Statistical Physics and the Ising Model: Summary and Exercises

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Contents

Introduction

This summary is intended to serve as a quick reference of the material discussed during the Ising model Nesin 2026 course. We provide the main definitions, theorems, and high level outlines of how these results are organized and proved. At the end of each section, we provide exercises and questions, as well as detailed references to downloadable lecture notes containing full proofs of the exposed results. In particular, we refer to:

- [3] `https://www.ihes.fr/~duminil/publi/2017PIMS.pdf`
- [6] `https://www.unige.ch/~velenik/smbook/Ising_Model.pdf`
- [7] `https://www.statslab.cam.ac.uk/~grg1000/papers/notes-reprint2012.pdf`
- [9] `https://www.imo.universite-paris-saclay.fr/~paul.melotti/ising.html`

1 Definitions and phase transition

On a finite graph Λ with vertices V and edges E, the Ising probability measure on the set of *spin configurations* $\Sigma_V := \{-\mathbf{1}, +\mathbf{1}\}^V$ at the *inverse temperature* $\beta > 0$ is:

$$\mu(\sigma) := \frac{1}{Z} \exp\left(\beta \sum_{e=\{u,v\}\in E} \sigma_u \sigma_v\right).$$

We denoted the *partition function*:

$$Z := \sum_{\sigma\in\Sigma_V} \exp\left(\beta \sum_{e=\{u,v\}\in E} \sigma_u \sigma_v\right),$$

and we will use the following notation for expectations:

$$\langle f(\sigma)\rangle := \int_{\Sigma_V} f d\mu = \sum_{\sigma\in\Sigma_V} f(\sigma)\mu(\sigma).$$

Sometimes there is also a *magnetic field* $h \in \mathbb{R}$, which introduces a term $h\sum_{v\in V}\sigma_v$ in the exponential, but we will mainly focus on the case $h = 0$.

For $\Lambda_n = [-n, n]^d \subset \mathbb{Z}^d$, the Ising measure on the induced subgraph is called the *free boundary condition* Ising measure and denoted $\mu^f_{\beta,n}$ (and the averages are denoted $\langle\cdot\rangle^f_{\beta,n}$). We can also consider + boundary conditions, where we include the neighbouring interaction with spins at distance 1 of Λ_n (that is $\partial\Lambda_n$), which are fixed to be equal to $+\mathbf{1}$. We denote this measure $\mu^+_{\beta,n}$ (and the average $\langle\cdot\rangle^+_{\beta,n}$).

We consider the average spin at the origin $\langle\sigma_0\rangle^+_{\beta,n}$. We will later show the existence of

$$m^+(\beta) = \lim_{n\to\infty} \langle\sigma_0\rangle^+_{\beta,n}$$

The main result of the beginning of the course is the *nontrivial phase transition* in dimension $d \geq 2$:

Theorem 1.1 (Phase transition)**.** *For $d \geq 2$, there exists $\beta_c \in (0, \infty)$ such that*

- $\forall\beta < \beta_c,\ m^+(\beta) = 0,$
- $\forall\beta > \beta_c,\ m^+(\beta) > 0.$

This is interpreted as the fact that under a certain temperature ($\beta > \beta_c$), the boundary conditions propagate through the material (even in the infinite volume limit), and the magnet retains a certain magnetization. But as soon as we go for a temperature above criticality ($\beta < \beta_c$), the effect of boundary conditions disappear, the spins average out and magnetization vanishes.

To prove the theorem at "low temperature" (big β), first in dimension 2, we use *Peierls argument.* We introduce the Low Temperature Expansion: a configuration is turned into its interfaces between $+\mathbf{1}$ and $-\mathbf{1}$ spins. This interface forms a subset Γ of the dual graph that has even degree at every dual vertex.

Lemma 1.2. *For any loop dual γ surrounding* 0*,*

$$\langle 1_{\gamma\subset\Gamma}\rangle^+_{\beta,n} \leq \exp(-2\beta|\gamma|).$$

Proposition 1.3. *In dimension $d = 2$, for all β* ***large enough****, there is a constant $c(\beta) > 0$ such that*

$$\forall n \geq 1,\ \langle\sigma_0\rangle^+_{\beta,n} \geq c(\beta).$$

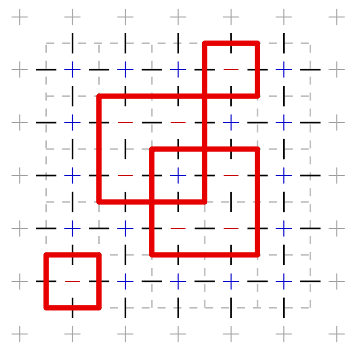

Figure 1: A spin configuration and its Low Temperature Expansion, with + boundary conditions.

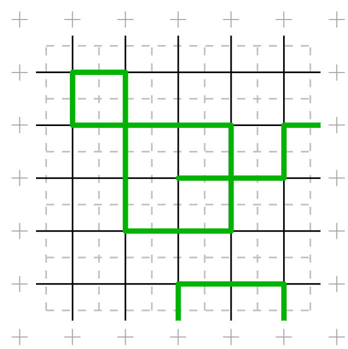

Figure 2: A subset $\eta \subset E$ arising in the High Temperature Expansion, such that $\partial\eta = \{0\}$.

This is the first step towards Theorem 1.1.

For the second step (high temperature, low β), we introduce the (combinatorial) High Temperature Expansion. For a subset of the primal edges $\eta \subset E$, we denote

$$\partial\eta = \{x \in V \mid x \text{ has odd degree in } \eta\}.$$

We show

$$Z = 2^{|V|} \cosh(\beta)^{|E|} \sum_{\eta \text{ s.t. } \partial\eta=\emptyset} \tanh(\beta)^{|\eta|}$$

and we deduce

Theorem 1.4. *For any $A \subset V$,*

$$\left\langle \prod_{x\in A} \sigma_x \right\rangle_{n,\beta}^{+} = \frac{\sum_{\eta \ s.t. \ \partial\eta=A} \tanh(\beta)^{|\eta|}}{\sum_{\eta \ s.t. \ \partial\eta=\emptyset} \tanh(\beta)^{|\eta|}}.$$

Corollary 1.5. *In dimension $d \geq 1$, for all β **small enough**, there is a $c(\beta) > 0$ such that*

$$\forall n \geq 1, \ \langle\sigma_0\rangle_{\beta,n}^{+} \leq \exp(-c(\beta)n).$$

Exercises

Exercise 1.1. Can you describe the model as $\beta \to 0$? As $\beta \to \infty$?

On $\mathbb{Z}^d$, for $\beta > 0$, can you relate μ_β and $\mu_{-\beta}$ *(Hint and generalization: $\mathbb{Z}^d$ is a bipartite graph).*

Exercise 1.2 (Dimension 1: three methods of resolution). Consider the Ising model in dimension 1, that is, on a discrete line segment $[-n, n] \subset \mathbb{Z}$, with + boundary conditions. Show that there is no phase transition: for any $\beta \in (0, \infty)$, we have $m^+(\beta) = 0$ (or, if you prefer, the critical inverse temperature is $\beta_c = +\infty$). To do so, you may either:

- prove that the change of spins happen independently along the line (conditionned on the total number being even), with a certain probability, that you may compute, and deduce that the origin has a probability of being + that tends to 1/2;
- show that Z can be seen as a matrix product (more precisely, a certain matrix element of a product of some 2 by 2 matrices) and that $\langle \sigma_0 \rangle^+_{\beta,n}$ can also be interpreted in this matrix representation. Deduce that the value of $\langle \sigma_0 \rangle^+_{\beta,n}$ can be explicitly computed via diagonalization, and do it if you feel brave;
- use the High Temperature Expansion to express $\langle \sigma_0 \rangle^+_{\beta,n}$ directly.

The second method is actually Ising's original proof in his PhD. It is called the *transfer matrix method*, and applies to many systems in statistical physics.

Exercise 1.3 (Exponential decay of correlations). Using the High Temperature Expansion, show for any $\beta > 0$ *small enough*,

$$\exists c > 0 \text{ s.t. } \forall x \in \mathbb{Z}^d, \langle \sigma_0 \sigma_x \rangle^+_{\Lambda_n,\beta} \leq \exp(-c\|x\|).$$

This property is called the *exponential decay of correlations*, and is in fact true for any $\beta < \beta_c$, although this requires some more work...

Exercise 1.4 (Ising and entropy). The Shannon entropy of a measure μ on a finite set Ω is defined as

$$S(\mu) = -\sum_{\omega \in \Omega} \mu(\omega) \log \mu(\omega).$$

Let Λ be a finite set, and $V : \Sigma_\Lambda \to \mathbb{R}$. We fix also $V_0 \in \mathbb{R}$. We consider the set of measures μ on Σ_Λ such that $\langle V \rangle_\mu = V_0$. Find the measure in this set that maximizes the entropy.

Proofs references

The content follows chapter 2 of [9]. See also chapter 3.7.2 of [6] for quantitative bounds.

2 Positive association, FKG, and comparisons

Since $\beta > 0$, spins tend to have the 'same orientation'. They are positively correlated. We prove several results to formalize the idea that *'positivity favors positivity'*. Thanks to the High Temperature Expansion, we can prove the following inequalities.

Theorem 2.1 (Griffith inequalities for + b.c.). *For any $A, B \subset V$, we have*

$$\left\langle \prod_{x \in A} \sigma_x \right\rangle^+_{n,\beta} \geq 0$$

$$\left\langle \prod_{x \in A \cup B} \sigma_x \right\rangle^+_{n,\beta} \geq \left\langle \prod_{x \in A} \sigma_x \right\rangle^+_{n,\beta} \left\langle \prod_{x \in B} \sigma_x \right\rangle^+_{n,\beta}$$

Comparing the magnetization for two values $0 < \beta < \beta'$ and using the Griffith inequalities, we prove that:

Corollary 2.2. *The function $\beta \mapsto \langle \sigma_0 \rangle^+_{\beta,n}$ is non-decreasing on $\mathbb{R}^+$.*

We use the natural partial ordering over spin configurations:

$$\sigma \leq \sigma' \iff \forall x \in V, \sigma_x \leq \sigma'_x$$

and say that a function $f : \Sigma_V \to \mathbb{R}$ is increasing if it is the case for this ordering, i.e. $\sigma \leq \sigma' \implies f(\sigma) \leq f(\sigma')$.

Theorem 2.3 (Fortuin-Kasteleyn-Ginibre (FKG) inequality)**.** *Let f, g be two increasing functions, then*

$$\langle fg \rangle^+_{n,\beta} \geq \langle f \rangle^+_{n,\beta} \langle g \rangle^+_{n,\beta}$$

Definition 2.4 (Stochastic domination.)**.** *Let μ, ν two probability measures on Σ_V. We say that μ is stochastically dominated by ν (and we denote $\mu \leq_{\mathrm{st}} \nu$) if, for any increasing function $f : \Sigma_V \to \mathbb{R}$,*

$$\mu[f] \leq \nu[f].$$

We will use the following characterization of stochastic domination.

Theorem 2.5 (Strassen)**.** *Let μ, ν two probability measures on Σ_V. then $\mu \leq_{\mathrm{st}} \nu$ if and only if there exists a coupling probability measure η on $\Sigma_V \times \Sigma_V$ such that*

(i) the first marginal of η is μ and the second is ν (which makes it a coupling)

(ii) $\eta(\{(\sigma, \sigma') \mid \sigma \leq \sigma'\}) = 1$

To prove Theorem 2.3, we can assume without loss of generality that $g > 0$. Then we define the measure

$$\nu(\sigma) := \frac{g(\sigma)\mu^+_{\beta,n}(\sigma)}{\int g \, \mathrm{d}\mu^+_{\beta,n}}$$

and prove the domination $\mu^+_{\beta,n} \leq_{\mathrm{st}} \nu$. To do so, we construct a Markov chain $(\sigma_n, \sigma'_n)_{n\geq 0}$ on Σ_V^2, such that $(\sigma_n)_{n\geq 0}$ (resp $(\sigma'_n)_{n\geq 0}$ is itself a Markov chain with invariant limiting distribution $\mu^+_{\beta,n}$ (resp. ν). The *coupled chain* is constructed so that almost surely, at every step, $\sigma_n \leq \sigma'_n$. In the limit, this yields an *increasing coupling* between the two measures.

More precisely, the chain is constructed using a *Glauber dynamic* : at every step, we randomly select a site $x \in V$ and resample $\sigma_{n,x}$ and $\sigma'_{n,x}$ according to their marginal distributions w.r.t μ, ν, while ensuring that after resampling, $\sigma_{n,x} \leq \sigma'_{n,x}$.

Definition 2.6 (Boundary Conditions)**.** *Let $\Lambda \subset \mathbb{Z}^d$ a finite subset and $\xi : \partial\Lambda \to \{-1, 1\}$ some function that we call a* boundary condition*. For $\beta \geq 0$, We define the measure $\mu^{\Lambda,\xi}_\beta$ to be the Ising measure on Λ conditioned on being equal to ξ on $\partial\Lambda$.*

For example, $\mu^+_{\beta,n}$ corresponds to $\Lambda = \Lambda_n$ and ξ being the constant function equal to $+1$. Using the same techniques as in the proof of the FKG inequality, we can prove the following comparison:

Proposition 2.7 (Comparison of Boundary Conditions)**.** *Let $\Lambda \subset \mathbb{Z}^d$ and ξ, ξ' two boundary conditions such that $\xi \leq \xi'$. Then $\mu^{\Lambda,\xi}_\beta \leq_{\mathrm{st}} \mu^{\Lambda,\xi'}_\beta$.*

By *exploring* the spin configuration from the outside, we deduce a monotonicity result in the domain : 'if the positive boundary is closer, spins tend to be more positive'.

Corollary 2.8 (Monotonicity in the Domain for + b.c.). *Let $\Lambda \subset \Lambda' \subset \mathbb{Z}^d$ two finite subsets, and $\beta \geq 0$. Then*

$$\mu_\beta^{\Lambda,+} \geq_{\text{st}} \mu_\beta^{\Lambda',+}.$$

This implies that for any increasing function f depending on a finite number of sites of $\mathbb{Z}^d$, $(\langle f \rangle_{n,\beta}^+)_{n\geq 0}$ is a decreasing sequence in n. In particular, this prove the existence of the magnetization $m^+(\beta) := \lim_{n\to\infty} \langle \sigma_0 \rangle_{\beta,n}^+$.

With this limit in hand, together with Corollary 2.2, Proposition 1.3 and Corollary 1.5, we deduce the existence of a critical β_c and hence prove Theorem 1.1.

Exercises

Exercise 2.1 (Griffith better than FKG inequality?). Are products of spins appearing in Theorem 2.1 increasing functions?

Exercise 2.2 (FKG better than Griffith inequality?). Can you find increasing functions f, g that cannot be rewritten as sums of products of spins with positive coefficients?

Exercise 2.3. Let $0 < \beta < \beta'$. Do we have stochastic ordering between $\mu_{\beta,n}^+$ and $\mu_{\beta',n}^+$?

Proofs references

To prove the Griffith inequalities (Theorem 2.1) from the High Temperature Expansion, we followed chapter 4 of [9]. The proof of FKG inequalities can be found in section 3.10.3 of [6]. Results on domain monotonicity and boundary conditions can be found in [6], section 3.6.3.

3 Infinite volume measures and their properties

From the previous section, we get the existence of a probability measure on the infinite set of spin configurations on $\mathbb{Z}^d$:

Corollary 3.1 (Infinite Volume Measure). *For any $\beta \geq 0$, there exists a measure μ_β^+ on $\Sigma_{\mathbb{Z}^d} = \{-\mathbf{1}, +\mathbf{1}\}^{\mathbb{Z}^d}$ such that for any event E depending on a finite number of sites,*

$$\mu_\beta^+(E) = \lim_{n\to\infty} \mu_{n,\beta}^+(E).$$

Similarly, for minus boundary conditions, and we get a measure μ_β^-. This section is devoted to the description of these measures.

In fact, the limit $m^+(\beta)$ that we found by monotonicity of $\langle \sigma_0 \rangle_{\beta,n}^+$ in n, is in fact equal to the average $\langle \sigma_0 \rangle_\beta^+$ for the infinite volume measure μ_β^+.

Proposition 3.2 (FKG for infinite volume measures). *For any $\beta > 0$, he measures μ_β^+, μ_β^- satisfy the FKG inequality: for any increasing* local *functions f, g,*

$$\langle fg \rangle_\beta^\pm \geq \langle f \rangle_\beta^\pm \langle g \rangle_\beta^\pm.$$

More generally, what we would like is the following, called the "DLR condition" for Dobrushin, Lanford and Ruelle:

Definition 3.3 (Gibbs Measure). *A* Gibbs measure *for the Ising model on $\mathbb{Z}^d$ at inverse temperature β is a probability measure on $\Sigma_{\mathbb{Z}^d} = \{-\mathbf{1}, +\mathbf{1}\}^{\mathbb{Z}^d}$ such that, for any finite set $\Lambda \subset \mathbb{Z}^d$ and $\omega \in \{-\mathbf{1}, +\mathbf{1}\}^{\partial\Lambda}$, and any event E depending only on the sites $\mathbb{Z}^d \setminus \Lambda$,*

$$\mu\left(\cdot \mid \{\sigma|_{\partial\Lambda} = \omega\} \cap E\right) = \mu_{\Lambda,\beta}^\omega(\cdot).$$

Theorem 3.4. *The measures $\mu_\beta^\pm$ are Gibbs measures. Moreover, if μ is another Gibbs measure for the Ising model on $\mathbb{Z}^d$ at inverse temperature β, then*

$$\mu_\beta^- \leq_{\mathrm{st}} \mu \leq_{\mathrm{st}} \mu_\beta^+.$$

This gives the first of a series of equivalences:

Theorem 3.5. *The following are equivalent:*

i) There exists a unique Gibbs measure for the Ising model at inverse temperature $\beta > 0$;

ii) $\mu_\beta^+ = \mu_\beta^-$;

iii) $m^+(\beta) = m^-(\beta) \quad (= 0)$;

This happens for $\beta < \beta_c$, and at this point we do not know if it is the case for β_c. We will see later that it is: $m^+(\beta_c) = 0$, the phase transition is said to be *continuous*.

Recall that those are proved without a magnetic field: $h = 0$. When $h \neq 0$, there is, in fact, always a unique Gibbs measure, whatever $\beta > 0$.

Theorem 3.6 (Translation Invariance)**.** *The measures $\mu_\beta^\pm$ are* translation-invariant*: if A is an event and $\theta_x(A)$ is the same event translated by $x \in \mathbb{Z}^d$, then*

$$\mu_\beta^\pm(\theta_x(A)) = \mu_\beta^\pm(A).$$

Theorem 3.7 (Ergodicity)**.** *The measures $\mu_\beta^\pm$ are* ergodic*: for any event A that is translation-invariant ($\forall x \in \mathbb{Z}^d, \theta_x(A) = A$), we have $\mu_\beta^\pm(A) \in \{0, 1\}$.*

Exercises

Exercise 3.1. Show that $\mu_\beta^\pm$ are also invariant under reflections and rotations by multiples of $\frac{\pi}{2}$.

Do you think we can expect more symmetries for these measures?

Exercise 3.2 (The Curie-Weiss model). Consider the Ising model on the complete graph with n vertices, but rescale the inverse temperature into $\frac{\beta}{n}$:

$$Z_{\beta,n} = \sum_{\sigma \in \{-\mathbf{1},+\mathbf{1}\}^{\{1,\dots,n\}}} \exp\left(\frac{\beta}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\right).$$

Show that the probability of a given σ depends only on the total spin $M(\sigma) = \sum_{i=1}^n \sigma_i$. Rewrite $Z_{\beta,n}$ as a single sum over all possible values of $M(\sigma)$, and show that this sum is dominated by very few terms.

Deduce that the model undergoes a phase transition, and give the value of β_c and the typical value of $\frac{1}{n}M(\sigma)$ for any given $\beta > 0$.

Can one define a limiting infinite volume measure for this model?

Exercise 3.3. Show that for any $\beta > 0$,

$$\langle \sigma_0 \rangle_\beta^+ = \lim_{n\to\infty} \frac{1}{|\Lambda_n|}\left\langle \sum_{x\in\Lambda_n} \sigma_x \right\rangle_{\Lambda_n,\beta}^+.$$

How do you interpret this for magnets?

Proofs references

All of these and more are proved in Sections 3.4 to 3.7 of [6]. See also chapter 3 of [9]. For a discussion on Gibbs measures that are *not* of the form $t\mu^+ + (1-t)\mu^-$, which exist in dimension ≥ 3, see Section 3.10.7 of [6].

4 Continuity of the phase transition via percolation of spins

We know that $\beta \mapsto m^+(\beta)$ is a non-negative non-decreasing function. We have the following, for any $\beta_0 > 0$

$$\lim_{\beta \searrow \beta_0} \langle \sigma_0 \rangle_\beta^+ = \inf_{\beta > \beta_0} \inf_n \langle \sigma_0 \rangle_{\beta,n}^+ = \inf_n \inf_{\beta > \beta_0} \langle \sigma_0 \rangle_{\beta,n}^+ = \langle \sigma_0 \rangle_{\beta_0}^+ ,$$

Proposition 4.1. *The function $\beta \mapsto m^+(\beta)$ is right-continuous on $\mathbb{R}_+^*$.*

Theorem 4.2. *We have $m^+(\beta_c) = 0$, in particular m^+ is a continuous function at β_c.*

We prove this theorem in dimension $d = 2$, using a *percolation* argument due to Wendelin Werner. Percolation models are diverse, but their common focus is the study of *connectedness* properties (in graphs or continuous spaces). Here, we say that two sites $x, y \in \mathbb{Z}^2$ are connected in a spin configuration σ if there is a nearest neighbour path of $-\mathbf{1}$ spin from x to y. We focus on the presence of infinite connected component (or *cluster*) of $-\mathbf{1}$ spins, and make use of the following two propositions.

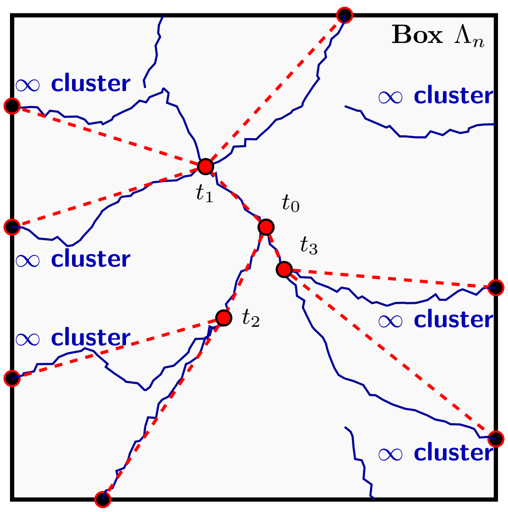

Figure 3: Illustration of the Burton-Keane argument: trifurcations $\mathcal{T}_x$ (red dots) and endpoints of infinite clusters (black dots) from a tree structure (red dashed lines).

Proposition 4.3. *Let μ be an ergodic Ising Gibbs measure, and $\sigma \sim \mu$. Either there is no infinite cluster a.s., or there is a unique infinite cluster a.s. in σ.*

The proof of this proposition relies on the so-called Burton-Keane argument. Ergodicity yields that there exists some $k \in \mathbb{N} \cup \infty$ such that there are k clusters almost surely. Thanks to *finite energy* of μ (the possibility of changing a spin with bounded multiplicative change to the probability), we exclude all the $2 \leq k < \infty$ by melting the clusters together. Finally, we exclude $k = \infty$ by showing that the probability of the event $\mathcal{T}_x$ that a site x is the junction of three infinite clusters is positive if $k = \infty$, but in the meantime it is bounded by $\frac{|\partial \Lambda_n|}{|\Lambda_n|} \xrightarrow{n \to \infty} 0$.

Proposition 4.4. *There is no infinite cluster in $\sigma \sim \mu_{\beta_c}^f$ with probability 1.*

To show the absence of infinite cluster, we rely on the crossing event $\mathcal{H}_n$ that there exists a path of -1 spins from left to right in the box $[0, n]^2$. On the one hand, a *duality* argument shows that $\mu_\beta^f(\mathcal{H}_n) \leq \frac{1}{2}$. On the other hand, using positive association and the uniqueness of the infinite cluster, we show that if there was an infinite cluster, then we would have $\mu_\beta^f(\mathcal{H}_n) \to 1$ as $n \to \infty$.

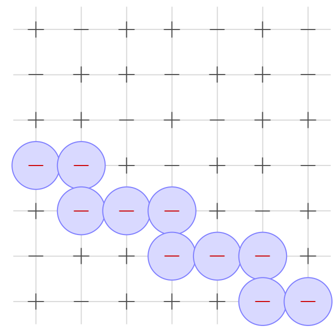

Figure 4: A realization of the event $\mathcal{H}_n$ with a left-right crossing of -1 spins.

Exercises

Exercise 4.1. We say that an (countable) infinite locally finite transitive graph G is amenable if

$$\inf_{A\subset G}\frac{|\partial A|}{|A|}=0.$$

Show that Proposition 4.3 still holds in this context. What about graphs which are not amenable, do we always have uniqueness of the infinite cluster?

Exercise 4.2. Are there increasing sequences of subsets Λ_n whose union is $\mathbb{Z}^d$ but such that $\frac{|\partial\Lambda_n|}{|\Lambda_n|}$ doesn't converge to 0?

Proofs references

This lecture corresponds to section 4.1 of [3]. The proof of the Burton-Keane argument (Proposition 4.3) can be found in section 7.1 of [7] (in the context of Bernoulli percolation).

5 The random cluster model

In the previous section, we took a 'percolation' point of view on the 2D Ising Model to prove the continuity of the magnetization at β_c by studying the spin clusters. In the beginning of this course, we also related the Ising Model to graphical representations (high and low temperature) to study the phase transition. Now, with the goal of computing the value of β_c, we will relate it to yet another probabilistic object, the *random cluster model*.

We consider $\mathbb{Z}^d$ endowed with its nearest-neighbour graph structures : two sites share an *edge* if they are at distance 1. Let $G=(V,E)$ be a finite subgraph of $\mathbb{Z}^d$. A (edge) percolation (or random cluster) configuration ω on G is a function $\omega: E\to\{0,1\}$. An edge $e\in E$ is said *open* if $\omega_e=1$ and *closed* otherwise. We are interested in the connective components of *open* edges.

A boundary condition ξ on G is a partition on ∂G (sites of G share an edge with an outside site), where two sites in the same partition are thought of as being connected outside of G.

Definition 5.1 (Random Cluster Measure)**.** *Let $G=(V,E)$ be a subraph of $\mathbb{Z}^d$, ξ a boundary condition on G. For a configuration $\omega: E\to\{0,1\}$, let $k^\xi(\omega)$ be the number of distinct connected components (considering ξ), $o(\omega)$ and $c(\omega)$ the number of* open *and* closed *edges. Then the random cluster measure for $q\geq 1$ and $p\in[0,1]$ is defined as:*

$$\Phi^\xi_{p,q}(\omega)=\frac{1}{Z_{G,\xi,p,q}}p^{o(\omega)}(1-p)^{c(\omega)}q^{k^\xi(\omega)}$$

where the denominator is a normalizing constant to obtain a probability measure.

We now fix $d = 2$. We can define the dual configuration ω^* corresponding to boundaries of connected component. Let E^* be the set of dual edges (in the planar graph meaning) of G, each edge $e^* \in E^*$ corresponds to a $\frac{\pi}{2}$ rotation of and edge $e \in E$. We define the dual configuration ω^* of a configuration ω where and edge e^* is open in ω^* if and only if it is closed in ω. For $p \in [0, 1]$, define p^* as the solution of

$$\frac{pp^*}{(1-p)(1-p^*)} = q.$$

Let 0 (resp. 1) be the boundary condition where no (resp. all) sites are connected outside of G.

Proposition 5.2 (Duality). *If $\omega \sim \Phi^0_{p,q}$ then $\omega \sim \Phi^1_{p^*,q}$.*

Similarly to the Ising Model, we can prove positive association FKG result, comparison of boundary conditions, and existence and convergence of infinite volume measures for $\Phi^0_{p,q}$ and $\Phi^1_{p,q}$. Using Peierls-type argument (exactly like for the high and low temperature expansion), we prove the existence of a phase transition at some $p_c(q) \in (0, 1)$.

Moreover, the transition is sharp:

Theorem 5.3 (Sharpness). *Fix $q \geq 1$ There exists $c > 0$ such that for $p > p_c$, $\Phi^1_{p,q}[0 \leftrightarrow \infty] \geq c(p - p_c)$. For every $p < p_c$, there exists $c_p > 0$ such that $\Phi^1_{p,q}[0 \leftrightarrow \partial\Lambda_n] \leq \exp(-c_p n)$.*

Using this sharpness result, the duality, together with arguments similar to those of Section 4, we can find the value of the critical parameter.

Theorem 5.4. *Let $p_{\mathrm{sd}} \in (0, 1)$ be the solution of the equation $p = p^*$. Then*

$$p_c = p_{\mathrm{sd}} = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

We now fix $q = 2$ and relate the random cluster and the Ising models. For a percolation configuration ω, define the spin configuration σ by assigning independently a spin to each cluster of ω, uniformly at random.

Proposition 5.5 (Edward-Sokal coupling). *If $\omega \sim \Phi^0_{p,2}$, then the spin configuration constructed above is distributed according to μ^f_β where $\beta = \frac{1}{2}\ln(1 - p)$.*

This way, observables of the Ising model can be expressed in terms of connectivity observable of the random cluster model. In particular, $m^+(\beta) = \Phi^1_{p,2}[0 \leftrightarrow \infty]$, which yields

Theorem 5.6.

$$\beta_c = \frac{1}{2}\ln(1 + \sqrt{2}).$$

Exercises

Exercise 5.1. Consider a finite graph $G = (V, E)$. Prove that the limit of $\Phi^0_{G,p,q}$ with $p \to 0$ and $q/p \to 0$ is the Uniform Spanning Tree on G, i.e. the uniform measure on connected subgraphs of the form $H = (V, F)$, with F not containing any cycle.

Exercise 5.2. In the Edward-Sokal coupling, what is the reverse procedure to obtain ω from some σ?

Exercise 5.3. Without using the connection with Ising, find a Markov chain whose limiting distribution is $\Phi^\xi_{p,q}$. How can you implement this in practice?

Proofs references

The definition of the random cluster model and the Edward-Sokal coupling is written in [3], section 1.2. The calculation of the critical point of the random cluster model can be found in Section 2.4 of [3]. The monotonicity and sharpness results on which this proof relies can be found in sections 1.3 and 2.3 of [3].

A Recent works and exploratory questions

We propose the following topics for the summer school second week research projects. For each topic, we tried to give a first bibliographic reference, questions that we don't know how to answer (which may or may not be addressed in the literature), and/or propose a computer simulation approach. These topics should be seen as a starting point for research and discussions, and in the end, students might have worked on something that hardly relates to the proposals below.

The first topics are variations of the Ising model, either by changing the underlying graph (A.1), the configuration space and interaction (A.2), or by conditioning with respect to an event that is unlikely to happen (A.3).

The next topics explore the links between couplings of the model and algorithmic aspects of simulations. An Ising model can be coupled to another one on a simpler reduced graph in A.4, A.5 is about exact simulation of the law of the Ising model and A.6 asks about the possibility of jointly simulating the Ising and the random cluster models.

We propose two topics on the random cluster model : what if we don't have FKG type positive correlations (A.7); and a way to compare the percolation probabilities on modified graphs (A.8).

In some statistical physics models (that are called *integrable*), we can compute exactly some quantities such as the partition function. The Ising model is one of them (A.9).

Finally, one of the goals in the study of discrete models of statistical physics is to prove the existence of and understand the continuous scaling limit when the underlying graph becomes finer and finer, A.10 introduces to this topic.

A.1 Ising model coupled with random graphs and maps

What if the graph itself is random, and then we consider the Ising model on it, as a second layer of randomness? A first model of a random graph is the one of Erdos-Renyi, where we have n vertices, and each of the possible $\binom{n}{2}$ edges is present independently with probability p. In the *dilute* regime $p = \frac{c}{n}$ and $n \to \infty$, it is possible to find the critical temperature of the Ising model in terms of c, see chapter 5.4 of [4]. Can you produce a full rigorous proof, and can you guess what happens in the *dense* regime $p \gg \frac{1}{n}$?

Another line of research concerns graphs with richer topology, such as *random planar maps*. Reading the introduction of Turunen's thesis [12] can be a nice introduction to what we know about the Ising model on such maps.

A.2 Interacting Ising models: the Ashkin-Teller model

The idea is to introduce an interaction between two spin models. For a *pair* of spin configurations σ, σ', we set the probability to be proportional to

$$\exp\left(\beta \sum_{uv} \sigma_u\sigma_v + \sigma'_u\sigma'_v \quad + K \sum_{uv} \sigma_u\sigma_v\sigma'_u\sigma'_v\right).$$

This is known as the Ashkin-Teller model, and its phase diagram in dimension 2 has been described rigorously in [1], which you are invited to discover. You can start by trying to find a solution analogous to Curie-Weiss computations from Exercise 3.2, and the solution in dimension 1 of this model.

A.3 Frustrating the model: Wulff crystal

Consider the Ising model on $\mathbb{Z}^2$ at low temperature ($\beta > \beta_c$), with + boundary condition, but conditioned so that the density of $+\mathbf{1}$ in Λ_n is smaller than some $m < m^+(\beta)$. That is, we force the density to be way smaller that what it wants to be (see Exercise 3.3). As $n \to \infty$, the best strategy that the model finds is to create a big 'droplet' of $-\mathbf{1}$, with a certain convex, asymptotically deterministic shape called the Wulff crystal.

You can read about this phenomenon in Section 5 of [11]. Can you condition the model (for certain parameters and boundary conditions) to have other non-typical behaviors and get interesting limiting measures? You may also explore these constrained models behavior through simulations.

A.4 Implementing the Feo-Provan algorithm

When we work with the Ising model on a finite planar graph, we can modify the graph locally in certain ways that don't change correlations or partition functions of the model: merging edges, removing useless vertices... and a special move called "star-triangle" or "wye-delta". In [5], the authors gave a way to efficiently reduce any planar graph to a single edge using these local transformations. Can you implement their algorithm and keep track of the Ising model observables along the way?

A.5 Efficient simulation

We discovered the Glauber dynamics, which converges to the Ising measure as time tends to infinity. But how fast is this convergence? And can we find a way to simulate exactly the measure, and not just hope to approximate it? It is, surprisingly, possible without enumerating all configurations, using the so-called *coupling from the past*. After researching this idea, you may implement it, and use it to explore certain observables of the model, such as the shape of interfaces at criticality. How computationally efficient is this method?

A.6 Glauber and coupling, at the same time

We discovered the Glauber dynamics of the Ising model, as well as similar Markovian processes for the random cluster model in Exercise 5.3. Moreover, we know from the course that the two models can be coupled together. Can we do all of these at the same time, that is, have a random process on the space of spins *and* edges configurations, that respect the coupling and evolves as Glauber on spins, and our Markov process on edges? This is the subject of [2].

Can you try the same question for Ising and its High Temperature Expansion? Can you implement those results in a simulation?

A.7 Percolation without positive association

For $q \in (0,1)$, the random cluster model $\Phi_{p,q}$ does not verify the FKG inequality (can you check where the proof breaks?). The question of the existence and nature of phase transitions is itself intricate. Can you explore, through literature review and numerical simulations, what we can say about phase transitions in these models?

A.8 Transfer matrices for percolation

In Exercise 1.2, we saw that the Ising model can be studied through the transfer matrices formalism. At p_c, the percolation models of the last lecture can be coupled to a loop model in the plane, the fully packed $O(n)$ model. In [10], the authors construct a transfer matrix representation for the case $q = 1$, and show the commutation of transfer matrices on a cylinder. Can you extend this result for any q?

A.9 The Kac-Ward formula and exact solution

The partition function of the Ising model on a planar graph can in fact be written as the determinant of a certain operator, which can be constructed geometrically: informally, $Z^2 = \det(I - K)$, where K is some kind of weighted adjacency matrix. An elementary proof of this fact can be found in [8]. Is it possible to use this fact to deduce other features of the exact solution of planar Ising models, such as the fact that on $\mathbb{Z}^2$, the mysterious magnetization is actually

$$m^+(\beta) = \left(1 - \sinh(2\beta)^{-4}\right)^{1/8}?$$

A.10 Conformal invariance of critical Ising Models

On $\mathbb{Z}^2$, consider the LTE, that is, the interface between $+$ and $-$ spins, and let the mesh of the grid tend to 0. For a generic β nothing interesting happens in the limit, but if we are exactly at the critical point β_c, then the interfaces have an interesting limiting distribution as a random family of curves, whose distribution is invariant under conformal maps. You can discover the ideas underlying the proof in the following lecture notes: `https://www.unige.ch/~smirnov/papers/clay-j.pdf`

Can you simulate this in practice, and evaluate the Hausdorff dimension of interfaces? What happens if we try to do the same construction for different boundary conditions, for instance an alternation of four boundary blocks of alternating spin?

References

[1] Yacine Aoun, Moritz Dober, and Alexander Glazman. Phase diagram of the Ashkin–Teller model. *Communications in Mathematical Physics*, 405(2):37, 2024.

[2] Raphaël Cerf and Sana Louhichi. Dynamical coupling between Ising and FK percolation. *ALEA : Latin American Journal of Probability and Mathematical Statistics*, 17(1):23–49, 2020.

[3] Hugo Duminil-Copin. Lectures on the Ising and Potts models on the hypercubic lattice. In *PIMS-CRM Summer School in Probability*, pages 35–161. Springer, 2017.

[4] Rick Durrett. *Random Graph Dynamics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2006.

[5] Thomas A. Feo and J. Scott Provan. Delta-Wye transformations and the efficient reduction of two-terminal planar graphs. *Operations Research*, 41(3):572–582, 1993. https://arxiv.org/pdf/1502.04322.

[6] Sacha Friedli and Yvan Velenik. *Statistical mechanics of lattice systems: a concrete mathematical introduction*. Cambridge University Press, 2017.

[7] Geoffrey Grimmett. Percolation and disordered systems. In *Lectures on Probability Theory and Statistics: Ecole d'Eté de Probabilités de Saint-Flour XXVI-1996*, pages 153–300. Springer, 2006.

[8] Marcin Lis. A short proof of the Kac–Ward formula. *Annales de l'Institut Henri Poincaré D*, Tome 3 no. 1:45–53, 2016. https://arxiv.org/pdf/1502.04322.

[9] Paul Melotti. *Lecture notes on the Ising Model*. Université Paris-Saclay, 2023.

[10] Ron Peled and Dan Romik. Bijective combinatorial proof of the commutation of transfer matrices in the dense O(1) loop model. *Séminaire Lotharingien de Combinatoire*, 73:B73b, 2015.

[11] J. Picard and R. Cerf. *The Wulff Crystal in Ising and Percolation Models: Ecole d'Eté de Probabilités de Saint-Flour XXXIV - 2004*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2009.

[12] Joonas Turunen. *On Ising model coupled to random planar triangulations*. University of Helsinki, 2020. https://helda.helsinki.fi/server/api/core/bitstreams/7e996d4a-3b15-4c1e-ad75-1064df907e24/content.