On the hyperbolic orbital counting problem in conjugacy classes

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Abstract

Given a discrete group Γ of isometries of a negatively curved manifold $\widetilde{M}$, a nontrivial conjugacy class $\mathfrak{K}$ in Γ and $x_0 \in \widetilde{M}$, we give asymptotic counting results, as $t \to +\infty$, on the number of orbit points γx_0 at distance at most t from x_0, when γ is restricted to be in $\mathfrak{K}$, as well as related equidistribution results. These results generalise and extend work of Huber on cocompact hyperbolic lattices in dimension 2. We also study the growth of given conjugacy classes in finitely generated groups endowed with a word metric. [1]

1 Introduction

Given an infinite discrete group of isometries Γ of a proper metric space X, the *orbital counting problem* studies, for fixed $x_0, y_0 \in X$, the asymptotic as $t \to +\infty$ of

$$\mathrm{Card}\{\gamma \in \Gamma \ : \ d(x_0, \gamma y_0) \leq t\} \ .$$

Initiated by Gauss in the Euclidean plane and by Huber in the real hyperbolic plane, there is a huge corpus of works on this problem, including the seminal results of Margulis's thesis, see for instance [Bab2, Oh1, Oh2] and their references for historical remarks, as well as [ABEM, PPS, Qui, Sam] for variations.

Given an infinite subset of the orbit Γx_0, defined in either an algebraic, a geometric or a probabilistic way, it is interesting to study the asymptotic growth of this subset, see for example [PR, BKS] and Chapter 4 of [PPS] for recent examples. In this paper, we consider the orbit points under the elements of a fixed nontrivial conjugacy class $\mathfrak{K}$ in Γ. More precisely, we will study the asymptotic growth as $t \to +\infty$ of the counting function

$$N_{\mathfrak{K}, x_0}(t) = \mathrm{Card}\{\gamma \in \mathfrak{K} \ : \ d(x_0, \gamma x_0) \leq t\}$$

introduced by Huber [Hub1] in a special case. Although we will work in the framework of negative curvature in this paper, the counting problem in (infinite) conjugacy classes is interesting even for discrete isometry groups in (nonabelian) nilpotent or solvable Lie groups endowed with left-invariant distances. We refer to Section 2 for examples of computations of the growth of $N_{\mathfrak{K}, x_0}(t)$ when Γ is a finitely generated group and X is the set Γ endowed with a word metric. This paper opens a new field of research, studying which

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growth types (or relative growth types) fixed conjugacy classes may have in finitely generated groups. For word-hyperbolic groups and negatively curved manifolds, the conjugacy classes usually have constant exponential growth rate, as illustrated by the following result (see also Proposition 5 and Corollary 10 for generalisations).

Theorem 1 *If M is a compact negatively curved Riemannian manifold, if h is the topological entropy of its geodesic flow, if Γ is the covering group of a universal Riemannian cover $X \to M$, if $\mathfrak{K}$ is a nontrivial conjugacy class in Γ, then*

$$\lim_{t\to+\infty} \frac{1}{t} \ln N_{\mathfrak{K},\, x_0}(t) = \frac{h}{2} \,.$$

In this introduction from now on, we concentrate on the case when X is the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$, and we assume that x_0 is not fixed by any nontrivial element of Γ, see the main body of the text for more general statements. Given a nontrivial element γ of a discrete group of isometries Γ of $\mathbb{H}^2_{\mathbb{R}}$, we will denote by $C_\gamma, \tau_\gamma, \iota_\gamma$ the following objects:

- if γ is loxodromic, then C_γ is the translation axis of γ; with ℓ_γ the translation length of γ, we define $\tau_\gamma = (\frac{\cosh \ell_\gamma - 1}{2})^{1/2}$ if γ preserves the orientation and $\tau_\gamma = (\frac{\cosh \ell_\gamma + 1}{2})^{1/2}$ otherwise; ι_γ is 2 if there exists an element in Γ exchanging the two fixed points of γ, and 1 otherwise;
- if γ is parabolic, then C_γ is a horoball centred at the parabolic fixed point of γ; we set $\tau_\gamma = 2 \sinh \frac{d(x,\gamma x)}{2}$ for any $x \in \partial C_\gamma$; we define ι_γ as 2 if there exists a nontrivial elliptic element of Γ fixing the fixed point of γ, and 1 otherwise;
- if γ is elliptic, then C_γ is the fixed point set of γ in $\mathbb{H}^2_{\mathbb{R}}$; if γ is orientation-reversing, we assume in this introduction that the stabiliser of C_γ is infinite; we set $\tau_\gamma = \sin \frac{\theta}{2}$ if γ preserves the orientation with rotation angle θ, and $\tau_\gamma = 1$ otherwise; we define $\iota_\gamma = 1$, unless γ preserves the orientation with rotation angle different from π and the stabiliser in Γ of C_γ is dihedral, in which case $\iota_\gamma = 2$.

We refer for instance to [Rob2] for the definition of the critical exponent δ_Γ of Γ, the Patterson-Sullivan measures $(\mu_x)_{x\in\mathbb{H}^2_{\mathbb{R}}}$ of Γ, the Bowen-Margulis measure m_{BM} of Γ, and to [OhS1, PP2] for the definition of the skinning measure σ_C of Γ associated to a nonempty proper closed convex subset C of $\mathbb{H}^2_{\mathbb{R}}$ (see also Section 3). We denote by $\|\mu\|$ the total mass of a measure μ and by Δ_x the unit Dirac mass at a point x.

The following result says in particular that the exponential growth rate of the orbit under a conjugacy class is $\frac{\delta_\Gamma}{2}$ and that the unit tangent vectors at x_0 to these orbit points equidistribute to the pulled-back Patterson-Sullivan measure.

Theorem 2 *Let Γ be a nonelementary finitely generated discrete group of isometries of $\mathbb{H}^2_{\mathbb{R}}$, and let $\mathfrak{K}$ be the conjugacy class of a fixed nontrivial element $\gamma_0 \in \Gamma$.*

(1) As $t \to +\infty$, we have

$$N_{\mathfrak{K},\, x_0}(t) \sim \frac{\iota_{\gamma_0} \, \|\mu_{x_0}\| \, \|\sigma_{C_{\gamma_0}}\|}{\delta_\Gamma \, \|m_{\mathrm{BM}}\| \, {\tau_{\gamma_0}}^{\delta_\Gamma}} \; e^{\frac{\delta_\Gamma}{2}\, t} \,.$$

If Γ is arithmetic or if M is compact, then the error term is $\mathrm{O}(e^{(\frac{\delta_\Gamma}{2}-\kappa)t})$ for some $\kappa > 0$.

(2) Let v_γ be the unit tangent vector at x_0 to the geodesic segment $[x_0, \gamma x_0]$ for every nontrivial $\gamma \in \Gamma$, and let $\pi_+ : T^1_{x_0}\mathbb{H}^2_{\mathbb{R}} \to \partial_\infty \mathbb{H}^2_{\mathbb{R}}$ be the homeomorphism sending v to the

point at infinity of the geodesic ray with initial vector v. For the weak-star convergence of measures on $T^1_{x_0}\mathbb{H}^2_{\mathbb{R}}$, we have

$$\lim_{t\to+\infty} \frac{\delta_\Gamma \, \|m_{\mathrm{BM}}\| \, {\tau_{\gamma_0}}^{\delta_\Gamma}}{\|\mu_{x_0}\| \, \|\sigma_{C_{\gamma_0}}\|} \, e^{-\frac{\delta_\Gamma}{2} t} \sum_{\gamma\in\mathfrak{K},\, d(x_0,\gamma x_0)\le t} \Delta_{v_\gamma} = (\pi_+^{-1})_* \mu_{x_0} \ .$$

When Γ is a cocompact lattice in dimension 2 and γ_0 is loxodromic, the first claim is due to Huber [Hub1, Theorem B] with an improved error bound in [Hub2]. The following corollary, proved in Sections 5 and 6, is a generalisation of Huber's result for noncompact quotients and for parabolic conjugacy classes. A version for elliptic conjugacy classes follows from Corollary 20, we leave the formulation for the reader.

Corollary 3 *Let Γ be a torsion-free and orientation-preserving discrete group of isometries of $\mathbb{H}^2_{\mathbb{R}}$ such that the surface $\Gamma\backslash\mathbb{H}^2_{\mathbb{R}}$ has finite area, with genus g and p punctures. If $\mathfrak{K}$ is a conjugacy class of primitive loxodromic elements with translation length ℓ, then as $t\to+\infty$,*

$$N_{\mathfrak{K},\,x_0}(t) \sim \frac{\ell}{2\pi(2g+p-2)\sinh\frac{\ell}{2}} \, e^{\frac{t}{2}} \ .$$

If $\mathfrak{K}$ is a conjugacy class of primitive parabolic elements, then as $t\to+\infty$,

$$N_{\mathfrak{K},\,x_0}(t) \sim \frac{1}{2\pi(2g+p-2)} \, e^{\frac{t}{2}} \ .$$

When Γ is a uniform lattice, $\mathfrak{K}$ is a conjugacy class of loxodromic elements, and $\mathbb{H}^2_{\mathbb{R}}$ is replaced by a regular tree, the analog of Corollary 3 is due to [Dou]. See [BrPP] for the case of any locally finite tree and more general discrete groups of isometries.

The main tool of this paper (see Section 3) is a counting and equidistribution result for the common perpendiculars between locally convex subsets of simply connected negatively curved manifolds, proved in [PP4]. In Section 4, we will use this tool in order to prove our abstract main result, Theorem 8, on the counting of the orbit points by the elements in a given conjugacy class. In Sections 5, 6 and 7, we give the elementary computations concerning the geometry of loxodromic, parabolic and elliptic isometries of a simply connected negatively curved manifold required to apply our abstract main result, proving as a special case the above Theorem 2. Finally, in Section 8, we give some results on the counting problem of subgroups of Γ in a given conjugacy class of subgroups.

2 Counting in conjugacy classes in finitely generated groups

In this section, we study the growth of a given conjugacy class in a finitely generated group endowed with a word metric, by giving three examples. We thank E. Breuillard, Y. Cornulier, S. Grigorchuk, D. Osin and R. Tessera for discussions on this topic.

Let Γ be a finitely generated group, endowed with a finite generating set S. For every $\gamma\in\Gamma$, we denote by $\|\gamma\|$ the smallest length of a word in $S\cup S^{-1}$ representing γ. We endow Γ with the left-invariant word metric d_S associated to S, that is, $d_S(\gamma,\gamma') = \|\gamma^{-1}\gamma'\|$ for all $\gamma,\gamma'\in\Gamma$. Given a conjugacy class $\mathfrak{K}$ in Γ, we want to study the growth as $n\to+\infty$ of

$$N_{\mathfrak{K}}(n) = N_{\mathfrak{K},\,S}(n) = \mathrm{Card}\ \mathfrak{K}\cap B_{d_S}(e,n) \ ,$$

the cardinality of the intersection of the conjugacy class $\mathfrak{K}$ with the ball of radius n centered at the identity element e for the word metric d_S.

Given two maps $f, g : \mathbb{N} \to]0, +\infty[$, we write $f \asymp g$ if there exists $c \in \mathbb{N} - \{0\}$ such that $g(n) \leq f(cn)$ and $f(n) \leq g(cn)$ for every $n \in \mathbb{N}$. Note that if S' is another finite generating set of Γ, then $N_{\mathfrak{K}, S'} \asymp N_{\mathfrak{K}, S}$.

The growth of a given conjugacy class in Γ is at most the growth of Γ, and we refer for instance to [Gri, Man] and their references for information on the growth of groups. The growth of the trivial conjugacy class is trivial ($N_{\{e\}}(n) = 1$ for every $n \in \mathbb{N}$). It would be interesting to know what are the possible growths of given conjugacy classes, between these two extremal bounds, and for which group all nontrivial conjugacy classes have the same growth. We only study two examples below.

The counting problem introduced in this paper is dual to the study of the asymptotic as $n \to +\infty$ either of the number of translation axes of (primitive) loxodromic elements meeting the ball of center x_0 and radius n, in the negatively curved manifold case, or of the number of (primitive) conjugacy classes meeting the ball of radius n, in the finitely generated group case. These asymptotics have been studied a lot, for instance by Bowen and Margulis in the manifold case, and by Hull-Osin [HuO] (see also the references of [HuO]) in the finitely generated group case. In particular, Ol'shanskii [Ols, Theo. 41.2] has constructed groups with exponential growth rate and only finitely many conjugacy classes: at least one of them has the same growth rate as the whole group, contrarily to the examples below.

First, let $\Gamma = F(S)$ be the free group on a finite set S of cardinality $|S| \geq 2$. Let $\mathfrak{K}$ be the conjugacy class in Γ of a nonempty reduced and cyclically reduced word in $S \cup S^{-1}$, denoted by γ_0, of length $\ell_{\mathfrak{K}} = \inf_{\gamma \in \mathfrak{K}} \|\gamma\|$. Let $m_{\mathfrak{K}}$ be the number of cyclic conjugates of γ_0 (for instance $m_{\mathfrak{K}} = 1$ if $\gamma_0 = s^\ell$ for some $s \in S \cap S^{-1}$). We denote by $\lfloor x \rfloor$ the integral part of a real number x.

Proposition 4 *For every $n \in \mathbb{N}$ with $n \geq \ell_{\mathfrak{K}} + 2$, we have*

$$N_{\mathfrak{K}, S}(n) = m_{\mathfrak{K}} \, (2\,|S| - 2) \, (2\,|S| - 1)^{\left\lfloor \frac{n - \ell_{\mathfrak{K}} - 2}{2} \right\rfloor} .$$

In particular, $\lim_{n \to +\infty} \frac{1}{n} \ln N_{\mathfrak{K}, S}(n) = \frac{\ln(2|S|-1)}{2}$ does not depend on the nontrivial conjugacy class $\mathfrak{K}$, and is half the exponential growth rate of Γ with respect to the generating set S (see Proposition 5 for a generalisation).

Proof. Let $k = |S|$ and $\ell = \ell_{\mathfrak{K}}$. Every element γ in $\mathfrak{K}$ can be written uniquely as $\alpha \gamma_0' \alpha^{-1}$, where α is a reduced word in $S \cup S^{-1}$ and γ_0' is a cyclic conjugate of γ_0, and where the writing is reduced, that is, the last letter of α is different from the inverse of the first letter s_1 of γ_0' and from the last letter s_ℓ of γ_0'. In particular,

$$\|\gamma\| = 2\,\|\alpha\| + \ell \, .$$

Note that $s_1^{-1} \neq s_\ell$, since γ_0 is cyclically reduced. For every $m \in \mathbb{N}$ with $m \geq 1$, there are $(2k-2)(2k-1)^{m-1}$ reduced words of length at most m whose last letter is different from s_1^{-1} and s_ℓ. The result follows. □

Remark. The group $\Gamma = F(S)$ acts faithfully on its Cayley graph associated to S by left multiplication, and $N_{\mathfrak{K}, S}(n) = \mathrm{Card}(\mathfrak{K} \cdot e \cap B(e, n))$. Proposition 4 gives an exact expression

for this orbit count, improving [Dou, Thm. 1] in this special case for $(q+1)$-regular trees with q odd.

The following result says in particular that in a torsion-free word-hyperbolic group, the nontrivial conjugacy classes have constant exponential growth rate, equal to half the one of the ambient group. Recall (see for instance [Cha, §5.1]) that the *virtual center* $Z^{\rm virt}(\Gamma)$ of a nonelementary word-hyperbolic group Γ is the finite subgroup of Γ consisting of the elements $\gamma \in \Gamma$ acting by the identity on the boundary at infinity $\partial_\infty\Gamma$ of Γ, or, equivalently, having finitely many conjugates in Γ, or, equivalently, whose centraliser in Γ has finite index in Γ. Note that $N_{\mathfrak{K}}(n)$ is bounded if (and only if) $\mathfrak{K}$ is the conjugacy class of an element in the virtual center.

Proposition 5 *Let Γ be a nonelementary word-hyperbolic group, S a finite generating set of Γ, and $\mathfrak{K}$ the conjugacy class of an element in $\Gamma - Z^{\rm virt}(\Gamma)$. Then*

$$\limsup_{n\to+\infty} \frac{1}{n} \ln N_{\mathfrak{K},\,S}(n) = \frac{1}{2}\ \limsup_{n\to+\infty} \frac{1}{n} \ln \mathrm{Card}\ B_{d_S}(e,n)\ .$$

Proof. Let $\gamma_0 \in \Gamma - Z^{\rm virt}(\Gamma)$ and $\delta = \limsup_{n\to+\infty} \frac{1}{n} \ln \mathrm{Card}\ B_{d_S}(e,n)$. Let C_{γ_0} be a quasi-translation axis of γ_0 if γ_0 has infinite order, and let C_{γ_0} be the set of quasi-fixed points of γ_0 otherwise. Note that C_{γ_0} is quasi-convex, that $Z_\Gamma(\gamma_0)$ preserves C_{γ_0}, and that C_{γ_0} is at bounded distance from $Z_\Gamma(\gamma_0)$ in Γ. In particular, $\Gamma_0 = Z_\Gamma(\gamma_0)$ is a quasi-convex-cocompact subgroup with infinite index in the nonelementary word hyperbolic group Γ.

It is well-known that the exponential growth rate of Γ/Γ_0 is then equal to the exponential growth rate δ of Γ. Indeed, the limit set $\Lambda\Gamma_0$ of Γ_0 is then a proper subset of $\partial_\infty\Gamma$ and Γ_0 acts properly discontinuously on $\Gamma \cup (\partial_\infty\Gamma - \Lambda\Gamma_0)$. Let $\xi \in \partial_\infty\Gamma - \Lambda\Gamma_0$. If U is a small enough neighbourhood of ξ in $\Gamma \cup \partial_\infty\Gamma$, then there exists $N \in \mathbb{N}$ such that U meets at most N of its images by the elements of Γ_0, and for every $x \in U \cap \Gamma$, if $p : \Gamma \to \Gamma/\Gamma_0$ is the canonical projection, then $|d(x,e) - d(p(x),p(e))|$ is uniformly bounded. It is well-known (see for instance the proof of [Rob1, Coro. 1]) that the (sectorial) exponential growth rate $\limsup_{n\to+\infty} \frac{1}{n} \ln \mathrm{Card}\ \big(U \cap B_{d_S}(e,n)\big)$ of Γ in U is equal to δ. This proves the above claim.

Up to a bounded additive constant, the distance between e and $\gamma^{-1}\gamma_0\gamma$ is equal to twice the distance from γ to C_{γ_0}, by hyperbolicity. Hence the exponential growth rate of $\mathfrak{K}$ is half the exponential growth rate of $\Gamma/Z_\Gamma(\gamma_0)$, that is $\delta/2$. □

Now, let A be a free abelian group of rank $2k$, let $\langle\cdot,\cdot\rangle$ be an integral symplectic form on A, and let Γ be the associated Heisenberg group, that is, the group with underlying set $A \times \mathbb{Z}$ and group law

$$(a,z)(a',z') = (a+a', z+z'+\langle a,a'\rangle)\ .$$

Note that Γ is finitely generated, and we have an exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } \Gamma \xrightarrow{\ \pi\ } A \longrightarrow 0\ ,$$

where $i : z \mapsto (0,z)$ has image the center of Γ and $\pi : (a,z) \mapsto a$. Let $\mathfrak{K}$ be a nontrivial conjugacy class (that is, the conjugacy class of a noncentral element) in Γ.

Proposition 6 *We have*

$$N_{\mathfrak{K}}(n) \asymp n^2 \, .$$

In particular, the growth of any nontrivial conjugacy class in the Heisenberg group Γ is quadratic. Note that Card $B_\Gamma(e,n) \asymp n^{2k+2}$ and that the number of (primitive or not) conjugacy classes meeting the ball of radius n is $\asymp n^2 \ln n$ if $k=1$, see [GS, Ex. 2.4].

Proof. Let $\gamma_0 = (a_0, z_0)$ be a noncentral element in Γ, so that $a_0 \neq 0$, and let $\|\gamma_0\|$ be its distance to the identity element e for a given word metric on Γ.

Since $\pi : \Gamma \to A$ is the abelianisation map, whose kernel is the center Z of Γ, we have $\pi(\mathfrak{K}) = \{\pi(\gamma_0)\}$ and

$$\mathfrak{K} \subset \pi^{-1}(\{\pi(\gamma_0)\}) = Z\,\gamma_0 \, .$$

Since $\langle \cdot, \cdot \rangle$ is nondegenerate and $a_0 \neq 0$, there exists $b_0 \in A$ such that $n_0 = 2\langle a_0, b_0 \rangle \neq 0$. For every $(a,z) \in \Gamma$, since $(a,z)^{-1} = (-a,-z)$, it is easy to compute that

$$(a,z)(a_0,z_0)(a,z)^{-1} = (a_0, z_0 + 2\langle a, a_0 \rangle) \, .$$

Hence, with $Z^{n_0} = \{(0, n_0 n) \ : \ n \in \mathbb{Z}\}$, which is a finite index subgroup of Z, we have

$$Z^{n_0}\,\gamma_0 \subset \mathfrak{K} \, .$$

We have

$$\text{Card}\, \mathfrak{K} \cap B(e,n) \leq \text{Card}\left(Z \cap B(e,n)\,\gamma_0^{-1}\right) \leq \text{Card}\, Z \cap B(e, n + \|\gamma_0\|) \, ,$$

and similarly, Card $\mathfrak{K} \cap B(e,n) \geq$ Card $Z^{n_0} \cap B(e, n - \|\gamma_0\|)$. We hence only have to prove that for every finite index subgroup Z' of Z, we have Card $Z \cap B(e,n) \asymp n^2$. This is well known (see for instance [Har, VII.21] when A has rank 2): for instance, we have $[(a_0,0)^p, (b_0,0)^q] = (0, pq\,\langle a_0, b_0 \rangle)$ for all $p, q \in \mathbb{Z}$, and the distance to e of the commutator on the left hand side of this equality is at most $c(p+q)$, for some constant $c > 0$. □

3 Counting common perpendicular arcs

In this section, we briefly review a simplified version of the geometric counting and equidistribution result proved in [PP4], which is the main tool in this paper (see also [PP3] for related references, [PP5] for arithmetic applications in real hyperbolic spaces and [PP6] for the case of locally symmetric spaces). We refer to [BrH] for the background definitions and properties concerning the isometries of CAT(-1) spaces.

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with (dimension at least 2 and) pinched negative sectional curvature $-b^2 \leq K \leq -1$, let $x_0 \in \widetilde{M}$, and let $T^1\widetilde{M}$ be the unit tangent bundle of $\widetilde{M}$. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$ and let $M = \Gamma\backslash\widetilde{M}$ and $T^1M = \Gamma\backslash T^1\widetilde{M}$ be the quotient orbifolds.

We denote by $\partial_\infty \widetilde{M}$ the boundary at infinity of $\widetilde{M}$, by $\Lambda\Gamma$ the limit set of Γ and by $(\xi, x, y) \mapsto \beta_\xi(x,y)$ the Busemann cocycle on $\partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M}$ defined by

$$(\xi, x, y) \mapsto \beta_\xi(x,y) = \lim_{t \to +\infty} d(\rho_t, x) - d(\rho_t, y) \, ,$$

where $\rho : t \mapsto \rho_t$ is any geodesic ray with point at infinity ξ and d is the Riemannian distance.

For every $v \in T^1\widetilde{M}$, let $\pi(v) \in \widetilde{M}$ be its origin, and let v_-, v_+ be the points at infinity of the geodesic line in $\widetilde{M}$ whose tangent vector at time $t = 0$ is v. We denote by $\pi_+ : T^1_{x_0}\widetilde{M} \to \partial_\infty \widetilde{M}$ the homeomorphism $v \mapsto v_+$.

Let D^- and D^+ be nonempty proper closed convex subsets in $\widetilde{M}$, with stabilisers Γ_{D^-} and Γ_{D^+} in Γ, such that the families $(\gamma D^-)_{\gamma\in\Gamma/\Gamma_{D^-}}$ and $(\gamma D^+)_{\gamma\in\Gamma/\Gamma_{D^+}}$ are locally finite in $\widetilde{M}$. We denote by $\partial^1_\pm D^\mp$ the *outer/inner unit normal bundle* of $\partial D^\mp$, that is, the set of $v \in T^1\widetilde{M}$ such that $\pi(v) \in \partial D^\mp$, $v_\pm \in \partial_\infty \widetilde{M} - \partial_\infty D^\mp$ and the closest point projection on $D^\mp$ of $v_\pm$ is $\pi(v)$. For every γ, γ' in Γ such that γD^- and $\gamma' D^+$ have a common perpendicular (that is, if the closures $\overline{\gamma D^-}$ and $\overline{\gamma' D^+}$ in $\widetilde{M} \cup \partial_\infty \widetilde{M}$ are disjoint), we denote by $\alpha_{\gamma,\gamma'}$ this common perpendicular (starting from γD^- at time $t = 0$), by $\ell(\alpha_{\gamma,\gamma'})$ its length, and by $v^-_{\gamma,\gamma'} \in \gamma\partial^1_+ D^-$ its initial tangent vector. The *multiplicity* of $\alpha_{\gamma,\gamma'}$ is

$$m_{\gamma,\gamma'} = \frac{1}{\mathrm{Card}(\gamma\Gamma_{D^-}\gamma^{-1} \cap \gamma'\Gamma_{D^+}\gamma'^{-1})}\,,$$

which equals 1 when Γ acts freely on $T^1\widetilde{M}$ (for instance when Γ is torsion-free). Let

$$\mathscr{N}_{D^-,D^+}(s) = \sum_{\substack{(\gamma,\gamma')\in\Gamma\backslash((\Gamma/\Gamma_{D^-})\times(\Gamma/\Gamma_{D^+}))\\ \overline{\gamma D^-}\cap\overline{\gamma' D^+}=\emptyset,\ \ell(\alpha_{\gamma,\gamma'})\le s}} m_{\gamma,\gamma'} = \sum_{\substack{[\gamma]\in\Gamma_{D^-}\backslash\Gamma/\Gamma_{D^+}\\ \overline{D^-}\cap\overline{\gamma D^+}=\emptyset,\ \ell(\alpha_{e,\gamma})\le s}} m_{e,\gamma}\ ,$$

where Γ acts diagonally on $(\Gamma/\Gamma_{D^-}) \times (\Gamma/\Gamma_{D^+})$. When Γ is torsion-free, $\mathscr{N}_{D^-,D^+}(s)$ is the number of the common perpendiculars of length at most s between the images of D^- and D^+ in M, with multiplicities coming from the fact that $\Gamma_{D^\pm}\backslash D^\pm$ is not assumed to be embedded in M. We refer to [PP4, §4] for the use of Hölder-continuous potentials on $T^1\widetilde{M}$ to modify this counting function by adding weights.

Recall the following notions (see for instance [Rob2]). The *critical exponent* of Γ is

$$\delta_\Gamma = \limsup_{N\to+\infty} \frac{1}{N} \ln \mathrm{Card}\{\gamma \in \Gamma : d(x_0, \gamma x_0) \le N\}\,,$$

which is positive, finite, independent of x_0 (and equal to the topological entropy h if Γ is cocompact and torsion-free). Let $(\mu_x)_{x\in\widetilde{M}}$ be a *Patterson-Sullivan density* for Γ, that is, a family $(\mu_x)_{x\in\widetilde{M}}$ of nonzero finite measures on $\partial_\infty\widetilde{M}$ whose support is $\Lambda\Gamma$, such that $\gamma_*\mu_x = \mu_{\gamma x}$ and

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-\delta_\Gamma \beta_\xi(x,y)}$$

for all $\gamma \in \Gamma$, $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M}$. The *Bowen-Margulis measure* $\widetilde{m}_{\mathrm{BM}}$ for Γ on $T^1\widetilde{M}$ is defined, using Hopf's parametrisation $v \mapsto (v_-, v_+, \beta_{v_+}(x_0, \pi(v)))$ of $T^1\widetilde{M}$, by

$$d\widetilde{m}_{\mathrm{BM}}(v) = e^{-\delta_\Gamma(\beta_{v_-}(\pi(v),x_0)+\beta_{v_+}(\pi(v),x_0))}\, d\mu_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt\,.$$

The measure $\widetilde{m}_{\mathrm{BM}}$ is nonzero and independent of x_0. It is invariant under the geodesic flow, the antipodal map $\iota : v \mapsto -v$ and the action of Γ, and thus defines a nonzero measure m_{BM} on $T^1M = \Gamma\backslash T^1\widetilde{M}$, called the *Bowen-Margulis measure* on M, which is invariant under the geodesic flow of M and the antipodal map. If $\widetilde{M}$ is symmetric and if

Γ is geometrically finite (for instance if $\widetilde{M} = \mathbb{H}^2_{\mathbb{R}}$ and Γ is finitely generated), then m_{BM} is finite. See for instance [DOP] for many more examples. If m_{BM} is finite, then m_{BM} is mixing under the geodesic flow if $\widetilde{M}$ is symmetric or if $\Lambda\Gamma$ is not totally disconnected (hence if M is compact), see for instance [Bab1, Dal].

Using the endpoint homeomorphisms $v \mapsto v_\pm$ from $\partial^1_\pm D^\mp$ to $\partial_\infty \widetilde{M} - \partial_\infty D^\mp$, the *skinning measure* $\widetilde{\sigma}_{D^\mp}$ of Γ on $\partial^1_\pm D^\mp$ is defined by

$$d\widetilde{\sigma}_{D^\mp}(v) = e^{-\delta\, \beta_{v_\pm}(\pi(v),\, x_0)}\, d\mu_{x_0}(v_\pm)\,,$$

see [OhS1, §1.2] when $D^\mp$ is a horoball or a totally geodesic subspace in $\widetilde{M}$ and [PP2], [PP4] for the general case of convex subsets in variable curvature and with a potential.

The measure $\widetilde{\sigma}_{D^\mp}$ is independent of $x_0 \in \widetilde{M}$, it is nonzero if $\Lambda\Gamma$ is not contained in $\partial_\infty D^\mp$, and satisfies $\widetilde{\sigma}_{\gamma D^\mp} = \gamma_* \widetilde{\sigma}_{D^\mp}$ for every $\gamma \in \Gamma$. Since the family $(\gamma D^\mp)_{\gamma \in \Gamma/\Gamma_{D^\mp}}$ is locally finite in $\widetilde{M}$, the measure $\sum_{\gamma \in \Gamma/\Gamma_{D^\mp}} \gamma_* \widetilde{\sigma}_{D^\mp}$ is a well defined Γ-invariant locally finite (nonnegative Borel) measure on $T^1\widetilde{M}$, hence it induces a locally finite measure $\sigma_{D^\mp}$ on T^1M, called the *skinning measure* of $D^\mp$ in T^1M. If $\Gamma_{D^\mp}\backslash \partial D^\mp$ is compact, then $\sigma_{D^\mp}$ is finite. We refer to [OhS2, §5] and [PP2, Theo. 9] for finiteness criteria of the skinning measure $\sigma_{D^\mp}$.

The following result on the asymptotic behaviour of the counting function $\mathscr{N}_{D^-,\, D^+}$ is a special case of more general results [PP4, Coro. 20, 21, Theo. 28]. We refer to [PP3] for a survey of the particular cases known before [PP4], due to Huber, Margulis, Herrmann, Cosentino, Roblin, Eskin-McMullen, Oh-Shah, Martin-McKee-Wambach, Pollicott, and the authors for instance.

Theorem 7 *Let Γ, D^-, D^+ be as above. Assume that the measures $m_{\mathrm{BM}}, \sigma_{D^-}, \sigma_{D^+}$ are nonzero and finite, and that m_{BM} is mixing for the geodesic flow of T^1M. Then*

$$\mathscr{N}_{D^-,\, D^+}(s) \;\sim\; \frac{\|\sigma_{D^-}\|\, \|\sigma_{D^+}\|}{\delta_\Gamma\, \|m_{\mathrm{BM}}\|}\, e^{\delta_\Gamma\, s}\,,$$

as $s \to +\infty$. If Γ is arithmetic or if M is compact, then the error term is $\mathrm{O}(e^{(\delta_\Gamma - \kappa)s})$ for some $\kappa > 0$. Furthermore, the initial vectors of the common perpendiculars equidistribute in the outer unit normal bundle of D^-:

$$\lim_{s\to+\infty} \frac{\delta_\Gamma\, \|m_{\mathrm{BM}}\|}{\|\sigma_{D^-}\|\, \|\sigma_{D^+}\|}\, e^{-\delta_\Gamma\, s} \sum_{\substack{[\gamma] \in \Gamma_{D^-}\backslash\Gamma/\Gamma_{D^+} \\ \overline{D^-} \cap \overline{\gamma D^+} = \emptyset,\ \ell(\alpha_{e,\,\gamma}) \le s}} m_{e,\,\gamma}\, \Delta_{v^-_{e,\,\gamma}} \;=\; \frac{\widetilde{\sigma}_{D^-}}{\|\sigma_{D^-}\|} \qquad (1)$$

for the weak-star convergence of measures on the locally compact space $T^1\widetilde{M}$. □

4 Counting in conjugacy classes

Let $\widetilde{M}, x_0, \Gamma$ be as in the beginning of Section 3. For any nontrivial element γ in Γ, let C_γ be

- the translation axis of γ if γ is loxodromic,
- the fixed point set of γ if γ is elliptic,

- a horoball centered at the fixed point of γ if γ is parabolic,

which is a nonempty proper closed convex subset of $\widetilde{M}$. We assume (this condition is automatically satisfied unless γ' is parabolic) that $\gamma C_{\gamma'} = C_{\gamma\gamma'\gamma^{-1}}$ for all $\gamma \in \Gamma$ and $\gamma' \in \Gamma - \{e\}$.

By the equivariance properties of the skinning measures, the total mass of σ_{C_γ} depends only on the conjugacy class $\mathfrak{K}$ of γ, and will be denoted by $\|\sigma_{\mathfrak{K}}\|$. This quantity, called the *skinning measure* of $\mathfrak{K}$, is positive and finite for instance when γ is loxodromic, since $\Lambda\Gamma$ contains at least 3 points and the image of C_γ in M is compact. In Sections 6 and 7, we will give other classes of examples of conjugacy classes $\mathfrak{K}$ with positive and finite skinning measure $\|\sigma_{\mathfrak{K}}\|$, and prove in particular that this is always true if $\widetilde{M} = \mathbb{H}^2_{\mathbb{R}}$ except possibly when γ is elliptic and orientation-reversing.

We define

$$m_\gamma = \frac{1}{\mathrm{Card}(\Gamma_{x_0} \cap \Gamma_{C_\gamma})}\,,$$

which is a natural complexity of γ, independent on the choice of C_γ when γ is parabolic, and equals 1 if the stabiliser of x_0 in Γ is trivial (for instance if Γ is torsion-free). Note that for every $\alpha \in \Gamma$, the real number $m_{\alpha\gamma\alpha^{-1}}$ depends only on the double coset of α in $\Gamma_{x_0}\backslash\Gamma/\Gamma_{C_\gamma}$.

The centraliser $Z_\Gamma(\gamma)$ of γ in Γ is contained in the stabiliser of C_γ in Γ. The index $[\Gamma_{C_\gamma} : Z_\Gamma(\gamma)]$ depends only on the conjugacy class $\mathfrak{K}$ of γ; it will be denoted by $i_{\mathfrak{K}}$ and called the *index* of $\mathfrak{K}$. We assume in what follows that $i_{\mathfrak{K}}$ is finite, which is in particular the case if γ is loxodromic (the stabiliser of its translation axis C_γ is virtually cyclic). In Sections 6 and 7, we will give other classes of examples of conjugacy classes $\mathfrak{K}$ with finite index $i_{\mathfrak{K}}$, and prove in particular that this is always true if $\widetilde{M} = \mathbb{H}^2_{\mathbb{R}}$.

We define the counting function

$$N_{\mathfrak{K},\,x_0}(t) = \sum_{\alpha\in\mathfrak{K},\ d(x_0,\alpha x_0)\le t} m_\alpha\,.$$

When the stabiliser of x_0 in Γ is trivial, we recover the definition of the Introduction.

Let $\psi : [0,+\infty[\to [0,+\infty[$ be an eventually nondecreasing map such that $\lim_{t\to+\infty}\psi(t) = +\infty$. We will say that a nontrivial element $\gamma_0 \in \Gamma$ is *ψ-equitranslating* if for every $x \in \widetilde{M}$ at distance big enough from C_{γ_0}, we have

$$d(x, C_{\gamma_0}) = \psi(d(x, \gamma_0 x))\,.$$

Note that this condition depends only on the conjugacy class of γ_0. When γ_0 is parabolic, up to replacing ψ by $\psi + c$ for some constant $c \in \mathbb{R}$, this condition does not depend on the choice of the horoball C_{γ_0}. In Sections 5, 6 and 7, we will give several classes of examples of equitranslating isometries, and prove in particular that every nontrivial isometry of $\mathbb{H}^2_{\mathbb{R}}$ is equitranslating.

The following theorem is the main abstract result of this paper.

Theorem 8 *Assume that the Bowen-Margulis measure of Γ is finite and mixing for the geodesic flow on T^1M. Let $\mathfrak{K}$ be a conjugacy class of ψ-equitranslating elements of Γ with finite index $i_{\mathfrak{K}}$ and positive and finite skinning measure $\|\sigma_{\mathfrak{K}}\|$. Then, as $t \to +\infty$,*

$$N_{\mathfrak{K},\,x_0}(t) \sim \frac{i_{\mathfrak{K}}\,\|\mu_{x_0}\|\,\|\sigma_{\mathfrak{K}}\|}{\delta_\Gamma\,\|m_{\mathrm{BM}}\|}\,e^{\delta_\Gamma\,\psi(t)}\,.$$

If Γ is arithmetic or if M is compact, then the error term is $\mathrm{O}(e^{(\delta_\Gamma-\kappa)\psi(t)})$ for some $\kappa>0$. Furthermore, if v_α is the unit tangent vector at x_0 to the geodesic segment $[x_0,\alpha x_0]$ for every $\alpha\in\Gamma-\Gamma_{x_0}$, for the weak-star convergence of measures on $T^1\widetilde{M}$, we have

$$\lim_{t\to+\infty}\ \frac{\delta_\Gamma\ \|m_{\mathrm{BM}}\|}{i_{\mathfrak{K}}\,\|\sigma_{\mathfrak{K}}\|}\ e^{-\delta_\Gamma\,\psi(t)}\sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,\alpha x_0)\le t} m_\alpha\,\Delta_{v_\alpha}\ =\ (\pi_+^{-1})_*\mu_{x_0}\ .$$

Proof. Let γ_0 be a ψ-equitranslating element of $\Gamma-\{e\}$ and let $\mathfrak{K}=\{\gamma\gamma_0\gamma^{-1}\ :\ \gamma\in\Gamma\}$ be its conjugacy class. Since $\widetilde{\sigma}_{\{x_0\}}=(\pi_+^{-1})_*\mu_{x_0}$ (see [PP2, §3]), we have

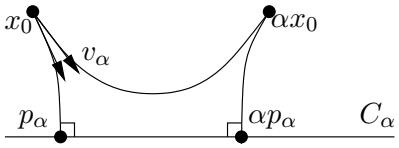
$$\|\sigma_{\{x_0\}}\|=\frac{\|\mu_{x_0}\|}{|\Gamma_{x_0}|}\,.$$

In particular, both $\|\sigma_{\{x_0\}}\|$ and $\|\sigma_{\mathscr{C}_{\gamma_0}}\|=\|\sigma_{\mathfrak{K}}\|$, are positive and finite. Hence, since ψ is eventually nondecreasing, by the definition of the counting function $\mathscr{N}_{D^-,\,D^+}$ for $D^-=\{x_0\}$ and $D^+=C_{\gamma_0}$, and by Theorem 7, we have, as $t\to+\infty$,

$$\begin{aligned}\sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,\alpha x_0)\le t} m_\alpha &\sim \sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,C_\alpha)\le\psi(t)} m_\alpha = \sum_{\gamma\in\Gamma/Z_\Gamma(\gamma_0),\ 0<d(x_0,\,\gamma C_{\gamma_0})\le\psi(t)} m_{\gamma\gamma_0\gamma^{-1}}\\
&= |\Gamma_{x_0}|\,i_{\mathfrak{K}}\sum_{\gamma\in\Gamma_{x_0}\backslash\Gamma/\Gamma_{C_{\gamma_0}},\ 0<d(x_0,\,\gamma C_{\gamma_0})\le\psi(t)} m_{\gamma\gamma_0\gamma^{-1}}\\
&= |\Gamma_{x_0}|\,i_{\mathfrak{K}}\,\mathscr{N}_{\{x_0\},\,C_{\gamma_0}}(\psi(t))\\
&\sim |\Gamma_{x_0}|\,i_{\mathfrak{K}}\,\frac{\|\sigma_{\{x_0\}}\|\ \|\sigma_{C_{\gamma_0}}\|}{\delta_\Gamma\ \|m_{\mathrm{BM}}\|}\ e^{\delta_\Gamma\,\psi(t)}\ .\end{aligned}$$

The first claim of Theorem 8 follows.

For every $\alpha\in\mathfrak{K}$, let p_α be the closest point to x_0 on C_α. Then αp_α is the closest point to αx_0 on C_α.



Since $\lim_{t\to+\infty}\psi(t)=+\infty$, when $d(x_0,\alpha x_0)$ is large enough, the distance $d(x_0,C_\alpha)$ becomes large. Hence the initial tangent vector v_α to the geodesic segment $[x_0,\alpha x_0]$ becomes arbitrarily close to the initial tangent vector to the geodesic segment $[x_0,p_\alpha]$, uniformly on $\alpha\in\mathfrak{K}$ such that $d(x_0,C_\alpha)$ is large enough, and independently on $d(p_\alpha,\alpha p_\alpha)$ which could be 0. Hence, using again and similarly Theorem 7 with $D^-=\{x_0\}$ and $D^+=C_{\gamma_0}$, we have, as $t\to+\infty$,

$$\begin{aligned}&\frac{\delta_\Gamma\ \|m_{\mathrm{BM}}\|}{i_{\mathfrak{K}}\,\|\sigma_{\mathfrak{K}}\|}\ e^{-\delta_\Gamma\,\psi(t)}\sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,\alpha x_0)\le t} m_\alpha\ \Delta_{v_\alpha}\\
\sim\ &\frac{\delta_\Gamma\ \|m_{\mathrm{BM}}\|}{i_{\mathfrak{K}}\,\|\sigma_{\mathfrak{K}}\|}\ e^{-\delta_\Gamma\,\psi(t)}\sum_{\gamma\in\Gamma/Z_\Gamma(\gamma_0),\ 0<d(x_0,\,\gamma C_{\gamma_0})\le\psi(t)} m_{\gamma\gamma_0\gamma^{-1}}\ \Delta_{v_{\gamma\gamma_0\gamma^{-1}}}\\
\sim\ &\frac{\delta_\Gamma\ \|m_{\mathrm{BM}}\|}{\|\sigma_{D^+}\|}\ e^{-\delta_\Gamma\,\psi(t)}\sum_{\gamma\in\Gamma/\Gamma_{D^+},\ 0<d(x_0,\,\gamma D^+)\le\psi(t)} m_{e,\,\gamma}\ \Delta_{v^-_{e,\,\gamma}}\\
\overset{*}{\rightharpoonup}\ &\widetilde{\sigma}_{\{x_0\}}=(\pi_+^{-1})_*\mu_{x_0}\ .\end{aligned}$$

This proves the second claim of Theorem 8. □

5 The geometry of loxodromic isometries

In this section, we fix a loxodromic isometry γ of a complete CAT(-1) geodesic metric space X. Let $\ell = \ell_\gamma = \inf_{x \in X} d(x, \gamma x) > 0$ be its translation length and let

$$C_\gamma = \{x \in X \ : \ d(x, \gamma x) = \ell\}$$

be its translation axis.

If $X = \mathbb{H}^2_{\mathbb{R}}$, if γ is orientation-preserving, and if $x \in \mathbb{H}^2_{\mathbb{R}}$ is at a distance s from the translation axis of γ, then

$$d(x, \gamma x) = 2 \operatorname{argsinh}(\cosh s \, \sinh \frac{\ell}{2}) \, . \tag{2}$$

Indeed, after a conjugation by a suitable isometry, we may assume that the translation axis of γ is the geodesic line with endpoints 0 and ∞ in the upper halfplane model of $\mathbb{H}^2_{\mathbb{R}}$, that $\gamma z = e^\ell z$ for all $z \in \mathbb{C}$ with Im $z > 0$, and that x is on the geodesic ray starting from i and ending at 1. Using the angle of parallelism formula [Bea, Thm. 7.9.1], we have $x = (\tanh s, \frac{1}{\cosh s})$, which gives $\gamma x = e^\ell (\tanh s, \frac{1}{\cosh s})$. From this, Equation (2) follows using the hyperbolic distance formula [Bea, Thm. 7.2.1 (iii)].

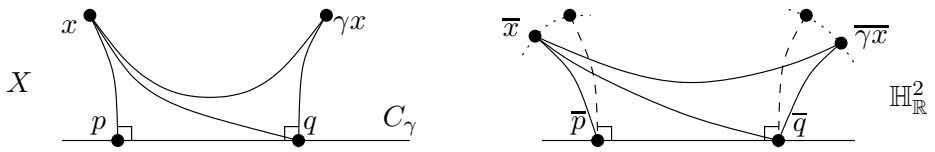
In the other extreme, if X is a tree and if $x \in X$ is at a distance s from the translation axis of γ, then $d(x, \gamma x) = \ell + 2s$. The general situation lies between these two cases.

Lemma 9 *If $x \in X$ is at distance s from the translation axis of γ, then*

$$2 \operatorname{argsinh}(\cosh s \sinh \frac{\ell}{2}) \leq d(x, \gamma x) \leq 2s + \ell \, .$$

Note that as $s \to +\infty$, the lower bound is equal to $2s + 2 \ln(\sinh \frac{\ell}{2}) + \mathrm{O}(e^{-2s})$, hence the difference of the upper and lower bounds is bounded by a constant that only depends on ℓ.

Proof. The upper bound follows from the triangle inequality. Let us prove the lower bound. Let p and $q = \gamma p$ be the closest points on C_γ to respectively x and γx. Let Q be the quadrilateral in X with vertices x, p, q and γx.



Let $\overline{Q}$ be the quadrilateral in $\mathbb{H}^2_{\mathbb{R}}$ with vertices $\overline{x}$, $\overline{p}$, $\overline{q}$ and $\overline{\gamma x}$, obtained by gluing together, along the geodesic segment $[\overline{x}, \overline{q}]$, the comparison triangles of the two triangles in X with sets of vertices $\{x, p, q\}$ and $\{x, q, \gamma x\}$. By comparison, the angles of $\overline{Q}$ at the vertices $\overline{p}$ and $\overline{q}$ are at least $\frac{\pi}{2}$. If we adjust these angles to $\frac{\pi}{2}$, keeping the lengths of the three sides $[\overline{x}, \overline{p}]$, $[\overline{p}, \overline{q}]$ and $[\overline{q}, \overline{\gamma x}]$ fixed, we obtain a quadrilateral $\overline{Q}'$ where the side that is not adjacent to the right angles has length less than $d(x, \gamma x)$. This gives the lower bound since the length of the side in question is given by Equation (2). □

The proof of the following result is then similar to that of Theorem 8.

Corollary 10 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with pinched negative sectional curvature, let $x_0 \in \widetilde{M}$ and let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Assume that the Bowen-Margulis measure of Γ is finite and mixing for the geodesic flow on T^1M. Let $\mathfrak{K}$ be a conjugacy class of loxodromic elements of Γ with translation length ℓ. Then, for every $\epsilon > 0$, if t is big enough,*

$$\frac{i_{\mathfrak{K}} \, \|\mu_{x_0}\| \, \|\sigma_{\mathfrak{K}}\|}{\delta_\Gamma \, \|m_{\mathrm{BM}}\| \, e^{\frac{\delta_\Gamma \, \ell}{2}}} \, e^{\frac{\delta_\Gamma}{2} t} \, (1-\epsilon) \leq N_{\mathfrak{K},\, x_0}(t) \leq \frac{i_{\mathfrak{K}} \, \|\mu_{x_0}\| \, \|\sigma_{\mathfrak{K}}\|}{\delta_\Gamma \, \|m_{\mathrm{BM}}\| \, (\sinh \frac{\ell}{2})^{\delta_\Gamma}} \, e^{\frac{\delta_\Gamma}{2} t} \, (1+\epsilon) \, . \quad \square$$

In particular, under the assumptions of this result, we have

$$\lim_{t \to +\infty} \frac{1}{t} \ln N_{\mathfrak{K},\, x_0}(t) = \frac{\delta_\Gamma}{2} \, .$$

Theorem 1 in the introduction follows from this, since if $M = \Gamma \backslash \widetilde{M}$ is a compact manifold, then any nontrivial element in Γ is loxodromic, and, as recalled in Section 4, the critical exponent δ_Γ is equal to the topological entropy h of the geodesic flow on M, and m_{BM} is finite and mixing.

Remark. With the notation and definitions of [PPS, §3.1], if $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a potential (that is, a Γ-invariant Hölder-continuous map), since the geodesic segment from x_0 to αx_0 passes at distance uniformly bounded (by a constant c_ℓ depending only on ℓ) from the translation axis C_α of α, with p_α the closest point to x_0 on C_α, the absolute value of the difference $\int_{x_0}^{\alpha x_0} \widetilde{F} - \int_{x_0}^{p_\alpha} \widetilde{F} - \int_{x_0}^{p_\alpha} \widetilde{F} \circ \iota$ is uniformly bounded (by a constant depending only on c_ℓ and on the maximum of $\widetilde{F}$ on the neighbourhood of C_α of radius c_ℓ). Hence using the version with potential of Theorem 7 in [PP4, Coro. 20] for $\widetilde{F}$ and $\widetilde{F} \circ \iota$, we have upper and lower bounds for the asymptotic of the counting function with weigths defined by the potential: Assume that the critical exponent $\delta_{\Gamma,\, F}$ of Γ for the potential $\widetilde{F}$ is finite and that the Gibbs measure of Γ for the potential $\widetilde{F}$ is finite and mixing for the geodesic flow on T^1M, then there exists $c > 0$ such that for all $t \geq 0$,

$$\frac{1}{c} \, e^{\frac{\delta_{\Gamma,\, F}}{2} t} \leq \sum_{\alpha \in \mathfrak{K},\, d(x_0, \alpha x_0) \leq t} m_\alpha \, e^{\int_{x_0}^{\alpha x_0} \widetilde{F}} \leq c \, e^{\frac{\delta_{\Gamma,\, F}}{2} t} \, .$$

Let us now consider the higher dimensional real hyperbolic spaces. If $X = \mathbb{H}^3_{\mathbb{R}}$, if γ is orientation-preserving, and if $x \in \mathbb{H}^3_{\mathbb{R}}$ is at a distance s from the translation axis of γ, then

$$\sinh^2 \frac{d(x, \gamma x)}{2} = \frac{\sinh^2 s \, |e^\lambda - 1|^2}{4 \, e^\ell} + \sinh^2(\frac{\ell}{2}) \, , \tag{3}$$

where $\lambda = \lambda_\gamma$ is the *complex translation length* of γ, defined as follows. The loxodromic isometry γ is conjugated in the upper halfspace model $\mathbb{C} \times]0, +\infty[$ of $\mathbb{H}^3_{\mathbb{R}}$ to a transformation $(z, r) \mapsto e^\ell (e^{i\theta} z, r)$, where $\theta = \theta_\gamma \in \mathbb{R}$ is uniquely defined modulo 2π, and we define $\lambda = \ell + i\theta \in]0, +\infty[\, + \, i \, \mathbb{R}/2\pi\mathbb{Z}$. Equation (3) follows from the distance formula in [Fen, pp. 37].

Let $n \in \mathbb{N} - \{0, 1\}$. A loxodromic isometry γ of $\mathbb{H}^n_{\mathbb{R}}$ is *uniformly rotating* if γ rotates all normal vectors to the translation axis of γ by the same angle, called the *rotation angle* of γ (which is 0 if and only if γ induces the parallel transport along its translation axis). This property is invariant under conjugation.

Clearly, all loxodromic isometries of $\mathbb{H}^2_{\mathbb{R}}$, all orientation-preserving loxodromic isometries $\mathbb{H}^3_{\mathbb{R}}$, and the loxodromic isometries of any $\mathbb{H}^n_{\mathbb{R}}$ with a trivial rotational part, are uniformly rotating. The orientation-reversing loxodromic isometries of $\mathbb{H}^3_{\mathbb{R}}$ are not uniformly rotating. More generally, by the normal form up to conjugation of the elements of $\mathrm{O}(n-1)$, uniformly rotating orientation-preserving loxodromic isometries with a nontrivial rotation angle exist in $\mathbb{H}^n_{\mathbb{R}}$ if and only if n is odd, and uniformly rotating orientation-reversing loxodromic isometries exist in $\mathbb{H}^n_{\mathbb{R}}$ if and only if n is even. For a fixed translation length and rotation angle $\theta \in (\mathbb{R} - 2\pi\mathbb{Z})/(2\pi\mathbb{Z})$, with $\theta = \pi$ in the orientation-reversing case, these elements form a unique conjugacy class.

Let γ be a uniformly rotating loxodromic isometry of $\mathbb{H}^n_{\mathbb{R}}$. Any configuration that consists of the translation axis of γ, a geodesic line L orthogonal to the axis and its image γL is contained in an isometrically embedded γ-invariant copy of $\mathbb{H}^3_{\mathbb{R}}$ in $\mathbb{H}^n_{\mathbb{R}}$ (unique if the rotation angle of γ is nonzero modulo $\pi\mathbb{Z}$). We then define the *complex translation length* of γ as the complex translation length of the restriction of γ to this subspace.

Lemma 11 *A uniformly rotating loxodromic isometry γ of $\mathbb{H}^n_{\mathbb{R}}$ with complex translation length $\lambda = \ell + i\theta$ is ψ-equitranslating with*

$$\psi(t) = \frac{1}{2}(t - \ln(\frac{\cosh \ell - \cos \theta}{2})) + \mathrm{O}(e^{-t})$$

as $t \to +\infty$.

Proof. Let x be a point in $\mathbb{H}^n_{\mathbb{R}}$ at a distance s from the translation axis of γ. We only have to prove that, as $s \to +\infty$,

$$d(x, \gamma x) = 2s + \ln\big(\frac{\cosh \ell - \cos \theta}{2}\big) + \mathrm{O}(e^{-2s})\,.$$

As noted above, it suffices to consider the case $n = 3$. By Equation (3), we have

$$\frac{e^{d(x,\,\gamma x)}}{4} = \frac{e^{2\,s}\,(e^{2\ell} - 2e^{\ell}\cos\theta + 1)}{16\,e^{\ell}} + \mathrm{O}(1)\,,$$

as $s \to +\infty$, which proves the claim after simplification and taking the logarithm. □

Corollary 12 *Let Γ be a nonelementary discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$, whose Bowen-Margulis measure is finite, and let $x_0 \in \mathbb{H}^n_{\mathbb{R}}$. Let $\mathfrak{K}$ be a conjugacy class of uniformly rotating loxodromic elements of Γ with complex translation length $\lambda = \ell + i\theta$. Then, as $t \to +\infty$,*

$$N_{\mathfrak{K},\,x_0}(t) \sim \frac{2^{\frac{\delta_\Gamma}{2}}\; i_{\mathfrak{K}}\, \|\mu_{x_0}\|\, \|\sigma_{\mathfrak{K}}\|}{\delta_\Gamma\, \|m_{\mathrm{BM}}\|\, (\cosh \ell - \cos\theta)^{\frac{\delta_\Gamma}{2}}}\; e^{\frac{\delta_\Gamma}{2}\,t}\,.$$

If Γ is arithmetic or if M is compact, then the error term is $\mathrm{O}(e^{(\frac{\delta_\Gamma}{2}-\kappa)t})$ for some $\kappa > 0$. Furthermore, if v_α is the unit tangent vector at x_0 to the geodesic segment $[x_0, \alpha x_0]$ for every $\alpha \in \Gamma - \Gamma_{x_0}$, for the weak-star convergence of measures on $T^1\widetilde{M}$, we have

$$\lim_{t\to+\infty} \frac{\delta_\Gamma\, \|m_{\mathrm{BM}}\|\, (\cosh \ell - \cos\theta)^{\frac{\delta_\Gamma}{2}}}{2^{\frac{\delta_\Gamma}{2}}\; i_{\mathfrak{K}}\, \|\sigma_{\mathfrak{K}}\|}\; e^{-\frac{\delta_\Gamma}{2}\,t} \sum_{\alpha\in\mathfrak{K},\; 0<d(x_0,\,\alpha x_0)\le t} m_\alpha\, \Delta_{v_\alpha} \;=\; (\pi_+^{-1})_*\mu_{x_0}\,.$$

Proof. As mentioned in Section 3, since $\mathbb{H}^n_{\mathbb{R}}$ has constant sectional curvature, the Bowen-Margulis measure of Γ, since finite, is mixing for the geodesic flow on T^1M. We have already seen that $i_{\mathfrak{K}}$ is finite and that $\|\sigma_{\mathfrak{K}}\|$ is positive and finite. The result follows from Theorem 8 and Lemma 11. □

Remark. Let Γ be a group of isometries of X and assume that γ is a loxodromic element of Γ. The element γ is Γ-*reciprocal* if there exists an element in Γ that switches the two fixed points of γ. If γ is reciprocal, then let $\iota_\Gamma(\gamma) = 2$, otherwise, we set $\iota_\Gamma(\gamma) = 1$. The stabiliser in Γ of the translation axis C_γ of γ is generated by the maximal cyclic subgroup of Γ containing γ, by an elliptic element that switches the two points at infinity of C_γ if γ is Γ-reciprocal, and a (possibly trivial) group of finite order, which is the pointwise stabiliser $\mathrm{Fix}_\Gamma(C_\gamma)$ of C_γ. Thus, if $\mathfrak{K}$ is the conjugacy class of γ in Γ,

$$\iota_{\mathfrak{K}} = \iota_\Gamma(\gamma)[\mathrm{Fix}_\Gamma(C_\gamma) : \mathrm{Fix}_\Gamma(C_\gamma) \cap Z_\Gamma(\gamma)] \,.$$

In particular, if $n = 2$, or if $n = 3$ and γ preserves the orientation, then $\iota_{\mathfrak{K}} = \iota_\Gamma(\gamma)$. Hence Theorem 2 in the Introduction when γ_0 is loxodromic follows from Corollary 12.

When Γ has finite covolume, the constant in Corollary 12 can be made more explicit.

Corollary 13 *Let Γ be a discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$ with finite covolume and let $x_0 \in \mathbb{H}^n_{\mathbb{R}}$. Let $\mathfrak{K}$ be the conjugacy class of a uniformly rotating loxodromic element γ_0 of Γ with complex translation length $\lambda = \ell + i\theta$, let m_{γ_0} be the order of γ_0 in the maximal cyclic group containing γ_0, and let n_{γ_0} be the order of the intersection of the pointwise stabiliser of the translation axis of γ_0 with the centraliser of γ_0. Then, as $t \to +\infty$,*

$$N_{\mathfrak{K},\, x_0}(t) \sim \frac{\mathrm{Vol}(\mathbb{S}^{n-2})\ \ell}{2^{\frac{n-1}{2}}\,(n-1)\,m_{\gamma_0}\,n_{\gamma_0}\ \mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})\,(\cosh\ell - \cos\theta)^{\frac{n-1}{2}}}\ e^{\frac{n-1}{2}\,t}\,.$$

If Γ is arithmetic or if M is compact, then the error term is $\mathrm{O}(e^{(\frac{n-1}{2}-\kappa)t})$ for some $\kappa > 0$. Furthermore, if Γ_{x_0} is trivial, if v_α is the unit tangent vector at x_0 to the geodesic segment $[x_0, \alpha x_0]$ for every $\alpha \in \Gamma - \{e\}$, with $\mathrm{Vol}_{T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}}$ the spherical measure on $T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}$, we have, for the weak-star convergence of measures on $T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}$,

$$\frac{(n-1)\,m_{\gamma_0}\,n_{\gamma_0}\ \mathrm{Vol}(\mathbb{S}^{n-1})\ \mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})\,(\cosh\ell - \cos\theta)^{\frac{n-1}{2}}}{2^{\frac{1-n}{2}}\ \mathrm{Vol}(\mathbb{S}^{n-2})\,\ell\ e^{\frac{n-1}{2}\,t}} \sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,\alpha x_0)\le t} \Delta_{v_\alpha}$$
$$\stackrel{*}{\rightharpoonup}\ \mathrm{Vol}_{T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}}\ .$$

Proof. Since Γ has finite covolume, we have $\delta_\Gamma = n-1$ and we can normalise the Patterson-Sullivan measure μ_{x_0} at x_0 to have total mass $\mathrm{Vol}(\mathbb{S}^{n-1})$, so that $(\pi_+^{-1})_*\mu_{x_0} = \mathrm{Vol}_{T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}}$. By [PP3, Prop. 10, 11], we have

$$\|m_{\mathrm{BM}}\| = 2^{n-1}\,\mathrm{Vol}(\mathbb{S}^{n-1})\,\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})$$

and

$$\|\sigma_{C_{\gamma_0}}\| = \mathrm{Vol}(\mathbb{S}^{n-2})\frac{\ell}{|\,\mathrm{Fix}_{\Gamma_0}(C_{\gamma_0})|\,\iota_\Gamma(\gamma_0)\,m_{\gamma_0}}\,,$$

since $\mathrm{Vol}(\Gamma_{C_{\gamma_0}}\backslash C_{\gamma_0}) = \frac{\ell}{\iota_\Gamma(\gamma_0)\,m_{\gamma_0}}$. The claims hence follow from the previous remark and from Corollary 12. □

The proof of the loxodromic case of Corollary 3 of the Introduction follows from Corollary 13 by taking $n = 2$, Γ torsion-free (so that $n_{\gamma_0} = 1$), and γ_0 primitive (so that $m_{\gamma_0} = 1$) and orientation-preserving (so that $\cos\theta = 1$). The area of a complete, connected, finite area hyperbolic surface with genus g and p punctures is $2\pi(2g + p - 2)$.

6 The geometry of parabolic isometries

In this section, we fix a parabolic isometry γ of a complete CAT(-1) geodesic metric space X. We fix a horoball C_γ centred at the fixed point of γ, and we call *horospherical translation length* of γ the quantity

$$\ell = \ell_\gamma = \inf_{y \in \partial C_\gamma} d(y, \gamma y) \, .$$

We will say that γ is *uniformly translating* if $d(y, \gamma y)$ is independent of $y \in \partial C_\gamma$. Note that being uniformly translating does not depend on the choice of C_γ, but the value of ℓ does (and can be fixed arbitrarily in $]0, +\infty[$ when X is a Riemannian manifold).

Every parabolic isometry of $X = \mathbb{H}^2_{\mathbb{R}}, \mathbb{H}^3_{\mathbb{R}}$ is uniformly translating, but using Euclidean screw motions, there exist parabolic isometries in $X = \mathbb{H}^4_{\mathbb{R}}$ which are not uniformly translating (and the map $y \mapsto d(y, \gamma y)$ is not even bounded). If $X = \mathbb{H}^n_{\mathbb{R}}$ and if γ induces a Euclidean translation on ∂C_γ, then γ is uniformly translating. Recall that by Bieberbach's theorem, any discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$, preserving a given horosphere and acting cocompactly on it, contains a finite index subgroup consisting of uniformly translating parabolic isometries and the identity.

If $X = \mathbb{H}^2_{\mathbb{R}}$, if $x \in X$ is at a distance s from the horoball C_γ, then

$$d(x, \gamma x) = 2 \operatorname{argsinh}(e^s \sinh \frac{\ell}{2}) \, . \tag{4}$$

This is immediate by considering the upper halfplane model and assuming that γ has fixed point ∞, by applying twice [Bea, Thm. 7.2.1 (iii)]. A similar triangle inequality and comparison argument as in the proof of Lemma 9 shows the following result.

Lemma 14 *If $x \in X$ is at distance $s > 0$ from the horoball C_γ, if p_γ is the closest point to x on C_γ, then*

$$2 \operatorname{argsinh}(e^s \sinh \frac{\ell}{2}) \leq d(x, \gamma x) \leq 2s + d(p_\gamma, \gamma \, p_\gamma) \, . \ \square$$

Corollary 15 *A uniformly translating parabolic isometry γ of $\mathbb{H}^n_{\mathbb{R}}$ with horospherical translation length ℓ is ψ-equitranslating with*

$$\psi(t) = \frac{t}{2} - \ln(\sinh \frac{\ell}{2}) - \ln 2 + \mathrm{O}(e^{-t})$$

as $t \to +\infty$.

Proof. Let x be a point in $\mathbb{H}^n_{\mathbb{R}}$ at a distance s from the horoball C_γ. We only have to prove that, as $s \to +\infty$,

$$d(x, \gamma x) = 2s + 2 \ln \big(\sinh \frac{\ell}{2} \big) + 2 \ln 2 + \mathrm{O}(e^{-2s}) \, .$$

It suffices to consider the case $n = 2$ (the points $x, \gamma x$ and the fixed point of γ are contained in a copy of $\mathbb{H}^2_{\mathbb{R}}$), in which case the result follows from Equation (4). □

Remark. If $X = \widetilde{M}$ and Γ are as in Section 4, if γ is a parabolic isometry of Γ and if $\mathfrak{K}$ is the conjugacy class of γ in Γ, the quantities $\|\sigma_{\mathfrak{K}}\|$ and $i_{\mathfrak{K}}$ defined in that Section are not always finite. Note that $\|\sigma_{\mathfrak{K}}\|$ is positive, since Γ is nonelementary.

- The mass $\|\sigma_{\mathfrak{K}}\|$ is finite for instance if the fixed point ξ_γ of γ is a bounded parabolic fixed point (that is, if its stabiliser Γ_{ξ_γ} in Γ acts cocompactly on $\Lambda\Gamma - \{\xi_\gamma\}$), which is in particular the case if Γ is a lattice or is geometrically finite.
- The index $i_{\mathfrak{K}}$ is equal to 1 if γ is central in the stabiliser Γ_{C_γ} of the horoball C_γ. This is in particular the case, up to passing to a finite index subgroup of Γ, if Γ is a lattice or is geometrically finite, as well as if X is a symmetric space and γ is in the center of the nilpotent Lie group of isometries of X acting simply transitively on the horosphere C_γ (see Proposition 18 below: in the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, this center consists of the vertical Heisenberg translations). If $X = \mathbb{H}^2_{\mathbb{R}}$, we have $i_{\mathfrak{K}} = 1$ if no nontrivial elliptic element of Γ fixes ξ_γ (in particular if Γ is torsion-free), and $i_{\mathfrak{K}} = 2$ otherwise. In the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, the stabilisers of horoballs are not abelian and $i_{\mathfrak{K}}$ is finite only if $\mathfrak{K}$ consists of vertical Heisenberg translations.

A proof similar to that of Corollary 12 gives the following result, which implies in particular Theorem 2 in the Introduction when γ_0 is parabolic.

Corollary 16 *Let Γ be a nonelementary discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$, whose Bowen-Margulis measure is finite, and let $x_0 \in \mathbb{H}^n_{\mathbb{R}}$. Let $\mathfrak{K}$ be a conjugacy class of uniformly translating parabolic elements of Γ with horospherical translation length ℓ, with $\|\sigma_{\mathfrak{K}}\|$ and $i_{\mathfrak{K}}$ finite. Then, as $t \to +\infty$,*

$$N_{\mathfrak{K},\, x_0}(t) \sim \frac{i_{\mathfrak{K}}\, \|\mu_{x_0}\|\, \|\sigma_{\mathfrak{K}}\|}{\delta_\Gamma\, \|m_{\mathrm{BM}}\|\, (2\sinh\frac{\ell}{2})^{\delta_\Gamma}}\, e^{\frac{\delta_\Gamma}{2} t}\,.$$

If Γ is arithmetic, then the error term is $\mathrm{O}(e^{(\frac{\delta_\Gamma}{2}-\kappa)t})$ for some $\kappa > 0$. Furthermore, if v_α is the unit tangent vector at x_0 to the geodesic segment $[x_0, \alpha x_0]$ for every $\alpha \in \Gamma - \Gamma_{x_0}$, for the weak-star convergence of measures on $T^1\widetilde{M}$, we have

$$\lim_{t\to+\infty} \frac{\delta_\Gamma\, \|m_{\mathrm{BM}}\|\, (2\,\sinh\frac{\ell}{2})^{\delta_\Gamma}}{i_{\mathfrak{K}}\, \|\sigma_{\mathfrak{K}}\|}\, e^{-\frac{\delta_\Gamma}{2} t} \sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,\alpha x_0)\le t} m_\alpha\, \Delta_{v_\alpha} \;=\; (\pi_+^{-1})_* \mu_{x_0}\,. \;\square$$

Corollary 17 *Let Γ be a discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$ with finite covolume and let $x_0 \in \mathbb{H}^n_{\mathbb{R}}$. Let $\mathfrak{K}$ be the conjugacy class of a uniformly translating parabolic element γ_0 of Γ with $i_{\mathfrak{K}}$ finite. Then, as $t \to +\infty$,*

$$N_{\mathfrak{K},\, x_0}(t) \sim \frac{i_{\mathfrak{K}}\ \mathrm{Vol}(\Gamma_{C_{\gamma_0}}\backslash C_{\gamma_0})}{\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})\, (2\sinh\frac{\ell}{2})^{n-1}}\, e^{\frac{n-1}{2} t}\,.$$

If Γ is arithmetic, then the error term is $\mathrm{O}(e^{(\frac{n-1}{2}-\kappa)t})$ for some $\kappa > 0$. Furthermore, if Γ_{x_0} is trivial, if v_α is the unit tangent vector at x_0 to the geodesic segment $[x_0, \alpha x_0]$ for every $\alpha \in \Gamma - \{e\}$, with $\mathrm{Vol}_{T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}}$ the spherical measure on $T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}$, we have, for the weak-star convergence of measures on $T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}$,

$$\frac{\mathrm{Vol}(\mathbb{S}^{n-1})\ \mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})\, (2\sinh\frac{\ell}{2})^{\frac{n-1}{2}}}{i_{\mathfrak{K}}\ \mathrm{Vol}(\Gamma_{C_{\gamma_0}}\backslash C_{\gamma_0})}\, e^{-\frac{n-1}{2} t} \sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,\alpha x_0)\le t} \Delta_{v_\alpha} \overset{*}{\rightharpoonup} \mathrm{Vol}_{T^1_{x_0}\mathbb{H}^n_{\mathbb{R}}}\,.$$

Proof. The claims are proved in the same way as Corollary 13, using the equality

$$\|\sigma_{\mathfrak{K}}\| = 2^{n-1}\,(n-1)\ \mathrm{Vol}(\Gamma_{C_{\gamma_0}}\backslash C_{\gamma_0})\,,$$

see [PP4, Prop. 29 (2)]. □

The parabolic case of Corollary 3 of the Introduction follows from Corollary 17. Consider the upper halfplane model of $\mathbb{H}^2_{\mathbb{R}}$ and normalise the group such that γ_0 is the translation $z \mapsto z+1$. We choose C_{γ_0} to be the horoball that consists of points with imaginary part at least 1. Since γ_0 is primitive and Γ is torsion-free, we have $\Gamma_{C_{\gamma_0}} = {\gamma_0}^{\mathbb{Z}}$ and $i_{\mathfrak{K}} = 1$. Hence $\mathrm{Vol}(\Gamma_{C_{\gamma_0}}\backslash C_{\gamma_0}) = 1$ by a standard computation of hyperbolic area. Now, $\sinh\frac{\ell}{2} = \frac{1}{2}$, and the claim follows as in the proof of the loxodromic case after Corollary 13.

We end this section by giving a necessary and sufficient criterion for a parabolic isometry of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ to be uniformly translating. We refer to [Gol], besides the reminder below, for the basic properties of $\mathbb{H}^n_{\mathbb{C}}$.

On $\mathbb{C}^{n+1} = \mathbb{C}\times\mathbb{C}^{n-1}\times\mathbb{C}$, consider the Hermitian product with signature $(1,n)$ defined by

$$\langle z, w\rangle = -z_0\overline{w}_n + z\cdot\overline{w} - z_n\overline{w}_0\,,$$

where $(z,w)\mapsto z\cdot\overline{w}$ is the standard Hermitian scalar product on $\mathbb{C}^{n-1}$. Let $q(z) = \langle z, z\rangle$ be the corresponding Hermitian form. The projective model of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ corresponding to this choice of q is the set

$$\{[w_0 : w : 1] \in \mathbb{P}_n(\mathbb{C})\ :\ q(w_0, w, 1) < 0\}\,,$$

endowed with the Riemannian metric, normalised to have sectional curvature between -4 and -1, whose Riemannian distance is given by

$$d(X,Y) = \mathrm{argcosh}\,\sqrt{\frac{\langle x,y\rangle\langle y,x\rangle}{q(x)\,q(y)}}$$

for any representatives x, y of X, Y in $\mathbb{C}^{n+1}$, see [Gol, p. 77], where the sectional curvature is normalised to be between -1 and $-\frac{1}{4}$. The boundary at infinity of $\mathbb{H}^n_{\mathbb{C}}$ is

$$\partial_\infty\mathbb{H}^n_{\mathbb{C}} = \{[w_0 : w : 1] \in \mathbb{P}_n(\mathbb{C})\ :\ q(w_0, w, 1) = 0\}\cup\{\infty\}\,,$$

where $\infty = [1:0:0]$. For every $s > 0$, the set

$$\mathscr{H}_s = \{[w_0 : w : 1] \in \mathbb{P}_n(\mathbb{C})\ :\ q(w_0 : w : 1) = -s\}$$

is a horosphere centred at ∞.

The parabolic isometries γ of $\mathbb{H}^n_{\mathbb{C}}$ fixing ∞ are the mappings induced by the projective action of the matrices

$$\widetilde{\gamma} = \begin{pmatrix} 1 & a^* & z_0 \\ 0 & A & b \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{5}$$

where $A \in U(n-1)$, $a^* = {}^t\overline{a}$ and $Aa = b$, see [CG, §4.1] and [PP1, p. 371]. For every $Z = [z_0 : z : 1] \in \partial_\infty\mathbb{H}^n_{\mathbb{C}} - \{\infty\}$, the isometry induced by the matrix

$$T_Z = \begin{pmatrix} 1 & z^* & z_0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

is called a *Heisenberg translation*, which is *vertical* if $z = 0$. The group of Heisenberg translations (which identifies with the Heisenberg group of dimension $2n-1$, see [Gol]) acts simply transitively on $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ and on each horosphere $\mathscr{H}_s$ for $s > 0$.

Proposition 18 *A parabolic isometry γ of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ is uniformly translating if and only if it is a vertical Heisenberg translation. Furthermore, if γ is not a vertical Heisenberg translation, then the map $y \mapsto d(y, \gamma y)$ is unbounded on any horosphere of $\mathbb{H}^n_{\mathbb{C}}$ centred at the fixed point of γ.*

Proof. For all $W = [w_0 : w : 1] \in \mathscr{H}_2$ and any parabolic isometry γ as given by Equation (5), we have

$$d(W, \gamma W) = \operatorname{argcosh} \frac{|w^*(A^* - I)w + \mathrm{O}(|w|)|}{2}.$$

If A is not the identity, then $w^*(A^* - I)w$ is equivalent to $|w|^2$ (up to a positive constant) on some line in $\mathbb{C}^{n-1}$, which makes the map $W \mapsto d(W, \gamma W)$ unbounded on $\mathscr{H}_2$. Thus we are reduced to considering Heisenberg translations. For all $Z = [z_0 : z : 1] \in \partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$, we have

$$d(W, T_Z W) = \operatorname{argcosh} \frac{|z \cdot \overline{w} - \overline{z} w - z_0 - 2|}{2}.$$

It is easy to see that this distance is independent of W if and only if $z = 0$, and is unbounded otherwise. □

7 The geometry of elliptic isometries

In this section, we fix $n \geq 2$ and a nontrivial elliptic isometry γ of $\mathbb{H}^n_{\mathbb{R}}$. We denote by C_γ the fixed point set of γ, which is a nonempty proper totally geodesic subspace of $\mathbb{H}^n_{\mathbb{R}}$ of dimension $k = k_\gamma$.

We will say that γ is *uniformly rotating* if there exists $\theta = \theta_\gamma \in]0, \pi]$ (called the *rotation angle* of γ) such that for every $v \in \partial^1_+ C_\gamma$, the (nonoriented) angle between v and γv is θ. This property is invariant under conjugation, and once k and θ are fixed, there exists only one conjugacy class of uniformly rotating nontrivial elliptic isometries. Note that when $n = 2$ or $n = 3$, every elliptic isometry γ is uniformly rotating, and $\theta = \pi$ if γ does not preserve the orientation. But there exist elliptic isometries in $\mathbb{H}^4_{\mathbb{R}}$ which are not uniformly rotating.

Assume that γ belongs to a nonelementary discrete group of isometries Γ of $\mathbb{H}^n_{\mathbb{R}}$, and let $\mathfrak{K}$ be the conjugacy class of γ in Γ.

- The skinning measure $\|\sigma_{\mathfrak{K}}\|$ is positive if and only if $\Lambda\Gamma$ is not contained in $\partial_\infty C_\gamma$. This is in particular the case if $n = 2$. Furthermore, $\|\sigma_{\mathfrak{K}}\|$ is finite for instance if $\Gamma_{C_\gamma} \backslash C_\gamma$ is compact or if $\partial_\infty C_\gamma \cap \Lambda\Gamma$ is empty. This is in particular the case if $n = 2$ and if γ preserves the orientation. But when $n = 2$ and γ does not preserve the orientation, the measure $\|\sigma_{\mathfrak{K}}\|$ is not necessary finite.

For instance, let $\Gamma = T(\infty, \infty, \infty)$ be the discrete group of isometries of $\mathbb{H}^2_{\mathbb{R}}$ generated by the reflexions s_1, s_2, s_3 on the sides of an ideal hyperbolic triangle. Then C_{s_1} is one of these sides. Let us prove that $\pi_* \widetilde{\sigma}_{C_{s_1}}$ is a constant multiple of the Lebesgue measure along C_{s_1}. Indeed, the Patterson-Sullivan measure at infinity of the disc model of $\mathbb{H}^2_{\mathbb{R}}$ based at its origin is a multiple of the Lebesgue measure $d\theta$ on the circle, since Γ has finite covolume. Since $d\theta$ is conformally invariant under every isometry of $\mathbb{H}^2_{\mathbb{R}}$, the measure $\pi_* \widetilde{\sigma}_{C_{s_1}}$ on C_{s_1}

is invariant under every loxodromic isometry preserving C_{s_1}, hence the result. Since C_{s_1} injects in $\Gamma\backslash\mathbb{H}^2_{\mathbb{R}}$ and since its stabiliser in Γ has order 2, the measure $\pi_*\sigma_{C_{s_1}}$ is the multiple by half the above constant of the Lebesgue measure on the image of C_{s_1} in $\Gamma\backslash\mathbb{H}^2_{\mathbb{R}}$, which is infinite.

- If $n = 2$ and $k_\gamma = 1$ (so that γ reverses the orientation), then every isometry preserving C_γ commutes with γ, hence $i_{\mathfrak{K}} = 1$. If $n = 2$ and $k_\gamma = 0$ (so that γ preserves the orientation), then the finite group Γ_{C_γ} is either cyclic, in which case $\Gamma_{C_\gamma} = Z_\Gamma(\gamma)$ and $i_{\mathfrak{K}} = 1$, or it is dihedral. Assume the second case holds. If the rotation angle of γ is π, then again $\Gamma_{C_\gamma} = Z_\Gamma(\gamma)$ and $i_{\mathfrak{K}} = 1$. Otherwise, $i_{\mathfrak{K}} = 2$.

Lemma 19 *A uniformly rotating elliptic isometry γ of $\mathbb{H}^n_{\mathbb{R}}$ with rotation angle θ is ψ-equitranslating with*

$$\psi(t) = \frac{t}{2} - \ln \frac{\sin\theta}{2} + \mathrm{O}(e^{-\frac{t}{2}})$$

as $t \to +\infty$.

Proof. By the formulas in right-angled hyperbolic triangles (see [Bea, Theo. 7.11.2 (ii)]), if $x \in \mathbb{H}^n_{\mathbb{R}}$ is at distance s from the fixed point set C_γ of γ, we have

$$\sinh \frac{d(x,\gamma x)}{2} = \sinh s \, \frac{\sin\theta}{2} \, .$$

The result follows as in Lemma 11. □

The next result follows from this lemma in the same way as Corollary 12 follows from Lemma 11. It implies Theorem 2 in the Introduction when γ_0 is elliptic.

Corollary 20 *Let Γ be a nonelementary discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$, whose Bowen-Margulis measure is finite, and let $x_0 \in \mathbb{H}^n_{\mathbb{R}}$. Let $\mathfrak{K}$ be a conjugacy class of uniformly rotating nontrivial elliptic elements of Γ with rotation angle θ, such that $\|\sigma_{\mathfrak{K}}\|$ and $i_{\mathfrak{K}}$ are positive and finite. Then, as $t \to +\infty$,*

$$N_{\mathfrak{K},\,x_0}(t) \sim \frac{i_{\mathfrak{K}}\,\|\mu_{x_0}\|\,\|\sigma_{\mathfrak{K}}\|}{\delta_\Gamma\,\|m_{\mathrm{BM}}\|\,(\sin\frac{\theta}{2})^{\delta_\Gamma}}\; e^{\frac{\delta_\Gamma}{2}\,t} \, .$$

If Γ is arithmetic or if M is compact, then the error term is $\mathrm{O}(e^{(\frac{\delta_\Gamma}{2}-\kappa)t})$ for some $\kappa > 0$. Furthermore, if v_α is the unit tangent vector at x_0 to the geodesic segment $[x_0, \alpha x_0]$ for every $\alpha \in \Gamma - \Gamma_{x_0}$, for the weak-star convergence of measures on $T^1\widetilde{M}$, we have

$$\lim_{t\to+\infty} \frac{\delta_\Gamma\,\|m_{\mathrm{BM}}\|\,(\sin\frac{\theta}{2})^{\delta_\Gamma}}{i_{\mathfrak{K}}\,\|\sigma_{\mathfrak{K}}\|}\; e^{-\frac{\delta_\Gamma}{2}\,t} \sum_{\alpha\in\mathfrak{K},\; 0<d(x_0,\alpha x_0)\le t} m_\alpha\,\Delta_{v_\alpha} \;=\; (\pi_+^{-1})_*\mu_{x_0} \, . \quad \square$$

8 Counting conjugacy classes of subgroups

Let $\widetilde{M}, x_0, \Gamma$ be as in the beginning of Section 3. Let Γ_0 be a subgroup of Γ, and let

$$\mathfrak{K} = \{\gamma\Gamma_0\gamma^{-1} \;:\; \gamma \in \Gamma\}$$

be its conjugacy class in Γ. In this Section, we will study the asymptotic growth, as $t \to +\infty$, of the cardinality of

$$\{A \in \mathfrak{K} \, : \, \inf_{\alpha \in A - \{e\}} d(x_0, \alpha x_0) \leq t\} \, ,$$

the set (assumed to be finite) of the conjugates of Γ_0 in Γ whose minimal displacement of x_0 is at most t.

We will assume the following conditions on Γ_0:

$(*)$ There exists a nonempty proper closed convex subset C_0 in $\widetilde{M}$ such that the normaliser $N_\Gamma(\Gamma_0)$ of Γ_0 in Γ is a subgroup of the stabiliser Γ_{C_0} of C_0 in Γ, with finite index, denoted by i_0, and such that the family $(\gamma C_0)_{\gamma \in \Gamma / \Gamma_{C_0}}$ is locally finite in $\widetilde{M}$;

$(**)$ There are $c_-, c_+ \in \,]0, +\infty[$ such that $c_- \leq \inf_{\gamma \in \Gamma_0 - \{e\}} d(y, \gamma y) \leq c_+$ for every $y \in \partial C_0$.

For instance, Γ_0 could be an infinite index malnormal torsion-free cocompact stabiliser of a proper totally geodesic subspace C_0 of dimension at least 1 in $\widetilde{M}$, or a torsion-free cocompact stabiliser of a horosphere centered at a parabolic fixed point of Γ (with C_0 the horoball bounded by this horosphere), in which cases $i_0 = 1$ and $\|\sigma_{C_0}\|$ is positive and finite.

For every $A = \gamma \Gamma_0 \gamma^{-1} \in \mathfrak{K}$, let

$$m_A = (\mathrm{Card}(\Gamma_{x_0} \cap \Gamma_{\gamma C_0}))^{-1} \, ,$$

which is well-defined since the normaliser of Γ_0 in Γ stabilises C_0. We define the counting function

$$N_{\mathfrak{K}, x_0}(t) = \sum_{A \in \mathfrak{K}, \, \inf_{\alpha \in A - \{e\}} d(x_0, \alpha x_0) \leq t} m_A = \sum_{\gamma \in \Gamma / N_\Gamma(\Gamma_0), \, \inf_{\alpha \in \Gamma_0 - \{e\}} d(x_0, \gamma \alpha \gamma^{-1} x_0) \leq t} m_A \, .$$

Proposition 21 *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with pinched negative sectional curvature, let $x_0 \in \widetilde{M}$, and let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Assume that the Bowen-Margulis measure of Γ is finite and mixing for the geodesic flow on T^1M. Let Γ_0 be a subgroup of Γ and let C_0 be a subset of $\widetilde{M}$ satisfying the conditions $(*)$ and $(**)$, such that the skinning measure $\|\sigma_{C_0}\|$ is positive and finite. Let $\mathfrak{K}$ be the conjugacy class of Γ_0 in Γ. Then, for every $\epsilon > 0$, if t is big enough,*

$$\frac{i_0 \, \|\mu_{x_0}\| \, \|\sigma_{C_0}\|}{\delta_\Gamma \, \|m_{\mathrm{BM}}\| \, e^{\frac{\delta_\Gamma \, c_+}{2}}} \, e^{\frac{\delta_\Gamma}{2} t} \, (1 - \epsilon) \leq N_{\mathfrak{K}, x_0}(t) \leq \frac{i_0 \, \|\mu_{x_0}\| \, \|\sigma_{C_0}\|}{\delta_\Gamma \, \|m_{\mathrm{BM}}\| \, (\sinh \frac{c_-}{2})^{\delta_\Gamma}} \, e^{\frac{\delta_\Gamma}{2} t} \, (1 + \epsilon) \, .$$

Proof. Let $\gamma \in \Gamma$. By the local finiteness assumption, except for finitely many cosets of γ in Γ / Γ_{C_0}, the point x_0 does not belong to γC_0. As in Lemma 9, if $x_0 \in X$ is at distance s from γC_0, we have

$$2 \, \mathrm{argsinh}(\cosh s \sinh \frac{c_-}{2}) \leq \inf_{\alpha \in \Gamma_0 - \{e\}} d(x_0, \gamma \alpha \gamma^{-1} x_0) \leq 2s + c_+ \, .$$

The proof is then similar to the proof of Corollary 10. □

We have the following more precise result under stronger assumptions on Γ_0, with a proof similar to those of Corollaries 12 and 16.

Theorem 22 *Let Γ be a nonelementary discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$ with finite Bowen-Margulis measure, and let $x_0 \in \mathbb{H}^n_{\mathbb{R}}$. Let Γ_0 be the stabiliser in Γ of a bounded parabolic fixed point of Γ, acting purely by translations on the boundary of any horoball C_0 centred at this fixed point. Let $\mathfrak{K}$ be the conjugacy class of Γ_0 in Γ and let $\ell = \min_{\gamma\in\Gamma_0-\{e\}} d(y,\gamma y)$ for any $y \in \partial C_0$. Then, as $t \to +\infty$,*

$$N_{\mathfrak{K},\,x_0}(t) \sim \frac{\|\mu_{x_0}\|\,\|\sigma_{C_0}\|}{\delta_\Gamma\,\|m_{\mathrm{BM}}\|\,(2\sinh\frac{\ell}{2})^{\delta_\Gamma}}\, e^{\frac{\delta_\Gamma}{2}\,t}\,.$$

If Γ is arithmetic, then the error term is $\mathrm{O}(e^{(\frac{\delta_\Gamma}{2}-\kappa)t})$ for some $\kappa > 0$. □

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