Ising model - Exam 1

You may use your handwritten and printed notes. No book is allowed.

9:00 - 12:00

Exercise 1 (from the lectures): transfer matrix and magnetization

Consider the Ising model on $\{-n, \dots, n\} \subset \mathbb{N}$, with $\beta > 0, h = 0, J \equiv 1$, and $+$ boundary conditions. Quickly define the transfer matrix and express $\langle \sigma_0 \rangle^+_{\beta,0,n}$ in terms of this transfer matrix (you're **not** expected to compute this further with eigenvalues...)

Exercise 2: positive magnetization for positive magnetic field

We consider the infinite-volume Ising measure μ^+ on $\mathbb{Z}^d$ for $d \geq 1$, at inverse temperature $\beta > 0$ and magnetic field $h > 0$. Show that $m^+(\beta, h) > 0$.

Hint: on a finite domain $\Lambda \Subset \mathbb{Z}^d$, define a measure ν with the magnetic field h **only affecting** 0. Show that $\mu^+_{\Lambda,\beta,0} \leq_{st} \nu \leq_{st} \mu^+_{\Lambda,\beta,h}$ and compute $\nu(\sigma_0)$.

Exercise 3: Ising model on the regular tree

Let $p \geq 2$ and $n \geq 1$, we consider the p-regular tree cut at height n, that is, the connected acyclic graph containing a vertex 0 such that every vertex v with $d(0, v) \leq n$ has degree p, and the ones at distance n have degree 1; see the figure below for the example of $p = 3, n = 4$. We consider p as fixed, and we denote this graph by T_n.

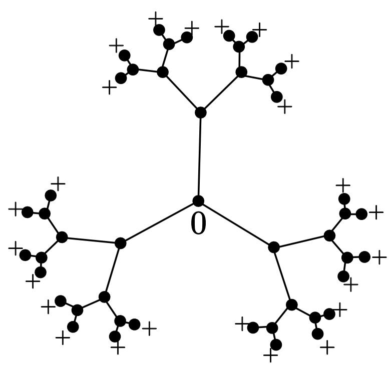
The vertices of T_n are denoted V_n and its edges E_n. Moreover, the set of *leaves* (vertices of degree 1) is denoted by ∂V_n; note that it is included in V_n. For $\beta > 0$, we consider the probability measure $\mu^+_{\beta,n}$ on $\Omega^+_n := \{\sigma \in \{-1, +1\}^{V_n} \mid \forall x \in \partial V_n, \sigma_x = +1\}$, given by

$$\forall \sigma \in \Omega^+_n,\ \mu^+_{\beta,n}(\sigma) = \frac{1}{Z^+_{\beta,n}} \exp\left(\beta \sum_{\{x,y\} \in E_n} \sigma_x \sigma_y\right),$$

with

$$Z^+_{\beta,n} = \sum_{\sigma \in \Omega^+_n} \exp\left(\beta \sum_{\{x,y\} \in E_n} \sigma_x \sigma_y\right).$$

The expectation relative to $\mu^+_{\beta,n}$ is denoted $\langle \cdot \rangle^+_{\beta,n}$.



1. (Lecture question) Show that $\langle \sigma_0 \rangle^+_{\beta,n}$ is monotonic in n and in β. You may use the Griffith inequalities freely.

2. Deduce the fact that $\langle \sigma_0 \rangle^+_\beta := \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\beta,n}$ is well-defined, and define the critical temperature β_c.

3. Show that

$$\langle \sigma_0 \rangle^+_{\beta,n} \le p(p-1)^{n-1} \tanh(\beta)^n$$

and deduce a bound on β_c.

4. In this question, we drop the β in notations for simplicity.
   (a) Write

   $$Z_n^+ = \sum_{\sigma_0 \in \{-1,+1\}} (Y_{n-1}(\sigma_0))^p$$

   where $Y_{n-1}(\sigma_0)$ is a sum depending on σ_0 that you shall write explicitly.

   (b) Show that the sequence defined for $n \in \mathbb{N}$ by $x_n := \frac{Y_n(-1)}{Y_n(+1)}$ satisfies the recursion relation $x_n = F(x_{n-1})$, where

   $$F(x) = \frac{1 + e^{2\beta} x^{p-1}}{e^{2\beta} + x^{p-1}}.$$

   (c) Show that $\langle \sigma_0 \rangle^+_\beta = 0$ *iff* $\lim_{n\to\infty} x_n = 1$.

   (d) Find the value of β_c.

5. *(Bonus, to try only if you are bored)* Show that as $\beta \to \beta_c$ with $\beta > \beta_c$,

$$\langle \sigma_0 \rangle^+_\beta \sim C(\beta - \beta_c)^b$$

for a certain constant C and a *critical exponent* b, and compute them. Compare with critical exponents you know.

Exercise 4: Curie-Weiss magnetization with Stein's method

Recall that in the Curie-Weiss model, we consider the Ising model on the complete graph K_n, whose vertices are denoted $\{0, \dots, n-1\}$, and for $\beta > 0$ we define the measure $\mu_{\beta,n}$ by

$$\forall \sigma \in \Omega = \{\pm 1\}^{\{0,\dots,n-1\}},\ \mu_{\beta,n}(\sigma) = \frac{1}{Z_{\beta,n}} \exp\left(\frac{\beta}{n} \sum_{i,j} \sigma_i \sigma_j\right)$$

where the sum is over all $(i,j) \in \{0, \dots, n-1\}^2$, possibly equal. We want to establish the behaviour of the magnetization $M_n := \frac{1}{n}\sum_{i=0}^{n-1} \sigma_i$, by proving the "spontaneous magnetization" theorem of the course, without using large deviation estimates.

We consider a probability measure P on $\Omega \times \Omega$ obtained as follows: σ is sampled with distribution $\mu_{\beta,n}$, and σ' is obtained by doing one step of the Glauber dynamics from σ (that is, by chosing one vertex uniformly at random and flipping it with the conditional probability defined in the lectures).

1. Write down $P(\sigma, \sigma')$ explicitly (you may only write it for $\sigma \neq \sigma'$), and check that it is exchangeable: $P(\sigma, \sigma') = P(\sigma', \sigma)$.

For $(\sigma, \sigma') \in \Omega^2$, let $F(\sigma, \sigma') = \sum_{i=0}^{n-1} \sigma_i - \sigma'_i$, and denoting $E[\cdot]$ the expectation for P, let $f(\sigma) = E[F(\sigma, \sigma') \mid \sigma]$.

2. Show that
$$f(\sigma) = \frac{1}{n} \sum_{i=0}^{n-1} \left(\sigma_i - \tanh\left(\frac{2\beta}{n} \sum_{j \neq i} \sigma_j \right) \right)$$
and deduce that P-alomst surely on (σ, σ'),
$$|f(\sigma) - f(\sigma')| \leq \frac{2 + 4\beta}{n}.$$

3. Let g be a function on Ω, show that
$$E[f(\sigma) g(\sigma)] = \frac{1}{2} E\left[F(\sigma, \sigma') \left(g(\sigma) - g(\sigma') \right) \right].$$

4. Deduce that $f \to 0$ in probability for $\mu_{\beta,n}$, that is,
$$\forall \epsilon > 0, \;\; \mu_{\beta,n}\left(|f| > \epsilon \right) \to_{n \to \infty} 0.$$

5. Show that $\mu_{\beta,n}$-almost surely,
$$|f(\sigma) - (M_n - \tanh(2\beta M_n))| \leq \frac{4\beta}{n}.$$

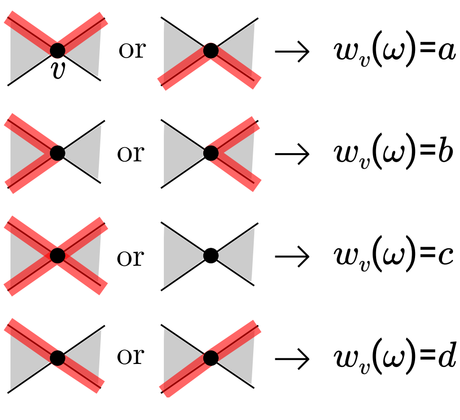
6. Conclude by proving the statement of the spontaneous magnetization theorem for the Curie-Weiss model.

Exercice 5 (*look at this only if you've tried all the rest*): duality of the eight-vertex model

Let G be a finite, connected, **planar** graph where every vertex has degree 4. Let V, E, F be its set of vetices, edges and faces respectively. We can color faces of G in black and white in a bipartite way (that is, adjacent faces always having different colors), see an example below.

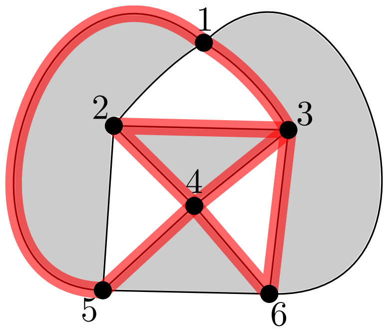
Let $\Omega = \{-1, +1\}^E$, and let $\Omega_e = \{\omega \in \Omega \mid \forall v \in V, \;\; \sum_{e \sim v} \omega_e \in \{-4, 0, 4\}\}$, where the sum is over the four edges adjacent to v. We identify $\omega \in \Omega_e$ with the set of edges where it is $+1$, see the examples below.

Let $a, b, c, d \in \mathbb{R}$. For $\omega \in \Omega_e$ and $v \in V$, let $w_v(\omega)$ be a *local weight* equal to either a, b, c or d depending on the local configuration of ω on the four edges around v, according to the following rule (ω is shown in thick lines):



Let $Z(a,b,c,d) = \sum_{\omega\in\Omega_e} \prod_{v\in V} w_v(\omega)$.

Example: a graph G with a configuration $\omega \in \Omega_e$ (in thick lines).



In that case, identifying V with $\{1,\ldots,6\}$, we have $w_1(\omega) = d,\ w_2(\omega) = b,\ w_3(\omega) = c, \ldots$. The total weight of ω is abc^2d^2.

1. Getting inspiration from the High Temperature Expansion, show that

$$Z(a,b,c,d) = Z(a',b',c',d')$$

where

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

2. Are there other transformations of (a,b,c,d) that leave Z invariant?

3. What does this model have to do with the Ising model?