The Ising model

M2 Mathématiques de l'aléatoire

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References:

- "Statistical mechanics of Lattice Systems",
  Friedli & Velenik.

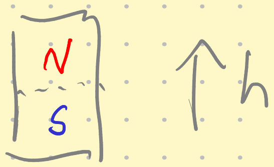
- "Percolation et Modèle d'Ising",
  Werner.

physics: - "Exactly solved Models in Statistical Mechanics",
  Baxter.

chap. 0 - Introduction

Pierre Curie's experiment: ($\simeq$1895)

- Put a piece of iron in a high magnetic field. It becomes a magnet.
- Remove the magnetic field. Depending on the temperature,



hase transition! (cf Percolation course)

↳ If $T > T_c$ ($\simeq 770°C$ for iron), the iron loses its magnetisation

↳ If $T < T_c$, its stays magnetic.

Idea: The piece is made of many small magnetic dipoles

subject to ① local alignment forces (nearby dipoles tend to be close to parallel)

② variations due to heat.

Depending on T, either ① or ② dominates.

Rk: this depiction is actually physically misleading, as it was shown later that exchange of electrons play a key role in ferromagnetism...

In 1920, Lenz introduces a model for Ferromagnetism that takes (1) and (2) into account, and tries to analyse the phase transition using it. In 1925 he gives the 1-dimensional case to his student Ising, who solves it, shows that there is no phase transition...

However, it was later found (Peierls, 1936) that there is a phase transition in dimension ≥ 2. The Lenz-Ising model gained huge interest throughout the development of modern statistical mechanics during the 20th century and still does. Parts of the reason for this success are:

- it is relatively simple and yet provides rich phenomena & behaviours
- it has several algebraic properties that make it possible to study rigorously, (cf. exact solution in 2D: Kaufman-Onsager, 1950
  - continuity of phase transition for any dimension: Aizenman, Duminil-Copin, Sidoravicius 2015
  - conformal invariance in 2D: Smirnov, ... $\simeq 2010$
  - ...)
- it is related to many other systems (percolation / random clusters, spanning trees, dimer model, 6V/8V ...)
- it is extremely beautiful.

<u>The model</u>:

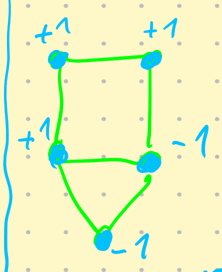
Let $G=(V,E)$ be a finite (For now...) graph, with vertices V and edges E (non-oriented).

Let $\Omega = \{-1,+1\}^V$ be the "configuration space". An element $\sigma \in \Omega$ is a function $\sigma : V \to \{-1,+1\}$, $v \mapsto \sigma_v$ and is called a spin configuration.

Let $J=(J_e)_{e\in E}$ be a family of numbers in $\mathbb{R}_+$ called coupling constants, and let $\beta > 0$ be the inverse temperature.

We consider the probability $\mu_{\beta,J}$ on Ω given by:

$$\forall \sigma \in \Omega, \quad \boxed{\mu_{\beta,J}(\sigma) = \frac{1}{Z_{\beta,J}} \exp\Big(\beta \sum_{e=\{u,v\}\in E} J_e \sigma_u \sigma_v\Big)}$$



a spin conf.

we will often drop the dependencies in β, J...

where $Z_{\beta,J} = \sum_{\sigma\in\Omega} \exp\big(\beta \sum_{e=\{u,v\}} J_e \sigma_u \sigma_v\big)$ is called the partition function.

Remarks: • as $\beta > 0$ and $J_e \geq 0$, this proba "favors" neighbouring sites to be equal ($+1 \; +1$ or $-1 \; -1$).

This effect is stronger if β is large (low temp.)

~~stronger~~ weaker — small (high temp)

• the previous "dipoles" orientation are discretized to ± 1 (makes study easier)

• This is a Boltzmann probability:

in general, $\mathbb{P}(\sigma) \propto \exp\Big(-\frac{1}{k_B T} \cdot H(\sigma)\Big)$ in Boltzmann's formalism.

$\frac{1}{k_B T}$: β

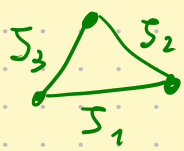
H: "energy" (an extensive function)

rough idea: the energy tends to favor order, while T tends to favor disorder. Which one wins?

• In stat. mechanics, we often work directly with Ω being the actual space of configurations... contrary to Kolmogorov's formalism.

• We will often take $\forall e, J_e = 1$ and work with β only.

Q What does σ typically look like depending on β?

Exercise: we take $G =$ 

with $J_1, J_2, J_3 > 0$.

Show that as $\beta \to 0$, $\mu_\beta \xrightarrow[\beta\to 0]{d} \mathrm{Unif}(\Omega)$

and as $\beta \to \infty$, $\mu_\beta \xrightarrow[\beta\to\infty]{} \frac{1}{2}\delta_{(+++)} + \frac{1}{2}\delta_{(---)}$

in general, as $\beta \to 0$ ("∞ temperature") we get site Bernoulli perco $(\frac{1}{2})$,
& as $\beta \to \infty$ ("zero temperature") we get frozen config.

Q Can we study average magnetization?

We would like to consider $\frac{1}{|V|}\sum_{v\in V}\sigma_v$ the "global spin",

and its average value $m = \langle \frac{1}{|V|}\sum_{v\in V}\sigma_v \rangle$

expectation for μ, that is, for $f:\Omega\to\mathbb{R}$,

$$\langle f\rangle := \sum_{\sigma\in\Omega} f(\sigma)\,\mu(\sigma).$$

we can also specify: $\langle\cdot\rangle_\beta$, $\langle\cdot\rangle_{\beta,J}$ ...

<u>Problem</u>: by symmetry, $m=0$.

<u>Possible solutions</u>:

① consider $\langle |\frac{1}{|V|}\sum_{v\in V}\sigma_v| \rangle$. Can work but not always practical.

② (the physics way) Add a magnetic field $h\in\mathbb{R}$ and consider instead

$$\mu_{\beta,J,h}(\sigma) = \frac{1}{Z_{\beta,J,}}\exp\left[\beta\sum_{\substack{e=\{u,v\}\\ \in E}} J\sigma_u\sigma_v + h\sum_{u\in V}\sigma_u\right]$$

if $h>0$, $+1$ spins are favored.

Magnetization: $m_{\beta,h} = \langle \frac{1}{|V|}\sum_{v\in V}\sigma_v \rangle_{\beta,h}$.

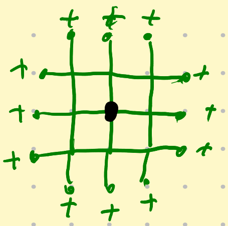
or simply σ_0 for a given site 0 "deep into the graph"

How does $m_{\beta,h}$ behave as $h\to 0$, for a "big graph"?

③ (The most common in math)

Introduce boundary conditions.

For instance, let $d\geq 1$, and let $\Lambda_n = [\![-n,n]\!]^d \subset \mathbb{Z}^d$ be a piece of $\mathbb{Z}^d$.



Consider the model with $J\equiv 1$ on Λ_n, with $+$ boundary conditions: we fix the spins on $\partial\Lambda_n$ to be $+1$.

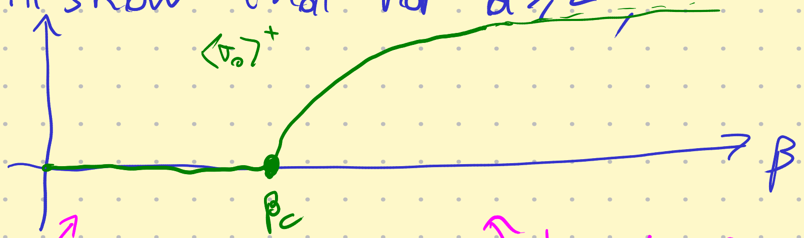
Let $\mu^+_{\Lambda_n}$ be the corresponding proba, and $\langle\cdot\rangle^+_{\Lambda_n}$ its expectation. We consider $\langle\sigma_0\rangle^+_{\Lambda_n} = \mu^+_{\Lambda_n}(\sigma_0=1) - \mu^+_{\Lambda_n}(\sigma_0=-1)$

We want to consider $\langle \sigma_0 \rangle^+ := \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n}$.

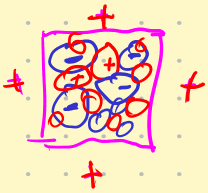
Q : • does this limit exist?

- Can we define a satisfying measure on the infinite graph $\mathbb{Z}^d$?
- How does it depend on β ?

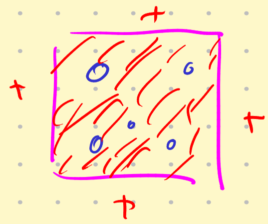
We will show that for $d \geq 2$



high temp: "$\langle \sigma_0 \rangle^+ = 0$, the system "does not see" the boundary condition:



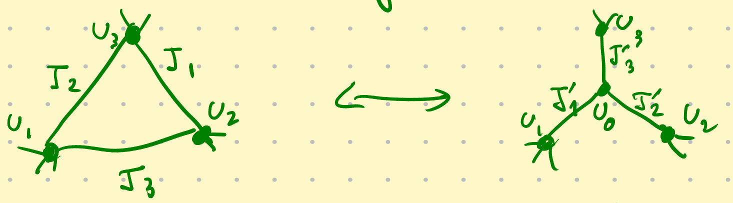
low temp: $\langle \sigma_0 \rangle^+ > 0$



mostly $+$.

We take $\beta = 1$ (imagine that β is absorbed in J)

Exercise: Consider a finite graph $G = (V, E)$ with coupling constants J, all > 0. Suppose that G contains a triangle. Transform this triangle into a star to get a new graph G':



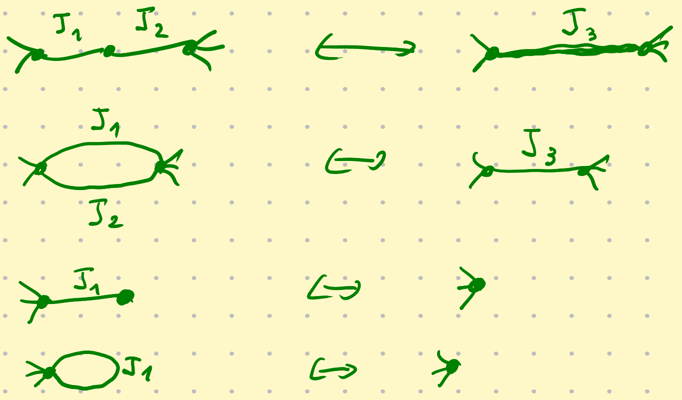
$G = (V, E)$ $\qquad\qquad$ $G' = (V', E')$.

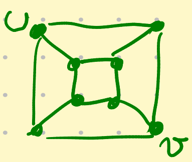
($J_e' = J_e$ for any non-represented edge e)

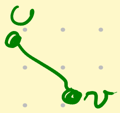
1) Show that one can find J_1', J_2', J_3' s.t. the distribution of the Ising model is unchanged. That is, if σ in $\{\pm 1\}^V$ has distribution $\mu_{G,J}$, and σ' in $\{\pm 1\}^{V'}$ has distri. $\mu_{G', J'}$, then $\sigma \overset{d}{=} \sigma'|_{V' \setminus \{v_0\}}$.

2) Show that the converse transformation is possible as well. (while keeping the distribution)

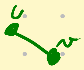
3) Show in addition that the following transformations can be made:



4) Transform the graph 

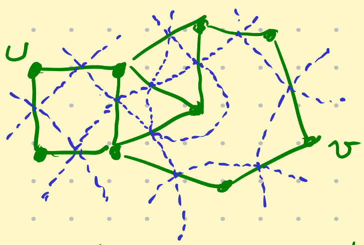
into 

with such transformations, keeping u,v untouched.

5) (*) Show that any connected planar graph with 2 marked vertices (u,v) on the boundary can be transformed into 

by such transformations. (Epifanov, 1966)

Hint: Consider the medial graph, here in dashed lines:



(Feo & Provan 1993) This is actually doable in polynomial time (in the size of the graph)

6) Deduce that there is a polynomial algorithm that takes as entry G, J, u, v satisfying 5) and gives $\langle \sigma_u \sigma_v \rangle_{G,J}$ and $Z_{G,J}$.

This is surprising: quantities like Z should be exponential in nature. It is a first glimpse at the fact that the Ising model is exactly solvable in dimension 2.