On the representation of integers by indefinite binary Hermitian forms

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Abstract

Given an integral indefinite binary Hermitian form f over an imaginary quadratic number field, we give a precise asymptotic equivalent to the number of nonequivalent representations, satisfying some congruence properties, of the rational integers with absolute value at most s by f, as s tends to $+\infty$. [1]

1 Introduction

Though less thoroughly developed than the real case of binary quadratic forms initiated by Gauss, the problem of the representation of integers by integral binary Hermitian forms, along with their reduction theory, initiated by Hermite, Bianchi and especially Humbert, has been much studied (see for instance [EGM1, Sect. 9] and references therein). The average, over the representatives of the binary Hermitian forms with a given discriminant, of the number of their nonequivalent representations of a given integer has been computed by Elstrodt, Grunewald and Mennicke [EGM2]. In this paper, we concentrate on a given form, and our result gives a precise asymptotic on the number of nonequivalent proper representations of rational integers with absolute value at most s by a given integral indefinite binary Hermitian form.

A binary Hermitian form naturally gives rise to a quaternary quadratic form. The representations of integers by positive definite quaternary quadratic forms have been studied for a long time (including Lagrange's four square theorem, see also the work of Ramanujan as in [Klo]). In the case of indefinite forms, the counting problem is complicated by the presence of an infinite group of automorphs of the form. General formulas are known (by Siegel's mass formula, see for instance [ERS]), but it does not seem to be easy (or even doable) to deduce our asymptotic formulae from them. Our proof is geometric, while the result of [EGM2] quoted above is based on a number-theoretical computation of the number of certain cosets associated with the representations of a fixed integer. There are numerous results on counting integer points with bounded norm on quadrics, see for instance [dL, DRS, EM, BR], the excellent survey [Bab], and [EO] which counts integer points with bounded norm on various homogeneous varieties. In this paper, we consider a problem of a somewhat different nature, and we count appropriate orbits of integer points on which a fixed integral binary Hermitian form is constant.

[1]**Keywords:** Binary Hermitian form, representation of integers, group of automorphs, Bianchi group. **AMS codes:** 11E39, 11N45, 20H10, 30F40

Let K be an imaginary quadratic number field, with discriminant D_K, ring of integers $\mathscr{O}_K$, and Dedekind zeta function ζ_K. Let $\mathfrak{m}$ be a nonzero fractional ideal of $\mathscr{O}_K$, with norm $N\mathfrak{m}$. For every $u, v \in K$, let $\langle u, v\rangle$ be the $\mathscr{O}_K$-module they generate. Fix an indefinite binary Hermitian form $f : \mathbb{C}^2 \to \mathbb{R}$ with

$$f(u,v) = a|u|^2 + 2\,\mathrm{Re}(bu\overline{v}) + c|v|^2 \tag{1}$$

which is integral over K (its coefficients satisfy $a, b \in \mathbb{Z}$ and $b \in \mathscr{O}_K$). The discriminant $\Delta(f) = |b|^2 - ac$ of the form f is positive. The group $\mathrm{SU}_f(\mathscr{O}_K)$ of automorphs of f consists of those elements $g \in \mathrm{SL}_2(\mathscr{O}_K)$ for which $f \circ g = f$.

For every $s > 0$, we consider the integer

$$\psi_{f,\mathfrak{m}}(s) = \mathrm{Card}\ {}_{\mathrm{SU}_f(\mathscr{O}_K)}\backslash\big\{(u,v) \in \mathfrak{m}\times\mathfrak{m}\ :\ (N\mathfrak{m})^{-1}|f(u,v)| \leq s,\quad \langle u,v\rangle = \mathfrak{m}\big\}\ ,$$

which is the number of nonequivalent $\mathfrak{m}$-primitive representations by f of rational integers with absolute value at most s. The finiteness of $\psi_{f,\mathfrak{m}}(s)$ follows from general results on orbits of algebraic groups defined over number fields [BHC, Lem. 5.3], see also [Shi, Theo. 11.1 (i)]. We prove the following theorem, as a special case of Theorem 4 below. This more general result applies also to representations that satisfy additional congruence properties, see Corollary 8.

Theorem 1 *As s tends to $+\infty$, we have the equivalence*

$$\psi_{f,\mathfrak{m}}(s)\ \sim\ \frac{\pi\ \mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K))}{2\ |D_K|\ \zeta_K(2)\ \Delta(f)}\ \ s^2\ .$$

Note that the image of $\mathrm{SU}_f(\mathscr{O}_K)$ in $\mathrm{PSL}_2(\mathbb{C})$ is an arithmetic Fuchsian subgroup and, by definition, $\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K))$ is the area of the quotient of the real hyperbolic plane $\mathscr{C}$ with constant curvature -1 preserved by $\mathrm{SU}_f(\mathscr{O}_K)$. The following corollary follows immediately by taking $\mathfrak{m} = \mathscr{O}_K$.

Corollary 2 *Let $\mathscr{P}_K$ be the set of relatively prime pairs of integers of K. Then*

$$\mathrm{Card}\ {}_{\mathrm{SU}_f(\mathscr{O}_K)}\backslash\big\{(u,v) \in \mathscr{P}_K\ :\ |f(u,v)| \leq s\big\}\ \sim\ \frac{\pi\ \mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K))}{2\ |D_K|\ \zeta_K(2)\ \Delta(f)}\ \ s^2\ ,$$

as s tends to $+\infty$.

The main input to prove Theorem 1 is the work [PP] (building on [EM]), where we proved an equidistribution result for the boundaries of big tubular neighbourhoods of a finite volume totally geodesic submanifold (here the image of $\mathscr{C}$) in the quotient of a real hyperbolic space by a lattice (here the Bianchi group $\mathrm{PSL}_2(\mathscr{O}_K)$).

The covolume of the group of automorphs could be computed using Prasad's very general formula in [Pra]. Using the work of Maclachlan and Reid [Mac, MR1, MR2], building on results of Humbert and Vigneras, we give an expression for $\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K))$ at the end of Section 2 (Remark 1).

As the final result of the note, we indicate how the results of [MR1] can be used to obtain an even more precise expression of the asymptotic formula in Theorem 1 when $K = \mathbb{Q}(i)$. A constant $\iota(f) \in \{1,2,3,6\}$ is defined as follows. If $\Delta(f) \equiv 0 \mod 4$, let $\iota(f) = 2$. If the coefficients a and c of the form f as in Equation (1) are both even, let $\iota(f) = 3$ if $\Delta(f) \equiv 1 \mod 4$, and let $\iota(f)$ be the remainder modulo 8 of $\Delta(f)$ if $\Delta(f) \equiv 2 \mod 4$. In all other cases, let $\iota(f) = 1$.

Corollary 3 *Let f be an indefinite binary Hermitian form with Gaussian integral coefficients. Then as s tends to $+\infty$,*

$$\psi_f(s) \sim \frac{\pi^2}{8\,\iota(f)\,\zeta_{\mathbb{Q}(i)}(2)} \prod_{p|\Delta(f)} \Big(1 + \Big(\frac{-1}{p}\Big) p^{-1}\Big)\ s^2\ .$$

Here p ranges over the odd positive rational primes and $\big(\frac{-1}{p}\big)$ is the Legendre symbol of -1 modulo p.

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2 Representing integers by indefinite binary Hermitian forms

Let $K, D_K, \mathscr{O}_K, \zeta_K$ and $\mathfrak{m}$ be as in the introduction.

Let us first recall some facts about binary Hermitian forms. The Lie group $\mathrm{SL}_2(\mathbb{C})$ acts linearly on the left on $\mathbb{C}^2$, and it acts on the right on the set of binary Hermitian forms f by precomposition, that is, by $f \mapsto f \circ g$ for every $g \in \mathrm{SL}_2(\mathbb{C})$. Note that

$$\Delta(f \circ g) = \Delta(f) \tag{2}$$

for every $g \in \mathrm{SL}_2(\mathbb{C})$. The (nonuniform) lattice $\Gamma_K = \mathrm{SL}_2(\mathscr{O}_K)$ of $\mathrm{SL}_2(\mathbb{C})$ preserves the set of integral indefinite binary Hermitian forms over K. The stabilizer in Γ_K of such a form f is the group of automorphs $\mathrm{SU}_f(\mathscr{O}_K)$ defined in the introduction.

For every indefinite binary Hermitian form f as in Equation (1) with discriminant $\Delta = \Delta(f)$, let

$$\mathscr{C}_\infty(f) = \{[u:v] \in \mathbb{P}^1(\mathbb{C})\ :\ f(u,v) = 0\}$$

and

$$\mathscr{C}(f) = \{(z,t) \in \mathbb{C}\times\ ]0,+\infty[\ :\ f(z,1) + |a|\,t^2 = 0\}\ .$$

Identifying, as usual, $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$ where $\infty = [1:0]$, the set $\mathscr{C}_\infty(f)$ is the circle of center $-\frac{\overline{b}}{a}$ and radius $\frac{\sqrt{\Delta}}{|a|}$ if $a \neq 0$, and it is the union of a real line with $\{\infty\}$ otherwise. The map $f \mapsto \mathscr{C}_\infty(f)$ induces a bijection between the set of indefinite binary Hermitian forms up to multiplication by a nonzero real factor and the set of circles and real lines in $\mathbb{C} \cup \{\infty\}$. The linear action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ induces a left action of $\mathrm{SL}_2(\mathbb{C})$ by homographies on the set of circles and real lines in $\mathbb{P}^1(\mathbb{C})$, and the map $f \mapsto \mathscr{C}_\infty(f)$ is anti-equivariant for the two actions of $\mathrm{SL}_2(\mathbb{C})$, in the sense that, for every $g \in \mathrm{SL}_2(\mathbb{C})$,

$$\mathscr{C}_\infty(f \circ g) = g^{-1}\,\mathscr{C}_\infty(f)\ . \tag{3}$$

Given a finite index subgroup G of Γ_K, an integral binary Hermitian form f is called G-reciprocal if there exists an element g in G such that $f \circ g = -f$. We define $R_G(f) = 2$ if f is G-reciprocal, and $R_G(f) = 1$ otherwise. The values of $f(z,1)$ are positive on one of the two components of $\mathbb{P}^1(\mathbb{C}) - \mathscr{C}_\infty(f)$ and negative on the other. As the signs are switched by precomposition by an element g as above, the reciprocity of the form f is equivalent to saying that there exists an element of G preserving $\mathscr{C}_\infty(f)$ and exchanging the two complementary components of $\mathscr{C}_\infty(f)$.

Let us now introduce the general counting function we will study. For every integral binary Hermitian form f over K, for every finite index subgroup G of Γ_K, for every x, y in $\mathscr{O}_K$ not both zero, and for every $s > 0$, let

$$\psi_{f,G,x,y}(s) = \operatorname{Card}\ {}_{\mathrm{SU}_f(\mathscr{O}_K)\cap G}\backslash\big\{(u,v) \in G(x,y)\ :\ (N\langle x,y\rangle)^{-1}|f(u,v)| \leq s\big\}\ .$$

Remarks. (1) Let us study the dependence of this counting function $\psi_{f,G,x,y}$ on x and y. Recall that the set of parabolic fixed points of $\Gamma_K = \mathrm{SL}_2(\mathscr{O}_K)$ is exactly $\mathbb{P}^1(K)$. Since G has finite index in Γ_K, the set of parabolic fixed points of G in $\mathbb{P}^1(\mathbb{C})$ is still $\mathbb{P}^1(K)$, and there are only finitely many orbits of G on $\mathbb{P}^1(K)$. It is easy to see that the map $\psi_{f,G,x,y}$ depends only on K, f, G and the orbit of $[x : y]$ under G in $\mathbb{P}^1(K)$. In particular, there are only finitely many maps $\psi_{f,G,x,y}$, for fixed f and G, as x and y vary in $\mathscr{O}_K$.

(2) Let $\mathscr{M}_K$ be the group of nonzero fractional ideals of $\mathscr{O}_K$, and let $\mathscr{I}_K = \mathscr{M}_K/K^*$ be the ideal class group of K. The map from $\Gamma_K\backslash\mathbb{P}^1(K)$ to $\mathscr{I}_K$, which associates the ideal class of $\langle x, y\rangle$ to the Γ_K-orbit of $[x : y]$, is a bijection, see for example Theorem 2.4 in Chapter 7 of [EGM1]. One easily checks that the counting function $s \mapsto \psi_{f,\mathfrak{m}}(s)$ defined in the introduction depends only on K, f and on the ideal class of $\mathfrak{m}$. Therefore we may assume that $\mathfrak{m}$ is contained in $\mathscr{O}_K$. Let then $x_\mathfrak{m}, y_\mathfrak{m}$ be elements of $\mathscr{O}_K$ such that $\mathfrak{m} = \langle x_\mathfrak{m}, y_\mathfrak{m}\rangle$. For instance by Lemma 2.1 in Chapter 7 of [EGM1], for all $x, y, x', y' \in \mathscr{O}_K$, we have $\langle x, y\rangle = \langle x', y'\rangle$ if and only if there exists $\gamma \in \Gamma_K$ such that $(x, y) = \gamma(x', y')$ in $\mathbb{C}^2$. Hence

$$\psi_{f,\mathfrak{m}}(s) = \psi_{f,\Gamma_K,x_\mathfrak{m},y_\mathfrak{m}}\ ,$$

so that Theorem 1 follows from Theorem 4 below.

Let $\Gamma_{K,x,y}$ and $G_{x,y}$ be the stabilizers of $(x,y) \in \mathbb{C}^2$ in Γ_K and G respectively; let $\iota_G = 1$ if $-\mathrm{id} \in G$, and $\iota_G = 2$ otherwise. Note that the image of $\mathrm{SU}_f(\mathscr{O}_K) \cap G$ in $\mathrm{PSL}_2(\mathbb{C})$ is again an arithmetic Fuchsian subgroup.

Theorem 4 *Let f be an integral indefinite binary Hermitian form over an imaginary quadratic number field K, let x and y be elements in $\mathscr{O}_K$ not both zero, and let G be a finite index subgroup of* $\mathrm{SL}_2(\mathscr{O}_K)$. *Then, as s tends to $+\infty$, we have the equivalence*

$$\psi_{f,G,x,y}(s) \sim \frac{\pi\ \iota_G\ [\Gamma_{K,x,y} : G_{x,y}]\ \operatorname{Covol}(\mathrm{SU}_f(\mathscr{O}_K)\cap G)}{2\ |D_K|\ \zeta_K(2)\ \Delta(f)\ [\Gamma_K : G]}\quad s^2\ .$$

Proof. Let us first recall a geometric result from [PP] that will be used to prove this theorem. A subset A of a set endowed with an action of a group G is said to be precisely invariant under this group if for every $g \in G$, if $gA \cap A$ is nonempty, then $gA = A$.

Let $n \geq 2$ and let $\mathbb{H}^n_\mathbb{R}$ be the upper halfspace model of the real hyperbolic space of dimension n, with (constant) sectional curvature -1. Let F be a finite covolume discrete group of isometries of $\mathbb{H}^n_\mathbb{R}$. Let $1 \leq k \leq n-1$ and let $\mathscr{C}$ be a real hyperbolic subspace of dimension k of $\mathbb{H}^n_\mathbb{R}$, whose stabilizer $F_\mathscr{C}$ in F has finite covolume. Let $\mathscr{H}$ be a horoball in $\mathbb{H}^n_\mathbb{R}$, which is precisely invariant under F, with stabilizer $F_\mathscr{H}$.

For every $\alpha, \beta \in F$, denote by $\delta_{\alpha,\beta}$ the common perpendicular geodesic arc between $\alpha\mathscr{C}$ and the horosphere $\beta\partial\mathscr{H}$, and let $\ell(\delta_{\alpha,\beta})$ be its length, counted positively if $\delta_{\alpha,\beta}$ exits

$\beta\mathscr{H}$ at its endpoint on $\beta\partial\mathscr{H}$, and negatively otherwise. By convention, $\ell(\delta_{\alpha,\beta}) = -\infty$ if the boundary at infinity of $\alpha\mathscr{C}$ contains the point at infinity of $\beta\mathscr{H}$. Also define the multiplicity of $\delta_{\alpha,\beta}$ as $m(\alpha,\beta) = 1/\mathrm{Card}(\alpha\, F_{\mathscr{C}}\alpha^{-1} \,\cap\, \beta\, F_{\mathscr{H}}\beta^{-1})$. Its denominator is finite, if the boundary at infinity of $\alpha\mathscr{C}$ does not contain the point at infinity of $\beta\mathscr{H}$, since then the subgroup $\alpha\, F_{\mathscr{C}}\alpha^{-1} \,\cap\, \beta\, F_{\mathscr{H}}\beta^{-1}$ that preserves both $\beta\mathscr{H}$ and $\alpha\mathscr{C}$, consists in elliptic elements. In particular, there are only finitely many elements $[g] \in F_{\mathscr{C}}\backslash F/F_{\mathscr{H}}$ such that $m(g^{-1}, \mathrm{id})$ is different from 1, or equivalently such that $g^{-1}F_{\mathscr{C}}g \,\cap\, F_{\mathscr{H}} \neq \{1\}$. For every $t \geq 0$, define $\mathscr{N}(t) = \mathscr{N}_{F,\mathscr{C},\mathscr{H}}(t)$ as the number, counted with multiplicity, of the orbits under F in the set of the common perpendicular arcs $\delta_{\alpha,\beta}$ for $\alpha, \beta \in F$ with length at most t:

$$\mathscr{N}(t) = \mathscr{N}_{F,\mathscr{C},\mathscr{H}}(t) = \sum_{(\alpha,\beta)\in F\backslash(F/F_{\mathscr{C}}\times F/F_{\mathscr{H}})\,:\,\ell(\delta_{\alpha,\beta})\leq t} m(\alpha,\beta)\ .$$

For every $m \in \mathbb{N}$, denoting by $\mathbb{S}_m$ the unit sphere of the Euclidean space $\mathbb{R}^{m+1}$, endowed with its induced Riemannian metric, we have the following result:

Theorem 5 ([PP, Coro. 4.9]) *As $t \to +\infty$, we have*

$$\mathscr{N}(t) \sim \frac{\mathrm{Vol}(\mathbb{S}_{n-k-1})\,\mathrm{Vol}(F_{\mathscr{H}}\backslash\mathscr{H})\,\mathrm{Vol}(F_{\mathscr{C}}\backslash\mathscr{C})}{\mathrm{Vol}(\mathbb{S}_{n-1})\,\mathrm{Vol}(F\backslash\mathbb{H}^n_{\mathbb{R}})}\, e^{(n-1)t}\ . \ \square$$

Now, let f, K, G, x and y be as in the statement of Theorem 4. By the first remark above it, since any orbit of G on $\mathbb{P}^1(K)$ is dense in $\mathbb{P}^1(\mathbb{C})$, we may assume that x and y are both nonzero. If $D_K = -4, -3$, then the class number of K is one, hence there is only one orbit of parabolic fixed points, and we choose $x = y = 1$.

We write f as in Equation (1), and denote its discriminant by Δ. In order to apply Theorem 5, we first define the various objects n, k, F, $\mathscr{H}$, and $\mathscr{C}$ that appear in its statement.

Let $n = 3$ and $k = 2$, so that $\mathrm{Vol}(\mathbb{S}_{n-1}) = 4\pi$, $\mathrm{Vol}(\mathbb{S}_{n-k-1}) = 2$, the boundary at infinity of $\mathbb{H}^n_{\mathbb{R}}$ is $\partial\mathbb{H}^3_{\mathbb{R}} = \mathbb{C}\cup\{\infty\}$, and $\mathrm{PSL}_2(\mathbb{C})$ acts faithfully and isometrically on $\mathbb{H}^n_{\mathbb{R}}$ by the Poincaré extension of homographies.

For any subgroup H of $\mathrm{SL}_2(\mathbb{C})$, we denote by $\overline{H}$ its image in $\mathrm{PSL}_2(\mathbb{C})$, except that the image of $\mathrm{SU}_f(\mathscr{O}_K)$ is denoted by $\mathrm{PSU}_f(\mathscr{O}_K)$. We will apply Theorem 5 to $F = \overline{G}$.

The Bianchi group $\overline{\Gamma_K} = \mathrm{PSL}_2(\mathscr{O}_K)$ acts discretely on $\mathbb{H}^3_{\mathbb{R}}$, with finite covolume. By a formula essentially due to Humbert (see for instance the sections 8.8 and 9.6 of [EGM1]), we have $\mathrm{Vol}(\overline{\Gamma_K}\backslash\mathbb{H}^3_{\mathbb{R}}) = \frac{1}{4\pi^2}|D_K|^{3/2}\zeta_K(2)$. Note that $\mathrm{Vol}(\overline{G}\backslash\mathbb{H}^3_{\mathbb{R}}) = [\,\overline{\Gamma_K} : \overline{G}\,]\,\mathrm{Vol}(\overline{\Gamma_K}\backslash\mathbb{H}^3_{\mathbb{R}})$ and $[\,\overline{\Gamma_K} : \overline{G}\,] = \frac{1}{\iota_G}[\Gamma_K : G]$ by the definition of ι_G. Thus,

$$\mathrm{Vol}(\overline{G}\backslash\mathbb{H}^3_{\mathbb{R}}) = \frac{1}{4\pi^2\iota_G}|D_K|^{3/2}\zeta_K(2)[\Gamma_K : G]\ . \tag{4}$$

Let $\rho = \frac{x}{y} \neq 0, \infty$, which as said above is a parabolic fixed point of $\overline{\Gamma_K}$ hence of $\overline{G}$. Let $\tau \in\,]0,1]$ be small enough so that the horoball $\mathscr{H}$ in $\mathbb{H}^3_{\mathbb{R}}$ centered at $\rho \in \partial\mathbb{H}^3_{\mathbb{R}}$, with Euclidean height τ, is precisely invariant under $\overline{\Gamma_K}$ hence under $\overline{G}$. Such a τ exists by the standard properties of parabolic fixed points. The stabilizer in the Bianchi group of the point at infinity ρ is equal to $(\,\overline{\Gamma_K}\,)_{\mathscr{H}}$.

Let $\mathscr{O}_K^\times$ be the group of units of $\mathscr{O}_K$, and let ω_K be the number of roots of unity in $\mathscr{O}_K$, which in our case is the cardinality of $\mathscr{O}_K^\times$. Recall that $\omega_K = 4$ if $D_K = -4$, $\omega_K = 6$ if $D_K = -3$ and $\omega_K = 2$ if $D_K \neq -3, -4$.

Lemma 6 *Let $\mathfrak{b}$ be the integral ideal $\mathscr{O}_K \cap \rho^{-1}\mathscr{O}_K \cap \rho^{-2}\mathscr{O}_K$. Then*

$$\mathrm{Vol}(\overline{G}_{\mathscr{H}}\backslash\mathscr{H}) = \frac{\tau^2\sqrt{|D_K|}}{2\,\omega_K}\; N\mathfrak{b}\; [(\overline{\Gamma_K})_{\mathscr{H}} : \overline{G}_{\mathscr{H}}]\;. \tag{5}$$

Proof. We only have to prove that $\mathrm{Vol}((\overline{\Gamma_K})_{\mathscr{H}}\backslash\mathscr{H}) = \frac{\tau^2 N\mathfrak{b}\,\sqrt{|D_K|}}{2\,\omega_K}$. Let

$$\gamma_\rho = \begin{pmatrix} \rho & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(K)\;.$$

Note that γ_ρ^{-1} maps ρ to ∞ and $\mathscr{H}$ to the horoball $\mathscr{H}_\infty$ consisting of the points in $\mathbb{H}^3_{\mathbb{R}}$ with Euclidean height at least $\frac{1}{\tau}$. Let Γ_∞ be the stabilizer of ∞ in $\gamma_\rho^{-1}\,\Gamma_K\,\gamma_\rho$. We claim that

$$\Gamma_\infty = \left\{\begin{pmatrix} a' & -c \\ 0 & d' \end{pmatrix} \;:\; c\in\mathfrak{b},\;\; a',d'\in\mathscr{O}_K^\times,\; a'd'=1\right\}. \tag{6}$$

This implies the result, since (using the fact that an isometry preserves the volume for the first equality, and an easy hyperbolic volume computation for the second one)

$$\begin{aligned}\mathrm{Vol}\big((\overline{\Gamma_K})_{\mathscr{H}}\backslash\mathscr{H}\big) &= \mathrm{Vol}(\overline{\Gamma_\infty}\backslash\mathscr{H}_\infty) = \frac{1}{2}\;\mathrm{Vol}(\overline{\Gamma_\infty}\backslash\partial\mathscr{H}_\infty) = \frac{\tau^2}{2}\;\mathrm{Vol}(\overline{\Gamma_\infty}\backslash\mathbb{C})\\ &= \frac{\tau^2}{\omega_K}\;\mathrm{Vol}(\mathfrak{b}\backslash\mathbb{C}) = \frac{\tau^2 N\mathfrak{b}}{\omega_K}\;\mathrm{Vol}(\mathscr{O}_K\backslash\mathbb{C}) = \frac{\tau^2 N\mathfrak{b}\sqrt{|D_K|}}{2\,\omega_K}\;.\end{aligned}$$

The last two equations hold since $N(\mathfrak{b}) = [\mathscr{O}_K : \mathfrak{b}]$ and since $\mathscr{O}_K$ is generated as a $\mathbb{Z}$-module by 1 and $\frac{D_K+i\sqrt{|D_K|}}{2}$.

To prove the claim (6), note that if $D_K = -4, -3$, then, by the choice $x = y = 1$, we have $\rho = 1$ (hence $\mathfrak{b} = \mathscr{O}_K$), and the claim is satisfied. Therefore, we assume that $D_K \neq -4, -3$, hence that $\mathscr{O}_K^\times = \{\pm 1\}$.

For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ_K, the element

$$\gamma_\rho^{-1}\gamma\gamma_\rho = \begin{pmatrix} \rho c + d & -c \\ \rho^2 c + \rho(d-a) - b & a - \rho c \end{pmatrix}$$

fixes ∞ if and only if

$$\rho^2 c + \rho(d-a) - b = 0\;. \tag{7}$$

If this equation holds, we have $(\rho c + d)(a - \rho c) = 1$. In particular, ρc is an algebraic integer which belongs to K, hence is an element of $\mathscr{O}_K$. Therefore $a' = \rho c + d$ and $d' = a - \rho c$ belong to $\mathscr{O}_K^\times = \{\pm 1\}$, since $a'd' = 1$. Hence $a' = d' = \pm 1$. Multiplying by ρ the equation $\rho c + d = a - \rho c$, and substracting it from Equation (7), we get $\rho^2 c = -b \in \mathscr{O}_K$. Hence $c, \rho c, \rho^2 c \in \mathscr{O}_K$, and this proves the inclusion from the left to the right in Equation (6).

Conversely, let $a' = d' \in \{\pm 1\}$ and $c \in \mathfrak{b}$. Define $a = d' + \rho c$, $d = a' - \rho c$, $b = -\rho^2 c$, which belong to $\mathscr{O}_K$ by the definition of $\mathfrak{b}$. One easily checks that $ad - bc = 1$ and that Equation (7) holds, therefore $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to Γ_K, and $\gamma_\rho^{-1}\gamma\gamma_\rho$ fixes ∞. This proves the inclusion from the right to the left in Equation (6). $\square$

Let us resume the proof of Theorem 4. Let $\mathscr{C} = \mathscr{C}(f)$, which is indeed a real hyperbolic plane in $\mathbb{H}^3_{\mathbb{R}}$, whose set of points at infinity is $\mathscr{C}_\infty(f)$ (hence ∞ is a point at infinity of $\mathscr{C}(f)$ if and only if $a = 0$). Note that $\mathscr{C}$ is invariant under the group $\mathrm{SU}_f(\mathscr{O}_K)$ by Equation (3) (which implies that $\mathscr{C}(f \circ g) = g^{-1}\mathscr{C}(f)$ for every $g \in \mathrm{SL}_2(\mathscr{O}_K)$). The arithmetic group $\mathrm{SU}_f(\mathscr{O}_K)$ acts with finite covolume on $\mathscr{C}(f)$, its finite subgroup $\{\pm\,\mathrm{id}\}$ acting trivially. By definition,

$$\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K) \cap G) = \mathrm{Vol}\big(\,{}_{\mathrm{PSU}_f(\mathscr{O}_K)\cap \overline{G}}\backslash \mathscr{C}(f)\big)\ .$$

Note that $\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K) \cap G)$ depends only on the G-orbit of f, by Equation (3) and since $\mathrm{SU}_{f\circ g}(\mathscr{O}_K) = g^{-1}\,\mathrm{SU}_f(\mathscr{O}_K) g$ for every $g \in \mathrm{SL}_2(\mathscr{O}_K)$. By its definition, $R_G(f)$ is the index of the subgroup $\mathrm{PSU}_f(\mathscr{O}_K) \cap \overline{G}$ in $\overline{G}_{\mathscr{C}}$, hence

$$\mathrm{Vol}(\,\overline{G}_{\mathscr{C}}\backslash\mathscr{C}) = \frac{1}{R_G(f)}\,\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K) \cap G)\ . \tag{8}$$

Now that we have defined all the objects needed to apply Theorem 5, let us pause in the proof and recall the following easy exercise in group theory.

Lemma 7 *Let C be a group and let A, B, A', B' be subgroups of C, such that $A \subset A'$ and $B \subset B'$, both with finite indices. Let D be the set of elements $g \in C$ such that $g^{-1}A'g \cap B' = \{1\}$. Then the fibers of the canonical map from $A\backslash D/B$ to $A'\backslash D/B'$ all have cardinality $[A' : A][B' : B]$.*

Proof. Note that the subset D of C, being invariant under left translation by A' and under right translation by B', is a disjoint union of double cosets $D = \coprod_{i\in I} A' g_i B'$. Write $A' = \coprod_{j=1}^{m} A a_j$ and $B' = \coprod_{k=1}^{n} b_k B$. Let us prove that $D = \coprod_{i\in I, 1\le j\le m, 1\le k\le n} A a_j g_i b_k B$, which yields the result. It is clear that D is the union of the double cosets $A a_j g_i b_k B$. Let us prove that for every $a \in A$ and $b \in B$, if the equality $a a_j g_i b_k b = a_{j'} g_{i'} b_{k'}$ holds, then $i = i', j = j', k = k'$, which implies the disjointness of these double cosets. That equality implies first that $i = i'$ by the definition of the double coset representatives g_i's, and thus that $g_i^{-1} a_{j'}^{-1} a a_j g_i = b_{k'} b^{-1} b_k^{-1}$. Since $a_{j'}^{-1} a a_j$ and $b_{k'} b^{-1} b_k^{-1}$ belong to A' and B' respectively, the assumptions defining D imply that they are both equal to the identity element. Hence $a a_j = a_{j'}$ and $b_k b = b_{k'}$. By the definition of the right coset representatives a_j's and the left coset representatives b_k's, we hence have $j = j'$ and $k = k'$. $\square$

The last step of the proof of Theorem 4 consists in relating the two counting functions $\psi_{f,G,x,y}$ and $\mathscr{N}_{\overline{G},\mathscr{C},\mathscr{H}}$, in order to apply Theorem 5.

For every $g \in \mathrm{SL}_2(\mathbb{C})$, let us compute the hyperbolic length of the common perpendicular geodesic arc $\delta_{g^{-1},\mathrm{id}}$ between the real hyperbolic plane $g^{-1}\mathscr{C}$ and the horoball $\mathscr{H}$, assuming that they do not meet. We use the notation $\gamma_\rho, \mathscr{H}_\infty$ introduced in the proof of Lemma 6. Since γ_ρ^{-1} sends the horoball $\mathscr{H}$ to the horoball $\mathscr{H}_\infty$, it sends the common perpendicular geodesic arc between $g^{-1}\mathscr{C}$ and $\mathscr{H}$ to the (vertical) common perpendicular geodesic arc between $\gamma_\rho^{-1} g^{-1}\mathscr{C}$ and $\mathscr{H}_\infty$. Let r be the Euclidean radius of the circle $\mathscr{C}_\infty(f \circ g \circ \gamma_\rho)$, which is the image by γ_ρ^{-1} of the boundary at infinity of $g^{-1}\mathscr{C}$ by Equation (3). Denoting by $a(f \circ g \circ \gamma_\rho)$ the coefficient of $|u|^2$ in $f \circ g \circ \gamma_\rho(u, v)$, we have, by Equation (2),

$$r = \frac{\sqrt{\Delta(f \circ g \circ \gamma_\rho)}}{|a(f \circ g \circ \gamma_\rho)|} = \frac{\sqrt{\Delta}}{|f \circ g \circ \gamma_\rho(1, 0)|} = \frac{\sqrt{\Delta}}{|f \circ g(\rho, 1)|} = \frac{|y|^2\sqrt{\Delta}}{|f \circ g(x, y)|}\ .$$

An immediate computation gives

$$\ell(\delta_{g^{-1},\mathrm{id}}) = \ell(\gamma_\rho^{-1}\delta_{g^{-1},\mathrm{id}}) = \ln\frac{1}{\tau} - \ln r = \ln\frac{|f\circ g(x,y)|}{\tau|y|^2\sqrt{\Delta}}\ . \tag{9}$$

With the conventions that we have taken, this formula is also valid if $g^{-1}\mathscr{C}$ and $\mathscr{H}$ meet.

For every $s>0$, using Equation (9), we have

$$\begin{aligned}\psi_{f,G,x,y}(s) &= \mathrm{Card}\,\big\{[g]\in(\mathrm{SU}_f(\mathscr{O}_K)\cap G)\backslash G/G_{x,y}\ :\ (N\langle x,y\rangle)^{-1}|f\circ g(x,y)|\le s\big\}\\ &= \mathrm{Card}\,\Big\{[g]\in(\mathrm{PSU}_f(\mathscr{O}_K)\cap\overline{G})\backslash\overline{G}/\,\overline{G_{x,y}}\ :\ \ell(\delta_{g^{-1},\mathrm{id}})\le\ln\frac{s\,N\langle x,y\rangle}{\tau|y|^2\sqrt{\Delta}}\Big\}\ .\end{aligned}$$

We apply Lemma 7 to $C=\overline{G}$, $A=\mathrm{PSU}_f(\mathscr{O}_K)\cap\overline{G}$, $A'=\overline{G}_{\mathscr{C}}$, $B=\overline{G_{x,y}}$ and $B'=\overline{G}_{\mathscr{H}}$. Since there are only finitely many elements $[g]\in\overline{G}_{\mathscr{C}}\backslash\overline{G}/\overline{G}_{\mathscr{H}}$ such that $g^{-1}\overline{G}_{\mathscr{C}}\,g\ \cap\ \overline{G}_{\mathscr{H}}$ is different from $\{1\}$, we have

$$\psi_{f,G,x,y}(s)\sim R_G(f)\ [\overline{G}_{\mathscr{H}}:\overline{G_{x,y}}\,]\ \mathrm{Card}\,\big\{[g]\in\overline{G}_{\mathscr{C}}\backslash\overline{G}/\,\overline{G}_{\mathscr{H}}\ :\ \ell(\delta_{g^{-1},\mathrm{id}})\le\ln\frac{s\,N\langle x,y\rangle}{\tau|y|^2\sqrt{\Delta}}\big\}\ .$$

By the definition of the counting function $\mathscr{N}_{\overline{G},\mathscr{C},\mathscr{H}}$, since there are only finitely many elements $[g]\in\overline{G}_{\mathscr{C}}\backslash\overline{G}/\overline{G}_{\mathscr{H}}$ such that the multiplicity $m(g^{-1},\mathrm{id})$ is different from 1, we have

$$\psi_{f,G,x,y}(s)\sim R_G(f)\ [\overline{G}_{\mathscr{H}}:\overline{G_{x,y}}\,]\ \mathscr{N}_{\overline{G},\mathscr{C},\mathscr{H}}\Big(\ln\frac{s\,N\langle x,y\rangle}{\tau|y|^2\sqrt{\Delta}}\Big)\ .$$

By Theorem 5 and the equations (8), (5) and (4), the number $\psi_{f,G,x,y}(s)$ is equivalent to

$$R_G(f)\ [\overline{G}_{\mathscr{H}}:\overline{G_{x,y}}\,]\ \frac{2}{4\pi}\,\frac{\dfrac{\tau^2\sqrt{|D_K|}}{2\,\omega_K}\ N\mathfrak{b}\ [(\overline{\Gamma_K})_{\mathscr{H}}:\overline{G}_{\mathscr{H}}]\ \dfrac{\mathrm{Covol}\big(\mathrm{SU}_f(\mathscr{O}_K)\cap G\big)}{R_G(f)}}{\dfrac{1}{4\pi^2\iota_G}\ |D_K|^{3/2}\ \zeta_K(2)\ [\Gamma_K:G]}\ \Big(\frac{s\,N\langle x,y\rangle}{\tau|y|^2\sqrt{\Delta}}\Big)^2\ , \tag{10}$$

as s tends to $+\infty$.

In order to simplify the expression (10), let us make two remarks. Firstly,

$$[(\overline{\Gamma_K})_{\mathscr{H}}:\overline{G}_{\mathscr{H}}][\overline{G}_{\mathscr{H}}:\overline{G_{x,y}}\,]=[(\overline{\Gamma_K})_{\mathscr{H}}:\overline{\Gamma_{K,x,y}}\,][\,\overline{\Gamma_{K,x,y}}:\overline{G_{x,y}}\,]=\frac{\omega_K}{2}[\Gamma_{K,x,y}:G_{x,y}]\ .$$

Secondly, let $(x)=\prod_{\mathfrak{p}}\mathfrak{p}^{\nu_{\mathfrak{p}}(x)}$ and $(y)=\prod_{\mathfrak{p}}\mathfrak{p}^{\nu_{\mathfrak{p}}(y)}$ be the prime decompositions of the principal ideals (x) and (y). By the formulas of the prime decompositions of intersections, sums and products of ideals (see for instance [Coh, page 124]), we have $(x^2)\cap(xy)\cap(y^2)=(x^2)\cap(y^2)=\prod_{\mathfrak{p}}\mathfrak{p}^{2\max\{\nu_{\mathfrak{p}}(x),\nu_{\mathfrak{p}}(y)\}}$ and $\langle x,y\rangle=\prod_{\mathfrak{p}}\mathfrak{p}^{\min\{\nu_{\mathfrak{p}}(x),\nu_{\mathfrak{p}}(y)\}}$. By the definition of the ideal $\mathfrak{b}$ and the multiplicativity of the norm, we hence have

$$\frac{N\mathfrak{b}\ (N\langle x,y\rangle)^2}{|y|^4}=N\Big(\big((x^2)\cap(xy)\cap(y^2)\big)\langle x,y\rangle^2(x)^{-2}(y)^{-2}\Big)=1\ .$$

Theorem 4 follows by simplifying the expression (10) using the above two observations. □

We now state the precise asymptotic result of the number of nonequivalent representations of rational integers, satisfying some congruence relations and having absolute value at most s, by a given integral indefinite binary Hermitian form. Given a nonzero ideal $\mathfrak{a}$ in $\mathscr{O}_K$, let $\iota_{\mathfrak{a}} = 1$ if $2 \in \mathfrak{a}$, and $\iota_{\mathfrak{a}} = 2$ otherwise; consider the (full and Hecke respectively) congruence subgroups

$$\Gamma_K(\mathfrak{a}) = \Big\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_K : \alpha - 1, \delta - 1, \gamma, \beta \in \mathfrak{a} \Big\}, \quad \Gamma_{K,0}(\mathfrak{a}) = \Big\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_K : \gamma \in \mathfrak{a} \Big\} .$$

Both $\Gamma_{K,0}(\mathfrak{a})$ and $\Gamma_K(\mathfrak{a})$ coincide with Γ_K when $\mathfrak{a} = \mathscr{O}_K$.

Corollary 8 *Let f be an integral indefinite binary Hermitian form over an imaginary quadratic number field K, and let $\mathfrak{a}$ be a nonzero ideal in $\mathscr{O}_K$. As s tends to $+\infty$, we have*

$$\operatorname{Card}\ {}_{\mathrm{SU}_f(\mathscr{O}_K) \cap \Gamma_K(\mathfrak{a})} \backslash \big\{ (u,v) \in \mathscr{P}_K \ :\ u-1, v \in \mathfrak{a},\ \ |f(u,v)| \le s \big\}$$
$$\sim \frac{\pi\ \iota_{\mathfrak{a}}\ \operatorname{Covol}\big(\mathrm{SU}_f(\mathscr{O}_K) \cap \Gamma_K(\mathfrak{a})\big)}{2\,\mathrm{N}(\mathfrak{a})^2 \prod_{\mathfrak{p}|\mathfrak{a}} \big(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^2}\big)\ |D_K|\ \zeta_K(2)\ \Delta(f)}\quad s^2 \ ,$$

and

$$\operatorname{Card}\ {}_{\mathrm{SU}_f(\mathscr{O}_K) \cap \Gamma_{K,0}(\mathfrak{a})} \backslash \big\{ (u,v) \in \mathscr{P}_K \ :\ v \in \mathfrak{a},\ \ |f(u,v)| \le s \big\}$$
$$\sim \frac{\pi\ \operatorname{Covol}\big(\mathrm{SU}_f(\mathscr{O}_K) \cap \Gamma_{K,0}(\mathfrak{a})\big)}{2\,\mathrm{N}(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} \big(1 + \frac{1}{\mathrm{N}(\mathfrak{p})}\big)\ |D_K|\ \zeta_K(2)\ \Delta(f)}\quad s^2 \ .$$

Proof. The orbits of $(1,0) \in \mathbb{C}^2$ under the linear action of the groups $\Gamma_K(\mathfrak{a})$ and $\Gamma_{K,0}(\mathfrak{a})$ are precisely the sets $\{(u,v) \in \mathscr{P}_K : u-1, v \in \mathfrak{a}\}$ and $\{(u,v) \in \mathscr{P}_K : v \in \mathfrak{a}\}$, respectively.

The indices of $\Gamma_K(\mathfrak{a})$ and $\Gamma_{K,0}(\mathfrak{a})$ in Γ_K, as computed for example in Theorems VII.16 and VII.17 of [New], are

$$[\Gamma_K : \Gamma_K(\mathfrak{a})] = \mathrm{N}(\mathfrak{a})^3 \prod_{\mathfrak{p}|\mathfrak{a}} \big(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^2}\big) \quad \text{and} \quad [\Gamma_K : \Gamma_{K,0}(\mathfrak{a})] = \mathrm{N}(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} \big(1 + \frac{1}{\mathrm{N}(\mathfrak{p})}\big) \ ,$$

where the products are taken over the prime ideals $\mathfrak{p}$ of $\mathscr{O}_K$ dividing $\mathfrak{a}$. The index in the stabilizer of $(1,0) \in \mathbb{C}^2$ in Γ_K of the stabilizer of $(1,0)$ in $\Gamma_K(\mathfrak{a})$ is $\mathrm{N}(\mathfrak{a})$:

$$[\Gamma_{K,1,0} : (\Gamma_K(\mathfrak{a}))_{1,0}] = \mathrm{N}(\mathfrak{a}) \ .$$

The index in the stabilizer of $(1,0) \in \mathbb{C}^2$ in Γ_K of the stabilizer of $(1,0)$ in $\Gamma_{K,0}(\mathfrak{a})$ is 1:

$$[\Gamma_{K,1,0} : (\Gamma_{K,0}(\mathfrak{a}))_{1,0}] = 1 \ .$$

Note that $-\mathrm{id}$ belongs to $\Gamma_{K,0}(\mathfrak{a})$, and it belongs to $\Gamma_K(\mathfrak{a})$ if and only if $2 \in \mathfrak{a}$, so that $\iota_{\Gamma_{K,0}(\mathfrak{a})} = 1$ and $\iota_{\Gamma_K(\mathfrak{a})} = \iota_{\mathfrak{a}}$. The corollary now follows from Theorem 4, applied with $x = 1$, $y = 0$, $G = \Gamma_K(\mathfrak{a})$ and $G = \Gamma_{K,0}(\mathfrak{a})$. □

Corollary 2 also follows from Corollary 8, by taking in both results $\mathfrak{a} = \mathscr{O}_K$. Note that from the techniques of [EGM1, Sect. 9], only the considerably weaker result $\psi_{f,\mathscr{O}_K}(s) = O(s^2 \log s)$ seems to be obtainable (see [EGM2, Coro. 2.12]).

In the following concluding remarks, for any positive integer Δ, let

$$f_\Delta(u,v) = |u|^2 - \Delta |v|^2,$$

which is an integral indefinite binary Hermitian form with discriminant Δ.

Remark 1. Here is a computation of $\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K))$ for f an integral indefinite binary Hermitian form over K, with discriminant Δ, following [Mac, MR2] instead of [Pra].

Maclachlan has proved in [Mac] that $\mathrm{SU}_f(\mathscr{O}_K)$ and $\mathrm{SU}_{f_\Delta}(\mathscr{O}_K)$ are commensurable up to conjugation, in the following way. Since the limit set of $\mathrm{PSL}_2(\mathscr{O}_K)$ is $\partial\mathbb{H}^3_\mathbb{R} = \mathbb{C}\cup\{\infty\}$ and since $\mathrm{SU}_f(\mathscr{O}_K) = \mathrm{SU}_{-f}(\mathscr{O}_K)$, we may assume, up to replacing f by an element in its Γ_K-orbit or its opposite, that $a = a(f) > 0$. Let G_a be the congruence subgroup of $\mathrm{SU}_f(\mathscr{O}_K)$ which is the preimage of the upper triangular subgroup by the morphism $\mathrm{SU}_f(\mathscr{O}_K) \to \mathrm{SL}_2(\mathscr{O}_K/a\mathscr{O}_K)$ of reduction modulo a of the coefficients. Let $g = \begin{pmatrix} \frac{1}{\sqrt{a}} & -\frac{\overline{b}}{\sqrt{a}} \\ 0 & \sqrt{a} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. By an easy computation, we have that $f\circ g = f_\Delta$, and that $g^{-1}G_a g$ is contained in $\mathrm{SL}_2(\mathscr{O}_K)$. Hence $g^{-1}G_a g$ is a finite index subgroup of $\mathrm{SU}_{f_\Delta}(\mathscr{O}_K)$. Therefore, we have

$$\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K)) = \frac{[\mathrm{SU}_{f_\Delta}(\mathscr{O}_K) : g^{-1}G_a g]}{[\mathrm{SU}_f(\mathscr{O}_K) : G_a]}\ \mathrm{Covol}(\mathrm{SU}_{f_\Delta}(\mathscr{O}_K))\ .$$

Maclachlan has also proved in [Mac] that $\mathrm{SU}_{f_\Delta}(\mathscr{O}_K)$ is commensurable with a lattice derived from a quaternion algebra, in the following way. Let $d_K = \frac{D_K}{4}$ if $D_K \equiv 0 \mod 4$, and $d_K = D_K$ otherwise. Let A be the quaternion algebra with Hilbert symbol $\left(\frac{d_K, \Delta}{\mathbb{Q}}\right)$ over $\mathbb{Q}$, which splits over K. Let $\Delta(A)$ be the reduced discriminant of A. Let $\mathscr{O}$ be the order $\mathbb{Z}[i,j,ij]$ in A for the standard basis $1,i,j,ij$ of A, let $\mathscr{O}_{\max}$ be the maximal order containing $\mathscr{O}$, and let $\mathscr{O}^1, \mathscr{O}^1_{\max}$ be their groups of elements of norm 1. Let $\varphi : A \to A\otimes_\mathbb{Q} K = \mathscr{M}_2(K)$ be the natural embedding, given by

$$\alpha+\beta i+\gamma j+\delta ij \mapsto \begin{pmatrix} \alpha+\beta\sqrt{d_K} & \Delta(\gamma+\delta\sqrt{d_K}\,) \\ \gamma-\delta\sqrt{d_K} & \alpha-\beta\sqrt{d_K} \end{pmatrix}\ .$$

An easy computation shows that $\varphi(\mathscr{O}^1)$ is a subgroup of $\mathrm{SU}_{f_\Delta}(\mathscr{O}_K)$. Then by [MR2, Theo. 11.1.1], we have

$$\mathrm{Covol}(\mathrm{SU}_{f_\Delta}(\mathscr{O}_K)) = \frac{\pi\ [\mathscr{O}^1_{\max} : \mathscr{O}^1]}{3\ [\mathrm{PSU}_{f_\Delta}(\mathscr{O}_K) : \overline{\varphi(\mathscr{O}^1)}\,]} \prod_{p\,|\,\Delta(A)} (p-1)\ ,$$

where p ranges over the positive rational primes. This gives a formula for $\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K))$.

Remark 2. Assume that $K = \mathbb{Q}(i)$ in this remark. Let us give in this case, following [MR1], a more precise formula for $\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K))$ where f is an integral indefinite binary Hermitian form over K, with discriminant Δ. Combined with Corollary 2, Corollary 3 will follow.

Since $\mathrm{SU}_f(\mathscr{O}_K) = \mathrm{SU}_{kf}(\mathscr{O}_K)$ for every $k \in \mathbb{N}-\{0\}$, we may assume, as required in [MR1], that f is primitive, that is, with f as in Equation (1), the coefficients a, c, and the real and imaginary parts of b have no common divisor in the rational integers. Note that the subgroup $\mathrm{PSU}_f(\mathscr{O}_K)$ of $\overline{\Gamma_K}$ is denoted by $\mathrm{Stab}(\mathscr{C}(f), \overline{\Gamma_K})$ in [MR1, p. 161], and it is a maximal Fuchsian subgroup of $\overline{\Gamma_K}$ (loc. cit.).

The hyperbolic plane $\mathscr{C}(f_\Delta)$ associated to the form f_Δ is the halfsphere of Euclidean radius $\sqrt{\Delta}$ centered at 0. A formula due to Humbert (see for instance [MR1, p. 169]) gives

$$\mathrm{Covol}(\mathrm{SU}_{f_\Delta}(\mathscr{O}_K)) = \eta\,\pi\,\Delta \prod_{\substack{p|\Delta \\ p \text{ odd}}} \Big(1+\Big(\frac{-1}{p}\Big)p^{-1}\Big)\ , \tag{11}$$

where p ranges over the positive rational primes, $\left(\frac{-1}{p}\right)$ is the Legendre symbol of -1 modulo p, and $\eta = 1/2$ if $\Delta \equiv 0 \mod 4$ and $\eta = 1$ otherwise.

The maximal Fuchsian subgroups of $\overline{\Gamma_K}$ are classified in [MR1], yielding the following cases.

- If $\Delta \equiv 0, 3 \mod 4$, then $\mathrm{PSU}_f(\mathscr{O}_K)$ is a conjugate in $\overline{\Gamma_K}$ of $\mathrm{PSU}_{f_\Delta}(\mathscr{O}_K)$, and its covolume is given by Equation (11) (see [MR1, p.170]).
- If $\Delta \equiv 1 \mod 4$, there are two cases: If the coefficients a and c are even, then

$$\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K)) = \frac{1}{3}\,\mathrm{Covol}(\mathrm{SU}_{f_\Delta}(\mathscr{O}_K))\ ;$$

otherwise, $\mathrm{PSU}_f(\mathscr{O}_K)$ is a conjugate in $\overline{\Gamma_K}$ of $\mathrm{PSU}_{f_\Delta}(\mathscr{O}_K)$ (see [MR1, p. 171-172]).

- If $\Delta \equiv 2 \mod 4$, there are two cases: If the coefficients a and c are even, then

$$\mathrm{Covol}(\mathrm{SU}_f(\mathscr{O}_K)) = \frac{1}{\eta'}\,\mathrm{Covol}\,\mathrm{SU}_{f_\Delta}(\mathscr{O}_K)\ ,$$

where $\eta' \in \{2, 6\}$ satisfies $\eta' = \Delta \mod 8$; otherwise, $\mathrm{PSU}_f(\mathscr{O}_K)$ is a conjugate in $\overline{\Gamma_K}$ of $\mathrm{PSU}_{f_\Delta}(\mathscr{O}_K)$ (see [MR1, p. 173]).

This proves Corollary 3 of the introduction.

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