chap 3 Infinite Volume Measures

The goal of this chapter is to construct probability measures for the Ising model on $\mathbb{Z}^d$ ($d \geq 1$), that is, probabilities on $\Omega = \{\pm 1\}^{\mathbb{Z}^d}$. There are two approches:

- Construct a "limit in distribution" ← not really, because the sets of definitions are different

  of our measures $\mu^+_{\Lambda_n,\beta}$. As $n \to \infty$, do we get a proba μ^+_β? Does it depend on boundary conditions?

  For instance, for $\ominus$ b.c., do we have $\mu^- = \mu^+$?

- Find an "axiomatic description" for what we want the measure to be, and show that they can be satisfied (the "DLR" conditions, or a "Gibbs measure")

  Dobrushin, Lanford, Ruelle

In fact we will prove:

will be constructed from $\mu^{+/-}_{\Lambda_n,\beta}$.

Theo $\forall \beta > 0$, there exists probabilities μ^+, μ^- on Ω such that:

and even for $h \in \mathbb{R}$ magnetic field

- μ^+, μ^- are Gibbs measures (satisfy DLR)
- μ^+, μ^- are translation-invariant, ergodic, positively associated (FKG)
- If ν is a Gibbs measure on Ω,

$$\mu^- \leq_{st} \nu \leq_{st} \mu^+$$

Moreover, the following are equivalent:

(i) $\mu^+ = \mu^-$

(ii) $\exists!$ Gibbs measure

(iii) $\langle \sigma_0 \rangle^+ = \langle \sigma_0 \rangle^-$.

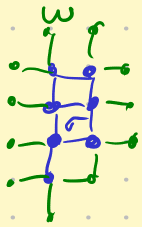
(All of that will be defined)

I- Construction

A) Generic boundary conditions

Recall that for $\Lambda \subset\subset \mathbb{Z}^d$, we had $\partial\Lambda := \dots$, $\overline{\Lambda} = \Lambda \cup \partial\Lambda$,
$E = \{\{xy\} / x, y \in \Lambda$ or $x \in \Lambda, y \in \partial\Lambda\}$, $\Omega = \{\pm 1\}^{\Lambda}$.
Let also $\mathring{E} = \{\{xy\} / x, y \in \Lambda\}$.

Let $\omega \in \{-1, 0, +1\}^{\partial\Lambda}$, $\beta > 0$, $h \in \mathbb{R}$. We define



$$\forall \sigma \in \Omega, \quad \mu^{\omega}_{\Lambda,\beta,h}(\sigma) = \frac{1}{Z^{\omega}_{\beta,n}} \exp\Big(\beta \sum_{\{xy\} \in \mathring{E}} \sigma_x \sigma_y + \beta \sum_{\substack{\{xy\} \in E \\ y \in \partial\Lambda}} \sigma_x \omega_y + h \sum_{x \in \Lambda} \sigma_x \Big)$$

For $\omega \equiv +1$, we recover $\mu^{+}_{\Lambda,\beta,h}$.

For $\omega \equiv -1$, we denote $\mu^{-}_{\Lambda,\beta,h} := \mu^{\omega}_{\Lambda,\beta,h}$.

Finally, for $\omega \equiv 0$, we get "free boundary conditions" denoted $\mu^{f}_{\Lambda,\beta,h}$.

B) Topology

Now let $\Lambda_n = [\![-n, n]\!]^d$, $\Omega_n = \{\pm 1\}^{\overline{\Lambda}_n}$,
and $\Omega = \{\pm 1\}^{\mathbb{Z}^d}$.

We want to construct measures on Ω (from those on Ω_n).
This usually requires a bit of topology, and an existence argument like Kolmogorov's extension theorem, or Riesz theorem.

We endow Ω with the product topology. By Tychonoff, Ω is compact. Moreover, this topology is consistent with a metric such as $d(\sigma, \sigma') = \frac{1}{r}$ where $r = \sup\{k \geq 1 / \sigma = \sigma'$ on $\Lambda_k\}$.
We also equip it with the product σ-algebra, which is generated by local events (or cylinders):

$$\{\sigma \in \Omega / \sigma|_{\overline{\Lambda}_n} \in A\} \quad \text{where } A \subset \Omega_n,\ n \geq 1.$$

A function $f : \Omega \to \mathbb{R}$ is said to be a local function if there exists $n \geqslant 1$ s.t. $f(\sigma)$ only depends on $\sigma|_{\Lambda_n}$.

Let $Loc(\Omega)$ be the space of local functions.

Prop: The closure of $Loc(\Omega)$ for the $\|\cdot\|_\infty$ norm contains $\mathcal{C}(\Omega)$ (the continuous functions on Ω).

Cor: If $\phi : Loc(\Omega) \to \mathbb{R}$ is a continuous, positive linear form, then it can be extended into a continuous positive linear form on $\mathcal{C}(\Omega)$.

$f \geqslant 0 \Rightarrow \phi(f) \geqslant 0$

Riesz: Then, there exists a unique measure μ on Ω st
$$\forall f \in \mathcal{C}(\Omega), \ \phi(f) = \int f \, d\mu .$$

proof of Prop: Let $f \in \mathcal{C}(\Omega)$. As Ω is compact (Tychonoff), f is uniformly continuous.

Let $\varepsilon > 0$, take δ st $d(\sigma, \sigma') \leq \delta \Rightarrow |f(\sigma) - f(\sigma')| \leq \varepsilon$.

Let $n = \lceil \frac{1}{\delta} \rceil$. Let $g(\sigma) = f(\sigma_n)$ where $\sigma_n = \begin{cases} \sigma & \text{on } \Lambda_n \\ +1 & \text{on } \mathbb{Z}^d \setminus \Lambda_n \end{cases}$

Then $\forall \sigma \in \Omega$, $d(\sigma, \sigma_n) \leq \underbrace{\tfrac{1}{n}}_{\leq \delta}$ by def.

so $|f(\sigma) - g(\sigma)| \leq \varepsilon$, so $\|f - g\|_\infty \leq \varepsilon$. ☒

Proof of Cor: The first part is clear.

For the second part, see Riesz theo:

A continuous linear form on $\mathcal{C}_c(\Omega)$ corresponds to a unique signed measure st $\phi(f) = \int f \, d\mu$.

Here, $\mathcal{C}_c(\Omega) = \mathcal{C}(\Omega)$ as Ω is compact, and $\mu \geqslant 0$ by positivity of ϕ. ☒

Remark: We want to prove that μ is a proba, that is $\phi(1) = 1$.

C) Definition of μ^+, μ^- via limits

Remark: For $f \in Loc(\Omega)$, its expectation $\mu^{+/-}_{\Lambda_n, \beta, h}(f)$ is well-defined for n large enough. $\leftarrow$ Check!

$\forall \beta > 0,\ h \in \mathbb{R},$

Theo There exists probability measures $\mu^+_{\beta,h}, \mu^-_{\beta,h}$ on Ω characterized by

$$\forall f \in Loc(\Omega),\quad \mu^+_{\Lambda_n,\beta,h}(f) \xrightarrow[n\to\infty]{} \mu^+_{\beta,h}(f)$$

$$\mu^-_{\Lambda_n,\beta,h}(f) \xrightarrow[n\to\infty]{} \mu^-_{\beta,h}(f).$$

These measures don't depend on the choice of $(\Lambda_n)_{n\geq 0}$ (with $(\Lambda_n)\nearrow$ and $\bigcup_n \Lambda_n = \mathbb{Z}^d$)

For $h \neq 0$, we don't have Griffith's ineq (which we used for monotonicity). Prove the theo using FKG ineq (next chapter) mimicking the proof.

Proof: We prove this for $h=0$.

Let $f \in Loc(\Omega)$. We claim that there exists $n \in \mathbb{N}^*$ st f can be written as

$$f = \sum_{A \subset \Lambda_n} \hat{f}_A \, \sigma_A$$

recall: $\prod_{x \in A} \sigma_x$.

Discrete Fourier transform!

Indeed, equip the space of functions on $\{\pm 1\}^{\Lambda_n}$ with the scalar product $(f,g) := \frac{1}{|\Lambda_n|} \sum_{\sigma \in \{\pm 1\}^{\Lambda_n}} f(\sigma) g(\sigma)$, then it is easy to see that the $(\sigma_A)_{A \subset \Lambda_n}$ are orthonormal, and there is $2^{|\Lambda_n|}$ of them, so it is a basis.

So $\mu^+_{\Lambda_n}(f) = \sum_{A \subset \Lambda_n} \hat{f}_A \ \langle \sigma_A \rangle^+_{\Lambda_n}$

We know that $\langle \sigma_A \rangle^+_{\Lambda_n}$ is $\searrow$ in n (cf prev. chapter, monotonicity in domain)

so $\lim_n \mu^+_{\Lambda_n}(f)$ exists, we denote it by $\Phi(f)$.

Then $|\Phi(f)| = \lim_n |\mu^+_{\Lambda_n}(f)| \leq \|f\|_\infty$

So Φ is a continuous linear form on $Loc(\Omega)$, and clearly ≥ 0. By section B), it is represented by a unique measure μ^+. Moreover, $\Phi(1) = 1$ so μ^+ is a proba.

The independance on Λ_n, and adaptation to μ^-, are left as exercises. $\square$

II - Positive association (FKG)

A) Stochastic domination

A function $f : \Omega \to \mathbb{R}$ is said to be **increasing** if
either $\{\pm 1\}^\Lambda$ or $\{\pm 1\}^{\mathbb{Z}^d}$, both work!

$$\sigma \leq \sigma' \implies f(\sigma) \leq f(\sigma')$$

($\forall x, \sigma_x \leq \sigma'_x$)

we should say "nondecreasing" but it's too long.

Example: • $\sigma \mapsto \sigma_x$ is increasing.

• ⚠ $\sigma \mapsto \sigma_A$ for $|A| \geq 2$ is not!

• $\sigma \mapsto \mathbb{1}_{\forall x \in A,\, \sigma_x = +1}$ is increasing. Note that it is $\prod_{x \in A} \frac{1+\sigma_x}{2}$.

$n_A :=$

Let μ, ν be two probabilities on Ω. We say that **μ is stochastically dominated by ν**, denoted $\mu \leq_{st} \nu$, if $\forall f$ increasing measurable bounded,

("μ tends to have more $\oplus$ spins than ν")

$$\int f \, d\mu \leq \int f \, d\nu.$$

Example: (Site Bernoulli percolation).

Let $p \in [0,1]$, and $\mathrm{Rad}_p = p\,\delta_1 + (1-p)\,\delta_{-1}$,

then for $p \leq p'$, $\quad \mathrm{Rad}_p^{\otimes \Lambda} \leq_{st} \mathrm{Rad}_{p'}^{\otimes \Lambda}$.

Indeed, let $(U_x)_{x \in \Lambda}$ be iid $\mathcal{U}nif([0,1])$, then

$$X : \Lambda \to \{-1, 1\} \qquad\qquad Y : \Lambda \to \{-1, +1\}$$

$$x \mapsto \begin{cases} 1 & \text{if } U_x \leq p \\ -1 & \text{if } U_x > p \end{cases} \qquad\qquad x \mapsto \begin{cases} 1 & \text{if } U_x \leq p' \\ -1 & \text{if } U_x > p' \end{cases}$$

We have $X \sim \mu = \mathrm{Rad}_p^{\otimes \Lambda}$, $\quad Y \sim \nu = \mathrm{Rad}_{p'}^{\otimes \Lambda}$, and

$X \leq Y$ a.s.

So for $f \nearrow$, $f(X) \leq f(Y)$ a.s. so $\underbrace{\mathbb{E}[f(X)]}_{= \int f d\mu} \leq \underbrace{\mathbb{E}[f(Y)]}_{= \int f d\nu}$.

The method of the example (coupling) is in fact generic:

Theo: Let μ, ν be probabilities on Ω. ← here $\Omega = \{\pm1\}^\Lambda$ or $\{\pm1\}^{\mathbb{Z}^d}$ but works for Ω Polish ordered space.

The following are equivalent:

(i) $\mu \leq_{st} \nu$

(ii) There exists random variables X, Y defined on the same proba. space such that

$$X \sim \mu, \quad Y \sim \nu \quad \text{and} \quad X \leq Y \text{ a.s.}$$

The law of (X,Y) is called a coupling

(Not proved, see Werner's "Percolation et Modèle d'Ising" for Ω finite, Lindvall '99 for generic case)

B) Glauber dynamics

For the moment, we take $\Lambda \subset\subset \mathbb{Z}^d$, and $\Omega = \{\pm1\}^\Lambda$. Also, $\beta > 0$, $h \in \mathbb{R}$, $\omega \in \{\pm1\}^{\partial\Lambda}$.

Q: How does one simulate the measure $\mu^\omega_{\Lambda,\beta,h}$?

(a priori we have $2^{|\Lambda|}$ configs to consider, so it may be impossible to list them all and compute their proba! Also Z itself requires all configs to compute, so we can't even know the proba of a single config σ!)

Idea: Create a Markov chain on Ω whose invariant distri is $\mu^\omega_{\Lambda,\beta,h}$. ← denoted μ later on.

Starting from a config $\sigma \in \Omega$, select a $x \in \Lambda$ uniformly at random. Let σ^{+x} be the config σ with a $+1$ at x, σ^{-x} — σ — -1 — .

easy to compute: Z and most of the exponential cancel out!

Then $\dfrac{\mu(\sigma^{+x})}{\mu(\sigma^{+x}) + \mu(\sigma^{-x})}$ is the conditional proba of seeing $+1$ at x conditionnally on $\sigma|_{\Lambda\setminus\{x\}}$

So we set σ_x to be $+1$ with proba $\dfrac{\mu(\sigma^{+x})}{\mu(\sigma^{+x}) + \mu(\sigma^{-x})}$

and to be -1 — $\dfrac{\mu(\sigma^{-x})}{\mu(\sigma^{+x}) + \mu(\sigma^{-x})}$.

(regardless of the previous σ_x).

In other words, we consider the Markov Chain on Ω with transition matrix

$\forall \sigma \in \Omega$, $\forall x \in \Lambda$, $\quad P(\sigma, \sigma^{+x}_{(-x)}) = \frac{1}{|\Lambda|} \frac{\mu(\sigma^{+x}_{(-x)})}{\mu(\sigma^{+x}) + \mu(\sigma^{-x})}$

Prop: This MC (is irreducible, recurrent > 0, aperiodic) has (unique) invariant distri μ.

Proof: • Clearly it is irred; since Ω is finite it is rec. > 0; $P(\sigma,\sigma)>0$ so aper.

• Moreover, it is reversible for μ!
Let $\sigma, \sigma' \in \Omega$. Then either $P(\sigma,\sigma')=0$ (then $P(\sigma',\sigma)=0$)
or $\exists x \in \Lambda$ / $\sigma' = \sigma^{\pm x}$ and then

$$\mu(\sigma)\, P(\sigma,\sigma') = \frac{1}{|\Lambda|} \frac{\mu(\sigma)\,\mu(\sigma')}{\mu(\sigma^{+x}) + \mu(\sigma^{-x})}$$

$$= \frac{1}{|\Lambda|} \left(\frac{\mu(\sigma')\,\mu(\sigma)}{\mu(\sigma'^{+x}) + \mu(\sigma'^{-x})}\right)$$

$$= \mu(\sigma')\, P(\sigma',\sigma).$$ $\boxtimes$

This may not look like much, but we deduce a "local criterion" for stochastic domination:

Prop Let μ, ν be probabilities on $\Omega = \{\pm 1\}^\Lambda$, such that for any $\sigma \leq \sigma' \in \Omega$ and $x \in \Lambda$,

"Holley criterion"

$$\frac{\mu(\sigma^{+x})}{\mu(\sigma^{-x})} \leq \frac{\nu(\sigma'^{+x})}{\nu(\sigma'^{-x})} \qquad \left(\Leftrightarrow \frac{\mu(\sigma^{+x})}{\mu(\sigma^{+x}) + \mu(\sigma^{-x})} \leq \frac{\nu(\sigma'^{+x})}{\nu(\sigma'^{+x}) + \nu(\sigma'^{-x})} \right)$$

by direct check.

Then $\mu \leq_{st} \nu$.

Proof Let $(X_n)_{n\geq 0}$, $(Y_n)_{n \geq 0}$ be two Markov chains on Ω, with X_n / Y_n having the previous transition matrix for μ / ν.

They are coupled s.t. at each step we take the same site $x \in \Lambda$, and whenever we set it to (+1) in X_n, we also set it to (+1) in Y_n. By the asumption, this is possible (cf direct check).

Then starting with $X_0 \le Y_0$, clearly, a.s., $\forall n$, $X_n \le Y_n$.

So if f is $\nearrow$ and bounded,

$$\mathbb{E}[f(X_n)] \le \mathbb{E}[f(Y_n)]$$

$$\downarrow \qquad\qquad\qquad \searrow$$

$$\int f d\mu \quad (\text{since } X_n \xrightarrow{d} \mu) \qquad\qquad \int f d\nu \quad (Y_n \xrightarrow{d} \mu)$$

$\boxtimes$

§ Finite volume FKG (Fortuin, Kasteleyn, Ginibre; 1971)

Thm For any $\Lambda \subset\subset \mathbb{Z}^d$, $\beta > 0$, $h \in \mathbb{R}$, $\omega \in \{\pm 1\}^{\partial \Lambda}$, the measure $\mu^{\omega}_{\Lambda,\beta,h}$ satisfies the FKG inequality:

"positive association"

$$\forall f, g : \Omega \to \mathbb{R} \text{ increasing}$$

$$\langle f \cdot g \rangle^{\omega}_{\Lambda,\beta,h} \ge \langle f \rangle^{\omega}_{\Lambda,\beta,h} \langle g \rangle^{\omega}_{\Lambda,\beta,h}.$$

For instance, take $f = 1_A$, $g = 1_B$, where $A, B \subset \Omega$ are increasing events ($\sigma \in A$ and $\sigma \le \sigma' \Rightarrow \sigma' \in A$). Then $\langle 1_A 1_B \rangle \ge \langle 1_A \rangle \langle 1_B \rangle$: A, B are positively correlated.

Proof: Let f, g be $\nearrow$. Adding a constant to g does not change the inequality, so we suppose $g > 0$ ($\forall \sigma \in \Omega, g(\sigma) > 0$).

Let $\mu = \mu^{\omega}_{\Lambda,\beta,h}$, and $\nu(\sigma) = \dfrac{g(\sigma)}{\langle g \rangle^{\omega}_{\Lambda,\beta,h}} \mu(\sigma)$.

One checks that ν is also a proba on Ω.

We will show $\mu \le_{st} \nu$.

This gives the result because $\langle f \rangle = \int f \, d\mu \leq \int f \, d\nu = \frac{1}{\langle g \rangle} \sum_\sigma f(\sigma) g(\sigma) \mu(\sigma)$

$= \frac{\langle fg \rangle}{\langle g \rangle}$.

We use the property: it is enough to show that for $\sigma \leq \sigma'$ and $x \in \Lambda$, $\frac{\mu(\sigma^{+x})}{\mu(\sigma^{-x})} \overset{?}{\leq} \frac{\nu(\sigma'^{+x})}{\nu(\sigma'^{-x})}$.

But $\frac{\nu(\sigma'^{+x})}{\nu(\sigma'^{-x})} = \frac{g(\sigma'^{+x}) \mu(\sigma'^{+x})}{g(\sigma'^{-x}) \mu(\sigma'^{-x})} \geq \frac{\mu(\sigma'^{+x})}{\mu(\sigma'^{-x})}$

So it is enough to show that $\sigma \mapsto \frac{\mu(\sigma^{+x})}{\mu(\sigma^{-x})}$ is $\nearrow$ in σ (then apply at $\sigma \leq \sigma'$)

But it is $\sigma \mapsto \exp\left(\beta \sum_{y \sim x} \left((+1)\sigma_y - (-1)\sigma_y\right) + 2h\right)$

$= \exp\left(+2\beta \sum_{y \sim x} \sigma_y + 2h\right)$, clearly $\nearrow$ in σ. $\boxtimes$

<u>Remark</u>: We see that if μ is a measure on Ω s.t. $\forall \sigma \leq \sigma'$, $\frac{\mu(\sigma^{+x})}{\mu(\sigma^{-x})} \leq \frac{\mu(\sigma'^{+x})}{\mu(\sigma'^{-x})}$, then μ satisfies FKG.

In fact it is enough to have $\forall \sigma$, $\forall x, y$, $\frac{\mu(\sigma^{+x-y})}{\mu(\sigma^{-x-y})} \leq \frac{\mu(\sigma^{+x+y})}{\mu(\sigma^{-x+y})}$ also "Holley"

<u>Application</u>: For $n_A = \prod_{x \in A} \frac{1+\sigma_x}{2}$, we get

$$\langle n_A n_B \rangle^\omega_{\Lambda,\beta,h} \geq \langle n_A \rangle^\omega_{\Lambda,\beta,h} \langle n_B \rangle^\omega_{\Lambda,\beta,h}.$$

<u>Exercise</u> Construct $\mu^{+/-}_{\beta,h}$ as limits using this inequality, for $h \in \mathbb{R}$.

For + b.c. (or "free" b.c., ie no constraint on $\partial\Lambda$)

<u>Remark</u>: Griffith: $\langle \sigma_A \sigma_B \rangle^+_{\Lambda,\beta} \geq \langle \sigma_A \rangle^+_{\Lambda,\beta} \langle \sigma_B \rangle^+_{\Lambda,\beta}$. $\leftarrow h = 0$ (works for $h \geq 0$)

not $\nearrow$! any b.c.

FKG: $\langle n_A n_B \rangle^\omega_{\Lambda,\beta,h} \geq \langle n_A \rangle^\omega_{\Lambda,\beta,h} \langle n_B \rangle^\omega_{\Lambda,\beta}$. $\leftarrow h \in \mathbb{R}$

$\nearrow$ functions.

Prop: Let $\Lambda \subset\subset \mathbb{Z}^d$, $\beta \geq 0$. Let $h \leq h'$ and $\omega \leq \omega' \in \{\pm 1\}^{\partial\Lambda}$.
Then $\mu^\omega_{\Lambda,\beta,h} \leq_{st} \mu^{\omega'}_{\Lambda,\beta,h'}$ Monotonicity (stochastic dom.) in b.c and in h.

<u>Proof</u>: <u>Exercise</u> (use Holley criterion).

D) Infinite volume FKG

Cor: The measures $\mu^{+/-}_{\beta,h}$ on $\Omega = \{\pm 1\}^{\mathbb{Z}^d}$ satisfy the FKG inequality:

$\forall f, g : \Omega \to \mathbb{R}$ increasing and in $Loc(\Omega)$ or $\mathcal{C}(\Omega)$,

$$\langle f g \rangle^{+/-}_{\beta,h} \geq \langle f \rangle^{+/-}_{\beta,h} \langle g \rangle^{+/-}_{\beta,h}.$$

Proof: We just put together finite FKG and limits ... ☒

III - Gibbs measures

A) Domain Markov property

Prop Let $\Delta \subset \Lambda \subset\subset \mathbb{Z}^d$, $\beta \geq 0$ and $h \in \mathbb{R}$.
Let $\omega \in \{\pm 1\}^{\partial\Lambda}$ and $\omega' \in \{\pm 1\}^{\partial\Delta}$ be compatible:
$\forall x \in \partial\Lambda \cap \partial\Delta$, $\omega_x = \omega'_x$.

Then $\forall \eta \in \{\pm 1\}^{\Delta}$,

$$\mu^{\omega}_{\Lambda,\beta,h}(\sigma|_\Delta = \eta \mid \forall x \in \partial\Delta \cap \Lambda, \sigma_x = \omega'_x) = \mu^{\omega'}_{\Delta,\beta,h}(\eta).$$

Proof. Suppose for simplicity $\partial\Delta \subset \Lambda$.
The left-hand side is

$$\frac{\displaystyle\sum_{\sigma \in \Omega_\Lambda \text{ st } \sigma|_\Delta = \eta,\ \sigma|_{\partial\Delta} = \omega'} \mu^{\omega}_\Lambda(\sigma)}{\displaystyle\sum_{\sigma \in \Omega_\Lambda \text{ st } \sigma|_{\partial\Delta} = \omega'} \mu^{\omega}_\Lambda(\sigma)}.$$

By a direct computation, if $\sigma \in \Omega_\Lambda$ st $\sigma|_{\partial\Delta} = \omega'$,

$$\mu^{\omega}_\Lambda(\sigma) \propto \underbrace{\mu^{\omega'}_\Delta(\sigma|_\Delta)}_{\substack{\text{terms } \Delta - \Delta \\ \text{and } \Delta - \partial\Delta}} \cdot \underbrace{\mu^{\omega,\omega'}_{\Lambda\setminus\bar\Delta}(\sigma|_{\Lambda\setminus\bar\Delta})}_{\substack{\text{terms } \Lambda\setminus\bar\Delta - \Lambda\setminus\bar\Delta \\ \text{and } \Lambda\setminus\bar\Delta - \partial\Delta \\ \text{and } \partial\Lambda - \dots}} \cdot \underbrace{\exp\Big(\beta \sum_{\substack{x \sim y \\ x,y \in \partial\Delta}} \omega'_x \omega'_y + h \sum_{x \in \partial\Delta} \omega'_x\Big)}_{\text{terms } \partial\Delta - \partial\Delta}$$

indep of σ

So we may write it as
(σ|_Δ and σ|_{Λ∖Δ̄} are disjoint)

$$\frac{\mu^{\omega'}_{\Delta}(\eta)\cdot \cancel{\sum_{\sigma\in\Omega_{\Lambda\setminus\bar\Delta}} \mu^{\omega,\omega'}_{\Lambda\setminus\bar\Delta}(\sigma)}}{\underbrace{\left(\sum_{\sigma\in\Omega_\Delta}\mu^{\omega'}_{\Delta}(\sigma)\right)}_{=1}\cdot\cancel{\left(\sum_{\sigma\in\Omega_{\Lambda\setminus\bar\Delta}}\mu^{\omega,\omega'}_{\Lambda\setminus\bar\Delta}(\sigma)\right)}} = \mu^{\omega'}_{\Delta}(\eta)$$

$\boxed{\Omega}$.

__Exercise:__ Using the domain Markov prop and the prop $\mu^{\omega}_{\Lambda}\leq_{st}\mu^{\omega'}_{\Lambda}\ldots$ show that for $\Delta\subset\Lambda\subset\subset\mathbb{Z}^d$,

$$\mu^+_\Lambda \leq_{st} \mu^+_\Delta . \quad (\text{for } \nearrow \text{ functions on } \{\pm1\}^{\Delta})$$

and $\mu^-_\Lambda \geq_{st} \mu^-_\Delta$.

Stochastic domination in domain

B) DLR conditions

Let $\beta>0$, $h\in\mathbb{R}$. A measure μ on $\Omega=\{\pm1\}^{\mathbb{Z}^d}$ is said to be a Gibbs measure (for the Ising model at β,h) if $\forall\,\Lambda\subset\subset\mathbb{Z}^d$, $\omega\in\{\pm1\}^{\partial\Lambda}$, and any event E depending only on the sites in $\mathbb{Z}^d\setminus\Lambda$,

"DLR conditions"

$$\boxed{\mu(\cdot\mid\{\sigma|_{\partial\Lambda}=\omega\}\cap E) = \mu^{\omega}_{\Lambda,\beta,h}(\cdot)}$$

In other words, conditionally on $\sigma|_{\partial\Lambda}$, $\sigma|_{\Lambda}$ and $\sigma|_{\Lambda^c}$ are indep and $\sigma|_{\Delta}\sim\mu^{\omega}_{\Lambda,\beta,h}$.

__Prop__ The measures μ^+ and μ^- are Gibbs measures.

__Proof:__ Let $\Lambda\subset\subset\mathbb{Z}^d$, and let A,B be local events st A depends only on Λ and B depends only on Λ^c.

$$\mu^+(A\mid\{\sigma|_{\partial\Lambda}=\omega\}\cap B) = \frac{\mu^+(A \text{ and } \sigma|_{\partial\Lambda}=\omega \text{ and } B)}{\mu^+(\sigma|_{\partial\Lambda}=\omega \text{ and } B)}$$

$$= \lim_{n\to\infty} \frac{\mu^+_{\Lambda_n}(A \text{ and } \sigma|_{\partial\Lambda}=\omega \text{ and } B)}{\mu^+_{\Lambda_n}(\sigma|_{\partial\Lambda}=\omega \text{ and } B)}$$

with n large enough s.t. B depends only on $\Lambda_n \setminus \Lambda$.

By Spatial Markov Prop, this is

$$\mu^+_{\Lambda_n}(A \mid \sigma|_{\partial\Lambda}=\omega \text{ and } B) = \mu^\omega_\Lambda(A).$$

As local events generate the σ-algebra, this is enough. □.

c) Extremality

It is natural to wonder if there are more Gibbs measures. There can be but we have found the extremal ones:

if $\mu^- \neq \mu^+$, consider $t\mu^- + (1-t)\mu^+$ for $t \in]0,1[$...

Prop If μ is a Gibbs measure,

$$\mu^- \leq_{st} \mu \leq_{st} \mu^+.$$

(For bounded ↗ functions, local or continuous..)

Proof: Let f be a local (dep. only on Λ_r), increasing function. For all $n \geq r$, and $\omega \in \{\pm 1\}^{\partial\Lambda_n}$,

$$\langle f\rangle^-_{\Lambda_n} \leq \langle f\rangle^\omega_{\Lambda_n} \leq \langle f\rangle^+_{\Lambda_n} \quad \text{(by a proposition)}$$

So

$$\langle f\rangle^-_{\Lambda_n} \leq \underbrace{\sum_{\omega\in\{\pm1\}^{\partial\Lambda_n}} \mu(\sigma|_{\partial\Lambda_n}=\omega)\langle f\rangle^\omega_{\Lambda_n}}_{\int f\, d\mu} \leq \langle f\rangle^+_{\Lambda_n}.$$

Then we let $n\to\infty$.

For continuous f, we approximate with local functions. □

Cor: The following are equivalent:

(i) $\mu^+ = \mu^-$

(ii) $\exists!$ Gibbs measure

For $h=0$, we will see that this holds when $\beta < \beta_c$. (high temp)

IV - Translation invariance

For $x \in \mathbb{Z}^d$, let $\theta_x : \begin{array}{l}\mathbb{Z}^d \to \mathbb{Z}^d \\ y \mapsto y+x\end{array}$ be the translation by x.

We extend it to events: $\sigma \in \theta_x(A)$ iff $\sigma \circ \theta_x^{-1} \in A$.

<u>Example</u>: For $A = \{\sigma_0 = +1\}$, $\theta_x(A) = \{\sigma_x = +1\}$.

<u>Theo</u> The measures μ^+, μ^- are translation invariant.

$(\forall x \in \mathbb{Z}^d,\ \mu^\pm \circ \theta_x = \mu^\pm)$

<u>Proof</u>: Let $n \in \mathbb{N}^*$. Then $\mu^+_{\Lambda_n} \circ \theta = \mu^+_{x+\Lambda_n}$.

But $x + \Lambda_n \subset \Lambda_{n+\|x\|_\infty}$, so

$$\mu^+_{x+\Lambda_n} \geq_{st} \mu^+_{\Lambda_{n+\|x\|_\infty}} \quad \text{(by an exercise!)}$$

As $n \to \infty$, we get $\mu^+ \circ \theta \geq_{st} \mu^+$.

Moreover, for n large enough, $\Lambda_{n-\|x\|_\infty} \subset x + \Lambda_n$, so similarly $\mu^+ \circ \theta \leq_{st} \mu^+$.

This implies $\mu^+ \circ \theta = \mu^+$. The same works for μ^-. $\square$

<u>Theo</u> The measures μ^+, μ^- are ergodic:

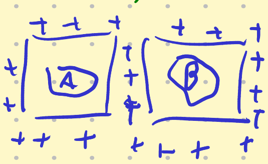
if A is an event that is invariant under all translations of $\mathbb{Z}^d$, then $\mu^\pm(A) \in \{0, 1\}$.

<u>Proof</u>: We first show that if A, B are <u>local</u>, <u>increasing</u> events, then they <u>mix</u>: $\mu^+(A \cap \theta_n(B)) \xrightarrow[n\to\infty]{} \mu^+(A)\mu^+(B)$ "asympt. indep"

translation for $x = (n, 0, \dots, 0)$

Indeed, $\mu^+(A \cap \theta_n(B)) \leq \mu^+\left(A \cap \theta_n(B) \mid \sigma|_{\partial(\Lambda_{n/2})} \equiv + \text{ and } \sigma|_{\partial(n+\Lambda_{n/2})} \equiv +\right)$

FKG for μ^+.



$$= \mu^+_{\Lambda_{n/2}}(A)\, \mu^+_{n+\Lambda_{n/2}}(\Theta_n(B)) \quad \text{by Gibbs}$$
$$= \mu^+_{\Lambda_{n/2}}(A)\, \mu^+_{\Lambda_{n/2}}(B)$$
$$\xrightarrow[n\to\infty]{} \mu^+(A)\mu^+(B)$$

So $\liminf_n \mu^+(A \cap \Theta_n(B)) \leq \mu^+(A)\mu^+(B)$,

but also $\mu^+(A \cap \Theta_n(B)) \geq \mu^+(A)\, \mu^+(\Theta_n(B))$ by FKG
$= \mu^+(A)\,\mu^+(B)$ by translation invariance

So $\mu^+(A \cap \Theta_n(B)) \xrightarrow[n\to\infty]{} \mu^+(A)\mu^+(B)$. So all <u>local</u>, $\nearrow$ events mix.

By monotone class lemmas, we deduce that any two events mix.

Show that for $A \nearrow$ local, $\mathcal{M}_A = \{B \text{ st } A, B \text{ mix}\}$ is a monotone class, and that $\nearrow$, local events generate the whole σ-algebras, then for any event B consider $\mathcal{M}_B = \ldots$

Now for A translation invariant, we get

$$\mu^+(A \cap \Theta_n(A)) \xrightarrow[n\to\infty]{} \mu^+(A)^2$$
$$\| \qquad \mu^+(A \cap A) = \mu(A)$$

So $\mu^+(A) = \mu^+(A)^2$ so $\mu^+(A) \in \{0,1\}$

Same for μ^-. ∎

<u>Exercise</u>: Show that for $f, g \in Loc(\Omega)$,

"short range correlations"

$$\langle f \cdot g \circ \theta_x \rangle^+ \xrightarrow[\|x\|\to+\infty]{} \langle f \rangle^+ \langle g \rangle^+.$$

Deduce $\langle \sigma_0 \sigma_x \rangle^+ \xrightarrow[\|x\|\to+\infty]{} (\langle \sigma_0 \rangle^+)^2$.

We conclude with the last missing part of the theorem announced at the beginning of this chapter:

<u>Theo</u> Let $\beta > 0$, $h \in \mathbb{R}$. The following are equivalent:

(i) $\mu^+ = \mu^-$

(iii) $\langle \sigma_0 \rangle^+ = \langle \sigma_0 \rangle^-$

average of σ_0 for the infinite volume measure $\mu^+_{\beta,h}$.
We already had a definition in the previous chapter for $h=0$. Check that they are the same:

$$\langle \sigma_0 \rangle^+_{\beta,h} = \lim_n \searrow \langle \sigma_0 \rangle^+_{\beta,h,\Lambda_n} \qquad \langle \sigma_0 \rangle^-_{\beta,h} = \lim_n \nearrow \langle \sigma_0 \rangle^-_{\beta,h,\Lambda_n}.$$

Proof: (i) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i): Let $A \subset\subset \mathbb{Z}^d$.

The function $\sum_{x\in A} \overset{\frac{1+\sigma_x}{2}}{n_x} - \overset{\prod_{x\in A}\frac{1+\sigma_x}{2}}{n_A}$ is increasing,

so $\langle \sum_{x\in A} n_x - n_A \rangle^- \leq \langle \sum_{x\in A} n_x - n_A \rangle^+$.

So $\langle n_A\rangle^+ - \langle n_A\rangle^- \leq \sum_{x\in A} \langle n_x\rangle^+ - \langle n_x\rangle^- \underset{\text{by translation invariance}}{=} \frac{|A|}{2}\left(\langle\sigma_0\rangle^+ - \langle\sigma_0\rangle^-\right)$

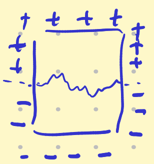
So (iii) implies $\forall A \subset\subset \mathbb{Z}^d$, $\langle n_A\rangle^+ = \langle n_A\rangle^-$.

This gives $\forall A \subset\subset \mathbb{Z}^d$, $\langle\sigma_A\rangle^+ = \langle\sigma_A\rangle^-$.

As the (σ_A) are a basis of local functions, we get $\mu^+ = \mu^-$. $\boxtimes$

Remark: At low temp ($\beta > \beta_c$) and $h=0$, there exists Gibbs measures $\neq \mu^\pm$, and in particular there might exist some that are not translation-invariant (in particular this implies that they are not mixtures $t\mu^- + (1-t)\mu^+$).

For instance, taking the limit of Dobrushin b.c.,



- in dim 2, we get $\frac{1}{2}\mu^- + \frac{1}{2}\mu^+$ (translation invariant)
- in dim ≥ 3, we get another Gibbs measure that is not translation invariant.

See Friedli-Velenik, 3.10.7.

V- Free energy and magnetization

An object related to the uniqueness of Gibbs measures is the free energy, or pressure. We will see that its derivative in h is the magnetization

A) Existence of the free energy

we will take (Λ_n) st
- $\forall n$, $\Lambda_n \subset\subset \mathbb{Z}^d$
- $\forall n$, $\Lambda_n \subset \Lambda_{n+1}$
- $\bigcup_n \Lambda_n = \mathbb{Z}^d$
- $\frac{|\partial\Lambda_n|}{|\Lambda_n|} \xrightarrow[n\to\infty]{} 0$

$(\Lambda_n) \nearrow \mathbb{Z}^d$

Theo Let $\beta \geq 0$, $h \in \mathbb{R}$. Let $(\Lambda_n) \nearrow \mathbb{Z}^d$ be such that $\frac{|\partial \Lambda_n|}{|\Lambda_n|} \to 0$.
Let (ω_n) be a sequence of b.c. $(\forall n, \omega_n \in \{-1,0,+1\}^{\partial \Lambda_n})$.
Then the limit

$$f(\beta, h) = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \log\left(Z^{\omega_n}_{\Lambda_n, \beta, h}\right)$$

exists, in $\mathbb{R}$, and is independent of the b.c. (ω_n), and of the domains (Λ_n).

The function f is called the free energy.

Proof • (ω_n) doesn't matter: Let (ω'_n) be another seq. of b.c, then

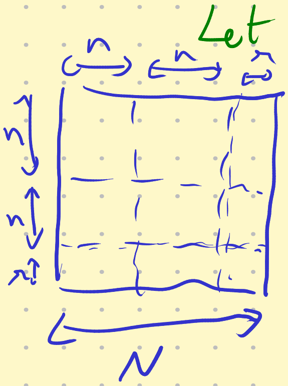
$$e^{-2\beta|\partial\Lambda_n|} \leq \frac{Z^{\omega_n}_{\Lambda_n, \beta, h}}{Z^{\omega'_n}_{\Lambda_n, \beta, h}} \leq e^{2\beta|\partial\Lambda_n|}$$

(because true for $\mu(\sigma)$ for any σ!)

so $$-2\beta\frac{|\partial\Lambda_n|}{|\Lambda_n|} \leq \frac{1}{|\Lambda_n|}\left(\log Z^{\omega_n}_{\Lambda_n} - \log Z^{\omega'_n}_{\Lambda_n}\right) \leq 2\beta\frac{|\partial\Lambda_n|}{|\Lambda_n|}.$$

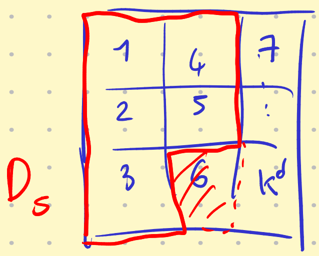
If the limit exists, it is indeed indep of (ω_n).

• We take (ω_n) all free b.c, and $\Lambda_n = [\![-n, n]\!]^d$
Let $1 \leq n \leq N$. We write $N = kn + r$, $k \geq 0$, $r \in [\![0; n-1]\!]$



Let us compare $Z^f_{\Lambda_N}$ with $\left(Z^f_{\Lambda_n}\right)^{k^d}$

For $r = 0$: We number the boxes of size n as $B_1, B_2, \ldots, B_{k^d}$.



Let $D_m = \bigcup_{i=1}^{m} B_i$.

Then

$$Z^f_{D_{m+1}} = \sum_{\sigma \in \Omega_{D_{m+1}}} \exp\left(\beta \sum_{\{x,y\} \in E_{m+1}} \sigma_x \sigma_y + h \sum_{x \in D_{m+1}} \sigma_x\right)$$

$$= \sum_{\sigma \in \Omega_{D_m}} \exp\left(\beta \sum_{\{x,y\} \in E_m} \sigma_x \sigma_y + h \sum_{x \in D_m} \sigma_x\right) \cdot Z^{\omega_\sigma}_{\Lambda_n}$$

by beginning of proof $\longrightarrow Z^f_{\Lambda_n} + \exp[O(|\partial\Lambda_n|)]$

This gives $Z^\beta_{\Lambda_N} = \left(Z^\beta_{\Lambda_n}\right)^{k^d} \exp\Big(\underbrace{O(k^d|\partial\Lambda_n|)}_{= O(k^d n^{d-1}) = O(k N^{d-1})}\Big)$

So we have, more precisely, $\exists c>0$ st

$$-c k N^{d-1} \leq \log Z^\beta_{\Lambda_N} - k^d \log Z^\beta_{\Lambda_n} \leq c k N^{d-1}$$

$|\Lambda_N| = N^d = k^d n^d$

$|\Lambda_n| = n^d$

So $$\frac{1}{|\Lambda_N|}\log Z^\beta_{\Lambda_N} \leq \underbrace{\frac{k^d}{|\Lambda_N|}}_{=\frac{1}{|\Lambda_n|}} \log Z^\beta_{\Lambda_n} + \underbrace{\frac{c k N^{d-1}}{|\Lambda_N|}}_{=\frac{c}{n}}$$

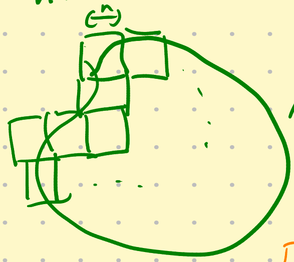
check that $h>0 \rightsquigarrow$ still works

So $\limsup_N \frac{1}{|\Lambda_N|}\log Z^\beta_{\Lambda_N} \leq \frac{1}{|\Lambda_n|}\log Z^\beta_{\Lambda_n} + \frac{c}{n}$

$$\leq \liminf_n \frac{1}{|\Lambda_n|}\log Z^\beta_{\Lambda_n}.$$

So the lim exists.

- Independence on (Λ_n): idea:



Λ'_N

- For n large enough, $\frac{1}{n^d}\log(Z_{\square})$ close to $f(\beta,h)$
- So (same as D_m before), $[\Lambda'_N] := \square\square$ also close to f.
- But $\Lambda'_N \,\Delta\, [\Lambda'_N] \lesssim \partial\Lambda'_N \cdot n^d$ ... so close together.

left as exercise, see Friedli–Velenik, 3.2.2.

$\boxed{}$

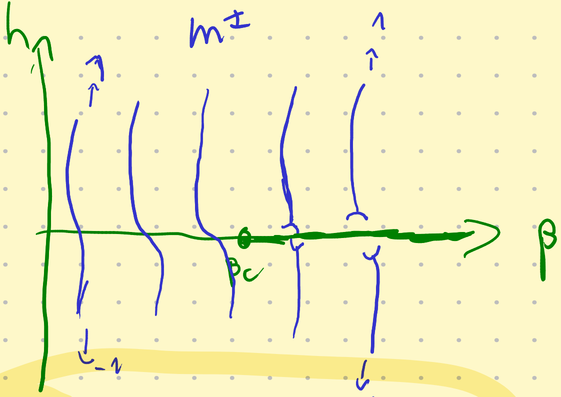
Exercise: Find a sequence $(\Lambda_n) \nearrow \mathbb{Z}^d$ and (ω_n) bc st $\frac{1}{|\Lambda_n|}\log Z^{\omega_n}_{\Lambda_n} \not\to f(\beta,h)$

B) Analytic properties of the magnetization

We now work towards seeing the critical phase (where the phase transition occurs) as the point where the magnetization (and the free energy) is non-analytic.

Let $m^+(\beta,h) := \langle\sigma_0\rangle^+_{\beta,h}$ and $m^-(\beta,h) := \langle\sigma_0\rangle^-_{\beta,h}$.

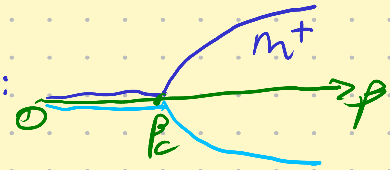
We would like to show that the profile of these functions is the same as in the Curie-Weiss model:



← $m^\pm$ is analytic on $(0,\infty)\times\mathbb{R} \setminus [\beta_c,+\infty)\times\{0\}$,

it is ↗ in h,

For $h=0$ it looks like:



$\forall(\beta,h)$ / $h\neq 0$ or $\beta<\beta_c$,
$m^+(\beta,h)=m^-(\beta,h)$

↖ so by the main theo of section, $\exists!$ Gibbs measure there!

So $\begin{cases} m^+ \text{ is right-continuous in } h. \\ m^- \text{ left} \end{cases}$

It is hard to get all these prop, in particular that $m^\pm$ are continuous at $(\beta_c, 0)$ (so they are both zero), but we will have the tools for it in the next chapter. Let us prove parts of this.

Remark: $m^+(\beta,h) = -m^-(\beta,-h)$ by symmetry.

Prop:
- For any fixed $\beta\geq 0$, $h\mapsto \begin{cases} m^+(\beta,h) \text{ is } \nearrow \text{ and right-continuous} \\ m^-(\beta,h) \text{ is } \nearrow \text{ and left-continuous} \end{cases}$
- For any fixed $h\geq 0$, $\beta\mapsto m^+(\beta,h)$ is $\nearrow$.
  For any fixed $h\leq 0$, $\beta\mapsto m^-(\beta,h)$ is $\searrow$.

Proof: • By the remark, we just have to prove the props for m^+.

• Let $\beta\geq 0$. We know that there is monotonicity (stochastic domination) in h, so we have $\forall h\leq h'$, $m^+(\beta,h)\leq m^+(\beta,h')$. (passing to the limit in Λ)

(other proof: for $\Lambda \subset\subset \mathbb{Z}^d$)

$$\frac{\partial}{\partial h}\langle\sigma_0\rangle^+_{\Lambda,\beta,h} = \frac{\partial}{\partial h}\left(\frac{\sum_\sigma \sigma_0 \exp(\beta\sum\sigma_x\sigma_y + h\sum\sigma_x)}{\sum_\sigma \exp(\beta\sum\sigma_x\sigma_y + h\sum\sigma_x)}\right)$$

$$= \frac{\sum_{\sigma,\sigma'}\left(\sigma_0\sum\sigma_x - \sigma_0\sum\sigma'_x\right)\exp(\beta\sum\sigma_x\sigma_y + h\sum\sigma_x)\exp(\beta\sum\sigma'_x\sigma'_y + \dots)}{Z^2}$$

$$= \sum_x\left(\langle\sigma_0\sigma_x\rangle^+_{\Lambda,\beta,h} - \langle\sigma_0\rangle^+_{\Lambda,\beta,h}\langle\sigma_x\rangle^+_{\Lambda,\beta,h}\right)$$

≥ 0 by FKG.)

For right-continuity, we use the lemma:

Lemma: Let $(a_{n,m})$ be a double sequence of real numbers which is bounded above and increasing.

$n \leq n'$ and $m \leq m' \Rightarrow a_{n,m} \leq a_{n',m'}$.

Then

(Exercise) $\lim_n \lim_m a_{n,m} = \lim_m \lim_n a_{n,m} = \sup_{n,m} a_{n,m}$.

So if $(h_m) \searrow h$,

$$\lim_m m(\beta, h_m) = \lim_m \lim_n \langle \sigma_0 \rangle^+_{\Lambda_n, \beta, h_m}$$

$$= \lim_n \lim_m \langle \sigma_0 \rangle^+_{\Lambda_n, \beta, h_m} \quad \text{by Lemma}$$

$$= \lim_n \langle \sigma_0 \rangle^+_{\Lambda_n, \beta, h} \quad \text{because for fixed } \Lambda_n, \ h \mapsto \langle \sigma_0 \rangle^+_{\beta, h_m} \text{ is contin. (because differentiable!)}$$

$$= m(\beta, h).$$

(Careful, for $\beta \leq \beta'$, we do not necessarily have $\mu^w_{\Lambda, \beta, h} \leq_{st} \mu^w_{\Lambda, \beta', h} \dots$)

Let $h \geq 0$. $\frac{\partial}{\partial \beta} \langle \sigma_0 \rangle^+_{\Lambda, \beta, h} = \dots$ as before

$$= \sum_{\{x,y\} \in E} \underbrace{\langle \sigma_0 \sigma_x \sigma_y \rangle^+_{\Lambda, \beta, h} - \langle \sigma_0 \rangle^+ \langle \sigma_x \sigma_y \rangle^+_{\Lambda, \beta, h}}_{\geq 0 \text{ By Griffith with } h \geq 0}$$

(not proved before, but we will show it in next chapter ...)

☒.

Prop: $\forall \beta > 0, h \in \mathbb{R}$,

$$\lim_{h' \nearrow h} m^+(\beta, h') = m^-(\beta, h).$$

As a result:

Theo: The following are equivalent:

(i) $\mu^+_{\beta,h} = \mu^-_{\beta,h}$ ($\exists!$ Gibbs measure)

(iv) $m^+(\beta, \cdot)$ is continuous at h.

Proof of theo: We know that (i) $\Leftrightarrow$ $m^+(\beta,h) = m^-(\beta,h)$, and since m^+ is right-cont, with the Prop,

m^+ is continuous at $h \Leftrightarrow m^-(\beta,h) = m^+(\beta,h)$

Proof of Prop: • $m^-(\beta,h) = \lim_{h'\nearrow h} m^-(\beta,h')$ by left-continuity

$$\le \lim_{h'\nearrow h} m^+(\beta,h') \quad \text{by stoch. dom. in b.c.}$$

clearly enough to have equality.

• Let $h'<h$. We show in fact $a := m^+(\beta,h') \le m^-(\beta,h) =: b$.

Let $n\in\mathbb{N}$. We write $S=\sum_{x\in\Lambda_n}\sigma_x$, $S'=\sum_{\substack{\{x,y\}\in E \\ y\in\partial\Lambda_n}}\sigma_x$.

Then $\langle S\rangle^+_{\Lambda_n,\beta,h'} = \sum_{x\in\Lambda_n}\langle\sigma_x\rangle^+_{\Lambda_n,\beta,h'} \le |\Lambda_n|\, a$.

$\forall x$, in n this $\searrow$ to $m^+(\beta,h)=a$

Let $\varepsilon>0$.

$$\mu^+_{\Lambda_n,\beta,h'}(S\le(a-\varepsilon)|\Lambda_n|) = \mu^+_{\Lambda_n,\beta,h'}(|\Lambda_n|-S \ge (1-a+\varepsilon)|\Lambda_n|)$$

$$\overset{\text{Markov}}{\le} \frac{|\Lambda_n| - \langle S\rangle^+_{\Lambda_n,\beta,h}}{(1-a+\varepsilon)|\Lambda_n|} \le \frac{1-a}{1-a+\varepsilon} \le 1-\frac{\varepsilon}{2}.$$

Similarly $\langle S\rangle^-_{\Lambda_n,\beta,h} \ge |\Lambda_n|\, b$

and $\mu^-_{\Lambda_n,\beta,h}(S\ge(b+\varepsilon)|\Lambda_n|) = \mu^-_{\Lambda_n,\beta,h}(|\Lambda_n|+S \ge (1+b+\varepsilon)|\Lambda_n|)$

$$\le \frac{1+b}{1+b+\varepsilon} \le 1-\frac{\varepsilon}{2}.$$

So $\frac{\varepsilon}{2}\le \mu^+_{\Lambda_n,\beta,h'}(S>(a-\varepsilon)|\Lambda_n|)$

$$= \frac{\langle 1\!\!1_{S>(a-\varepsilon)|\Lambda_n|}\exp((h'-h)S+2\beta S')\rangle^-_{\Lambda_n,\beta,h}}{\langle \exp((h'-h)S+2\beta S')\rangle^-_{\Lambda_n,\beta,h}}$$

$|S'|\le d|\partial\Lambda_n|$

$$\le \frac{\exp\left((h'-h)(a-\varepsilon)|\Lambda_n| + 4\beta d|\partial\Lambda_n|\right)}{\langle 1\!\!1_{S<(b+\varepsilon)|\Lambda_n|}\exp((h'-h)S)\rangle^-_{\Lambda_n,\beta,h}}$$

$$\leq \frac{\exp\left((h'-h)(a-b-2\varepsilon)|\Lambda_n| + 4\beta d|\partial\Lambda_n|\right)}{\mu^-_{\Lambda_n,\beta,h}(S \leq (b+\varepsilon)|\Lambda_n|)}$$

So $\left(\frac{\varepsilon}{2}\right)^2 \leq \exp\left[(h'-h)(a-b-2\varepsilon)|\Lambda_n|\right] \cdot \exp(4\beta d|\partial\Lambda_n|)$

If we had $a-b-2\varepsilon > 0$, then as $n\to\infty$, as $h'-h<0$ and $|\Lambda_n| \gg |\partial\Lambda_n|$, the right-hand-side would go to 0. Contradiction.

So $a-b-2\varepsilon \leq 0$, and $a \leq b+2\varepsilon$ ... so $a \leq b$, as claimed. $\square$

This allows us to prove the announced profile for $h \neq 0$:

Theo: $\forall \beta > 0$, $\forall h \neq 0$, there is a unique Gibbs measure ($\mu^- = \mu^+$ ..., $m^- = m^+$ ...)

Proof: We also need an inequality that will be proved in the next chapter, the GHS inequality:
Griffith-Hurst-Sherman

For $\Lambda \subset\subset \mathbb{Z}^d$, $\beta > 0$, $h \geq 0$, $\forall x, y, z \in \Lambda$,

$$\langle \sigma_x \sigma_y \sigma_z \rangle^+_{\Lambda,\beta,h} - \langle \sigma_x \rangle^+_{..} \langle \sigma_y \sigma_z \rangle^+_{..} - \langle \sigma_y \rangle^+_{..} \langle \sigma_x \sigma_z \rangle - \langle \sigma_z \rangle \langle \sigma_x \sigma_y \rangle$$
$$+ 2\langle \sigma_x \rangle \langle \sigma_y \rangle \langle \sigma_z \rangle \leq 0.$$

We assume it for now.

We prove that $\forall \Lambda \subset \mathbb{Z}^d$, $h \mapsto \langle \sigma_0 \rangle^+_{\Lambda,h,\beta}$ is concave (on $[0,\infty)$)

This implies that the limit $h \mapsto m(\beta,h)$ is concave (on $[0,\infty)$), so it is continuous (on $(0,\infty)$).

We get the theo for $h>0$, but also for $h<0$ as $m^+(\beta,h) = -m^-(\beta,-h)$ ...

Let's go: For $\Lambda \subset\subset \mathbb{Z}^d$, $\beta > 0$, $h > 0$, we differentiate twice:

$$\forall A \subset\subset \mathbb{Z}^d, \quad \frac{\partial}{\partial h}\left(\langle \sigma_A \rangle^+_{\Lambda,\beta,h}\right) = \dots = \sum_{x\in\Lambda} \langle \sigma_x \sigma_A \rangle^+_{\Lambda,\beta,h} - \langle \sigma_x \rangle^+_{\Lambda,\beta,h} \langle \sigma_A \rangle^+_{\Lambda,\beta,h}$$

and so, $\frac{\partial^2}{\partial h^2}\left(\langle\sigma_0\rangle^+_{1,\beta,h}\right) = \frac{\partial}{\partial h}\left(\sum_x \langle\sigma_x\sigma_0\rangle - \langle\sigma_x\rangle\langle\sigma_0\rangle\right)$

$$= \sum_x \frac{\partial}{\partial h}\langle\sigma_0\sigma_x\rangle - \langle\sigma_0\rangle\frac{\partial}{\partial h}\langle\sigma_x\rangle - \langle\sigma_x\rangle\frac{\partial}{\partial h}\langle\sigma_0\rangle$$

$$= \sum_{x,y} \langle\sigma_0\sigma_x\sigma_y\rangle - \langle\sigma_0\sigma_x\rangle\langle\sigma_y\rangle - \langle\sigma_0\rangle\langle\sigma_x\sigma_y\rangle + \langle\sigma_0\rangle\langle\sigma_x\rangle\langle\sigma_y\rangle$$
$$- \langle\sigma_x\rangle\langle\sigma_0\sigma_y\rangle + \langle\sigma_x\rangle\langle\sigma_0\rangle\langle\sigma_y\rangle$$

≤ 0 By GHS! ☒.

5) Analytic prop. of the Free energy

Prop: The function f is convex on $(0,\infty)\times\mathbb{R}$.

Proof: Let (β,h), $(\beta',h')\in(0,\infty)\times\mathbb{R}$ and $t\in[0,1]$, and $\beta_t = t\beta+(1-t)\beta'$, $h_t = th+(1-t)h'$.

We want to show
$$f(\beta_t,h_t)\leq tf(\beta,h)+(1-t)f(\beta',h').$$

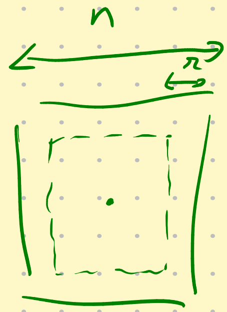
But $Z^f_{\Lambda_n,\beta_t,h_t} = \sum_\sigma \exp\left(\beta\sum\sigma_x\sigma_y + h\sum\sigma_x\right)^t \exp\left(\beta'\sum\ldots+h'\sum.\right)^{1-t}$

$\overset{\text{Hölder}}{\leq}\left(Z^f_{\Lambda_n,\beta,h}\right)^t\left(Z^f_{\Lambda_n,\beta',h'}\right)^{1-t}$. We take the log and voilà! ☒

Cor:
- $\forall\beta>0$, the function $h\mapsto f(\beta,h)$ has left- and right-derivatives at every point, and
$$\frac{\partial f}{\partial h}(\beta,h^-) = \langle\sigma_0\rangle^-_{\beta,h}, \qquad \frac{\partial f}{\partial h}(\beta,h^+) = \langle\sigma_0\rangle^+_{\beta,h}.$$
- $\forall\beta>0$, $\exists!$ Gibbs measure at (β,h) iff $h\mapsto f(\beta,h)$ is differentiable at h.

We need a lemma for the proof:

Lemma (average magnetization):
$$\forall\beta,h,\ \langle\sigma_0\rangle^+_{\beta,h} = \lim_{n\to\infty}\frac{1}{|\Lambda_n|}\left\langle\sum_{x\in\Lambda_n}\sigma_x\right\rangle^+_{\Lambda_n,\beta,h}$$



Proof of lemma:

- Let $r > 0$ be s.t. $\left|\langle\sigma_0\rangle^+_{\Lambda_r} - \langle\sigma_0\rangle^+\right| \le \varepsilon$.

  Then for $n \geq 0$ large,

$$\sum_{x\in\Lambda_n}\langle\sigma_x\rangle^+_{\Lambda_n,\beta,h} \le \sum_{x\in\Lambda_{n-r}}\langle\sigma_x\rangle^+_{\Lambda_r,\beta,h} \quad \leftarrow \text{(monotonicity in domain)}$$

$$+ \sum_{x\in\Lambda_n\setminus\Lambda_{n-r}} 1$$

$$\le \underbrace{|\Lambda_{n-r}|}_{\sim_{n\to\infty} |\Lambda_n|}\left(\langle\sigma_0\rangle^+ + \varepsilon\right) + \underbrace{|\Lambda_n\setminus\Lambda_{n-r}|}_{O(n^{d-1}r)}.$$

So $\limsup_n \frac{1}{|\Lambda_n|}\sum\langle\sigma_x\rangle^+_{\Lambda_n,\beta,h} \le \dots \langle\sigma_0\rangle^+ + \varepsilon$ so $\le \langle\sigma_0\rangle^+$.

- On the other hand, $\langle\sigma_x\rangle^+_{\Lambda_n} \geq \langle\sigma_0\rangle^+_{\Lambda_{n+\|x\|}} \geq \langle\sigma_0\rangle^+_{\Lambda_{2n}}$.

  So $\frac{1}{|\Lambda_n|}\sum\langle\sigma_x\rangle^+_{\Lambda_n} \geq \langle\sigma_0\rangle^+ - \varepsilon$ for n large enough.

  So $\liminf_n$ '' $\geq \langle\sigma_0\rangle^+ - \varepsilon$ so $\dots \geq \langle\sigma_0\rangle^+$. $\boxtimes$

Proof of Cor: • By convexity we have left & right derivatives

$$\left(\frac{\partial f}{\partial h}(\beta, h^+)\right) = \lim_{\delta\to 0^+}\frac{f(\beta,h+\delta)-f(\beta,h)}{\delta}$$

h

Moreover, $\frac{\partial}{\partial h}\left(\log Z^+_{\Lambda_n,\beta,h}\right) = \left\langle\sum_{x\in\Lambda_n}\sigma_x\right\rangle^+_{\Lambda_n,\beta,h}$ (we see the average magn!)

$$\text{So } \frac{f(\beta,h+\delta)-f(\beta,h)}{\delta} = \lim_n \frac{1}{\delta|\Lambda_n|}\left(\log\left(Z^+_{\Lambda_n,\beta,h}\right) - \log\left(Z^+_{\Lambda_n,\beta,h+\delta}\right)\right)$$

$$= \lim_n \frac{1}{\delta}\int_h^{h+\delta} \underbrace{\frac{1}{|\Lambda_n|}\left\langle\sum_{x\in\Lambda_n}\sigma_x\right\rangle^+_{\Lambda_n,\beta,g}}_{\geq \frac{1}{|\Lambda_n|}\langle\sum\sigma_x\rangle^+_{\Lambda_n,\beta,h} \text{ as } g\geq h} dg$$

and $\le \frac{1}{|\Lambda_n|}\langle\sigma_x\rangle^+_{\Lambda_n,\beta,h+\delta}$ as $g \le h+\delta$

So $\lim_n \frac{1}{|\Lambda_n|} \langle \sum \sigma_x \rangle^+_{\Lambda_n, \beta, h} \leq \frac{f(\beta, h+\delta) - f(\beta, h)}{\delta} \leq \lim_n \frac{1}{|\Lambda_n|} \langle \sum \sigma_x \rangle^+_{\Lambda_n, \beta, h+\delta}$

$$\lim_n \frac{1}{|\Lambda_n|} \langle \sum \sigma_x \rangle^+_{\Lambda_n, \beta, h} = m^+(\beta, h), \qquad \lim_n \frac{1}{|\Lambda_n|} \langle \sum \sigma_x \rangle^+_{\Lambda_n, \beta, h+\delta} = m^+(\beta, h+\delta)$$

We let $\delta \to 0$ and use right-continuity of m^+ to get

$$\frac{\partial}{\partial h} f(\beta, h^+) = m^+(\beta, h).$$

Similar for left-deriv. $\square$

<u>Exercise</u> Show that $m^+(\beta, h) > 0$ for $\beta > 0$ and $h > 0$.