function $E(z)$ of Euler zigzag numbers E_n is given by*

$$E(z) = \sum_{n \geq 0} E_n \frac{z^n}{n!} = \sec z + \tan z\,.$$

Proof. First, note that a permutation p is alternating if and only if its complement $p^c := (n+1-p_1)\ (n+1-p_2)\ \dots$ is reverse alternating. This suggests that the number of alternating and reverse alternating permutations are the same. We can now decompose each alternating permutation of length $n+1$ into left L and right R permutations (possibly of 0 length) such that $p = L(n+1)R$. Note that since $n+1$ is the maximal element of p, R and the reverse of L are both reverse alternating permutations. Choosing the position k of $(n+1)$ then concatenating it with two reverse alternating permutations gives us all alternating *and* reverse alternating permutations of length $n+1$, once summed over k. Hence for $n \geq 1$ we can write

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}\,.$$

This is equivalent to (check it now by comparing the coefficients of $\frac{z^n}{n!}$ terms of both sides!)

$$2E'(z) = E^2(z) + 1$$

with the initial condition $E(0) = 1$. This differential equation can be easily solved to obtain the desired result. □

The numbers E_n can be computed by an interesting algorithm, called the "boustrophedon" algorithm, which proceeds in a way similar to an ox plowing a field. This description is better followed by looking at Figure 1.1. We start on row 0, where we write a 1. Then on row 1, we write down a 0 on the left, and we go towards the right by adding the number we see on top every time. Then we proceed to row 2, by writing a 0 on the right and going right-to-left. In general, we go left-to-right on odd rows, and right-to-left on even rows, adding the number on top of our step every time.

Then, a remarkable thing happens: the last number computed on row n is exactly E_n. This is a quick way to find the expansion of tan and sec in case you forget them!

To prove this, we generalize the Euler zigzag number by denoting E_n^k for the number of alternating n-permutations such that $p_1 \geq n-k+1$. Here $0 \leq n$, and clearly $E_n^0 = 0$ and $E_n^n = E_n$.

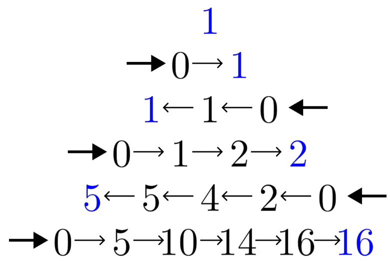

Figure 1.1: The boustrophedon algorithm for the computation of Euler zigzag numbers

Proposition 1.17. *For any* $1 \leq k \leq n$*,*

$$E_n^k = E_n^{k-1} + E_{n-1}^{n-k}$$

Proof. Consider an alternating n-permutation such that $p_1 \geq n-k+1$. Two things can occur:

- either $p_1 > n-k+1$. There are E_n^{k-1} such permutations.
- either $p_1 = n-k+1$. Then we remove the first entry and reduce by 1 all entries that are larger than p_1 to obtain a valid permutation of length $n-1$. We hence get a *reverse* alternating permutation of length $n-1$, whose first term is *smaller than or equal* to $n-k$. The complement of this permutation is alternating and has first entry $\geq k$. These are counted by E_{n-1}^{n-k}, and the transformation was clearly bijective.

□

Proposition 1.17 and careful index-checking should be enough to convince yourself that the boustrophedon algorithm of Figure 1.1 produce the numbers E_n^k, on row n at position k in the boustrophedon order. Therefore, at the end of each row computation, we have E_n.

1.6 Exercises

Exercise 1.18. *We have* n *boxes numbered from* 1 *to* n*, and* n *balls numbered from* 1 *to* n*. We drop the balls successively in order, starting from ball* 1*, with the constraint that for all* $i \in [n]$*, the ball* i *has to be dropped into a box with label* $\leq i$*.*

Let $B(n,k)$ *be the number of total choices for this procedure, such that at the end,* $k-1$ *boxes are left empty. Prove that* $B(n,k) = A(n,k)$*.*

Try to find a bijective proof of this formula.

Exercise 1.19. *We consider the square grid* $\mathbb{Z}^2$ *as a graph, with edges between nearest-neighbours. We put a* weight *on every edge: the edge* $\{(i,j),(i+1,j)\}$ *gets weight* j*, and the edge* $\{(i,j),(i,j+1)\}$ *gets weight* i*.*

Let s *be a lattice path going from* $(1,1)$ *to* $(n-k+1,k)$ *by going every step to the east or to the north (in particular it must have* $n-1$ *steps). Define its weight* P_s *as the product of all edge weights on* s*. Show that*

$$A(n,k) = \sum_s P_s.$$

Exercise 1.20. *A* decreasing non-plane 1-2 tree *is a rooted tree on vertex set* $[n]$ *in which each non-leaf vertex has at most two children, and the label of each vertex is smaller than that of its parent. Let* T_n *denote the number of decreasing non-plane trees on vertex set* $[n]$*. Prove that* $T_n = E_n$*.*

Try to find a bijective proof as well.

2 Inversions

In Section 1, we counted permutations through descents, which is a local move: an entry being greater than the one following it. We shall now try to enumerate them by inversions, which is a more global move: an entry being smaller than some entry(ies) following it.

Definition 2.1. *Let* $p = p_1 \dots p_n$ *be a permutation. We say that* (p_i, p_j) *is an inversion of* p *if* $i < j$ *and* $p_i > p_j$*.*

On the go. List the inversions of 31425.

Let us denote the number of inversions of p by $i(p)$. Clearly, $p = 12\dots n$ has no inversions, i.e. $i(p) = 0$. What would be the obvious upper bound for $i(p)$ and what permutation would realize it?

Theorem 2.2. *For all* $n \geq 2$

$$I_n(z) := \sum_{p \in S_n} z^{i(p)} = (1+z)(1+z+z^2)\dots(1+z+\dots+z^{n-1}).$$

Proof. Note that the right hand side has $n!$ terms. We shall prove that each of these terms correspond to a unique permutation in S_n where its contribution to the sum $z^{a_1}z^{a_2}\dots z^{a_{n-1}}$ is such that for each $i \in [n-1]$, the entry $i+1$ is followed by exactly a_i entries that are smaller than itself.

Check the statement for $n = 2$ **and find** $I_2(z)$. Suppose now, for induction, that the statement is true for $n-1$ and and let p be an $(n-1)$-permutation represented by $z^{a_1}\dots z^{a_{n-2}}$. Inserting n into p results in an (n)-permutation, name it q. If it is inserted so that it precedes i entries, than q shall be represented by $z^{a_1}\dots z^{a_{n-2}}z^i$. Depending on where we insert n, q has 0 or 1 or … or $n-1$ more inversions than p. Since p was arbitrary

$$I_n(z) = (1+z+z^2+\dots+z^{n-1})I_{n-1}(z) = (1+z)(1+z+z^2)\dots(1+z+\dots+z^{n-1}).$$

□

Denoting the number of n-permutations with k inversions by $b(n,k)$, we can write

$$I_n(z) = \sum_{k=0}^{\binom{n}{2}} b(n,k) z^k .$$

On the go. With Theorem 2.2, convince yourself that for any $0 \leq k \leq \binom{n}{2}$, $b(n,k) \neq 0$.

It is possible to obtain a closed formula for $b(n,k)$ when $k \leq n$, but it is cumbersome. We will not cover it in this course but refer the reader to [Bon22][Theorem 2.15] (and the machinery developed therein from Lemma 2.5 on). We will be contented with a recursion relation.

Proposition 2.3. *Let $n \geq k$. Then*

$$b(n+1,k) = b(n+1,k-1) + b(n,k)\,.$$

Proof. Start by an $(n+1)$-permutation $p_1 \dots p_{n+1}$ with k inversions. If $n+1$ is at the last position, removing it results in a (n)-permutation with k inversions. If $n+1$ is at the i'th position with $i \leq n$, we can swap it with the entry following it and obtain an $(n+1)$-permutation with $k-1$ descents, where $n+1$ is not at the first position. Note that $n+1$ being at the first position results in at least n descents, and since $n > k-1$ by assumption, no $(n+1)$-permutation with $k-1$ descents can have $n+1$ at the first position. This concludes the proof. □

2.1 Some applications

2.1.1 Determinants

Everyone who had some undergraduate math courses is familiar with the following definition of the determinant of an $n \times n$ matrix $A = (a_{ij})$:

$$\det A = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det A_{1j}$$

where A_{1j} is the $(n-1) \times (n-1)$ matrix obtained by removing the first row and the jth column of A. Here is another formula:

Theorem 2.4. *Let A be an $n \times n$ matrix. Then*

$$\det A = \sum_{p \in S_n} (-1)^{i(p)} a_{1p_1} a_{2p_2} \dots a_{np_n}\,. \tag{2.1}$$

Proof. Check it for $n = 2$. We now proceed by induction. Assuming that the statement holds for $(n-1) \times (n-1)$ matrices we have

$$\det A_{1j} = \sum_{q} (-1)^{i(q)} a_{2q_2} a_{3q_3} \dots a_{nq_n}$$

where $q = q_2 q_3 \dots q_n$ is a permutation of the integers from 1 to n **except** j. Setting $p_1 = j$ we see that $p = j q_2 \dots q_n$ has exactly $j-1$ more inversions than q, i.e. $i(p) = i(q) + j - 1$. This is true for any j, hence inserting this last expression into Eq. (2.1), we obtain the desired result. □

2.1.2 Perfect matchings of bipartite graphs

The previous expansion of determinants as a sum over permutations has another useful application. Let G be a finite *bipartite* graph, that is, a graph whose vertex set is the disjoint union of two sets X and Y, such that every edge joins a vertex in X and a vertex in Y. A *perfect matching* of G is a subset M of edges such that every vertex of G belongs to exactly one edge in M.

Given a bipartite graph G, can we easily know if it admits at least one perfect matching? Clearly, it is necessary that $|X| = |Y|$. When this is the case, we label the elements of X and Y: $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. We consider the square matrix $B(G)$, such that $B(G)_{i,j} = 1$ when x_i, y_j are adjacent, and 0 otherwise. We call it the *truncated adjacency matrix*, as its size is half that of the usual adjacency matrix.

On the go. Write down the matrix $B(G)$ for G a cycle graph of length 6, or hexagon.

The following statement gives a sufficient condition for G to have a perfect matching:

Theorem 2.5. *Let G be a bipartite graph such that $|X| = |Y|$. If G does not have a perfect matching, then* $\det B(G) = 0$*.*

By contrapositive, if $\det B(G) \neq 0$, then G has to admit a perfect matching. This is a simple, polynomial-time computable sufficient condition for our initial question.

Proof. By (2.1),

$$\det B(G) = \sum_{p \in S_n} (-1)^{i(p)} b_{1,p_1} \cdots b_{n,p_n}.$$

But the product of entries of $B(G)$ is non-zero if, and only if, the set of edges $\{x_i, y_{p_i}\}$ is a perfect matching (in that case, it is a product of 1's, and in the other case, it is simply 0). Therefore, when G has no perfect matching, the determinant must be 0. □

Again, the fact that $\det B(G) \neq 0$ is only a sufficient condition for the existence of a perfect matching. In fact:

On the go. Find a graph bipartite graph G that admits a perfect matching but such that $\det B(G) = 0$.

Let us mention that the problem of knowing wether a generic bipartite graph has a perfect matching or not can in fact be completely solved by a polynomial algorithm (more precisely, there is a polynomial algorithm that outputs a matching of maximum cardinality). Much harder is the question of *counting* how many perfect matchings G has. This problem can be proven to be *harder* than NP-complete problems. In terms of the matrix $B(G)$, it amounts to computing the *permanent*

$$\text{perm } B(G) := \sum_{p \in S_n} b_{1,p_1} \cdots b_{n,p_n}.$$

Despite its simpler expression, this quantity is significantly harder to compute than determinants.

There is, however, a polynomial-time algorithm restricted to the case of *planar* bipartite graphs. This method is due to Kasteleyn, Temperley and Fisher, physicists who discovered it in the 70s. It consists in choosing a specific orientation of the edges of G, such that every face has an odd number of clockwise edges (as G is planar, it has well-defined faces). Then, one modifies the matrix $B(G)$ to take this orientation into account: $\tilde{B}(G)_{ij}$ is set to be $+1$ if the edge x_i, y_j is oriented in that direction, to -1 if it is in the opposite direction, and 0 again if they are not linked. Then, one can prove that the signs coming from the orientation exactly cancel the term $(-1)^{i(p)}$, which gives that $|\det \tilde{B}(G)|$ is the number of perfect matchings of G.

2.2 Exercises

Exercise 2.6. *Let $n < k \leq \binom{n}{2}$. Prove that*

$$b(n+1, k) = b(n+1, k-1) + b(n, k) - b(n, k-n-1). \tag{2.2}$$

Exercise 2.7. *Let p be a permutation on $[n]$. The* inversion graph *of p is the graph G with vertex set $[n]$, such that $\{i, j\}$ is an edge of G if and only if (i, j) is an inversion of p. Find an example of an* ***unlabeled*** *graph that is not the inversion graph of any permutation (for any labeling of its vertices).*

Exercise 2.8. *Let T be a rooted tree with root 0 and non-root vertex set $[n]$. Define an inversion of T to be a pair (i, j) of vertices so that $i > j$ and i is an ancestor of j (that is, the unique path from 0 to j goes through i). How many such trees have zero inversions?*

3 In cycles

One can also see an n-permutation as a function $f : [n] \to [n]$ that is a bijection. We can then define the product $f \cdot g$ of two permutations f, g as $f \cdot g(i) := f(g(i))$. You can check that this is a group and the set of all permutations S_n is called the symmetric group as well.

Another way of writing down a permutation p is through its decomposition into its *cycles*. For instance, if $f = 34152$, we see that the consecutive applications of f will keep permuting 1 and 3 among each other, and 2,4 and 5 among themselves. We then write down f as (31)(524). Each element in a parenthesis is mapped to the one on its right, and the last one goes to the first one.

On the go. Show that such a decomposition exists.

We start writing down each cycle by its largest element to have a canonical notation. The cycles one obtains are obviously disjoint. Note also that the order in which we write down the cycles does not matter, hence (31)(524) = (524)(31).

On the go. Show that this decomposition is unique.

3.1 Trees and transpositions

It is good to think of the consecutive parentheses as individual mappings leaving the numbers that it does not contain invariant. We now relax the condition that each element should be in one cycle only i.e. we allow cycles that are not disjoint. We can then write any permutation p as a product of not necessarily disjoint 2-cycles that we call *transpositions*. For instance, 312 = (12)(23). The order now matters and this decomposition is not unique: 312 = (23)(13).

On the go. Show (if you want, by induction) that such a decomposition exists for any n.

Exercise 3.1. *Use now the previous result to show that any permutation can be written as a product of adjacent transpositions i.e. transpositions interchanging two consecutive entries.*

Lemma 3.2. *Let $s_2, s_3, \ldots, s_n$ be transpositions on $[n]$. Then $s_n s_{n-1} \ldots s_2$ is a cyclic permutation if and only if the graph G with vertices on $[n]$ and edges $s_2, s_3, \ldots, s_n$ is a tree.*

Proof. Suppose $S := s_n s_{n-1} \ldots s_2$ is a cyclic permutation. For any $i \in [n]$ and $m \in \{2, \ldots, n\}$, there is a path between i and $s_m s_{m-1} \ldots s_2(i)$ in G. Since S is cyclic, for all $i, j \in [n]$, there exist $k \leq n$ such that $S^k(i) = j$. Hence G is connected. Being a connected graph with n vertices and $n - 1$ edges implies that it is a tree (a connected graph with no cycles).

On the go. Show that a connected graph with n vertices and $n - 1$ edges is a tree.

Suppose now that G is a tree with vertex set $[n]$ and edges $s_n \dots s_2$. We shall proceed by induction and assume that a tree on $< n$ vertices gives a cyclic permutation. Removing the edge s_n gives us two trees that are cycles by the induction hypothesis. Calling these cycles that are disjoint C_1 and C_2, we have

$$s_n s_{n-1} \dots s_2 = (a_1 a_2) \cdot C_1 \cdot C_2$$

where $s_n = (a_1 a_2)$ with $a_2 \in C_2$ and $a_1 \in C_1$. This gives us a cycle on $[n]$ and completes the proof. $\square$

References

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