Ising model - Exam 2

You may use your handwritten and printed notes. No book is allowed.

9:00 - 12:00

The marking scale is not fixed, but Exercises 1,2,3 and the beginning of 4 will be enough to get a good grade.

Exercise 1: exponential decay of correlations for subcritical Ising

For the Ising model on $\mathbb{Z}^d$ at $h = 0$ and $\beta < \beta_c$, show that for the (unique) Gibbs measure, there is a constant c (depending on β) such that

$$\forall x, y \in \mathbb{Z}^d, \langle \sigma_x \sigma_y \rangle_\beta \leq \exp\left(-c\|x - y\|_\infty\right).$$

Hint: show that we may suppose $x = O$, then condition on the configuration on $\partial\Lambda_n$ for $n = \|y\|_\infty$, and use that $\langle \sigma_0 \rangle^+_{\Lambda_n,\beta} \leq \exp(-cn)$.

Exercise 2: average edge density in FK-percolation

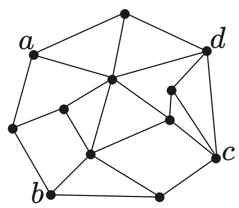
We consider the FK-percolation model on $\Lambda_n = [-n, n]^d \subset \mathbb{Z}^d$ with wired boundary conditions, with parameters $q \geq 1$ and $p \in [0, 1]$ a constant. Show the convergence of the expectation of the number of edges in ω:

$$\frac{1}{|E(\Lambda_n)|}\phi^1_{\Lambda_n,p,q}(|\omega|) \to_{n\to\infty} d^1(p, q) := \phi^1_{\mathbb{Z}^d,p,q}(e \in \omega)$$

for any fixed edge e. You may use any theorem/proposition from the course.

Exercise 3: a "Plücker" identity

Let $G = (V, E)$ be a connected finite **planar** graph equipped with Ising coupling constants $(J_e)_{e\in E}$ (you may take a piece of $\mathbb{Z}^2$ if you prefer). Let a, b, c, d be four distinct vertices on the **outer face** of G, in that counter-clockwise order.



Using the switching lemma, show that

$$\begin{aligned}\langle \sigma_a\sigma_b\rangle\langle \sigma_c\sigma_d\rangle + \langle \sigma_a\sigma_d\rangle\langle \sigma_b\sigma_c\rangle &= \langle \sigma_a\sigma_b\sigma_c\sigma_d\rangle + \langle \sigma_a\sigma_c\rangle\langle \sigma_b\sigma_d\rangle \\ &\geq 2\langle \sigma_a\sigma_c\rangle\langle \sigma_b\sigma_d\rangle.\end{aligned}$$

Exercise 4: The Lupu-Werner coupling

Let $G = (V, E)$ be a finite graph, and let $\beta > 0$. We consider the Ising model on G with temperature β, and its partition function:

$$Z_{\mathrm{I}} = \sum_{\sigma \in \{\pm 1\}^V} \exp\left(\beta \sum_{\{x,y\} \in E} \sigma_x \sigma_y \right).$$

1. Write Z_{I} as a sum on random currents, justifying quickly. That is, show that

$$Z_{\mathrm{I}} = C Z_{\mathrm{rc}},$$

where Z_{rc} is a partition function of certain currents, and C is a constant, that you shall all make explicit.

2. Recall the Edwards-Sokal coupling, in particular how to sample a certain model of FK-percolation conditional on an Ising configuration σ (without proof). Show that

$$Z_{\mathrm{I}} = C' Z_{\mathrm{FK}},$$

and make these explicit, similarly (in particular the values of p and q for the FK-percolation model).

In the previous two questions, the second equality is based on a probabilistic coupling, while the first is not (it is a purely combinatorial argument). The goal of this exercise is to find a coupling[1] between currents and FK-percolation.

Let N be a random current configuration whose distribution is the probability associated to Z_{rc} from Question 1. Let ξ be an independent Bernoulli percolation[2] on G with parameter $p' = 1 - e^{-\beta}$. Let W be the random variable taking values in $\{0, 1\}^E$ defined by

$$\forall e \in E, W_e = 1 - 1_{N_e = \xi_e = 0}.$$

The goal is to show that W is in fact distributed as the FK percolation measure of Question 2.

3. Let U be the random variable in $\{0, 1, 2\}^E$ defined by

$$\forall e \in E, U_e = \begin{cases} 0 \text{ if } N_e = 0, \\ 1 \text{ if } N_e \text{ is odd}, \\ 2 \text{ if } N_e \text{ is even and } \neq 0. \end{cases}$$

What is, more precisely, the set of possible values for U? For a fixed possible u, give its probability as a product of certain edge weights (up to a global constant).

4. Let $\bar{W}$ be the random variable in $\{-1, 0, 1\}^E$ defined by

$$\forall e \in E, \bar{W}_e = \begin{cases} W_e \text{ if } N_e \text{ is even}, \\ -W_e \text{ if } N_e \text{ is odd}. \end{cases}$$

For a given $\bar{w} \in \{0, 1\}^E$, show that the probability of $\bar{W} = \bar{w}$ is proportional to

$$\prod_{e \in E} (\sinh \beta)^{|\bar{w}_e|} \exp(-\beta)^{1 - |\bar{w}_e|}.$$

[1] more precisely, and *interesting* coupling: the independent coupling for instance is not a satisfying solution

[2] In case you don't know, it is an FK-percolation with $q = 1$!

5. For a given $w \in \{0,1\}^E$, show that the number of choices of $\bar{w}$ compatible with w is $2^{|w|+k(w)-|V|}$.

   Hint: show that this is the number of even subgraphs of w, which you can compute using identities from the beginning of the course.

6. Conclude by checking that the distribution of W is the FK-percolation measure from Question 2.

Another case where we had a combinatorial identity but no probabilistic coupling was High Temperature Expansion. In the following questions, for $n \in \mathbb{N}^E$, we use the notation $\widehat{n} = \{e \in E \mid n_e \geq 1\}$. We just found the distribution of $W = \widehat{N + \xi}$.

7. Give the distribution of the random subgraph H associated to the High Temperature Expansion of Z_{I}, and find similarly a certain Bernoulli percolation ξ' such that $\widehat{H + \xi'}$ has the same distribution as $\widehat{N}$.

8. Conclude that there is a joint coupling such that a.s.,

$$H \subset \widehat{N} \subset W$$

   and deduce a sandwich of stochastic dominations.

9. Convince yourself that this would all work if we started with edge-dependent coupling constants $(J_e)_{e \in E}$. Write "Yes" when you are convinced.

Exercice 5: FKG for a height function

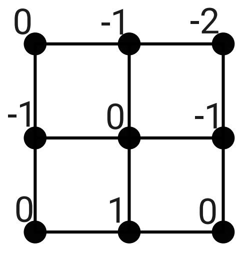
Let $\Lambda_n = \{0, \ldots, n-1\}^2 \subset \mathbb{Z}^2$. A *height function* is a function $h : \Lambda_n \to \mathbb{Z}$ such that

$$\forall x, y \in \Lambda_n, \|x - y\| = 1 \Longrightarrow |h(x) - h(y)| = 1.$$

(with the Euclidian norm on the left, that is, x, y are nearest-neighbours). To get a finite family, we also require $h(0,0) = 0$. Let Ω be the set of all such height functions.

For $h \in \Omega$, and a unit square $(i,j), (i+1,j), (i+1,j+1), (i,j+1)$ included in Λ_n (that is, $0 \leq i, j \leq n-2$), we say that h is a *saddle* at this unit square if $h(i,j) = h(i+1,j+1)$ and $h(i+1,j) = h(i,j+1)$. Let $N(h)$ be the number of unit squares where h is a saddle.

Example: a height function with $N(h) = 1$:



Let $c \geq 1$, we define a probability on Ω by

$$\mu(h) = \frac{1}{Z} c^{N(h)}.$$

1. Show that this measure satisfies the FKG inequality. You may first describe quickly an irreducible Markov chain on Ω to get a criterion for stochastic domination, and use it to get FKG as we did in the course.

We now want to show that if H has distribution μ, then the distribution of $|H|$ also has FKG.

2. Let $h \in \Omega$ be fixed. Show that conditionally on $|H| = |h|$, the ditribution of the signs of H is the same as the spins of a certain Ising model on a certain graph constructed from $|h|$.

3. (*) Show that $|H|$ also satisfies the FKG inequality.

Exercise 6

Invent a model of statistical mechanics, and say something non-trivial about it.