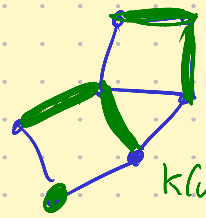
chap 5 - FK-percolation

Also called the random cluster model.

I - Definitions

Let again $G=(V,E)$ be a finite graph.
Let $(p_e)_{e\in E}$ be a family of "edge weights" in $[0,1]$.
Let $q>0$.
Let $\Omega=\{0,1\}^E$. "closed" "open"



$K(\omega)=3$

For $\omega\in\Omega$, we identify it with the subset $\{e\in E/\omega_e=1\}$.
We say that $x\overset{\omega}{\leftrightarrow}y$ if they are connected in this subset ("by a path of open edges")
Let $K(\omega)$ be the number of connected components in ω.

The FK-percolation measure is the proba. on Ω given by:

$$\forall\omega\in\Omega=\{0,1\}^E,\quad \phi(\omega)=\phi_{p,q}(\omega)=\frac{1}{Z}\Big(\prod_{\substack{e\in E/\\ \omega_e=1}}p_e\Big)\cdot\Big(\prod_{\substack{e\in E/\\ \omega_e=0}}1-p_e\Big)\cdot q^{K(\omega)}$$

$$\text{where } Z=\sum_{\omega}\prod_{\omega_e=1}p_e\prod_{\omega_e=0}1-p_e\; q^{K(\omega)}$$

Examples:
- For $q=1$, we get Bernoulli percolation: the measure is $\bigotimes_{e\in E}\mathrm{Ber}(p_e)$

  Studied extensively in 1st semester course!

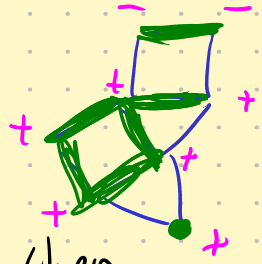
- As $q\to 0$, if $p_e\equiv\frac{1}{2}$, $\phi\xrightarrow{d}$ Uniform connected subgraph.

  if $p_e\equiv p$ and $p\to 0$ and $q\ll p$, $\phi\xrightarrow{d}$ Uniform Spanning Tree.

Exercise: Show that for $q\neq 1$ and $p_e\equiv p$, the edges are not independent.

We will see now that the case $q=2$ is something we know...

II – The Edwards–Sokal coupling



Let us study the case $q=2$.

Let us sample ω with distribution ϕ, and then conditionally on ω, we take each of the clusters of ω and decide to color its vertices with all $\oplus$ or with all $\ominus$, in an iid $\mathrm{Ber}(\frac{1}{2})$ way.

This gives a measure on $\{0,1\}^E \times \{\pm 1\}^V$, precisely:

$$\nu(\omega,\sigma) = \frac{1}{Z}\, \cancel{2^{K(\omega)}} \cdot \prod_{e/\omega_e=1} p_e \prod_{e/\omega_e=0} 1-p_e \cdot \cancel{\left(\frac{1}{2}\right)^{K(\omega)}}\ \mathbb{1}_{\forall \{x,y\}\in E,\ \omega_{xy}=1 \Rightarrow \sigma_x=\sigma_y}$$

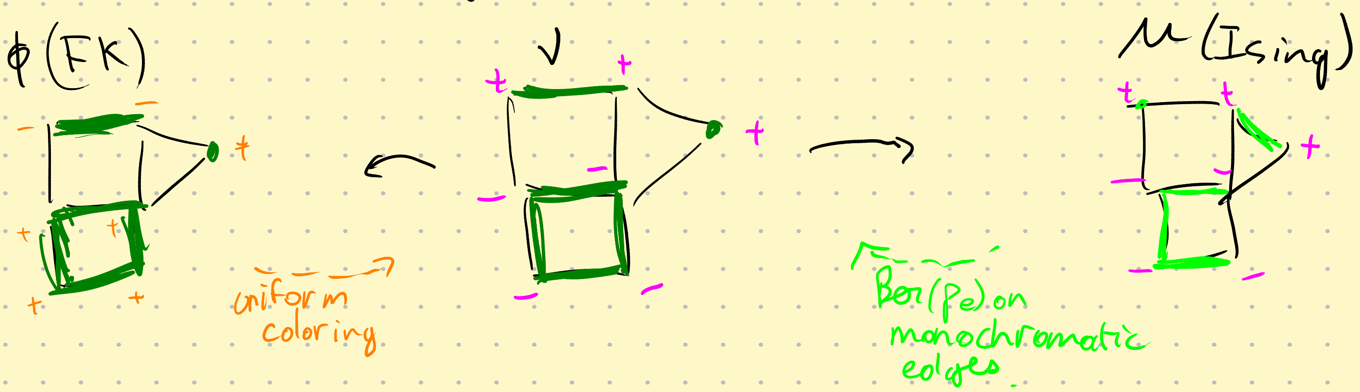
Theo (Edwards–Sokal coupling)

(i) The first marginal of ν is $FK(q=2, (p_e))$

(ii) The second marginal of ν is $\mu_{G,J}$, with $p_e = 1-e^{-2J_e}$.
Ising!

(iii) Conditionally on ω, the law of σ is a uniform assignment of ±1 to the clusters of ω

(iv) Conditionally on σ, the $(\omega_e)_{e\in E}$ are independent with, for $e=\{x,y\}$,
- If $\sigma_x \neq \sigma_y$, $\omega_{xy}=0$
- If $\sigma_x=\sigma_y$, $\omega_{xy} \sim \mathrm{Ber}(p_e)$



Proof: • By construction, (i) and (iii) ✓.

• (iv) Fix σ. Then $\nu(\omega,\sigma)$ is 0 if ω is incompatible ($\exists \{x,y\}\in E / \sigma_x \neq \sigma_y$ and $\omega_{xy}=1$)

Otherwise,

$$\frac{\nu(\omega,\sigma)}{\nu(\sigma)} = \frac{\prod_{e\in\omega} p_e \prod_{e\notin\omega} 1-p_e \cancel{1_{\sigma_x=\sigma_y \Rightarrow \omega_{xy}=1}}}{Z_\sigma}$$

$$= \frac{\prod_{e\in\omega\cap E_\sigma} p_e \prod_{e\in\omega^c\cap E_\sigma} 1-p_e}{Z'_\sigma} \cdot \left(\prod_{e\in E_\sigma^c} 1-p_e\right)$$

where $E_\sigma = \{\{x,y\} / \sigma_x=\sigma_y\}$. $(\omega\subset E_\sigma)$

ω is a subset of E_σ, summing on compat ω's gives 1.

$$= \prod_{e\in\omega\cap E_\sigma} p_e \prod_{e\in\omega^c\cap E_\sigma} 1-p_e .$$

(ii) Let $\sigma\in\{\pm1\}^V$. $\nu(\sigma) = \frac{1}{Z}\sum_{\omega \text{ compat with } \sigma} \prod_{e\in\omega} p_e \prod_{e\notin\omega} 1-p_e$

$$= \frac{1}{Z}\prod_{e/\sigma_x\neq\sigma_y}(1-p_e)\cdot \cancel{\prod_{e/\sigma_x=\sigma_y}(p_e+1-p_e)}$$

$$= \frac{1}{Z}\prod_{e/\sigma_x\neq\sigma_y} e^{-2J_e}$$

$$= \frac{1}{Z}\exp\left(\sum_{e=\{x,y\}} J_e(\sigma_x\sigma_y-1)\right)$$

$$= \frac{1}{Z'}\exp\left(\sum_{e=\{x,y\}} J_e\sigma_x\sigma_y\right) \qquad \blacksquare$$

Exercise Let $q\geq 2$ be an integer.

Let $G=(V,E)$ be a graph, with positive coupling constants $(J_e)_{e\in E}$.

The **Potts model** with q colours is the measure on $\Omega=\{1,\dots,q\}^V$ given by

$$\forall\sigma\in\Omega,\ \mu(\sigma) = \frac{1}{Z}\exp\left(-\sum_{e=\{xy\}\in E} J_e 1_{\sigma_x\neq\sigma_y}\right).$$

Find a similar coupling between Potts and FK.

Applications • For $A\subset V$ event,

$$\langle\sigma_A\rangle_{G,J} = \phi(\text{all vertices of } A \text{ are connected})\ !$$

Indeed, with the coupling ν,

$$\langle \sigma_A \rangle = \underbrace{\nu(\sigma_A \mid \text{all connected})}_{=1} \, \phi(\text{all connected})$$
$$+ \underbrace{\nu(\sigma_A \mid \text{not all connected})}_{=0 \text{ since CC are iid}} \, \phi(\text{not all connected})$$

In particular, Griffith $\Leftrightarrow$ $\forall A, B$ even,

$$\phi(A \Delta B \text{ connected}) \geq \phi(A \text{ connected}) \cdot \phi(B \text{ connected})$$

$\nearrow$ events on $\{0,1\}^E$!

It is a case of FKG (here at $q=2$, we will see that it holds for $q \geq 1$)

III - Infinite volume measures

A) FKG inequality

On $\{0,1\}^E$, we still have a partial order ($\omega \leq \omega'$ iff $\forall e \in E$, $\omega_e \leq \omega'_e$) so it makes sense to talk about positive association (FKG).

Theo $\forall q \geq 1$, $p = (p_e)_{e \in E}$, $\phi_{p,q}$ satisfies FKG. That is, for all $f, g \nearrow$,

$$\phi_{p,q}(f \cdot g) \geq \phi_{p,q}(f) \, \phi_{p,q}(g).$$

Proof: Like we did before, it is enough to show that $\dfrac{\phi_{p,q}(\omega^{+e})}{\phi_{p,q}(\omega^{-e})}$ is $\nearrow$ in ω (see proof of Ising FKG).

$$\Vert \quad \frac{p_e}{1-p_e} \, \frac{1}{q} \, q^{\mathbb{1}_{x \overset{\omega^{-e}}{\longleftrightarrow} y}}, \quad \text{clearly} \nearrow \text{in } \omega! \quad \square$$

B) Monotonicity in parameters

Theo If $p \leq p'$, $q \geq q' \geq 1$, then

$$\phi_{p,q} \leq_{st} \phi_{p',q'}$$

Proof: $\forall \omega \leq \omega'$, and $e \in E$,

$$\frac{\phi_{p,q}(\omega^{+e})}{\phi_{p,q}(\omega^{-e})} = \frac{p_e}{1-p_e} \frac{1}{q} q^{1_{x \overset{\omega_{-e}}{\longleftrightarrow} y}} \qquad \left(p \mapsto \frac{p}{1-p} \nearrow\right)$$

$$\leq \frac{p'_e}{1-p'_e} \frac{1}{q'} q'^{1_{x \overset{\omega_{-e}}{\longleftrightarrow} y}} \quad \big\downarrow q' \geq 1$$

$$\leq \frac{p'_e}{1-p'_e} \frac{1}{q'} q'^{1_{x \overset{\omega'_{-e}}{\longleftrightarrow} y}} = \frac{\phi_{p',q'}(\omega'^{+e})}{\phi_{p',q'}(\omega'^{-e})}$$

and we conclude by a previous criterion.

Other proof: via FKG! For $f \nearrow$,

$$\phi_{p',q'}(f) = \frac{\phi_{p,q}\left(f \left(\frac{q'}{q}\right)^{k(\omega)} \prod_{e \in \omega} \frac{p'_e}{p_e} \prod_{e \notin \omega} \frac{1-p'_e}{1-p_e}\right)}{\phi_{p,q}\left(\left(\frac{q'}{q}\right)^{k(\omega)} \prod_{e \in \omega} \frac{p'_e}{p_e} \prod_{e \notin \omega} \frac{1-p'_e}{1-p_e}\right)}$$

$\nearrow$ in ω:

- If $\omega \nearrow$, $k(\omega) \searrow$ so $\left(\frac{q'}{q}\right)^{k(\omega)} \nearrow$.
- $\frac{p'_e}{p_e} \geq 1$
- $\frac{1-p'_e}{1-p_e} \leq 1$

$$\overset{FKG}{\geq} \phi_{p,q}(f).$$

$\boxtimes$

Exercise: Show that if $1 \leq q \leq q'$ and $\frac{p}{(1-p)q} \leq \frac{p'}{(1-p')q'}$ $\forall e \in E \dots$

Then $\phi_{p,q} \leq_{st} \phi_{p',q'}$.

$\rightsquigarrow$ Application: $\phi_{\hat{p},1} \leq_{st} \phi_{p,q} \leq_{st} \phi_{p,1}$

where $\hat{p} = \frac{p}{p+q(1-p)}$

Bernoulli percol

C) Boundary conditions

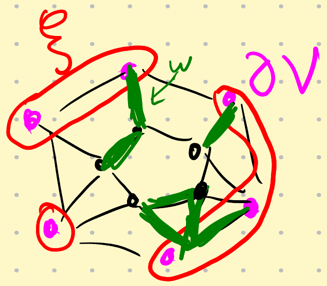
Recall $G=(V,E)$, $p=(p_e)_{e\in E}$.

Let $\partial V \subset V$.

A boundary condition is a partition ξ of ∂V.

For a given ξ, and $\omega \in \{0,1\}^E$, we define $k_\xi(\omega)$ to be the number of cc of the subgraph ω quotiented by ξ.

Ex:



$k_\xi(\omega)=3$.

We define on $\{0,1\}^E$:

$$\phi^\xi_{p,q}(\omega) = \frac{1}{Z^\xi_{p,q}} q^{k_\xi(\omega)} \prod_{e\in\omega} p_e \prod_{e\notin\omega} 1-p_e$$

similar to quotienting ∂V by the ξ equivalence relation

In particular, for ξ the whole set ∂V, we denote it $\phi^1_{p,q}$ (wired b.c.)

and for ξ singletons, we denote $\phi^0_{p,q}$ (free b.c.)

We say that $\xi \le \xi'$ if ξ is finer than ξ'

(all vertices of ∂V wired in ξ are wired in ξ')

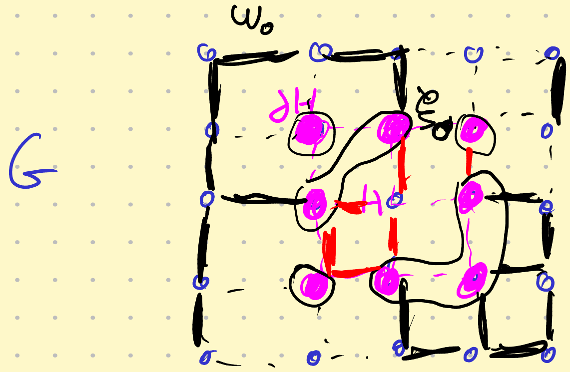
Theo (Domain Markov prop.)

For $G=(V,E)$, Let $H=(V_H,E_H)$ be a subgraph ($V_H \subset V$, $E_H=\{\{x,y\}\in E / x,y\in V_H\}$) and $\partial H = \{x\in V_H / \exists y \notin V_H, \{x,y\}\in E\}$.

Then for any $\omega_0 \in \{0,1\}^{E\setminus E_H}$, let ξ_0 be the partition of ∂H into classes for $\overset{\omega_0}{\longleftrightarrow}$. Then

$$\phi_{G,p,q}(\cdot \mid \omega|_{E\setminus E_H} = \omega_0) = \phi^{\xi_0}_{H,p,q}(\cdot)$$

+ conditional indep..



Proof exercise
(cut $w\in\{0,1\}^E$ into its part in E_H and outside...)

Theo (Monotonicity in b.c and domain)

For any $G=(V,E)$, $\partial V\subset V$, $q\geq 1$ and $p=(p_e)_{e\in E}$,

if $\xi\leq\xi'$,

$$(\phi^0\leq)\qquad \phi^{\xi}_{p,q}\leq_{st}\phi^{\xi'}_{p,q}\qquad(\leq\phi^1).$$

For $H\subset G$ a subgraph (as in the previous theo)

$$\phi^1_{H,p,q}\geq_{st}\phi^1_{G,p,q}\quad,\quad\phi^0_{H,p,q}\leq_{st}\phi^0_{G,p,q}$$

Proof: 2nd ineq:

- $\phi^1_{H,p,q}(\cdot)=\phi^1_{G,p,q}(\cdot\mid \underbrace{w=1 \text{ on } E_H^c})$ by DMP
 $\geq_{st}\phi^1_{G,p,q}(\cdot)$ by FKG!

- $\phi^0_{H,p,q}(\cdot)=\phi^0_{G,p,q}(\cdot\mid \underbrace{w\equiv 0 \text{ on } E_H^c})$
 $\leq_{st}\phi^0_{G,p,q}(\cdot)$

(FKG $\Rightarrow$ for $f\nearrow$, $g\searrow$, $\langle fg\rangle\leq\langle f\rangle\langle g\rangle$)

1st ineq: Consider $G'=G/\xi$ (glue all vertices of ∂V that are in a ξ cluster).
and put edges E' between vertices that are joined in ξ'.

Then $\phi^{\xi}_{G,p,q}=\phi^0_{G',p,q}(\cdot\mid w_{E'}\equiv 0)$

$\leq_{st}\phi^0_{G',p,q}$ (FKG)

$$\leq_{st} \phi^{0}_{G',p,q}(\cdot \mid w_{E'} \equiv 1) \quad \text{(FKG)}.$$

$$= \phi^{\xi'}_{G,p,q}.$$

$\boxed{\text{Ce}}$

D) Infinite volume measure

We now define FK-percolation measures on $\mathbb{Z}^d$. Let $q \geq 1$ (FKG...), $p \in (0,1)$ be fixed $(p_e \equiv p)$.

As usual, $\Lambda_n = [-n,n]^d$.

By Monotonicity in domain, for f local $\nearrow$,

$\phi^{1}_{\Lambda_n,p,q}(f) \searrow$ in n

$\phi^{0}_{\Lambda_n,p,q}(f) \nearrow$ in n

Just like in Ising this allows for the construction of probas on $\{0,1\}^{E(\mathbb{Z}^d)}$, denoted $\phi^{0/1}_{p,q}$.

Theo • $\exists\ \phi^{0}_{p,q},\ \phi^{1}_{p,q}$ probas on $\{0,1\}^{E(\mathbb{Z}^d)}$ characterized by $\forall f$ local, $\phi^{0/1}_{\Lambda_n,p,q}(f) \xrightarrow[n\to\infty]{} \phi^{0/1}_{p,q}$ $(\forall (\Lambda_n)_n \nearrow \mathbb{Z}^d !)$

• They are translation-invariant, ergodic, FKG, Gibbs $\leftarrow$ DLR: For $F \subset E(\mathbb{Z}^d)$ finite, ξ bc on $\partial F = \{x \in \mathbb{Z}^d / \exists y / \{x,y\} \in F \text{ and } \exists z / \{x,z\} \in F^c\}$ and B event dep only on F^c,

$$\phi^{0/1}_{p,q}(\cdot \mid \{\omega \text{ induces } \xi \text{ on } F^c\} \cap B) = \phi^{\xi}_{F,p,q}(\cdot)$$

• For any Gibbs measure ν of $\text{FK}_{p,q}$-perco on $\mathbb{Z}^d$,

$$\phi^{0}_{p,q} \leq_{st} \nu \leq_{st} \phi^{1}_{p,q}$$

We don't write the proof; see also Grimmett's "The random cluster model" (Free online)

Application (of the application):

Fix $q \geq 1$, $d \geq 2$.

For $\omega \in \{0,1\}^{E(\mathbb{Z}^d)}$, we say that $0 \xleftrightarrow{\omega} \infty$ if the c.c. of 0 in ω is infinite.

Using the perco. course, show $\exists\, p_c^1(q) \in (0,1)$ / $p_c^0(q)$ s.t. $\forall p < p_c(q)$, $\phi^1_{p,q}$ / ϕ^0 $(0 \leftrightarrow \infty) = 0$

$\forall p > p_c(q)$, $\phi^1_{p,q}$ / ϕ^0 $(0 \leftrightarrow \infty) > 0$.

($p \mapsto \phi^{0/1}_{p,q}(0 \leftrightarrow \infty)$ is $\nearrow$ in p by $\phi_{p,q} \leq_{st} \phi_{p',q}$ for $p < p'$ ...)

Since $\phi^0_{p,q} \leq_{st} \phi^1_{p,q}$, we have $p_c^1(q) \leq p_c^0(q)$. We will show equality.

Exercise: Let $\beta > 0$, $h = 0$, $\Lambda_n = [\![-n,n]\!]^d$. Using ϕ^0, show the existence of an Ising Gibbs measure which is the limit of $\mu^{\text{free}}_{\Lambda_n,\beta}$.

E] Analogy with Ising; density/magnetization

For $q \geq 1$, $p \in [0,1]$, let $d^{0/1}(p,q) := \phi^{0/1}_{p,q}(e \in \omega)$ where e is any fixed edge; this is well defined by translation invariance (& $\frac{\pi}{2}$-rotation invariance). We call this the density function.

Exercise: Show that $\frac{1}{|E_{\Lambda_n}|} \phi^{0/1}_{\Lambda_n,p,q}(|\omega|) \xrightarrow[n\to\infty]{} d^{0/1}(p,q)$.

$|\omega|$: numb. of edges in ω

We can make a full analogy with the Ising model's magnetiz.° functions $m^{\pm}(\beta,h)$. Namely:

Theo • $\forall q \geq 1$, $p \mapsto d^1(p,q)$ (resp. $d^0(p,q)$) is $\nearrow$ and right-contin. (resp. left-contin.)

Also, $\lim_{\substack{p' \to p \\ p' < p}} d^1(p',q) = d^0(p,q)$.

• The following are equivalent:

(i) $\phi^0_{p,q} = \phi^1_{p,q}$

(ii) $\exists!$ Gibbs measure for $FK_{p,q}$ on $\mathbb{Z}^d$

(iii) $d^0(p,q) = d^1(p,q)$

(iv) $d^1(\cdot,q)$ is continuous at p.

All the proofs are the same as we did for Ising. All are good exercises! See also Grimmett.

Remark: For the analogy of the free energy, it is better to define $Y^{\xi}_{\Lambda,p,q} := (1-p)^{-|E_\Lambda|} Z^{\xi}_{\Lambda,p,q}$

and $f(p,q) = \lim_{\substack{\Lambda_n \nearrow (\mathbb{Z}^d), \\ \frac{|\partial\Lambda_n|}{|\Lambda_n|} \to 0}} \frac{1}{|E_{\Lambda_n}|} \log(Y^{\xi_n}_{\Lambda_n,p,q})$

can be shown to be indep. of (Λ_n), (ξ_n).

Then f is a convex function of $\pi = \log \frac{p}{1-p}$ $Y = \sum_\omega e^{\pi|\omega|} q^{k(\omega)}$ $\simeq$ Ising...

and its right-left π-derivatives are d^1/d^0.

Application $\forall q \geq 1$, $p_c^1(q) = p_c^0(q)$.

That is, $\exists p_c(q) \in (0,1)$ s.t.

$\forall p < p_c(q)$, $\phi^1_{p,q}(0 \leftrightarrow \infty) = 0$ $\leftarrow$ a.s., no ∞ cluster

$\forall p > p_c(q)$, $\phi^0_{p,q}(0 \leftrightarrow \infty) > 0$. $\leftarrow$ a.s., $\exists$ an ∞ cluster

Proof: let $p > p_c^1$, we want to show $p \geq p_c^0$. (so $p_c^0 \leq p_c^1$ and we had $\geq$)

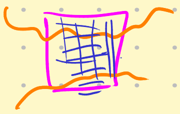
Since $\phi^1(\cdot,q)$ is $\nearrow$, it is continuous except on a set that is at most countable. So $\exists p' \in (p_c^1, p)$ st $\phi^0_{p',q} = \phi^1_{p',q}$.

So $\phi^0_{p',q}(0 \leftrightarrow \infty) = \phi^1_{p',q}(0\leftrightarrow \infty) > 0$ as $p' > p_c^1$.

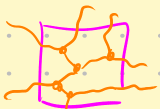
So $p' \geq p_c^0$. $\square$.

p''

Remark. Like in the percolation course, one can wonder about how many ∞ clusters there are. Let N be this number, then $\forall p,q$, $\forall k \in \mathbb{N} \cup \{\infty\}$, the event $\{N = k\}$ is translation-invariant so $\phi^{0/1}_{p,q}(N=k) = 0$ or 1.
So $\exists!\, k^{0/1}_{p,q}$ st $N = k^{0/1}_{p,q}$, $\phi^{0/1}_{p,q}$-as.



The same arguments from percolation apply to show that for $k \in \{2,3,\dots\}$, $\phi^{0/1}_{p,q}(N=k) = 0$,



trifurcations

then Burton-Keane to show $\phi^{0/1}_{p,q}(N=\infty) = 0$.

So in fact, there is a.s. no ∞ cluster, $\Leftarrow$ it is the case for $p < p_c(q)$
or there is a.s. a unique ∞ cluster. $\Leftarrow$ $p > p_c(q)$.

At criticality, in fact it depends on q!

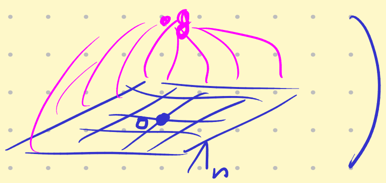
IV- Application to Ising

We now fix $q = 2$. ("FK-Ising model")

A) Generic dimension ($\mathbb{Z}^d$, $d \geq 2$)

Prop: $\forall \beta > 0$, let $p = 1 - e^{-2\beta}$. Then
$$m^+(\beta) \left(= \langle \sigma_0 \rangle^+_{\beta, h=0, \mathbb{Z}^d} \right) = \phi^1_{p,2}(0 \leftrightarrow \infty).$$
In particular, $p_c(2) = 1 - e^{-2\beta_c}$.

Proof: $m^+(\beta) = \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\beta, \Lambda_n}$

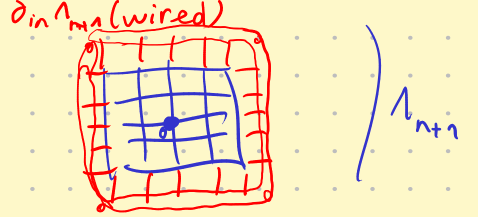


G_n

$= \lim_{n\to\infty} \langle \sigma_0 \sigma_g \rangle_{G_n}$

$= \lim_{n \to \infty} \phi_{G_n, p, 2}(0 \leftrightarrow g)$ for $p = 1 - e^{-2\beta}$, by Edwards-Sokal (see last Application of Edwards-Sokal section)

$$= \lim_n \phi^1_{\Lambda_{n+1},p,2}(0 \leftrightarrow \partial_{in}\Lambda_{n+1})$$



But $\phi^1_{\mathbb{Z}^d,p,2}(0 \leftrightarrow \infty) = \phi^1_{\mathbb{Z}^d,p,2}\left(\bigcap_K \{0 \leftrightarrow \partial\Lambda_K\}\right)$

$$= \lim_K \downarrow \phi^1_{\mathbb{Z}^d,p,2}(0 \leftrightarrow \partial\Lambda_K)$$

$$= \lim_K \downarrow \lim_{\substack{n \\ (n \geq K)}} \downarrow \underbrace{\phi^1_{\Lambda_n,p,2}(0 \leftrightarrow \partial\Lambda_K)}_{a_{K,n}}$$

$\phi^1 = \lim \downarrow \phi^1_{\Lambda_K}$ stochastically

$a_{K,n}$: For $K > n$, we set $a_{K,n} = \phi^1_{\Lambda_n,p,2}(0 \leftrightarrow \partial_{in}\Lambda_n)$

Then a is $\downarrow$ in K and in n (and bounded). By a Lemma (see proof that $h \mapsto m^+(\beta,h)$ right-cont)

$$= \lim_n \lim_K a_{K,n}$$

$$= \lim_n \phi^1_{\Lambda_n,p,2}(0 \leftrightarrow \partial_{in}\Lambda_n)$$

$$= m^+(\beta). \qquad \square$$

Cor $\forall p < 1 - e^{-2\beta_c}$, $\exists c > 0$ st $\forall n \geq 1$, $\phi^1_{p,2}(0 \leftrightarrow \partial\Lambda_n) \leq e^{-cn}$.

Proof: We have $\phi^1_{p,2}(0 \leftrightarrow \partial\Lambda_n) \leq \phi^1_{\Lambda_n,p,2}(0 \leftrightarrow \partial\Lambda_n) = \langle\sigma_0\rangle^+_{\Lambda_{n-1},\beta,0}$

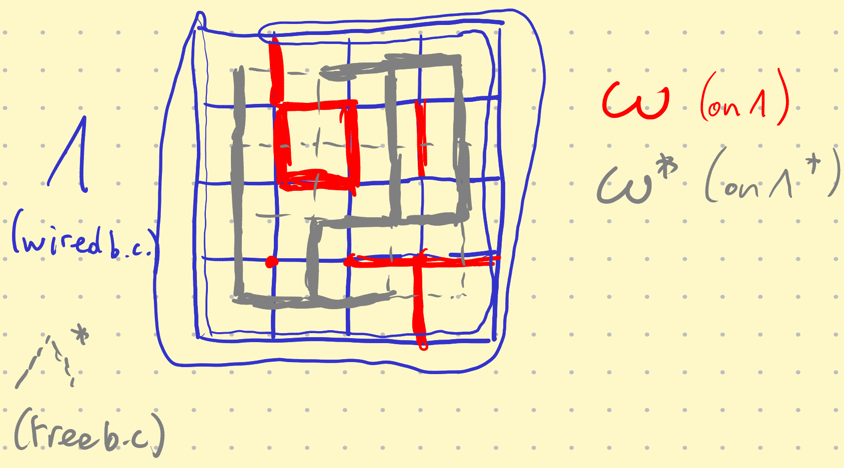
where β is s.t. $p = 1 - e^{-2\beta}$, so $\beta < \beta_c$.

By exp decay from Ising, $\langle\sigma_0\rangle^+_{\Lambda_{n-1},\beta,0} < e^{-cn}$. $\square$

B) On $\mathbb{Z}^2$: duality

Here $\Lambda = [\![-n,n]\!]^2$ (we drop the n in notation)

Idea: consider $\phi^1_{\Lambda,p,q}$. For a config ω, let ω^* be its dual config, defined on Λ^*, by $\omega_{e^*} = 1 - \omega_e$:



Q: If $\omega \sim \phi^1_{\Lambda,p,q}$, what is the distri of ω^*?

Theo: Let $q>0$, $p\in(0,1)$.
If $\omega \sim \phi^1_{\Lambda,p,q}$, then $\omega^* \sim \phi^0_{\Lambda^*,p^*,q}$
where $p^*\in(0,1)$ is s.t. $\dfrac{pp^*}{(1-p)(1-p^*)} = q$.

Proof: $\phi^1_{\Lambda,p,q}(\omega) \propto p^{|\omega|}(1-p)^{|\omega^c|} q^{k^1(\omega)}$ ← recall : clusters for wired b.c.

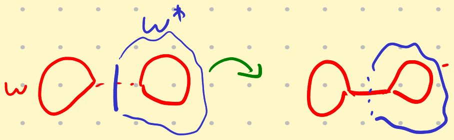
$$\propto \left(\frac{p}{1-p}\right)^{|\omega|} q^{k^1(\omega)}.$$

Claim: $k^1(\omega) - k^0(\omega^*) + |\omega| = |\Lambda| - |\partial\Lambda|$.

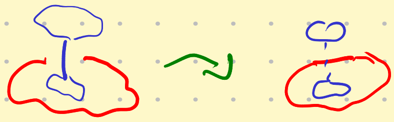
Indeed, true for $\omega\equiv 0$ $\left(\begin{array}{l} k^1(\omega) = |\Lambda| - |\partial\Lambda| + 1 \\ k^0(\omega^*) = 1 \\ |\omega| = 0 \end{array}\right)$

and if true for ω, let's add an edge e to ω.

- If e joins 2 clusters of ω, $k^1(\omega) \searrow$ by 1, $k^0(\omega^*)$ unchanged, $|\omega| \nearrow$ by 1.



- Otherwise, $k^1(\omega)$ unchanged, $k^0(\omega^*) \searrow$ by 1, $|\omega| \nearrow$ by 1.



With the claim,

$$\phi^1_{\Lambda,p,q}(\omega) \propto \left(\frac{p}{1-p}\right)^{|\omega|} q^{k^0(\omega^*) - |\omega|}$$

$$\propto \left(\frac{p}{q(1-p)}\right)^{|\omega|} q^{k^0(\omega^*)}$$

$|\omega| = \text{cst} - |\omega^*|$.

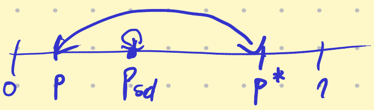
$$\propto \left(\frac{q(1-p)}{p}\right)^{|\omega^*|} q^{k^0(\omega^*)}$$

$$\propto \left(\frac{p^*}{1-p^*}\right)^{|\omega^*|} q^{k^0(\omega^*)}$$

$\boxtimes$.

Remark. This works for $q>0$; in the limit $q\to 0$, $p\to 0$, $q \ll p$, we find for instance tree duality...

We define $p_{sd}(q) = \dfrac{\sqrt{q}}{1+\sqrt{q}}$. It is the self-dual point (that is, $p_{sd} = p_{sd}^*$).

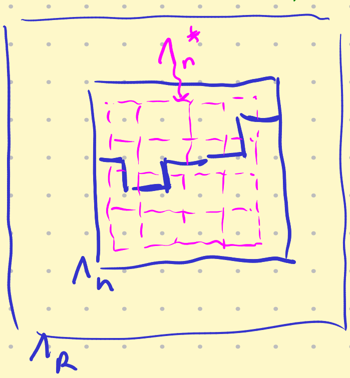


Theo (Beffara, Duminil-Copin '12)
For $q \geq 1$, on $\mathbb{Z}^2$, $p_c(q) = p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$.

Corollary on $\mathbb{Z}^2$, $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$

We will prove the theo only for $q=2$, which is what we want for the corollary $\left(p_c(2) = \frac{\sqrt{2}}{1+\sqrt{2}} = 1 - e^{-2\beta_c} \text{ so } \beta_c = \dots\right)$

proof: • $p_{sd} \geq p_c$: First, we show that at p_{sd},



$$\forall n < R, \quad \phi^1_{\Lambda_R, p_{sd}} \Big(\underset{\Lambda_n}{\boxed{\sim}} \Big) \geq \frac{1}{2}$$

a left-right crossing that doesnt use the edges on $\partial\Lambda_n$.

Indeed, either $\underset{\Lambda_n}{\boxed{\sim}}$, or there is a $\boxed{\wr}$ in Λ_n^*.

So $\phi^1_{\Lambda_R, p_{sd}}\Big(\underset{\Lambda_n}{\boxed{\sim}}\Big) + \phi^1_{\Lambda_R, p_{sd}}\Big(\underset{\Lambda_n^*}{\boxed{\wr}}\Big) = 1$.

$$\phi^1_{\Lambda_R, p_{sd}}\Big(\underset{\Lambda_{n-1}}{\boxed{\sim}}\Big) \;\geq\; \phi^1_{\Lambda_R, p_{sd}}\Big(\underset{\Lambda_n}{\boxed{\sim}}\Big)$$

$$\phi^1_{\Lambda_R, p_{sd}}\Big(\underset{\Lambda_n^*}{\boxed{\wr}}\Big) = \phi^0_{\Lambda_R^*, p_{sd}}\Big(\underset{\Lambda_{n-1}}{\boxed{\wr}}\Big) \quad \leftarrow \text{by duality}$$

$$\leq \phi^0_{\Lambda_R, p_{sd}}\Big(\underset{\Lambda_{n-1}}{\boxed{\wr}}\Big) \quad \leftarrow \text{by stoch. dom. in domain } (\Lambda_R^* \text{ smaller than } \Lambda_R)$$

$$\leq \phi^1_{\Lambda_R, p_{sd}}\Big(\underset{\Lambda_{n-1}}{\boxed{\sim}}\Big) \quad \leftarrow \text{as } \phi^0 \leq_{st} \phi^1.$$

This gives $\phi^1_{\mathbb{Z}^d, p_{sd}}\Big(\underset{\Lambda_n}{\boxed{\sim}}\Big) \geq \frac{1}{2}$.

This implies that one of the left-vertices of Λ_n $v\!\boxed{\;}$ is st $\phi^1_{\mathbb{Z}^d, p_{sd}}\Big(v\!\underset{\Lambda_n}{\boxed{\sim}}\Big) \geq \frac{1}{2n}$.

By translation invariance, $\phi^1_{\mathbb{Z}^d, p_{sd}}(0 \leftrightarrow \partial\Lambda_n) \geq \frac{1}{2n}$.

But for $p < p_c$, $\phi^1_{\mathbb{Z}^d, p}(0 \leftrightarrow \partial\Lambda_n) \leq \exp(-cn)$, so we have $p_{sd} \geq p_c$.

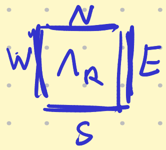
• $p_{sd} \leq p_c$: We prove in fact $\phi^0_{\mathbb{Z}^2, p_{sd}}(\exists \infty \text{ cc}) = 0$. ← but still at crit, for ϕ^1 there might be an ∞ cluster a.s. (On $\mathbb{Z}^2$ this happens for $q > 4$).

(This will imply $p_{sd} \leq p_c$, as for $p > p_c$, $\phi^0_{\mathbb{Z}^2, p}(\exists \infty \text{ cc}) = 1$).

Suppose the contrary : $\phi^0_{\mathbb{Z}^2, p_{sd}}(\underbrace{\exists \infty \text{ cc}}_{\bigcup_R \{\Lambda_R \leftrightarrow \infty\}}) = 1$ (recall $N=0$ a.s. or $N=1$ a.s.)

Let $\varepsilon > 0$ (to be fixed later), Then $\exists R > 0$ st

$\phi^0(\Lambda_R \leftrightarrow \infty) \geq 1-\varepsilon$ ($\mathbb{Z}^2, p_{sd}, 1,2$ implicit), so $\varepsilon \geq \phi^0(\Lambda_R \not\leftrightarrow \infty) = \phi^0(N \not\leftrightarrow \infty \cap S \not\leftrightarrow \infty \cap E \ldots \cap W \ldots)$



$\geq \phi^0(N \not\leftrightarrow \infty)\phi^0(S \ldots)\phi^0(E \ldots)\phi^0(\ldots)$ by FKG for $\downarrow$ events

$= \phi^0(N \not\leftrightarrow \infty)^4$

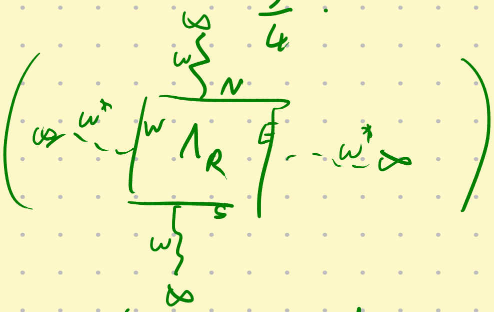
So $\phi^0(N \not\leftrightarrow \infty) \leq \varepsilon^{1/4} < \frac{1}{4}$. (we fix at the beginning $\varepsilon = \frac{1}{4^4+1}$)

Then $\phi^0(N \leftrightarrow \infty) > \frac{3}{4}$

Also $\phi^0_{p_{sd}}(N \overset{w^*}{\not\leftrightarrow} \infty) = \phi^1_{p_{sd}}(N \overset{w}{\leftrightarrow} \infty)$ by duality

$\geq \phi^0(N \leftrightarrow \infty)$

$\geq \frac{3}{4}$.

We get $\phi^0_{p_{sd}}\Big($  $\Big) > 0$ by union bound.

But then, closing all edges in Λ_R, we have $\phi^0_{p_{sd}}(\text{at least 2 } \infty \text{ cc}) > 0$.

Contradiction. $\boxtimes$

This finally concludes the proof that $\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$, and the content of the course.

Opening remarks

We have been able to show important features of nearest-neighbour Ising model at temp. $\beta>0$ (and sometimes $h\neq 0$) on $\mathbb{Z}^d$.
Many directions are still available, and are in fact vast fields of research, like:

- What about long-range interactions? Even for $d=1$, rich phenomena can appear!
- What about $\beta<0$: spins want to be $\neq$? This is the anti-ferromagnetic Ising model. It also has a phase transition, and is in general much harder to study!
- What about other graphs? Random graphs?

  In particular, on $\mathbb{Z}^2$ we could compute β_c exactly and had many "exact" tools, like duality. These are features of integrability, and arise for (at least) vast classes of planar graphs!
- What happens exactly at criticality? Continuity? Shape of interfaces? In particular, for planar graphs, this is where some self-similar/fractal features appear, and where "the model becomes conformal invariant". Key objects: SLE / CLE random curves, GFF...
- Can we wonder again about the physics relevance of the model? Can we incorporate quantum mechanics effect?...

And many more! (at least 650 000 articles to read on the subject...)