"Solutions"

Ex 1: By the same argument as in the lectures,

For $P = \begin{array}{cc} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} & \begin{array}{c} \ominus \\ \oplus \end{array} \\ \begin{array}{cc} \ominus & \oplus \end{array} & \end{array}$,

Z_n^+ is the "bottom right" coef of P^{2n+2}.

$$\left(Z_n^+ = \sum_{\substack{\sigma_{-n-1}=+1, \\ \sigma_{-n},\dots,\sigma_n \in \{\pm 1\}, \\ \sigma_{n+1}=+1}} P_{\sigma_{-n-1}\sigma_{-n}} \cdots P_{\sigma_n \sigma_{n+1}} = P^{2n+2}[+,+] \right),$$

This can be written as

$$Z_n^+ = (0\ 1)\, P^{2n+2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly, $\langle \sigma_0 \rangle^+ Z_n^+ = \sum_{\text{same}} \sigma_0 P_{\sigma_{-n-1}\sigma_{-n}} \cdots P_{\sigma_n \sigma_{n+1}}$

$$= \sum_{\substack{\sigma_{-n-1}=+1, \\ \sigma_{-n},\dots,\sigma_{-1} \in \{\pm 1\}, \\ \sigma_0 = +1, \\ \sigma_1,\dots,\sigma_n \in \{\pm 1\}, \\ \sigma_{n+1}=+1}} P_{\sigma_{-n-1}\sigma_{-n}} \cdots P_{\sigma_{-1}\sigma_0} P_{\sigma_0 \sigma_1} \cdots P_{\sigma_n \sigma_{n+1}}$$

$$- \sum_{\text{same with } \sigma_0 = -1} \quad ''$$

$$= P^{n+1}[+,+]\, P^{n+1}[+,+] - P^{n+1}[+,-]\, P^{n+1}[-,+].$$

$$= \left[(0\ 1) P^{n+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^2 - \left[(0\ 1) P^{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]^2 \quad \text{as } P \text{ is symmetric.}$$

② So $\langle \sigma_0 \rangle^+ = \dfrac{\left[(0\;1)\, P^{n+1} \binom{0}{1}\right]^2 - \left[(0\;1)\, P^{n+1} \binom{1}{0}\right]^2}{(0\;1)\, P^{2n+2} \binom{0}{1}}$.

other solution: put a magnetic field h at 0, then differentiate with resp. to h...

Ex 2: We follow the hint.

For $\Lambda \subset\subset \mathbb{Z}^d$, $\beta > 0$, $h > 0$, let

$$\forall \sigma \in \Omega = \{\pm 1\}^\Lambda, \quad \nu(\sigma) = \frac{\exp\Big(\beta \sum\limits_{\substack{x\sim y \\ x,y\in\Lambda}} \sigma_x \sigma_y + \beta \sum\limits_{\substack{x \sim y \\ x\in\Lambda \\ y\in\partial\Lambda}} \sigma_x \overset{+1}{\overset{\shortparallel}{\sigma_y}} + \underline{\underline{h\sigma_0}}\Big)}{Z_\nu}.$$

where $Z_\nu := \ldots$

We want to show $\mu^+_{\Lambda,\beta,0} \leq_{st} \nu \leq_{st} \mu^+_{\Lambda,\beta,h}$.

We use a criterion from the course.

Let $\sigma \leq \sigma' \in \Omega$, and $x \in \Lambda$.

$$\frac{\mu^+_{\Lambda,\beta,0}(\sigma^{+x})}{\mu^+_{\Lambda,\beta,0}(\sigma^{-x})} = \exp\Big(2\beta \sum_{y\sim x} \overset{+1 \text{ if } y\in\partial\Lambda}{\sigma_y}\Big)$$

$$\leq \quad \text{since } \sigma \leq \sigma' \text{ and } 0 \leq h.$$

$$\frac{\nu(\sigma'^{+x})}{\nu(\sigma'^{-x})} = \exp\Big(2\beta \sum_{y\sim x} \sigma'_y + 2h\, \mathbb{1}_{x=0}\Big)$$

so $\mu^+_{\Lambda,\beta,0} \leq_{st} \nu$.

Similarly, $\dfrac{\nu(\sigma^{+x})}{\nu(\sigma^{-x})} = \exp\Big(2\beta \sum\limits_{y\sim x} \sigma_y + 2h\,\mathbb{1}_{x=0}\Big)$

$$\leq$$

$$\frac{\mu^+_{\Lambda,\beta,h}(\sigma'^{+x})}{\mu^+_{\Lambda,\beta,h}(\sigma'^{-x})} = \exp\Big(2\beta \sum_{y\sim x} \sigma'_y + 2h\Big)$$

so $\nu \leq_{st} \mu^+_{\Lambda,\beta,h}$.

③ Since σ_0 is $\nearrow$, or $\searrow(\sigma_0)$.

$$\langle\sigma_0\rangle^+_{\Lambda,\beta,0} \leq \langle\sigma_0\rangle_\searrow \leq \langle\sigma_0\rangle^+_{\Lambda,\beta,h}$$

But $$\searrow(\sigma_0) = \frac{\langle\sigma_0 e^{h\sigma_0}\rangle^+_{\Lambda,\beta,0}}{\langle e^{h\sigma_0}\rangle^+_{\Lambda,\beta,0}} = \frac{\langle\sigma_0 \sum_{n\geq 0}\frac{h^n\sigma_0^n}{n!}\rangle^+_{\Lambda,\beta,0}}{\langle\sum\frac{h^n\sigma_0^n}{n!}\rangle^+_{\Lambda,\beta,0}}$$

$$= \frac{\langle\sigma_0\rangle^+_{\Lambda,\beta,0}\cosh h + \sinh h}{\langle\sigma_0\rangle^+_{\Lambda,\beta,0}\sinh h + \cosh h}$$

$$\geq \frac{\sinh(h)}{\sinh(h)+\cosh(h)}$$

So $\langle\sigma_0\rangle^+_{\Lambda,\beta,h} \geq$ ⤸. As $n\to\infty$, we get that it is >0.

Ex3: 1) $\langle\sigma_0\rangle^+_{\beta,n}$ is $\nearrow$ in β:

For $\beta\leq\beta'$,

$$\langle\sigma_0\rangle^+_{\beta',n} = \frac{\langle\sigma_0 g\rangle^+_{\beta,n}}{\langle g\rangle^+_{\beta,n}} \qquad \text{where } g = \exp\Big((\beta'-\beta)\sum_{x\sim y}\sigma_x\sigma_y\Big).$$

$$= \sum_{k\geq 0}\frac{1}{k!}(\beta'-\beta)^k\Big(\sum\sigma_x\sigma_y\Big)^k$$

$$= \sum_{S\subset V_n}\beta_S\sigma_S \quad , \beta_S\geq 0.$$

So $$\langle\sigma_0 g\rangle^+_{\beta,n} = \sum\beta_S\langle\sigma_0\sigma_S\rangle^+_{\beta,n}$$
$$\geq \sum\beta_S\langle\sigma_0\rangle^+_{\beta n}\langle\sigma_S\rangle^+_{\beta n} \quad \text{by Griffith}$$
$$\geq \langle\sigma_0\rangle^+_{\beta,n}\langle g\rangle^+_{\beta,n}.$$

This gives $\langle\sigma_0\rangle^+_{\beta,n} \leq \langle\sigma_0\rangle^+_{\beta',n}$.

$\boxed{4}$ • $\langle \sigma_0 \rangle^+_{\beta,n}$ is $\searrow$ in n:

$$\langle \sigma_0 \rangle^+_{\beta,n} = \frac{\langle \sigma_0 \mathbb{1}_{\forall x / d(0,x)=n,\ \sigma_x = +1} \rangle^+_{\beta, n+1}}{\langle \mathbb{1}_{\forall x / d(0,x)=n,\ \sigma_x=+1} \rangle^+_{\beta, n+1}}$$

by domain Markov prop.

Since $\mathbb{1}_{\dots} = \prod_{x / d(0,x)=n} \frac{1+\sigma_x}{2} = \frac{1}{2^{p(p-1)^{n-1}}} \sum_{T \subset \{x / d(0,x)=n\}} \sigma_T$,

$$\langle \sigma_0 \rangle^+_{\beta,n} = \frac{\sum_{T \dots} \langle \sigma_0 \sigma_T \rangle^+_{\beta,n+1}}{\sum_{T \dots} \langle \sigma_T \rangle^+_{\beta,n+1}} \underset{G.}{\geq} \langle \sigma_0 \rangle^+_{\beta,n+1}.$$

2) By $\searrow$ in n, $\langle \sigma_0 \rangle^+_\beta$ is well defined.

By $\nearrow$ in β, $\beta \mapsto \langle \sigma_0 \rangle^+_\beta$ is $\nearrow$.

Let $\beta_c = \sup \{\beta > 0 / \langle \sigma_0 \rangle^+_\beta = 0\}$

$= \inf \{\beta > 0 / \langle \sigma_0 \rangle^+_\beta > 0\}$.

3) By HTE,

$$\langle \sigma_0 \rangle^+_{\beta,n} = \frac{\sum_{\eta / \partial\eta = \{0\}} \operatorname{th}\beta^{|\eta|}}{\sum_{\eta / \partial \eta = \emptyset} \operatorname{th}\beta^{|\eta|}}$$

where $\eta \subset E_n$, $\partial\eta := \{x \in V_n \setminus \partial V_n / \deg_\eta(x) \text{ odd}\}$

If $\partial\eta = \{0\}$, then ... η contains a path γ from 0 to ∂V_n.

For a given path γ,

$$\frac{\sum_{\eta / \gamma \subset \eta,\ \partial\eta = \{0\}} \operatorname{th}\beta^{|\eta|}}{\sum_{\eta / \partial\eta = \emptyset} \operatorname{th}\beta^{|\eta|}} = \operatorname{th}\beta^{|\gamma|} \frac{\sum_{\eta \dots} \operatorname{th}\beta^{|\eta \setminus \gamma|}}{\sum_{\eta / \partial\eta = \emptyset} \operatorname{th}\beta^{|\eta|}}.$$

5

$\leq \operatorname{th}\beta^{|\gamma|}$ by the same argument as in the lectures

So $\langle\sigma_0\rangle^+_{\beta,n} \leq \sum_\gamma \operatorname{th}\beta^{|\gamma|}$.

Moreover, $|\gamma| = n$ and there are (exactly!) $p\cdot(p-1)^{n-1}$ paths $0 \rightsquigarrow \partial V_n$.

So $\langle\sigma_0\rangle^+_{\beta,n} \leq p(p-1)^{n-1}\operatorname{th}\beta^n$.

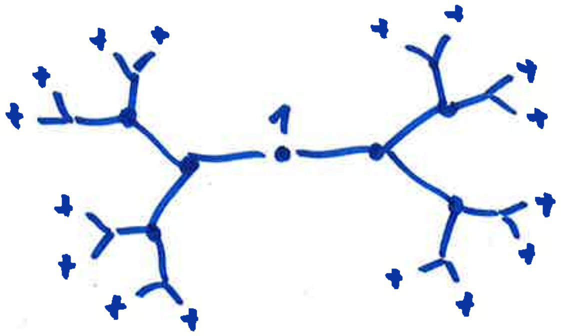
Therefore, if $\operatorname{th}\beta < \frac{1}{p-1}$ $\left(\Leftrightarrow \beta < \operatorname{argth}\left(\frac{1}{p-1}\right)\right)$,

$$\langle\sigma_0\rangle^+_{\beta,n} \to 0 \quad \text{and} \quad \langle\sigma_0\rangle^+_\beta = 0.$$

So $\boxed{\beta_c \geqslant \operatorname{argth}\left(\frac{1}{p-1}\right)}$.

4) a) $$Z_n^+ = \sum_{\sigma_0\in\{\pm1\}} \sum_{\substack{\sigma : V_n\setminus\{0\}\to\{\pm1\}\\ \partial V_n \to +1}} \exp\Big(\beta \sum_{\{x,y\}\in E_n} \sigma_x\sigma_y\Big)$$

Let T_n' be the tree whose root 1 has degree $p-1$ and the other non-leaves vertices have degree p:



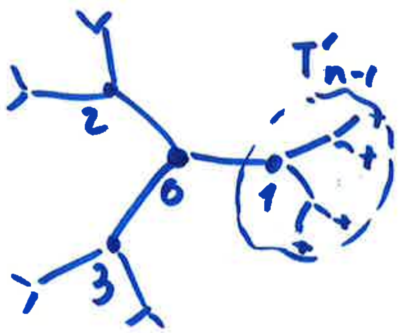
T_4' (For $p=3$)

Let V_n', $\partial V_n'$, E_n' vertices and edges of T_n'..

Let $$Y_n(\sigma_0) = \sum_{\substack{\sigma\in\{\pm1\}^{V_n'}\\ \sigma|_{\partial V_n'}\equiv +1}} e^{\beta\sigma_0\sigma_1} \exp\Big(\beta \sum_{\{x,y\}\in E_n'} \sigma_x\sigma_y\Big)$$

a) Then $Z_n^+ = \sum_{\sigma_0 \in \{\pm 1\}} Y_{n-1}(\sigma_0)^p$.

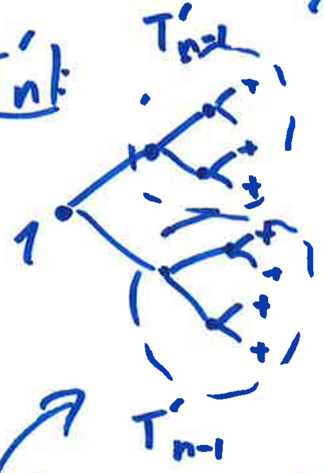
T_n



b) Let $n \geq 1$.

$$Y_n(\sigma_0) = \sum_{\substack{\sigma : V'_n \to \{\pm 1\} \\ \partial V'_n \to 1}} e^{\beta \sigma_0 \sigma_1} \exp\Big(\beta \sum_{\{x,y\} \in E'_n} \sigma_x \sigma_y\Big)$$

T'_n:



$$= \sum_{\sigma_1 \in \{\pm 1\}} e^{\beta \sigma_0 \sigma_1} \big(Y_{n-1}(\sigma_1)\big)^{p-1}.$$

$p-1$ copies of T'_{n-1}

so $$x_n = \frac{Y_n(-1)}{Y_n(+1)} = \frac{e^{\beta} Y_{n-1}(-1)^{p-1} + e^{-\beta} Y_{n-1}(+1)^{p-1}}{e^{-\beta} Y_{n-1}(-1)^{p-1} + e^{\beta} Y_{n-1}(+1)^{p-1}}$$

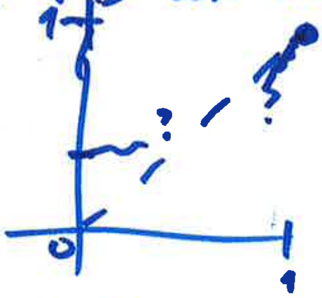
$$= \frac{e^{\beta} x_{n-1}^{p-1} + e^{-\beta}}{e^{-\beta} x_{n-1}^{p-1} + e^{\beta}}$$

$$= \frac{1 + e^{2\beta} x_{n-1}^{p-1}}{e^{2\beta} + x_{n-1}^{p-1}} = F(x_{n-1})$$

c) $$\langle \sigma_0 \rangle_{\beta,n}^+ = \frac{\sum_{\sigma_0 \in \{\pm 1\}} \sigma_0 \big(Y_{n-1}(\sigma_0)\big)^p}{\sum_{\sigma_0 \in \{\pm 1\}} \big(Y_{n-1}(\sigma_0)\big)^p} = \frac{1 - x_{n-1}^p}{1 + x_{n-1}^p}.$$

7 Moreover, F is continuous on $[0,1]$ and $F(0) = e^{-2\beta}$,
$F(1) = 1$,
$\forall x \in [0,1]$, $F(x) \geqslant 0$ and $F(x) \leq 1$.

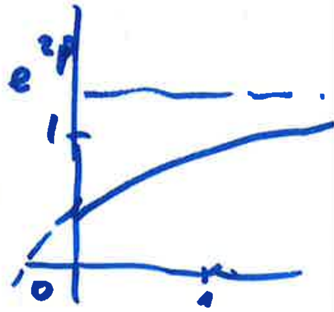
So



so $(x_n)_n$ stays in $[0,1]$.

~~Therefore~~

Also, F is $\nearrow$ on $[0,1]$ (by composing $\dfrac{1+e^{2\beta}t}{e^{2\beta}+t}$:



and $x \mapsto x^{p-1}$)!

so x_n is $\nearrow$ so $\ell := \lim_n x_n$ exists (and is the smallest fixed point of F in $[0,1]$)

We get $\langle \sigma_0 \rangle^+_\beta = \dfrac{1-\ell^p}{1+\ell^p}$. This is 0 iff $\ell = 1$.

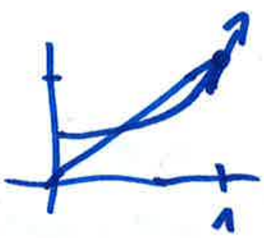
d) Quick way: let $\beta \geqslant \operatorname{argth}\left(\frac{1}{p-1}\right)$.

We want to show that $\ell \neq 1$ in that case.

This will show that $\beta \geqslant \beta_c$, so that $\beta_c \leq \operatorname{argth}\left(\frac{1}{p-1}\right)$,
and with 3) we get equality.

$$F'(x) = \dots = (p-1) \frac{x^{p-2}(e^{4\beta}-1)}{(e^{2\beta}+x^{p-1})^2}$$

$$\text{so } F'(1) = (p-1)\frac{e^{4\beta}-1}{(e^{2\beta}+1)^2} = (p-1)\frac{e^{2\beta}-1}{e^{2\beta}+1} = (p-1)\operatorname{th}\beta > 1$$



So F has a fixed point < 1 in $[0,1]$.
so $\ell \neq 1$!

⑧) 5) The idea is to Taylor expand $\ell(\beta) = F(\ell(\beta))$ around β_c, but I would need more time...

Ex4: 1) For $\sigma \neq \sigma'$,

if $\exists i \in \{0,\dots,n-1\}$ / $\forall j \neq i$, $\sigma_j = \sigma'_j$ and $\sigma_i \neq \sigma'_i$,

~~s.t. $\sigma_i = +1, \sigma'_i = -1$~~

then

$$P(\sigma,\sigma') = \underbrace{\frac{1}{Z}\exp\left(\frac{\beta}{n}\sum_{k,\ell}\sigma_k\sigma_\ell\right)}_{\mu(\sigma)} \frac{1}{n} \frac{\mu(\sigma')}{\mu(\sigma)+\mu(\sigma')}$$

$$= \frac{1}{n} \frac{\mu(\sigma)\mu(\sigma')}{\mu(\sigma)+\mu(\sigma')} = P(\sigma';\sigma).$$

otherwise, $P(\sigma,\sigma') = 0 = P(\sigma';\sigma)$

(we didn't treat $\sigma = \sigma'$, for which symmetry is obvious)

2) P-as on (σ,σ'), $F(\sigma,\sigma')$ is $+2$ if we chose a -1 spin in σ and make it a $+1$,

-2 ——— $+1$ ——— -1,

0 otherwise.

so $f(\sigma) = E[F(\sigma,\sigma')|\sigma]$

$$= \sum_{\substack{0\leq i\leq n-1 \\ \sigma_i=+1}} \frac{1}{n}\cdot 2\cdot \frac{\mu(\sigma^{-i})}{\mu(\sigma)+\mu(\sigma^{-i})} - \sum_{\substack{0\leq i\leq n-1 \\ \sigma_i=-1}} \frac{1}{n}\cdot 2 \frac{\mu(\sigma^{+i})}{\mu(\sigma)+\mu(\sigma^{+i})}$$

(9) $= \frac{2}{n}\sum_{i/\sigma_i=+1} \frac{\exp(-4\sum_{j\neq i}\frac{\beta}{n}\sigma_j)}{1+\exp(-4\sum_{j\neq i}\frac{\beta}{n}\sigma_j)} - \frac{2}{n}\sum_{i/\sigma_i=-1} \frac{\exp(4\sum_{j\neq i}\frac{\beta}{n}\sigma_j)}{1+\exp(4\sum_{j\neq i}\frac{\beta}{n}\sigma_j)}$

changing σ_i affects the term (i,j) and the term (j,i)

For $\sigma_i = +1$, $\frac{\exp(-2\frac{\beta}{n}\sum_{j\neq i}\sigma_j)}{\exp(2\frac{\beta}{n}\sum_{j\neq i}\sigma_j)+\exp(2\frac{\beta}{n}\sum_{j\neq i}\sigma_j)}$

$$\sigma_i - th\left(\frac{2\beta}{n}\sum_{j\neq i}\sigma_j\right) = 1 - th(\ldots) = \frac{2e^{-2\frac{\beta}{n}\sum\ldots}}{e^{2\frac{\beta}{n}\sum\ldots}+e^{-2\frac{\beta}{n}\sum\ldots}}$$

$$= \frac{\exp(-4\sum_{j\neq i}\frac{\beta}{n}\sigma_j)}{1+\exp(-4\sum\frac{\beta}{n}\sigma_j)} \checkmark$$

For $\sigma_i = -1$, similar.

So $F(\sigma) = \frac{1}{n}\sum_{i=0}^{n-1}\left[\sigma_i - th\left(\frac{2\beta}{n}\sum_{j\neq i}\sigma_j\right)\right]$.

So P-a.s, as $\sum\sigma_i$ and $\sum\sigma'_i$ differ by at most 2,

$$|f(\sigma)-f(\sigma')| \leq \frac{1}{n}\underbrace{\left|\sum_{i=0}^{n-1}\sigma_i - \sum_{i=0}^{n-1}\sigma'_i\right|}_{\leq 2} + \frac{1}{n}\sum_{i=0}^{n-1}\underbrace{\left|th\left(\frac{2\beta}{n}\sum_{j\neq i}\sigma_j\right) - th\left(\frac{2\beta}{n}\sum_{j\neq i}\sigma'_j\right)\right|}$$

$$\leq \left|\frac{2\beta}{n}\sum_{j\neq i}\sigma_j - \frac{2\beta}{n}\sum_{j\neq i}\sigma'_j\right|$$

as $|th'(x)| \leq 1$ so $|th(x)-th(y)| \leq |x-y|$.

$$\leq \frac{4\beta}{n}.$$

$$\leq \frac{2}{n} + \frac{1}{n}\cdot n\cdot\frac{4\beta}{n}$$

$$\leq \frac{2+4\beta}{n}.$$

3) $E[f(\sigma)g(\sigma)] = E[E[F(\sigma,\sigma')|\sigma]\,g(\sigma)] = E[F(\sigma,\sigma')g(\sigma)]$

$= E[F(\sigma',\sigma)g(\sigma')]$ — 1

$= -E[F(\sigma,\sigma')g(\sigma')]$ — $F(\sigma',\sigma) = -F(\sigma,\sigma')$

$E[f(\sigma)g(\sigma)] = \frac{1}{2}E[F(\sigma,\sigma')(g(\sigma)-g(\sigma'))]$.

(10) 4) We apply for $g = f$:

$$\mathbb{E}[f^2(\sigma)] = \frac{1}{2} E\left[F(\sigma,\sigma')(f(\sigma) - f(\sigma'))\right]$$

$$\leq \frac{1}{2} E\left[|F(\sigma,\sigma')|\,|f(\sigma) - f(\sigma')|\right]$$

$$\leq \frac{2+4\beta}{n}$$

$|F(\sigma,\sigma')| \leq 2$ P-a.s. and 2).

Moreover, $E[f(\sigma)] = E[F(\sigma,\sigma')] = E[F(\sigma',\sigma)]$

$= -E[F(\sigma,\sigma')]$

$= 0$

So f is centered and has variance $\leq \frac{2+4\beta}{n} \xrightarrow[n\to\infty]{} 0$.

By Tchebitchev,

$$\mu_{\beta,n}(|f| > \varepsilon) \leq \frac{1}{\varepsilon^2} \frac{2+4\beta}{n} \xrightarrow[n\to\infty]{} 0.$$

5) $\left| f(\sigma) - (M_n - th(2\beta M_n)) \right|$

$$= \frac{1}{n} \left| \sum_{i=0}^{n-1} th(2\beta M_n) - th\left(\frac{2\beta}{n} \sum_{j\neq i} \sigma_j\right) \right|$$

$$\leq \frac{1}{n} \sum_{i=0}^{n-1} \underbrace{\left| th\left(\frac{2\beta}{n} \sum_{j=0}^{n-1} \sigma_j\right) - th\left(\frac{2\beta}{n} \sum_{j\neq i} \sigma_j\right) \right|}_{\leq \left|\frac{2\beta}{n} \sum_{j=0}^{n-1} \sigma_j - \frac{2\beta}{n} \sum_{j\neq i} \sigma_j\right| \leq \frac{4\beta}{n}}$$

$$\leq \frac{4\beta}{n}.$$

11) 6) $\forall \varepsilon > 0$,

$$\mu_{\beta,n}\left(|M_n - \operatorname{th}(2\beta M_n)| > 2\varepsilon\right)$$

$$\leq \mu_{\beta,n}\left(|f(\sigma)| > 2\varepsilon - \frac{4\beta}{n}\right) \text{ by } 5)$$

$$\leq \mu_{\beta,n}\left(|f(\sigma)| > \varepsilon\right) \text{ for } n \text{ large enough}$$

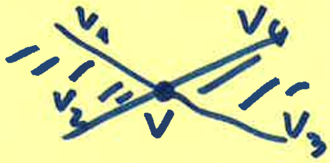
$$\xrightarrow[n\to\infty]{} 0 \quad \text{by } 4).$$

This implies that with high probability, M_n is close to one of the ~~fixed points~~ zeros of $x \mapsto x - \operatorname{th}(2\beta x)$ ... But it doesn't tell us which one! For $\beta \leq \frac{1}{2}$, we do get that M_n is close to 0 (there is only one fixed point).

For $\beta > \frac{1}{2}$, the fixed point are 0 and $\pm m^*(\beta)$.

One idea would be to re-do this with $h \neq 0$ (everything works), as then there is only one fixed point with the same sign as h ...

Ex 5 : 1) For $v \in V$, let v_1, v_2, v_3, v_4 be the 4 edges surrounding v:



in c-cw order starting from a white→black transition.

We try to write

$\omega_{v_1} := \omega_1 \ldots$

$$\varpi_v(\omega) = \hat{a}^{\mathbb{1}}(\omega_1\omega_4 + \omega_2\omega_3) + \hat{b}^{\mathbb{1}}(\omega_1\omega_2 + \omega_3\omega_4)$$
$$+ \hat{c}^{\mathbb{1}}(\omega_1\omega_2\omega_3\omega_4 + 1) + \hat{d}^{\mathbb{1}}(\omega_1\omega_3 + \omega_2\omega_4)$$

12 This gives $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 2\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \\ \hat{d} \end{pmatrix}$

test every possibility of w_1, w_2, w_3, w_4

$$\Leftrightarrow \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \\ \hat{d} \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

so it works.

Then, $Z(a,b,c,d) = \sum_{\omega \in \Omega_e} \prod_{v \in V} w_v(\omega)$

$$= \sum_{\omega \in \underline{\underline{\Omega}}} \prod_{v \in V} w_v(\omega)$$

we take all $\Omega = \{\pm 1\}^E$ but we set $w_v(\omega) = 0$ when ω is wrong at v.

$$= \sum_{\omega \in \Omega} \prod_{v \in V} \hat{a}(\omega_{v_1}\omega_{v_0} + \omega_{v_2}\omega_{v_3}) + \hat{b}(\ \ldots\ ) + \hat{c}(\ \ldots\ ) + \hat{d}(\ \ldots\ )$$

We develop the product. This gives a sum on subsets of edges, where every edge can possibly be taken twice (once for each of its endpoints). Switching with $\sum_\omega$, we see that each edge has to be taken 0 or 2 times (otherwise the sum is 0).

The remaining terms from the expansion can therefore be seen as subsets of edges (the ones taken twice, let's say) and since at every $v \in V$ we made a choice in the product that is compatible with the local rule, we get

(13)

that the remaining terms are in fact elements of $\underline{\underline{\Omega_e}}$.

All in all,

$$Z(a,b,c,d) = 2^{|E|} \, Z(\hat{a},\hat{b},\hat{c},\hat{d})$$

sum on Ω for surviving terms.

~~As~~ By the "handshake lemma", $\underbrace{4|V|}_{=\sum_{v\in V} \deg v} = 2|E|$.

So $2^{|E|} = 2^{2|V|}$, and ~~since~~ so

$$Z(a,b,c,d) = \sum_{w\in\Omega_e} \prod_{v\in V} 4\,\hat{w}_v(w)$$

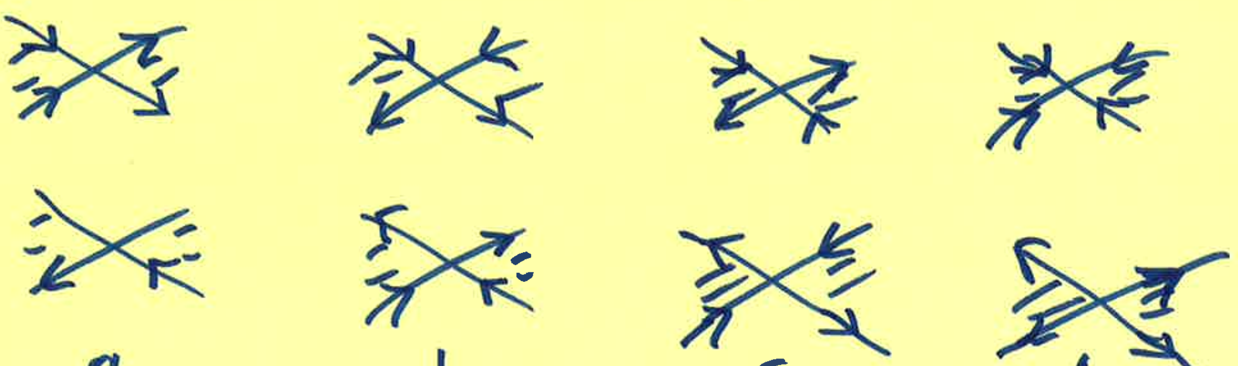
$$= Z(4\hat{a}, 4\hat{b}, 4\hat{c}, 4\hat{d})$$

$$= Z(a', b', c', d').$$

2) Yes. The two "d" config have to appear the same number of times.

To see that, orient every edge $e\in E$ with black on its right when $w_e = +1$, and white on its right when $w_e = -1$.

The local possibilities become



a b c d

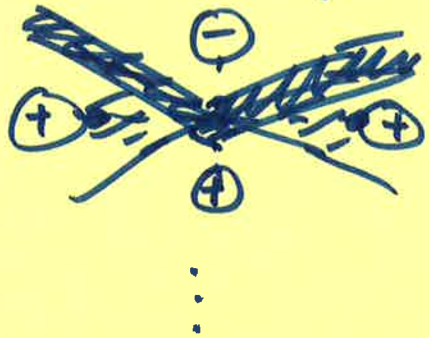
Then the two "d" config are "sinks" and "sources", there has to be the same number of these.

So the transformation $(a,b,c,d) \mapsto (a,b,c,-d)$ also leaves Z invariant.

And of course, all combinations of these two (you can check that they generate a group isomorphic to $\mathfrak{S}_3$).

3) For instance, take an Ising model at temperature β on black faces, and β' on white faces.

Then the +/- interface are an eight-vertex model, with



$$a = \exp(\beta - \beta')$$
$$b = \exp(-\beta + \beta')$$
$$c = \exp(\beta + \beta')$$
$$d = \exp(-\beta - \beta').$$

You can event couple the two models, with 4-spins interactions ...