chap 2 Non trivial phase transition on $\mathbb{Z}^d$, $d \geq 2$

We saw that in dimension 1, the critical inverse temperature is "$\beta_c = +\infty$"; there is no phase transition. In this chapter, we show that for $d \geq 2$, $\beta_c \in (0, \infty)$.

More precisely, recall that we consider Ising with + boundary conditions, no magnetic field ($h=0$) and we want to study

$$\langle \sigma_0 \rangle_\beta^+ := \lim_{n \to \infty} \langle \sigma_0 \rangle_{\beta, \Lambda_n}^+ \qquad \text{where } \Lambda_n = [\![-n, n]\!]^d .$$

The goal is to prove • that this limit exists, is indep of the sequence $(\Lambda_n) \nearrow \mathbb{Z}^d$, and is $\nearrow$ in β.

• the main theorem:

Theo: For $d \geq 2$, $\exists \beta_c(d) \in (0, \infty)$ s.t.

- $\forall \beta < \beta_c(d)$, $\langle \sigma_0 \rangle_\beta^+ = 0$,
- $\forall \beta > \beta_c(d)$, $\langle \sigma_0 \rangle_\beta^+ > 0$.

I - Ising with + boundary conditions

Let us define this more carefully.

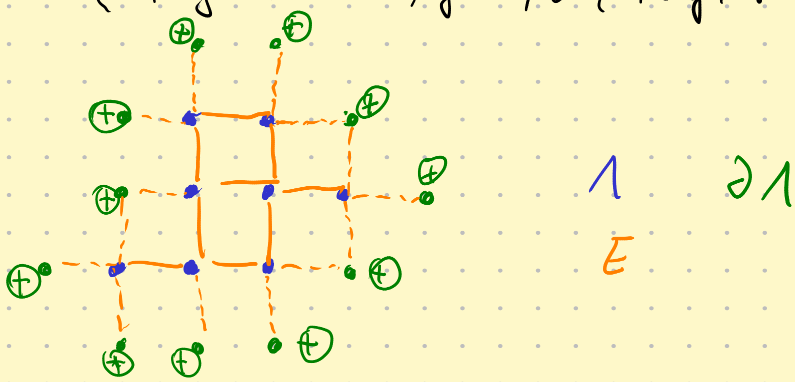
For $x, y \in \mathbb{Z}^d$, we denote by $x \sim y$ the fact that $\|x-y\|_1 = 1$. The corresponding edge of $\mathbb{Z}^d$ is denoted $\{xy\}$.

Let $\Lambda \subset\subset \mathbb{Z}^d$ (a finite subset of vertices).

Let $\partial \Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda \text{ st } \exists y \in \Lambda, x \sim y\}$.

$\bar{\Lambda} = \Lambda \cup \partial \Lambda$

$E := E(\bar{\Lambda}) = \{ \{xy\} \text{ st } x \in \Lambda, y \in \Lambda \} \cup \{ \{xy\} \text{ st } x \in \Lambda, y \in \partial \Lambda \}$.



Let $\Omega^+ = \{\sigma \in \{-1,+1\}^{\bar{\Lambda}}$ st $\forall x \in \partial\Lambda,\ \sigma_x = +1\}$.

Let $\beta \geq 0$. $\forall \sigma \in \Omega^+$, $\mu^+_{\Lambda,\beta}(\sigma) := \frac{1}{Z^+_{\Lambda,\beta}} \exp\left(\beta \sum_{\{x,y\}\in E} \sigma_x \sigma_y\right)$.

where $Z^+_{\Lambda,\beta} = \sum_{\sigma \in \Omega^+} \exp\left(\beta \sum_{\{x,y\}\in E} \sigma_x \sigma_y\right)$

As usual, let $\langle \cdot \rangle^+_{\Lambda,\beta}$ be the expectation associated to $\mu^+_{\Lambda,\beta}$.

II - Low temperature expansion (Peierls' "droplet" argument)

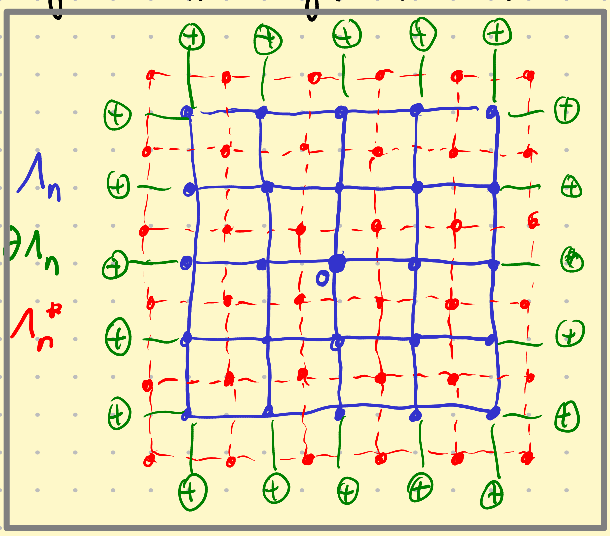
We start by showing that in dimension 2, for β large enough (low temperature), the sequence $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta}$ is bounded from below. Later we will extend this to $d \geq 3$, and prove that the limit $\langle \sigma_0 \rangle^+_\beta$ exists, so it will imply $\langle \sigma_0 \rangle^+_\beta > 0$.

The idea is famously due to Peierls in his 1936 article.
If $\sigma_0 = -1$, there has to be an interface of spins separating +/- surrounding 0. If β is very large, the cost of a single such "droplet" is very high, and their number doesn't grow fast enough for it to be reasonable to expect even one "droplet".

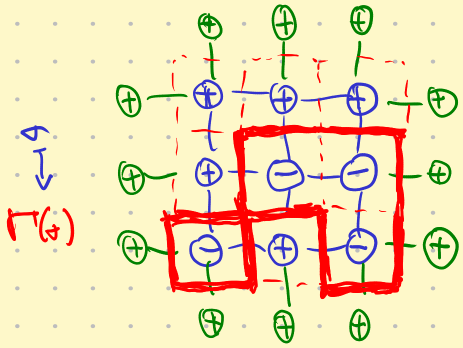
Let us make this more precise.

We are in $\Lambda_n = [\![-n, n]\!]^2$.

Let Λ_n^* be the dual domain:
it is a subgraph of $\mathbb{Z}^2 + \left(\frac{1}{2}, \frac{1}{2}\right)$
with vertices $V(\Lambda_n^*) = \{-n-\frac{1}{2}, -n+\frac{1}{2}, \dots, n+\frac{1}{2}\}^2$
and edges $E(\Lambda_n^*)$ between nearest neighbours.



For $\sigma \in \Omega^+$, we define $\Gamma(\sigma)$ to be the +/- interfaces of σ, that is, the subset of $E(\Lambda_n^*)$ made of edges separating a $\oplus$ spin from a $\ominus$ spin.



Let Ω^{LT} be the set of all subsets $\Gamma \subset E(\Lambda_n^*)$ such that, for every $v \in V(\Lambda_n^*)$, the degree of v in Γ is even.

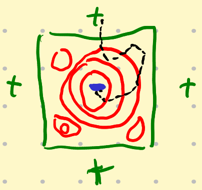
Clearly, $\Gamma(\sigma) \in \Omega^{LT}$, and in fact this is a bijection between Ω^+ and Ω^{LT}.

Moreover, $\exp\left(\beta \sum_{\{x,y\}} \sigma_x \sigma_y\right) = \exp\left[\beta\left(\sum_{e \in \Gamma(\sigma)} -1 + \sum_{e \notin \Gamma(\sigma)} 1\right)\right]$

$$= \exp\left(\beta \underset{\substack{\uparrow \\ \text{coordinal}}}{|E(\Lambda_n^*)|} - 2\beta|\Gamma(\sigma)|\right)$$

Therefore, $Z^+_{\Lambda_n,\beta} = e^{\beta|E(\Lambda_n^*)|} \sum_{\Gamma \in \Omega^{LT}} e^{-2\beta|\Gamma|}$

We wont to show that $\langle \sigma_0 = -1 \rangle^+_{\Lambda_n,\beta}$ is small. For a configuration σ s.t. $\sigma_0 = -1$, what does $\Gamma(\sigma)$ satisfy?



Clearly this means that on any path from 0 to $\partial\Lambda_n$ (on $V(\Lambda_n)$), we cross an odd number of edges in $\Gamma(\sigma)$. Let Ω^{LT}_- be the set of such elements of Ω^{LT}, we get

$$\langle \sigma_0 = -1 \rangle^+_{\Lambda_n,\beta} = \frac{1}{Z^+_{\Lambda_n,\beta}} \cdot \sum_{\Gamma \in \Omega^{LT}_-} e^{\beta|E^*| - 2\beta|\Gamma|}$$

$$= \frac{\sum_{\Gamma \in \Omega^{LT}_-} \exp(-2\beta|\Gamma|)}{\sum_{\Gamma \in \Omega^{LT}} \exp(-2\beta|\Gamma|)}$$

a "new" partition function!

Let μ_{HT} be the measure on Ω^{LT} defined by $\mu_{LT}(\Gamma) = \frac{1}{Z^{LT}} e^{-2\beta|\Gamma|}$, and $\langle \cdot \rangle_{HT}$ its expectation. Then $\boxed{\langle \sigma_0 = -1 \rangle_{\Lambda_n,\beta} = \langle \Gamma \in \Omega^{LT}_- \rangle_{LT}}$.

Note that when $\Gamma \in \Omega^{LT}_-$, there has to be at least one loop surrounding 0 in Γ.

γ set of edges $\{x_i x_{i+1}\} \in E(\Lambda_n^*)$ where $x_0, \dots, x_{n-1}, x_n = x_0$ are distinct vertices of Λ_n^*, and $0 \in \gamma$ as a real curve

By the union bound,

$$\langle \sigma_0 = -1\rangle^+_{\Lambda_n,\beta} \leq \sum_{\gamma \text{ surr. } 0} \langle \gamma \subset \Gamma \rangle_{LT}$$

Lemma: For any loop surrounding 0, γ, we have $\langle \gamma \subset \Gamma\rangle_{LT} \leq e^{-2\beta|\gamma|}$.

Proof:
$$\langle \gamma \subset \Gamma \rangle_{LT} = e^{-2\beta|\gamma|} \frac{\sum_{\Gamma \text{ s.t. } \gamma \subset \Gamma} \exp[-2\beta(|\Gamma| - |\gamma|)]}{\sum_{\Gamma \in \Omega^{LT}} \exp(-2\beta|\Gamma|)}. \quad (*)$$

from the point of view of σ this amounts to flipping all spins inside γ

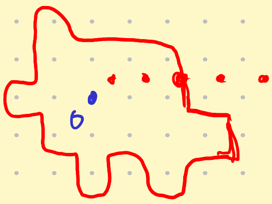
If $\Gamma \in \Omega^{LT}$ is such that $\gamma \subset \Gamma$, we may remove γ from Γ and get a new element of Ω^{LT} (as γ is a loop!). The weight of this element $\Gamma \setminus \gamma$ is

$$\exp(-2\beta|\Gamma \setminus \gamma|) = \exp(-2\beta|\Gamma| + 2\beta|\gamma|).$$

This means that any term in the numerator of $(*)$ also appears in the denominator! As they are all ≥ 0, the ratio is ≤ 1. $\square$.

We deduce $\langle \sigma_0 = -1\rangle_{\Lambda_n,\beta} \leq \sum_{k \geq 4} e^{-\beta k} \, \#\{\gamma \text{ loop of length } k \text{ surrounding } 0\}$.

It remains to bound this cardinal. Such a loop γ has to touch at least one of the vertices $\{(\frac{1}{2},\frac{1}{2}), (\frac{3}{2},\frac{1}{2}), \dots, (\lfloor \frac{k}{2} \rfloor - \frac{1}{2}, \frac{1}{2})\}$.



Given one of these vertices, the number of loops (of length k) surrounding 0 passing through it is at most $4 \cdot 3^{k-1}$.

We get $\langle \sigma_0 = -1\rangle_{\Lambda_n,\beta} \leq \sum_{k \geq 4} \frac{k}{2} \cdot 4 \cdot 3^{k-1} \cdot e^{-\beta k}$.

For β large enough, the series is convergent. Moreover, as $\beta \to +\infty$, the sum goes to 0. Note also that this bound is independent of n. We just proved:

For the moment $d = 2$.

Prop: $\exists \beta_1 > 0$ st $\forall \beta > \beta_1$, $\exists c(\beta) > 0$ st $\forall n$, $\langle \sigma_0 \rangle^+_{\Lambda_n(\mathbb{Z}^2),\beta} \geq c(\beta)$.
Moreover, $c(\beta) \xrightarrow[\beta \to \infty]{} 1$.

$\rightsquigarrow \langle \sigma_0 \rangle^+_\beta \geq c(\beta)$ if we show the limit exists.

proof Write $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} = \langle \sigma_0 = +1 \rangle^+_{\Lambda_n,\beta} - \langle \sigma_0 = -1 \rangle^+_{\Lambda_n,\beta}$
$= 1 - 2 \langle \sigma_0 = -1 \rangle^+_{\Lambda_n,\beta}$
and use the previous computation. $\square$.

Exercise: Still in dimension 2, show that for β large enough, the typical loop in Γ has size at most $O(\log n)$.
That is,
$\exists \beta_2 / \forall \beta > \beta_2, \forall c > 0, \exists K_0(c) < \infty$ st $\forall K > K_0(c), \forall n,$
$\langle \exists \gamma \text{ loop} \subset \Gamma \text{ with } |\gamma| \geq K \log n \rangle_{\Lambda_n,\beta} \leq n^{-c}$.

III - High temperature expansion

For β small, we want to show that $\langle \sigma_0^+ \rangle_{\Lambda_n,\beta}$ is very close to 0. The technique of the previous ~~paragraph~~ section seem hopeless. However, there is a trick to create similar contours, but in a "dual" way.

We no longer suppose $d=2$. Let $\Lambda \subset\subset \mathbb{Z}^d$, $d \geq 1$.

For $\sigma \in \Omega^+$, we rewrite its weight:

$$\exp\Big(\beta \sum_{\{x,y\} \in E} \sigma_x \sigma_y\Big) = \prod_{\{x,y\}} \exp(\beta \sigma_x \sigma_y)$$

We "linearize" this term, by writing it as $a + \sigma_x \sigma_y b$.
We find... $a = \cosh \beta$, $b = \sinh(\beta)$.

$$= \prod_{\{x,y\} \in E} \cosh\beta \big(1 + \sigma_x \sigma_y \tanh(\beta)\big)$$

$$= \cosh(\beta)^{|E|} \cdot \prod_{\{x,y\} \in E} \big(1 + \sigma_x \sigma_y \tanh(\beta)\big)$$

Therefore, the partition function is

$$Z^+_{\Lambda,\beta} = \big(\cosh(\beta)^{|E|}\big) \cdot \sum_{\sigma \in \Omega^+} \prod_{\{x,y\} \in E} \big(1 + \sigma_x \sigma_y \tanh(\beta)\big).$$

We expand the product (!) to get a sum on subsets $\eta \subset E$:

$$Z^+_{\Lambda,\beta} = (\mathrm{ch}\,\beta^{|E|}) \sum_{\sigma\in\Omega^+} \sum_{\eta\subset E} \prod_{\substack{e\in\eta \\ "\{x,y\}}} \sigma_x\sigma_y \tanh(\beta)$$

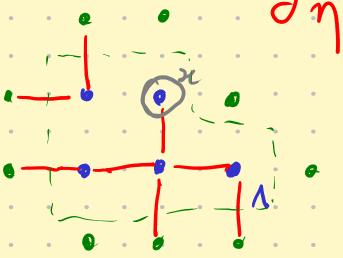
$$= (\mathrm{ch}\,\beta^{|E|}) \sum_{\eta\subset E} (\tanh(\beta)^{|\eta|}) \cdot \left(\sum_{\sigma\in\Omega^+} \prod_{e=\{x,y\}\in\eta} \sigma_x\sigma_y\right)$$

For $\eta\subset E$, imagine that there is a vertex $x\in\Lambda$ with odd degree in η. Then in the sum, consider the involution on Ω^+ that flips the spin σ_x. This has the effect of negating the product, as σ_x appears with an odd exponent. So the sum is then 0.

For $\eta\subset E$, we define

$$\partial\eta := \left\{x\in\Lambda \;/\; \sum_{y\in\bar\Lambda} 1_{\{x,y\}\in\eta} \text{ is odd}\right\}.$$

we don't care about vertices in $\partial\Lambda$!



η st $\partial\eta=\{x\}$.

We just saw that only terms η st $\partial\eta=\emptyset$ will survive in our expansion of $Z^+_{\Lambda,\beta}$.

Moreover, if $\partial\eta=\emptyset$,

$$\sum_{\sigma\in\Omega^+} \underbrace{\prod_{\{x,y\}\in\eta} \sigma_x\sigma_y}$$

$= 1$ as $\forall x\in\bar\Lambda$, either $x\in\partial\Lambda$ so $\sigma_x=+1$ or $x\in\Lambda$ and it appears with an even power, and $\sigma_x^{2n}=1$.

$|\Omega^+| = 2^{|\Lambda|}$.

Therefore, $\boxed{Z^+_{\Lambda,\beta} = \left(2^{|\Lambda|}(\mathrm{ch}\,\beta)^{|E|}\right) \cdot \sum_{\eta \text{ st } \partial\eta=\emptyset} \mathrm{th}(\beta)^{|\eta|}}$

Given what we did, it is tempting to wonder if we can put a probability on the set of subsets of E (the η's), with weight $\mathrm{th}(\beta)^{|\eta|}$, so that expectations like $\langle\sigma_0\rangle^+$ would have a meaning on this measure. However, we did formal computations, and we see no direct link between spin configurations σ and subsets η...

Theo (High Temperature Expansion)

Let $A \subset \Lambda$. Then

$$\Big\langle \prod_{x\in A} \sigma_x \Big\rangle^+_{\Lambda,\beta} = \frac{\sum_{\eta \subset E / \partial\eta = A} w(\eta)}{\sum_{\eta \subset E/\partial\eta=\emptyset} w(\eta)} \qquad \text{where } w(\eta) = \tanh(\beta)^{|\eta|}.$$

Proof: $\Big\langle \prod_{x\in A} \sigma_x \Big\rangle^+_{\Lambda,\beta} = \dfrac{1}{Z^+_{\Lambda,\beta}} \sum_{\sigma\in\Omega^+} \Big(\prod_{x\in A}\sigma_x\Big) \prod_{\{uv\}\in E} e^{\beta\sigma_u\sigma_v}$.

By the same computation as before

$$\sum_{\sigma} \prod_{x\in A}\sigma_x \prod_{\{uv\}} e^{\beta\sigma_u\sigma_v} = (\mathrm{ch}\,\beta)^{|E|} \sum_{\eta\subset E} (\tanh\beta)^{|\eta|} \cdot \Big(\underbrace{\sum_{\sigma\in\Omega^+} \prod_{x\in A}\sigma_x \prod_{\{uv\}\in\eta} \sigma_u\sigma_v} \Big)$$

Similarly, the sum vanishes if $\partial\eta \neq A$, and if $\partial\eta = A$ it gives $|\Omega^+| = 2^{|\Lambda|}$.

Taking the ratio with $Z^+_{\Lambda,\beta}$ we get the Theo. ☐.

Remark: The right-hand side in the Theo does not correspond to the expectation of an event for a proba. on η's. Rather, we "change configuration space" between numerator and denominator.

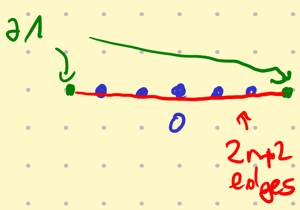
Sometimes we still use probabilistic vocabulary for it, like saying that at small β (high temp), we "expect" η to have few edges, as $\mathrm{th}(\beta)$ is close to 0 ...

Exercise: For $d=1$, let $\Lambda = [\![-n,n]\!]$.

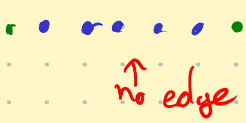
Show that $\langle\sigma_0\rangle^+_{\Lambda,\beta} = 2\dfrac{(\tanh\beta)^{n+1}}{1+(\tanh\beta)^{2n+2}} \leq 2(\tanh\beta)^{n+1} \quad \Big(\xrightarrow[n\to\infty]{} 0\Big)$

Solution: Let's count subsets $\eta\subset E$ st. $\partial\eta=\emptyset$, and $\partial\eta=\{0\}$.

For $\partial\eta=\emptyset$:

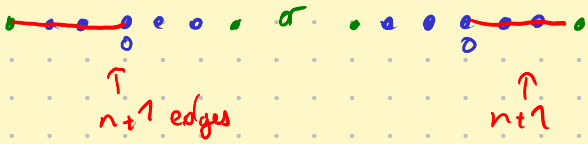


or



$\rightsquigarrow$ denominator is $1+\mathrm{th}\,\beta^{2n+2}$.

For $\partial\eta = \{0\}$,



$\rightsquigarrow$ numerator is $2 \cdot \operatorname{th}_\beta^{\,n+1}$. $\boxtimes$.

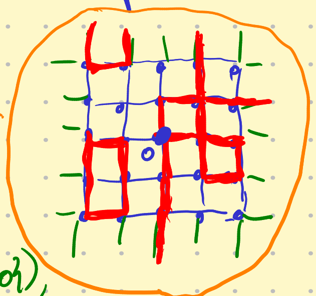
We can now use the previous theorem and an adaptation of Peierls' argument to show:

Prop: $\forall d \geq 1$, $\exists \beta_0(d) > 0$ s.t. $\forall \beta < \beta_0(d)$, $\exists c (= c(\beta, d)) > 0$ st $\langle \sigma_0 \rangle^+_{\Lambda_n(\mathbb{Z}^d), \beta} \leq e^{-cn}$ for all $n \geq 1$.

In particular,
$$\langle \sigma_0 \rangle^+_{\Lambda_n(\mathbb{Z}^d), \beta} \xrightarrow[n\to\infty]{} 0.$$

Remark: For $d=1$, the exercise above shows that "$\beta_0(1) = +\infty$" works.

Proof: By the HTE Theorem,



$$\langle \sigma_0 \rangle^+_{\Lambda_n, \beta} = \frac{\sum_{\eta \subset E_n / \partial\eta = \{0\}} \operatorname{th}\beta^{|\eta|}}{\sum_{\eta \subset E_n / \partial\eta = \emptyset} \operatorname{th}\beta^{|\eta|}}.$$

For η contributing to the numerator ($\partial\eta = \{0\}$), let η_0 be the connected component of 0 in η. (it has to connect 0 with $\partial\Lambda_n$!)

Then η_0 has to contain at least n edges.

We have
$$\langle \sigma_0 \rangle^+_{\Lambda_n, \beta} = \sum_{\substack{\eta_0 \subset E_n \\ \text{st } 0 \text{ touches } \eta_0}} (\operatorname{th}\beta)^{|\eta_0|} \cdot \frac{\sum_{\eta \subset E_n / \partial\eta = \{0\} \text{ and } \eta_0 \subset \eta} (\operatorname{th}\beta)^{|\eta| - |\eta_0|}}{\sum_{\eta \subset E_n / \partial\eta = \emptyset} (\operatorname{th}\beta)^{|\eta|}}$$

If η is s.t. $\begin{cases} \partial\eta = \{0\} \\ \eta_0 \subset \eta \end{cases}$, then $\eta \setminus \eta_0$ is a subset of E_n with $\partial(\eta \setminus \eta_0) = \emptyset$.

So the term $(\operatorname{th}\beta)^{|\eta| - |\eta_0|}$ that appears in the numerator also appears in the denominator and they are all > 0 so:

$$\langle \sigma_0 \rangle^+_{\Lambda_n, \beta} \leq \sum_{\substack{\eta_0 \subset E_n \\ \text{st } 0 \text{ touches } \eta_0 \\ \text{and } \eta_0 \text{ connected}}} (\operatorname{th}\beta)^{|\eta_0|}.$$

To adapt Peierls' argument, we want to count the connected subsets of edges in Λ_n that touch 0 and have $k \geq n$ edges.

Lemma: Let G be a connected graph.

Then, starting from any vertex of G, there exists a path that uses each edge of G exactly twice.

(in Exercise: do an induction on the number of edges).

Using the lemma, we have an injection from the set of η_0's of length k to the set of paths on $\mathbb{Z}^d$ of length $2k$ starting from 0, of which there are $(2d)^{2k}$. Therefore,

$$\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \lesssim \sum_{k \geq n} (2d)^{2k} (\operatorname{th}_\beta)^k \lesssim C(\beta,d)\, (2d)^{2n} (\operatorname{th}_\beta)^n$$

$$\lesssim \exp(-cn)$$

for a certain c whenever $4d^2 \operatorname{th}\beta < 1$ ⚡

(in particular, as $\operatorname{th}\beta < \beta$, if $\beta < \frac{1}{4d^2}$ then the conclusion holds.)

Exercise Show that in the same setting, for β small enough, $\exists c > 0 /\ \forall x \in \mathbb{Z}^d$,

"Exponential decay of correlations"

$$\langle \sigma_0 \sigma_x \rangle^+_{\Lambda_n,\beta} \leq \exp(-c\|x\|_\infty) \text{ for all } n \text{ large enough.}$$

IV - Griffith inequalities and monotonicity

We have the main ingredients to prove the main theorem of this chapter, however we still have to

- show existence of the limit of $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta}$ in n
  $\rightsquigarrow$ compare $\langle \sigma_0 \rangle^+_{\Lambda,\beta}$ for different domains
- Extend the "large β" result on $\mathbb{Z}^2$ (section II) to $\mathbb{Z}^d$, $d \geq 2$
  $\rightsquigarrow$ compare different dimensions.
  Particular case of comparing domains, as $\Lambda_n(\mathbb{Z}^d) \subset \Lambda_n(\mathbb{Z}^{d+1})$
- Show that $\langle \sigma_0 \rangle^+_\beta$ is $\nearrow$ in β
  $\rightsquigarrow$ compare different β's.

A first step towards these "monotonicity" results is Griffith's inequalities (sometimes called GKS inequalities)
Griffith Kelly Sherman

Theo (Griffith inequalities for + b.c.)

Let $\Lambda \subset\subset \mathbb{Z}^d$ and $A, B \subset \Lambda$.

Then $\forall \beta > 0$,
- $\langle \sigma_A \rangle^+_{\Lambda,\beta} \geq 0$
- $\langle \sigma_A \sigma_B \rangle^+_{\Lambda,\beta} \geq \langle \sigma_A \rangle^+_{\Lambda,\beta} \langle \sigma_B \rangle^+_{\Lambda,\beta}$

$\sigma_A := \prod_{x \in A} \sigma_x$

Remarks:
- The first ineq. is very natural: for + b.c. (and $h=0$, or even $h\geq 0$), ⊕ spins are favored so we expect σ_A to be more frequently ≥ 0.
- The second is about positive correlations:

as a direct consequence, for $A = \{x\}$ and $B = \{y\}$, we get

$$\mu^+_{\Lambda,\beta}(\sigma_x = +1 \mid \sigma_y = +1) = \frac{\left\langle \frac{\sigma_x+1}{2} \frac{\sigma_y+1}{2} \right\rangle^+_{\Lambda,\beta}}{\left\langle \frac{\sigma_y+1}{2} \right\rangle^+_{\Lambda,\beta}} \geq \frac{\frac{\langle\sigma_x\rangle^+ \langle\sigma_y\rangle^+}{4} + \frac{\langle\sigma_x\rangle^+ + \langle\sigma_y\rangle^+}{4} + \frac{1}{2}}{\frac{\langle\sigma_y\rangle^+ + 1}{2}}$$

$$\geq \frac{\left\langle \frac{\sigma_x+1}{2} \right\rangle^+ \cancel{\left\langle \frac{\sigma_y+1}{2} \right\rangle^+}}{\cancel{\left\langle \frac{\sigma_y+1}{2} \right\rangle^+}} \geq \mu^+_{\Lambda,\beta}(\sigma_x = +1)$$

$\mathbb{1}_{\sigma_x=+1} = \frac{\sigma_x+1}{2}$

"spins are positively correlated"

Proof: By HTE, $\langle \sigma_A \rangle^+_{\Lambda,\beta} = \dfrac{\sum_{\eta / \partial\eta = A} \operatorname{th}\beta^{|\eta|}}{\sum_{\eta/\partial\eta=\emptyset} \operatorname{th}\beta^{|\eta|}} \geq 0$.

- $\langle \sigma_A \sigma_B \rangle^+_{\Lambda,\beta} - \langle \sigma_A \rangle^+_{\Lambda,\beta} \langle \sigma_B \rangle^+_{\Lambda,\beta}$

$$= \frac{1}{(Z_{\Lambda,\beta})^2} \cdot \sum_{\sigma,\sigma' \in \Omega^+} (\sigma_A\sigma_B - \sigma_A\sigma'_B) \exp\Big(\beta \sum_{\{x,y\}\in E} (\sigma_x\sigma_y + \sigma'_x\sigma'_y)\Big)$$

$$= \text{"} \sum_{\sigma,\sigma' \in \Omega^+} \sigma_A\sigma_B (1 - \sigma_B\sigma'_B) \exp\Big(\beta \sum_{\{x,y\}} \sigma_x\sigma_y (1 + \sigma_x\sigma'_x\sigma_y\sigma'_y)\Big)$$

$\omega = \sigma\sigma'$

$$= \text{"} \sum_{\omega\in\Omega^+} (1-\omega_B) \cdot \Big[\sum_{\sigma\in\Omega^+} \sigma_A\sigma_B \exp\Big(\beta \sum_{\{x,y\}} \sigma_x\sigma_y \underbrace{(1+\omega_x\omega_y)}\Big)\Big]$$

0 on edges st $\omega_x \neq \omega_y$
2 on " " $\omega_x = \omega_y$

Let Λ_ω be the subgraph of edges $\{x,y\}$ s.t. $\omega_x = \omega_y$.

Then $\left[\sum_{\sigma\in\Omega^+} \cdots\right]$ is $\langle\sigma_A\sigma_B\rangle^+_{\Lambda_w,2\beta} \cdot Z^+_{\Lambda_w,2\beta}$!

By the first inequality, this is ≥ 0. $\qquad (\sigma_A\sigma_B = \sigma_{A\Delta B})$

symmetric difference

$\boxtimes$.

Corollary: (Monotonicity in β)

Let $\Lambda \subset\subset \mathbb{Z}^d$ and $0<\beta\leq\beta'$.
Then $\forall A\subset\Lambda$,

$$\langle\sigma_A\rangle^+_{\beta,\Lambda} \leq \langle\sigma_A\rangle^+_{\beta',\Lambda}$$

proof:
$$\langle\sigma_A\rangle^+_{\beta',\Lambda} = \frac{\sum_{\sigma\in\Omega^+} \sigma_A \exp\left(\beta'\sum_{\{x,y\}}\sigma_x\sigma_y\right)}{\sum_{\sigma\in\Omega^+} \exp\left(\beta'\sum_{\{x,y\}}\sigma_x\sigma_y\right)}$$

$$= \frac{\sum_{\sigma\in\Omega^+} \sigma_A \exp\left((\beta'-\beta)\sum\sigma_x\sigma_y\right)\exp\left(\beta\sum\sigma_x\sigma_y\right)}{\sum_{\sigma\in\Omega^+} \exp\left((\beta'-\beta)\sum\sigma_x\sigma_y\right)\exp\left(\beta\sum\sigma_x\sigma_y\right)}$$

$$= \frac{\langle\sigma_A g\rangle^+_{\beta,\Lambda}}{\langle g\rangle^+_{\beta,\Lambda}} \quad \checkmark$$

where $g(\sigma) = \exp\left((\beta'-\beta)\sum_{\{x,y\}\in E}\sigma_x\sigma_y\right) = \sum_{k\geq 0}\frac{1}{k!}\underbrace{(\beta'-\beta)^k}_{\geq 0}\underbrace{\left(\sum_{\{x,y\}\in E}\sigma_x\sigma_y\right)^k}_{\text{a certain sum } \sum_{S\subset\Lambda}\alpha_S\sigma_S}$, with $\alpha_S \geq 0$

$$= \sum_{S\subset\Lambda}\beta_S\sigma_S. \qquad \beta_S \geq 0$$

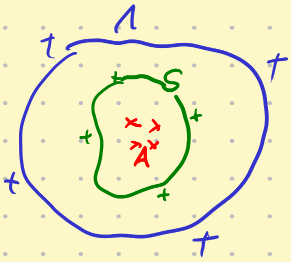
Therefore, $\langle\sigma_A g\rangle^+_{\beta,\Lambda} = \sum_{S\subset\Lambda}\beta_S\langle\sigma_A\sigma_S\rangle^+_{\beta,\Lambda} \underset{\text{Griffith's 2nd ineq}}{\geq} \sum_{S\subset\Lambda}\beta_S\langle\sigma_A\rangle^+_{\beta,\Lambda}\langle\sigma_S\rangle^+_{\beta,\Lambda}$

$$= \langle\sigma_A\rangle^+_{\beta,\Lambda}\langle g\rangle^+_{\beta,\Lambda}.$$

As $g\geq 0$, we get $\langle\sigma_A\rangle^+_{\beta',\Lambda} \geq \langle\sigma_A\rangle^+_{\beta,\Lambda}$ $\boxtimes$.

Cor: (Monotonicity in domain)

Let $\Lambda \subset\subset \mathbb{Z}^d$, and let $A \subset S \subset \Lambda$. Then $\forall \beta > 0$,
$$\langle \sigma_A \rangle^+_{S,\beta} \geq \langle \sigma_A \rangle^+_{\Lambda,\beta}$$



Proof: To be clear, we take E_Λ to be the edges on $\bar\Lambda = \Lambda \cup \partial\Lambda$ and $\Omega^+_\Lambda = \{\sigma \in \{\pm 1\}^{\bar\Lambda} \text{ s.t. } \forall x \in \partial\Lambda,\ \sigma_x = +1\}$.

As $\bar S \subset \bar\Lambda$, we can express $\langle \sigma_A \rangle^+_{S,\beta}$ with sums on this space:

"domain Markov property"

$$\langle \sigma_A \rangle^+_{S,\beta} = \frac{\sum_{\sigma \in \Omega^+_\Lambda /\ \forall x \notin S, \sigma_x = +1} \sigma_A \exp\left[\beta \sum_{\{x,y\} \in E_\Lambda} \sigma_x \sigma_y - \beta |E'|\right]}{\sum_{\sigma \in \Omega^+_\Lambda /\ \forall x \notin S, \sigma_x = +1} \exp\left[\beta \sum_{\{x,y\} \in E_\Lambda} \sigma_x \sigma_y - \beta |E'|\right]}$$

with $E' = E_\Lambda \setminus E_S$.

$$= \langle \sigma_A \mid \forall x \in \Lambda \setminus S,\ \sigma_x = +1 \rangle^+_{\Lambda,\beta}$$

$$= \frac{\langle \sigma_A \mathbb{1}_{\forall x \in \Lambda\setminus S, \sigma_x = +1} \rangle^+_{\Lambda,\beta}}{\langle \mathbb{1}_{\forall x \in \Lambda\setminus S, \sigma_x = +1} \rangle^+_{\Lambda,\beta}}$$

Moreover, $\mathbb{1}_{\forall x \in \Lambda\setminus S, \sigma_x = +1} = \prod_{x \in \Lambda\setminus S} \frac{1+\sigma_x}{2} = \frac{1}{2^{|\Lambda\setminus S|}} \sum_{T \subset \Lambda\setminus S} \sigma_T$.

So
$$\langle \sigma_A \rangle^+_{S,\beta} = \frac{\sum_{T \subset \Lambda\setminus S} \langle \sigma_A \sigma_T \rangle^+_{\Lambda,\beta}}{\sum_{T \subset \Lambda\setminus S} \langle \sigma_T \rangle^+_{\Lambda,\beta}} \underset{\text{Griffith}}{\geq} \frac{\sum_{T \subset \Lambda\setminus S} \langle \sigma_A \rangle^+_{\Lambda,\beta} \langle \sigma_T \rangle^+_{\Lambda,\beta}}{\sum_{T \subset \Lambda\setminus S} \langle \sigma_T \rangle^+_{\Lambda,\beta}} = \langle \sigma_A \rangle^+_{\Lambda,\beta}.$$

$\boxtimes$

Cor: $\forall d \geq 1$, $\forall \beta > 0$, the sequence $\langle \sigma_0 \rangle^+_{[\![-n,n]\!]^d,\beta}$ is decreasing in n. (so it converges)

"Magnetization on $\mathbb{Z}^d$ for β."

Moreover, if $(\Lambda_n)_{n\geq 0}$ is another increasing sequence of finite subsets of $\mathbb{Z}^d$ s.t. $\bigcup_{n\geq 0} \Lambda_n = \mathbb{Z}^d$, then the sequence $(\langle \sigma_0 \rangle^+_{\Lambda_n,\beta})_n$ converges to the same limit.

We denote this limit by $\langle \sigma_0 \rangle^+_\beta$, or $\langle \sigma_0 \rangle^+_{\beta,\mathbb{Z}^d}$.
It is an increasing function of β.

Proof: The first part is a direct consequence of the monotonicity in the domain.

For the sequence $\langle\sigma_0\rangle^+_{\Lambda_n,\beta}$, simply check that for any n, $\exists\, k_0(n), k_1(n)$ s.t. $[\![-k_0(n), k_0(n)]\!]^d \subset \Lambda_n \subset [\![-k_1(n), k_1(n)]\!]^d$ and we can take k_0, k_1 going to ∞ as $n\to\infty$; and use monotonicity in domain.

Finally, if $\beta \leq \beta'$, by monotonicity in β we have
$$\forall n,\quad \langle\sigma_0\rangle^+_{\beta,\Lambda_n} \leq \langle\sigma_0\rangle^+_{\beta',\Lambda_n}$$
So in the limit, $\langle\sigma_0\rangle^+_\beta \leq \langle\sigma_0\rangle^+_{\beta'}$. ∎

For any $d \geq 1$, we may now define the critical inverse temperature $\beta_c(d)$ as:
$$\beta_c(d) = \inf\{\beta\geq 0 \,/\, \langle\sigma_0\rangle^+_{\beta,\mathbb{Z}^d} > 0\}$$
$$= \sup\{\beta\geq 0 \,/\, \langle\sigma_0\rangle^+_{\beta,\mathbb{Z}^d} = 0\}.$$

Those are equal by monotonicity in β. In particular, $\forall \beta<\beta_c$, $\langle\sigma_0\rangle^+_\beta = 0$ and $\forall\beta>\beta_c$, $\langle\sigma_0\rangle^+_\beta > 0$.

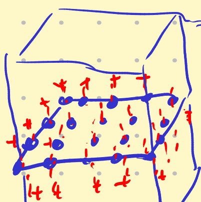
Now that this is defined, we know:

- By section II, $\beta_c(2)<\infty$ (and $\langle\sigma_0\rangle^+_{\beta,\mathbb{Z}^2} \xrightarrow[\beta\to\infty]{} 1$)
- By section III, $\forall d\geq 1$, $\beta_c(d) > 0$.

To complete the proof of the main theorem of this chapter, it remains to prove that $\forall d\geq 3$, $\beta_c(d)<\infty$. In fact:

Prop: (Monotonicity in the dimension)
The sequence $(\beta_c(d))_{d\geq 1}$ is decreasing in d.

Proof: We just show $\beta_c(3)\leq\beta_c(2)$, as the generic case is a direct adaptation. For $n\geq 1$, even though $[\![-n,n]\!]^2$ may be seen as a subset of $[\![-n,n]\!]^3$, we cannot really use monotonicity in the domain, as the boundary conditions for $[\![-n,n]\!]^2$ as a subset of $\mathbb{Z}^3$ and those as a subset of $\mathbb{Z}^2$ are not the same.



Consider $\Lambda = [\![-n,n]\!]^3$, now putting coupling constants $(J_e)_{e\in E}$.

<u>Exercise:</u> Check that Griffith inequalities still hold with a non constant family of nonnegative couplings $(J_e)_{e\in E}$. Extend the monotonicity in β into monotonicity in J, that is, if $J' \leq J$ (meaning $\forall e \in E$, $J'_e \leq J_e$) then $\langle \sigma_0 \rangle^+_{\Lambda, J', \beta} \leq \langle \sigma_0 \rangle^+_{\Lambda, J, \beta}$.

We start with $J \equiv 1$, and we define J' to be the same except we set $J'_e = 0$ for all edges e that touch $[\![-n,n]\!]^2 \times \{0\}$ but are not contained in it. By the exercise,

check as well

$$\langle \sigma_0 \rangle^+_{\Lambda_n, J', \beta} \leq \langle \sigma_0 \rangle^+_{\Lambda_n, J, \beta} \left(= \langle \sigma_0 \rangle^+_{[\![-n,n]\!]^3, \beta} \right)$$

$$\langle \sigma_0 \rangle^+_{[\![-n,n]\!]^2, \beta}$$ for the usual $[\![-n,n]\!]^2 \subset \mathbb{Z}^2$.

Therefore, $\langle \sigma_0 \rangle^+_{\beta, \mathbb{Z}^2} \leq \langle \sigma_0 \rangle^+_{\beta, \mathbb{Z}^3}$ which implies

$$\beta_c(2) \geq \beta_c(3) \qquad \boxtimes$$

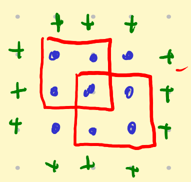
This completes the proof that the Ising model undergoes a nontrivial phase transition on $\mathbb{Z}^d$ for $d \geq 2$.

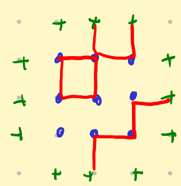
IV - On $\mathbb{Z}^2$: Kramers-Wannier duality (and $\frac{1}{2}\log(1+\sqrt{2})$)

We conclude with an intriguing relation in dimension 2. There, we showed two expansions as subsets of edges, one on the dual (LTE) and one on the primal lattice (HTE). They look surprisingly the same. Can we push this relation?

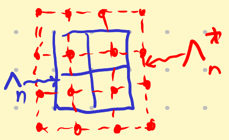
Recall that on $\Lambda_n = [\![-n;n]\!]^2$, $\forall \beta > 0$,

$$Z^+_{\Lambda_n, \beta} = e^{\beta |E^*|} \sum_{\substack{\Gamma \subset E^* / \\ \partial \Gamma = \emptyset}} e^{-2\beta |\Gamma|} = 2^{|\Lambda|} \operatorname{ch}(\beta)^{|E|} \sum_{\substack{\eta \subset E / \\ \partial \eta = \emptyset}} \operatorname{th}(\beta)^{|\eta|}.$$

Γ: 

η: 

In 1941, Kramers and Wannier used this to show that the 2D Ising model satisfies a duality relation. Indeed, consider another Ising model with spins on Λ^*, at inverse temperature β^*. It turns out that we can chose β^* s.t. the following are satisfied:



Lemma: For $\beta, \beta^* > 0$, the following are equivalent:

- $e^{-2\beta} = \tanh(\beta^*)$
- $e^{-2\beta^*} = \tanh(\beta)$
- $\sinh(2\beta)\sinh(2\beta^*) = 1$.

By taking this value of β^*, up to boundary issues, the LTE of $Z^+_{\Lambda,\beta}$ "is" the HTE of $Z^+_{\Lambda^*,\beta^*}$, and the HTE of $Z^+_{\Lambda,\beta}$ "is" the LTE of $Z^+_{\Lambda^*,\beta^*}$. !

Then they use this relation to show that the free energy

$$f(\beta) := \lim_{n\to\infty} -\frac{1}{4n^2} \log Z^+_{\Lambda_n,\beta}$$

$4n^2$ ≃ number of vertices

this limit exists, and is in fact indep of the choice of boundary conditions... (see later)

satisfies a functional equation:

$$Z^+_{\Lambda_n,\beta} = 2^{|\Lambda_n|} ch(\beta)^{|E|} \sum_{\substack{\eta \subset E \\ \partial\eta = \emptyset}} th\beta^{|\eta|}$$

$$Z^+_{\Lambda_n^*,\beta^*} = e^{\beta|E^{**}|} \sum_{\substack{\Gamma \subset E^{**} \\ \partial\Gamma = \emptyset}} \left(e^{-2\beta^*}\right)^{|\Gamma|}$$

$e^{-2\beta^*} = th\beta$

$$\text{so} \ldots -\frac{1}{4n^2}\log\left(\frac{Z^+_{\Lambda_n,\beta}}{2^{|\Lambda_n|} ch\beta^{|E|}}\right) \underset{n\to\infty}{\sim} -\frac{1}{4n^2}\log\left(\frac{Z^+_{\Lambda^*_n,\beta^*}}{e^{\beta|E^{**}|}}\right)$$

$\rightsquigarrow \ldots$

$$\boxed{f(\beta) = f(\beta^*) - \log(sh\, 2\beta)}$$

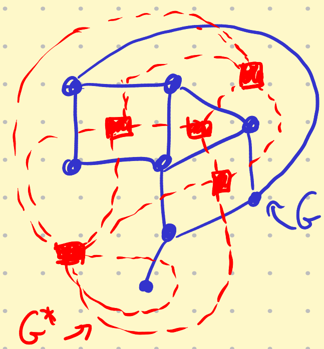
Now, if we assume that f is non-analytic at the critical point, and that this point is unique, we must have $\beta_c = \beta_c^*$. So $(sh\, 2\beta)^2 = 1$.

This can be made rigorous, as we will see later:

Theo $\beta_c(\mathbb{Z}^2) = \operatorname{argsh}\left(\frac{1}{2}\right) = \frac{1}{2}\log(1+\sqrt{2})$.

In fact on $\mathbb{Z}^2$ we can give an explicit expression for $f(\beta)$ at any $\beta > 0$.

We have been sloppy in this section with boundary conditions. The duality relation can in fact be made exact in some settings, like finite planar graphs:



Exercise: Consider $G=(V,E)$ a finite planar graph, with coupling constants $J=(J_e)_{e\in E}$ and $\beta=1$. Express the LTE and HTE for $Z_{G,J}$ (no b.c.) (via subgraphs on G and its dual graph G^*).

Find a dual Ising model on G^*, with coupling constants $J^*=(J^*_e)_{e\in E^*}$, and write an explicit relation between $Z_{G,J}$ and Z_{G^*,J^*}.