Equidistribution and counting under equilibrium states in negative curvature and trees. Applications to non-Archimedean Diophantine approximation

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February 13, 2019

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Chapter 1

Introduction

In this book, we study equidistribution and counting problems concerning locally geodesic arcs in negatively curved spaces endowed with potentials, and we deduce, from the special case of tree quotients, various arithmetic applications to equidistribution and counting problems in non-Archimedean local fields.

For several decades, tools in ergodic theory and dynamical systems have been used to obtain geometric equidistribution and counting results on manifolds, both inspired by and with applications to arithmetic and number theoretic problems, in particular in Diophantine approximation. Especially pioneered by Margulis, this field has produced a huge corpus of works, by Bowen, Cosentino, Clozel, Dani, Einseidler, Eskin, Gorodnik, Ghosh, Guivarc'h, Kim, Kleinbock, Kontorovich, Lindenstraus, Margulis, McMullen, Michel, Mohammadi, Mozes, Nevo, Oh, Pollicott, Roblin, Shah, Sharp, Sullivan, Ullmo, Weiss and the last two authors, just to mention a few contributors. We refer for now to the surveys [Bab2, Oh, PaP16, PaP17c] and we will explain in more details in this introduction the relation of our work with previous works.

In this text, we consider geometric equidistribution and counting problems weighted with a potential function in quotient spaces of CAT(-1) spaces by discrete groups of isometries. The CAT(-1) spaces form a huge class of metric spaces that contains (but is not restricted to) metric trees, hyperbolic buildings and simply connected complete Riemannian manifolds with sectional curvature bounded above by -1. In Chapter 2, we review some basic properties of these spaces and we refer to [BridH] for more details. Although some of the equidistribution and counting results with potentials on negatively curved manifolds are known,[1] as well as some of such results on CAT(-1) spaces without potential,[2] bringing together these two aspects and producing new results and applications is one of the goals of this book.

We extend the theory of Patterson-Sullivan, Bowen-Margulis and skinning measures to CAT(-1) spaces with potentials, with a special emphasis on trees endowed with a system of conductances. We prove that under natural nondegeneracy, mixing and finiteness assumptions, the pushforward under the geodesic flow of the skinning measure of properly immersed locally convex closed subsets of locally CAT(-1) spaces equidistributes to the Gibbs measure, generalising the main result of [PaP14a].

We also prove that the (appropriate generalisations of) the initial and terminal tangent vectors of the common perpendiculars to any two properly immersed locally convex closed

[1]See for instance [PauPS].
[2]See for instance [Rob2]

subsets jointly equidistribute to the skinning measures when the lengths of the common perpendiculars tend to $+\infty$. This result is then used to prove asymptotic results on weighted counting functions of common perpendiculars whose lengths tend to $+\infty$. Common perpendiculars have been studied, in various particular cases, sometimes not explicitly, by Basmajian, Bridgeman, Bridgeman-Kahn, Eskin-McMullen, Herrmann, Huber, Kontorovich-Oh, Margulis, Martin-McKee-Wambach, Meyerhoff, Mirzakhani, Oh-Shah, Pollicott, Roblin, Shah, the last two authors and many others. See the comments after Theorem 1.5 below, and the survey [PaP16] for references.

In Part III of this book, we apply the geometric results obtained for trees to deduce arithmetic applications in non-Archimedean local fields. In particular, we prove equidistribution and counting results for rationals and quadratic irrationals in any completion of any function field over a finite field.

Let us now describe more precisely the content of this book, restricted to special cases for the sake of the exposition.

Geometric and dynamical tools

Let Y be a geodesically complete connected proper locally CAT(-1) space (or good orbispace), which is nonelementary, that is, whose fundamental group is not virtually nilpotent. In this introduction, we will mainly concentrate on the cases where Y is either a metric graph (or graph of finite groups in the sense of Bass and Serre, see [Ser3]) or a Riemannian manifold (or good orbifold) of dimension at least 2 with sectional curvature at most -1. Let $\mathscr{G}Y$ be the space of locally geodesic lines of Y, on which the geodesic flow $(\mathsf{g}^t)_{t\in\mathbb{R}}$ acts by real translations on the source. When Y is a simplicial[3] graph (of finite groups), we consider the discrete time geodesic flow $(\mathsf{g}^t)_{t\in\mathbb{Z}}$, see Section 2.6. If Y is a Riemannian manifold, then $\mathscr{G}Y$ is naturally identified with the unit tangent bundle T^1Y by the map that associates to a locally geodesic line its tangent vector at time 0. In general, we define T^1Y as the space of germs of locally geodesic lines in Y, and $\mathscr{G}Y$ maps onto T^1Y with possibly uncountable fibers.

Let $F : T^1Y \to \mathbb{R}$ be a continuous map, called a *potential*, which plays the same role in the construction of Gibbs measures/equilibrium states as the energy function in Bowen's treatment of the thermodynamic formalism of symbolic dynamical systems in [Bowe2, Sect. 1]. We define in Section 3.3 the *critical exponent* δ_F associated with F, which describes the logarithmic growth of an orbit of the fundamental group on the universal cover of Y weighted by the (lifted) potential F, and which coincides with the classical critical exponent when $F = 0$. When Y is a metric graph, we associate in Section 3.5 a potential F_c to a *system of conductances* c (that is, a map from the set of edges of Y to $\mathbb{R}$), in such a way that the correspondence $c \mapsto F_c$ is bijective at the level of cohomology classes, and we denote δ_{F_c} by δ_c.

In this introduction, we assume that F is bounded and that δ_F is finite and positive in order to simplify the statements.

We say that the pair (Y, F) satisfies the *HC-property* if the integral of F on compact locally geodesic segments of Y varies in a Hölder-continuous way on its extremities (see Definition 3.13). The pairs which have the HC-property include complete Riemannian manifolds with

[3] that is, if its edges all have lengths 1

pinched sectional curvature at most -1 and Hölder-continuous potentials, and metric graphs with any potential. This HC-property is the new technical idea compared to [PauPS] which allows the extensions to our very general framework. See also [ConLT], under the very strong assumption that Y is compact.

In Chapter 4, building on the works of [Rob2][4] when $F = 0$ and of [PauPS][5] when Y is a Riemannian manifold, we generalise, to locally CAT(-1) spaces Y endowed with a potential F satisfying the HC-property, the construction and basic properties of the *Patterson densities* at infinity of the universal cover of Y associated with F and the *Gibbs measure* m_F on $\mathscr{G}Y$ associated with F.

Using the Patterson-Sullivan-Bowen-Margulis approach, the Patterson densities are limits of renormalised measures on the orbit points of the fundamental group on the universal cover of Y, weighted by the potential, and the Gibbs measures on $\mathscr{G}Y$ are local products of Patterson densities on the endpoints of the geodesic line, with the Lebesgue measure on the time parameter, weighted by the Gibbs cocycle defined by the potential.

Generalising a result of [CoP2], we prove in Section 6.2 that when Y is a regular simplicial graph and c is an antireversible system of conductances, then the Patterson measures, normalised to be probability measures, are harmonic measures (or hitting measures) on the boundary at infinity of the universal cover of Y for a transient random walk on the vertices, whose transition probabilities are constructed using the total mass of the Patterson measures.

Gibbs measures were first introduced in statistical mechanics, and are naturally associated via the thermodynamic formalism[6] with symbolic dynamics. We prove in Section 4.2 that our Gibbs measures satisfy a Gibbs property analogous to the one in symbolic dynamics. If $F = 0$, the Gibbs measure m_F is the *Bowen-Margulis measure* $m_{\rm BM}$. If Y is a compact Riemannian manifold and F is the *strong unstable Jacobian* $v \mapsto -\frac{d}{dt}_{|t=0} \ln \operatorname{Jac}\big(\mathtt{g}^t{}_{|W^-(v)}\big)(v)$, then m_F is the Liouville measure and $\delta_F = 0$ (see [PauPS, Chap. 7] for more general assumptions on Y). Thus, one interesting aspect of Gibbs measure is that they form a natural family of measures invariant under the geodesic flow that interpolates between the Liouville measure and the Bowen-Margulis measure (which in variable curvature are in general not in the same measure class). Another interesting point is that such measures are plentiful: a recent result of Belarif [Bel] proves that when Y is a geometrically finite Riemannian manifold with pinched negative curvature and topologically mixing geodesic flow, the finite and mixing Gibbs measures associated with bounded Hölder-continuous potentials are, once normalised, dense (for the weak-star topology) in the whole space of probability measures invariant under the geodesic flow.

The Gibbs measures are remarkable measures for CAT(-1) spaces endowed with potentials due to their unique ergodic-theoretic properties. Let $(Z, (\phi_t)_{t\in\mathbb{R}})$ be a topological space endowed with a continuous one-parameter group of homeomorphisms and let $\psi : Z \to \mathbb{R}$ be a bounded continuous map. Let $\mathscr{M}$ be the set of Borel probability measures on Z invariant under the flow $(\phi_t)_{t\in\mathbb{R}}$. Let $h_m(\phi^1)$ be the (metric) entropy of the geodesic flow with respect to $m \in \mathscr{M}$. The *metric pressure* for ψ of a measure $m \in \mathscr{M}$ and the *pressure* of ψ are respectively

$$P_\psi(m) = h_m(\phi_1) + \int_Z \psi \, dm \quad \text{and} \quad P_\psi = \sup_{m\in\mathscr{M}} P_\psi(m)\ .$$

An element $m \in \mathscr{M}$ is an *equilibrium state* for ψ if the least upper bound defining P_ψ is

[4] itself building on the works of Patterson, Sullivan, Coornaert, Burger-Mozes, ...

[5] itself building on the works of Ledrappier [Led], Hamenstädt, Coudène, Mohsen

[6] See for instance [Rue3, Kel, Zin].

attained on m.

Let $F^\sharp : \mathscr{G}Y \to \mathbb{R}$ be the composition of the canonical map $\mathscr{G}Y \to T^1Y$ with F, and note that $F^\sharp = F$ if Y is a Riemannian manifold. When $F = 0$ and Y is a Riemannian manifold, whose sectional curvatures and their first derivatives are bounded, by [OtaP, Theo. 2], the pressure P_F coincides with the entropy of the geodesic flow, it is equal to the critical exponent of the fundamental group of Y, and the Bowen-Margulis measure $m_F = m_{\rm BM}$, normalised to be a probability measure, is the measure of maximal entropy. When Y is a Riemannian manifold whose sectional curvatures and their first derivatives are bounded and F is Hölder-continuous, by [PauPS, Theo. 6.1], we have $P_F = \delta_F$. If furthermore the Gibbs measure m_F is finite and normalised to be a probability measure, then m_F is an equilibrium state for F.

In Section 5.4, we prove an analog of these results for the potential $F^\sharp$ when Y is a metric graph of groups. The case when Y is a finite simplicial graph[7] is classical by the work of Bowen [Bowe2], as it reduces to arguments of subshifts of finite type (see for instance [CoP1]). When Y is a compact[8] locally CAT(-1)-space,[9] a complete statement about existence, uniqueness and Gibbs property of equilibrium states for any Hölder-continuous potential is given in [ConLT].

Theorem 1.1 (The variational principle for metric graphs of groups)**.** *Assume that Y is a metric graph of finite groups, with a positive lower bound and finite upper bound on the lengths of edges. If the critical exponent δ_F is finite, if the Gibbs measure m_F is finite, then $P_{F^\sharp} = \delta_F$ and the Gibbs measure normalised to be a probability measure is the unique equilibrium state for $F^\sharp$.*

The main tool is a natural coding of the discrete time geodesic flow by a topological Markov shift (see Section 5.1). This coding is delicate when the vertex stabilisers are nontrivial, in particular as it does not satisfy in general the Markovian property of dependence only on the immediate past (see Section 5.2). We then apply results of Buzzi and Sarig in symbolic dynamics over a countable alphabet (see Appendix A written by J. Buzzi), and suspension techniques introduced in Section 5.3. See also [Kemp].

Let Y be any geodesically complete connected proper locally CAT(-1) space, and let D be any connected proper nonempty *properly immersed*[10] closed locally convex subset of Y. In Chapter 7, we generalise for nonconstant potentials on Y the construction of the *skinning measures* σ^+_D and σ^-_D on the outer and inner unit normal bundles of D in Y. We refer to Section 2.4 for the appropriate definition of the outer and inner unit normal bundles of D when the boundary of D is not smooth. We construct these measures σ^+_D and σ^-_D as pushforwards of the Patterson densities associated with the potential F to the outer and inner unit normal bundles of the lift of D in the universal cover of Y. This construction generalises the one in [PaP14a] when $F = 0$, which itself generalises the one in [OhS1, OhS2] when M has constant curvature and D is a ball, a horoball or a totally geodesic submanifold.

In Section 10.1, we prove the following result on the equidistribution of equidistant hypersurfaces in CAT(-1) spaces. This result is a generalisation of [PaP14a, Theo. 1] (valid in Riemannian manifolds with zero potential) which itself generalised the ones in [Mar2, EM, PaP12]

[7] that is, a finite graph of trivial groups with edge lengths 1

[8] a very strong assumption that we do not want to make in this text

[9] not in the orbifold sense, hence this excludes for instance the case of graphs of groups with some nontrivial vertex stabiliser

[10] By definition, D is the image in Y, by the universal covering map, of a proper nonempty closed convex subset of the universal cover of Y, whose family of images under the universal covering group is locally finite.

when Y has constant curvature, $F = 0$ and D is a ball, a horoball or a totally geodesic submanifold. See also [Rob2] when Y is a CAT(-1) space, $F = 0$ and D is a ball or a horoball.

Theorem 1.2. *Let Y, D be as above, and let F be a potential of Y satisfying the HC-property. Assume that the Gibbs measure m_F on $\mathscr{G}Y$ is finite and mixing for the geodesic flow $(\mathtt{g}^t)_{t\in\mathbb{R}}$, and that the skinning measure σ^+_D is finite and nonzero. Then, as t tends to $+\infty$, the pushforwards $(\mathtt{g}^t)_*\sigma^+_D$ of the skinning measure of D by the geodesic flow weak-star converge towards the Gibbs measure m_F (after normalisation as probability measures).*

We prove in Theorem 10.4 an analog of Theorem 1.2 for the discrete time geodesic flow on simplicial graphs and, more generally, simplicial graphs of groups. As a special case, we recover known results on nonbacktracking simple random walks on regular graphs. The equidistribution of the pushforward of the skinning measure of a subgraph is a weighted version of the following classical result, see for instance [AloBLS], which under further assumptions on the spectral properties on the graph gives precise rates of convergence.

Corollary 1.3. *Let $\mathbb{Y}$ be a finite regular graph which is not bipartite. Let $\mathbb{Y}'$ be a nonempty connected subgraph. Then the n-th vertex of the nonbacktracking simple random walk on $\mathbb{Y}$ starting transversally to $\mathbb{Y}'$ converges in distribution to the uniform distribution as $n \to +\infty$.*

See Chapter 10 for more details and for the extensions to nonzero potential and to graphs of groups, as well as Section 10.4 for error terms.

The distribution of common perpendiculars

Let D^- and D^+ be connected proper nonempty properly immersed locally convex closed subsets of Y. A *common perpendicular* from D^- to D^+ is a locally geodesic path in Y starting perpendicularly from D^- and arriving perpendicularly to D^+.[11] We denote the length of a common perpendicular α from D^- to D^+ by $\lambda(\alpha)$, and its initial and terminal unit tangent vectors by v^-_α and v^+_α. In the general CAT(-1) case, $v^\pm_\alpha$ are two different parametrisations (by $\mp[0, \lambda(\alpha)]$) of α, considered as elements of the space $\breve{\mathscr{G}}Y$ of generalised locally geodesic lines in Y, see [BartL] or Section 2.2. For all $t > 0$, we denote by $\operatorname{Perp}(D^-, D^+, t)$ the set of common perpendiculars from D^- to D^+ with length at most t (considered with multiplicities), and we define the counting function with weights by

$$\mathscr{N}_{D^-,\,D^+,\,F}(t) = \sum_{\alpha\in\operatorname{Perp}(D^-,\,D^+,\,t)} e^{\int_\alpha F}\;,$$

where $\int_\alpha F = \int_0^{\lambda(\alpha)} F(\mathtt{g}^t v^-_\alpha)\,dt$. We refer to Section 12.1 for the definition of the multiplicities in the manifold case, which are equal to 1 if D^- and D^+ are embedded and disjoint. Higher multiplicities for common perpendiculars α can occur for instance when D^- is a nonsimple closed geodesic and the initial point of α is a multiple point of D^-.

Let $\operatorname{Perp}(D^-, D^+)$ be the set of all common perpendiculars from D^- to D^+ (considered with multiplicities). The family $(\lambda(\alpha))_{\alpha\in\operatorname{Perp}(D^-,\,D^+)}$ is called the *marked ortholength spectrum* from D^- to D^+. The set of lengths (with multiplicities) of elements of $\operatorname{Perp}(D^-, D^+)$ is called the *ortholength spectrum* of D^-, D^+. This second set has been introduced by Basmajian [Basm] (under the name "full orthogonal spectrum") when M has constant curvature,

[11]See Section 2.5 for explanations when the boundary of D^- or D^+ is not smooth.

and D^- and D^+ are disjoint or equal embedded totally geodesic hypersurfaces or embedded horospherical cusp neighbourhoods or embedded balls. We refer to the paper [BridK] which proves that the ortholength spectrum with $D^- = D^+ = \partial M$ determines the volume of a compact hyperbolic manifold M with totally geodesic boundary (see also [Cal] and [MasaM]).

We prove in Chapter 12 that the critical exponent δ_F of F is the exponential growth rate of $\mathscr{N}_{D^-,\,D^+,\,F}(t)$, and we give an asymptotic formula of the form $\mathscr{N}_{D^-,\,D^+,\,F}(t) \sim c\,e^{\delta_F t}$ as $t \to +\infty$, with error term estimates in appropriate situations. The constants c that will appear in such asymptotic formulas will be explicit, in terms of the measures naturally associated with the (normalised) potential F: the Gibbs measure m_F and the skinning measures of D^- and D^+.

When $F = 0$ and Y is a Riemannian manifold with pinched sectional curvature and finite and mixing Bowen-Margulis measure, the asymptotics of the counting function $\mathscr{N}_{D^-,\,D^+,\,0}(t)$ are described in [PaP17b, Theo. 1]. The only restriction on the two convex sets $D^\pm$ is that their skinning measures are finite. Here, we generalise that result by allowing for nonzero potential and more general CAT(-1) spaces than just manifolds.

The counting function $\mathscr{N}_{D^-,\,D^+,\,0}(t)$ has been studied in negatively curved manifolds since the 1950's and in a number of more recent works, sometimes in a different guise. A number of special cases (all with $F = 0$ and covered by the results of [PaP17b]) were known:

- D^- and D^+ are reduced to points, by for instance [Hub2], [Mar1] and [Rob2],
- D^- and D^+ are horoballs, by [BeHP], [HeP3], [Cos] and [Rob2] without an explicit form of the constant in the asymptotic expression,
- D^- is a point and D^+ is a totally geodesic submanifold, by [Herr], [EM] and [OhS3] in constant curvature,
- D^- is a point and D^+ is a horoball, by [Kon] and [KonO] in constant curvature, and [Kim] in rank one symmetric spaces,
- D^- is a horoball and D^+ is a totally geodesic submanifold, by [OhS1] and [PaP12] in constant curvature, and
- D^- and D^+ are (properly immersed) locally geodesic lines in constant curvature and dimension 3, by [Pol2].

We refer to the survey [PaP16] for more details on the manifold case.

When X is a compact metric or simplicial graph and $D^\pm$ are points, the asymptotics of $\mathscr{N}_{D^-,\,D^+,\,0}(t)$ as $t \to +\infty$ is treated in [Gui], as well as [Rob2]. Under the same setting, see also the work of Kiro-Smilansky-Smilansky announced in [KiSS] for a counting result of paths (not assumed to be locally geodesic) in finite metric graphs with rationally independent edge lengths and vanishing potential.

The proofs of the asymptotic results on the counting function $\mathscr{N}_{D^-,\,D^+,\,F}$ are based on the following simultaneous equidistribution result that shows that the initial and terminal tangent vectors of the common perpendiculars equidistribute to the skinning measures of D^- and D^+. We denote the unit Dirac mass at a point z by Δ_z and the total mass of any measure m by $\|m\|$.

Theorem 1.4. *Assume that Y is a nonelementary Riemannian manifold with pinched sectional curvature at most -1 or a metric graph. Let $F : T^1Y \to \mathbb{R}$ be a potential, with finite and positive critical exponent δ_F, which is bounded and Hölder-continuous when Y is a manifold. Let $D^\pm$ be as above. Assume that the Gibbs measure m_F is finite and mixing for the* *geodesic flow. For the weak-star convergence of measures on $\check{\mathscr{G}}Y \times \check{\mathscr{G}}Y$, we have*

$$\lim_{t\to+\infty} \delta_F \, \|m_F\| \, e^{-\delta_F t} \sum_{\alpha\in \operatorname{Perp}(D^-,\,D^+,\,t)} e^{\int_\alpha F} \, \Delta_{v_\alpha^-} \otimes \Delta_{v_\alpha^+} \;=\; \sigma^+_{D^-} \otimes \sigma^-_{D^+} \,.$$

There is a similar statement for nonbipartite simplicial graphs and for more general graphs of groups on which the discrete time geodesic flow is mixing for the Gibbs measure, see the end of Chapter 11 and Section 12.4. Again, the results can then be interpreted in terms of nonbacktracking random walks.

In Section 12.2, we deduce our counting results for common perpendiculars between the subsets D^- and D^+ from the above simultaneous equidistribution theorem.

Theorem 1.5. *(1) Let $Y, F, D^\pm$ be as in Theorem 1.4. Assume that the Gibbs measure m_F is finite and mixing for the continuous time geodesic flow and that the skinning measures $\sigma^+_{D^-}$ and $\sigma^-_{D^+}$ are finite and nonzero. Then, as $s \to +\infty$,*

$$\mathscr{N}_{D^-,\,D^+,\,F}(s) \sim \frac{\|\sigma^+_{D^-}\|\,\|\sigma^-_{D^+}\|}{\|m_F\|}\,\frac{e^{\delta_F\, s}}{\delta_F}\,.$$

(2) If Y is a finite nonbipartite simplicial graph, then

$$\mathscr{N}_{D^-,\,D^+,F}(n) \sim \frac{e^{\delta_F}\;\|\sigma^+_{D^-}\|\;\|\sigma^-_{D^+}\|}{(e^{\delta_F}-1)\;\|m_F\|}\;\; e^{\delta_F\, n}\,.$$

The above Assertion (1) is valid when Y is a good orbifold instead of a manifold or a metric graph of finite groups instead of a metric graph (for the appropriate notion of multiplicities), and when D^- and D^+ are replaced by locally finite families. See Section 12.4 for generalisations of Assertion (2) to (possibly infinite) simplicial graphs of finite groups and Sections 12.3 and 12.6 for error terms.

We avoid any compactness assumption on Y, we only assume that the Gibbs measure m_F of F is finite and that it is mixing for the geodesic flow. By Babillot's theorem [Bab1], if the length spectrum of Y is not contained in a discrete subgroup of $\mathbb{R}$, then m_F is mixing if finite. If Y is a Riemannian manifold, this condition is satisfied for instance if the limit set of a fundamental group of Y is not totally disconnected, see for instance [Dal1, Dal2]. When Y is a metric graph, Babillot's mixing condition is in particular satisfied if the lengths of the edges of Y are rationally independent.

As in [PaP17b], we have very weak finiteness and curvature assumptions on the space and the convex subsets we consider. Furthermore, the space Y is no longer required to be a manifold and we extend the theory to nonconstant weights using equilibrium states. Such a weighted counting has only been used in the orbit-counting problem in manifolds with pinched negative curvature in [PauPS]. The approach is based on ideas from Margulis's thesis to use the mixing of the geodesic flow. Our skinning measures are much more general than the Patterson measures appearing in earlier works. As in [PaP17b], we push simultaneously the unit normal vectors to the two convex sets D^- and D^+ in opposite directions.

Classically, an important characterisation of the Bowen-Margulis measure on closed negatively curved Riemannian manifolds ($F = 0$) is that it coincides with the weak-star limit of properly normalised sums of Lebesgue measures supported on periodic orbits. The result was extended to CAT(-1) spaces with zero potential in [Rob2] and to Gibbs measures in the manifold case in [PauPS, Theo. 9.11]. As a corollary of the simultaneous equidistribution result Theorem 1.4, we obtain a weighted version for simplicial and metric graphs of groups. The following is a simplified version of such a result for Gibbs measures of metric graphs.

Let $\mathbf{Per}'(t)$ be the set of prime periodic orbits of the geodesic flow on Y. Let $\lambda(g)$ denote the length of a closed orbit g. Let $\mathscr{L}_g$ be the Lebesgue measure along g and let $\mathscr{L}_g(F)$ be the period of g for the potential F.

Theorem 1.6. *Assume that Y is a finite metric graph, that the critical exponent δ_F is positive and that the Gibbs measure m_F is mixing for the continuous time geodesic flow. As $t \to +\infty$, the measures*

$$\delta_F\, e^{\delta_F t} \sum_{g \in \mathbf{Per}'(t)} e^{\mathscr{L}_g(F)}\, \mathscr{L}_g$$

and

$$\delta_F\, t\, e^{\delta_F t} \sum_{g \in \mathbf{Per}'(t)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g}{\lambda(g)}$$

converge to $\frac{m_F}{\|m_F\|}$ for the weak-star convergence of measures.

See Section 13.2 for the proof of the full result and for a similar statement for (possibly infinite) simplicial graphs of finite groups. As a corollary, we obtain counting results of simple loops in metric and simplicial graphs, generalising results of [ParP], [Gui].

Corollary 1.7. *Assume that Y is a finite metric graph whose vertices have degrees at least* 3*, such that the critical exponent δ_F is positive.*

(1) *If the Gibbs measure is mixing for the continuous time geodesic flow, then*

$$\sum_{g \in \mathbf{Per}'(t)} e^{\mathscr{L}_g(F)} \sim \frac{e^{\delta_F\, t}}{\delta_F\, t}$$

as $t \to +\infty$.

(2) *If Y is simplicial and if the Gibbs measure is mixing for the discrete time geodesic flow, then*

$$\sum_{g \in \mathbf{Per}'(t)} e^{\mathscr{L}_g(F)} \sim \frac{e^{\delta_F}}{e^{\delta_F} - 1}\, \frac{e^{\delta_F\, t}}{t}$$

as $t \to +\infty$.

In the cases when error bounds are known for the mixing property of the continuous time or discrete time geodesic flow on $\mathscr{G}Y$, we obtain corresponding error terms in the equidistribution result of Theorem 1.2 generalising [PaP14a, Theo. 20] (where $F = 0$) and in the approximation of the counting function $\mathscr{N}_{D^-,\,D^+,\,0}$ by the expression introduced in Theorem 1.5. In the manifold case, see [KM1], [Clo], [Dol1], [Sto], [Live], [GLP], and Section 12.3 for definitions and precise references. Here is an example of such a result in the manifold case.

Theorem 1.8. *Assume that Y is a compact Riemannian manifold and m_F is exponentially mixing under the geodesic flow for the Hölder regularity, or that Y is a locally symmetric space, the boundary of $D^\pm$ is smooth, m_F is finite, smooth, and exponentially mixing under the geodesic flow for the Sobolev regularity. Assume that the strong stable/unstable ball masses by the conditionals of m_F are Hölder-continuous in their radius.*

(1) *As t tends to $+\infty$, the pushforwards $(\mathsf{g}^t)_*\sigma^+_{D^-}$ of the skinning measure of D^- by the geodesic flow equidistribute towards the Gibbs measure m_F with exponential speed.*

(2) *If the skinning measures $\sigma^+_{D^-}$ and $\sigma^-_{D^+}$ are finite and nonzero, there exists $\kappa > 0$ such that, as $t \to +\infty$,*

$$\mathscr{N}_{D^-,\,D^+,\,F}(t) = \frac{\|\sigma^+_{D^-}\|\;\|\sigma^-_{D^+}\|}{\delta_F\;\|m_F\|}\;e^{\delta_F t}\big(1 + \mathrm{O}(e^{-\kappa t})\big)\;.$$

See Section 12.3 for a discussion of the assumptions and the dependence of $\mathrm{O}(\cdot)$ on the data. Similar (sometimes more precise) error estimates were known earlier for the counting function in special cases of $D^\pm$ in constant curvature geometrically finite manifolds (often in small dimension) through the work of Huber, Selberg, Patterson, Lax and Phillips [LaxP], Cosentino [Cos], Kontorovich and Oh [KonO], Lee and Oh [LeO].

When Y is a finite volume hyperbolic manifold and the potential F is constant 0, the Gibbs measure is proportional to the Liouville measure and the skinning measures of totally geodesic submanifolds, balls and horoballs are proportional to the induced Riemannian measures of the unit normal bundles of their boundaries. In this situation, there are very explicit forms of the counting results in finite-volume hyperbolic manifolds, see [PaP17b, Cor.21], [PaP16]. These results are extended to complex hyperbolic space in [PaP17a].

As an example of this result, if D^- and D^+ are closed geodesics of Y of lengths ℓ_- and ℓ_+, respectively, then the number $\mathscr{N}(s) = \mathscr{N}_{D^-,\,D^+,\,0}(s)$ of common perpendiculars (counted with multiplicity) from D^- to D^+ of length at most s satisfies, as $s \to +\infty$,

$$\mathscr{N}(s) \sim \frac{\pi^{\frac{n}{2}-1}\Gamma(\frac{n-1}{2})^2}{2^{n-2}(n-1)\Gamma(\frac{n}{2})}\;\frac{\ell_-\ell_+}{\mathrm{Vol}(Y)}\;e^{(n-1)s}\;. \tag{1.1}$$

Counting in weighted graphs of groups

From now on in this introduction, we only consider metric or simplicial graphs or graphs of groups.

Let $\mathbb{Y}$ be a connected finite graph with set of vertices $V\mathbb{Y}$ and set of edges $E\mathbb{Y}$ (see [Ser3] for the conventions). We assume that the degree of the graph at each vertex is at least 3. Let $\lambda : E\mathbb{Y} \to\,]0, +\infty[$ with $\lambda(\overline{e}) = \lambda(e)$ for every $e \in E\mathbb{Y}$ be an *edge length map*, let $Y = |\mathbb{Y}|_\lambda$ be the geometric realisation of $\mathbb{Y}$ where the geometric realisation of every edge $e \in E\mathbb{Y}$ has length $\lambda(e)$, and let $c : E\mathbb{Y} \to \mathbb{R}$ be a map, called a (logarithmic) *system of conductances* in the analogy between graphs and electrical networks, see for instance [Zem].

Let $\mathbb{Y}^\pm$ be proper nonempty subgraphs of $\mathbb{Y}$. For every $t \geqslant 0$, we denote by $\mathrm{Perp}(\mathbb{Y}^-, \mathbb{Y}^+, t)$ the set of edge paths $\alpha = (e_1, \dots, e_k)$ in $\mathbb{Y}$ without backtracking, of length $\lambda(\alpha) = \sum_{i=1}^k \lambda(e_i)$ at most t, of conductance $c(\alpha) = \sum_{i=1}^k c(e_i)$, starting from a vertex of $\mathbb{Y}^-$ but not by an edge of $\mathbb{Y}^-$, ending at a vertex of $\mathbb{Y}^+$ but not by an edge of $\mathbb{Y}^+$. Let

$$\mathscr{N}_{\mathbb{Y}^-,\mathbb{Y}^+}(t) = \sum_{\alpha\in\mathrm{Perp}(\mathbb{Y}^-,\mathbb{Y}^+,\,t)} e^{c(\alpha)}$$

be the number of paths without backtracking from $\mathbb{Y}^-$ to $\mathbb{Y}^+$ of length at most t, counted with weights defined by the system of conductances.

Recall that a real number x is *Diophantine* if it is badly approximable by rational numbers, that is, if there exist $\alpha, \beta > 0$ such that $|x - \frac{p}{q}| \geqslant \alpha\, q^{-\beta}$ for all $p, q \in \mathbb{Z}$ with $q > 0$. We obtain the following result, which is a very simplified version of our results for the sake of this introduction.

Theorem 1.9. *(1) If Y has two cycles whose ratio of lengths is Diophantine, then there exists $C > 0$ such that for every $k \in \mathbb{N} - \{0\}$, as $t \to +\infty$,*

$$\mathscr{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(t) = C\, e^{\delta_c\, t}\big(1 + \mathrm{O}(t^{-k})\big)\ .$$

(2) If $\lambda \equiv 1$, then there exist $C', \kappa > 0$ such that, as $n \in \mathbb{N}$ tends to $+\infty$,

$$\mathscr{N}_{\mathbb{Y}^-, \mathbb{Y}^+}(n) = C'\, e^{\delta_c\, n}\big(1 + \mathrm{O}(e^{-\kappa\, n})\big)\ .$$

Note that the Diophantine assumption on Y in Theorem 1.9 (1) is standard in the theory of quantum graphs (see for instance [BerK]).

The constants $C = C_{\mathbb{Y}^\pm, c, \lambda} > 0$ and $C' = C'_{\mathbb{Y}^\pm, c} > 0$ in the above asymptotic formulas are explicit. When $c \equiv 0$ and $\lambda \equiv 1$, the constants can often be determined concretely, as indicated in the two examples below.[12] Among the ingredients in these computations are the explicit expressions of the total mass of many Bowen-Margulis measures and skinning measures obtained in Chapter 8.

See Sections 12.4, 12.5 and 12.6 for generalisations of Theorem 1.9 when the graphs $\mathbb{Y}^\pm$ are not embedded in $\mathbb{Y}$, and for versions in (possibly infinite) metric graphs of finite groups. In particular, Assertion (2) remains valid if Y is the quotient of a uniform simplicial tree by a geometrically finite lattice in the sense of [Pau4], such as an arithmetic lattice in PGL_2 over a non-Archimedean local field, see [Lub1]. Recall that a locally finite metric tree X is *uniform* if it admits a discrete and cocompact group of isometries, and that a *lattice* Γ of X is a lattice in the locally compact group of isometries of X preserving without edge inversions the simplicial structure. We refer for instance to [BasK, BasL] for uncountably many examples of tree lattices.

Example 1.10. (1) When $\mathbb{Y}$ is a $(q+1)$-regular finite graph with constant edge length map $\lambda \equiv 1$ and vanishing system of conductances $c \equiv 0$, then $\delta_c = \ln q$, and if furthermore $\mathbb{Y}^+$ and $\mathbb{Y}^-$ are vertices, then (see Equation (12.11))

$$C' = \frac{q+1}{(q-1)\,\mathrm{Card}(V\mathbb{Y})}\ .$$

(2) When $\mathbb{Y}$ is biregular of degrees $p+1$ and $q+1$ with $p, q \geqslant 2$, when $\lambda \equiv 1$ and $c \equiv 0$, then $\delta_c = \ln \sqrt{pq}$, and if furthermore the subgraphs $\mathbb{Y}^\pm$ are simple cycles of lengths $L^\pm$, then (see Equation (12.12)) the number of common perpendiculars of even length at most $2N$ from $\mathbb{Y}^-$ to $\mathbb{Y}^+$ as $N \to +\infty$ is asymptotic to

$$\frac{(p+q)\ L^-\ L^+}{2\,(pq-1)\ \mathrm{Card}(E\mathbb{Y})}\ (pq)^{N+1}$$

The main tool in order to obtain the error terms in Theorem 1.9 and its more general versions is to study the error terms in the mixing property of the geodesic flow. Using the

[12] See Section 12.4 for more examples.

already mentioned coding (given in Section 5.2) of the discrete time geodesic flow by a two-sided topological Markov shift, classical reduction to one-sided topological Markov shift, and results of Young [You1] on the decay of correlations for Young towers with exponentially small tails, we in particular obtain the following simple criteria for the exponential decay of correlation of the discrete time geodesic flow, where we only assume $\mathbb{Y}$ to be locally finite (and maybe not finite). See Theorem 9.1 for the complete result.

Theorem 1.11. *Assume that the Gibbs measure m_F is finite and mixing for the discrete time geodesic flow on $\mathbb{Y}$. Assume moreover that there exist a finite subset E of $V\mathbb{Y}$ and $C', \kappa' > 0$ such that for all $n \in \mathbb{N}$, we have*

$$m_F\big(\{\ell \in \mathscr{G}\mathbb{Y} \;:\; \ell(0) \in E \text{ and } \forall\, k \in \{1, \dots, n\},\; \ell(k) \notin E\}\big) \leqslant C'\, e^{-\kappa' n}\;.$$

Then the discrete time geodesic flow has exponential decay of Hölder correlations for m_F.

The assumption of having exponentially small mass of geodesic lines which have a big return time to a given finite subset of $V\mathbb{Y}$ is in particular satisfied (see Section 9.2) if $\mathbb{Y}$ is the quotient of a uniform simplicial tree by a geometrically finite lattice,[13] such as an arithmetic lattice in PGL_2 over a non-Archimedian local field, see [Lub1], but also by many other examples of $\mathbb{Y}$. This statement corrects the mistake in [Kwo], as indicated in its erratum.

These results allow to prove in Section 9.3, under Diophantine assumptions, the rapid mixing property for the continuous time geodesic flow, that leads to Assertion (1) of Theorem 1.9, see Section 12.6. The proof uses suspension techniques due to Dolgopyat [Dol2] when Y is a compact metric tree, and to Melbourne [Mel1] otherwise.

As a corollary of the general version of the counting result Theorem 1.5, we have the following asymptotic for the orbital counting function in conjugacy classes for groups acting on trees. Given $x_0 \in X$ and a nontrivial conjugacy class $\mathfrak{K}$ in a discrete group Γ of isometries of X, we consider the counting function

$$N_{\mathfrak{K},\, x_0}(t) = \mathrm{Card}\{\gamma \in \mathfrak{K} \;:\; d(x_0, \gamma x_0) \leqslant t\}\;,$$

introduced by Huber [Hub1] when X is replaced by the real hyperbolic plane and Γ is a lattice. We refer to [PaP15] for many results on the asymptotic growth of such orbital counting functions in conjugacy classes, when X is replaced by a finitely generated group with a word metric, or a complete simply connected pinched negatively curved Riemannian manifold. See also [ChaP, ArCT, Pol3].

Theorem 1.12. *Let X be a uniform metric tree with vertices of degree $\geqslant 3$, let δ be the Hausdorff dimension of its space of ends, let Γ be a discrete group of isometries of X, let x_0 be a vertex of X with trivial stabiliser in Γ, and let $\mathfrak{K}$ be a loxodromic conjugacy class in Γ.*

(1) If the metric graph $\Gamma\backslash X$ is compact and has two cycles whose ratio of lengths is Diophantine, then there exists $C > 0$ such that for every $k \in \mathbb{N} - \{0\}$, as $t \to +\infty$,

$$\mathscr{N}_{\mathfrak{K},\, x_0}(t) = C\, e^{\frac{\delta}{2}\, t}\big(1 + \mathrm{O}(t^{-k})\big)\;.$$

(2) If X is simplicial and Γ is a geometrically finite lattice of X, then there exist $C', \kappa > 0$ such that, as $n \in \mathbb{N}$ tends to $+\infty$,

$$\mathscr{N}_{\mathfrak{K},\, x_0}(n) = C'\, e^{\delta\, \lfloor (n - \lambda(\gamma))/2 \rfloor}\big(1 + \mathrm{O}(e^{-\kappa\, n})\big)\;.$$

[13] See for instance [Pau4].

We refer to Theorem 13.1 for a more general version, including a version with a system of conductances in the counting function, and when $\mathfrak{K}$ is elliptic. When $\Gamma\backslash X$ is compact and Γ is torsion free,[14] Assertion (1) of this result is due to Kenison and Sharp [KeS], who proved it using transfer operator techniques for suspensions of subshifts of finite type. Up to strengthening the Diophantine assumption, using work of Melbourne [Mel1] on the decay of correlations of suspensions of Young towers, we are able to extend Assertion (1) to all geometrically finite lattices Γ of X in Section 13.1.

The constants $C = C_{\mathfrak{K},x_0}$ and $C' = C'_{\mathfrak{K},x_0}$ are explicit. For instance in Assertion (2), if X is the geometric realisation of a regular simplicial tree $\mathbb{X}$ of degree $q+1$, if x_0 is a vertex of $\mathbb{X}$, if $\mathfrak{K}$ is the conjugacy class of γ_0 with translation length $\lambda(\gamma_0)$ on X, if

$$\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}) = \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{1}{|\Gamma_x|}$$

is the volume[15] of the quotient graph of groups $\Gamma\backslash\backslash\mathbb{X}$, then

$$C' = \frac{\lambda(\gamma_0)}{[Z_\Gamma(\gamma_0) : \gamma_0^{\mathbb{Z}}]\ \mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X})}\ ,$$

where $Z_\Gamma(\gamma_0)$ is the centraliser of γ_0 in Γ. When furthermore Γ is torsion free, γ_0 is not a proper power and $\Gamma\backslash\mathbb{X}$ is finite, as $\delta = \ln q$, we get that there exists $\kappa > 0$ such that

$$\mathscr{N}_{\mathfrak{K},\,x_0}(n) = \frac{\lambda(\gamma_0)}{\mathrm{Card}(\Gamma\backslash\mathbb{X})}\ q^{\lfloor (n-\lambda(\gamma_0))/2\rfloor} + \mathrm{O}(q^{(1-\kappa')\,n/2})$$

as $n \in \mathbb{N}$ tends to $+\infty$, thus recovering the result of [Dou] who used the spectral theory of the discrete Laplacian.

Selected arithmetic applications

We end this introduction by giving a sample of our arithmetic applications (see Part III of this book) of the ergodic and dynamical results on the discrete time geodesic flow on simplicial trees described in Part II of this book, as summarized above. Our equidistribution and counting results of common perpendiculars between subtrees indeed produce equidistribution and counting results of rationals and quadratic irrationals in non-Archimedean local fields. We refer to [BrPP] for an announcement of the results of Part III, with a presentation different from the one in this introduction.

To motivate what follows, consider $R = \mathbb{Z}$ the ring of integers, $K = \mathbb{Q}$ its field of fractions, $\widehat{K} = \mathbb{R}$ the completion of $\mathbb{Q}$ for the usual Archimedean absolute value $|\cdot|$, and $\mathrm{Haar}_{\widehat{K}}$ the Lebesgue measure of $\mathbb{R}$ (which is the Haar measure of the additive group $\mathbb{R}$ normalised so that $\mathrm{Haar}_{\widehat{K}}([0,1]) = 1$).

The following equidistribution result of rationals, due to Neville [Nev], is a quantitative statement on the density of K in $\widehat{K}$: For the weak-star convergence of measures on $\widehat{K}$, as $s \to +\infty$, we have

$$\lim_{s\to+\infty} \frac{\pi}{6}\, s^{-2} \sum_{p,q\in R\,:\, pR+qR=R,\ |q|\leqslant s} \Delta_{\frac{p}{q}} = \mathrm{Haar}_{\widehat{K}}\ .$$

[14] In particular, Γ then has the very restricted structure of a free group.
[15] See for instance [BasK, BasL].

Furthermore, there exists $\ell \in \mathbb{N}$ such that for every smooth function $\psi : \widehat{K} \to \mathbb{C}$ with compact support, there is an error term in the above equidistribution claim evaluated on ψ, of the form $\operatorname{O}(s(\ln s)\|\psi\|_\ell)$ where $\|\psi\|_\ell$ is the Sobolev norm of ψ. The following counting result due to Mertens on the asymptotic behaviour of the average of Euler's totient function $\varphi : k \mapsto \operatorname{Card}(R/kR)^\times$, follows from the above equidistribution one:

$$\sum_{k=1}^{n} \varphi(k) = \frac{3}{\pi}\, n^2 + \operatorname{O}(n \ln n)\ .$$

See [PaP14b] for an approach using methods similar to the ones in this text, and for instance [HaW, Theo. 330] for a more traditional proof, as well as [Walf] for a better error term.

Let us now switch to a non-Archimedean setting, restricting to positive characteristic in this introduction. See Part III for analogous applications in characteristic zero.

Let $\mathbb{F}_q$ be a finite field of order q. Let $R = \mathbb{F}_q[Y]$ be the ring of polynomials in one variable Y with coefficients in $\mathbb{F}_q$. Let $K = \mathbb{F}_q(Y)$ be the field of rational fractions in Y with coefficients in $\mathbb{F}_q$, which is the field of fractions of R. Let $\widehat{K} = \mathbb{F}_q((Y^{-1}))$ be the field of formal Laurent series in the variable Y^{-1} with coefficients in $\mathbb{F}_q$, which is the completion of K for the (ultrametric) absolute value $|\frac{P}{Q}| = q^{\deg P - \deg Q}$. Let $\mathscr{O} = \mathbb{F}_q[[Y^{-1}]]$ be the ring of formal power series in Y^{-1} with coefficients in $\mathbb{F}_q$, which is the ball of centre 0 and radius 1 in $\widehat{K}$ for this absolute value.

Note that $\widehat{K}$ is locally compact, and we endow the additive group $\widehat{K}$ with the Haar measure $\operatorname{Haar}_{\widehat{K}}$ normalised so that $\operatorname{Haar}_{\widehat{K}}(\mathscr{O}) = 1$. The following results extend (with appropriate constants) when K is replaced by any function field of a nonsingular projective curve over $\mathbb{F}_q$ and $\widehat{K}$ any completion of K, see Part III.

The following equidistribution result[16] of elements of K in $\widehat{K}$ gives an analog of Neville's equidistribution result for function fields. Note that when $G = \operatorname{GL}_2(R)$, we have $(P,Q) \in G(1,0)$ if and only if $\langle P, Q\rangle = R$. We denote by H_x the stabiliser of any element x of any set endowed with any action of any group H.

Theorem 1.13. *Let G be any finite index subgroup of $\operatorname{GL}_2(R)$. For the weak-star convergence of measures on $\widehat{K}$, we have*

$$\lim_{t\to+\infty} \frac{(q+1)\,[\operatorname{GL}_2(R):G]}{(q-1)\,q^2\,[\operatorname{GL}_2(R)_{(1,0)}:G_{(1,0)}]}\, q^{-2\,t} \sum_{(P,Q)\in G(1,0),\ \deg Q\leqslant t} \Delta_{\frac{P}{Q}} \;=\; \operatorname{Haar}_{\widehat{K}}\ .$$

We emphasize the fact that we are not assuming G to be a congruence subgroup of $\operatorname{GL}_2(R)$. This is made possible by our geometric and ergodic methods.

The following variation of this result is more interesting when the class number of the function field K is larger than 1 (see Corollary 16.7 in Chapter 16).

Theorem 1.14. *Let $\mathfrak{m}$ be a nonzero fractional ideal of R with norm $\mathtt{N}(\mathfrak{m})$. For the weak-star convergence of measures on $\widehat{K}$, we have*

$$\lim_{t\to+\infty} \frac{q+1}{(q-1)\,q^2}\, s^{-2} \sum_{\substack{(x,y)\in\mathfrak{m}\times\mathfrak{m}\\ \mathtt{N}(\mathfrak{m})^{-1}\,\mathtt{N}(y)\leqslant s,\ Rx+Ry=\mathfrak{m}}} \Delta_{\frac{x}{y}} \;=\; \operatorname{Haar}_{\widehat{K}}\ .$$

If $\alpha \in \widehat{K}$ is quadratic irrational over K,[17] let α^σ be the Galois conjugate of α,[18] let

[16] See Theorem 16.4 in Chapter 16 for a more general version.

[17] that is, α does not belong to K and satisfies a quadratic equation with coefficients in K

[18] that is, the other root in $\widehat{K}$ of the irreducible quadratic polynomial over K defining α

$\mathtt{tr}(\alpha) = \alpha + \alpha^\sigma$ and $\mathtt{n}(\alpha) = \alpha\alpha^\sigma$, and let

$$h(\alpha) = \frac{1}{|\alpha - \alpha^\sigma|}\;.$$

This is an appropriate complexity for quadratic irrationals in a given orbit by homographies under $\mathrm{PGL}_2(R)$. See Section 17.2 and for instance [HeP4, §6] for motivations and results. Note that although there are only finitely many orbits by homographies of $\mathrm{PGL}_2(R)$ on K (and exactly one in the particular case of this introduction), there are infinitely many orbits of $\mathrm{PGL}_2(R)$ in the set of quadratic irrationals in $\widehat{K}$ over K. The following result gives in particular that any orbit of quadratic irrationals under $\mathrm{PGL}_2(R)$ equidistributes in $\widehat{K}$, when the complexity tends to infinity. See Theorem 17.6 in Section 17.2 for a more general version. We denote by $\cdot$ the action by homographies of $\mathrm{GL}_2(\widehat{K})$ on $\mathbb{P}_1(\widehat{K}) = \widehat{K} \cup \{\infty = [1:0]\}$.

Theorem 1.15. *Let G be a finite index subgroup of $\mathrm{GL}_2(R)$. Let $\alpha_0 \in \widehat{K}$ be a quadratic irrational over K. For the weak-star convergence of measures on $\widehat{K}$, we have*

$$\lim_{s\to+\infty} \frac{(\ln q)\;(q+1)\;m_0\;[\mathrm{GL}_2(R):G]}{2\;q^2\;(q-1)^3\;\big|\ln|\operatorname{tr} g_0|\big|}\;s^{-1} \sum_{\alpha\in G\cdot\alpha_0,\;h(\alpha)\leqslant s} \Delta_\alpha \;=\; \mathrm{Haar}_{\widehat{K}}\;.$$

where $g_0 \in G$ fixes α_0 with $|\operatorname{tr} g_0| > 1$, and m_0 is the index of $g_0^{\mathbb{Z}}$ in G_{α_0}.

Another equidistribution result of an orbit of quadratic irrationals under $\mathrm{PGL}_2(R)$ is obtained by taking another complexity, constructed using crossratios with a fixed quadratic irrational. We denote by $[a,b,c,d] = \frac{(c-a)(d-b)}{(c-b)(d-a)}$ the crossratio of four pairwise distinct elements in $\widehat{K}$. If $\alpha,\beta \in \widehat{K}$ are two quadratic irrationals over K such that $\alpha \notin \{\beta,\beta^\sigma\}$,[19] let

$$h_\beta(\alpha) = \max\{|[\alpha,\beta,\beta^\sigma,\alpha^\sigma]|,\;|[\alpha^\sigma,\beta,\beta^\sigma,\alpha]|\}\;,$$

which is also an appropriate complexity when α varies in a given orbit of quadratic irrationals by homographies under $\mathrm{PGL}_2(R)$. See Section 18.1 and for instance [PaP14b, §4] for motivations and results in the Archimedean case.

Theorem 1.16. *Let G be a finite index subgroup of $\mathrm{GL}_2(R)$. Let $\alpha_0,\beta \in \widehat{K}$ be two quadratic irrationals over K. For the weak-star convergence of measures on $\widehat{K} - \{\beta,\beta^\sigma\}$, we have, with g_0 and m_0 as in the statement of Theorem 1.15,*

$$\lim_{s\to+\infty} \frac{(\ln q)\;(q+1)\;m_0\;[\mathrm{GL}_2(R):G]}{2\;q^2\;(q-1)^3\;|\beta-\beta^\sigma|\;\big|\ln|\operatorname{tr} g_0|\big|}\;s^{-1} \sum_{\alpha\in G\cdot\alpha_0,\;h_\beta(\alpha)\leqslant s} \Delta_\alpha$$
$$= \frac{d\,\mathrm{Haar}_{\widehat{K}}(z)}{|z-\beta|\,|z-\beta^\sigma|}\;.$$

The fact that the measure towards which we have an equidistribution is only absolutely continuous with respect to the Haar measure is explained by the invariance of $\alpha \mapsto h_\beta(\alpha)$ under the (infinite) stabiliser of β in $\mathrm{PGL}_2(R)$. See Theorem 18.4 in Section 18.1 for a more general version.

[19] See Section 18.1 when this condition is not satisfied.

The last statement of this introduction is an equidistribution result for the integral representations of quadratic norm forms

$$(x, y) \mapsto \mathtt{n}(x - y\alpha)$$

on $K \times K$, where $\alpha \in \widehat{K}$ is a quadratic irrational over K. See Theorem 19.1 in Section 19 for a more general version, and for instance [PaP14b, §5.3] for motivations and results in the Archimedean case.

There is an extensive bibliography on the integral representation of norm forms and more generally decomposable forms over function fields, see for instance [Sch1, Maso1, Gyo, Maso2]. These references mostly consider higher degrees, with an algebraically closed ground field of characteristic 0, instead of $\mathbb{F}_q$.

Theorem 1.17. *Let G be a finite index subgroup of $\mathrm{GL}_2(R)$ and let $\beta \in \widehat{K}$ be a quadratic irrational over K. For the weak-star convergence of measures on $\widehat{K} - \{\beta, \beta^\sigma\}$, we have*

$$\lim_{s \to +\infty} \frac{(q+1)\,[\mathrm{GL}_2(R_v) : G]}{q^2\,(q-1)^3\,[\mathrm{GL}_2(R_v)_{(1,0)} : G_{(1,0)}]}\; s^{-1} \sum_{\substack{(x,y) \in G(1,0),\\ |x^2 - xy\,\mathtt{tr}(\beta) + y^2\,\mathtt{n}(\beta)| \leqslant s}} \Delta_{\frac{x}{y}}$$

$$= \frac{d\,\mathrm{Haar}_{\widehat{K}}(z)}{|z-\beta|\,|z-\beta^\sigma|}\,.$$

Furthermore, we have error estimates in the arithmetic applications: There exists $\kappa > 0$ such that for every locally constant function with compact support $\psi : \widehat{K} \to \mathbb{C}$ in Theorems 1.13, 1.14 and 1.15, or $\psi : \widehat{K} - \{\beta, \beta^\sigma\} \to \mathbb{C}$ in Theorems 1.16 and 1.17, there are error terms in the above equidistribution claims evaluated on ψ, of the form $\mathrm{O}(s^{-\kappa})$ where $s = q^t$ in Theorem 1.13, with for each result an explicit control on the test function ψ involving only some norm of ψ, see in particular Section 15.4.

The link between the geometry described in the first part of this introduction and the above arithmetic statements is provided by the Bruhat-Tits tree of $(\mathrm{PGL}_2, \widehat{K})$, see [Ser3] and Section 15.1 for background. We refer to Part III for more general arithmetic applications.

Acknowledgements: This work was partially supported by the NSF grants no 093207800 and DMS-1440140, while the third author was in residence at the MSRI, Berkeley CA, during the Spring 2015 and Fall 2016 semesters. The second author thanks Université Paris-Sud, Forschungsinstitut für Mathematik of ETH Zürich, and Vilho, Yrjö ja Kalle Väisälän rahasto for their support during the preparation of this work. This research was supported by the CNRS PICS n^0 6950 "Equidistribution et comptage en courbure négative et applications arithmétiques". We thank, for interesting discussions on this text, Y. Benoist, J. Buzzi (for his help in Sections 5.2 and 9.2, and for kindly agreeing to insert Appendix A used in Section 5.4), N. Curien, S. Mozes, M. Pollicott (for his help in Section 9.3), R. Sharp and J.-B. Bost (for his help in Sections 14.2 and 16.1). We especially thank O. Sarig for his help in Section 9.2: In a long email, he explained to us how to prove Theorem 9.2.

General notation

In this preamble, we introduce some general notation that will be used throughout the book. We recommend the use of the List of symbols (mostly in alphabetical order by the first letter) and of the Index for easy references to the definitions in the text.

Let A be a subset of a set E. We denote by $\mathbb{1}_A : E \to \{0, 1\}$ the *characteristic* (or indicator) *function* of A: $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$ otherwise. We denote by ${}^cA = E - A$ the complementary subset of A in E.

We denote by $\lfloor x \rfloor = \sup\{n \in \mathbb{N} \;:\; n \leqslant x\}$ the lower integral part of any $x \in \mathbb{R}$ and by $\lceil x \rceil = \inf\{n \in \mathbb{N} \;:\; x \leqslant n\}$ its upper integral part.

We denote by $\ln$ the natural logarithm (with $\ln(e) = 1$).

We denote by $\operatorname{Card}(E)$ or by $|E|$ the order of a finite set E.

We denote by $\|\mu\|$ the total mass of a finite positive measure μ.

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are measurable spaces, $f : X \to Y$ a measurable map, and μ a measure on X, we denote by $f_*\mu$ the image measure of μ by f, with $f_*\mu(B) = \mu(f^{-1}(B))$ for every $B \in \mathscr{B}$.

If (X, d) is a metric space, then $B(x, r)$ is the closed ball with centre $x \in X$ and radius $r > 0$.

For every subset A of a metric space and for every $\epsilon > 0$, we denote by $\mathscr{N}_\epsilon A$ the closed ϵ-neighbourhood of A, and by convention $\mathscr{N}_0 A = \overline{A}$. We denote by $\mathscr{N}_{-\epsilon} A$ the set of points of A at distance at least ϵ from the complement of A.

Given a topological space Z, we denote by $\mathscr{C}_{\rm c}(Z)$ the vector space of continuous maps from Z to $\mathbb{R}$ with compact support.

Given a locally compact topological space Z, we denote by $\overset{*}{\rightharpoonup}$ the weak-star convergence of (Borel, positive) measures on Z: We have $\mu_n \overset{*}{\rightharpoonup} \mu$ if and only if $\lim_{n\to+\infty} \mu_n(f) = \mu(f)$ for every $f \in \mathscr{C}_c(Z)$.

The *negative part* of a real-valued map f is $f^- = \max\{0, -f\}$.

We denote by Δ_x the unit Dirac mass at a point x in any measurable space.

Finally, the symbol $\Box$ right at the end of a statement indicates that this statement will not be given a proof, either since a reference is given or since it is an immediate consequence of previous statements.

Part I

Geometry and dynamics in negative curvature

Chapter 2

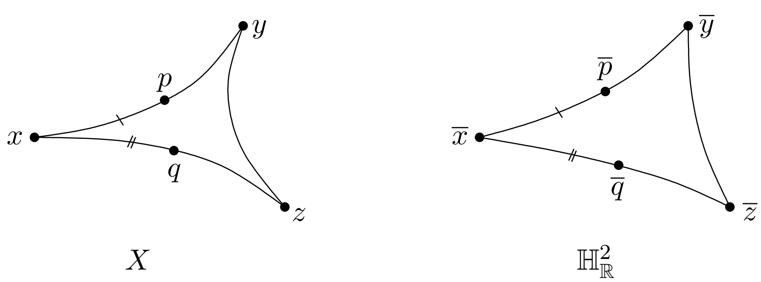
Negatively curved geometry

2.1 Background on CAT(-1) spaces

Let X be a geodesically complete proper CAT(-1) space, let $x_0 \in X$ be an arbitrary basepoint, and let Γ be a nonelementary discrete group of isometries of X.

We refer for example to [BridH] for the relevant terminology, proofs and complements on these notions. In this Section, we recall some definitions and notation for the sake of completeness.

A metric space is *proper* if its closed balls are compact. A *geodesic* in a metric space X' is an isometric map c from an interval I of $\mathbb{R}$ into X'.[1] A metric space X' is *geodesic* if for all $x, y \in X'$, there exists a geodesic segment $c : [a, b] \to X'$ from $x = c(a)$ to $y = c(b)$. A geodesic metric space X' is *geodesically complete* (or has *extendible geodesics*) if any isometric map from an interval in $\mathbb{R}$ to X' extends to at least one isometric map from $\mathbb{R}$ to X'. A *comparison triangle* of a triple of points (x, y, z) in a metric space X' is a (unique up to isometry) triple of points $(\overline{x}, \overline{y}, \overline{z})$ in the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ such that $d(x, y) = d(\overline{x}, \overline{y})$, $d(y, z) = d(\overline{y}, \overline{z})$ and $d(z, x) = d(\overline{z}, \overline{x})$.



A metric space X' is CAT(-1) if it is geodesic and if for every triple of points (x, y, z) in X', for all geodesic segments a, b respectively from x to y and from x to z, and for all points p, q in the image of a, b respectively, if $(\overline{x}, \overline{y}, \overline{z})$ is a comparison triangle of (x, y, z), if $\overline{p}$ (resp. $\overline{q}$) is the point on the geodesic segment from $\overline{x}$ to $\overline{y}$ (resp. $\overline{z}$) at distance $d(x, p)$ (resp. $d(x, q)$) from $\overline{x}$, then $d(p, q) \leqslant d(\overline{p}, \overline{q})$.

We will put a special emphasis on the case when X is a (proper, geodesically complete) $\mathbb{R}$-*tree*, that is, a uniquely arcwise connected geodesic metric space. In the Introduction, we

[1] We say that c is a *geodesic segment* if I is compact, a *geodesic ray* if I is a half-infinite interval, and a *geodesic line* if $I = \mathbb{R}$.

have denoted by Y the geodesically complete proper locally CAT(-1) good orbispace $\Gamma\backslash X$, see for instance [GH, Ch. 11] for the terminology.

Two geodesic rays $\rho, \rho' : [0, +\infty[\to X$ are *asymptotic* if their images are at finite Hausdorff distance, or equivalently if there exists $a \in \mathbb{R}$ such that $\lim_{t\to+\infty} d(\rho(t), \rho'(t+a)) = 0$. We denote by $\partial_\infty X$ the *space at infinity* of X, which consists of the asymptotic classes of geodesic rays in X, and we endow it with the quotient topology of the compact-open topology. It coincides with the space of (Freudenthal's) ends of X when X is an $\mathbb{R}$-tree.

Let $x, y \in X$ and $\xi, \xi' \in \partial_\infty X$. We denote by $[x, y] = [x, y]$ the unique image of a geodesic segment from x to y. We denote by $[x, \xi[$ the image of the unique geodesic ray $\rho : [0, +\infty[\to X$ in the asymptotic class ξ with $\rho(0) = x$, and we say that ρ *starts from* x and *ends at* ξ. We denote by $]\xi, \xi'[\; = \;]\xi', \xi[$ the unique image of a geodesic line $\ell : \mathbb{R} \to X$ with $t \mapsto \ell(t)$ and $t \mapsto \ell(-t)$ in the asymptotic classes ξ and ξ' respectively, and we say that ℓ *starts from* ξ' and *ends at* ξ.

We endow the disjoint union $X \cup \partial_\infty X$ with the unique metrisable compact topology (independent of x_0) inducing the above topologies on X and $\partial_\infty X$, such that a sequence $(y_i)_{i\in\mathbb{N}}$ in X converges to $\xi \in \partial_\infty X$ if and only if $\lim_{i\to+\infty} d(x_0, y_i) = +\infty$ and, with $c_i : [0, d(x_0, y_i)] \to X$ the geodesic segment from x_0 to y_i and $\rho : [0, +\infty[\to X$ the geodesic ray from x_0 to ξ, we have $\lim_{i\to+\infty} d(c_i(t), \rho(t)) = 0$ for every $t \geqslant 0$.

We denote by $\mathrm{Isom}(X)$ the isometry group of X, and we endow it with the compact-open topology. Its action on X uniquely extends to a continuous action on $X \cup \partial_\infty X$. We say that a discrete subgroup Γ' of $\mathrm{Isom}(X)$ is *nonelementary* if it does not fix a point or an unordered pair of points in $X \cup \partial_\infty X$. We denote by $\Lambda\Gamma$ the *limit set* of Γ, which is the set of accumulation points in $\partial_\infty X$ of any orbit of Γ in X. It is the smallest closed nonempty Γ-invariant subset of $\partial_\infty X$.

A subset D of $X \cup \partial_\infty X$ is *convex* if for all $u, v \in D$, the image of the unique geodesic segment, ray or line from u to v is contained in D. We denote by $\mathscr{C}\Lambda\Gamma$ the *convex hull* in X of $\Lambda\Gamma$, which is the intersection of the closed convex subsets of $X \cup \partial_\infty X$ containing $\Lambda\Gamma$. When X is an $\mathbb{R}$-tree, then a subset D of X is convex if and only if it is connected, and we will call it a *subtree*. In particular, if X is an $\mathbb{R}$-tree, then $\mathscr{C}\Lambda\Gamma$ is equal to the union of the geodesic lines between pairs of distinct points in $\Lambda\Gamma$, since this union is connected and contained in $\mathscr{C}\Lambda\Gamma$.

A point $\xi \in \partial_\infty X$ is called a *conical limit point* of Γ if there exists a sequence of orbit points of x_0 under Γ converging to ξ while staying at bounded distance from a geodesic ray ending at ξ. The set of conical limit points of Γ is the *conical limit set* $\Lambda_{\rm c}\Gamma$ of Γ.

A point $p \in \Lambda\Gamma$ is a *bounded parabolic limit point* of Γ if its stabiliser Γ_p in Γ acts properly discontinuously with compact quotient on $\Lambda\Gamma - \{p\}$. The discrete nonelementary group of isometries Γ of X is said to be *geometrically finite* if every element of $\Lambda\Gamma$ is either a conical limit point or a bounded parabolic limit point of Γ. See for instance [Bowd], as well as [Pau4] when X is an $\mathbb{R}$-tree, and [DaSU] for a very interesting study of equivalent conditions in an even greater generality.

For all $x \in X \cup \partial_\infty X$ and $A \subset X$, the *shadow of A seen from x* is the subset $\mathscr{O}_x A$ of $\partial_\infty X$ consisting of the endpoints towards $+\infty$ of the geodesic rays starting at x and meeting A if $x \in X$, and of the geodesic lines starting at x and meeting A if $x \in \partial_\infty X$.

The *translation length* of an isometry $\gamma \in \mathrm{Isom}(X)$ is

$$\lambda(\gamma) = \inf_{x\in X} d(x, \gamma x)\,.$$

An element $\gamma \in \mathrm{Isom}(X)$ is *elliptic* if it fixes a point in X, and then $\lambda(\gamma) = 0$. An element

$\gamma \in \mathrm{Isom}(X)$ is *parabolic* if it is not elliptic and fixes a unique point in $\partial_\infty X$, and then $\lambda(\gamma) = 0$. An element $\gamma \in \mathrm{Isom}(X)$ is *loxodromic* if $\lambda(\gamma) > 0$, and then

$$\mathrm{Ax}_\gamma = \{x \in X \ : \ d(x, \gamma x) = \lambda(\gamma)\}$$

is (the image of) a geodesic line in X, called the *translation axis* of γ. In particular, the restriction of γ to Ax_γ is conjugated, by any isometry between Ax_γ and $\mathbb{R}$, to a translation of $\mathbb{R}$ of the form $t \mapsto t \pm \lambda(\gamma)$. For all $\beta \in \mathrm{Isom}(X)$ and $n \in \mathbb{Z} - \{0\}$, we have

$$\mathrm{Ax}_{\beta\gamma\beta^{-1}} = \beta\,\mathrm{Ax}_\gamma, \quad \lambda(\beta\gamma\beta^{-1}) = \lambda(\gamma), \quad \mathrm{Ax}_{\gamma^n} = \mathrm{Ax}_\gamma, \quad \lambda(\gamma^n) = |n|\lambda(\gamma)\ . \tag{2.1}$$

A loxodromic element $\gamma \in \mathrm{Isom}(X)$ has exactly two fixed points γ_-, γ_+ in $X \cup \partial_\infty X$, with $\gamma_- \in \partial_\infty X$ its *repulsive* fixed point[2] and $\gamma_+ \in \partial_\infty X$ its *attractive* fixed point.[3]

We will need the following well-known lemma later on. An element of Γ is *primitive* in Γ if there is no $\gamma_0 \in \Gamma$ and $k \in \mathbb{N} - \{0, 1\}$ such that $\gamma = \gamma_0^k$. Note that there might exist a primitive loxodromic element γ in Γ, whose translation length is not minimal amongst the translation lengths of the loxodromic elements $\gamma' \in \Gamma$ with $\mathrm{Ax}_\gamma = \mathrm{Ax}_{\gamma'}$.[4]

Lemma 2.1. *(1) For every loxodromic element $\gamma \in \Gamma$, there exist $k' \in \mathbb{N} - \{0\}$ and a primitive loxodromic element $\gamma_1 \in \Gamma$ such that $\gamma = \gamma_1^{k'}$, and there exist $k \in \mathbb{N} - \{0\}$, a primitive loxodromic element $\gamma_0 \in \Gamma$ whose translation length is minimal amongst the translation lengths of the loxodromic elements $\gamma' \in \Gamma$ with $\mathrm{Ax}_\gamma = \mathrm{Ax}_{\gamma'}$, and an element $\gamma' \in \Gamma$ pointwise fixing Ax_γ such that $\gamma = \gamma_0^k \gamma'$.*

(2) For every compact subset K of X, for all $A > 0$ and $r > 0$, there exists $L > 0$ such that for all loxodromic elements $\gamma, \alpha \in \Gamma$, if Ax_γ meets K, if $\lambda(\alpha) = \lambda(\gamma) \leqslant A$ and if Ax_γ and Ax_α have segments of length at least L at Hausdorff distance at most r, then $\mathrm{Ax}_\alpha = \mathrm{Ax}_\gamma$.

(3) For every compact subset K of X and for every $A > 0$, there exists $N \in \mathbb{N}$ such that for every loxodromic element $\gamma \in \Gamma$ whose translation axis meets K, the cardinality of the set of loxodromic elements $\alpha \in \Gamma$ with $\mathrm{Ax}_\alpha = \mathrm{Ax}_\gamma$ and $\lambda(\alpha) \leqslant A$ is at most N.

Proof. (1) If $\gamma \in \Gamma$ is loxodromic, then the group of restrictions to Ax_γ of the elements of Γ preserving Ax_γ is conjugated, by any isometry between Ax_γ and $\mathbb{R}$, to a discrete group of isometries Λ of $\mathbb{R}$. Since replacing γ by an element of Γ having a power at least 2 equal to γ strictly decreases the translation length and by Equation (2.1), the first claim of (1) holds. The normal subgroup of Λ consisting of translations is isomorphic to $\mathbb{Z}$, generated by the conjugate of the restriction to Ax_γ of an element $\gamma_0 \in \Gamma$. Any such element has minimal translation length, hence is primitive since if there exist $\gamma_1 \in \Gamma$ and $n \in \mathbb{N} - \{0, 1\}$ with $\gamma_0 = \gamma_1^n$, by Equation (2.1), we would have $\mathrm{Ax}_{\gamma_1} = \mathrm{Ax}_{\gamma_0}$ and $\lambda(\gamma_1) = \frac{1}{n}\lambda(\gamma_0) < \lambda(\gamma_0)$. There exists $k \in \mathbb{Z} - \{0\}$ such that the restrictions of γ and γ_0^k to Ax_γ coincide. Hence $\gamma' = \gamma_0^{-k}\gamma$ pointwise fixes Ax_γ, and up to replacing γ_0 by its inverse, Assertion (1) of Lemma 2.1 holds.

(2) Since the action of Γ on X is properly discontinuous, there exists $N = N(K, A, r) \geqslant 1$ such that for every loxodromic element $\gamma \in \Gamma$ whose translation axis meets K and whose translation length is at most A, for every $x \in \mathrm{Ax}_\gamma$, the cardinality of $\{\beta \in \Gamma \ : \ d(x, \beta x) \leqslant 2r\}$

[2] For every $x \in X \cup (\partial_\infty X - \{\gamma_+\}$, we have $\lim_{n\to+\infty} \gamma^{-n} x = \gamma_-$.

[3] For every $x \in X \cup (\partial_\infty X - \{\gamma_-\})$, we have $\lim_{n\to+\infty} \gamma^{+n} x = \gamma_+$.

[4] For instance, if X is the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$ and if Γ contains an orientation preserving loxodromic element γ such that the stabiliser in Γ of Ax_γ is generated by γ and by the symmetry σ with respect to Ax_γ, then $\gamma^{2^n}\sigma$ is primitive for all $n \in \mathbb{Z}$.

is at most N. Let $L = AN$. For every loxodromic element $\alpha \in \Gamma$ with $\lambda(\alpha) = \lambda(\gamma) \leqslant A$, assume that $[x, y]$ and $[x', y']$ are segments in Ax_γ and Ax_α respectively, with length exactly L such that $d(x, x'), d(y, y') \leqslant r$. We may assume, up to replacing γ and α by their inverses, that γ translates from x towards y and α translates from x' towards y'. In particular for $k = 0, \dots, N$, we have $d(\alpha^{-k}\gamma^k x, x) \leqslant d(\gamma^k x, \alpha^k x') + d(x', x) \leqslant 2r$ since $\gamma^k x$ and $\alpha^k x'$ are respectively the points at distance $k\lambda(\gamma) \leqslant kA \leqslant L$ from x and x' on the segments $[x, y]$ and $[x', y']$. Hence by the definition of N, there exists $k \neq k'$ such that $\alpha^{-k}\gamma^k = \alpha^{-k'}\gamma^{k'}$. Therefore $\gamma^{k-k'} = \alpha^{k-k'}$, which implies by Equation (2.1) that $\mathrm{Ax}_\gamma = \mathrm{Ax}_\alpha$.

(3) Since the action of Γ on X is properly discontinuous, there exists $N' \in \mathbb{N}$ such that the cardinality of the stabiliser in Γ of a point of K is at most N', and there exists $\eta > 0$ such that $\lambda(\gamma) \geqslant \eta$ for every loxodromic element $\gamma \in \Gamma$ whose translation axis meets K. Let us fix such an element $\gamma \in \Gamma$. We may assume that its translation length is minimal amongst the translation lengths of the loxodromic elements $\gamma' \in \Gamma$ with $\mathrm{Ax}_\gamma = \mathrm{Ax}_{\gamma'}$. Then as seen in the proof of Assertion (1), for every loxodromic element $\alpha \in \Gamma$ with $\mathrm{Ax}_\alpha = \mathrm{Ax}_\gamma$, there exist $k \in \mathbb{N} - \{0\}$ and $\alpha' \in \Gamma$ pointwise fixing Ax_γ such that $\alpha = \gamma^k \alpha'$. Thus if $\lambda(\alpha) \leqslant A$, then $|k| = \frac{\lambda(\alpha)}{\lambda(\gamma)} \leqslant \frac{A}{\eta}$, and there are at most $N = N'(2\lceil \frac{A}{\eta} \rceil + 1)$ such elements α. □

For every $x \in X$, recall that the Gromov-Bourdon *visual distance* d_x on $\partial_\infty X$ seen from x (see [Bou]) is defined by

$$d_x(\xi, \eta) = \lim_{t \to +\infty} e^{\frac{1}{2}(d(\xi_t, \eta_t) - d(x, \xi_t) - d(x, \eta_t))} , \tag{2.2}$$

where $\xi, \eta \in \partial_\infty X$ and $t \mapsto \xi_t, \eta_t$ are any geodesic rays ending at ξ, η respectively. If X is an $\mathbb{R}$-tree, if $\xi, \eta \in \partial_\infty X$ are distinct, if $p \in X$ is such that $[x, p] = [x, \xi[\cap [x, \eta[$, then

$$d_x(\xi, \eta) = e^{-d(x, p)} . \tag{2.3}$$

For all $x \in X$, $\xi, \eta \in \partial_\infty X$ and $\gamma \in \mathrm{Isom}(X)$, we have

$$d_{\gamma x}(\gamma\xi, \gamma\eta) = d_x(\xi, \eta) .$$

By the triangle inequality, for all $x, y \in X$ and $\xi, \eta \in \partial_\infty X$, we have

$$e^{-d(x, y)} \leqslant \frac{d_x(\xi, \eta)}{d_y(\xi, \eta)} \leqslant e^{d(x, y)} . \tag{2.4}$$

In particular, the identity map from $(\partial_\infty X, d_x)$ to $(\partial_\infty X, d_y)$ is a bilipschitz homeomorphism. Under our assumptions, $(\partial_\infty X, d_{x_0})$ is hence a compact metric space, on which $\mathrm{Isom}(X)$ acts by bilipschitz homeomorphisms. The following well-known result compares shadows of balls to balls for the visual distance.

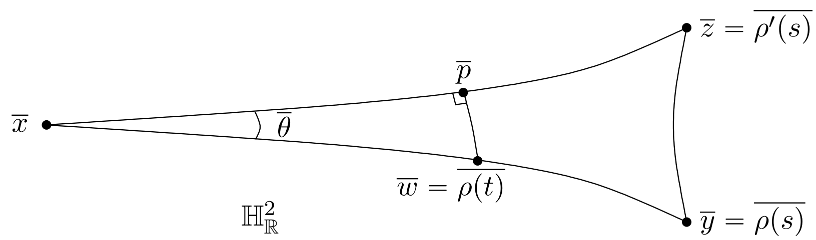
Lemma 2.2. *For every geodesic ray ρ in X, starting from $x \in X$ and ending at $\xi \in \partial_\infty X$, for all $R \geqslant 0$ and $t \in \,]R, +\infty[$, we have*

$$B_{d_x}(\xi, R\,e^{-t}) \subset \mathscr{O}_x B(\rho(t), R) \subset B_{d_x}(\xi, e^R\,e^{-t}) .$$

Proof. In order to prove the left inclusion, we adapt the proof of the left inclusion in [HeP2, Lem. 3.1] (which only uses the CAT(-1) property). Let $\xi' \in B_{d_x}(\xi, R\,e^{-t}) - \{\xi\}$, let ρ' be the geodesic ray from x to ξ' and let p be the closest point to $w = \rho(t)$ on the image of ρ'.

For every $s > t$, let $y = \rho(s)$ and $z = \rho'(s)$. Let $(\overline{x}, \overline{y}, \overline{z})$ be a comparison triangle in $\mathbb{H}^2_{\mathbb{R}}$ of (x, y, z), and let $\overline{\theta} \in [0, \pi]$ be its angle at $\overline{x}$. Let $\overline{w}$ be the point on $[\overline{x}, \overline{y}]$ at distance t from $\overline{x}$, and let $\overline{p}$ be its orthogonal projection on $[\overline{x}, \overline{z}]$. By the CAT$(-1)$ property, we have

$$d(w, p) \leqslant d(\overline{w}, \overline{p}) .$$



By the hyperbolic sine rule for right angled triangles in $\mathbb{H}^2_{\mathbb{R}}$, we have

$$\sin \overline{\theta} = \frac{\sinh d(\overline{w}, \overline{p})}{\sinh t} \quad \text{and} \quad \sin \frac{\overline{\theta}}{2} = \frac{\sinh \frac{1}{2} d(\overline{y}, \overline{z})}{\sinh d(\overline{x}, \overline{y})} = \frac{\sinh \frac{1}{2} d(y, z)}{\sinh s} .$$

Hence

$$d(w, p) \leqslant d(\overline{w}, \overline{p}) \leqslant \frac{e^{d(\overline{x}, \overline{w})}}{2 \sinh d(\overline{x}, \overline{w})} \sinh d(\overline{w}, \overline{p}) = \frac{1}{2} e^t \sin \overline{\theta} \leqslant e^t \sin \frac{\overline{\theta}}{2} .$$

Since

$$\lim_{s \to +\infty} \sin \frac{\overline{\theta}}{2} = \lim_{s \to +\infty} e^{\frac{1}{2} d(y,z) - s} = d_x(\xi, \xi') \leqslant R\, e^{-t} ,$$

we hence have $d(w, p) \leqslant R$, so that $\xi' \in \mathscr{O}_x B(w, R)$, as wanted.

In order to prove the right inclusion in Lemma 2.2, let $\xi' \in \mathscr{O}_x B(\rho(t), R)$ and let ρ' be the geodesic ray from x to ξ'. The closest point p to $\rho(t)$ on the image of ρ' satisfies $d(p, \rho(t)) \leqslant R$, hence $d(x, p) \geqslant t - R$. Therefore, for t' large enough, $d(\rho'(t'), p) \leqslant t' - (t - R)$, and

$$\begin{aligned} d_x(\xi', \xi) &\leqslant \limsup_{t' \to +\infty}\; e^{\frac{1}{2}\big(d(\rho(t'), \rho(t)) + d(\rho(t), p) + d(p, \rho'(t')) \big) - t'} \\ &\leqslant \lim_{t' \to +\infty} e^{\frac{1}{2}\big((t' - t) + R + (t' - t + R)\big) - t'} = e^R e^{-t} . \end{aligned}$$

Therefore $\xi' \in B_{d_x}(\xi, e^R\, e^{-t})$, as wanted. □

The *Busemann cocycle* of X is the map $\beta : \partial_\infty X \times X \times X \to \mathbb{R}$ defined by

$$(\xi, x, y) \mapsto \beta_\xi(x, y) = \lim_{t \to +\infty} d(\xi_t, x) - d(\xi_t, y) ,$$

where $t \mapsto \xi_t$ is any geodesic ray ending at ξ. If X is an $\mathbb{R}$-tree, if $p \in X$ is such that $[x, \xi[\, \cap\, [y, \xi[\, = [p, \xi[$, then

$$\beta_\xi(x, y) = d(x, p) - d(y, p) . \tag{2.5}$$

The triangle inequality gives immediately the upper bound

$$|\, \beta_\xi(x, y)\, | \leqslant d(x, y) . \tag{2.6}$$

The *horosphere* with centre $\xi \in \partial_\infty X$ through $x \in X$ is $\{y \in X \;:\; \beta_\xi(x, y) = 0\}$, and $\{y \in X \;:\; \beta_\xi(x, y) \leqslant 0\}$ is the (closed) *horoball* centred at ξ bounded by this horosphere. Horoballs are nonempty proper closed (strictly) convex subsets of X. Given a horoball $\mathscr{H}$ and $t \geqslant 0$, we denote by $\mathscr{H}[t] = \{x \in \mathscr{H} \;:\; d(x, \partial \mathscr{H}) \geqslant t\}$ the horoball contained in $\mathscr{H}$ (hence centred at the same point at infinity as $\mathscr{H}$) whose boundary is at distance t from the boundary of $\mathscr{H}$.

2.2 Generalised geodesic lines

Let $\breve{\mathscr{G}}X$ be the space of 1-Lipschitz maps $w : \mathbb{R} \to X$ which are isometric on a closed interval and locally constant outside it.[5] This space has been introduced by Bartels and Lück in [BartL], to which we refer for the following basic properties. The elements of $\breve{\mathscr{G}}X$ are called the *generalised geodesic lines* of X. Any geodesic segment or ray of X will be considered as an element of $\breve{\mathscr{G}}X$, by extending it continuously to $\mathbb{R}$ as locally constant outside its domain of definition.

We endow $\breve{\mathscr{G}}X$ with the distance $d = d_{\breve{\mathscr{G}}X}$ defined by

$$\forall\, w, w' \in \breve{\mathscr{G}}X, \quad d(w, w') = \int_{-\infty}^{+\infty} d(w(t), w'(t))\, e^{-2|t|}\, dt\,. \tag{2.7}$$

The group $\operatorname{Isom}(X)$ acts isometrically on $\breve{\mathscr{G}}X$ by postcomposition. The distance d induces the topology of uniform convergence on compact subsets on $\breve{\mathscr{G}}X$, and $\breve{\mathscr{G}}X$ is a proper metric space.

The *geodesic flow* $(\mathtt{g}^t)_{t\in\mathbb{R}}$ on $\breve{\mathscr{G}}X$ is the one-parameter group of homeomorphisms of the space $\breve{\mathscr{G}}X$ defined by $\mathtt{g}^t w : s \mapsto w(s+t)$ for all $w \in \breve{\mathscr{G}}X$ and $t \in \mathbb{R}$. It commutes with the action of $\operatorname{Isom}(X)$. If w is isometric exactly on the interval I, then $\mathtt{g}^{-t}w$ is isometric exactly on the interval $t + I$.

The *footpoint projection* is the $\operatorname{Isom}(X)$-equivariant $\frac{1}{2}$-Hölder-continuous[6] map $\pi : \breve{\mathscr{G}}X \to X$ defined by $\pi(w) = w(0)$ for all $w \in \breve{\mathscr{G}}X$. The *antipodal map* of $\breve{\mathscr{G}}X$ is the $\operatorname{Isom}(X)$-equivariant isometric map $\iota : \breve{\mathscr{G}}X \to \breve{\mathscr{G}}X$ defined by $\iota w : s \mapsto w(-s)$ for all $w \in \breve{\mathscr{G}}X$, which satisfies $\iota \circ \mathtt{g}^t = \mathtt{g}^{-t} \circ \iota$ for every $t \in \mathbb{R}$ and $\pi \circ \iota = \pi$.

The *positive* and *negative* endpoint maps are the continuous maps from $\breve{\mathscr{G}}X$ to $X \cup \partial_\infty X$ defined by

$$w \mapsto w_\pm = \lim_{t\to\pm\infty} w(t)\,.$$

The space $\mathscr{G}X$ of geodesic lines in X is the $\operatorname{Isom}(X)$-invariant closed metric subspace of $\breve{\mathscr{G}}X$ consisting of the elements $\ell \in \breve{\mathscr{G}}X$ with $\ell_\pm \in \partial_\infty X$. Note that the distances on $\mathscr{G}X$ considered in [BartL] and [PauPS] are topologically equivalent to, although slightly different from, the restriction to $\mathscr{G}X$ of the distance defined in Equation (2.7). The factor $e^{-2|t|}$ in this equation, sufficient in order to deal with Hölder-continuity issues, is replaced by $e^{-t^2}/\sqrt{\pi}$ in [PauPS] and by $e^{-|t|}/2$ in [BartL].

Note that for all $w \in \breve{\mathscr{G}}X$ and $s \in \mathbb{R}$, we have

$$d(w, \mathtt{g}^s w) \leqslant |s|\,, \tag{2.8}$$

with equality if $w \in \mathscr{G}X$.

We will also consider the $\operatorname{Isom}(X)$-invariant closed subspaces

$$\mathscr{G}_\pm X = \{w \in \breve{\mathscr{G}}X : w_\pm \in \partial_\infty X\}\,,$$

[5] that is, constant on each complementary component

[6] See Section 3.1 for the definition of the (locally uniform) Hölder-continuity used in this book, and Proposition 3.2 for a proof of this claim.

and their $\operatorname{Isom}(X)$-invariant closed subspaces $\mathscr{G}_{\pm,0}X$ consisting of the elements $\rho \in \mathscr{G}_\pm X$ which are isometric exactly on $\pm[0,+\infty[$.

The subspaces $\mathscr{G}X$ and $\mathscr{G}_\pm X$ satisfy $\mathscr{G}_- X \cap \mathscr{G}_+ X = \mathscr{G}X$ and they are invariant under the geodesic flow. The antipodal map ι preserves $\mathscr{G}X$, and maps $\mathscr{G}_\pm X$ to $\mathscr{G}_\mp X$ as well as $\mathscr{G}_{\pm,0}X$ to $\mathscr{G}_{\mp,0}X$. We denote again by $\iota : \Gamma\backslash\breve{\mathscr{G}}X \to \Gamma\backslash\breve{\mathscr{G}}X$ and by $(\mathsf{g}^t)_{t\in\mathbb{R}}$ with $\mathsf{g}^t : \Gamma\backslash\breve{\mathscr{G}}X \to \Gamma\backslash\breve{\mathscr{G}}X$ the quotient maps of ι and g^t, for every $t \in \mathbb{R}$.

Let $w \in \breve{\mathscr{G}}X$ be isometric exactly on an interval I of $\mathbb{R}$. If I is compact then w is a *(generalised) geodesic segment*, and if $I =]-\infty, a]$ or $I = [a, +\infty[$ for some $a \in \mathbb{R}$, then w is a *(generalised) (negative or positive) geodesic ray* in X. Any geodesic line $\widehat{w} \in \mathscr{G}X$ such that $\widehat{w}|_I = w|_I$ is an *extension* of w. Note that $\widehat{w}$ is an extension of w if and only if $\gamma\widehat{w}$ is an extension of γw for any $\gamma \in \operatorname{Isom}(X)$, if and only if $\iota\widehat{w}$ is an extension of ιw, and if and only if $\mathsf{g}^s\widehat{w}$ is an extension of $\mathsf{g}^s w$ for any $s \in \mathbb{R}$. For any subset Ω' of $\breve{\mathscr{G}}X$ and any subset A of $\mathbb{R}$, let

$$\Omega'|_A = \{w|_A : w \in \Omega'\}\,.$$

Remark 2.3. Let $(\ell_i)_{i\in\mathbb{N}}$ be a sequence of generalised geodesic lines such that $[t_i^-, t_i^+]$ is the maximal segment on which ℓ_i is isometric. Let $(s_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $t_i^\pm - s_i \to \pm\infty$ as $i \to +\infty$ and $\ell_i(s_i)$ stays in a compact subset of X, then $d(\ell_i, \mathscr{G}X) \to 0$ as $i \to +\infty$. Furthermore if $(s_i)_{i\in\mathbb{N}}$ is bounded, then up to extracting a subsequence, $(\ell_i)_{i\in\mathbb{N}}$ converges to an element in $\mathscr{G}X$.

This conceptually important observation explains how it is conceivable that long common perpendicular segments may equidistribute towards measures supported on geodesic lines. See Chapter 11 for further developments of these ideas.

2.3 The unit tangent bundle

In this book, we define the *unit tangent bundle* T^1X of X as the space of germs at 0 of the geodesic lines in X. It is the quotient space

$$T^1X = \mathscr{G}X/\sim$$

where $\ell \sim \ell'$ if and only if there exists $\epsilon > 0$ such that $\ell|_{[-\epsilon,\epsilon]} = \ell'|_{[-\epsilon,\epsilon]}$. The canonical projection from $\mathscr{G}X$ to T^1X will be denoted by $\ell \mapsto v_\ell$. When X is a Riemannian manifold, the spaces $\mathscr{G}X$ and T^1X canonically identify with the usual unit tangent bundle of X, but in general, the map $\ell \mapsto v_\ell$ has infinite fibers.

We endow T^1X with the quotient distance $d = d_{T^1X}$ of the distance of $\mathscr{G}X$, defined by:

$$\forall\, v, v' \in T^1X, \quad d_{T^1X}(v, v') = \inf_{\ell,\,\ell' \in \mathscr{G}X\,:\, v = v_\ell,\, v' = v_{\ell'}} d(\ell, \ell')\,. \tag{2.9}$$

It is easy to check that this distance is indeed Hausdorff, hence that T^1X is locally compact, and that it induces on T^1X the quotient topology of the compact-open topology of $\mathscr{G}X$. The map $\ell \mapsto v_\ell$ is 1-Lipschitz.

The action of $\operatorname{Isom}(X)$ on $\mathscr{G}X$ induces an isometric action of $\operatorname{Isom}(X)$ on T^1X. The antipodal map and the footpoint projection restricted to $\mathscr{G}X$ respectively induce an $\operatorname{Isom}(X)$-equivariant isometric map $\iota : T^1X \to T^1X$ and an $\operatorname{Isom}(X)$-equivariant $\frac{1}{2}$-Hölder-continuous map $\pi : T^1X \to X$ called the *antipodal map* and *footpoint projection* of T^1X. The canonical projection from $\mathscr{G}X$ to T^1X is $\mathrm{Isom}(X)$-equivariant and commutes with the antipodal map: For all $\gamma \in \mathrm{Isom}(X)$ and $\ell \in \mathscr{G}X$, we have $\gamma v_\ell = v_{\gamma\ell}$, $\iota v_\ell = v_{\iota\ell}$ and $\pi(v_\ell) = \pi(\ell)$. We denote again by $\iota : \Gamma\backslash T^1X \to \Gamma\backslash T^1X$ the quotient map of ι.

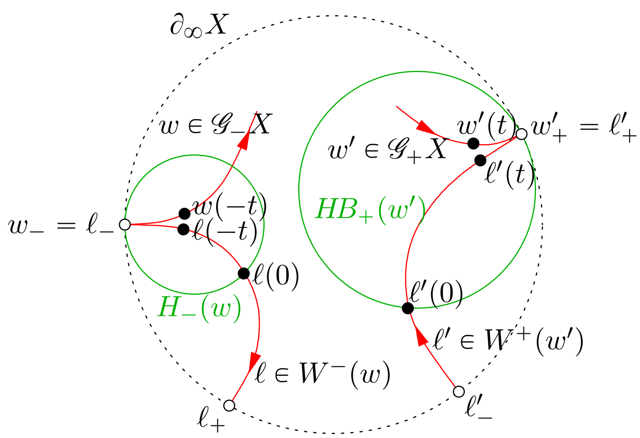
Let $\partial^2_\infty X$ be the subset of $\partial_\infty X \times \partial_\infty X$ which consists of pairs of distinct points at infinity of X. *Hopf's parametrisation* of $\mathscr{G}X$ is the homeomorphism which identifies $\mathscr{G}X$ with $\partial^2_\infty X \times \mathbb{R}$, by the map $\ell \mapsto (\ell_-, \ell_+, t)$, where t is the signed distance from the closest point to the basepoint x_0 on the geodesic line ℓ to $\ell(0)$.[7] We have $\mathtt{g}^s(\ell_-, \ell_+, t) = (\ell_-, \ell_+, t+s)$ for all $s \in \mathbb{R}$, and for all $\gamma \in \Gamma$, we have $\gamma(\ell_-, \ell_+, t) = (\gamma\ell_-, \gamma\ell_+, t + t_{\gamma,\ell_-,\ell_+})$ where $t_{\gamma,\ell_-,\ell_+} \in \mathbb{R}$ depends only on γ, ℓ_- and ℓ_+. In Hopf's parametrisation, the restriction of the antipodal map to $\mathscr{G}X$ is the map $(\ell_-, \ell_+, t) \mapsto (\ell_+, \ell_-, -t)$.

The *strong stable leaf* of $w \in \mathscr{G}_+X$ is

$$W^+(w) = \left\{\ell \in \mathscr{G}X \ : \lim_{t\to+\infty} d(\ell(t), w(t)) = 0\right\},$$

and the *strong unstable leaf* of $w \in \mathscr{G}_-X$ is

$$W^-(w) = \iota W^+(\iota w) = \left\{\ell \in \mathscr{G}X \ : \lim_{t\to-\infty} d(\ell(t), w(t)) = 0\right\}.$$



For every $w \in \mathscr{G}_\pm X$, let $d_{W^\pm(w)}$ be *Hamenstädt's distance* on $W^\pm(w)$ defined as follows:[8] for all $\ell, \ell' \in W^\pm(w)$, let

$$d_{W^\pm(w)}(\ell, \ell') = \lim_{t\to+\infty} e^{\frac{1}{2}d(\ell(\mp t),\, \ell'(\mp t))-t}.$$

The above limits exist, and Hamenstädt's distances are distances inducing the original topology on $W^\pm(w)$. For all $\ell, \ell' \in W^\pm(w)$ and $\gamma \in \mathrm{Isom}(X)$, we have

$$\gamma W^\pm(w) = W^\pm(\gamma w)$$

and

$$d_{W^\pm(\gamma w)}(\gamma\ell, \gamma\ell') = d_{W^\pm(w)}(\ell, \ell') = d_{W^\mp(\iota w)}(\iota\ell, \iota\ell').$$

[7]More precisely, $\ell(t)$ is the closest point to x_0 on ℓ.
[8]See [HeP1, Appendix] and compare with [Ham1].

Furthermore, for every $s \in \mathbb{R}$, we have

$$\mathtt{g}^s W^{\pm}(w) = W^{\pm}(\mathtt{g}^s w)$$

and for all $\ell, \ell' \in W^{\pm}(w)$

$$d_{W^{\pm}(\mathtt{g}^s w)}(\mathtt{g}^s \ell, \mathtt{g}^s \ell') = e^{\mp s} d_{W^{\pm}(w)}(\ell, \ell') \;. \tag{2.10}$$

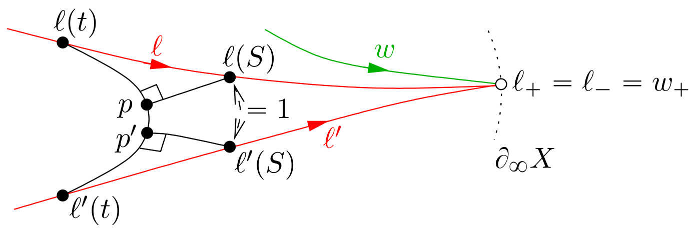
If X is an $\mathbb{R}$-tree, for all $w \in \mathscr{G}_+ X$ and $\ell, \ell' \in W^+(w)$, if $[s, +\infty[$ is the maximal interval on which ℓ and ℓ' agree, then $d_{W^+(w)}(\ell, \ell') = e^s$.

The following lemma compares the distance in $\mathscr{G}X$ with Hamenstädt's distance for two geodesic lines in the same strong (un)stable leaf.

Lemma 2.4. *There exists a universal constant $c > 0$ such that for all $w \in \mathscr{G}_\pm X$ and $\ell, \ell' \in W^{\pm}(w)$, we have*

$$d(\ell, \ell') \leqslant c\, d_{W^{\pm}(w)}(\ell, \ell') \quad \text{and} \quad d(\pi(\ell), \pi(\ell')) \leqslant d_{W^{\pm}(w)}(\ell, \ell') \;.$$

Proof. We could refer to [PaP14a, Lem. 3] (see also [PauPS, Lem. 2.4]) for a proof of the first result. Note that the distance on $\mathscr{G}X$ considered in loc. cit. is slightly different from the one in this book, hence we give a full proof for the sake of completeness. We assume that $w \in \mathscr{G}_+ X$, the proof when $w \in \mathscr{G}_- X$ is similar.



Let $\ell, \ell' \in W^+(w)$. We may assume that $\ell \neq \ell'$. By the convexity properties of the distance in X, the map from $\mathbb{R}$ to $\mathbb{R}$ defined by $t \mapsto d(\ell(t), \ell'(t))$ is decreasing, with image $]0, +\infty[$. Let $S \in \mathbb{R}$ be such that $d(\ell(S), \ell'(S)) = 1$. For every $t \leqslant S$, let p and p' be the closest point projections of $\ell(S)$ and $\ell'(S)$ on the geodesic segment $[\ell(t), \ell'(t)]$. We have $d(p, \ell(S)), d(p', \ell'(S)) \leqslant 1$ by comparison. Hence, by convexity and the triangle inequality,

$$\begin{aligned} d(\ell(t), \ell'(t)) &\geqslant d(\ell(t), p) + d(p', \ell'(t)) \\ &\geqslant d(\ell(t), \ell(S)) - 1 + d(\ell'(t), \ell'(S)) - 1 = 2(S - t - 1) \;. \end{aligned}$$

Thus by the definition of the Hamenstädt distance $d_{W^+(w)}$, we have

$$d_{W^+(w)}(\ell, \ell') \geqslant e^{S-1} \;. \tag{2.11}$$

By the triangle inequality, if $t \leqslant S$, then

$$d(\ell(t), \ell'(t)) \leqslant d(\ell(t), \ell(S)) + d(\ell(S), \ell'(S)) + d(\ell'(S), \ell'(t)) = 2(S - t) + 1 \;.$$

Since X is CAT(-1), if $t \geqslant S$, we have by comparison

$$d(\ell(t), \ell'(t)) \leqslant e^{S-t} \sinh d(\ell(S), \ell'(S)) = (\sinh 1)\, e^{S-t} \;.$$

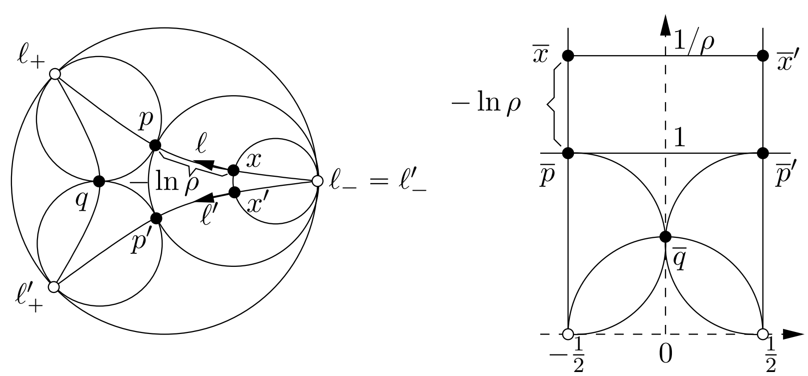
Therefore, by the definition of the distance d on $\mathscr{G}X$ (see Equation (2.7)),

$$d(\ell,\ell') \leqslant \int_{-\infty}^{S} (2(S-t)+1)\, e^{-2|t|}\, dt + (\sinh 1)\int_{S}^{+\infty} e^{S-t}\, e^{-2|t|}\, dt = \mathrm{O}(e^{S})\ .$$

The first inequality of Lemma 2.4 hence follows from Equation (2.11).

The second one is proved in [PauPS, Lem. 2.4], and we again only give a proof for the sake of completeness.

Let $x=\pi(\ell)$, $x'=\pi(\ell')$ and $\rho = d_{W^{\pm}(w)}(\ell,\ell')$. Consider the ideal triangle Δ with vertices ℓ_+,ℓ'_+ and $\ell_-=\ell'_-$ (see the picture below on the left). Let $p\in\ell(\mathbb{R})$, $p'\in\ell'(\mathbb{R})$ and $q\in\,]\ell_+,\ell'_+[$ be the tangency points of the unique triple of pairwise tangent horospheres centred at the vertices of Δ: $\beta_{\ell_-}(p,p')=0$, $\beta_{\ell_+}(p,q)=0$ and $\beta_{\ell'_+}(p',q)=0$. By the definition of the Hamenstädt distance, we have $p=\ell(-\ln\rho)$.



Consider the ideal triangle $\overline{\Delta}$ in the hyperbolic upper half-plane $\mathbb{H}^2_{\mathbb{R}}$, with vertices $-\frac{1}{2}$, $\frac{1}{2}$ and ∞ (see the above picture on the right). Let $\overline{p}=(-\frac{1}{2},1)$, $\overline{p}'=(\frac{1}{2},1)$ and $\overline{q}=(0,\frac{1}{2})$ be the pairwise tangency points of horospheres centred at the vertices of $\overline{\Delta}$. Let $\overline{x}$ and $\overline{x}'$ be the point at algebraic (hyperbolic) distance $-\ln\rho$ from $\overline{p}$ and $\overline{p}'$, respectively, on the upwards oriented vertical lines through them. By comparison, we have $d(x,x')\leqslant d(\overline{x},\overline{x}')\leqslant 1/e^{-\ln\rho}=\rho$. □

Let $\mathscr{H}$ be a horoball in X, centred at $\xi\in\partial_\infty X$. The strong stable leaves $W^+(w)$ are equal for all geodesic rays w starting at time $t=0$ from a point of $\partial\mathscr{H}$ and converging to ξ. Using the homeomorphism $\ell\mapsto\ell_-$ from $W^+(w)$ to $\partial_\infty X-\{\xi\}$, Hamenstädt's distance on $W^+(w)$ defines a distance $d_{\mathscr{H}}$ on $\partial_\infty X-\{\xi\}$ that we also call *Hamenstädt's distance*. For all $\ell,\ell'\in W^+(w)$, we have

$$d_{\mathscr{H}}(\ell_-,\ell_+) = d_{W^+(w)}(\ell,\ell')\ ,$$

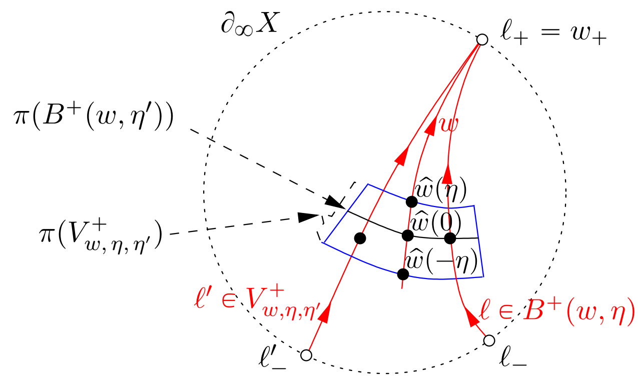
and for all $\eta,\eta'\in\partial_\infty X-\{\xi\}$, we have

$$d_{\mathscr{H}}(\eta,\eta') = \lim_{t\to+\infty} e^{\frac{1}{2}d(\ell_\eta(-t),\,\ell_{\eta'}(-t))-t}\ , \tag{2.12}$$

where $\ell_\eta,\ell_{\eta'}$ are the geodesic lines starting from η,η' respectively, ending at ξ, and passing through the boundary of $\mathscr{H}$ at time $t=0$. Note that for every $t\geqslant 0$, if $\mathscr{H}[t]$ is the horoball contained in $\mathscr{H}$ whose boundary is at distance t from the boundary of $\mathscr{H}$, then we have

$$d_{\mathscr{H}[t]} = e^{-t}\, d_{\mathscr{H}}\ . \tag{2.13}$$

Let $w \in \mathscr{G}_\pm X$ and $\eta' > 0$. We define $B^\pm(w, \eta')$ as the set of $\ell \in W^\pm(w)$ such that there exists an extension $\widehat{w} \in \mathscr{G}X$ of w with $d_{W^\pm(w)}(\ell, \widehat{w}) < \eta'$. In particular, $B^\pm(w, \eta')$ contains all the extensions of w, and is the union of the open balls centred at the extensions of w, of radius η', for Hamenstädt's distance on $W^\pm(w)$.



The union over $t \in \mathbb{R}$ of the images under g^t of the strong stable leaf of $w \in \mathscr{G}_+X$ is the *stable leaf*

$$W^{0+}(w) = \bigcup_{t \in \mathbb{R}} \mathsf{g}^t W^+(w)$$

of w, which consists of the elements $\ell \in \mathscr{G}X$ with $\ell_+ = w_+$. Similarly, the *unstable leaf* of $w \in \mathscr{G}_-X$

$$W^{0-}(w) = \bigcup_{t \in \mathbb{R}} \mathsf{g}^t W^-(w)\,,$$

consists of the elements $\ell \in \mathscr{G}X$ with $\ell_- = w_-$. Note that the (strong) (un)stable leaves are subsets of the space of geodesic lines $\mathscr{G}X$. The (un)stable leaves are invariant under the geodesic flow, and for all $w \in \mathscr{G}_\pm X$ and $\gamma \in \mathrm{Isom}(X)$, we have

$$\iota\, W^{0\pm}(w) = W^{0\mp}(\iota\, w) \quad \text{and} \quad \gamma W^{0\pm}(w) = W^{0\pm}(\gamma w)\;.$$

The *unstable horosphere* $H_-(w)$ of $w \in \mathscr{G}_-X$ is the horosphere in X centred at w_- and passing through $\widehat{w}(0)$ for any extension $\widehat{w} \in \mathscr{G}X$ of w (see the picture above the definition of Hamenstädt's distance). The *stable horosphere* $H_+(w)$ of $w \in \mathscr{G}_+X$ is the horosphere in X centred at w_+ and passing through $\widehat{w}(0)$ for any extension $\widehat{w} \in \mathscr{G}X$ of w. These horospheres $H_\pm(w)$ do not depend on the chosen extensions $\widehat{w}$ of $w \in \mathscr{G}_\pm X$. The *unstable horoball* $HB_-(w)$ of $w \in \mathscr{G}_-X$ and *stable horoball* $HB_+(w)$ of $w \in \mathscr{G}_+X$ are the horoballs bounded by these horospheres. Note that

$$\pi(W^\pm(w)) = H_\pm(w) \tag{2.14}$$

for every $w \in \mathscr{G}_\pm X$, and that $w(0)$ belongs to $H_\pm(w)$ if and only if w is isometric at least on $\pm[0, +\infty[$.

2.4 Normal bundles and dynamical neighbourhoods

In this Section, adapting [PaP17b, §2.2] to the present context, we define spaces of geodesic rays that generalise the unit normal bundles of submanifolds of negatively curved Riemannian manifolds. When X is a manifold, these normal bundles are submanifolds of the unit tangent bundle of X, which identifies with $\mathscr{G}X$. In general and in particular in trees, it is essential to use geodesic rays to define normal bundles, and not geodesic lines.

Let D be a nonempty *proper*[9] closed convex subset in X. We denote by ∂D its boundary in X and by $\partial_\infty D$ its set of points at infinity. Let

$$P_D : X \cup (\partial_\infty X - \partial_\infty D) \to D$$

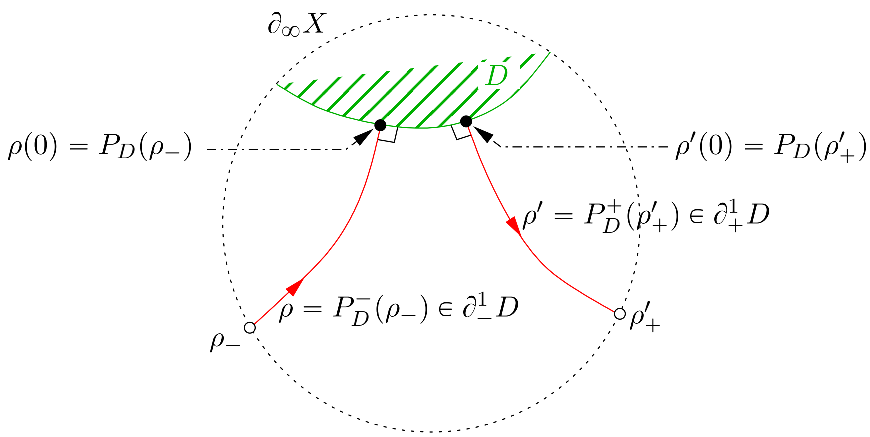
be the (continuous) *closest point map to* D, defined on $\xi \in \partial_\infty X - \partial_\infty D$ by setting $P_D(\xi)$ to be the unique point in D that minimises the function $y \mapsto \beta_\xi(y, x_0)$ from D to $\mathbb{R}$. The *outer unit normal bundle* $\partial^1_+ D$ of (the boundary of) D is

$$\partial^1_+ D = \{\rho \in \mathscr{G}_{+,0}X \ : \ P_D(\rho_+) = \rho(0)\}\,.$$

The *inner unit normal bundle* $\partial^1_- D$ of (the boundary of) D is

$$\partial^1_- D = \iota\partial^1_+ D = \{\rho \in \mathscr{G}_{-,0}X \ : \ P_D(\rho_-) = \rho(0)\}.$$

Note that $\partial^1_+ D$ and $\partial^1_- D$ are spaces of geodesic rays. If X is a smooth manifold, then these spaces have a natural identification with subsets of $\mathscr{G}X$ because every geodesic ray is the restriction of a unique geodesic line. But this does not hold in general.



Remark 2.5. As X is assumed to be proper with extendible geodesics, we have

$$\pi(\partial^1_\pm D) = \partial D\ .$$

To see this, let $x \in \partial D$ and let $(x_k)_{k\in\mathbb{N}}$ be a sequence of points in the complement of D converging to x. For all $k \in \mathbb{N}$, let $\rho_k \in \partial^1_+ D$ be a geodesic ray with $\rho_k(0) = P_D(x_k)$ and such that the image of ρ_k contains x_k. As the closest point map does not increase distances, the sequence $(P_D(x_k))_{k\in\mathbb{N}}$ converges to x. Since X is proper, the space $\partial_\infty X$ is compact and the sequence $((\rho_k)_+)_{k\in\mathbb{N}}$ has a subsequence that converges to a point $\xi \in \partial_\infty X$. The claim follows from the continuity of the closest point map.

The possible failure of this equality when X is not proper is easy to see. For example, let X be the $\mathbb{R}$-tree constructed by starting with the Euclidean line $D = \mathbb{R}$ and attaching a copy of the halfline $[0, +\infty[$ at each $x \in D$ such that $x > 0$. Then $0 \in \partial D - \pi(\partial^1_\pm D)$.

[9]that is, different from X

The restriction of the endpoint map $\rho \mapsto \rho_\pm$ to $\partial^1_\pm D$ is a homeomorphism to its image $\partial_\infty X - \partial_\infty D$. We denote its inverse map by $P^\pm_D$ (see the above picture): for every $\rho \in \partial^1_\pm D$, we have

$$\rho = P^\pm_D(\rho_\pm)\ .$$

Note that $P_D = \pi \circ P^\pm_D$. For every isometry γ of X, we have $\partial^1_\pm(\gamma D) = \gamma\, \partial^1_\pm D$ and $P^\pm_{\gamma D} \circ \gamma = \gamma \circ P^\pm_D$. In particular, $\partial^1_\pm D$ is invariant under the isometries of X that preserve D.

For every $w \in \mathscr{G}_\pm X$, we have a canonical homeomorphism

$$N^\pm_w : W^\pm(w) \to \partial^1_\mp HB_\pm(w)\ ,$$

that associates to each geodesic line $\ell \in W^\pm(w)$ the unique geodesic ray $\rho \in \partial^1_\mp HB_\pm(w)$ such that $\ell_\mp = \rho_\mp$, or, equivalently, such that $\ell(t) = \rho(t)$ for every $t \in \mathbb{R}$ with $\mp t > 0$. It is easy to check that $N^\pm_{\gamma w} \circ \gamma = \gamma \circ N^\pm_w$ for every $\gamma \in \operatorname{Isom}(X)$.

We define

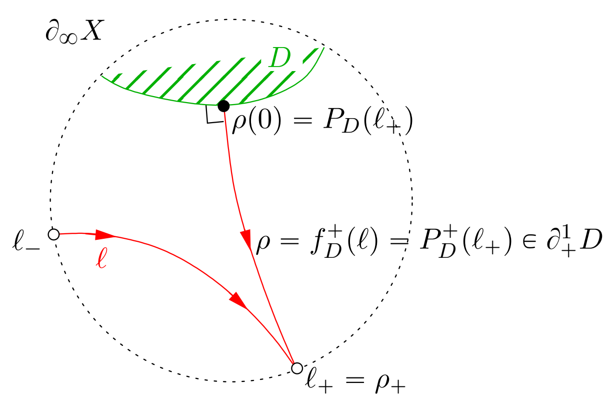
$$\mathscr{U}^\pm_D = \{\ell \in \mathscr{G}X\ :\ \ell_\pm \notin \partial_\infty D\}\ . \tag{2.15}$$

Note that $\mathscr{U}^\pm_D$ is an open subset of $\mathscr{G}X$, invariant under the geodesic flow. We have $\mathscr{U}^\pm_{\gamma D} = \gamma \mathscr{U}^\pm_D$ for every isometry γ of X and, in particular, $\mathscr{U}^\pm_D$ is invariant under the isometries of X preserving D. Define

$$f^\pm_D : \mathscr{U}^\pm_D \to \partial^1_\pm D$$

as the composition of the continuous endpoint map $\ell \mapsto \ell_\pm$ from $\mathscr{U}^\pm_D$ onto $\partial_\infty X - \partial_\infty D$ and the homeomorphism $P^\pm_D$ from $\partial_\infty X - \partial_\infty D$ to $\partial^1_\pm D$ (see the picture below). The continuous map $f^\pm_D$ takes $\ell \in \mathscr{U}^\pm_D$ to the unique element $\rho \in \partial^1_\pm D$ such that $\rho_\pm = \ell_\pm$. The fiber of $\rho \in \partial^1_+ D$ for f^+_D is exactly the stable leaf $W^{0+}(\rho)$, and the fiber of $\rho \in \partial^1_- D$ for f^-_D is the unstable leaf $W^{0-}(\rho)$. For all $\gamma \in \operatorname{Isom}(X)$ and $t \in \mathbb{R}$, we have

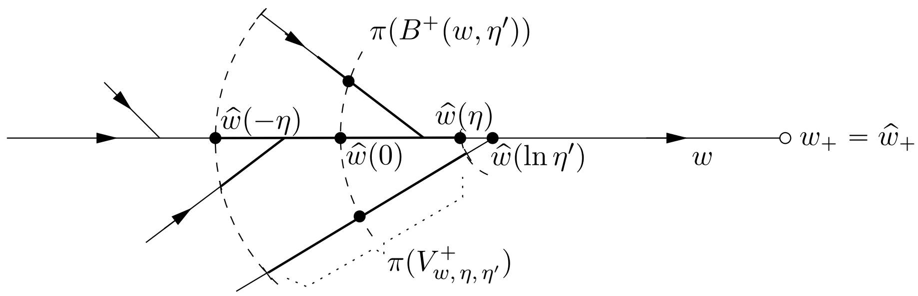
$$f^\pm_{\gamma D} \circ \gamma = \gamma \circ f^\pm_D \quad \text{and} \quad f^\pm_D \circ \mathtt{g}^t = f^\pm_D\ . \tag{2.16}$$



Let $w \in \mathscr{G}_\pm X$ and $\eta, \eta' > 0$. We define (see the picture after Equation (2.13)) the *dynamical (η, η')-neighbourhood* of w by

$$V^\pm_{w,\,\eta,\,\eta'} = \bigcup_{s \in\,]-\eta,\, \eta[} \mathtt{g}^s B^\pm(w, \eta')\ . \tag{2.17}$$

Example 2.6. If X is an $\mathbb{R}$-tree, $w \in \mathscr{G}_+X$ and $\eta < \ln \eta'$, then $V^+_{w,\,\eta,\,\eta'}$, which is the set of $\mathsf{g}^s\ell$ where $s \in \,]-\eta, \eta[$ and $\ell \in \mathscr{G}X$ is such that there exists an extension $\widehat{w}$ of w with $\inf\{t \in \mathbb{R} \,:\, \ell(t) = \widehat{w}(t)\} \leqslant \ln \eta'$, is as in the following picture.



Clearly, $B^\pm(w, \eta') = \iota B^\mp(\iota w, \eta')$, and hence we have $V^\pm_{w,\,\eta,\,\eta'} = \iota V^\mp_{\iota w,\,\eta,\,\eta'}$. Furthermore, for every $s \in \mathbb{R}$,

$$\mathsf{g}^s B^\pm(w, \eta') = B^\pm(\mathsf{g}^s w, e^{\mp s}\eta')\,, \;\text{ hence }\; \mathsf{g}^s V^\pm_{w,\,\eta,\,\eta'} = V^\pm_{\mathsf{g}^s w,\,\eta,\,e^{\mp s}\eta'}\,. \tag{2.18}$$

For every $\gamma \in \mathrm{Isom}(X)$, we have $\gamma B^\pm(w, \eta') = B^\pm(\gamma w, \eta')$ and $\gamma V^\pm_{w,\,\eta,\,\eta'} = V^\pm_{\gamma w,\,\eta,\,\eta'}$. The map from $]-\eta, \eta[\, \times B^\pm(w, \eta')$ to $V^\pm_{w,\,\eta,\,\eta'}$ defined by $(s, \ell') \mapsto \mathsf{g}^s\ell'$ is a homeomorphism.

For all subsets Ω^- of $\mathscr{G}_+X$ and Ω^+ of $\mathscr{G}_-X$, let

$$\mathscr{V}^\pm_{\eta,\,\eta'}(\Omega^\mp) = \bigcup_{w \in \Omega^\mp} V^\pm_{w,\,\eta,\,\eta'}\,, \tag{2.19}$$

that we call the *dynamical neighbourhoods* of $\Omega^\mp$. Note that they are subsets of $\mathscr{G}X$, not of $\mathscr{G}_\pm X$. The families $(\mathscr{V}^\pm_{\eta,\,\eta'}(\Omega^\mp))_{\eta,\eta'>0}$ are nondecreasing in η and in η'. For every $\gamma \in \mathrm{Isom}(X)$, we have $\gamma \mathscr{V}^\pm_{\eta,\,\eta'}(\Omega^\mp) = \mathscr{V}^\pm_{\eta,\,\eta'}(\gamma\Omega^\mp)$ and for every $t \geqslant 0$, we have

$$\mathsf{g}^{\pm t}\mathscr{V}^\pm_{\eta,\,\eta'}(\Omega^\mp) = \mathscr{V}^\pm_{\eta,\,e^{-t}\eta'}(\mathsf{g}^{\pm t}\Omega^\mp)\,. \tag{2.20}$$

Note that

$$\bigcup_{\eta\,,\,\eta'>0} \mathscr{V}^\pm_{\eta,\,\eta'}(\partial^1_\pm D) = \mathscr{U}^\pm_D\,,$$

and that $\bigcap_{\eta,\,\eta'>0} \mathscr{V}^\pm_{\eta,\,\eta'}(\partial^1_\pm D)$ is the set of all extensions in $\mathscr{G}X$ of the elements of $\partial^1_\pm D$. Assume that $\Omega^\mp$ is a subset of $\partial^1_\pm D$. The restriction of $f^\pm_D$ to $\mathscr{V}^\pm_{\eta,\,\eta'}(\Omega^\mp)$ is a continuous map onto $\Omega^\mp$, with fiber over $w \in \Omega^\mp$ the open subset $V^\pm_{w,\,\eta,\,\eta'}$ of $W^{0\pm}(w)$.

We will need the following elementary lemma in Section 10.4.

Lemma 2.7. *There exists a universal constant $c' > 0$ such that for every $w \in \mathscr{G}_+X$ which is isometric on $[s_w, +\infty[$ and every $\ell \in V^+_{w,\,\eta,\,\eta'}$, we have*

$$d(\ell, w) \leqslant c'(\eta + \eta' + e^{s_w})\,.$$

Proof. By Equation (2.17) and by the definition of $B^+(w,\eta')$ in Section 2.2, there exist $s\in\,]-\eta,+\eta[$ and an extension $\widehat{w}\in\mathscr{G}X$ of w such that $\mathsf{g}^s\ell\in W^+(w)$ and $d_{W^+(w)}(\mathsf{g}^s\ell,\widehat{w})\leqslant\eta'$. By Equation (2.8), we have $d(\ell,\mathsf{g}^s\ell)\leqslant|s|\leqslant\eta$. By Lemma 2.4, we have $d(\mathsf{g}^s\ell,\widehat{w})\leqslant c\;d_{W^+(w)}(\mathsf{g}^s\ell,\widehat{w})\leqslant c\;\eta'$. By the definition of the distance on $\breve{\mathscr{G}}X$ (see Equation (2.7)), we have

$$d(\widehat{w},w)\leqslant\int_{-\infty}^{s_w}|s_w-t|\;e^{-2|t|}\;dt=\operatorname{O}(e^{s_w})\;.$$

Therefore the result follows from the triangle inequality

$$d(\ell,w)\leqslant d(\ell,\mathsf{g}^s\ell)+d(\mathsf{g}^s\ell,\widehat{w})+d(\widehat{w},w)\;.\quad\square$$

2.5 Creating common perpendiculars

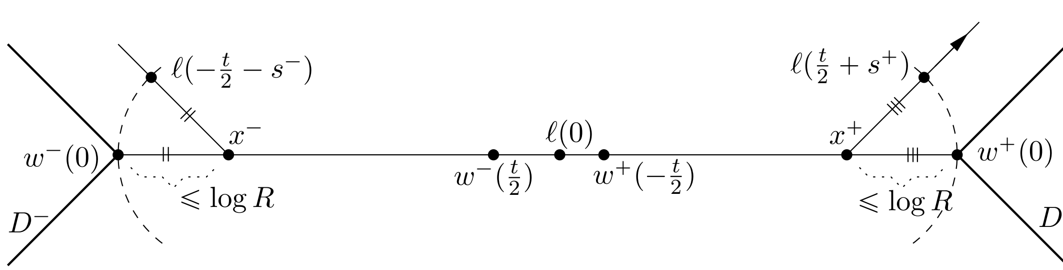
Let D^- and D^+ be two nonempty proper closed convex subsets of X, where X is as in the beginning of Section 2.1. A geodesic arc $\alpha:[0,T]\rightarrow X$, where $T>0$, is a *common perpendicular of length T from D^- to D^+* if there exists $w^\mp\in\partial^1_\pm D^\mp$ such that $w^-|_{[0,T]}=\mathsf{g}^{-T}w^+|_{[0,T]}=\alpha$. Since X is CAT(-1), this geodesic arc α is the unique shortest geodesic segment from a point of D^- to a point of D^+. There exists a common perpendicular from D^- to D^+ if and only if the closures of D^- and D^+ in $X\cup\partial_\infty X$ are disjoint. When X is an $\mathbb{R}$-tree, then two closed subtrees of X have a common perpendicular if and only if they are nonempty and disjoint.

One of the aims of this book is to count orbits of common perpendiculars between two equivariant families of closed convex subsets of X. The crucial remark is that two nonempty proper closed convex subsets D^- and D^+ of X have a common perpendicular α of length a given $T>0$ if and only if the subsets $\mathsf{g}^{T/2}\partial^1_+D^-\big|_{[-\frac{T}{2},\frac{T}{2}]}$ and $\mathsf{g}^{-T/2}\partial^1_-D^+\big|_{[-\frac{T}{2},\frac{T}{2}]}$ of $\breve{\mathscr{G}}X$ intersect. This intersection then consists of the common perpendicular from D^- to D^+ reparametrised by $[-\frac{T}{2},\frac{T}{2}]$. As a controlled perturbation of this remark, we now give an effective creation result of common perpendiculars in $\mathbb{R}$-trees. It has a version satisfied for X in the generality of Section 2.1, see the end of this Section.

Lemma 2.8. *Assume that X is an $\mathbb{R}$-tree. For all $R>1$, $\eta\in\,]0,1]$ and $t\geqslant 2\ln R+4$, for all nonempty closed connected subsets D^-,D^+ in X, and for every geodesic line $\ell\in\mathsf{g}^{t/2}\mathscr{V}^+_{\eta,R}(\partial^1_+D^-)\cap\mathsf{g}^{-t/2}\mathscr{V}^-_{\eta,R}(\partial^1_-D^+)$, there exist $s\in\,]-2\eta,2\eta[$ and a common perpendicular $\widetilde{c}$ from D^- to D^+ such that*

- *the length of $\widetilde{c}$ is $t+s$,*
- *the endpoint of $\widetilde{c}$ in $D^\mp$ is $w^\mp(0)$ where $w^\mp=f^\pm_{D^\mp}(\ell)$,*
- *the footpoint $\ell(0)$ of ℓ lies on $\widetilde{c}$, and*

$$\max\Big\{\;d\big(w^-(\frac{t}{2}),\ell(0)\big),\;d\big(w^+(-\frac{t}{2}),\ell(0)\big)\;\Big\}\leqslant\eta\;.$$



Proof. Let $R, \eta, t, D^\pm, \ell$ be as in the statement. By the definition of the sets $\mathscr{V}^{\mp}_{\eta,R}(\partial^1_{\mp} D^\pm)$, there exist geodesic rays $w^\pm \in \partial^1_{\mp} D^\pm$, geodesic lines $\widehat{w}^\pm \in \mathscr{G}X$ extending $w^\pm$, and $s^\pm \in]-\eta, +\eta[$, such that $\ell_\pm = (w^\mp)_\pm$ and

$$d_{W^\pm(w^\mp)}(\mathsf{g}^{\mp\frac{t}{2}\mp s^\mp}\ell, \widehat{w}^\mp) \leqslant R\,.$$

Let $x_\pm$ be the closest point to $w^\pm(0)$ on ℓ. By the definition of Hamenstädt's distances, we have

$$d(w^\pm(0), x^\pm) = d(\ell(\pm\frac{t}{2} \pm s^\pm), x^\pm) \leqslant \ln R\,,$$

and in particular $x_\pm = w^\pm(0)$ if and only if $\ell(\pm\frac{t}{2} \pm s^\pm) = w^\pm(0)$. As $t \geqslant 2\ln R + 4$ and $|s^\pm| \leqslant 2\eta \leqslant 2$, the points $\ell(-\frac{t}{2} - s^-)$, x^-, $\ell(0)$, x^+, $\ell(\frac{t}{2} + s^+)$ are in this order on ℓ. In particular, the segment $[w^-(0), x^-] \cup [x^-, x^+] \cup [x^+, w^+(0)]$ is a nontrivial geodesic segment from a point of D^- to a point of D^+ that meets $D^\mp$ only at an endpoint. Hence, D^- and D^+ are disjoint, and $[w^-(0), w^+(0)]$ is the image of the common perpendicular from D^- to D^+.

Let $s = s_- + s_+$. The length of $\widetilde{c}$ is $(\frac{t}{2} + s^+) - (-\frac{t}{2} - s^-) = t + s$. The point $\ell(0)$ lies on $\widetilde{c}$, we have $w^\mp = f^\pm_{D^\mp}(\ell)$ and the endpoints of $\widetilde{c}$ are $w^\pm(0)$. Furthermore,

$$d(w^\mp(\pm\frac{t}{2}), \ell(0)) = \big|\, d(w^\mp(\pm\frac{t}{2}), w^\mp(0)) - d(\ell(0), \ell(\mp\frac{t}{2} \mp s^\mp))\,\big| = |s^\mp| \leqslant \eta\,. \qquad \square$$

When X is as in the beginning of Section 2.1, the statement and the proof of the following analog of Lemma 2.8 is slightly more technical. We refer to [PaP17b, Lem. 7] for a proof in the Riemannian case, and we leave the extension to the reader, since we will not need it in this book.

Lemma 2.9. *Let X be as in the beginning of Section 2.1. For every $R > 0$, there exist $t_0, c_0 > 0$ such that for all $\eta \in]0,1]$ and all $t \in [t_0, +\infty[$, for all nonempty closed convex subsets D^-, D^+ in X, and for all $w \in \mathsf{g}^{t/2}\mathscr{V}^+_{\eta,R}(\partial^1_+ D^-) \cap \mathsf{g}^{-t/2}\mathscr{V}^-_{\eta,R}(\partial^1_- D^+)$, there exist $s \in]-2\eta, 2\eta[$ and a common perpendicular $\widetilde{c}$ from D^- to D^+ such that*

- *the length of $\widetilde{c}$ is contained in $[t + s - c_0\, e^{-\frac{t}{2}}, t + s + c_0\, e^{-\frac{t}{2}}]$,*
- *if $w^\mp = f^\pm_{D^\mp}(w)$ and if $p^\pm$ is the endpoint of $\widetilde{c}$ in $D^\pm$, then $d(\pi(w^\pm), p^\pm) \leqslant c_0\, e^{-\frac{t}{2}}$,*
- *the basepoint $\pi(w)$ of w is at distance at most $c_0\, e^{-\frac{t}{2}}$ from a point of $\widetilde{c}$, and*

$$\max\{\, d(\pi(\mathsf{g}^{\frac{t}{2}} w^-), \pi(w)),\ d(\pi(\mathsf{g}^{-\frac{t}{2}} w^+), \pi(w))\,\} \leqslant \eta + c_0\, e^{-\frac{t}{2}}\,. \qquad \square$$

2.6 Metric and simplicial trees, and graphs of groups

Metric and simplicial trees and graphs of groups are important examples throughout this book. In this Section, we recall the definitions and basic properties of these objects.

Using Serre's definitions in [Ser3, §2.1], a *graph* $\mathbb{X}$ is the data consisting of two sets $V\mathbb{X}$ and $E\mathbb{X}$, called the *set of vertices* and the *set of edges* of $\mathbb{X}$, of two maps $o, t : E\mathbb{X} \to V\mathbb{X}$ and of a fixed point free involution $e \mapsto \overline{e}$ of $E\mathbb{X}$, such that $t(\overline{e}) = o(e)$ for every $e \in E\mathbb{X}$. The elements $o(e)$, $t(e)$ and $\overline{e}$ are called the *initial vertex*, the *terminal vertex* and the *opposite edge* of an edge $e \in E\mathbb{X}$. The quotient of $E\mathbb{X}$ by the involution $e \mapsto \overline{e}$ is called the set of *nonoriented edges* of $\mathbb{X}$. Recall that a connected graph is *bipartite* if it is endowed with a partition of its set of vertices into two nonempty subsets such that any two elements of either subset are not related by an edge.

The *degree* of a vertex $x \in V\mathbb{X}$ is the cardinality of the set $\{e \in E\mathbb{X} : o(e) = x\}$. For all $j, k \in \mathbb{N}$, a graph $\mathbb{X}$ is *k-regular* if the degree of each vertex $x \in V\mathbb{X}$ is k, and it is *(j,k)-biregular* if it is bipartite with the elements of the partition of its vertices into two subsets having degree j and k respectively.

A *metric graph* $(\mathbb{X}, \lambda)$ is a pair consisting of a graph $\mathbb{X}$ and a map $\lambda : E\mathbb{X} \to]0, +\infty[$ with a positive lower bound[10] such that $\lambda(\overline{e}) = \lambda(e)$, called its *edge length map.* A *simplicial graph* $\mathbb{X}$ is a metric graph whose edge length map is constant equal to 1.

The *topological realisation* of a graph $\mathbb{X}$ is the topological space obtained from the family $(I_e)_{e \in E\mathbb{X}}$ of closed unit intervals I_e for every $e \in E\mathbb{X}$ by the finest equivalence relation that identifies intervals corresponding to an edge and its opposite edge by the map $t \mapsto 1 - t$ and identifies the origins of the intervals I_{e_1} and I_{e_2} if and only if $o(e_1) = o(e_2)$, see [Ser3, Sect. 2.1].

The *geometric realisation* of a metric tree $(\mathbb{X}, \lambda)$ is the topological realisation of $\mathbb{X}$ endowed with the maximal geodesic metric that gives length $\lambda(e)$ to the topological realisation of each edge $e \in E\mathbb{X}$, and we denote it by $X = |\mathbb{X}|_\lambda$. We identify $V\mathbb{X}$ with its image in X. The metric space X determines $(\mathbb{X}, \lambda)$ up to subdivisions of edges, hence we will often not make a strict distinction between X and $(\mathbb{X}, \lambda)$. In particular, we will refer to convex subsets of $(\mathbb{X}, \lambda)$ as convex subsets of X, etc.

If $\mathbb{X}$ is a tree, the metric space X is an $\mathbb{R}$-tree, hence it is a CAT(-1) space. Since λ is bounded from below by a positive constant, the $\mathbb{R}$-tree X is geodesically complete if and only if $\mathbb{X}$ is not reduced to one vertex and has no *terminal vertex* (that is, no vertex of degree 1).

We will denote by $\operatorname{Aut}(\mathbb{X}, \lambda)$, and $\operatorname{Aut} \mathbb{X}$ in the simplicial case, the group of edge-preserving isometries of X that have no inversions.[11] Since the edge length map has a positive lower bound, the metric space X is proper if and only if $\mathbb{X}$ is locally finite. In this case, the nonelementary discrete subgroups Γ of isometries of X we will consider will always be edge-preserving and without inversion. If Γ is a subgroup of $\operatorname{Aut}(\mathbb{X}, \lambda)$, we will again denote by $\lambda : \Gamma \backslash E\mathbb{X} \to]0, +\infty[$ the map induced by $\lambda : E\mathbb{X} \to]0, +\infty[$.

A locally finite metric tree $(\mathbb{X}', \lambda)$ is *uniform* if there exists some discrete subgroup Γ' of $\operatorname{Aut}(\mathbb{X}', \lambda)$ such that $\Gamma' \backslash \mathbb{X}'$ is a finite graph. See [BasK, BasL] for characterisations of this property in the case of simplicial trees.

Discrete time geodesic flow on trees Let $\mathbb{X}$ be a locally finite simplicial tree. The *space of generalised discrete geodesic lines* of $\mathbb{X}$ is the locally compact space $\check{\mathscr{G}}\mathbb{X}$ of 1-Lipschitz mappings w from $\mathbb{R}$ to the geometric realisation $X = |\mathbb{X}|_1$ which are isometric on a closed interval with endpoints in $\mathbb{Z} \cup \{-\infty, +\infty\}$ and locally constant outside it, such that $w(0) \in V\mathbb{X}$ (or equivalently $w(\mathbb{Z}) \subset V\mathbb{X}$). Note that $\check{\mathscr{G}}\mathbb{X}$ is hence a proper subset of $\check{\mathscr{G}}X$, unless $\mathbb{X}$ is reduced to one vertex.

By restriction to $\check{\mathscr{G}}\mathbb{X}$, or intersection with $\check{\mathscr{G}}\mathbb{X}$, of the objects defined in Sections 2.2 and 2.4 for $\check{\mathscr{G}}X$, we define the distance d on $\check{\mathscr{G}}\mathbb{X}$, the subspaces $\mathscr{G}_\pm\mathbb{X}$, $\mathscr{G}\mathbb{X}$, $\mathscr{G}_{\pm,0}\mathbb{X}$, the strong stable/unstable leaves $W^\pm(w)$ of $w \in \mathscr{G}_\pm\mathbb{X}$ and their Hamenstädt distances $d_{W^\pm(w)}$, the stable/unstable leaves $W^{0\pm}(w)$ of $w \in \mathscr{G}_\pm\mathbb{X}$, the outer and inner unit normal bundles $\partial^1_\pm\mathbb{D}$

[10] This assumption, though not necessary at this stage, will be used repeatedly in this book, hence we prefer to add it to the definition.

[11] An automorphism g of a graph has an *inversion* if there exists an edge e of the graph such that $ge = \overline{e}$. The assumption that the elements of $\operatorname{Aut}(\mathbb{X})$ have no inversion ensures that, for every subgroup Γ' of $\operatorname{Aut}(\mathbb{X})$, the quotient map $\Gamma' \backslash E\mathbb{X} \to \Gamma' \backslash E\mathbb{X}$ of $e \mapsto \overline{e}$ is still a fixed point free involution, so that with the quotient maps $\Gamma' \backslash E\mathbb{X} \to \Gamma' \backslash V\mathbb{X}$ defined by o and t, we do have a quotient graph structure $\Gamma' \backslash \mathbb{X}$.

of a nonempty proper simplicial subtree $\mathbb{D}$ of $\mathbb{X}$, the dynamical neighbourhoods $\mathscr{V}^{\pm}_{\eta,\eta'}(\Omega^{\mp})$ of subsets $\Omega^{\mp}$ of $\partial^1_{\pm}\mathbb{D}$ as well as the fibrations

$$f^{\pm}_{\mathbb{D}} : \mathscr{U}^{\pm}_{\mathbb{D}} = \{\ell \in \mathscr{G}\mathbb{X} \ : \ \ell_{\pm} \notin \partial_\infty |\mathbb{D}|_1\} \to \partial^1_{\pm}\mathbb{D} \ ,$$

whose fiber over $\rho \in \partial^1_{\pm}\mathbb{D}$ is $W^{0\pm}(\rho)$. Note that some definitions actually simplify when considering generalised discrete geodesic lines. For instance, for all $w \in \mathscr{G}_{\pm}\mathbb{X}$, $\eta' > 0$ and $\eta \in \,]0,1[\,$, the dynamical neighbourhood $V^{\pm}_{w,\eta,\eta'}$ is equal to $B^{\pm}(w,\eta')$, and is hence independent of $\eta \in \,]0,1[\,$.

Besides the map $\pi : \mathscr{G}\mathbb{X} \to V\mathbb{X}$ defined as in the continuous case by $\ell \mapsto \ell(0)$, we have another natural map $T\pi : \mathscr{G}\mathbb{X} \to E\mathbb{X}$, which associates to ℓ the edge e with $o(e) = \ell(0)$ and $t(e) = \ell(1)$. This map is equivariant under the group of automorphisms (without inversions) $\operatorname{Aut}(\mathbb{X})$ of $\mathbb{X}$, and we also denote by $T\pi : \Gamma\backslash\mathscr{G}\mathbb{X} \to \Gamma\backslash E\mathbb{X}$ its quotient map, for every subgroup Γ of $\operatorname{Aut}(\mathbb{X})$.

If $\mathbb{X}$ has no terminal vertex, for every $e \in E\mathbb{X}$, let

$$\partial_e\mathbb{X} = \{\ell_+ \ : \ \ell \in \mathscr{G}\mathbb{X},\ T\pi(\ell) = e\}$$

be the set of points at infinity of the geodesic rays whose initial (oriented) edge is e.

Given $x_0 \in V\mathbb{X}$, the *discrete Hopf parametrisation* now identifies $\mathscr{G}\mathbb{X}$ with $\partial^2_\infty X \times \mathbb{Z}$ by the map $\ell \mapsto (\ell_-, \ell_+, t)$ where $t \in \mathbb{Z}$ is the signed distance from the closest vertex to the basepoint x_0 on the geodesic line ℓ to the vertex $\ell(0)$.

The *discrete time geodesic flow* $(\mathsf{g}^t)_{t\in\mathbb{Z}}$ on $\check{\mathscr{G}}\mathbb{X}$ is the one-(discrete-)parameter group of homeomorphisms of $\check{\mathscr{G}}\mathbb{X}$ consisting of (the restriction to $\check{\mathscr{G}}\mathbb{X}$ of) the integral time maps of the continuous time geodesic flow of the geometric realisation of $\mathbb{X}$: we have $\mathsf{g}^t w : s \mapsto w(s+t)$ for all $w \in \check{\mathscr{G}}\mathbb{X}$ and $t \in \mathbb{Z}$.

Crossratios of ends of trees Let $\mathbb{X}$ be a locally finite simplicial tree, with geometric realisation $X = |\mathbb{X}|_1$. Recall[12] that if $(\xi_1, \xi_2, \xi_3, \xi_4)$ is an ordered quadruple of pairwise distinct points in $\partial_\infty X$, then their (logarithmic) *crossratio* is

$$[\![\xi_1,\xi_2,\xi_3,\xi_4]\!] = \lim_{x_i\to\xi_i,\, x_i\in V\mathbb{X}} \ \frac{1}{2}\big(d(x_1,x_4) - d(x_4,x_3) + d(x_3,x_2) - d(x_2,x_1)\big) \ . \tag{2.21}$$

A similar definition is valid for general CAT(-1)-spaces, but we will only need the case of simplicial trees in this book.

If x and y are the closest points on the geodesic line $]\xi_1,\xi_3[$ to ξ_2 and ξ_4 respectively, then

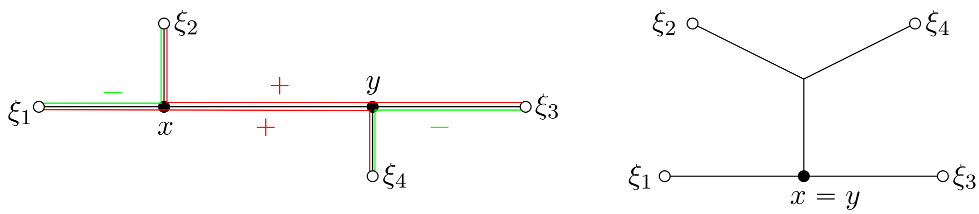
$$[\![\xi_1,\xi_2,\xi_3,\xi_4]\!] = d(x,y)$$

if ξ_1, x, y, ξ_3 are in this order on $]\xi_1,\xi_3[$ and

$$[\![\xi_1,\xi_2,\xi_3,\xi_4]\!] = -d(x,y)$$

otherwise. In particular, $[\![\xi_1,\xi_2,\xi_3,\xi_4]\!] = 0$ if the geodesic lines $]\xi_1,\xi_3[$ and $]\xi_2,\xi_4[$ are disjoint.

[12] See [Pau2], as well as [Ota] in the case of Riemannian manifolds, and note that the convention on the order varies in the literature.



We have the following properties:
• $[\![\xi_1, \xi_2, \xi_3, \xi_4]\!] = [\![\xi_4, \xi_3, \xi_2, \xi_1]\!] = -[\![\xi_3, \xi_2, \xi_1, \xi_4]\!]$.
• the crossratio is continuous, and even locally constant on the space of pairwise distinct quadruples of elements of $\partial_\infty X$,
• if γ is a loxodromic element of $\mathrm{Aut}(\mathbb{X})$, with repulsive and attractive fixed points γ_- and γ_+ in $\partial_\infty X$ respectively, then for every $\xi \in \partial_\infty X - \{\gamma_-, \gamma_+\}$, the translation length of γ satisfies

$$\lambda(\gamma) = [\![\gamma_-, \xi, \gamma\xi, \gamma_+]\!] \ .$$

Bass-Serre's graphs of groups Recall (see for instance [Ser3, BasL]) that a *graph of groups* $(\mathbb{Y}, G_*)$ consists of

- a graph $\mathbb{Y}$, which is connected unless otherwise stated,
- a group G_v for every vertex $v \in V\mathbb{Y}$,
- a group G_e for every edge $e \in E\mathbb{Y}$ such that $G_e = G_{\overline{e}}$,
- an injective group morphism $\rho_e : G_e \to G_{t(e)}$ for every edge $e \in V\mathbb{Y}$.

We will identify G_e with its image in $G_{t(e)}$ by ρ_e, unless the meaning is not clear (which might be the case for instance if $o(e) = t(e)$). We refer to op. cit.[13] for the definition of the *Bass-Serre tree* $T(\mathbb{Y}, G_*)$ of $(\mathbb{Y}, G_*)$, of its *fundamental group* $\pi_1(\mathbb{Y}, G_*)$ when a basepoint in $V\mathbb{Y}$ is chosen, and of the simplicial action of $\pi_1(\mathbb{Y}, G_*)$ on $T(\mathbb{Y}, G_*)$. Note that the fundamental group of $(\mathbb{Y}, G_*)$ does not always act faithfully on its Bass-Serre tree $T(\mathbb{Y}, G_*)$, that is, the kernel of its action might be nontrivial.

A *subgraph of subgroups* of $(\mathbb{Y}, G_*)$ is a graph of groups $(\mathbb{Y}', G'_*)$ where
• $\mathbb{Y}'$ is a subgraph of $\mathbb{Y}$,
• for every $v \in V\mathbb{Y}'$, the group G'_v is a subgroup of G_v,
• for every $e \in E\mathbb{Y}'$, the group G'_e is a subgroup of G_e,
• the monomorphism $\rho'_e : G'_e \to G'_{t(e)}$ is the restriction to G'_e of the monomorphism $\rho_e : G_e \to G_{t(e)}$, and

$$G'_{t(e)} \cap \rho_e(G_e) = \rho'_e(G'_e) \ .$$

This condition, first introduced in [Bass, Coro. 1.14], is equivalent to the injectivity of the natural map $G'_{t(e)}/\rho'_e(G'_e) \to G_{t(e)}/\rho_e(G_e)$ for every $e \in E\mathbb{Y}$. It implies by [Bass, 2.15] when the underlying basepoint is chosen in $\mathbb{Y}'$, that
• the fundamental group $\Gamma' = \pi_1(\mathbb{Y}', G'_*)$ of $(\mathbb{Y}', G'_*)$ injects into the fundamental group $\Gamma = \pi_1(\mathbb{Y}, G_*)$ of $(\mathbb{Y}, G_*)$,
• the Bass-Serre tree $\mathbb{X}'$ of $(\mathbb{Y}', G'_*)$ injects in an equivariant way into the Bass-Serre tree $\mathbb{X}$ of $(\mathbb{Y}, G_*)$ so that the stabiliser of $\mathbb{X}'$ in Γ is Γ', and

[13] though see Example 2.10 for the main example encountered in this book

• the map $(\Gamma'\backslash\mathbb{X}') \rightarrow (\Gamma\backslash\mathbb{X})$ induced by the inclusion map $\mathbb{X}' \rightarrow \mathbb{X}$ by taking quotient is injective:

$$\forall\, \gamma \in \Gamma,\ \forall\, z \in V\mathbb{X}' \cup E\mathbb{X}', \quad \text{if}\ \ \gamma z \in V\mathbb{X}' \cup E\mathbb{X}',\ \text{then}\ \exists\, \gamma' \in \Gamma', \quad \gamma' z = \gamma z\,. \tag{2.22}$$

The *edge-indexed graph* $(\mathbb{Y}, i)$ of the graph of groups $(\mathbb{Y}, G_*)$ is the graph $\mathbb{Y}$ endowed with the map $i : E\mathbb{Y} \rightarrow \mathbb{N} - \{0\}$ defined by $i(e) = [G_{o(e)} : G_e]$ (see for instance [BasK, BasL]).

In Section 12.4, we will consider *metric graphs of groups* $(\mathbb{Y}, G_*, \lambda)$ which are graphs of groups endowed with an edge length function $\lambda : E\mathbb{Y} \rightarrow\,]0, +\infty[$ (with $\lambda(\overline{e}) = \lambda(e)$ for every $e \in E\mathbb{Y}$).

Example 2.10. The main examples of graphs of groups that we will consider in this book are the following ones. Let $\mathbb{X}$ be a simplicial tree and let Γ be a subgroup of $\mathrm{Aut}(\mathbb{X})$. The *quotient graph of groups* $\Gamma\backslash\backslash\mathbb{X}$ is the following graph of groups $(\mathbb{Y}, G_*)$. Its underlying graph $\mathbb{Y}$ is the quotient graph $\Gamma\backslash\mathbb{X}$. Fix a lift $\widetilde{z} \in V\mathbb{X} \cup E\mathbb{X}$ for every $z \in V\mathbb{Y} \cup E\mathbb{Y}$. For every $e \in E\mathbb{Y}$, assume that $\widetilde{\overline{e}} = \overline{\widetilde{e}}$, and fix an element $g_e \in \Gamma$ such that $g_e\,\widetilde{t(e)} = t(\widetilde{e})$. For every $y \in V\mathbb{Y} \cup E\mathbb{Y}$, take as G_y the stabiliser $\Gamma_{\widetilde{y}}$ in Γ of the fixed lift $\widetilde{y}$. Take as monomorphism $\rho_e : G_e \rightarrow G_{t(e)}$ the restriction to $\Gamma_{\widetilde{e}}$ of the conjugation $\gamma \mapsto g_e^{-1}\gamma g_e$ by g_e^{-1}. Note that $\Gamma\backslash\backslash\mathbb{X}$ has finite vertex groups if $\mathbb{X}$ is locally finite and Γ is discrete. For every choice of basepoint in $V\mathbb{Y}$, there exist a group isomorphism $\theta : \pi_1(\mathbb{Y}, G_*) \rightarrow \Gamma$ and a θ-equivariant simplicial isomorphism from the Bass-Serre tree $T(\mathbb{Y}, G_*)$ to $\mathbb{X}$ (see for instance [Ser3, BasL]).

The *volume form* of a graph of finite groups $(\mathbb{Y}, G_*)$ is the measure $\mathrm{vol}_{(\mathbb{Y},\,G_*)}$ on the discrete set $V\mathbb{Y}$, such that for every $y \in V\mathbb{Y}$,

$$\mathrm{vol}_{(\mathbb{Y},\,G_*)}(\{y\}) = \frac{1}{|G_y|}\ ,$$

where $|G_y|$ is the order of the finite group G_y. Its total mass, called the *volume* of $(\mathbb{Y}, G_*)$, is

$$\mathrm{Vol}(\mathbb{Y},\,G_*) = \|\,\mathrm{vol}_{(\mathbb{Y},\,G_*)}\,\| = \sum_{y\in V\mathbb{Y}} \frac{1}{|G_y|}\ .$$

We denote by $\mathbb{L}^2(\mathbb{Y}, G_*) = \mathbb{L}^2(V\mathbb{Y}, \mathrm{vol}_{(\mathbb{Y},G_*)})$ the complex Hilbert space of square integrable maps $V\mathbb{Y} \rightarrow \mathbb{C}$ for this measure $\mathrm{vol}_{(\mathbb{Y},G_*)}$, and by $f \mapsto \|f\|_2$ and $(f, g) \mapsto \langle f, g\rangle_2$ its norm and (antilinear on the right) scalar product. Let

$$\mathbb{L}^2_0(\mathbb{Y}, G_*) = \big\{f \in \mathbb{L}^2(\mathbb{Y}, G_*)\ :\ \int f\ d\,\mathrm{vol}_{(\mathbb{Y},\,G_*)} = 0\big\}\,.$$

When $\mathrm{Vol}(\mathbb{Y}, G_*)$ is finite, $\mathbb{L}^2_0(\mathbb{Y}, G_*)$ is the orthogonal subspace to the constant functions.

We also consider a *(edge-)volume form* $\mathrm{Tvol}_{(\mathbb{Y},\,G_*)}$ on the discrete set $E\mathbb{Y}$ such that for every $e \in E\mathbb{Y}$,

$$\mathrm{Tvol}_{(\mathbb{Y},\,G_*)}(\{e\}) = \frac{1}{|G_e|}\ ,$$

with total mass

$$\mathrm{TVol}(\mathbb{Y},\,G_*) = \|\mathrm{Tvol}_{(\mathbb{Y},\,G_*)}\| = \sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}\ .$$

The *(edge-)volume form* of a metric graph of groups $(\mathbb{Y}, G_*, \lambda)$ is given by

$$\mathrm{Tvol}_{(\mathbb{Y},\, G_*, \lambda)} = \frac{ds}{|G_e|}$$

on each edge e of $\mathbb{Y}$ parameterised by its arclength s, so that its total mass is

$$\mathrm{TVol}(\mathbb{Y},\, G_*, \lambda) = \|\mathrm{Tvol}_{(\mathbb{Y},\, G_*, \lambda)}\| = \sum_{e \in E\mathbb{Y}} \frac{\lambda(e)}{|G_e|}\ .$$

For $\lambda \equiv 1$, this total mass agrees with that of the discrete definition above.

Remark 2.11. Note that $\mathrm{TVol}(\mathbb{Y},\, G_*) = \mathrm{Card}(E\mathbb{Y})$ when the edge groups are trivial. We have

$$\mathrm{TVol}(\mathbb{Y},\, G_*) = \sum_{e \in E\mathbb{Y}} \frac{1}{|G_e|} = \sum_{y \in V\mathbb{Y}} \frac{1}{|G_y|} \sum_{e \in E\mathbb{Y},\, o(e)=y} \frac{|G_y|}{|G_e|} = \sum_{y \in V\mathbb{Y}} \frac{\deg(\widetilde{y})}{|G_y|}\ ,$$

where $\widetilde{y}$ is any lift of y in the Bass-Serre tree of $(\mathbb{Y}, G_*)$. In particular, if $\mathbb{X}$ is a uniform simplicial tree and Γ is discrete subgroup of $\mathrm{Aut}\,\mathbb{X}$, then the finiteness of $\mathrm{Vol}(\Gamma \backslash\!\backslash \mathbb{X})$ and of $\mathrm{TVol}(\Gamma \backslash\!\backslash \mathbb{X})$ are equivalent. Defining the volume form on $V\mathbb{Y}$ by $\{y\} \mapsto \frac{\deg(\widetilde{y})}{|G_y|}$ sometimes makes formulas simpler, but we will follow the convention which occurs in the classical references (see for instance [BasL]).

If the Bass-Serre tree of $(\mathbb{Y},\, G_*)$ is $(q+1)$-regular, then

$$\pi_* \mathrm{Tvol}_{\mathbb{Y},\, G_*} = (q+1)\, \mathrm{vol}_{\mathbb{Y},\, G_*} \quad \text{and} \quad \mathrm{TVol}(\mathbb{Y},\, G_*) = (q+1)\, \mathrm{Vol}(\mathbb{Y},\, G_*)\,. \tag{2.23}$$

We say that a discrete group of (inversion-free) automorphisms Γ of a locally finite metric or simplicial tree $(\mathbb{X}, \lambda)$ is a *(tree) lattice* of $(\mathbb{X}, \lambda)$ if the quotient graph of groups $\Gamma \backslash\!\backslash \mathbb{X}$ has finite volume:

$$\mathrm{Vol}(\Gamma \backslash\!\backslash \mathbb{X}) < +\infty\ .$$

This implies by [BasK, Prop. 4.5][14] that Γ is a lattice[15] in the locally compact group $\mathrm{Aut}(\mathbb{X}, \lambda)$ (hence that $\mathrm{Aut}(\mathbb{X}, \lambda)$ is unimodular, see for instance [Rag, Chap. I, Rem. 1.9]), the converse being true for instance if $(\mathbb{X}, \lambda)$ is uniform. If Γ is a *uniform lattice* of $(\mathbb{X}, \lambda)$, that is, if Γ is a discrete subgroup of $\mathrm{Aut}(\mathbb{X}, \lambda)$ and if the quotient graph $\Gamma \backslash \mathbb{X}$ is finite, then Γ is clearly a (tree) lattice of $(\mathbb{X}, \lambda)$.

A graph of finite groups $(\mathbb{Y}, G_*)$ is a *cuspidal ray* if $\mathbb{Y}$ is a simplicial ray such that the homomorphisms $G_{e_n} \to G_{o(e_n)}$ are surjective for its sequence of consecutive edges $(e_i)_{i \in \mathbb{N}}$ oriented towards the unique end of $\mathbb{Y}$. By [Pau3], a discrete group Γ' of $\mathrm{Aut}(\mathbb{X})$ (hence of $\mathrm{Isom}(|\mathbb{X}|_1)$) is geometrically finite if and only if it is nonelementary and if the quotient graph of groups by Γ' of its minimal nonempty invariant subtree is the union of a finite graph of groups and a finite number of cuspidal rays attached to the finite graph at their finite endpoints.

Remark 2.12. If $\mathbb{X}$ is a locally finite simplicial tree and if Γ' is a geometrically finite discrete group of $\mathrm{Aut}(\mathbb{X})$ such that the convex hull of its limit set $\mathscr{C}\Lambda\Gamma'$ is (the geometric realisation of) a uniform tree, then Γ' is a lattice of the simplicial tree $\mathscr{C}\Lambda\Gamma'$.

[14] using the fact that $\mathrm{Aut}(\mathbb{X}, \lambda)$ is a closed subgroup of $\mathrm{Aut}(\mathbb{X})$

[15] Recall that a *lattice* in a locally compact group G is a discrete subgroup Γ' of G such that the left quotient space $\Gamma' \backslash G$ admits a probability measure invariant under translations on the right by G.

Proof. Since $\mathscr{C}\Lambda\Gamma'$ is uniform, there is a uniform upper bound on the length of an edge path in $\mathscr{C}\Lambda\Gamma'$ which injects in $\Gamma'\backslash\mathscr{C}\Lambda\Gamma'$ and such that the stabiliser of each edge of this edge path is equal to the stabilisers of both endpoints of this edge. It is hence easy to see that the volume of each of the (finitely many) cuspidal rays in $\Gamma'\backslash\backslash\mathscr{C}\Lambda\Gamma'$ is finite, by a geometric series argument. Hence the volume of $\Gamma'\backslash\backslash\mathscr{C}\Lambda\Gamma'$ is finite. □

Note that contrarily to the case of Riemannian manifolds, there are many more (tree) lattices than there are geometrically finite (tree) lattices, even in regular trees, see for instance [BasL].

In Part III of this book, we will consider simplicial graphs of groups that arise from the arithmetic of non-Archimedean local fields. We say that a discrete group Γ of (inversion-free) automorphisms of a simplicial tree $\mathbb{X}$ is *algebraic* if there exist a non-Archimedean local field $\widehat{K}$ (a finite extension of $\mathbb{Q}_p$ for some prime p or the field of formal Laurent series over a finite field) and a connected semi-simple algebraic group $\underline{G}$ with finite centre defined over $\widehat{K}$, of $\widehat{K}$-rank one, such that $\mathbb{X}$ identifies with the Bruhat-Tits tree of $\underline{G}$ in such a way that Γ identifies with a lattice of the locally compact group $\underline{G}(\widehat{K})$. If Γ is algebraic, then Γ is geometrically finite by [Lub1]. Note that $\mathbb{X}$ is then bipartite, see Section 2 of op. cit. for a discussion and references. See Sections 14 and 15.1 for more details, and their subsequent Sections for arithmetic applications arising from algebraic lattices.

Chapter 3

Potentials, critical exponents and Gibbs cocycles

Let X be a geodesically complete proper CAT(-1) space, let $x_0 \in X$ be an arbitrary basepoint, and let Γ be a nonelementary discrete group of isometries of X.

In this Chapter, we define potentials on T^1X, which are new data on X in addition to its geometry. We introduce the fundamental tools associated with potentials, and we give some of their basic properties. The development follows [PauPS] with modifications to fit the present more general context.

In Section 3.5, given a simplicial or metric tree $(\mathbb{X}, \lambda)$, with geometric realisation X, we introduce a natural method to associate a (Γ-invariant) potential $\widetilde{F}_c : T^1X \to \mathbb{R}$ to a Γ-invariant function $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ defined on the set of edges of $\mathbb{X}$, that we call a system of conductances on $\mathbb{X}$. This construction gives a nonsymmetric generalisation of electric networks.

3.1 Background on (uniformly local) Hölder-continuity

In this preliminary Section, we recall the notion of Hölder-continuity we will use in this book, which needs to be defined appropriately in order to deal with noncompactness issues. The Hölder-continuity will be used on the one hand for potentials when X is a Riemannian manifold in Section 3.2, and on the other hand for error term estimates in Chapters 9, 10 and 11.

As in [PauPS], we will use the following uniformly local definition of Hölder-continuous maps. Let E and E' be two metric spaces, and let $\alpha \in \,]0, 1]$. A map $f : E \to E'$ is

- *α-Hölder-continuous* if there exist $c, \epsilon > 0$ such that for all $x, y \in E$ with $d(x, y) \leqslant \epsilon$, we have
$$d(f(x), f(y)) \leqslant c\, d(x, y)^\alpha \,.$$
- *locally α-Hölder-continuous* if for every $x \in E$, there exists a neighbourhood U of x such that the restriction of f to U is α-Hölder-continuous;
- *Hölder-continuous* (respectively *locally Hölder-continuous*) if there exists $\alpha \in \,]0, 1]$ such that f is α-Hölder-continuous (respectively locally α-Hölder-continuous);
- *Lipschitz* if it is 1-Hölder-continuous and *locally Lipschitz* if it is locally 1-Hölder-continuous.

Let E be a set. Two distances d and d' on E are (uniformly locally) *Hölder-equivalent* if the identity map from (E,d) to (E,d') and the identity map from (E,d') to (E,d) are Hölder-continuous. This is an equivalence relation on the set of distances on E, and a *Hölder structure* on E is the choice of such an equivalence class. For a map between two metric spaces, to be Hölder-continuous depends only on the Hölder structures on the source and target spaces.

Let E and E' be two metric spaces. We say that a map $f : E \to E'$ has

- *at most linear growth* if there exist $a, b \geqslant 0$ such that $d(f(x), f(y)) \leqslant a\, d(x,y) + b$ for all $x, y \in E$,
- *subexponential growth* if for every $a > 0$, there exists $b \geqslant 0$ such that $d(f(x), f(y)) \leqslant b\, e^{a\, d(x,y)}$ for all $x, y \in E$.

Remark 3.1. When E is a geodesic space, a consequence of the (uniformly local) Hölder-continuous property of $f : E \to E'$ is that f then has at most linear growth: the definition implies that

$$d(f(x), f(y)) \leqslant c\, \epsilon^{\alpha-1}\, d(x,y) + c\, \epsilon^{\alpha}$$

for all x, y in X, by subdividing the geodesic segment in E from x to y into $\lceil \frac{d(x,y)}{\epsilon} \rceil$ segments of equal lengths at most ϵ and using the triangle inequality in E'.

The following result (due to Bartels-Lück [BartL] with a different distance on $\check{\mathscr{G}}X$) proves in particular that the footpoint projection $\pi : \check{\mathscr{G}}X \to X$ is $\frac{1}{2}$-Hölder-continuous, as claimed in Section 2.2. Recall that $\check{\mathscr{G}}X$ is endowed with the distance d defined by Equation (2.7).

Proposition 3.2. *For every $t \in \mathbb{R}$, the map from $\check{\mathscr{G}}X$ to X defined by $\ell \mapsto \ell(t)$ is $\frac{1}{2}$-Hölder-continuous.*

Proof. Let $\ell, \ell' \in \check{\mathscr{G}}X$ be such that $d(\ell, \ell') \leqslant 1$. Assume that $t \geqslant 0$, otherwise the argument is similar, replacing $\int_t^{t+\epsilon}$ by $\int_{t-\epsilon}^{t}$. For every $\epsilon > 0$, we have by the triangle inequality

$$d(\ell, \ell') \geqslant \int_t^{t+\epsilon} d(\ell(s), \ell'(s))\; e^{-2s}\, ds \geqslant \big(d(\ell(t), \ell'(t)) - 2\epsilon\big)\, \epsilon\; e^{-2t-2\epsilon}\;.$$

If $d(\ell(t), \ell'(t)) \geqslant 4$, let $\epsilon = 1$, so that $d(\ell, \ell') \geqslant \frac{e^{-2t-2}}{2}\; d(\ell(t), \ell'(t))$, hence

$$d(\ell(t), \ell'(t)) \leqslant 2\; e^{2t+2}\; d(\ell, \ell')^{\frac{1}{2}}\;.$$

If $d(\ell(t), \ell'(t)) \leqslant 4$, let $\epsilon = \frac{1}{4}\; d(\ell(t), \ell'(t)) \leqslant 1$. Hence $d(\ell, \ell') \geqslant \frac{1}{8}\; d(\ell(t), \ell'(t))^2\, e^{-2t-2}$, so that

$$d(\ell(t), \ell'(t)) \leqslant 2\sqrt{2}\; e^{t+1}\; d(\ell, \ell')^{\frac{1}{2}}\;. \quad \square$$

When X is an $\mathbb{R}$-tree, the regularity property of the footpoint projection $\pi : \mathscr{G}X \to X$ is stronger than the one given by Proposition 3.2 (see Lemma 3.4 (2)). The results below, that will be needed in Sections 10.4 and 12.6, say that not only the evaluation maps $\ell \mapsto \ell(t)$ are $\frac{1}{2}$-Hölder-continuous, but so are the endpoint maps $\ell \mapsto \ell_{\pm}$. As we will only need these facts in the tree case, we prove them only when X is an $\mathbb{R}$-tree, and we start by a simplicial version of it.

Lemma 3.3. *Let $\mathbb{X}$ be a simplicial tree. There are universal constants $\epsilon_0 > 0, c_0 \geqslant 1$ such that for all $\epsilon \in \,]0, \epsilon_0[$ and $\ell \in \mathscr{G}\mathbb{X}$, the ball $B_d(\ell, \epsilon)$ is contained in*

$$\{\ell' \in \mathscr{G}\mathbb{X} \;:\; \ell'(0) = \ell(0),\ \ell'_\pm \in B_{d_{\ell(0)}}(\ell_\pm, c_0\sqrt{\epsilon}\,)\}$$

and contains $\{\ell' \in \mathscr{G}\mathbb{X} \;:\; \ell'(0) = \ell(0),\ \ell'_\pm \in B_{d_{\ell(0)}}(\ell_\pm, \frac{1}{c_0}\sqrt{\epsilon}\,)\}$.

In particular, the endpoint maps $\ell \mapsto \ell_\pm$ from $\mathscr{G}\mathbb{X}$ to $\partial_\infty\mathbb{X}$ are $\frac{1}{2}$-Hölder-continuous.

Proof. If $\ell, \ell' \in \mathscr{G}\mathbb{X}$ have distinct footpoints, then $d(\ell(0), \ell'(0)) \geqslant 1$, so that $d(\ell(t), \ell'(t)) \geqslant \frac{1}{2}$ if $|t| \leqslant \frac{1}{4}$, so that $d(\ell, \ell') \geqslant \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{2}\, e^{-2|t|} = \epsilon_0 > 0$.

Conversely, assume that $\ell, \ell' \in \mathscr{G}\mathbb{X}$ have equal footpoints, so that they coincide on $[-N, N']$ for some $N, N' \in \mathbb{N}$. By the definition of the visual distances (see Equation (2.3)), we have

$$d_{\ell(0)}(\ell_+, \ell'_+) = e^{-N'}$$

and similarly $d_{\ell(0)}(\ell_-, \ell'_-) = e^{-N}$. By the definition of the distance on $\check{\mathscr{G}}X$ (see Equation (2.7)), we have, by an easy change of variables,

$$\begin{aligned} d(\ell, \ell') &= \int_{N'}^{+\infty} 2\,|t - N'|\; e^{-2t}\; dt + \int_{-\infty}^{-N} 2\,|-N - t|\; e^{2t}\; dt \\ &= (e^{-2N'} + e^{-2N})\; \int_0^{+\infty} 2\,u\; e^{-2u}\; du = \frac{1}{2}(e^{-2N'} + e^{-2N})\;. \end{aligned}$$

The result follows. □

Let us now give a (more technical) version of this lemma for $\mathbb{R}$-trees, also proving that the footpoint projection is Lipschitz. If a and b are positive functions of some parameters, we write $a \asymp b$ if there exists a universal constant $C > 0$ such that $\frac{1}{C}\, b \leqslant a \leqslant C\, b$.

Lemma 3.4. *Assume that X is an $\mathbb{R}$-tree.*

(1) *There exists a universal constant $c_1 > 0$ such that for all $\ell, \ell' \in \mathscr{G}X$, if $d(\ell, \ell') \leqslant c_1$, then $\ell'(0) \in \ell(\mathbb{R})$, the intersection $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ is not reduced to a point, the orientations of ℓ and ℓ' coincide on this intersection, and*

$$d(\ell, \ell') \asymp d_{\ell(0)}(\ell_-, \ell'_-)^2 + d_{\ell(0)}(\ell_+, \ell'_+)^2 + d(\ell(0), \ell'(0))\;.$$

(2) *The footpoint map $\pi : \mathscr{G}X \to X$ defined by $\ell \mapsto \ell(0)$ is (uniformly locally) Lipschitz.*

(3) *There are universal constants $\epsilon_0 > 0, c_0 \geqslant 1$ such that for all $\epsilon \in \,]0, \epsilon_0[$ and $\ell \in \mathscr{G}X$, the ball $B_d(\ell, \epsilon)$ in $\mathscr{G}X$ is contained in*

$$\{\ell' \in \mathscr{G}X \;:\; \ell'(0) \in \ell(\mathbb{R}),\ d(\ell'(0), \ell(0)) \leqslant c_0\,\epsilon,\ \ell'_\pm \in B_{d_{\ell(0)}}(\ell_\pm, c_0\sqrt{\epsilon}\,)\}$$

and contains

$$\{\ell' \in \mathscr{G}X \;:\; \ell'(0) \in \ell(\mathbb{R}),\ d(\ell'(0), \ell(0)) \leqslant \frac{1}{c_0}\,\epsilon,\ \ell'_\pm \in B_{d_{\ell(0)}}(\ell_\pm, \frac{1}{c_0}\sqrt{\epsilon}\,)\}\;.$$

(4) *The endpoint maps $\ell \mapsto \ell_\pm$ from $\mathscr{G}X$ to $\partial_\infty X$ are $\frac{1}{2}$-Hölder-continuous.*

Proof. Note that Assertion (2) follows from Assertion (1) and that Assertion (4) follows from Assertion (3).

(1) Let $\ell, \ell' \in \mathscr{G}X$. If $\ell'(0) \notin \ell(\mathbb{R})$, then $\ell'(t) \notin \ell(\mathbb{R})$ for all $t \geqslant 0$ or $\ell'(t) \notin \ell(\mathbb{R})$ for all $t \leqslant 0$, since X is an $\mathbb{R}$-tree. In the first case, we hence have $d(\ell(t), \ell'(t)) \geqslant t$ for all $t \geqslant 0$, thus $d(\ell, \ell')$ is at least $c_2 = \int_0^{+\infty} te^{-2\,t}\, dt = \frac{1}{4} > 0$. The same estimate holds in the second case. By symmetry, if $\ell(0) \notin \ell'(\mathbb{R})$, then $d(\ell, \ell') \geqslant c_2 > 0$. This argument furthermore shows that if the geodesic segment (or ray or line) $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ is reduced to a point, then $d(\ell, \ell')$ is at least $c_2 > 0$.

If $d(\ell'(0), \ell(0)) \geqslant 1$, then $d(\ell(t), \ell'(t)) \geqslant \frac{1}{2}$ for $|t| \leqslant \frac{1}{4}$, thus

$$d(\ell, \ell') \geqslant \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{2}\, e^{-2|t|}\, dt\ ,$$

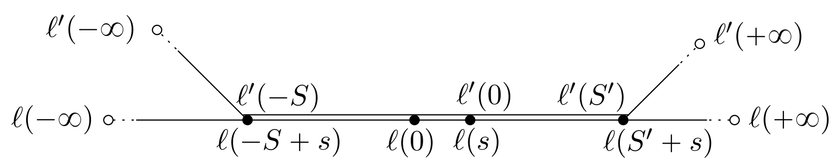
which is a positive universal constant.

If $d(\ell'(0), \ell(0)) \leqslant 1$, if $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ contains $\ell'(0)$ and is not reduced to a point, but if the orientations of ℓ and ℓ' do not coincide on this intersection, then

$$d(\ell(t), \ell'(t)) \geqslant 2t - d(\ell(0), \ell'(0)) \geqslant 2t - 1$$

for all $t \geqslant 1$, so that $d(\ell, \ell')$ is at least $\int_1^{+\infty} (2t-1)\, e^{-2t}\, dt$, which is a positive constant.

Assume now that $\ell'(0) \in \ell(\mathbb{R})$ and $\ell(0) \in \ell'(\mathbb{R})$, that $d(\ell'(0), \ell(0)) \leqslant 1$, that $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ is not reduced to a point and that the orientations of ℓ and ℓ' coincide on this intersection. Then there exists $s \in \mathbb{R}$ such that $\ell'(0) = \ell(s)$, so that $|s| = d(\ell(0), \ell'(0)) \leqslant 1$. Assume for instance that $s \geqslant 0$, the other case being treated similarly. Then there exist $S, S' \geqslant 0$ maximal such that $\ell'(t) = \ell(t+s)$ for all $t \in [-S, S']$. We use the conventions that $S = +\infty$ if $\ell'_- = \ell_-$, that $S' = +\infty$ if $\ell'_+ = \ell_+$, and that $e^{-\infty} = 0$. Since $\ell(0) \in \ell'(\mathbb{R})$, we have $-S + s \leqslant 0$.



By the definition of the visual distances (see Equation (2.3)), we have, for t big enough,

$$d_{\ell(0)}(\ell_+, \ell'_+) = e^{-d(\ell(0),\, \ell(S'+s))} = e^{-S'-s} \asymp e^{-S'}\ .$$

Similarly $d_{\ell(0)}(\ell_-, \ell'_-) \asymp e^{-S}$.

As can be seen in the above picture, we have

$$d(\ell(t), \ell'(t)) = \begin{cases} -2t - 2S + s & \text{if } \ t \leqslant -S \\ s & \text{if } \ -S \leqslant t \leqslant S' + s \\ 2t - 2S' - s & \text{if } \ t \geqslant S' + s\ . \end{cases}$$

By the definition of the distance on $\check{\mathscr{G}}X$ (see Equation (2.7)), by easy changes of variables, assuming that at least one of S, S' is at least 1 for the last line (otherwise the previous line shows that $d(\ell,\ell')$ is larger than a positive constant), we have

$$\begin{aligned} d(\ell,\ell') &= \int_{-\infty}^{-S} (-2t-2S+s)\, e^{-2|t|}\, dt + \int_{-S}^{S'+s} s\, e^{-2|t|}\, dt + \int_{S'+s}^{+\infty} (2t-2S'-s)\, e^{-2t}\, dt \\ &= \frac{1}{2}\, e^{-2S+s} \int_{s}^{+\infty} u\, e^{-u}\, du + s \int_{-S}^{S'+s} e^{-2|t|}\, dt + \frac{1}{2}\, e^{-2S'-s} \int_{s}^{+\infty} u\, e^{-u}\, du \\ &\asymp e^{-2S} + e^{-2S'} + s\ . \end{aligned}$$

Assertion (1) of Lemma 3.4 follows.

(3) The first inclusion in Assertion (3) follows easily from Assertion (1). The second inclusion follows from the argument of its proof, and the fact that if $d_{\ell(0)}(\ell_+,\ell'_+)$, $d_{\ell(0)}(\ell_-,\ell_-)$ and $d(\ell(0),\ell'(0))$ are at most some small positive constant, then $\ell(\mathbb{R}) \cap \ell'(\mathbb{R})$ contains $\ell'(0)$ and is not reduced to a point. □

When X is a Riemannian manifold with pinched sectional curvature, the following result says that the Hölder structure defined by the distance d given in Equation (2.7) on the unit tangent bundle of X, identified with $\mathscr{G}X$ and T^1X as explained previously, is the usual one.

Recall that *Sasaki's metric* on T^1X is the Riemannian metric induced on the submanifold T^1X by the canonical Riemannian metric on the tangent bundle TX, such that for every $v \in TX$, if $T_vTX = V_v \oplus H_v$ is the direct sum decomposition defined by the Levi-Civita connection of the Riemannian manifold X, then

- the direct sum $V_v \oplus H_v$ is orthogonal for Sasaki's metric,
- the canonical isomorphism $V_v \simeq T_{\pi(v)}X$ is an isometry, when V_v is endowed with Sasaki's scalar product,
- the restriction of the tangent map of the footpoint projection $T\pi : H_v \to T_{\pi(v)}X$ is an isometry, when H_v is endowed with Sasaki's scalar product.

Proposition 3.5. *When X is a Riemannian manifold with pinched sectional curvature, the following distances on T^1X are Hölder-equivalent:*

(1) *the Riemannian distance on T^1X defined by Sasaki's metric,*

(2) *the distance δ_1 on T^1X defined, for all $\ell,\ell' \in T^1X$, by*

$$\delta_1(\ell,\ell') = \exp(-\sup\{t \geq 0 \ :\ \sup_{s\in[-t,t]} d(\ell(s),\ell'(s)) \leq 1\})\ ,$$

with the convention $\delta_1(\ell,\ell') = 1$ if $d(\ell(0),\ell'(0)) > 1$ and $\delta_1(\ell,\ell') = 0$ if $\ell = \ell'$,

(3) *the distance δ_2 on T^1X defined, for all $\ell,\ell' \in T^1X$, by*

$$\delta_2(\ell,\ell') = \sup_{s\in[0,1]} d(\ell(s),\ell'(s))\ ,$$

(4) *the distance d defined by Equation (2.7).*

Note that we could replace the interval $[0,1]$ in the definition of δ_2 by any interval $[a,b]$ with $a<b$.

Proof. The fact that the first three distances on T^1X are Hölder-equivalent is already known, see for instance [Bal, p. 70].

Let us hence prove that d and δ_2 are Hölder-equivalent. By the convexity of the distance function between two geodesic segments in a CAT(-1)-space, we have

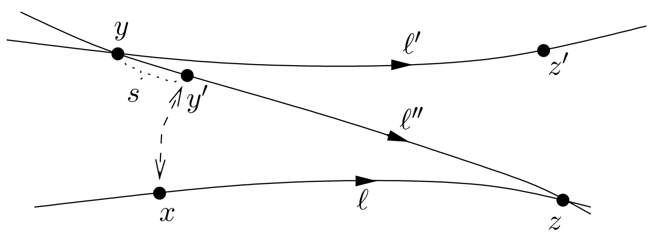
$$\delta_2(\ell,\ell') = \max\{d(\ell(0),\ell'(0)),\ d(\ell(1),\ell'(1))\}\ .$$

Hence by Proposition 3.2 applied twice (with $t=0$ and $t=1$), there exists $c>0$ such that if $d(\ell,\ell')\leqslant 1$, then $\delta_2(\ell,\ell')\leqslant c\, d(\ell,\ell')^{\frac{1}{2}}$. Therefore the identity map from (T^1X,d) to (T^1X,δ_2) is Hölder-continuous.

Let us now prove conversely that there exist two constants $c_1'\geqslant 1$ and $c_2'\in\,]0,1]$ such that for all $\ell,\ell'\in\mathscr{G}X$, if $\delta_2(\ell,\ell')\leqslant\frac{1}{2}$, then

$$d(\ell,\ell')\leqslant c_1'\ \delta_2(\ell,\ell')^{c_2'}\ . \tag{3.1}$$

Let $x=\ell(0)$, $y=\ell'(0)$, $z=\ell(1)$, $z'=\ell'(1)$. Note that $d(y,z)\geqslant\frac{1}{2}$ by the triangle inequality, since $d(x,y)\leqslant\delta_2(\ell,\ell')\leqslant\frac{1}{2}$ and $d(x,z)=1$. Let ℓ'' be the geodesic line through y and z, oriented from y to z with $\ell''(0)=y$. Let us prove that $d(\ell,\ell'')\leqslant\frac{c_1'}{2}\ d(x,y)^{c_2'}$ for appropriate constants c_1' and c_2'. A similar argument proves that $d(\ell'',\ell')\leqslant\frac{c_1'}{2}\ d(z,z')^{c_2'}$, and the triangle inequality for the distance d gives Equation (3.1).



If $d(y,z)\leqslant d(x,z)$, let $x'\in[x,z]$ be such $d(x',z)=d(y,z)\geqslant\frac{1}{2}$. We have $s=d(x,x')=d(x,z)-d(x',z)=d(x,z)-d(y,z)\leqslant d(x,y)$ by the triangle inequality, and $d(x',y)\leqslant d(x,y)$ by convexity. By Equation (2.8), we have

$$d(\ell,\ell'')\leqslant d(\ell,\mathsf{g}^s\ell)+d(\mathsf{g}^s\ell,\ell'')\leqslant s+d(\mathsf{g}^s\ell,\ell'')\leqslant d(x,y)+d(\mathsf{g}^s\ell,\ell'')\ .$$

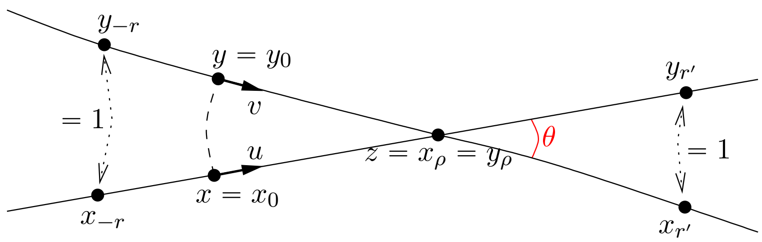
If $d(y,z)\geqslant d(x,z)$, let $y'\in[y,z]$ be such $d(y',z)=d(x,z)=1\geqslant\frac{1}{2}$ (see the above picture). We similarly have $d(x,y')\leqslant d(x,y)$ and $d(\ell,\ell'')\leqslant d(x,y)+d(\mathsf{g}^s\ell'',\ell'')$ if $s=d(y,y')$.

We hence only have to prove that for all $x,y,z\in X$ with $0<d(x,y)\leqslant\frac{1}{2}$ and $d(x,z)=d(y,z)\geqslant\frac{1}{2}$, if u and v are the unit tangent vectors at x and y respectively pointing to z, then $d(u,v)\leqslant c_1''\ d(x,y)^{c_2''}$ for appropriate constants c_1'' and c_2''. This follows from the following lemma with $t=0$, since $\frac{1}{2}\geqslant\frac{1}{2}\sqrt{d(x,y)}$. We will need its more general version in the proof of Lemma 3.15.

Lemma 3.6. *There exist two constants $c_8\geqslant 1$ and $c_3\in\,]0,1]$ such that for all $x,y,z\in X$ with $0<d(x,y)\leqslant 1$ and $\rho=d(x,z)=d(y,z)\geqslant\frac{1}{2}\sqrt{d(x,y)}$, for every $t\in[0,\rho]$, if u and v are the unit tangent vectors at x and y respectively pointing to z, then*

$$d(\mathsf{g}^tu,\mathsf{g}^tv)\leqslant c_8\ d(x,y)^{c_3}(e^{-t}+e^{2t-2\rho})\ . \tag{3.2}$$

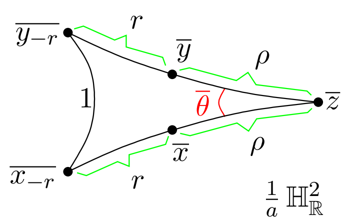
Proof. For every $s \in \mathbb{R}$, let $x_s = \pi(\mathsf{g}^s u)$ and $y_s = \pi(\mathsf{g}^s v)$, so that $x_0 = x$, $y_0 = y$ and $x_\rho = y_\rho = z$. Let $-r < r'$ in $\mathbb{R}$ be such that $d(x_{-r}, y_{-r}) = d(x_{r'}, y_{r'}) = 1$, which exist since $x \neq y$. We have $r > 0$ by convexity since $d(x_0, y_0) \leqslant 1$, and $r' > \rho$.



Claim 1 : There exists a constant $c_3 \in \,]0,1]$ depending only on the lower bound of the curvature of X such that

$$e^{-r} \leqslant 2\, d(x,y)^{c_3} \quad \text{and} \quad e^{-r'} \leqslant d(x,y)^{c_3}\, e^{-\rho}\,.$$

Proof. Let $a \geqslant 1$ be such that the sectional curvature of X is at least $-a^2$. Consider the comparison triangle $(\overline{x_{-r}}, \overline{y_{-r}}, \overline{z})$, in the real hyperbolic space $\frac{1}{a}\,\mathbb{H}^2_{\mathbb{R}}$ with constant sectional curvature $-a^2$, to the triple of points (x_{-r}, y_{-r}, z) in X. Let $\overline{\theta}$ be its angle at $\overline{z}$, and let $\overline{x}, \overline{y}$ be the points corresponding to x, y (so that we have $d(\overline{x}, \overline{z}) = d(\overline{y}, \overline{z}) = \rho$).



Since the geodesic triangles in X are less pinched than the geodesic triangles in $\frac{1}{a}\,\mathbb{H}^2_{\mathbb{R}}$, we have $d(\overline{x}, \overline{y}) \leqslant d(x,y)$. By the hyperbolic sine rule in $\frac{1}{a}\,\mathbb{H}^2_{\mathbb{R}}$, we have

$$\frac{\sinh \frac{a}{2}}{\sinh(a(r+\rho))} = \frac{\sinh\big(\frac{a}{2}\, d(x_{-r}, y_{-r})\big)}{\sinh\big(a\, d(x_{-r}, z)\big)} = \sin\frac{\overline{\theta}}{2} = \frac{\sinh\big(\frac{a}{2}\, d(\overline{x}, \overline{y})\big)}{\sinh\big(a\, d(x, z)\big)} \leqslant \frac{\sinh\big(\frac{a}{2}\, d(x,y)\big)}{\sinh(a\,\rho)}\,.$$

Since the map $t \mapsto (\ln 2)t + \ln(1 - \frac{t}{2})$ is nonnegative on $[0,1]$, and since $\rho \geqslant \frac{1}{2}\sqrt{d(x,y)}$, we have

$$e^{2a\rho} \geqslant e^{\ln 2 \sqrt{d(x,y)}} \geqslant \frac{1}{1 - \frac{1}{2}\sqrt{d(x,y)}}\,.$$

Hence $\sinh(a\rho) = \frac{e^{a\rho} - e^{-a\rho}}{2} \geqslant \frac{1}{4}\, e^{a\rho}\sqrt{d(x,y)}$. Therefore, since $d(x,y) \leqslant 1$,

$$e^{-ar} = \frac{e^{a\rho}}{e^{ar + a\rho}} \leqslant \frac{4 \sinh(a\rho) \,/\, \sqrt{d(x,y)}}{2 \sinh(ar + a\rho)} \leqslant \frac{2 \sinh\big(\frac{a}{2}\, d(x,y)\big)}{\sqrt{d(x,y)}\, \sinh \frac{a}{2}} \leqslant 2\sqrt{d(x,y)}\,.$$

So that the first assertion of Claim 1 follows, since $2^{\frac{1}{a}} \leqslant 2$, with $c_3 = \frac{1}{2a}$.

In order to prove the second assertion of Claim 1, let θ be the angle at z between the unit tangent vectors $\mathsf{g}^\rho u$ and $\mathsf{g}^\rho v$ (see the picture before the statement of Claim 1). Since the geodesic triangles in X are less pinched than their comparison triangles in $\frac{1}{a}\,\mathbb{H}^2_{\mathbb{R}}$, and again by the hyperbolic sine rule in $\frac{1}{a}\,\mathbb{H}^2_{\mathbb{R}}$, we have

$$\sin\frac{\theta}{2} \geqslant \frac{\sinh\big(\frac{a}{2}\, d(x_{r'}, y_{r'})\big)}{\sinh\big(a\, d(x_{r'}, z)\big)} = \frac{\sinh \frac{a}{2}}{\sinh(a(r' - \rho))}\,.$$

Since the geodesic triangles in X are more pinched than their comparison triangles in $\mathbb{H}^2_{\mathbb{R}}$, since $d(x,y) \leqslant 1$ and since $\sinh\rho \geqslant \rho \geqslant \frac{1}{2}\sqrt{d(x,y)}$, we have

$$\sin\frac{\theta}{2} \leqslant \frac{\sinh\big(\frac{1}{2}\,d(x,y)\big)}{\sinh d(x,z)} \leqslant \frac{(\sinh\frac{1}{2})d(x,y)}{\sinh\rho} \leqslant 2\,(\sinh\frac{1}{2})\,\sqrt{d(x,y)}\;.$$

Let $c_4 = \Big(\frac{\sinh\frac{1}{2}}{\sinh\frac{a}{2}}\Big)^{\frac{1}{a}}$, which is positive and strictly less than 1 since $a > 1$. Then

$$\begin{aligned} e^{-ar'} &= \frac{e^{-a\rho}}{e^{ar'-a\rho}} \leqslant \frac{e^{-a\rho}}{2\sinh(a(r'-\rho))} \leqslant e^{-a\rho}\,\frac{\sinh\frac{1}{2}}{2\sinh\frac{a}{2}}\,\frac{d(x,y)}{\sinh\rho} \\ &\leqslant e^{-a\rho}\,{c_4}^a\,\sqrt{d(x,y)}\;. \end{aligned} \tag{3.3}$$

So that the second assertion of Claim 1 follows, since $c_4 \leqslant 1$, again with $c_3 = \frac{1}{2a}$. □

Let us remark that since $d(x,y) \leqslant 1$, Equation (3.3) also gives that

$$r' - \rho \geqslant -\ln c_4 > 0\;.$$

Since θ is also the angle between the unit tangent vectors $-\mathsf{g}^\rho u$ and $-\mathsf{g}^\rho v$, a similar argument gives that

$$\frac{\sinh\frac{1}{2}}{\sinh(r'-\rho)} = \frac{\sinh\big(\frac{1}{2}\,d(x_{r'},y_{r'})\big)}{\sinh d(x_{r'},z)} \geqslant \sin\frac{\theta}{2} \geqslant \frac{\sinh\big(\frac{a}{2}\,d(x,y)\big)}{\sinh\big(a\,d(x,z)\big)} \geqslant \frac{a\,d(x,y)}{2\,\sinh(a\,\rho)}\;.$$

Let $c_5 = \frac{2\,\sinh\frac{1}{2}}{a(1-c_4^2)} > 0$. Since the map $t \mapsto \frac{\sinh(t)}{e^t}$ is nondecreasing on $\mathbb{R}$, we hence have

$$e^{r'-\rho} \leqslant \frac{e^{-\ln c_4}}{\sinh(-\ln c_4)}\sinh(r'-\rho) \leqslant \frac{2}{1-c_4^2}\,\frac{2\,\sinh\frac{1}{2}}{a\,d(x,y)}\,\sinh(a\,\rho) \leqslant \frac{c_5}{d(x,y)}\,e^{a\rho}\;.$$

Therefore

$$1 + r' \leqslant (a+1)\rho + (1+\ln c_5) - \ln d(x,y)\;, \tag{3.4}$$

a formula which be useful later on.

Let $t \in [0,\rho]$. By the definition of the distance d on $\mathscr{G}X$ (see Equation (2.7)), we have

$$d(\mathsf{g}^t u, \mathsf{g}^t v) = \int_{-\infty}^{+\infty} d(\pi(\mathsf{g}^s\mathsf{g}^t u), \pi(\mathsf{g}^s\mathsf{g}^t v))\,e^{-2|s|}\,ds = \int_{-\infty}^{+\infty} d(x_s,y_s)\,e^{-2|s-t|}\,ds\;. \tag{3.5}$$

We subdivide the integral $\int_{-\infty}^{+\infty}$ as $\int_{-\infty}^{-r} + \int_{-r}^{0} + \int_0^\rho + \int_\rho^{r'} + \int_{r'}^{+\infty}$.

Claim 2 : We have

$$I_1 = \int_{-\infty}^{-r} d(x_s,y_s)\,e^{-2|s-t|}\,ds \leqslant 2\,e^{-2t}\,d(x,y)^{c_3}\;.$$

Proof. By the triangle inequality, for every $s \in [r,+\infty[$, we have

$$d(x_{-s},y_{-s}) \leqslant d(x_{-s},x_{-r}) + d(x_{-r},y_{-r}) + d(y_{-r},y_{-s}) \leqslant 2s+1\;.$$

Hence

$$I_1 = \int_r^{+\infty} d(x_{-s}, y_{-s})\, e^{-2(s+t)}\, ds \leqslant e^{-2t} \int_r^{+\infty} (2s+1)\, e^{-2s}\, ds \leqslant e^{-2t} e^{-r} ,$$

and the result follows from the first assertion of Claim 1. □

Claim 3 : We have

$$I_2 = \int_{-r}^{0} d(x_s, y_s)\, e^{-2|s-t|}\, ds \leqslant 2\,(\sinh 1)\, e^{-2t}\, d(x,y)^{c_3} .$$

Proof. Recall that since X is CAT(-1) and by an easy exercise in hyperbolic geometry (see for instance [PauPS, Lem. 2.5 (i)]), for all x', y', z' in X such that $d(x', z') = d(y', z')$, for every $t' \in [0, d(x', z')]$, if x'_t (respectively y'_t) is the point on $[x', z']$ (respectively $[y', z']$) at distance t' from x' (respectively y'), then

$$d(x'_t, y'_t) \leqslant e^{-t'} \sinh d(x', y') . \tag{3.6}$$

Hence for all $s \in [0, r]$, we have $d(x_{-s}, y_{-s}) \leqslant e^{-(r-s)} \sinh d(x_{-r}, y_{-r}) = e^{-r+s} \sinh 1$. Thus

$$\begin{aligned} I_2 &= \int_0^r d(x_{-s}, y_{-s})\, e^{-2(s+t)}\, ds \\ &\leqslant (\sinh 1)\, e^{-2t} \int_0^{+\infty} e^{-r+s}\, e^{-2s}\, ds = (\sinh 1)\, e^{-2t}\, e^{-r} , \end{aligned}$$

and the result also follows from the first assertion of Claim 1. □

Claim 4 : There exists a universal constant $c_6 > 0$ such that

$$I_3 = \int_0^{\rho} d(x_s, y_s)\, e^{-2|s-t|}\, ds \leqslant c_6\, e^{-t}\, d(x,y) .$$

Proof. By Equation (3.6) and since $d(x, y) \leqslant 1$, for every $s \in [0, \rho]$, we have

$$d(x_s, y_s) \leqslant e^{-s} \sinh d(x,y) \leqslant (\sinh 1)\, e^{-s}\, d(x,y) .$$

Therefore

$$\begin{aligned} I_3 &= \int_0^t d(x_s, y_s)\, e^{-2(t-s))}\, ds + \int_t^{\rho} d(x_s, y_s)\, e^{-2(s-t))}\, ds \\ &\leqslant (\sinh 1)\, d(x,y) \Big(e^{-2t} \int_0^t e^s\, ds + e^{2t} \int_t^{+\infty} e^{-3s}\, ds \Big) \leqslant \frac{4 \sinh 1}{3}\, d(x,y)\, e^{-t} . \quad \square \end{aligned}$$

Claim 5 : We have

$$I_4 = \int_{\rho}^{r'} d(x_s, y_s)\, e^{-2|s-t|}\, ds \leqslant (\sinh 1)\, e^{2t-2\rho}\, d(x,y)^{c_3} .$$

Proof. By Equation (3.6), for every $s \in [\rho, r']$, we have

$$d(x_s, y_s) \leqslant e^{-(r'-s)} \sinh d(x_{r'}, y_{r'}) = (\sinh 1)\, e^{-r'+s} .$$

Hence

$$I_4 = \int_\rho^{r'} d(x_s, y_s)\, e^{-2(s-t))}\, ds \leqslant (\sinh 1)\, e^{2t}\, e^{-r'} \int_\rho^{r'} e^{-s}\, ds \leqslant (\sinh 1)\, e^{2t-\rho}\, e^{-r'}\ .$$

The result follows from the second assertion of Claim 1. □

Claim 6 : There exists a constant $c_7 > 0$ depending only on the lower bound of the curvature of X such that

$$I_5 = \int_{r'}^{+\infty} d(x_s, y_s)\, e^{-2|s-t|}\, ds \leqslant c_7\, e^{2t-2\rho}\, d(x,y)^{c_3}\ .$$

Proof. By the triangle inequality, for every $s \in [r', +\infty[$, we have

$$d(x_s, y_s) \leqslant d(x_s, x_{r'}) + d(x_{r'}, y_{r'}) + d(y_{r'}, y_s) \leqslant 2s + 1\ .$$

Hence, using the second assertion of Claim 1, we have

$$I_5 = \int_{r'}^{+\infty} d(x_s, y_s)\, e^{-2(s-t)}\, ds \leqslant e^{2t} \int_{r'}^{+\infty} (2s+1)\, e^{-2s}\, ds = e^{2t}(1+r')e^{-2r'}\ .$$

By Equation (3.3), since $\sinh \rho \geqslant \rho \geqslant \frac{1}{2}\sqrt{d(x,y)}$ and $c_3 = \frac{1}{2a}$, we have

$$e^{-2r'} \leqslant e^{-2\rho}\, c_4^2 \left(\frac{d(x,y)}{2\sinh \rho}\right)^{2/a} \leqslant e^{-2\rho}\, d(x,y)^{2c_3}\ .$$

Since $c \leqslant e^c$ for every $c \geqslant 0$ and $d(x,y) \leqslant 1$, we have $-\ln d(x,y) \leqslant \frac{1}{c_3}\, d(x,y)^{-c_3}$. Hence by Equation (3.4), we have

$$I_5 \leqslant e^{2t-2\rho}\Big((a+1)c_4^2\, d(x,y)^{2/a} \frac{\rho}{(2\sinh \rho)^{2/a}} + (1 + \ln c_5)\, d(x,y)^{2c_3} + \frac{1}{c_3}\, d(x,y)^{c_3}\Big)\ .$$

Assuming, as we may, that $a \geqslant 2$, the map $\rho' \mapsto \frac{\rho'}{(\sinh \rho')^{2/a}}$ is bounded on $[0, +\infty[$. Since $c_3 = \frac{1}{2a} \leqslant \frac{2}{a}$, this proves Claim 6. □

Since $d(x,y) \leqslant 1$, it follows from Equation (3.5) and from Claims 2 to 6 that there exists a constant $c_8 > 0$ depending only on the lower bound of the curvature of X such that

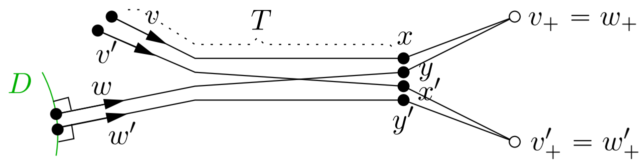
$$d(\mathtt{g}^t u, \mathtt{g}^t v) = I_1 + I_2 + I_3 + I_4 + I_5 \leqslant c_8\, d(x,y)^{c_3}(e^{-t} + e^{2t-2\rho})\ .$$

This proves Lemma 3.6, hence concludes the proof of Proposition 3.5. □ □

Let D be a nonempty proper closed convex subset of X. The regularity property in the Riemannian manifold case of the fibrations $f_D^\pm : \mathscr{U}_D^\pm \to \partial_\pm^1 D$ defined in Section 2.4, that will be needed in Section 12.3, is the Hölder-continuity one, as proved in the following lemma (see also [PaP17b, Lem. 6]).

Lemma 3.7. *Assume that X is a Riemannian manifold with pinched negative curvature. The maps $f_D^\pm$ are Hölder-continuous on any set of elements $\ell \in \mathscr{U}_D^\pm$ such that $d(\pi(\ell), \pi(f_D^\pm(\ell)))$ is bounded.*

Proof. We prove the result for f_D^+, the one for f_D^- follows similarly. We will use the Hölder-equivalent distances δ_1 and δ_2, defined in the statement of Proposition 3.5, on the unit tangent bundle of X, identified with $\mathscr{G}X$ as explained previously.



Let $v, v' \in T^1X$ be such that $d(v(0), v'(0)) \leqslant 1$, let $w = f_D^+(v)$ and $w' = f_D^+(v')$. Let $T = \sup\{t \geqslant 0 \ : \ \sup_{s\in[0,t]} d(v(s), v'(s)) \leqslant 1\}$, so that $\delta_1(v, v') \geqslant e^{-T}$. We may assume that T is finite, otherwise $v_+ = v'_+$, hence $w = w'$. Let $x = v(T)$ and $x' = v'(T)$, which satisfy $d(x, x') \leqslant 1$. Let y (respectively y') be the closest point to x (respectively x') on the geodesic ray defined by w (respectively w'). By convexity, since $d(v(0), w(0))$ and $d(v'(0), w'(0))$ are bounded by a constant $c > 0$ and since $v_+ = w_+, v'_+ = w'_+$, we have $d(x, y) \leqslant c$ and $d(x', y') \leqslant c$. By the triangle inequality, we have $d(y, y') \leqslant 2c+1$, $d(y, w(1)) \geqslant T - 2c - 1$ and $d(y', w'(1)) \geqslant T - 2c - 1$. By convexity, and since closest point maps exponentially decrease the distances, there exists a constant $c' > 0$ such that

$$\delta_2(w, w') = d(w(1), w'(1)) \leqslant c'\, d(y, y')\, e^{-(T-2c-1)} \leqslant c'\,(2c+1)\, e^{2c+1}\, \delta_1(v, v')\,.$$

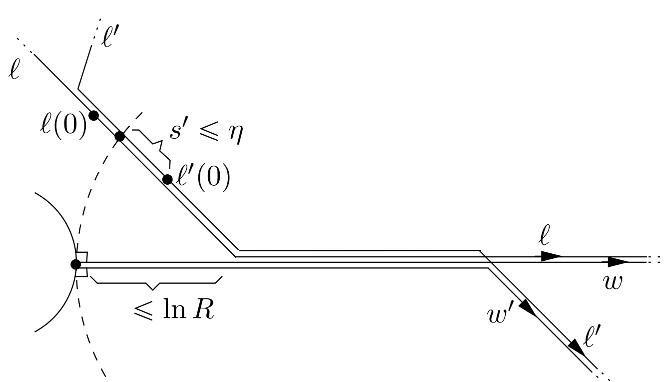
The result follows. □

When X is an $\mathbb{R}$-tree, we have a stronger version of Lemma 3.7, that will be needed in Section 12.6.

Lemma 3.8. *Assume that X is an $\mathbb{R}$-tree. Let $\eta, R > 0$ be such that $\eta \leqslant 1 \leqslant \ln R$, and let D be a nonempty closed convex subset of X. Then the restriction to the dynamical neighborhood $\mathscr{V}^{\pm}_{\eta,R}(\partial^1_{\pm} D)$ of the fibration $f^{\pm}_D$ is (uniformly locally) Lipschitz, with constants independent of η.*

Proof. We assume for instance that $\pm = +$. Let $\ell, \ell' \in \mathscr{V}^{+}_{\eta,R}(\partial^1_{+} D)$ and let $w = f^+_D(\ell), w' = f^+_D(\ell')$.

Since the fiber over $\rho \in \partial^1_+ D$ of the restriction to $\mathscr{V}^{+}_{\eta,R}(\partial^1_{+} D)$ of f^+_D is $V^+_{\rho,\eta,R}$, [1] there exist $s, s' \in \,]-\eta, \eta[$ such that $\mathtt{g}^s \ell \in B^+(w, R)$ and $\mathtt{g}^{s'} \ell' \in B^+(w', R)$, so that $\mathtt{g}^s \ell(t) = w(t)$ and $\mathtt{g}^{s'} \ell'(t) = w'(t)$ for all $t \geqslant \ln R$ by the definition of the Hamenstädt balls. Up to permuting ℓ and ℓ', we assume that $s' \geqslant s$.



[1]See the end of Section 2.4.

By (the proof of) Lemma 3.4 (1), there exists a constant $c_R > 0$ depending only on R such that if $d(\ell, \ell') \leqslant c_R$ and $s'' = d(\ell(0), \ell'(0))$, then $s'' = s' - s$ and the geodesic lines $\mathtt{g}^s \ell$ and $\mathtt{g}^{s'} \ell'$ coincide at least on $[-\ln R - 1, \ln R + 1]$. In particular, we have, since $|s|, |s'| \leqslant \eta \leqslant 1$,

$$w(\ln R) = \ell(s + \ln R) = \ell'(s' + \ln R) = w'(\ln R)\ .$$

Since the origin of w is the closest point on D to any point of $w([0, +\infty[)$, we hence have that $w(t) = w'(t)$ for all $t \in [0, \ln R]$. Therefore (using Equation (2.8) for the last inequality),

$$\begin{aligned} d(w, w') &= \int_{\ln R}^{+\infty} d(w(t), w'(t))\, e^{-2\,t}\, dt = \int_{\ln R}^{+\infty} d(\mathtt{g}^s \ell(t), \mathtt{g}^{s'} \ell'(t))\, e^{-2\,t}\, dt \\ &= e^{2s} \int_{\ln R + s}^{+\infty} d(\ell(u), \mathtt{g}^{s''} \ell'(u))\, e^{-2\,u}\, du \leqslant e^{2s}\, d(\ell, \mathtt{g}^{s''} \ell') \\ &\leqslant e^{2s} \big(d(\ell, \ell') + d(\ell', \mathtt{g}^{s''} \ell')\big) \leqslant e^{2s}\, (d(\ell, \ell') + s'') \\ &= e^{2s} \big(d(\ell, \ell') + d(\ell(0), \ell'(0))\big)\ , \end{aligned}$$

so that the result follows from Lemma 3.4 (2).

Note that when X is (the geometric realisation of) a simplicial tree, we have $s = s' = s'' = 0$ and the above computations simplify to give $d(w, w') \leqslant d(\ell, \ell')$. □

For any metric space Z and $\alpha \in\ ]0, 1]$, the *Hölder norm* of a bounded α-Hölder-continuous function $f : Z \to \mathbb{R}$ is

$$\|f\|_\alpha = \|f\|_\infty + \|f\|'_\alpha\,,$$

where

$$\|f\|'_\alpha = \sup_{\substack{x,\, y \,\in\, Z \\ 0 < d(x, y) \leqslant 1}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}\ .$$

When the diameter of Z is bounded by 1,[2] this coincides with the usual definition. Note that even if the constant ϵ in the definition of a α-Hölder-continuous map is less than 1, this norm is finite, since

$$\sup_{\substack{x,\, y \in Z \\ \epsilon \leqslant d(x,y) \leqslant 1}} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \leqslant 2\, \epsilon^{-\alpha}\, \|f\|_\infty\ .$$

Note that for all bounded α-Hölder-continuous maps $f, g : Z \to \mathbb{R}$, we have

$$\|fg\|_\alpha \leqslant \|f\|_\alpha\, \|g\|_\infty + \|f\|_\infty\, \|g\|_\alpha\ . \tag{3.7}$$

We denote by $\mathscr{C}_{\rm c}^\alpha(Z)$ (respectively $\mathscr{C}_{\rm b}^\alpha(Z)$) the space of α-Hölder-continuous real-valued functions with compact support (respectively which are bounded) on Z, endowed with this norm. Note that $\mathscr{C}_{\rm b}^\alpha(Z)$ is a real Banach space.[3]

The next two lemmas will be needed only in Part III of this book. The first one is a metric estimate on the extension of geodesic segments to geodesic rays, with its functional counterpart.

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[2]This is in particular the case for the sequence spaces of symbolic dynamical systems, see Sections 5.2 and 9.2.

[3]The standard proof using Arzela-Ascoli's theorem applies with our slightly different definition of the Hölder norms.

Lemma 3.9. *Let X be a geodesically complete proper* $\operatorname{CAT}(-1)$ *space, let $T \geqslant 1$, and let $\alpha \in \breve{\mathscr{G}}X$ be a generalised geodesic line which is isometric exactly on $[0,T]$. For every generalised geodesic line $\rho \in \breve{\mathscr{G}}X$ which is isometric exactly on $[0,+\infty[$, such that $\rho|_{[0,T]} = \alpha|_{[0,T]}$, we have*

$$d(\alpha,\rho) = \frac{e^{-2\,T}}{4} \;<1\;,$$

and hence for all $\beta \in \,]0,1]$ and $\widetilde{\psi} \in \mathscr{C}_b^{\beta}(\breve{\mathscr{G}}X)$,

$$|\widetilde{\psi}(\alpha)-\widetilde{\psi}(\rho)| \leqslant \frac{e^{-2\,\beta\,T}}{4^{\beta}}\;\|\widetilde{\psi}\,\|_{\beta}\;.$$

Proof. By Equation (2.7) defining the distance on $\breve{\mathscr{G}}X$, we have, since $d(\alpha(t),\rho(t)) = 0$ for all $t \in\,]-\infty,T]$ and $d(\alpha(t),\rho(t)) = t-T$ otherwise,

$$d(\alpha,\rho) = \int_T^{+\infty}(t-T)\,e^{-2t}\,dt = e^{-2\,T}\;\int_0^{+\infty}u\,e^{-2u}\,du = \frac{e^{-2\,T}}{4}\;.$$

The result follows. □

The second lemma gives a metric estimate, with its functional counterpart, on the map which associates to a geodesic ray in an outer normal unit bundle its point at infinity, emphasising the $\frac{1}{2}$-Hölder-continuity of endpoints maps (see Lemma 3.4 (4)). We start by giving some definitions.

Let X be a geodesically complete proper $\operatorname{CAT}(-1)$ space, and let D be a nonempty proper closed convex subset of X. The *distance-like map*

$$d_D : \big(\partial_\infty X - \partial_\infty D\big)^2 \rightarrow [0,+\infty[$$

associated with D is defined in [HeP4, §2.2] as follows. For $\xi,\xi' \in \partial_\infty X - \partial_\infty D$, let $\xi_t,\xi'_t : [0,+\infty[\; \rightarrow X$ be the geodesic rays starting at the closest points $P_D(\xi), P_D(\xi')$ to ξ,ξ' on D and converging to ξ,ξ' as $t \rightarrow \infty$. Let

$$d_D(\xi,\xi') = \lim_{t\rightarrow+\infty} e^{\frac{1}{2}d(\xi_t,\,\xi'_t)-t}\;. \tag{3.8}$$

The distance-like map d_D is invariant by the diagonal action of the isometries of X preserving D. If D consists of a single point x, then d_D is the visual distance[4] d_x on $\partial_\infty X$ based at x. If D is a horoball with point at infinity ξ_0, then d_D is Hamenstädt's distance[5] on $\partial_\infty X - \{\xi_0\}$. As seen in [HeP4, §2.2, Ex. (4)], if X is a metric tree, then

$$d_D(\xi,\xi') = \begin{cases} e^{\frac{1}{2}\,d(P_D(\xi),\,P_D(\xi'))} \;>1 & \text{if }\; P_D(\xi)\neq P_D(\xi') \\ d_x(\xi,\xi') = e^{-d(x,y)}\;\leqslant 1 & \text{if }\; P_D(\xi)=P_D(\xi')=x \\ & \quad\text{and }\; [x,\xi[\,\cap\,[x,\xi'[\,=[x,y]\;. \end{cases}$$

In particular, although in general it is not an actual distance on its whole domain $\partial_\infty X - \partial_\infty D$, the map d_D is locally a distance, and we can define with the standard formula the β-Hölder-continuity of maps with values in $(\partial_\infty X - \partial_\infty D, d_D)$ and the β-Hölder-norm of a function defined on $(\partial_\infty X - \partial_\infty D, d_D)$. In the next result, we endow $\partial_\infty X - \partial_\infty D$ with the distance-like map d_D.

[4]See Equation (2.2).
[5]See Equation (2.12).

Proposition 3.10. *Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, and let $\mathbb{D}$ be a proper nonempty simplicial subtree of $\mathbb{X}$. Let $X = |\mathbb{X}|_1$ and $D = |\mathbb{D}|_1$ be their geometric realisations. The homeomorphism $\partial^+ : \partial^1_+\mathbb{D} \to (\partial_\infty X - \partial_\infty D)$ defined by $\rho \mapsto \rho_+$ is $\frac{1}{2}$-Hölder-continuous, and for all $\beta \in \,]0,1]$ and $\psi \in \mathscr{C}_b^\beta(\partial_\infty X - \partial_\infty D)$, the map $\psi \circ \partial^+ : \partial^1_+\mathbb{D} \to \mathbb{R}$ is bounded and $\frac{\beta}{2}$-Hölder-continuous, with*

$$\|\psi \circ \partial^+\|_{\frac{\beta}{2}} \leqslant (1 + 2^{\frac{\beta}{2}+1})\, \|\psi\|_\beta \;.$$

Proof. Let us prove that for every $\rho, \rho' \in \partial^1_+\mathbb{D}$, if $d(\rho,\rho') \leqslant 1$, then $\rho(0) = \rho'(0)$, and

$$d_D(\rho_+, \rho'_+) = \sqrt{2}\, d(\rho,\rho')^{\frac{1}{2}} \;. \tag{3.9}$$

This proves that the map ∂^+ is $\frac{1}{2}$-Hölder-continuous. We may assume that $\rho \neq \rho'$.

Let $\rho, \rho' \in \partial^1_+\mathbb{D}$. If $\rho(0) \neq \rho'(0)$, then the images of ρ and ρ' are disjoint and their connecting segment in the tree $\mathbb{X}$ joins $\rho(0)$ and $\rho'(0)$; hence for every $t \in [0, +\infty[$, we have

$$d(\rho(t), \rho'(t)) = d(\rho(0), \rho'(0)) + d(\rho(t), \rho(0)) + d(\rho'(t), \rho'(0)) \geqslant 1 + 2\,t \;.$$

Thus

$$\begin{aligned} d(\rho,\rho') &= \int_{-\infty}^0 d(\rho(0), \rho'(0))\; e^{2\,t}\; dt + \int_0^{+\infty} d(\rho(t), \rho'(t))\; e^{-2\,t}\; dt \\ &\geqslant \int_{-\infty}^0 e^{2\,t}\; dt + \int_0^{+\infty} (1+2t)\; e^{-2\,t}\; dt > 2\int_0^{+\infty} e^{-2\,t}\; dt = 1 \;. \end{aligned}$$

Assume that $x = \rho(0) = \rho'(0)$ and let n be the length of the intersection of ρ and ρ'. Then

$$d_D(\rho_+, \rho'_+) = d_x(\rho_+, \rho'_+) = \lim_{t \to +\infty} \; e^{\frac{1}{2}\, d(\rho(t), \rho'(t)) - t} = e^{-n} \;.$$

Furthermore, since $\rho(t) = \rho'(t)$ for $t \leqslant n$ and $d(\rho(t), \rho'(t)) = 2(t-n)$ otherwise, we have, using the change of variables $u = 2(t-n)$,

$$d(\rho,\rho') = \int_n^{+\infty} \; 2\,(t-n)\; e^{-2\,t}\; dt = e^{-2\,n}\int_0^{+\infty} \; u\; e^{-u}\; \frac{du}{2} = \frac{e^{-2\,n}}{2} \;.$$

This proves Equation (3.9).

Let $\beta \in \,]0,1]$ and $\psi \in \mathscr{C}_b^\beta(\partial_\infty X - \partial_\infty D)$. We have $\|\psi \circ \partial^+\|_\infty = \|\psi\|_\infty$ since ∂^+ is a homeomorphism, and, by Equation (3.9),

$$\begin{aligned} \|\psi \circ \partial^+\|'_{\frac{\beta}{2}} &= \sup_{\rho,\,\rho' \in \partial^1_+\mathbb{D},\; 0 < d(\rho,\rho') \leqslant 1} \frac{|\psi \circ \partial^+(\rho) - \psi \circ \partial^+(\rho')|}{d(\rho,\rho')^{\frac{\beta}{2}}} \\ &\leqslant \sup_{\rho,\,\rho' \in \partial^1_+\mathbb{D},\; 0 < d(\rho,\rho') \leqslant \frac{1}{2}} \frac{|\psi \circ \partial^+(\rho) - \psi \circ \partial^+(\rho')|}{d(\rho,\rho')^{\frac{\beta}{2}}} + \frac{2\,\|\psi \circ \partial^+\|_\infty}{2^{-\frac{\beta}{2}}} \\ &\leqslant \sup_{\substack{\xi,\,\xi' \in \partial_\infty X - \partial_\infty D \\ 0 < d_D(\xi,\xi') \leqslant 1}} \frac{|\psi(\xi) - \psi(\xi')|}{2^{-\frac{\beta}{2}}\; d_D(\xi,\xi')^\beta} + 2^{\frac{\beta}{2}+1}\,\|\psi\|_\infty \\ &\leqslant 2^{\frac{\beta}{2}+1}\,\|\psi\|_\beta \;. \end{aligned}$$

Since $\|\psi \circ \partial^+\|_{\frac{\beta}{2}} = \|\psi \circ \partial^+\|_\infty + \|\psi \circ \partial^+\|'_{\frac{\beta}{2}}$, this proves the last claim of Proposition 3.10. □

A stronger assumption than the Hölder regularity is the locally constant regularity, that we now define. Alhough it is only useful for totally disconnected metric spaces, several error terms estimates in the literature use this stronger regularity (see for instance [AtGP, KemaPS] and Part III of this book).

Let $\epsilon > 0$. For every metric space E and every set E', we say that a map $f : E \to E'$ is *ϵ-locally constant* if f is constant on every closed ball of radius ϵ (or equivalently of radius at most ϵ) in E. We say that $f : E \to E'$ is *locally constant* if there exists $\epsilon > 0$ such that f is ϵ-locally constant.

Note that if E is a geodesic metric space and $f : E \to E'$ is locally constant, then f is constant. But when E is for instance an ultrametric space, since two distinct closed balls of the same radius are disjoint, the above definition turns out to be very interesting (and much used in representation theory in positive characteristic, for instance). For example, the characteristic function $\mathbb{1}_A$ of a subset A of E is ϵ-locally constant if and only if for every $x \in A$, the closed ball $B(x, \epsilon)$ is contained in A. In particular, the characteristic function of a closed ball of radius ϵ in an ultrametric space is ϵ-locally constant.

Remark 3.11. Let E and E' be two metric spaces. If a map $f : E \to E'$ is ϵ-locally constant, then it is α-Hölder-continuous for every $\alpha \in \,]0, 1]$. Indeed, for all $x, y \in E$, if $d(x, y) \leqslant \epsilon$ then $d(f(x), f(y)) = 0 \leqslant c\, d(x, y)^\alpha$ for all $c > 0$. If furthermore $E' = \mathbb{R}$ and f is bounded, then

$$\sup_{x,y\in E,\, x\neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} = \sup_{x,y\in E,\, d(x,y)>\epsilon} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \leqslant \frac{2}{\epsilon^\alpha}\; \|f\|_\infty \,.$$

For all $\epsilon \in \,]0, 1]$ and $\beta > 0$, we denote by $\mathscr{C}_b^{\epsilon\,\mathrm{lc},\,\beta}(E)$ the real vector space[6] of ϵ-locally constant functions $f : E \to \mathbb{R}$ endowed with the $\epsilon\,$lc-*norm of exponent* β defined by

$$\|f\|_{\epsilon\,\mathrm{lc},\,\beta} = \; \epsilon^{-\beta}\; \|f\|_\infty \,.$$

The above remark proves that if $\beta \in \,]0, 1]$, then $\|f\|_\beta \leqslant 3\|f\|_{\epsilon\,\mathrm{lc},\,\beta}$, so that the inclusion map from $\mathscr{C}_b^{\epsilon\,\mathrm{lc},\,\beta}(E)$ into $\mathscr{C}_b^{\beta}(E)$ is continuous.

3.2 Potentials

In this book, a *potential* for Γ is a continuous Γ-invariant function $\widetilde{F} : T^1X \to \mathbb{R}$. The quotient function $F : \Gamma\backslash T^1X \to \mathbb{R}$ of $\widetilde{F}$ is called a *potential* on $\Gamma\backslash T^1X$. Precomposing by the canonical projection $\mathscr{G}X \to T^1X$, the function $\widetilde{F}$ defines a continuous Γ-invariant function from $\mathscr{G}X$ to $\mathbb{R}$, also denoted by $\widetilde{F}$, by $\widetilde{F}(\ell) = \widetilde{F}(v_\ell)$ for every $\ell \in \mathscr{G}X$.

For all $x, y \in X$, and any geodesic line $\ell \in \mathscr{G}X$ such that $\ell(0) = x$ and $\ell(d(x, y)) = y$, let

$$\int_x^y \widetilde{F} = \int_0^{d(x,y)} \widetilde{F}(v_{\mathtt{g}^t\ell})\, dt \,.$$

Note that for all $t \in \,]0, d(x, y)[$, the germ $v_{\mathtt{g}^t\ell}$ is independent on the choice of such a line ℓ, hence $\int_x^y \widetilde{F}$ does not depend on the extension ℓ of the geodesic segment $[x, y]$. The following

[6] Note that a linear combination of ϵ-locally constant functions is again a ϵ-locally constant function.

properties are easy to check using the Γ-invariance of $\widetilde{F}$ and the basic properties of integrals: For all $\gamma \in \Gamma$

$$\int_{\gamma x}^{\gamma y} \widetilde{F} = \int_x^y \widetilde{F}\,, \tag{3.10}$$

for the antipodal map ι

$$\int_y^x \widetilde{F} = \int_x^y \widetilde{F} \circ \iota\,, \tag{3.11}$$

and, for any $z \in [x, y]$,

$$\int_x^y \widetilde{F} = \int_x^z \widetilde{F} + \int_z^y \widetilde{F}\,. \tag{3.12}$$

The *period* of a loxodromic isometry γ of X for the potential $\widetilde{F}$ is

$$\mathrm{Per}_F(\gamma) = \int_x^{\gamma x} \widetilde{F}$$

for any x in the translation axis of γ. Note that, for all $\alpha \in \Gamma$ and $n \in \mathbb{N} - \{0\}$, we have

$$\mathrm{Per}_F(\alpha\gamma\alpha^{-1}) = \mathrm{Per}_F(\gamma), \quad \mathrm{Per}_F(\gamma^n) = n\ \mathrm{Per}_F(\gamma) \quad \text{and} \quad \mathrm{Per}_F(\gamma^{-1}) = \mathrm{Per}_{F\circ\iota}(\gamma)\,. \tag{3.13}$$

In trees, we have the following Lipschitz-type control on the integrals of the potentials along segments.

Lemma 3.12. *When $\widetilde{F}$ is constant or when X is an $\mathbb{R}$-tree, for all $x, x', y, y' \in X$, we have*

$$\Big|\int_x^y \widetilde{F} - \int_{x'}^{y'} \widetilde{F}\Big| \leqslant\ d(x,x') \sup_{\pi^{-1}([x,x'])} |\widetilde{F}|\ +\ d(y,y') \sup_{\pi^{-1}([y,y'])} |\widetilde{F}|\,.$$

Proof. When $\widetilde{F}$ is constant, the result follows from the triangle inequality.

Assume that X is an $\mathbb{R}$-tree. Consider the case $x = x'$. Let $z \in X$ be such that $[x, z] = [x, y] \cap [x, y']$. Using Equation (3.12) and the fact that $d(y, z) + d(z, y') = d(y, y')$, the claim follows. The general case follows by combining this case $x = x'$ and a similar estimate for the case $y = y'$. □

Some form of uniform Hölder-type control of the potential, analogous to the Lipschitz-type one in the previous lemma, will be crucial throughout the present work. The following Definition 3.13 formalises this (weaker) assumption.

Definition 3.13. *The triple $(X, \Gamma, \widetilde{F})$ satisfies the* HC-property *(Hölder-type control) if $\widetilde{F}$ has subexponential growth when X is not an $\mathbb{R}$-tree and if there exist $\kappa_1 \geqslant 0$ and $\kappa_2 \in\,]0, 1]$ such that for all $x, y, x', y' \in X$ with $d(x, x'), d(y, y') \leqslant 1$, we have*

$$\Big|\int_x^y \widetilde{F} - \int_{x'}^{y'} \widetilde{F}\,\Big| \tag{HC}$$
$$\leqslant\ \big(\kappa_1 + 2 \max_{\pi^{-1}(B(x,1)\cup B(x',1))} |\widetilde{F}|\big)\ d(x,x')^{\kappa_2} + \big(\kappa_1 + 2 \max_{\pi^{-1}(B(y,1)\cup B(y',1))} |\widetilde{F}|\big)\ d(y,y')^{\kappa_2}\,.$$

By Equation (3.11), $(X, \Gamma, \widetilde{F} \circ \iota)$ satisfies the HC-property if and only if $(X, \Gamma, \widetilde{F})$ does. By the triangle inequality $|\,d(x,y) - d(x',y')\,| \leqslant d(x,x') + d(y,y')$, for every $\kappa \in \mathbb{R}$, the triple $(X, \Gamma, \widetilde{F} + \kappa)$ satisfies the HC-property (up to changing the constant κ_1) if and only if $(X, \Gamma, \widetilde{F})$ does.

When X is assumed to be a Riemannian manifold with pinched sectional curvature, requiring the potentials to be Hölder-continuous as in [PauPS] is sufficient to have the HC-property, as we will see below.

Proposition 3.14. *The triple $(X, \Gamma, \widetilde{F})$ satisfies the HC-property if one of the following conditions is satisfied:*

- *$\widetilde{F}$ is constant,*
- *X is an $\mathbb{R}$-tree,*
- *X is a Riemannian manifold with pinched sectional curvature and $\widetilde{F}$ is Hölder-continuous.*

Proof. The first two cases are treated in Lemma 3.12, and we may take for them $\kappa_1 = 0$ and $\kappa_2 = 1$ in the definition of the HC-property.

The claim for Riemannian manifolds follows from the property of at most linear growth of the Hölder-continuous maps (see Remark 3.1) and from the following lemma, so that the constants $\kappa_1 > 0$ and $\kappa_2 \in \,]0, 1]$ of the HC-property depend only on the Hölder-continuity constants of $\widetilde{F}$ and on the bounds on the sectional curvature of X.

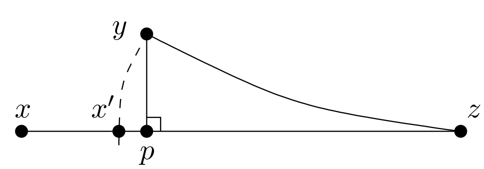
Lemma 3.15. *If X is a Riemannian manifold with pinched sectional curvature and $\widetilde{F}$ is Hölder-continuous, there exist two constants $c_1 > 0$ and $c_2 \in \,]0, 1]$ such that, for all x, y, z in X with $d(x,y) \leqslant 1$, we have*

$$\Big|\int_x^z \widetilde{F} - \int_y^z \widetilde{F}\,\Big| \leqslant c_1\, d(x,y)^{c_2} + 2\, d(x,y)^{\frac{1}{2}} \max_{\pi^{-1}(B(x,1) \cup B(y,1))} |\widetilde{F}|\;.$$

The constants c_1 and c_2 depend only on the Hölder-continuity constants of $\widetilde{F}$ and the bounds on the sectional curvature of X.

This lemma is similar to the second claim in [PauPS, Lem. 3.2], but the proof of this claim (and more precisely the proof of [PauPS, Lem. 2.3] used in the proof of [PauPS, Lem. 3.2]), which involves a different distance d on $\mathscr{G}X$, does not extend with the present definition of d.

Proof. By symmetry, we may assume that $d(x,z) \geqslant d(y,z)$. The result is true if $x = y$, hence we assume that $x \neq y$. Let x' be the point on $[x,z]$ at distance $d(y,z)$ from z.



The closest point p of y on $[x,z]$ lies in $[x',z]$ by convexity. Hence

$$d(x,x') \leqslant d(x,p) \leqslant d(x,y) \leqslant \sqrt{d(x,y)} \leqslant 1\;,$$

since closest point maps do not increase distances and $d(x,y) \leqslant 1$. Therefore

$$\Big|\int_x^{x'} \widetilde{F}\,\Big| \leqslant d(x,x') \max_{\pi^{-1}([x,x'])} |\widetilde{F}| \leqslant \sqrt{d(x,y)} \max_{\pi^{-1}(B(x,1))} |\widetilde{F}|\;, \tag{3.14}$$

Since $\int_x^z \widetilde{F} = \int_x^{x'} \widetilde{F} + \int_{x'}^z \widetilde{F}$ (see Equation (3.12)), we have

$$\Big| \int_x^z \widetilde{F} - \int_y^z \widetilde{F} \Big| \leqslant \Big| \int_{x'}^z \widetilde{F} - \int_y^z \widetilde{F} \Big| + \Big| \int_x^{x'} \widetilde{F} \Big| . \tag{3.15}$$

Assume first that $d(y,z) = d(x',z) \leqslant \frac{1}{2}\sqrt{d(x,y)}$. We have

$$\begin{aligned} \Big| \int_x^z \widetilde{F} - \int_y^z \widetilde{F} \Big| &\leqslant \Big| \int_{x'}^z \widetilde{F} \Big| + \Big| \int_y^z \widetilde{F} \Big| + \Big| \int_x^{x'} \widetilde{F} \Big| \\ &\leqslant d(x',z) \max_{\pi^{-1}([x',z])} |\widetilde{F}| + d(y,z) \max_{\pi^{-1}([y,z])} |\widetilde{F}| + \sqrt{d(x,y)} \max_{\pi^{-1}(B(x,1))} |\widetilde{F}| \\ &\leqslant 2\sqrt{d(x,y)} \max_{\pi^{-1}(B(x,1)\cup B(y,1))} |\widetilde{F}| , \end{aligned}$$

and Lemma 3.15 follows, for any $c_1 > 0$ and $c_2 \in \,]0,1]$.

Now assume that that $d(y,z) \geqslant \frac{1}{2}\sqrt{d(x,y)}$. Since the distance function from a given point to a point varying on a geodesic line is convex, we have $d(x',y) \leqslant d(x,y)$. By Equations (3.14) and (3.15), we may therefore assume that $x = x'$ and prove that

$$\Big| \int_x^z \widetilde{F} - \int_y^z \widetilde{F} \Big| \leqslant c_1 \, d(x,y)^{c_2}$$

for appropriate constants c_1, c_2.

Since X is a Riemannian manifold, we identify $\mathscr{G}X$ and T^1X with the usual unit tangent bundle of X as explained previously. Let u (respectively v) be the unit tangent vector at x (respectively y) pointing towards z. Let $\rho = d(x,z) = d(y,z) \geqslant \frac{1}{2}\sqrt{d(x,y)}$, and $t \in [0,\rho]$. We apply Lemma 3.6, whose hypotheses are indeed satisfied.

Since $t \in [0,\rho]$ and $d(x,y) \leqslant 1$, the term on the right hand side of Equation (3.2) is bounded by $2\,c_8$. Since $\widetilde{F}$ is Hölder-continuous, let $c > 0$ and $\alpha \in \,]0,1]$ be the Hölder-continuity constants such that $|\widetilde{F}(u') - \widetilde{F}(v')| \leqslant c \; d(u',v')^\alpha$ for all $u', v' \in T^1X$ such that $d(u',v') \leqslant 2\,c_8$. Then, by Lemma 3.6,

$$\begin{aligned} \Big| \int_x^z \widetilde{F} - \int_y^z \widetilde{F} \Big| &= \Big| \int_0^\rho \big(\widetilde{F}(\mathsf{g}^t u) - \widetilde{F}(\mathsf{g}^t v)\big) dt \Big| \leqslant \int_0^\rho c \; d(\mathsf{g}^t u, \mathsf{g}^t v)^\alpha \, dt \\ &\leqslant c \; {c_8}^\alpha \; d(x,y)^{\alpha c_3} \Big(\int_0^{+\infty} e^{-\alpha t} \, dt + \int_{-\infty}^{\rho} e^{2\alpha t - 2\alpha\rho} \, dt \Big) \\ &= \frac{3c}{2\alpha} \, {c_8}^\alpha \; d(x,y)^{\alpha c_3} \; . \end{aligned}$$

This concludes the proof of Lemma 3.15 with $c_2 = \alpha c_3$ and $c_1 = \frac{3c}{2\alpha}\, c_8^\alpha$, hence completes the proof of Proposition 3.14 in the Riemannian manifold case. □□

Remark 3.16. (1) If $X = \widetilde{M}$ is a Riemannian manifold, then T^1X is naturally identified with the usual Riemannian unit tangent bundle of $\widetilde{M}$. If the potential $\widetilde{F} : T^1\widetilde{M} \rightarrow \mathbb{R}$ is Hölder-continuous for Sasaki's Riemannian metric on $T^1\widetilde{M}$, it is a potential as in [Rue1] and [PauPS]. Furthermore, the definition of $\int_x^y \widetilde{F}$ coincides with the one in these references.

(2) The quotient function F is Hölder-continuous when $\widetilde{F}$ is Hölder-continuous.

Let $\widetilde{F}, \widetilde{F}^* : T^1X \to \mathbb{R}$ be potentials for Γ. We say that $\widetilde{F}^*$ is *cohomologous* to $\widetilde{F}$ (see for instance [Livš]) if there exists a continuous Γ-invariant function $\widetilde{G} : T^1X \to \mathbb{R}$, such that, for every $\ell \in \mathscr{G}X$, the map $t \mapsto \widetilde{G}(v_{\mathtt{g}^t\ell})$ is differentiable and

$$\widetilde{F}^*(v_\ell) - \widetilde{F}(v_\ell) = \frac{d}{dt}_{|t=0} \widetilde{G}(v_{\mathtt{g}^t\ell}) \,. \tag{3.16}$$

When working with Hölder-continuous potentials, the regularity requirement is for $\widetilde{G}$ to also be Hölder-continuous. Note that the right-hand side of Equation (3.16) does not depend on the choice of the representative ℓ of its germ v_ℓ. In particular, $\mathrm{Per}_F(\gamma) = \mathrm{Per}_{F^*}(\gamma)$ for any loxodromic isometry γ if $\widetilde{F}$ and $\widetilde{F}^*$ are cohomologous potentials.

A potential $\widetilde{F}$ is said to be *reversible* if $\widetilde{F}$ and $\widetilde{F} \circ \iota$ are cohomologous.

3.3 Poincaré series and critical exponents

Let us fix a potential $\widetilde{F} : T^1X \to \mathbb{R}$ for Γ, and $x, y \in X$.

The *critical exponent* of (Γ, F) is the element $\delta = \delta_{\Gamma,\,F}$ of the extended real line $[-\infty, +\infty]$ defined by

$$\delta = \limsup_{n \to +\infty} \frac{1}{n} \ln \sum_{\gamma \in \Gamma,\, n-1 < d(x,\gamma y) \leqslant n} e^{\int_x^{\gamma y} \widetilde{F}} \,.$$

The *Poincaré series* of (Γ, F) is the map $Q = Q_{\Gamma,\,F,\,x,\,y} : \mathbb{R} \to [0, +\infty]$ defined by

$$Q : s \mapsto \sum_{\gamma \in \Gamma} e^{\int_x^{\gamma y} (\widetilde{F} - s)} \,.$$

If $\delta < +\infty$, we say that (Γ, F) is *of divergence type* if the series $Q_{\Gamma,\,F,\,x,\,y}(\delta)$ diverges, and *of convergence type* otherwise.

When $F = 0$, the critical exponent $\delta_{\Gamma,\,0}$ is the usual critical exponent $\delta_\Gamma \in \,]0, +\infty]$ of Γ, the Poincaré series $Q_{\Gamma,\,0,\,x,\,y}$ is the usual Poincaré series of Γ, and we recover the usual notion of divergence or convergence type of Γ, see for instance [Rob2].

The Poincaré series of (Γ, F) and its critical exponent make sense even if Γ is elementary (see for instance Lemma 3.17 (10)). The following result collects some of the basic properties of the critical exponent.

Lemma 3.17. *Assume that $(X, \Gamma, \widetilde{F})$ satisfies the HC-property. Then*

(1) *the critical exponent $\delta_{\Gamma,\,F}$ and the divergence or convergence of $Q_{\Gamma,\,F,\,x,\,y}(s)$ are independent of the points $x, y \in X$; they depend only on the cohomology class of $\widetilde{F}$;*

(2) *$Q_{\Gamma,\,F\circ\iota,\,x,\,y} = Q_{\Gamma,\,F,\,y,\,x}$ and $\delta_{\Gamma,\,F\circ\iota} = \delta_{\Gamma,\,F}$; in particular, (Γ, F) is of divergence type if and only if $(\Gamma, F \circ \iota)$ is of divergence type ;*

(3) *the Poincaré series $Q(s)$ diverges if $s < \delta_{\Gamma,\,F}$ and converges if $s > \delta_{\Gamma,\,F}$;*

(4) *$\delta_{\Gamma,\,F+\kappa} = \delta_{\Gamma,\,F} + \kappa$ for any $\kappa \in \mathbb{R}$, and (Γ, F) is of divergence type if and only if $(\Gamma, F + \kappa)$ is of divergence type;*

(5) *if Γ' is a nonelementary subgroup of Γ, denoting by $F' : \Gamma' \backslash T^1X \to \mathbb{R}$ the map induced by $\widetilde{F}$, then $\delta_{\Gamma',\,F'} \leqslant \delta_{\Gamma,\,F}$;*

(6) *if* $\delta_\Gamma < +\infty$*, then* $\delta_\Gamma + \inf_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} \leqslant \delta_{\Gamma,\,F} \leqslant \delta_\Gamma + \sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F}$ *;*

(7) $\delta_{\Gamma,\,F} > -\infty$ *;*

(8) *the map* $\widetilde{F} \mapsto \delta_{\Gamma,\,F}$ *is convex, sub-additive, and* 1*-Lipschitz for the uniform norm on the vector space of real continuous maps on* $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$ *;*[7]

(9) *if* Γ'' *is a discrete cocompact group of isometries of* X *such that* $\widetilde{F}$ *is* Γ''*-invariant, denoting by* $F'' : \Gamma''\backslash T^1X \to \mathbb{R}$ *the map induced by* $\widetilde{F}$*, then*

$$\delta_{\Gamma,\,F} \leqslant \delta_{\Gamma'',\,F''} \;;$$

(10) *if* Γ *is infinite cyclic, generated by a loxodromic isometry* γ *of* X*, then* (Γ, F) *is of divergence type and*

$$\delta_{\Gamma,\,F} = \max\Big\{\frac{\mathrm{Per}_F(\gamma)}{\lambda(\gamma)}, \frac{\mathrm{Per}_{F\circ\iota}(\gamma)}{\lambda(\gamma)}\Big\} .$$

Proof. We give details of the proofs of the statements (1)–(7) and (10) which are the ones used in this book, only for the sake of completeness. The proofs from [PauPS, Lem. 3.3] generalise to the current setting, replacing the use of [PauPS, Lem. 3.2] by the following consequence of the HC-property: there exist $\kappa_1 \geqslant 0$ and $\kappa_2 \in \,]0,1]$ such that for every $N \in \mathbb{N} - \{0\}$, for all $x, y, x', y' \in X$ with $d(x,x'), d(y,y') \leqslant N$, we have

$$\Big|\int_x^y \widetilde{F} - \int_{x'}^{y'} \widetilde{F}\,\Big| \leqslant 2N\big(\kappa_1 + 2 \max_{\pi^{-1}(B(x,\,N)\cup B(x',\,N)\cup B(y,\,N)\cup B(y',\,N))} |\widetilde{F}|\,\big) \,. \tag{3.17}$$

This is obtained from Equation (HC) by subdividing the segments $[x, x']$ and $[y, y']$ into N subintervals of equal lengths at most 1, and by using the triangle inequality.

(1) For $x', y' \in X$, if $N = \max\{\lceil d(x,x')\rceil, \lceil d(y,y')\rceil\}$, by Equation (3.17) and by the Γ-invariance of $\widetilde{F}$, we have, for every $\gamma \in \Gamma$,

$$\Big|\int_x^{\gamma y} \widetilde{F} - \int_{x'}^{\gamma y'} \widetilde{F}\,\Big| \leqslant c = 2\,N\,\big(\kappa_1 + 2 \max_{\pi^{-1}(B(x,\,N)\cup B(x',N)\cup B(y,\,N)\cup B(y',N))} |\widetilde{F}|\,\big) \,,$$

which is finite since the continuous map $\widetilde{F}$ is bounded on compact subsets of T^1X. Hence by the triangle inequality, we have, for every $s \in \mathbb{R}$,

$$e^{-c-s\,d(x,x')-s\,d(y,y')}\;Q_{\Gamma,\,F,\,x,\,y}(s) \leqslant Q_{\Gamma,\,F,\,x',\,y'}(s) \leqslant e^{c+s\,d(x,x')+s\,d(y,y')}\;Q_{\Gamma,\,F,\,x,\,y}(s) \;.$$

The first claim of Assertion (1) follows.

Let $\widetilde{F}^* : T^1X \to \mathbb{R}$ be a potential for Γ which is cohomologous to $\widetilde{F}$. Let $\widetilde{G} : T^1X \to \mathbb{R}$ be a continuous Γ-invariant function satisfying Equation (3.16). Let $\kappa_x = \max_{\pi^{-1}(x)} |\widetilde{G}|$, which

[7] That is, if $\widetilde{F}, \widetilde{F^*} : T^1X \to \mathbb{R}$ are potentials for Γ satisfying the HC-property, inducing $F, F^* : \Gamma\backslash T^1X \to \mathbb{R}$, and if $\delta_{\Gamma,\,F}, \delta_{\Gamma,\,F^*} < +\infty$, then $\delta_{\Gamma,\,tF+(1-t)F^*} \leqslant t\,\delta_{\Gamma,\,F} + (1-t)\,\delta_{\Gamma,\,F^*}$ for every $t \in [0,1]$,

$$\delta_{\Gamma,\,F+F^*} \leqslant \delta_{\Gamma,\,F} + \delta_{\Gamma,\,F^*} \;,$$

$$|\;\delta_{\Gamma,\,F^*} - \delta_{\Gamma,\,F}\;| \leqslant \sup_{v\in\pi^{-1}(\mathscr{C}\Lambda\Gamma)} |\;\widetilde{F^*}(v) - \widetilde{F}(v)\;| \;.$$

is finite by continuity. By Γ-invariance, for every $\gamma \in \Gamma$, we have $\kappa_{\gamma y} = \kappa_y$. For every $\gamma \in \Gamma$, with $\ell \in \mathscr{G}X$ any geodesic line such that $\ell(0) = x$ and $\ell(d(x, \gamma y)) = \gamma y$, we have

$$\Big| \int_x^{\gamma y} \widetilde{F}^* - \int_x^{\gamma y} \widetilde{F} \;\Big| = \Big| \int_0^{d(x,\gamma y)} \frac{d}{dt}_{|t=0} \widetilde{G}(v_{\mathtt{g}^t \mathtt{g}^s \ell})\, ds \Big| = \Big| \int_0^{d(x,\gamma y)} \frac{d}{ds} \widetilde{G}(v_{\mathtt{g}^s \ell})\, ds \Big|$$
$$= \big|\; \widetilde{G}(v_\ell) - \widetilde{G}(v_{\mathtt{g}^{d(x,\gamma y)}\ell}) \;\big| \leqslant \kappa_x + \kappa_y \;.$$

Hence by the triangle inequality, we have, for every $s \in \mathbb{R}$,

$$e^{-\kappa_x - \kappa_y}\, Q_{\Gamma,\, F,\, x,\, y}(s) \leqslant Q_{\Gamma,\, F^*,\, x,\, y}(s) \leqslant e^{\kappa_x + \kappa_y}\, Q_{\Gamma,\, F,\, x,\, y}(s) \;.$$

The second claim of Assertion (1) follows.

(2) This assertion follows from Equations (3.11) and (3.10), by the change of variable $\gamma \mapsto \gamma^{-1}$ in the summation of the Poincaré series.

(3) This assertion is a standard argument of Poincaré series. For every $s \neq \delta_{\Gamma,\, F}$, let $\epsilon = \frac{1}{2}\,|\delta_{\Gamma,\, F} - s| > 0$. First assume that $s > \delta_{\Gamma,\, F}$. By the definition of the critical exponent $\delta_{\Gamma,\, F}$, there exists $N \in \mathbb{N}$ such that for every integer $n \geqslant N$,

$$\sum_{\gamma \in \Gamma,\, n-1 < d(x,\gamma y) \leqslant n} e^{\int_x^{\gamma y} \widetilde{F}} \leqslant e^{n(\delta_{\Gamma,\, F} + \epsilon)} \;.$$

Hence there exists $c > 0$ such that

$$Q(s) \leqslant c + \sum_{n \in \mathbb{N}} e^{n(\delta_{\Gamma,\, F} + \epsilon) - s\, n + |s|} = c + e^{|s|} \sum_{n \in \mathbb{N}} e^{-\epsilon n} < +\infty \;.$$

Now assume that $s < \delta_{\Gamma,\, F}$. By the definition of $\delta_{\Gamma,\, F}$, there exists an increasing sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that for every $k \in \mathbb{N}$, we have

$$\sum_{\gamma \in \Gamma,\, n_k - 1 < d(x,\gamma y) \leqslant n_k} e^{\int_x^{\gamma y} \widetilde{F}} \geqslant e^{n_k(\delta_{\Gamma,\, F} - \epsilon)} \;.$$

Hence

$$Q(s) \geqslant \sum_{k \in \mathbb{N}} e^{n_k(\delta_{\Gamma,\, F} - \epsilon) - s\, n_k - |s|)} = e^{-|s|} \sum_{k \in \mathbb{N}} e^{\epsilon\, n_k} = +\infty \;.$$

This proves Assertion (3).

Assertions (4) and (5) are immediate by Assertion (3), since with κ and Γ', F' as in these assertions, for every $s \in \mathbb{R}$, we have

$$Q_{\Gamma,\, F + \kappa,\, x,\, y}(s) = Q_{\Gamma,\, F,\, x,\, y}(s - \kappa) \quad \text{and} \quad Q_{\Gamma',\, F',\, x,\, y}(s) \leqslant Q_{\Gamma,\, F,\, x,\, y}(s) \;.$$

(6) If x is a point in the convex hull $\mathscr{C}\Lambda\Gamma$ of the limit set of Γ, then, for every $\gamma \in \Gamma$, the geodesic segment between x and γx is contained in $\mathscr{C}\Lambda\Gamma$. Hence

$$d(x, \gamma x) \Big(\inf_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} - s \Big) \leqslant \int_x^{\gamma x} (\widetilde{F} - s) \leqslant d(x, \gamma x) \Big(\sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} - s \Big) \;.$$

This proves Assertion (6) by taking the exponential, summing over $\gamma \in \Gamma$ with $n - 1 < d(x, \gamma y) \leqslant n$, taking the logarithm, dividing by n and taking the upper limit as n tends to $+\infty$.

(7) Let Γ' be a nonelementary convex-cocompact subgroup of Γ (for instance a Schottky subgroup of Γ, which exists since Γ is nonelementary). Denote by $F' : \Gamma' \backslash T^1 X \to \mathbb{R}$ the map induced by $\widetilde{F}$. Since $|\widetilde{F}\,|$ is Γ-invariant and bounded on compact subsets of $T^1 X$, by Assertion (6), we have $\delta_{\Gamma', F'} > -\infty$ as $\delta_\Gamma \geqslant 0$. Assertion (7) then follows from Assertion (5).

(10) For every $s \in \mathbb{R}$, if x belongs to the translation axis of γ, we have by Equation (3.13)

$$\sum_{\alpha\in\Gamma} e^{\int_x^{\alpha x}(\widetilde{F}-s)} = \sum_{n\in\mathbb{N}} e^{\int_x^{\gamma^n x}(\widetilde{F}-s)} + \sum_{n\in\mathbb{N}-\{0\}} e^{\int_x^{\gamma^{-n} x}(\widetilde{F}-s)}$$
$$= \sum_{n\in\mathbb{N}} e^{n(\mathrm{Per}_F(\gamma)-s\,\ell(\gamma))} + \sum_{n\in\mathbb{N}-\{0\}} e^{n(\mathrm{Per}_{F\circ\iota}(\gamma)-s\,\ell(\gamma))}\,.$$

Hence $Q_{\Gamma,\,F,\,x,\,x}(s)$ converges if and only if $\mathrm{Per}_F(\gamma) - s\,\ell(\gamma) < 0$ and $\mathrm{Per}_{F\circ\iota}(\gamma) - s\,\ell(\gamma) < 0$. Let $\overline{\delta} = \max\big\{\frac{\mathrm{Per}_F(\gamma)}{\lambda(\gamma)}, \frac{\mathrm{Per}_{F\circ\iota}(\gamma)}{\lambda(\gamma)}\big\}$. By Assertion (3), letting s tend to $\delta_{\Gamma,\,F}$ on the right gives that $\delta_{\Gamma,\,F} \geqslant \overline{\delta}$, and letting s tend to $\overline{\delta}$ on the left gives the reverse inequality. The above computation also gives that $Q_{\Gamma,\,F,\,x,\,x}(\,\overline{\delta}\,)$ diverges, which proves Assertion (10) and concludes the proof of Lemma 3.17. □

Examples 3.18. (1) If δ_Γ is finite and $\widetilde{F}$ is bounded, then the critical exponent δ is finite by Lemma 3.17 (6).

(2) If X is a Riemannian manifold with pinched negative curvature or when X has a compact quotient, then δ_Γ is finite. See for instance [Bou].

(3) There are examples of (X, Γ) with $\delta_\Gamma = +\infty$ (and hence $\delta = +\infty$ if $\widetilde{F}$ is constant), for instance when X is the complete ideal hyperbolic triangle complex with 3 ideal triangles along each edge, see [GaP], and Γ its isometry group. Hence the finiteness assumption of the critical exponent is nonempty in general. For the type of results treated in this book, it is however natural and essential.

We may replace upper limits by limits in the definition of the critical exponents, as follows.

Theorem 3.19. *Assume that $(X, \Gamma, \widetilde{F})$ satisfies the HC-property. If $c > 0$ is large enough, then*

$$\delta = \lim_{n\to+\infty} \frac{1}{n} \ln \sum_{\gamma\in\Gamma,\, n-c<d(x,\gamma y)\leqslant n} e^{\int_x^{\gamma y} \widetilde{F}}\,.$$

If $\delta > 0$, then

$$\delta = \lim_{n\to+\infty} \frac{1}{n} \ln \sum_{\gamma\in\Gamma,\, d(x,\gamma y)\leqslant n} e^{\int_x^{\gamma y} \widetilde{F}}\,.$$

Proof. The proofs of [PauPS, Theo. 4.2 and Theo. 4.3], either using the original arguments of [Rob1] valid when $\widetilde{F}$ is constant, or the super-multiplicativity arguments of [DaPS], extend, using the HC-property (see Definition 3.13) instead of [PauPS, Lem. 3.2]. □

In what follows, we fix a potential $\widetilde{F}$ for Γ such that $(X, \Gamma, \widetilde{F})$ satisfies the HC-property. We define $\widetilde{F}^+ = \widetilde{F}$ and $\widetilde{F}^- = \widetilde{F} \circ \iota$, we denote by $F^\pm : \Gamma\backslash T^1 X \to \mathbb{R}$ their induced maps, and we assume that $\delta = \delta_{\Gamma,\,F^+} = \delta_{\Gamma,\,F^-}$ is finite.

3.4 Gibbs cocycles

The *(normalised) Gibbs cocycle* associated with the group Γ and the potential $\widetilde{F}^{\pm}$ is the map $C^{\pm} = C^{\pm}_{\Gamma, F^{\pm}} : \partial_\infty X \times X \times X \to \mathbb{R}$ defined by

$$(\xi, x, y) \mapsto C^{\pm}_{\xi}(x, y) = \lim_{t \to +\infty} \int_y^{\xi_t} (\widetilde{F}^{\pm} - \delta) - \int_x^{\xi_t} (\widetilde{F}^{\pm} - \delta) \,,$$

where $t \mapsto \xi_t$ is any geodesic ray with endpoint $\xi \in \partial_\infty X$.

We will prove in Proposition 3.20 below that this map is well defined, that is, the above limits exist for all $(\xi, x, y) \in \partial_\infty X \times X \times X$ and they are independent of the choice of the geodesic rays $t \mapsto \xi_t$. If $\widetilde{F}^{\pm} = 0$, then $C^- = C^+ = \delta_\Gamma \beta$, where β is the Busemann cocycle. If X is an $\mathbb{R}$-tree, then

$$C^{\pm}_{\xi}(x, y) = \int_y^{p} (\widetilde{F}^{\pm} - \delta) - \int_x^{p} (\widetilde{F}^{\pm} - \delta) \,, \tag{3.18}$$

where $p \in X$ is the point for which $[\,p, \xi[\; = [x, \xi[\; \cap [y, \xi[\,$; in particular, the map $\xi \mapsto C^{\pm}_{\xi}(x, y)$ is locally constant on the totally discontinuous space $\partial_\infty X$.

The Gibbs cocycles satisfy the following equivariance and cocycle properties: For all $\xi \in \partial_\infty X$ and $x, y, z \in X$, and for every isometry γ of X, we have

$$C^{\pm}_{\gamma\xi}(\gamma x, \gamma y) = C^{\pm}_{\xi}(x, y) \;\text{ and }\; C^{\pm}_{\xi}(x, z) + C^{\pm}_{\xi}(z, y) = C^{\pm}_{\xi}(x, y)\,. \tag{3.19}$$

For every $\ell \in \mathscr{G}X$, for all x and y on the image of the geodesic line ℓ, if ℓ_-, x, y, ℓ_+ are in this order on ℓ, we have

$$C^{-}_{\ell_-}(x, y) = C^{+}_{\ell_+}(y, x) = -C^{+}_{\ell_+}(x, y) = \int_x^y (\widetilde{F}^{+} - \delta)\,. \tag{3.20}$$

Proposition 3.20. *Assume that $(X, \Gamma, \widetilde{F})$ satisfies the HC-property and that $\delta < +\infty$.*

(1) *The maps $C^{\pm} : \partial_\infty X \times X \times X \to \mathbb{R}$ are well-defined.*

(2) *With the constants κ_1, κ_2 of the HC-property, for all $x, y \in X$ and $\xi \in \partial_\infty X$, if we assume that $d(x, y) \leqslant 1$, then*

$$|\, C^{\pm}_{\xi}(x, y)\,| \leqslant (\kappa_1 + 2\,|\delta| + 2 \max_{\pi^{-1}(B(x,1) \cup B(y,1))} |\widetilde{F}|)\, d(x, y)^{\kappa_2} \,,$$

and, in general, if $N = \lceil d(x, y) \rceil$, then

$$|\, C^{\pm}_{\xi}(x, y)\,| \leqslant N\,(\kappa_1 + 2\,|\delta| + 2 \max_{\pi^{-1}(B(x,N) \cup B(y,N))} |\widetilde{F}|)\,.$$

If X is an $\mathbb{R}$-tree, then for all $x, y \in X$ and $\xi \in \partial_\infty X$, we have

$$|\, C^{\pm}_{\xi}(x, y)\,| \leqslant d(x, y) \max_{\pi^{-1}([x,y])} |\widetilde{F}^{\pm} - \delta|\,.$$

(3) *The maps $C^{\pm} : \partial_\infty X \times X \times X \to \mathbb{R}$ are locally Hölder-continuous (and locally Lipschitz when X is an $\mathbb{R}$-tree). In particular, they are continuous.*

(4) *For all $r > 0$, $x, y \in X$ and $\xi \in \partial_\infty X$, if ξ belongs to the shadow $\mathscr{O}_x B(y,r)$ of the ball $B(y,r)$ seen from x, then with the constants κ_1, κ_2 of the HC-property, if $r \leqslant 1$, we have*

$$\Big| C^{\pm}_{\xi}(x,y) + \int_x^y (\widetilde{F}^{\pm} - \delta) \Big| \leqslant 2(\kappa_1 + 2\,|\delta| + 2 \max_{\pi^{-1}(B(y,2))} |\widetilde{F}|)\, r^{\kappa_2} \;,$$

and in general

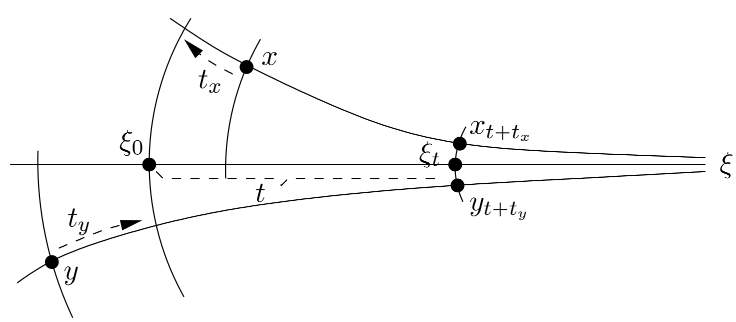
$$\Big| C^{\pm}_{\xi}(x,y) + \int_x^y (\widetilde{F}^{\pm} - \delta) \Big| \leqslant 2\,\lceil r \rceil\, (\kappa_1 + 2\,|\delta| + 2 \max_{\pi^{-1}(B(y,2\lceil r \rceil))} |\widetilde{F}|) \;.$$

If X is an $\mathbb{R}$-tree, then

$$\Big| C^{\pm}_{\xi}(x,y) + \int_x^y (\widetilde{F}^{\pm} - \delta) \Big| \leqslant 2r \max_{\pi^{-1}(B(y,r))} |\widetilde{F}^{\pm} - \delta|\,.$$

Proof. (1) The fact that $C^{\pm}_{\xi}(x,y)$ is well defined when X is an $\mathbb{R}$-tree follows from Equation (3.18).

When X is not an $\mathbb{R}$-tree, let $\rho : t \mapsto \xi_t$ be any geodesic ray with endpoint $\xi \in \partial_\infty X$, let $t \mapsto x_t$ (respectively $t \mapsto y_t$) be the geodesic ray from x (respectively y) to ξ. Let $t_x = \beta_\xi(x, \xi_0)$ and $t_y = \beta_\xi(y, \xi_0)$, so that the quantity $\beta = t_y - t_x$ is equal to $\beta_\xi(y,x)$ (which is independent of ρ), and for every t big enough, we have $\beta_\xi(\xi_t, x_{t+t_x}) = \beta_\xi(\xi_t, y_{t+t_y}) = 0$.



Since X is CAT(-1), if t is big enough, then the distances $d(\xi_t, x_{t+t_x})$ and $d(\xi_t, y_{t+t_y})$ are at most one, and converge, in a nonincreasing way, exponentially fast to 0 as $t \to +\infty$. For $s \geqslant 0$, let $a_s = \int_y^{y_s} (\widetilde{F}^{\pm} - \delta) - \int_x^{x_{s-\beta}} (\widetilde{F}^{\pm} - \delta)$ (which is independent of ρ). We have, using Equation (HC), and the fact that $B(x',1) \cup B(y',1) \subset B(x',2)$ if $d(x',y') \leqslant 1$,

$$\begin{aligned}
&\Big| \Big(\int_y^{\xi_t} (\widetilde{F}^{\pm} - \delta) - \int_x^{\xi_t} (\widetilde{F}^{\pm} - \delta) \Big) - a_{t+t_y} \Big| \\
&= \Big| \Big(\int_y^{\xi_t} (\widetilde{F}^{\pm} - \delta) - \int_y^{y_{t+t_y}} (\widetilde{F}^{\pm} - \delta) \Big) + \Big(\int_x^{x_{t+t_x}} (\widetilde{F}^{\pm} - \delta) - \int_x^{\xi_t} (\widetilde{F}^{\pm} - \delta) \Big) \Big| \\
&\leqslant 2(\kappa_1 + 2 \max_{\pi^{-1}(B(\xi_t,2))} |\widetilde{F} - \delta|\,)\; \max\{d(\xi_t, x_{t+t_x}), d(\xi_t, y_{t+t_y})\}^{\kappa_2} \;,
\end{aligned}$$

which converges to 0 since $\widetilde{F}$ has subexponential growth by the assumptions of the HC-property. Hence in order to prove Assertion (1), we only have to prove that $\lim_{s \to +\infty} a_s$ exists.

For all $s \geqslant t \geqslant |\beta|$, we have, by the additivity of the integral along geodesics (see Equation (3.12)) and by using again Equation (HC),

$$\begin{aligned}|a_t - a_s| &= \Big| \int_{y_t}^{y_s} (\widetilde{F}^{\pm} - \delta) - \int_{x_{t-\beta}}^{x_{s-\beta}} (\widetilde{F}^{\pm} - \delta) \Big| \\ &\leqslant (\kappa_1 + 2 \max_{\pi^{-1}(B(x_{t-\beta},\, 2))} |\widetilde{F} - \delta|)\, d(y_t, x_{t-\beta})^{\kappa_2} + (\kappa_1 + 2 \max_{\pi^{-1}(B(y_s,\, 2))} |\widetilde{F} - \delta|\,)\, d(y_s, x_{s+\beta})^{\kappa_2}\ .\end{aligned}$$

Again by the subexponential growth of $\widetilde{F}$, the above expression converges (exponentially fast, for future use) to 0 as $t \to +\infty$ uniformly in s, hence $\lim_{s\to+\infty} a_s$ exists by a Cauchy type argument.

(2) Let $(\xi, x, y) \in \partial_\infty X \times X \times X$. Assertion (2) of Proposition 3.20 follows from Equation (3.18) when X is an $\mathbb{R}$-tree, since $[x, y] = [x, p] \cup [p, y]$ where p is the closest point to y on $[x, \xi[$. When X is not an $\mathbb{R}$-tree, the first claim of Assertion (2) follows immediately from the HC-property of $(X, \Gamma, \widetilde{F}^{\pm} - \delta)$, and the second claim from this HC-property and the same subdivision argument of the geodesic segment $[x, y]$ into $\lceil d(x, y)\rceil$ subintervals of equal lengths (at most 1), as in the proof of Equation (3.17).

(3) Let $(\xi, x, y), (\xi', x', y') \in \partial_\infty X \times X \times X$. By the cocycle property (3.19), we have

$$|C^{\pm}_{\xi}(x, y) - C^{\pm}_{\xi'}(x', y')| \leqslant |C^{\pm}_{\xi}(x, y) - C^{\pm}_{\xi'}(x, y)| + |C^{\pm}_{\xi'}(x, x')| + |C^{\pm}_{\xi'}(y, y')|\ . \tag{3.21}$$

First assume that X is an $\mathbb{R}$-tree. Let K be a compact subset of X, and let

$$\epsilon_K = \inf_{x,y\in K} e^{-d(x,\,x_0) - d(x,\,y)} > 0\,.$$

Let p, q be the points in X such that $[x, \xi[\ \cap [y, \xi[\ =\ [p, \xi[$ and $[x, \xi[\ \cap [x, \xi'[\ =\ [x, q]$. If $d_{x_0}(\xi, \xi') \leqslant \epsilon_K$, then by the definition of the visual distance (see Equation (2.3)) and by Equation (2.4), we have

$$e^{-d(x,\,q)} = d_x(\xi, \xi') \leqslant e^{d(x,\,x_0)}\ d_{x_0}(\xi, \xi') \leqslant e^{-d(x,\,y)} \leqslant e^{-d(x,\,p)}\,.$$

In particular $q \in [p, \xi[$, so that $[x, \xi'[\ \cap [y, \xi'[\ =\ [p, \xi'[$. Thus by Equation (3.18), we have

$$|\, C^{\pm}_{\xi}(x, y) - C^{\pm}_{\xi'}(x, y)\,| = 0\ .$$

Therefore, by Equation (3.21) and by the $\mathbb{R}$-tree case of Assertion (2), if $d_{x_0}(\xi, \xi') \leqslant \epsilon_K$, if $x, y \in K$ and $d(x, x'), d(y, y') \leqslant 1$, then

$$|C^{\pm}_{\xi}(x, y) - C^{\pm}_{\xi'}(x', y')| \leqslant d(x, x') \max_{\pi^{-1}([x,\,x'])} |\widetilde{F}^{\pm} - \delta| + d(y, y') \max_{\pi^{-1}([y,\,y'])} |\widetilde{F}^{\pm} - \delta|\ .$$

Since $\widetilde{F}$ is bounded on compact subsets of T^1X, this proves that $C^{\pm}$ is locally Lipschitz.

Let us now consider the case when X is general. For all distinct $\xi, \xi' \in \partial_\infty X$, let $t \mapsto \xi_t$ and $t \mapsto \xi'_t$ be the geodesic rays from x_0 to ξ and ξ' respectively. By the end of the proof of Assertion (1), for every compact subset K of X, there exist $a_1, a_2 > 0$ such that for every $x, y \in K$, we have for all $\eta \in \{\xi, \xi'\}$ and $t \geqslant 0$,

$$\Big| C^{\pm}_{\eta}(x,y) - \Big(\int_y^{\eta_t} (\widetilde{F}^{\pm} - \delta) - \int_x^{\eta_t} (\widetilde{F}^{\pm} - \delta) \Big) \Big| \leqslant a_1\, e^{-a_2 t}\,.$$

Let $T = -\frac{1}{2} \ln d_{x_0}(\xi, \xi')$. If $T \geqslant 0$, by the properties of CAT(-1)-spaces (see Equation (3.6) for the second inequality), there exist two constants $a_3, a_4 > 0$ such that $d(\xi_{2T}, \xi'_{2T}) \leqslant a_3$ and $d(\xi_T, \xi'_T) \leqslant a_4\, e^{-T}$. Hence if $d_{x_0}(\xi, \xi') \leqslant \min\big\{\frac{1}{a_4^2}, 1\big\}$ (so that $T \geqslant 0$ and $d(\xi_T, \xi'_T) \leqslant 1$), we have, using Equation (HC) for the last inequality,

$$\begin{aligned}
&|\, C^{\pm}_{\xi}(x,y) - C^{\pm}_{\xi'}(x,y)\,| \\
&\leqslant \Big| \int_y^{\xi_T} (\widetilde{F}^{\pm} - \delta) - \int_y^{\xi'_T} (\widetilde{F}^{\pm} - \delta) \Big| + \Big| \int_x^{\xi_T} (\widetilde{F}^{\pm} - \delta) - \int_x^{\xi'_T} (\widetilde{F}^{\pm} - \delta) \Big| + 2\, a_1\, e^{-a_2 T} \\
&\leqslant 2\, (\kappa_1 + 2 \max_{\pi^{-1}(B(\xi_T, 2))} |\widetilde{F}^{\pm} - \delta|)\, d(\xi_T, \xi'_T)^{\kappa_2} + 2\, a_1\, e^{-a_2 T}\,.
\end{aligned}$$

By the subexponential growth of $\widetilde{F}$, there exists $a_5 > 0$ such that

$$|\, C^{\pm}_{\xi}(x,y) - C^{\pm}_{\xi'}(x,y)\,| \leqslant a_5\, e^{-\frac{\kappa_2}{2} T} + 2\, a_1\, e^{-a_2 T} \leqslant (a_5 + 2\, a_1)\, d_{x_0}(\xi, \xi')^{\min\{\frac{\kappa_2}{4}, \frac{a_2}{2}\}}\,.$$

We now conclude from Equation (3.21) and Assertion (2) as in the end of the above tree case that $C^{\pm}$ is locally Hölder-continuous.

(4) Let $r > 0$, $x, y \in X$ and $\xi \in \partial_\infty X$ be such that $\xi \in \mathscr{O}_x B(y, r)$. Let p be the closest point to y on $[x, \xi[$, so that $d(p, y) \leqslant r$. By Equations (3.20) and (3.19), we have

$$\begin{aligned}
\Big| C^{\pm}_{\xi}(x,y) + \int_x^y (\widetilde{F}^{\pm} - \delta) \Big| &= \Big| C^{\pm}_{\xi}(x,y) - C^{\pm}_{\xi}(x,p) - \int_x^p (\widetilde{F}^{\pm} - \delta) + \int_x^y (\widetilde{F}^{\pm} - \delta) \Big| \\
&\leqslant |\, C^{\pm}_{\xi}(p,y)| + \Big| \int_x^p (\widetilde{F}^{\pm} - \delta) - \int_x^y (\widetilde{F}^{\pm} - \delta) \Big|\,. \qquad (3.22)
\end{aligned}$$

First assume that X is an $\mathbb{R}$-tree. Then by Assertion (2) and by Lemma 3.12, we deduce from Equation (3.22) that

$$\Big| C^{\pm}_{\xi}(x,y) + \int_x^y (\widetilde{F}^{\pm} - \delta) \Big| \leqslant 2\, d(y,p) \max_{\pi^{-1}([y,p])} |\widetilde{F}^{\pm} - \delta| \leqslant 2r \max_{\pi^{-1}(B(y,r))} |\widetilde{F}^{\pm} - \delta|\,.$$

In the general case, the result then follows similarly from Equation (3.22) by using Assertion (2) and the HC-property if $r \leqslant 1$ of Equation (3.17) in general. □

3.5 Systems of conductances on trees and generalised electrical networks

Let $(\mathbb{X}, \lambda)$ be a locally finite metric tree without terminal vertices, let $X = |\mathbb{X}|_\lambda$ be its geometric realisation, and let Γ be a nonelementary discrete subgroup of $\mathrm{Isom}(\mathbb{X}, \lambda)$.

Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a Γ-invariant function, called a *system of (logarithmic) conductances* for Γ. We denote by $c : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$ the function induced by $\widetilde{c}$, which we also call a *system of conductances* on $\Gamma\backslash\mathbb{X}$.

Classically, an *electric network*[8] (without sources or reactive elements) is a pair (G, e^c), where G is a graph and $c : EG \to \mathbb{R}$ a function, such that c is *reversible*: $c(e) = c(\overline{e})$ for all $e \in EG$, see for example [NaW], [Zem]. In this text, we do not assume our system of conductances $\widetilde{c}$ to be reversible. In Chapter 6, we will even sometimes assume that the system of conductances is *antireversible*, that is, satisfying $c(\overline{e}) = -\,c(e)$ for every $e \in E\mathbb{X}$.

Two systems of conductances $\widetilde{c}, \widetilde{c'} : E\mathbb{X} \to \mathbb{R}$ are said to be *cohomologous*, if there exists a Γ-invariant map $f : V\mathbb{X} \to \mathbb{R}$ such that

$$\widetilde{c'} - \widetilde{c} = df \ ,$$

where for all $e \in E\mathbb{X}$, we have

$$df(e) = \frac{f(t(e)) - f(o(e))}{\lambda(e)} \ .$$

Proposition 3.21. *Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a system of conductances for Γ. There exists a potential $\widetilde{F}$ on T^1X for Γ such that for all $x, y \in V\mathbb{X}$, if $(e_1, \dots, e_n)$ is the edge path in $\mathbb{X}$ without backtracking such that $x = o(e_1)$ and $y = t(e_n)$, then*

$$\int_x^y \widetilde{F} = \sum_{i=1}^{n} \widetilde{c}(e_i)\,\lambda(e_i) \,.$$

Proof. Any germ $v \in T^1X$ determines a unique edge e_v of the tree $\mathbb{X}$, the first one into which it enters: if ℓ is any geodesic line whose class in T^1X is v, the edge e_v is the unique edge of $\mathbb{X}$ containing $\pi(v)$ whose terminal vertex is the first vertex of $\mathbb{X}$ encountered at a positive time by ℓ. The function $\widetilde{F} : T^1X \to \mathbb{R}$ defined by

$$\widetilde{F}(v) = \frac{4\,\widetilde{c}(e_v)}{\lambda(e_v)} \min\big\{d(\pi(v),\, o(e_v)), d(\pi(v), t(e_v))\big\} \qquad (3.23)$$

is a (indeed Γ-invariant) potential on the $\mathbb{R}$-tree X, with $\widetilde{F}(v) = 0$ if $\pi(v) \in V\mathbb{X}$.

Let us now compute $\int_x^y \widetilde{F}$, for all $x, y \in X$. For every $\lambda > 0$, let $\psi_\lambda : [0, \lambda] \to \mathbb{R}$ be the continuous map defined by $\psi_\lambda(t) = \frac{t^2}{2}$ if $t \in [0, \frac{\lambda}{2}]$ and $\psi_\lambda(t) = \frac{\lambda^2}{4} - \frac{(\lambda-t)^2}{2}$ if $t \in [\frac{\lambda}{2}, \lambda]$. Let $(e_0, e_1, \dots, e_n)$ be the edge path in $\mathbb{X}$ without backtracking such that $x \in e_0 - \{t(e_0)\}$ and $y \in e_n - \{o(e_n)\}$. An easy computation shows that

$$\int_x^y \widetilde{F} \;=\; \sum_{i=0}^{n-1} \widetilde{c}(e_i)\,\lambda(e_i) + \frac{4\,\widetilde{c}(e_n)}{\lambda(e_n)}\,\psi_{\lambda(e_n)}\big(d(y, o(e_n))\big) - \frac{4\,\widetilde{c}(e_0)}{\lambda(e_0)}\,\psi_{\lambda(e_0)}\big(d(x, o(e_0))\big)\,.$$

If x and y are vertices, the expression simplifies to the sum of the lengths of the edges weighted by the conductances. □

We denote by $\widetilde{F}_c$ the potential defined by Equation (3.23) in the above proof, and by $F_c : \Gamma\backslash T^1X \to \mathbb{R}$ the induced potential. Note that F_c is bounded if c is bounded. We call $\widetilde{F}_c$ and F_c the potentials *associated with* the system of conductances $\widetilde{c}$ and c. This is by no means the unique potential with the property required in Proposition 3.21. The following result proves that the choice is unimportant.

[8]A potential in this work is not the analog of a potential in an electric network, we follow the dynamical systems terminology as in for example [PauPS].

Given a potential $\widetilde{F} : T^1X \to \mathbb{R}$ for Γ, let us define a map $\widetilde{c}_F : E\mathbb{X} \to \mathbb{R}$ by

$$\widetilde{c}_F : e \mapsto \widetilde{c}_F(e) = \frac{1}{\lambda(e)} \int_{o(e)}^{t(e)} \widetilde{F} \ . \tag{3.24}$$

Note that $\widetilde{c}_F$ is Γ-invariant, hence it is a system of conductances for Γ. We denote by $c_F : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$ the function induced by $\widetilde{c}_F : E\mathbb{X} \to \mathbb{R}$. Note that $\widetilde{c}_{F+\kappa} = \widetilde{c}_F + \kappa$ for every constant $\kappa \in \mathbb{R}$, that $\widetilde{c}_F$ is bounded if $\widetilde{F}$ is bounded, and that $c_{F_c} = c$ by the above proposition.

Proposition 3.22.

(1) *Every potential (resp. bounded potential) for Γ is cohomologous to a potential (resp. bounded potential) associated with a system of conductances for Γ.*

(2) *If two systems of conductances $\widetilde{c'}$ and $\widetilde{c}$ are cohomologous, then their associated potentials $\widetilde{F}_{c'}$ and $\widetilde{F}_c$ are cohomologous.*

(3) *If $\mathbb{X}$ has no vertex of degree 2, if two potentials $\widetilde{F}^*$ and $\widetilde{F}$ for Γ are cohomologous, then the systems of conductances $\widetilde{c}_{F^*}$ and $\widetilde{c}_F$ for Γ are cohomologous.*

Hence if $\mathbb{X}$ has no vertex of degree 2, the map $[F] \mapsto [c_F]$ from the set of cohomology classes of potentials for Γ to the set of cohomology classes of systems of conductances for Γ is bijective, with inverse $[c] \mapsto [F_c]$.

Proof. (1) Let $\widetilde{F}$ be a potential for Γ, and let $\widetilde{F}^* = \widetilde{F}_{c_F}$ be the potential associated with the system of conductances $\widetilde{c}_F$. Note that if $\widetilde{F}$ is bounded, so is $\widetilde{c}_F$ by Equation (3.24), hence $\widetilde{F}^*$ is bounded by Equation (3.23). For all $e \in E\mathbb{X}$ and $t \in \,]0, \lambda(e)[\,$, let $v_{e,\,t} \in T^1X$ be the germ of any geodesic line passing at time 0 through the point of e at distance t from $o(e)$. Let $\widetilde{G} : T^1X \to \mathbb{R}$ be the map defined by $\widetilde{G}(v) = 0$ if $\pi(v) \in V\mathbb{X}$ and such that for all $e \in E\mathbb{X}$ and $t \in \,]0, \lambda(e)[\,$,

$$\widetilde{G}(v_{e,\,t}) = \int_0^t (\widetilde{F}^*(v_{e,\,s}) - \widetilde{F}(v_{e,\,s}))\; ds \ .$$

Since $\int_{o(e)}^{t(e)} \widetilde{F} = \lambda(e)\, \widetilde{c}_F(e)$ by the construction of $\widetilde{c}_F$ and $\lambda(e)\, \widetilde{c}_F(e) = \int_{o(e)}^{t(e)} \widetilde{F}^*$ by Proposition 3.21, the map $\widetilde{G} : T^1X \to \mathbb{R}$ is continuous. Let ℓ be a geodesic line. The map $t \mapsto \widetilde{G}(v_{\mathtt{g}^t\ell})$ is obviously differentiable at time $t = 0$ if $\pi(\ell) \notin V\mathbb{X}$, with derivative $\widetilde{F}^*(v_\ell) - \widetilde{F}(v_\ell)$. By considering right and left derivatives of this map at $t = 0$, by using the fact that $\int_{o(e)}^{t(e)} (\widetilde{F} - \widetilde{F}^*) = 0$ for every $e \in E\mathbb{X}$, and by the continuity of $\widetilde{F}$ and $\widetilde{F}^*$ at such a point, this is still true if $\pi(\ell) \in V\mathbb{X}$. Hence $\widetilde{F}^*$ and $\widetilde{F}$ are cohomologous, and this proves the first claim.

(2) Assume that $\widetilde{c'}$ and $\widetilde{c}$ are cohomologous systems of conductances for Γ, and let $f : V\mathbb{X} \to \mathbb{R}$ be a Γ-invariant function such that $\widetilde{c'} - \widetilde{c} = df$. Define $\widetilde{G}(v) = f(\pi(v))$ if $\pi(v) \in V\mathbb{X}$. For all $e \in E\mathbb{X}$ and $t \in \,]0, \lambda(e)[\,$, define

$$\widetilde{G}(v_{e,\,t}) = \int_0^t (\widetilde{F}_{c'}(v_{e,\,s}) - \widetilde{F}_c(v_{e,\,s}))\; ds + f(o(e)) \ ,$$

which is Γ-invariant. Its limit as $t \to 0$ is $f(o(e))$ (independent of the edge e with given origin), and its limit as $t \to \lambda(e)$ is, by the construction of $\widetilde{F}_c$ and $\widetilde{F}_{c'}$,

$$\lambda(e)\big(\widetilde{c'}(e) - \widetilde{c}(e)\big) + f(o(e)) = \lambda(e)\; df(e) + f(o(e)) = f(t(e))$$

(independent of the edge e with given extremity). This proves that $\widetilde{G}$ is continuous. One checks as in Assertion (1) that $\widetilde{F}_{c'}$ and $\widetilde{F}_c$ are cohomologous.

(3) In order to prove the third claim, assume that $\widetilde{F}^*$ and $\widetilde{F}$ are two cohomologous potentials for Γ, and let $\widetilde{G} : T^1X \to \mathbb{R}$ be as in the definition of cohomologous potentials, see Equation (3.16). By the continuity of $\widetilde{G}$, for all elements v and v' in T^1X such that $\pi(v) = \pi(v') \in V\mathbb{X}$, we have $\widetilde{G}(v) = \widetilde{G}(v')$, since (by the assumption on the degrees of vertices) the two edges (possibly equal) into which v and v' enter can be extended to geodesic lines with a common negative subray. Hence for every $x \in V\mathbb{X}$, the value $f(x) = \widetilde{G}(v_x)$ for every $v_x \in T^1X$ such that $\pi(v_x) = x$ does not depend on the choice of v_x. The map $f : V\mathbb{X} \to \mathbb{R}$ thus defined is Γ-invariant. With the above notation and by Equation (3.24), we hence have, for every $e \in E\mathbb{X}$,

$$\begin{aligned}\widetilde{c}_{F^*}(e) - \widetilde{c}_F(e) &= \frac{1}{\lambda(e)} \int_0^{\lambda(e)} \big(\widetilde{F}^*(v_{e,\,t}) - \widetilde{F}(v_{e,\,t})\big)\, dt = \frac{1}{\lambda(e)} \int_0^{\lambda(e)} \frac{d}{dt}\widetilde{G}(v_{e,\,t})\, dt \\ &= \frac{1}{\lambda(e)} \big(\widetilde{G}(v_{t(e)}) - \widetilde{G}(v_{o(e)})\big) = \frac{f(t(e)) - f(o(e))}{\lambda(e)} = df(e)\ .\end{aligned}$$

Hence $\widetilde{c}_{F^*}$ and $\widetilde{c}_F$ are cohomologous. □

Given a metric tree $(\mathbb{X}, \lambda)$, we define the *critical exponent* of a Γ-invariant system of conductances $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ (or of the induced system of conductances $c : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$) as the critical exponent of $(\Gamma, \widetilde{F}_c)$ where $\widetilde{F}_c$ is the potential for Γ associated with $\widetilde{c}$:

$$\delta_c = \delta_{\Gamma,\,F_c}\ .$$

By Proposition 3.22 (2) and Lemma 3.17 (1), this does not depend on the choice of a potential $\widetilde{F}_c$ satisfying Proposition 3.21.

Chapter 4

Patterson-Sullivan and Bowen-Margulis measures with potential on CAT(−1) spaces

Let X, x_0, Γ be as in the beginning of Section 2.1,[1] and let $\widetilde{F}$ be a potential for Γ. From now on, we assume that the triple $(X, \Gamma, \widetilde{F})$ satisfies the HC-property of Definition 3.13 and that the critical exponent $\delta = \delta_{\Gamma,\, F^\pm}$ is finite.

In this chapter, we discuss geometrically and dynamically relevant measures on the boundary at infinity of X and on the space of geodesic lines $\mathscr{G}X$. We extend the theory of Gibbs measures from the case of manifolds with pinched negative sectional curvature treated in [PauPS] [2] to CAT(−1) spaces with the HC-property.

4.1 Patterson densities

A family $(\mu^\pm_x)_{x\in X}$ of finite nonzero (positive Borel) measures on $\partial_\infty X$, whose support is $\Lambda\Gamma$, is a *(normalised) Patterson density* for the pair $(\Gamma, \widetilde{F}^\pm)$ if

$$\gamma_* \mu^\pm_x = \mu^\pm_{\gamma x} \tag{4.1}$$

for all $\gamma \in \Gamma$ and $x \in X$, and if the following Radon-Nikodym derivatives exist for all $x, y \in X$ and satisfy for (almost) all $\xi \in \partial_\infty X$

$$\frac{d\mu^\pm_x}{d\mu^\pm_y}(\xi) = e^{-C^\pm_\xi(x,\,y)}\,. \tag{4.2}$$

In particular, the measures $\mu^\pm_x$ are in the same measure class for all $x \in X$, and, by Proposition 3.20, they depend continuously on x for the weak-star convergence of measures. Note that a Patterson density for $(\Gamma, F^\pm)$ is also a Patterson density for $(\Gamma, F^\pm + s)$ for every $s \in \mathbb{R}$, since the definition involves only the normalised potential $\widetilde{F}^\pm - \delta$. If $F = 0$, we get the usual notion of a Patterson-Sullivan density (of dimension δ_Γ) for the group Γ, see for instance [Pat2, Sul2, Nic, Coo, Bou, Rob2].

[1] That is, X is a geodesically complete proper CAT(−1) space, $x_0 \in X$ is a basepoint, and Γ is a nonelementary discrete group of isometries of X.

[2] See also the previous works [Led, Ham2, Cou, Moh].

Proposition 4.1. *There exists at least one Patterson density for the pair* $(\Gamma, \widetilde{F}^{\pm})$.

Proof. The Patterson construction (see [Pat1], [Coo]) modified as in [Led] (with a multiplicative rather than additive parameter s), [Moh] and [PauPS, Section 3.6] (all in the Riemannian manifold case) gives the result, and we give a proof only for the sake of completeness.

We start by an independent lemma, generalizing [Pat2, Lem. 3.1] with a similar proof.

Lemma 4.2. *Let* $\delta' \in \mathbb{R}$. *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be sequences of positive real numbers such that* $\lim_{n\to+\infty} a_n = +\infty$ *and the generalised Dirichlet series* $\sum_{n\in\mathbb{N}} b_n\, a_n^{-s}$ *converges if* $s > \delta'$ *and diverges if* $s < \delta'$. *Then there exists a positive nondecreasing map* h *on* $]0, +\infty[$ *such that*

- *for every* $\epsilon > 0$, *there exists* $r'_\epsilon > 0$ *such that* $h(t'r') \leqslant t'^{\,\epsilon}\, h(r')$ *for all* $t' > 1$ *and* $r' \geqslant r'_\epsilon$;
- *the series* $\sum_{n\in\mathbb{N}} b_n\, a_n^{-s}\, h(a_n)$ *converges if and only if* $s > \delta'$.

Proof. Let $t_0 = 0$, $t_1 = 1$ and $h_1 :]0, 1] \to [1, +\infty[$ the constant map 1. Let us define by induction on $n \in \mathbb{N} - \{0\}$ a positive real number t_n and a continous map $h_n :]t_{n-1}, t_n] \to [1, +\infty[$. If t_n and h_n are constructed, let $t_{n+1} \in \mathbb{R}$ be such that

$$\frac{h_n(t_n)}{t_n^{1/n}} \sum_{k\in\mathbb{N}\,:\, t_n < a_k \leqslant t_{n+1}} b_k\, a_k^{-\delta'+1/n} \geqslant 1 \;,$$

which exists since the generalised Dirichlet series diverges at $s = \delta' - 1/n$. For every $t \in]t_n, t_{n+1}]$, let $h_{n+1}(t) = h_n(t_n)\,\big(\frac{t}{t_n}\big)^{1/n}$.

Note that the sequence $(t_n)_{n\in\mathbb{N}}$ is increasing. Let $h :]0, +\infty[\to [1, +\infty[$ be the map equal to h_n on $]t_{n-1}, t_n]$ for every $n \in \mathbb{N} - \{0\}$. The map h is positive, continuous and nondecreasing. For every $\epsilon > 0$, let $n = \lceil \frac{1}{\epsilon} \rceil$. Since $\ln h(t)$ is continuous and piecewise affine in $\ln t$ with slopes at most $\frac{1}{n}$ on $[t_n, +\infty[$, we have $\ln(h(t'r')) - \ln h(r') \leqslant \frac{1}{n} \ln t' \leqslant \epsilon \ln t'$ if $t' > 1$ and $r' \geqslant t_n$, which proves the first claim on h.

We have

$$\sum_{n\in\mathbb{N}} b_n\, a_n^{-\delta'}\, h(a_n) = \sum_{n\in\mathbb{N}} \sum_{\substack{k\in\mathbb{N}\\ t_n < a_k \leqslant t_{n+1}}} b_k\, a_k^{-\delta'}\, h(t_n)\big(\frac{a_k}{t_n}\big)^{1/n} \geqslant \sum_{n\in\mathbb{N}} 1 = +\infty \;.$$

If $s > \delta'$, let $\epsilon = \frac{s-\delta'}{2} > 0$. Since by construction $h(t) = \mathrm{O}(t^\epsilon)$ as $t \to +\infty$, there exists a constant $C > 0$ such that $h(a_n) \leqslant C\, a_n^\epsilon$ for every $n \in \mathbb{N}$, and the convergence of the series $\sum_{n\in\mathbb{N}} b_n\, a_n^{-s}\, h(a_n)$ follows from the convergence of the generalised Dirichlet series $\sum_{n\in\mathbb{N}} b_n\, a_n^{-\delta-\epsilon}$, thus proving the second claim on h. □

Now, for every $z \in X$, recall that Δ_z denotes the unit Dirac mass at z. Let $h^{\pm} : [0, +\infty[\;\to\;]0, +\infty[$ be a nondecreasing map such that

- for every $\epsilon > 0$, there exists $r_\epsilon \geqslant 0$ such that $h^{\pm}(t + r) \leqslant e^{\epsilon\, t} h^{\pm}(r)$ for all $t \geqslant 0$ and $r \geqslant r_\epsilon$;
- if $\overline{Q}_x(s) = \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x_0} (\widetilde{F}^{\pm} - s)}\, h^{\pm}(d(x, \gamma x_0))$, then $\overline{Q}_x(s)$ diverges if and only if the inequality $s \leqslant \delta$ holds.

If $(\Gamma, F^{\pm})$ is of divergence type, we may take $h^{\pm} = 1$ constant. Otherwise, the existence of $h^{\pm}$ follows from Lemma 4.2.[3] For all $s > \delta$ and $x \in X$, define the measure

$$\mu^{\pm}_{x,\,s} = \frac{1}{\overline{Q}_{x_0}(s)} \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x_0} (\widetilde{F}^{\pm} - s)}\, h^{\pm}(d(x, \gamma x_0))\, \Delta_{\gamma x_0}$$

[3] Let $(\gamma_n)_{n\in\mathbb{N}}$ be an enumeration of the elements of Γ, take $a_n = e^{d(x, \gamma_n x_0)}$ and $b_n = e^{\int_x^{\gamma_n x_0} \widetilde{F}^{\pm}}$, and then take $h^{\pm} = h \circ \exp$ for h the map given by Lemma 4.2.

on X. By compactness for the weak-star topology of the space of probability measures on the compact space $X \cup \partial_\infty X$, there exists a sequence $(s_k)_{k\in\mathbb{N}}$ in $]\delta, +\infty[$ converging to δ such that the sequence of probability measures $(\mu_{x_0,\, s_k})_{k\in\mathbb{N}}$ weak-star converges to a probability measure μ_{x_0} on $X \cup \partial_\infty X$. Since $\overline{Q}_{x_0}(\delta) = +\infty$ and since the support of $\mu_{x_0,\, s}$ in $X \cup \partial_\infty X$ is equal to $\overline{\Gamma x_0} = \Gamma x_0 \cup \Lambda\Gamma$, the support of μ_{x_0} is contained in $\Lambda\Gamma$, hence equal to $\Lambda\Gamma$ by minimality. The Radom-Nikodym derivative $\frac{d\mu^{\pm}_{x,\, s}}{d\mu^{\pm}_{x_0,\, s}}$ is the map with support Γx_0 defined by

$$\frac{d\mu^{\pm}_{x,\, s}}{d\mu^{\pm}_{x_0,\, s}}(\gamma x_0) = e^{\int_x^{\gamma x_0}(\widetilde{F}^{\pm}-s) - \int_{x_0}^{\gamma x_0}(\widetilde{F}^{\pm}-s)}\ \frac{h^{\pm}(d(x,\gamma x_0))}{h^{\pm}(d(x_0,\gamma x_0))}\,. \tag{4.3}$$

For every $k \in \mathbb{N}$, if $d(x_0, \gamma x_0)$ is large enough, then

$$h^{\pm}(d(x,\gamma x_0)) \leqslant h^{\pm}(d(x,x_0)+d(x_0,\gamma x_0)) \leqslant e^{(s_k-\delta)d(x,x_0)}\ h^{\pm}(d(x_0,\gamma x_0))\,.$$

By the HC-property, as $k \to +\infty$, the right hand side of Equation (4.3) with $s = s_k$ converges to $e^{-C^{\pm}_{\xi}(x,x_0)}$ uniformly in $\xi \in \Lambda\Gamma$ as γx_0 tends to ξ. Therefore, as $k \to +\infty$, the measures $\mu^{\pm}_{x,\, s_k}$ converge to a (finite nonzero) measure $\mu^{\pm}_x$ with support $\Lambda\Gamma$ such that $\frac{d\mu^{\pm}_x}{d\mu^{\pm}_{x_0}}(\xi) = e^{-C^{\pm}_{\xi}(x,x_0)}$. Let $\gamma \in \Gamma$. Since $\gamma_*\Delta_z = \Delta_{\gamma z}$ for every $z \in X$, a change of variable in the summation defining $\mu^{\pm}_{x,\, s}$ gives that $\gamma_*\mu^{\pm}_{x,\, s} = \mu^{\pm}_{\gamma x,\, s}$. By the continuity of pushforwards of measures, we have $\gamma_*\mu^{\pm}_x = \mu^{\pm}_{\gamma x}$. By the cocycle properties of the Radom-Nikodym derivatives and of the Gibbs cocycles, the family $(\mu^{\pm}_x)_{x\in X}$ is a (normalised) Patterson density for $(\Gamma, \widetilde{F}^{\pm})$. □

We refer to Theorem 4.6 for the uniqueness up to scalar multiple of the Patterson density when $(\Gamma, F^{\pm})$ is of divergence type and to [DaSU, Coro. 17.1.8] for a characterisation of the uniqueness when $F = 0$.

The Patterson densities satisfy the following extension of the classical Sullivan shadow lemma (which gives the claim when $\widetilde{F}$ is constant, see [Rob2]), and its corollaries.

If μ is a positive Borel measure on a metric space (X, d), the triple (X, d, μ) is called a *metric measure space*. A metric measure space (X, d, μ) is *doubling* if there exists $c \geqslant 1$ such that, for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2\,r)) \leqslant c\,\mu(B(x, r))\,.$$

Note that, up to changing c, the number 2 may be replaced by any constant larger than 1. See for instance [Hei] for more details on doubling metric measure spaces. We refer for instance to [DaSU, Ex. 17.4.12] for examples of nondoubling Patterson(-Sullivan) measures, and to [DaSU, Prop. 17.4.4] for a characterisation of the doubling property of the Patterson measures when Γ is geometrically finite and $F = 0$.

A family $((X, \mu_i, d_i))_{i\in I}$ of positive Borel measures μ_i and distances d_i on a common set X is called *uniformly doubling* if there exists $c \geqslant 1$ such that, for all $i \in I$, $x \in X$ and $r > 0$,

$$\mu_i(B_{d_i}(x, 2\,r)) \leqslant c\,\mu_i(B_{d_i}(x, r))\,.$$

Lemma 4.3. *Let $(\mu^{\pm}_x)_{x\in X}$ be a Patterson density for the pair $(\Gamma, F^{\pm})$, and let K be a compact subset of X.*

(1) [Mohsen's shadow lemma] *If R is large enough, there exists $C > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$,*

$$\frac{1}{C}\ e^{\int_x^{\gamma y}(\widetilde{F}^{\pm}-\delta)} \leqslant \mu^{\pm}_x\big(\mathscr{O}_x B(\gamma y, R)\big) \leqslant C\ e^{\int_x^{\gamma y}(\widetilde{F}^{\pm}-\delta)}\,.$$

(2) *For all $x, y \in X$, there exists $c > 0$ such that for every $n \in \mathbb{N}$*

$$\sum_{\gamma\in\Gamma\,:\, n-1<d(x,\gamma y)\leqslant n} e^{\int_x^{\gamma y}(\widetilde{F}^{\pm}-\delta)} \leqslant c\ .$$

(3) *For every $R > 0$ large enough, there exists $C = C(R) > 0$ such that for all $\gamma \in \Gamma$ and all $x, y \in K$*

$$\mu_x^{\pm}(\mathscr{O}_x B(\gamma y, 5R)) \leqslant C\, \mu_x^{\pm}(\mathscr{O}_x B(\gamma y, R))\ .$$

(4) *If Γ is convex-cocompact, then the metric measure space $(\Lambda\Gamma, d_x, \mu_x^{\pm})$ is doubling for every x inX, and the family of metric measure spaces $((\Lambda\Gamma, d_x, \mu_x^{\pm}))_{x\in C\Lambda\Gamma}$ is uniformly doubling.*

Proof. For the first assertion, the proof of [PauPS, Lem. 3.10] (see also [Cou, Lem. 4] with the multiplicative rather than additive convention, as well as [Moh]) extends, using Proposition 3.20 (2), (4) instead of [PauPS, Lem. 3.4 (i), (ii)]. The second assertion is similar to the one of [PauPS, Lem. 3.11 (i)], and the proof of the last two assertions is similar to the one of [PauPS, Prop. 3.12], using Lemma 2.2 instead of [HeP4, Lem. 2.1]. The uniformity in the last assertion follows from the compactness of $\Gamma\backslash C\Lambda\Gamma$ and the invariance and continuity properties of the Patterson densities. □

4.2 Gibbs measures

We fix from now on two Patterson densities $(\mu_x^{\pm})_{x\in X}$ for the pairs $(\Gamma, F^{\pm})$.

The *Gibbs measure* $\widetilde{m}_F$ on $\mathscr{G}X$ (associated with this ordered pair of Patterson densities) is the measure $\widetilde{m}_F$ on $\mathscr{G}X$ given by the density

$$d\widetilde{m}_F(\ell) = e^{C^-_{\ell_-}(x_0,\,\ell(0))\,+\,C^+_{\ell_+}(x_0,\,\ell(0))}\; d\mu^-_{x_0}(\ell_-)\, d\mu^+_{x_0}(\ell_+)\, dt \tag{4.4}$$

in Hopf's parametrisation with respect to the basepoint x_0. The Gibbs measure $\widetilde{m}_F$ is independent of x_0 by Equations (4.2) and (3.19). Hence it is invariant under the action of Γ by Equations (4.1) and (3.19). It is invariant under the geodesic flow by construction (and the invariance of Lebesgue's measure under translation). Thus,[4] it defines a measure m_F on $\Gamma\backslash\mathscr{G}X$ which is invariant under the quotient geodesic flow, called the *Gibbs measure* on $\Gamma\backslash\mathscr{G}X$ (associated with the above ordered pair of Patterson densities). If $F = 0$ and the Patterson densities $(\mu_x^{+})_{x\in X}$ and $(\mu_x^{-})_{x\in X}$ coincide, the Gibbs measure m_F coincides with the *Bowen-Margulis measure* $m_{\rm BM}$ on $\Gamma\backslash\mathscr{G}X$ (associated with this Patterson density), see for instance [Rob2].

Remark 4.4. (i) The (positive Borel) measure given by the density

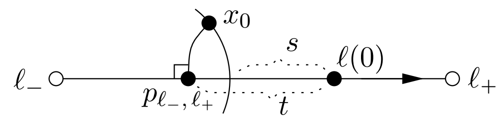
$$d\lambda(\xi,\eta) = e^{C^-_{\xi}(x_0,\,p)\,+\,C^+_{\eta}(x_0,\,p)}\; d\mu^-_{x_0}(\xi)\, d\mu^+_{x_0}(\eta) \tag{4.5}$$

on $\partial^2_\infty X$ is (by the same arguments as above) independent of $p \in\,]\xi,\eta[$, locally finite and invariant under the diagonal action of Γ on $\partial^2_\infty X$. It is a *geodesic current* for the action of

[4] See for instance [PauPS, §2.6] for the precautions in order to push locally forward an invariant measure by an orbifold covering, since the group Γ does not necessarily act freely on $\mathscr{G}X$.

Γ on the Gromov-hyperbolic proper metric space X in the sense of Ruelle-Sullivan-Bonahon, see for instance [Bon] and references therein.

(ii) Another parametrisation of $\mathscr{G}X$ also depending on the choice of the basepoint x_0 in X, is the map from $\mathscr{G}X$ to $\partial^2_\infty X \times \mathbb{R}$ sending ℓ to (ℓ_-, ℓ_+, s) where now $s = \beta_{\ell_-}(\pi(\ell), x_0)$ (one may also use the different time parameter $s = \beta_{\ell_+}(x_0, \pi(\ell))$). For every (η, ξ) in $\partial^2_\infty X$, let $p_{\eta,\xi}$ be the closest point to x_0 on the geodesic line between η and ξ.



For every $\ell \in \mathscr{G}X$, with (ℓ_-, ℓ_+, t) the original Hopf parametrisation, since

$$s - t = \beta_{\ell_-}(\ell(0), x_0) - \beta_{\ell_-}(\ell(0), p_{\ell_-,\ell_+}) = \beta_{\ell_-}(p_{\ell_-,\ell_+}, x_0)$$

depends only on ℓ_- and ℓ_+, the measures $d\mu^-_{x_0}(\ell_-)\, d\mu^+_{x_0}(\ell_+)\, dt$ and $d\mu^-_{x_0}(\ell_-)\, d\mu^+_{x_0}(\ell_+)\, ds$ are equal. Hence using the above variant of Hopf's parametrisation does not change the Gibbs measures $\widetilde{m}_F$ and m_F.

(iii) Since the time reversal map ι is $(\ell_-, \ell_+, t) \mapsto (\ell_+, \ell_-, -t)$ in Hopf's coordinates, the measure $\iota_* \widetilde{m}_F$ is the Gibbs measure on $\mathscr{G}X$ associated with the switched pair of Patterson densities $\big((\mu^+_x)_{x\in X}, (\mu^-_x)_{x\in X}\big)$ (and similarly on $\Gamma\backslash\mathscr{G}X$).

The Gibbs property of Gibbs measures Let us now indicate why the terminology of Gibbs measures is indeed appropriate. This explanation will be the aim of Proposition 4.5, but we need to give some definitions first.

For all $\ell \in \mathscr{G}X$ and $r > 0$, $T, T' \geqslant 0$, the *dynamical (or Bowen) ball* around ℓ is

$$B(\ell; T, T', r) = \big\{\ell' \in \mathscr{G}X \ :\ \sup_{t\in[-T', T]} d(\ell(t), \ell'(t)) < r\big\}\ .$$

Bowen balls have the following invariance properties: for all $s \in [-T', T]$ and $\gamma \in \Gamma$,

$$\mathsf{g}^s B(\ell; T, T', r) = B(\mathsf{g}^s \ell; T - s, T' + s, r) \quad \text{and} \quad \gamma B(\ell; T, T', r) = B(\gamma\ell; T, T', r)\ .$$

The following inclusion properties of the dynamical balls are immediate: If $r \leqslant s$, $T \geqslant S$, $T' \geqslant S'$, then $B(\ell; T, T', r)$ is contained in $B(\ell; S, S', s)$. The dynamical balls are almost independent on r : For all $r' \geqslant r > 0$, there exists $T_{r,r'} \geqslant 0$ such that for all $\ell \in \mathscr{G}X$ and $T, T' \geqslant 0$, the dynamical ball $B(\ell; T + T_{r,r'}, T' + T_{r,r'}, r')$ is contained in $B(\ell; T, T', r)$. This follows from the properties of long geodesic segments with endpoints at bounded distance in a CAT(-1)-space.

For every $\ell \in \Gamma\backslash\mathscr{G}X$, let us define $B(\ell; T, T', r')$ as the image by the canonical projection $\mathscr{G}X \to \Gamma\backslash\mathscr{G}X$ of $B(\widetilde{\ell}; T, T', r')$, for any preimage $\widetilde{\ell}$ of ℓ in $\mathscr{G}X$.

A $(\mathsf{g}^t)_{t\in\mathbb{R}}$-invariant measure m' on $\Gamma\backslash\mathscr{G}X$ *satisfies the Gibbs property* for the potential F with constant $c(F) \in \mathbb{R}$ if for every compact subset K of $\Gamma\backslash\mathscr{G}X$, there exist $r > 0$ and $c_{K,r} \geqslant 1$ such that for all large enough $T, T' \geqslant 0$, for every $\ell \in \Gamma\backslash\mathscr{G}X$ with $\mathsf{g}^{-T'}\ell, \mathsf{g}^T \ell \in K$, we have

$$\frac{1}{c_{K,r}} \leqslant \frac{m'(B(\ell; T, T', r))}{e^{\int_{-T'}^{T} (F(v_{\mathsf{g}^t \ell}) - c(F))\, dt}} \leqslant c_{K,r}\ .$$

We refer to [PauPS, Sect. 3.8] for equivalent variations on the definition of the Gibbs property m_F. The following result shows that the Gibbs measures indeed satisfy the Gibbs property on the dynamical balls of the geodesic flow, thereby justifying the name. We refer for instance to [PauPS, Sect. 3.8] for the explanations of the connection with symbolic dynamics mentioned in the introduction. See also Proposition 4.14 for a discussion of the case when X is a simplicial tree – here the correspondence with symbolic dynamics is particularly clear.

Proposition 4.5. *Let m_F be the Gibbs measure on $\Gamma\backslash\mathscr{G}X$ associated with a pair of Patterson densities $(\mu^{\pm}_x)_{x\in X}$ for $(\Gamma, \widetilde{F}^{\pm})$. Then m_F satisfies the Gibbs property for the potential F, with constant $c(F) = \delta$.*

Proof. The proof is similar to the one of [PauPS, Prop. 3.16] (in which the key Lemma 3.17 uses only CAT(−1) arguments), up to replacing [PauPS, Lem. 3.4 (1)] by Proposition 3.20 (2). □

The Hopf-Tsuji-Sullivan-Roblin theorem The basic ergodic properties of the Gibbs measures are summarised in the following result. The case when $\widetilde{F}$ is constant is due to [Rob2], see also [BuM, §6].

Theorem 4.6 (Hopf-Tsuji-Sullivan-Roblin)**.** *The following conditions are equivalent*

(i) *The pair (Γ, F) is of divergence type.*

(ii)$^-$ *The conical limit set of Γ has positive measure with respect to μ^-_x for some (equivalently every) $x \in X$.*

(ii)$^+$ *The conical limit set of Γ has positive measure with respect to μ^+_x for some (equivalently every) $x \in X$.*

(iii) *The dynamical system $(\partial^2_\infty X, \Gamma, (\mu^-_x \otimes \mu^+_x)_{|\partial^2_\infty X})$ is ergodic for some (equivalently every) $x \in X$.*

(iv) *The dynamical system $(\partial^2_\infty X, \Gamma, (\mu^-_x \otimes \mu^+_x)_{|\partial^2_\infty X})$ is conservative for some (equivalently every) $x \in X$.*

(v) *The dynamical system $(\Gamma\backslash\mathscr{G}X, (\mathtt{g}^t)_{t\in\mathbb{R}}, m_F)$ is ergodic.*

(vi) *The dynamical system $(\Gamma\backslash\mathscr{G}X, (\mathtt{g}^t)_{t\in\mathbb{R}}, m_F)$ is conservative.*

If one of the above conditions is satisfied, then

(1) *the measures $\mu^{\pm}_x$ have no atoms for any $x \in X$,*

(2) *the diagonal of $\partial_\infty X \times \partial_\infty X$ has measure 0 for $\mu^-_x \otimes \mu^+_x$,*

(3) *the Patterson densities $(\mu^{\pm}_x)_{x\in X}$ are unique up to a scalar multiple, and*

(4) *for all $x, y \in X$, as $n \to +\infty$,*

$$\max_{\gamma\in\Gamma,\, n-1<d(x,\gamma y)\leq n} e^{\int_x^{\gamma y} \widetilde{F}^{\pm}} = \mathrm{o}(e^{\delta\, n}) \,.$$

Proof. The proof[5] of the equivalence claim is similar to the one of [PauPS, Theo. 5.4], using

- Proposition 3.20 (2), (4) instead of [PauPS, Lem. 3.4],
- the HC-property instead of [PauPS, Lem. 3.2],
- Lemma 3.17 (2) instead of [PauPS, Lem. 3.3 (ii)],
- Lemma 4.3 (2) instead of [PauPS, Coro. 3.11 (i)],
- Lemma 4.3 (3) instead of [PauPS, Coro. 3.12 (i)],
- Remark 4.4 (ii) instead of [PauPS, Rem. (ii), §3.7],
- Lemma 4.3 (1) instead of [PauPS, Lem. 3.10].

Claims (1) and (4) are proved as in [PauPS, Prop. 5.13], Claim (2) is proved as in [PauPS, Prop. 5.5 (c)], and Claim (3) is proved as in [PauPS, Coro. 5.12]. □

The following corollary follows immediately from Poincaré's recurrence theorem and the Hopf-Tsuji-Sullivan-Roblin theorem, see [PauPS, Coro. 5.15, Theo. 5.4 (ii')-(iii')] for the arguments written for the manifold case, which extend.

Corollary 4.7. *If m_F is finite, then*

(1) *the pair $(\Gamma, F^\pm)$ is of divergence type,*

(2) *the Patterson densities $(\mu^\pm_x)_{x\in X}$ are unique up to a multiplicative constant and the Gibbs measure m_F is uniquely defined up to a multiplicative constant.*

(3) *the Gibbs measure m_F gives full measure to the image $\Omega_{\rm c}$ of*

$$\widetilde{\Omega}_{\rm c} = \{\ell \in \mathscr{G}X \ : \ \ell_\pm \in \Lambda_{\rm c}\Gamma\}$$

in $\Gamma\backslash\mathscr{G}X$, and

(4) *the geodesic flow is ergodic for m_F.* □

On the finiteness of Gibbs measures As the finiteness of the Gibbs measures will be a standing hypothesis in many of the following results, we now give criteria for Gibbs measures to be finite. Recall[6] that the discrete nonelementary group of isometries Γ of X is geometrically finite if every element of $\Lambda\Gamma$ is either a conical limit point or a bounded parabolic limit point of Γ.

Theorem 4.8. *Assume that Γ is a geometrically finite discrete group of isometries of X.*

(1) *If $(\Gamma, F^\pm)$ is of divergence type, then the Gibbs measure m_F is finite if and only if for every bounded parabolic limit point p of Γ, the series*

$$\sum_{\alpha\in\Gamma_p} d(x,\alpha y)\ e^{\int_x^{\alpha y}(\widetilde{F}^\pm-\delta)}$$

converges, where Γ_p is the stabiliser of p in Γ.

(2) *If we have $\delta_{\Gamma_p,\,F_p^\pm} < \delta$, for every bounded parabolic limit point p of Γ with stabiliser Γ_p in Γ and with $F_p^\pm : \Gamma_p\backslash X \to \mathbb{R}$ the map induced by $\widetilde{F}^\pm$, then (Γ, F) is of divergence type. In particular, m_F is finite.*

[5] The proof occupies about 16 pages in [PauPS], hence we cannot reproduce it in this book.
[6] See Section 2.1.

When X is a manifold, this result is due to [DaOP, Theo. B] for the case $F = 0$, and to [Cou] and [PauPS, Theo. 8.3, 8.4] for the general case of Hölder-continuous potentials. When $F = 0$ but on much more general assumptions on X with optimal generality, this result is due to [DaSU, Theo. 17.1.2].

Proof. The proof is similar to the manifold case in [PauPS], which follows closely the proof of [DaOP]. Note that the convergence or divergence of the above series does not depend on the choice of the sign $\pm$.

Let Par_Γ be the set of bounded parabolic limit points of Γ. By [Rob2, Lem. 1.9][7], there exists a Γ-equivariant family $(\mathscr{H}_p)_{p\in\mathrm{Par}_\Gamma}$ of pairwise disjoint closed horoballs, with $\mathscr{H}_p$ centred at p, such that the quotient

$$M_0 = \Gamma\backslash\big(\mathscr{C}\Lambda\Gamma - \bigcup_{p\in\mathrm{Par}_\Gamma} \mathscr{H}_p\big)$$

is compact. Using Proposition 3.20, Theorem 4.6 and Equation (3.17) instead of respectively [PauPS, Lem. 3.4, Coro. 5.10,Lem. 3.2], the HC-property, the proofs of [PauPS, Theo. 8.3, 8.4] then extend to our situation. □

Recall that the *length spectrum* of Γ on X is the additive subgroup of $\mathbb{R}$ generated by the translation lengths in X of the elements of Γ.

Recall that a continuous-time one-parameter group $(h^t)_{t\in\mathbb{R}}$ of homeomorphisms of a topological space Z is *topologically mixing* if for all nonempty open subsets U, V of Z, there exists $t_0 \in \mathbb{R}$ such that for every $t \geqslant t_0$, we have $U \cap h^t V \neq \varnothing$.

We have the following result, due to [Bab1, Theo. 1] in the manifold case, with developments by [Rob2] when $\widetilde{F} = 0$, and by [PauPS, Theo. 8.1] for manifolds with pinched negative curvature.

Theorem 4.9. *If the Gibbs measure is finite, then the following assertions are equivalent :*

(1) *the geodesic flow of $\Gamma\backslash X$ is mixing for the Gibbs measure,*

(2) *the geodesic flow of $\Gamma\backslash X$ is topologically mixing on its nonwandering set, which is the quotient under Γ of the space of geodesic lines in X both of whose endpoints belong to $\Lambda\Gamma$.*

(3) *the length spectrum of Γ on X is not contained in a discrete subgroup of $\mathbb{R}$.* □

In the manifold case, the third assertion of Theorem 4.9 is satisfied, for example, if Γ has a parabolic element, if $\Lambda\Gamma$ is not totally disconnected (hence if $\Gamma\backslash X$ is compact), or if X is a surface or a (rank-one) symmetric space, see for instance [Dal1, Dal2].

Error terms for the mixing property will be described in Chapter 9. The above result holds for the continuous time geodesic flow when X is a metric tree. See Theorem 4.17 for a version of this theorem for the discrete time geodesic flow on simplicial trees. At least when X is an $\mathbb{R}$-tree and Γ is a uniform lattice (so that $\Gamma\backslash X$ is a finite metric graph), we have a stronger result under additional regularity assumptions, see Section 9.2.

[7] See also [Pau4] for the case of simplicial trees and [DaSU, Theo. 12.4.5] for a greater generality on X.

Bowen-Margulis measure computations in locally symmetric spaces Assume in this subsection that the potential $\widetilde{F}$ is zero. The next result, Proposition 4.10, gathers computations done in [PaP16, PaP17a] of the Bowen-Margulis measures $m_{\rm BM}$ when X is a real or complex hyperbolic space and Γ is a lattice. We start by giving the notation necessary in order to state Proposition 4.10.

When X is a complete simply connected Riemannian manifold with dimension $m \geqslant 2$ and sectional curvature at most -1, we endow the usual unit tangent bundle T^1X with Sasaki's Riemannian metric.[8] Its Riemannian measure $d\,\mathrm{vol}_{T^1X}$, called *Liouville's measure*, disintegrates (equivariantly with respect to Γ) under the fibration $\pi : T^1X \to X$ over the Riemannian measure $d\,\mathrm{vol}_X$ of X, as

$$d\,\mathrm{vol}_{T^1X} = \int_{x\in X} d\,\mathrm{vol}_{T^1_xX}\;\; d\,\mathrm{vol}_X(x)\;,$$

where $d\,\mathrm{vol}_{T^1_xX}$ is the spherical measure on the fiber T^1_xX of π above $x \in X$. In particular,

$$\mathrm{Vol}(T^1(\Gamma\backslash X)) = \mathrm{Vol}(\mathbb{S}^{m-1})\,\mathrm{Vol}(\Gamma\backslash X)\;.$$

Assume furthermore that X is a symmetric space, and that Γ is a lattice in $\mathrm{Isom}(X)$. In particular, Γ is geometrically finite and the critical exponent of the stabiliser in Γ of every bounded parabolic fixed point of Γ is strictly less than the critical exponent of Γ, see for instance [Dal1, Dal2]. Then the Patterson density is independent of Γ and uniquely defined up to a multiplicative constant by Theorem 4.8 (2) and Lemma 4.7 (2). We take $\mu^-_x = \mu^+_x$ for every $x \in \mathbb{H}^n_{\mathbb{C}}$ and we will denote this measure simply by μ_x. By homogeneity reasons, the Bowen-Margulis measure of Γ on X is proportional to the Liouville measure vol_{T^1X}, and the main point of Proposition 4.10 is to compute the proportionality constant.

Let $n \geqslant 2$. We endow $\mathbb{R}^{n-1}$ with its usual Euclidean norm $\|\cdot\|$ and its usual Lebesgue measure λ_{n-1}. We use the upper halfspace model for the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ of dimension n, that is, $\mathbb{H}^n_{\mathbb{R}} = \{(x_1,\cdots,x_n) \in \mathbb{R}^n \;:\; x_n > 0\}$ endowed with the Riemannian metric

$$ds^2_{\mathbb{H}^n_{\mathbb{R}}} = \frac{1}{x_n^2}\,(dx_1^2 + \cdots + dx_n^2)\;.$$

We identify $\mathbb{R}^{n-1}$ with $\mathbb{R}^{n-1}\times\{0\}$ in $\mathbb{R}^n$, and again denote by $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^n$. The boundary at infinity of $\mathbb{H}^n_{\mathbb{R}}$ is $\partial_\infty\mathbb{H}^n_{\mathbb{R}} = \mathbb{R}^{n-1}\cup\{\infty\}$. Assuming that Γ is a lattice in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$, we normalise its Patterson density so that in the ball model of $\mathbb{H}^n_{\mathbb{R}}$ with center 0, the measure μ_0 is the spherical measure on the space at infinity $\mathbb{S}^{n-1}$.

Let $n \geqslant 2$. We refer to [Gol] and [PaP17a, §3] for background on the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$. We denote by $\zeta\cdot\overline{\zeta'} = \sum_{i=1}^{n-1}\zeta_i\,\overline{\zeta'_i}$ the standard Hermitian product and by $d\zeta$ the standard Lebesgue measure on $\mathbb{C}^{n-1}$. We denote by Heis_{2n-1} the *Heisenberg group* of dimension $2n-1$, which is the real Lie group structure on $\mathbb{C}^{n-1}\times\mathbb{R}$ with law

$$(\zeta,u)(\zeta',u') = (\zeta+\zeta', u+u'+2\,\mathrm{Im}\,\zeta\cdot\overline{\zeta'})\;.$$

We endow Heis_{2n-1} with the usual left-invariant Haar measure $d\lambda_{2n-1}(\zeta,u) = d\zeta\,du$ and with the *Cygan distance*[9] $d_{\rm Cyg}$ which is the unique left-invariant distance on Heis_{2n-1} with $d_{\rm Cyg}((\zeta,u),(0,0)) = (|\zeta|^4+u^2)^{\frac14}$.

[8]See Section 3.1.

[9]See [Gol, page 160]. It is called the *Korányi distance* by many people working in sub-Riemannian geometry, though Korányi [Kor] does attribute it to Cygan [Cyg].

We use the *Siegel domain* model of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ of dimension n, normalised to have maximum constant sectional curvature -1, hence to be CAT(-1). This is the manifold $\operatorname{Heis}_{2n-1} \times]0, +\infty[$ endowed with the Riemannian metric given, in the *horospherical coordinates* $(\zeta, u, t) \in \mathbb{C}^{n-1} \times \mathbb{R} \times]0, +\infty[$, by

$$ds^2_{\mathbb{H}^n_{\mathbb{C}}} = \frac{1}{4\,t^2}\big(dt^2 + (du + 2\operatorname{Im} d\zeta \cdot \overline{\zeta}\,)^2 + 4\,t\,d\zeta \cdot \overline{d\zeta}\,\big)\,,$$

so that its volume form is

$$d\operatorname{vol}_{\mathbb{H}^n_{\mathbb{C}}}(\zeta, u, t) = \frac{1}{4\,t^{n+1}}\;d\zeta\,du\,dt\;.$$

Note that the action of $\operatorname{Heis}_{2n-1}$ on $\mathbb{H}^n_{\mathbb{C}} = \operatorname{Heis}_{2n-1} \times]0, +\infty[$ by left translations on the first factor, preserving the second one, is isometric. We identify $\operatorname{Heis}_{2n-1}$ with $\operatorname{Heis}_{2n-1} \times \{0\}$ and we endow $\operatorname{Heis}_{2n-1} \times [0, +\infty[$ with the distance $d_{\rm Cyg}$ extending the Cygan distance on $\operatorname{Heis}_{2n-1}$, defined by

$$d_{\rm Cyg}((\zeta, u, t), (\zeta', u', t')) = \Big|\,|\zeta - \zeta'|^2 + |t - t'| + i(u - u' + 2\operatorname{Im}\zeta \cdot \overline{\zeta'})\Big|^{1/2}\,.$$

The space at infinity $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$ of $\mathbb{H}^n_{\mathbb{C}}$ is the Alexandrov compactification $\operatorname{Heis}_{2n-1} \cup \{\infty\}$, so that the extension at infinity of the isometric action of $\operatorname{Heis}_{2n-1}$ on $\mathbb{H}^n_{\mathbb{C}}$ fixes ∞ and is the left translation on $\operatorname{Heis}_{2n-1}$. We denote by $\mathscr{H}_\infty = \operatorname{Heis}_{2n-1} \times [1, +\infty[$ the horoball of $\mathbb{H}^n_{\mathbb{C}}$ centered at ∞ whose boundary contains the point $(0, 0, 1)$.

Assuming that Γ is a lattice in $\operatorname{Isom}(\mathbb{H}^n_{\mathbb{C}})$, we normalise its Patterson density $(\mu_x)_{x \in \mathbb{H}^n_{\mathbb{C}}}$ as follows. If $\mu_{\mathscr{H}_\infty}$ is the measure defined in Proposition 7.2 associated with the horoball $\mathscr{H}_\infty$, then

$$d\mu_{\mathscr{H}_\infty}(\zeta, u) = d\lambda_{2n-1}(\zeta, u) = d\zeta\,du\;.$$

This is possible since $\mu_{\mathscr{H}_\infty}$ is invariant under the isometric action of $\operatorname{Heis}_{2n-1}$ on $\mathbb{H}^n_{\mathbb{C}}$, which preserves $\mathscr{H}_\infty$, hence is a left-invariant Haar measure on $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\} = \operatorname{Heis}_{2n-1}$.

As the arguments of the following result are purely computational and rather long, we do not copy them in this book, but we refer respectively to the proofs of [PaP16, Eq. (19), Eq. (21), Prop. 10] and [PaP17a, Lem. 12 (i), (ii), (iii)]. Analogous computations can be done when X is the quaternionic hyperbolic n-space $\mathbb{H}^n_{\mathbb{H}}$.

Proposition 4.10. *(1) Let Γ be a lattice in $\operatorname{Isom}(\mathbb{H}^n_{\mathbb{R}})$, with Patterson density $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ normalised as above. For all $x = (x_0, \cdots, x_n)$ in $\mathbb{H}^n_{\mathbb{R}}$, $\xi \in \partial_\infty \mathbb{H}^n_{\mathbb{R}} - \{\infty\}$ and $v \in T^1\mathbb{H}^n_{\mathbb{R}}$ such that $v_\pm \neq \infty$, we have*

(i) $d\mu_x(\xi) = \big(\frac{2x_n}{\|x-\xi\|^2}\big)^{n-1}\,d\lambda_{n-1}(\xi)$,

(ii) using a Hopf parametrisation $v \mapsto (v_-, v_+, s)$,

$$d\widetilde{m}_{\rm BM}(v) = \frac{2^{2(n-1)} d\lambda_{n-1}(v_-)\;d\lambda_{n-1}(v_+)\;dt}{\|v_+ - v_-\|^{2(n-1)}}\;,$$

(iii)

$$\widetilde{m}_{\rm BM} = 2^{n-1}\;\operatorname{Vol}_{T^1\mathbb{H}^n_{\mathbb{R}}}\;,$$

and in particular,

$$\|m_{\rm BM}\| = 2^{n-1}\operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})\;.$$

(2) Let Γ *be a lattice in* $\operatorname{Isom}(\mathbb{H}^n_{\mathbb{C}})$*, with Patterson density* $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{C}}}$ *normalised as above. For all* $x=(\zeta,u,t)$ *in* $\mathbb{H}^n_{\mathbb{C}}$*,* $(\xi,r)\in\partial_\infty\mathbb{H}^n_{\mathbb{C}}-\{\infty\}$ *and* $v\in T^1\mathbb{H}^n_{\mathbb{C}}$ *such that* $v_\pm\neq\infty$*, we have*

(i) $d\mu_x(\xi,r)=\dfrac{t^n}{d_{\rm Cyg}(x,(\xi,r))^{4n}}\,d\xi\,dr$;

(ii) using a Hopf parametrisation $v\mapsto(v_-,v_+,s)$*,*

$$d\widetilde{m}_{\rm BM}(v)=\frac{d\lambda_{2n-1}(v_-)\,d\lambda_{2n-1}(v_+)\,ds}{d_{\rm Cyg}(v_-,v_+)^{4n}}\;;$$

(iii)

$$\widetilde{m}_{\rm BM}=\frac{1}{2^{2n-2}}\;\operatorname{vol}_{T^1\mathbb{H}^n_{\mathbb{C}}}\,,$$

and in particular

$$\|m_{\rm BM}\|=\frac{\pi^n}{2^{2n-3}\,(n-1)!}\;\operatorname{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{C}})\;.\quad\Box$$

On the cohomological invariance of Gibbs measures We end this Section by an elementary remark on the independence of Gibbs measures upon replacement of the potential F by a cohomologous one.

Remark 4.11. Let $\widetilde{F}^*:T^1X\to\mathbb{R}$ be a potential for Γ cohomologous to $\widetilde{F}$ and satisfying the HC-property. As usual, let $\widetilde{F}^{*+}=\widetilde{F}^*$ and $\widetilde{F}^{*-}=\widetilde{F}^*\circ\iota$, and let $F^*:\Gamma\backslash T^1X\to\mathbb{R}$ be the induced map. Let $\widetilde{G}:T^1X\to\mathbb{R}$ be a continuous Γ-invariant function such that, for every $\ell\in\mathscr{G}X$, the map $t\mapsto\widetilde{G}(v_{{\tt g}^t\ell})$ is differentiable and $\widetilde{F}^*(v_\ell)-\widetilde{F}(v_\ell)=\frac{d}{dt}_{|t=0}\widetilde{G}(v_{{\tt g}^t\ell})$. Furthermore assume that X is an $\mathbb{R}$-tree or that G is uniformly continuous (for instance Hölder-continuous).

For all $x\in X$ and $\xi\in\partial_\infty X$, let $\ell_{x,\xi}$ be any geodesic line with footpoint $\ell_{x,\xi}(0)=x$ and positive endpoint $(\ell_{x,\xi})_+=\xi$, and let $\ell_{\xi,x}$ be any geodesic line with $\ell_{\xi,x}(0)=x$ and origin $(\ell_{\xi,x})_-=\xi$. Note that the value $\widetilde{G}(v_{\ell_{x,\xi}})$ is independent of the choice of $\ell_{x,\xi}$, by the continuity of $\widetilde{G}$, and similarly for $\widetilde{G}(v_{\ell_{\xi,x}})$. In particular, for all $\gamma\in\Gamma$, by the Γ-invariance of $\widetilde{G}$, we have

$$\widetilde{G}(v_{\ell_{x,\,\gamma^{-1}\xi}})=\widetilde{G}(v_{\ell_{\gamma x,\,\xi}})\quad\text{and}\quad\widetilde{G}\circ\iota\,(v_{\ell_{x,\,\xi}})=\widetilde{G}(v_{\ell_{\xi,\,x}})\;.\tag{4.6}$$

Note that $\widetilde{F}^{*-}=\widetilde{F}^*\circ\iota$ and $\widetilde{F}^-=\widetilde{F}\circ\iota$ are cohomologous, since if $\widetilde{G}^*=-\widetilde{G}\circ\iota$, for every $\ell\in\mathscr{G}X$, we have

$$\begin{aligned}\widetilde{F}^*\circ\iota(v_\ell)-\widetilde{F}\circ\iota(v_\ell)=\widetilde{F}^*(v_{\iota\ell})-\widetilde{F}(v_{\iota\ell})&=\frac{d}{dt}_{|t=0}\widetilde{G}(v_{{\tt g}^t\iota\ell})\\&=\frac{d}{dt}_{|t=0}\widetilde{G}(\iota v_{{\tt g}^{-t}\ell})=\frac{d}{dt}_{|t=0}\widetilde{G}^*(v_{{\tt g}^t\ell})\;.\end{aligned}$$

As already seen in Lemma 3.17 (1) and (2), the critical exponent $\delta_{\Gamma,\,F^{*\pm}}$ is equal to the critical exponent $\delta_{\Gamma,\,F^\pm}$, and independent of the choice of $\pm$, and we denote by δ the common value in the definition of the Gibbs cocycle.

Let us prove that if $C^{*\pm}=C^\pm_{\Gamma,\,F^{*\pm}}$ is the Gibbs cocycle associated with $(\Gamma,F^{*\pm})$, then $C^{*\pm}$ and $C^\pm$ are *cohomologous*:

$$C^{*+}_\xi(x,y)-C^+_\xi(x,y)=\widetilde{G}(v_{\ell_{x,\,\xi}})-\widetilde{G}(v_{\ell_{y,\,\xi}})\;,\tag{4.7}$$

and similarly

$$C^{*-}_{\xi}(x,y) - C^{-}_{\xi}(x,y) = \widetilde{G}^*(v_{\ell_{x,\,\xi}}) - \widetilde{G}^*(v_{\ell_{y,\,\xi}})\ . \tag{4.8}$$

We only prove the first equality, the second one follows similarly, noting that $\widetilde{G}^*$ is uniformly continuous if $\widetilde{G}$ is, since ι is isometric. For all x, y in X and $\xi \in \partial_\infty X$, let $t \mapsto \xi_t$ be a geodesic ray with point at infinity ξ, let $a_t = d(x, \xi_t)$, let $b_t = d(y, \xi_t)$, and for $z = x, y$, let $\ell_{z,\,\xi_t}$ be any geodesic line with footpoint z passing through ξ_t. Then

$$\begin{aligned}
&\Big(\int_y^{\xi_t}(\widetilde{F}^{*+}-\delta) - \int_x^{\xi_t}(\widetilde{F}^{*+}-\delta)\Big) - \Big(\int_y^{\xi_t}(\widetilde{F}^{+}-\delta) - \int_x^{\xi_t}(\widetilde{F}^{+}-\delta)\Big)\\
&= \int_y^{\xi_t}(\widetilde{F}^{*+}-\widetilde{F}^{+}) - \int_x^{\xi_t}(\widetilde{F}^{*+}-\widetilde{F}^{+})\\
&= \int_0^{b_t}\frac{d}{dt}\widetilde{G}(v_{\mathtt{g}^s\ell_{y,\,\xi_t}})\,ds - \int_0^{a_t}\frac{d}{dt}\widetilde{G}(v_{\mathtt{g}^s\ell_{x,\,\xi_t}})\,ds\\
&= \widetilde{G}(v_{\ell_{x,\,\xi_t}}) - \widetilde{G}(v_{\ell_{y,\,\xi_t}}) + \big(\widetilde{G}(v_{\mathtt{g}^{b_t}\ell_{y,\,\xi_t}}) - \widetilde{G}(v_{\mathtt{g}^{a_t}\ell_{x,\,\xi_t}})\big)\ .
\end{aligned}$$

When t goes to $+\infty$, the first term of this series of equalities converges to $C^{*+}_{\xi}(x,y) - C^{+}_{\xi}(x,y)$ by the definition of the Gibbs cocycle (see Section 3.4). By continuity, $\widetilde{G}(v_{\ell_{y,\,\xi_t}})$ and $\widetilde{G}(v_{\ell_{x,\,\xi_t}})$ converge to $\widetilde{G}(v_{\ell_{y,\,\xi}})$ and $\widetilde{G}(v_{\ell_{x,\,\xi}})$ respectively. If X is an $\mathbb{R}$-tree, then if t is large enough, we have $v_{\mathtt{g}^{b_t}\ell_{y,\,\xi_t}} = v_{\mathtt{g}^{a_t}\ell_{x,\,\xi_t}}$, hence Equation (4.7) follows. Otherwise, by the uniform continuity of $\widetilde{G}$, since $v_{\mathtt{g}^{b_t}\ell_{y,\,\xi_t}}$ and $v_{\mathtt{g}^{a_t}\ell_{x,\,\xi_t}}$ are uniformly arbitrarily close as t tends to 0 by the CAT(-1) property, the result also follows.

Let $(\mu^{\pm}_x)_{x\in X}$ be a Patterson density for $(\Gamma, F^{\pm})$. In order to simplify the notation, let $\widetilde{G}^+ = \widetilde{G}$ and $\widetilde{G}^- = \widetilde{G}^*$. The family of measures $(\mu^{*\pm}_x)_{x\in X}$ defined by setting, for all $x \in X$ and $\xi \in \partial_\infty X$,

$$d\mu^{*\pm}_x(\xi) = e^{-\widetilde{G}^{\pm}(v_{\ell_{x,\,\xi}})}\, d\mu^{\pm}_x(\xi)\ , \tag{4.9}$$

is also a Patterson density for $(\Gamma, F^{*\pm})$. Indeed, the equivariance property (4.1) for $(\mu^{*\pm}_x)_{x\in X}$ follows from the one for $(\mu^{\pm}_x)_{x\in X}$ and from Equation (4.6). The absolutely continuous property (4.2) for $(\mu^{*\pm}_x)_{x\in X}$ follows from the one for $(\mu^{\pm}_x)_{x\in X}$ and Equations (4.7) and (4.8).

Assume that the Patterson density for $(\Gamma, F^{*\pm})$ defined by Equation (4.9) is chosen in order to construct the Gibbs measure $\widetilde{m}_{F^*}$ for (Γ, F^*) on $\mathscr{G}X$. Then using

- Hopf's parametrisation with respect to the base point x_0 and Equation (4.4) with F replaced by F^* for the first equality,
- Equations (4.7), (4.8), (4.9) and cancellations for the second equality,
- the definition of $\widetilde{G}^* = -\widetilde{G}\circ\iota$ and again Equation (4.4) for the third equality,
- Equation (4.6) and the fact that we may choose $\ell_{\ell_-,\,\ell(0)} = \ell$ and $\ell_{\ell(0),\,\ell_+} = \ell$ for the last equality,

we have

$$\begin{aligned}
d\widetilde{m}_{F^*}(\ell) &= e^{C^{*-}_{\ell_-}(x_0,\,\ell(0)) + C^{*+}_{\ell_+}(x_0,\,\ell(0))}\, d\mu^{*-}_{x_0}(\ell_-)\, d\mu^{*+}_{x_0}(\ell_+)\, dt\\
&= e^{C^{-}_{\ell_-}(x_0,\,\ell(0)) - \widetilde{G}^*(v_{\ell_{\ell(0),\,\ell_-}}) - C^{+}_{\ell_+}(x_0,\,\ell(0)) + \widetilde{G}(v_{\ell_{\ell(0),\,\ell_+}})}\, d\mu^{-}_{x_0}(\ell_-)\, d\mu^{+}_{x_0}(\ell_+)\, dt\\
&= e^{\widetilde{G}\circ\iota(v_{\ell_{\ell(0),\,\ell_-}}) - \widetilde{G}(v_{\ell_{\ell(0),\,\ell_-}})}\, d\widetilde{m}_F(\ell)\\
&= e^{\widetilde{G}(v_\ell) - \widetilde{G}(v_\ell)}\, d\widetilde{m}_F(\ell)\ ,
\end{aligned}$$

hence $\widetilde{m}_{F*} = \widetilde{m}_F$.

In particular, since the Gibbs measure, when finite, is independent up to a multiplicative constant on the choice of the Patterson densities by Corollary 4.7, we have that m_F is finite if and only if m_{F*} is finite, and then

$$\frac{m_{F*}}{\|m_{F*}\|} = \frac{m_F}{\|m_F\|} \,. \tag{4.10}$$

4.3 Patterson densities for simplicial trees

In this Section and the following one, we specialise and modify the general framework of the previous sections to treat simplicial trees. Recall[10] that a simplicial tree $\mathbb{X}$ is a metric tree whose edge length map is constant equal to 1. The time 1 map of the geodesic flow $(\mathtt{g}^t)_{t\in\mathbb{R}}$ on the space $\breve{\mathscr{G}}X$ of all generalised geodesic lines of the geometric realisation $X = |\mathbb{X}|_1$ of $\mathbb{X}$ preserves for instance its subset of generalised geodesic lines whose footpoints are at distance at most 1/4 from vertices. Since both this subset and its complement have nonempty interior in $\breve{\mathscr{G}}X$, the geodesic flow on $\breve{\mathscr{G}}X$ has no mixing or ergodic measure with full support. This is why we considered the discrete time geodesic flow $(\mathtt{g}^t)_{t\in\mathbb{Z}}$ on $\breve{\mathscr{G}}\mathbb{X}$ in Section 2.6.

Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, and let $X = |\mathbb{X}|_1$ be its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X})$. Let $\widetilde{F} : T^1X \to \mathbb{R}$ be a potential for Γ, and let $\widetilde{F}^+ = \widetilde{F}$, $\widetilde{F}^- = \widetilde{F}\circ\iota$. Let $\delta = \delta_{\Gamma, F^\pm}$ be the critical exponent of $(\Gamma, F^\pm)$. Let $C^\pm : \partial_\infty X \times X \times X \to \mathbb{R}$ be the associated (normalised) Gibbs cocycles. Let $(\mu^\pm_x)_{x\in X}$ be two Patterson densities on $\partial_\infty X$ for the pairs $(\Gamma, F^\pm)$.

Note that only the restrictions of the cocycles $C^\pm$ to $\partial_\infty X \times V\mathbb{X} \times V\mathbb{X}$ are useful and that it is often convenient and always sufficient to replace the cocycles by finite sums involving a system of conductances (as defined in Section 3.5), see below. Furthermore, only the restriction $(\mu^\pm_x)_{x\in V\mathbb{X}}$ of the family of Patterson densities to the set of vertices of $\mathbb{X}$ is useful.

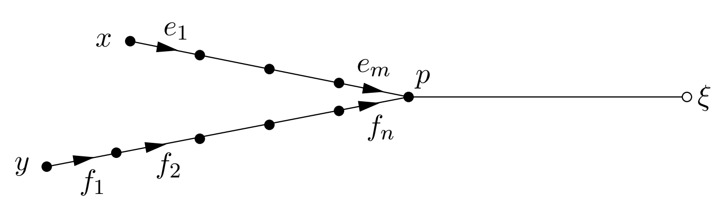
Example 4.12. Let $\mathbb{X}$ be a simplicial tree with geometric realisation X and let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a system of conductances on $\mathbb{X}$. For all x, y in $V\mathbb{X}$ and $\xi \in \partial_\infty X$, with the usual convention on the empty sums, let

$$c^+_\xi(x,y) = \sum_{i=1}^m \widetilde{c}(e_i) - \sum_{j=1}^n \widetilde{c}(f_j)$$

and

$$c^-_\xi(x,y) = \sum_{i=1}^m \widetilde{c}(\overline{e_i}) - \sum_{j=1}^n \widetilde{c}(\overline{f_j})\,,$$

where, if $p \in V\mathbb{X}$ is such that $[p,\xi[\,=\, [x,\xi[\,\cap\, [y,\xi[$, then $(e_1, e_2, \dots, e_m)$ is the geodesic edge path in $\mathbb{X}$ from $x = o(e_1)$ to $p = t(e_m)$ and $(f_1, f_2, \dots, f_n)$ is the geodesic edge path in $\mathbb{X}$ from $v = o(f_1)$ to $p = t(f_n)$.



[10] See Section 2.6.

With δ_c defined in the end of Section 3.5 and with $C^\pm$ the Gibbs cocycles for $(\Gamma, \widetilde{F}_c)$, by Equation (3.18) and by Proposition 3.21, we have, for all $\xi \in \partial_\infty X$ and $x, y \in V\mathbb{X}$,

$$C^\pm_\xi(x, y) = -\,c^\pm_\xi(x, y) + \delta_c\,\beta_\xi(x, y)\,,$$

and Equation (4.2) gives

$$d\mu^\pm_x(\xi) = e^{c^\pm_\xi(x,y) - \delta_c\,\beta_\xi(x,y)}\;d\mu^\pm_y(\xi)\,.$$

Using the particular structure of trees, we can prove a version of the Shadow Lemma 4.3 where one can take the radius R to be 0. When $F = 0$, this result is due to Coornaert [Coo].

Lemma 4.13 (Mohsen's shadow lemma for trees)**.** *Let K be a finite subset of $V\mathbb{X}$. There exists $C > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$ with $y \in \mathscr{C}\Lambda\Gamma$, we have*

$$\frac{1}{C}\;e^{\int_x^{\gamma y}(\widetilde{F}^\pm - \delta)} \leqslant \mu^\pm_x\big(\mathscr{O}_x\{\gamma y\}\big) \leqslant C\;e^{\int_x^{\gamma y}(\widetilde{F}^\pm - \delta)}\;.$$

Proof. The structure of the proof is the same one as for Lemma 4.3 (1) (that is, the one of [PauPS, Lem. 3.10]) with differences towards the end of the argument. Let us prove that there exists $C = C_K > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$ with $\gamma y \in \mathscr{C}\Lambda\Gamma$, we have

$$\frac{1}{C} \leqslant \mu^\pm_{\gamma y}\big(\mathscr{O}_x\{\gamma y\}\big) \leqslant C\;. \tag{4.11}$$

Assuming this, let us prove Lemma 4.13. By Equation (4.2), we have

$$\mu^\pm_x\big(\mathscr{O}_x\{\gamma y\}\big) = \int_{\xi \in \mathscr{O}_x\{\gamma y\}} e^{-C^\pm_\xi(x,\,\gamma y)}\;d\mu^\pm_{\gamma y}(\xi)\;.$$

Note that $C^\pm_\xi(x, \gamma y) + \int_x^{\gamma y}(\widetilde{F}^\pm - \delta) = 0$ if $\xi \in \mathscr{O}_x\{\gamma y\}$ (that is, if $\gamma y \in [x, \xi[\,$), by Equation (3.20). Hence

$$\mu^\pm_x(\mathscr{O}_x\{\gamma y\}) = e^{\int_x^{\gamma y}(\widetilde{F}^\pm - \delta)}\;\mu^\pm_{\gamma y}(\mathscr{O}_x\{\gamma y\})\;,$$

and Lemma 4.13 follows from Equation (4.11).

Let us now prove the upper bound in Equation (4.11). Fix $z_0 \in K$, and let

$$C' = \sup_{x,\,y \in K,\;\xi \in \partial_\infty X} |C^\pm_\xi(x, y)|\;,$$

which is finite by Proposition 3.20 (2), since K is compact and $\widetilde{F}^\pm$ continuous. Then, using Equation (4.1) for the equality and Equation (4.2) for the last inequality, we have

$$\mu^\pm_{\gamma y}(\mathscr{O}_x\{\gamma y\}) \leqslant \|\mu^\pm_{\gamma y}\| = \|\mu^\pm_y\| \leqslant e^{C'}\;\|\mu^\pm_{z_0}\|\;,$$

and the upper bound holds if $C \geqslant e^{C'}\;\|\mu^\pm_{z_0}\|$.

Finally, in order to prove the lower bound in Equation (4.11), we assume for a contradiction that there exist sequences $(x_i)_{i\in\mathbb{N}}$, $(y_i)_{i\in\mathbb{N}}$ in K with $y_i \in \mathscr{C}\Lambda\Gamma$ and $(\gamma_i)_{i\in\mathbb{N}}$ in Γ such that $\mu^\pm_{\gamma_i y_i}(\mathscr{O}_{x_i}\{\gamma_i y_i\})$ converges to 0 as $i \to +\infty$. Up to extracting a subsequence, since K is finite, we may assume that the sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are constant, say with value x and y respectively. Since $y \in \mathscr{C}\Lambda\Gamma$, as every point in $\mathscr{C}\Lambda\Gamma$ belongs to at least one geodesic line between two limit points of Γ, the geodesic segment from x to $\gamma_i y$ may be extended to a geodesic ray from x to a limit point of Γ. Since the support of $\mu^\pm_z$ is equal to $\Lambda\Gamma$ for any $z \in X$, we have $\mu^\pm_{\gamma_i y}(\mathscr{O}_x\{\gamma_i y\}) > 0$ for all $i \in \mathbb{N}$. Thus, up to taking a subsequence, we can assume that $\gamma_i^{-1}x$ converges to $\xi \in \Lambda\Gamma$ (otherwise by discreteness, we may extract a subsequence so that $(\gamma_i)_{i\in\mathbb{N}}$ is constant, and $\mu^\pm_{\gamma_i y}(\mathscr{O}_x\{\gamma_i y\})$ cannot converge to 0).

Since $\mathbb{X}$ is a tree, there exists a positive integer N such that $\mathscr{O}_{\gamma_i^{-1}x}\{y\} = \mathscr{O}_{\gamma_N^{-1}x}\{y\} = \mathscr{O}_\xi\{y\}$ for all $i \geqslant N$. As above, $\mathscr{O}_\xi\{y\}$ meets $\Lambda\Gamma$ since $y \in \mathscr{C}\Lambda\Gamma$, thus $\mu^\pm_y(\mathscr{O}_\xi\{y\}) > 0$. But for every $i \geqslant N$,

$$\mu^\pm_y\big(\mathscr{O}_\xi\{y\}\big) = (\gamma_i^{-1})_*\mu^\pm_{\gamma_i y}\big(\mathscr{O}_{\gamma_i^{-1}x}\{y\}\big) = \mu^\pm_{\gamma_i y}\big(\mathscr{O}_x\{\gamma_i y\}\big)$$

tends to 0 as $i \to +\infty$, a contradiction. □

Let $\widetilde{\phi}_{\mu^\pm} : V\mathbb{X} \to [0, +\infty[$ be the *total mass functions of the Patterson densities*:

$$\widetilde{\phi}_{\mu^\pm}(x) = \|\mu^\pm_x\|$$

for every $x \in V\mathbb{X}$. These maps are Γ-invariant by Equation (4.1), hence they induce maps $\phi_{\mu^\pm} : \Gamma\backslash V\mathbb{X} \to [0, +\infty[$. In the case of real hyperbolic manifolds and vanishing potentials, the total mass functions have important links to the spectrum of the hyperbolic Laplacian (see [Sul1]). See also [CoP3, CoP5] for the case of simplicial trees and the discrete Laplacian, Section 6.1 for a generalisation of the result of Coornaert and Papadopoulos, and for instance [BerK] for developments in the field of quantum graphs.

4.4 Gibbs measures for metric and simplicial trees

Let $(\mathbb{X}, \lambda)$ be a locally finite metric tree without terminal vertices, let $X = |\mathbb{X}|_\lambda$ be its geometric realisation, and let $x_0 \in V\mathbb{X}$ be a basepoint. Let Γ be a nonelementary discrete subgroup of $\mathrm{Aut}(\mathbb{X}, \lambda)$. Let $\widetilde{F} : T^1X \to \mathbb{R}$ be a potential for Γ, and let $\widetilde{F}^+ = \widetilde{F}$, $\widetilde{F}^- = \widetilde{F}\circ\iota$. Let $\delta = \delta_{\Gamma,F^\pm}$ be the critical exponent of $(\Gamma, F^\pm)$, assumed to be finite. Let $(\mu^\pm_x)_{x\in V\mathbb{X}}$ be two (normalised) Patterson densities on $\partial_\infty X$ for the pairs $(\Gamma, F^\pm)$.

The *Gibbs measure* $\widetilde{m}_F$ on the space of discrete geodesic lines $\mathscr{G}\mathbb{X}$ of $\mathbb{X}$, invariant under Γ and under the discrete time geodesic flow $(\mathsf{g}^t)_{t\in\mathbb{Z}}$ of $\check{\mathscr{G}}\mathbb{X}$, is defined analogously with the continuous time case, using the discrete Hopf parametrisation for any basepoint $x_0 \in V\mathbb{X}$, by

$$d\widetilde{m}_F(\ell) = e^{C^-_{\ell_-}(x_0,\ell(0)) + C^+_{\ell_+}(x_0,\ell(0))}\, d\mu^-_{x_0}(\ell_-)\, d\mu^+_{x_0}(\ell_+)\, dt\ , \tag{4.12}$$

where now dt is the counting measure on $\mathbb{Z}$. We again denote by m_F the measure that $\widetilde{m}_F$ induces on $\Gamma\backslash\mathscr{G}\mathbb{X}$.

In this Section, we prove that the Gibbs measures in the case of trees satisfy a Gibbs property even closer to the one in symbolic dynamics, we give an analytic finiteness criterion of the Gibbs measures for metric trees, and we recall the ergodic properties of tree lattices.

As recalled in the introduction, Gibbs measures were first introduced in statistical mechanics and consequently in symbolic dynamics, see for example [Bowe2], [ParP], [PauPS]. In order to motivate the terminology used in this book, we recall the definition of a Gibbs measure for the full two-sided shift on a finite alphabet:[11] Let $\Sigma_n = \{1, 2, \dots, n\}^{\mathbb{Z}}$ be the product space of sequences $x = (x_n)_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$ in the finite discrete set $\{1, 2, \dots, n\}$,

[11] See Section 5.1 for the appropriate definition when the alphabet is countable.

and let $\sigma : \Sigma_n \to \Sigma_n$ be the *shift map* defined by $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. A shift-invariant probability measure μ on Σ_n *satisfies the Gibbs property* for an energy function $\phi : \Sigma_n \to \mathbb{R}$ if

$$\frac{1}{C} \leqslant \frac{\mu([a_{-m_-}, a_{-m_-+1}, \dots, a_{m_+-1}, a_{m_+}])}{e^{-P(m_-+m_++1)+\sum_{k=-m_-}^{m_+} \phi(\sigma^k x)}} \leqslant C$$

for some constants $C \geqslant 1$ and $P \in \mathbb{R}$ (called the *pressure*) and for all $m_\pm$ in $\mathbb{N}$ with $m_- \leqslant m_+$ and x in the cylinder $[a_{-m_-}, a_{-m_-+1}, \dots, a_{m_+-1}, a_{m_+}]$ that consists of those $x \in \Sigma_n$ for which $x_k = a_k$ for all $k \in [-m_-, m_+]$.

Let $x_-, x_+ \in V\mathbb{X}$ and let $x_0 \in V\mathbb{X} \cap [x_-, x_+]$. Let us define the *tree cylinder* of the triple (x_-, x_0, x_+) by

$$[x_-, x_0, x_+] = \{\ell \in \mathscr{G}\mathbb{X} : \ell_\pm \in \mathscr{O}_{x_0}\{x_\pm\},\ \ell(0) = x_0\}\,.$$

These cylinders are close to the dynamical balls that have been introduced in Section 4.2, and the parallel with the symbolic case is obvious, as this cylinder is the set of geodesic lines which coincides on $[-m_-, m_+]$, where $m_\pm = d(x_0, x_\pm)$, with a given geodesic line passing through $x_\pm$ and through x_0 at time $t = 0$. The Gibbs measure $\widetilde{m}_F$ on the space of discrete geodesic lines $\mathscr{G}\mathbb{X}$ satisfies a variant of the Gibbs property which is even closer to the one in symbolic dynamics than the general case described in Proposition 4.5.

Proposition 4.14 (Gibbs property)**.** *Let K be a finite subset of $V\mathbb{X} \cap \mathscr{C}\Lambda\Gamma$. There exists $C' > 1$ such that for all $x_\pm \in \Gamma K$ and $x_0 \in V\mathbb{X} \cap [x_-, x_+]$,*

$$\frac{1}{C'} \leqslant \frac{\widetilde{m}_F([x_-, x_0, x_+])}{e^{-\delta\, d(x_-, x_+)+\int_{x_-}^{x_+} \widetilde{F}}} \leqslant C'\,.$$

Proof. The result is immediate if $d(x_-, x_+)$ is bounded, since the above denominator and numerator take only finitely many values by the finiteness of K and by Γ-invariance, and the numerator is nonzero since $x_\pm \in \mathscr{C}\Lambda\Gamma$, hence the tree cylinder $[x_-, x_0, x_+]$ meets the support of $\widetilde{m}_F$. We may hence assume that $d(x_-, x_+) \geqslant 2$. Using the invariance of $\widetilde{m}_F$ under the discrete time geodesic flow, we may thus assume that $x_0 \neq x_-, x_+$. By Γ-invariance, we may assume that x_- varies in the finite set K, and that x_0 is at distance 1 from x_-, hence also varies in a finite set.

Using the discrete Hopf parametrisation with respect to the vertex x_0, we have, by Lemma 4.13, for some $C > 0$,

$$\begin{aligned}\widetilde{m}_F([x_-, x_0, x_+]) &= \mu^-_{x_0}(\mathscr{O}_{x_0}\{x_-\})\,\mu^+_{x_0}(\mathscr{O}_{x_0}\{x_+\})\\ &\leqslant C^2\, e^{\int_{x_0}^{x_-}(\widetilde{F}\circ\iota-\delta)}\, e^{\int_{x_0}^{x_+}(\widetilde{F}-\delta)} = C^2 e^{\int_{x_-}^{x_+}(\widetilde{F}-\delta)}\,.\end{aligned}$$

This proves the upper bound in Proposition 4.14 with $C' = C^2$ and the lower bound follows similarly. □

Next, we give a finiteness criterion of the Gibbs measure for metric trees in terms of the total mass functions of the Patterson densities, extending the case when Γ is torsion free and $\widetilde{F} = 0$, due to [CoP4, Theo. 1.1].

Proposition 4.15. *Let $(\mathbb{X}, \lambda, \Gamma, \widetilde{F})$ be as in the beginning of this Section.*

(1) *If $(\mathbb{X}, \lambda)$ is simplicial and $\|\cdot\|_2$ is the Hilbert norm of $\mathbb{L}^2(\Gamma\backslash V\mathbb{X}, \mathrm{vol}_{\Gamma\backslash\backslash\mathbb{X}})$, we have*[12]

$$\|m_F\| \leqslant \|\phi_{\mu^+}\|_2\, \|\phi_{\mu^-}\|_2\,.$$

[12] The maps $\phi_{\mu^\pm}$ are defined at the end of Section 4.3.

(2) *In general, with* $\|\cdot\|_2$ *the Hilbert norm of* $\mathbb{L}^2(\Gamma\backslash E\mathbb{X}, \operatorname{Tvol}_{(\Gamma\backslash\backslash\mathbb{X},\lambda)})$, *we have*[13]

$$\|m_F\| \leqslant \|\phi_{\mu^+} \circ o\|_2 \, \|\phi_{\mu^-} \circ o\|_2 \; .$$

Proof. (1) The simplicial assumption on $(\mathbb{X}, \lambda)$ means that all edges have length 1. The space $\Gamma\backslash\mathscr{G}\mathbb{X}$ is the disjoint union of the subsets $\{\ell \in \Gamma\backslash\mathscr{G}\mathbb{X} \;:\; \pi(\ell) = \ell(0) = [x]\}$ as the orbit $[x] = \Gamma x$ of $x \in V\mathbb{X}$ ranges over $\Gamma\backslash V\mathbb{X}$. By Equation (4.12), using the discrete Hopf decomposition with respect to the basepoint x, we have

$$d(\widetilde{m}_F)\big|_{\{\ell\in\mathscr{G}\mathbb{X}\,:\,\ell(0)=x\}}(\ell) = d\mu_x^-(\ell_-)\, d\mu_x^+(\ell_+) \; .$$

Let $\Delta_{[x]}$ be the unit Dirac mass at $[x]$. By ramified covering arguments, we hence have the following equality of measures on the discrete set $\Gamma\backslash V\mathbb{X}$:

$$\pi_* m_F = \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{1}{|\Gamma_x|} \; \big(\mu_x^- \times \mu_x^+\big)\big(\{(\ell_-, \ell_+) \in \partial_\infty^2 X \;:\; x \in \,]\ell_-, \ell_+[\}\big)\; \Delta_{[x]} \; . \tag{4.13}$$

Thus, using the Cauchy-Schwarz inequality and by the definition of the measure $\operatorname{vol}_{\Gamma\backslash\backslash\mathbb{X}}$,

$$\begin{aligned}\|m_F\| = \|\pi_* m_F\| &\leqslant \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{1}{|\Gamma_x|}\; \|\mu_x^- \times \mu_x^+\| = \langle \phi_{\mu^-}, \phi_{\mu^+}\rangle_2 \\ &\leqslant \|\phi_{\mu^-}\|_2 \|\phi_{\mu^+}\|_2 \; .\end{aligned}$$

This proves Assertion (1) of Proposition 4.15.

(2) The argument is similar to the previous one. Since the singletons in $\mathbb{R}$ have zero Lebesgue measure, the space $\Gamma\backslash\mathscr{G}X$ is, up to a measure zero subset for m_F, the disjoint union for $[e] \in \Gamma\backslash E\mathbb{X}$ of the sets $A_{[e]}$ consisting of the elements $\ell \in \Gamma\backslash\mathscr{G}X$ such that $\ell(0)$ belongs to the interior of the edge $[e]$ and the orientations of ℓ and e coincide on e. We fix a representative e of each $[e] \in \Gamma\backslash E\mathbb{X}$. For every $t \in [0, \lambda(e)]$, let e_t be the point of e at distance t from $o(e)$. By Equation (4.4), using Hopf's decomposition with respect to the basepoint $o(e)$ in $A_{[e]}$, we have as above

$$\begin{aligned}\|m_F\| = \sum_{[e]\in\Gamma\backslash E\mathbb{X}} \frac{1}{|\Gamma_e|} \int_{\ell_-\in\partial_{\overline{e}}\mathbb{X}} \int_{\ell_+\in\partial_e\mathbb{X}} \int_0^{\lambda(e)} \\ e^{C^-_{\ell_-}(o(e),\, e_t) + C^+_{\ell_+}(o(e),\, e_t)}\; d\mu^-_{o(e)}(\ell_-)\; d\mu^+_{o(e)}(\ell_+)\; dt \; .\end{aligned}$$

Since $\ell_-, o(e), e_t, \ell_+$ are in this order on the geodesic line ℓ with $\ell_- \in \partial_{\overline{e}}\mathbb{X}$ and $\ell_+ \in \partial_e\mathbb{X}$, we have $C^-_{\ell_-}(o(e),\, e_t) + C^+_{\ell_+}(o(e),\, e_t) = 0$ by Equation (3.20). Hence, by the definition of the measure $\operatorname{Tvol}_{(\Gamma\backslash\backslash\mathbb{X},\lambda)}$,[14]

$$\begin{aligned}\|m_F\| &= \sum_{[e]\in\Gamma\backslash E\mathbb{X}} \frac{\lambda(e)}{|\Gamma_e|}\; \mu^-_{o(e)}(\partial_{\overline{e}}\mathbb{X})\; \mu^+_{o(e)}(\partial_e\mathbb{X}) \\ &\leqslant \sum_{[e]\in\Gamma\backslash E\mathbb{X}} \frac{\lambda(e)}{|\Gamma_e|}\; \|\mu^-_{o(e)}\|\; \|\mu^+_{o(e)}\| = \langle \phi_{\mu^-}\circ o, \phi_{\mu^+}\circ o\rangle_2 \\ &\leqslant \|\phi_{\mu^-}\circ o\|_2\; \|\phi_{\mu^+}\circ o\|_2 \; ,\end{aligned}$$

[13] Recall that $o : \Gamma\backslash E\mathbb{X} \to \Gamma\backslash V\mathbb{X}$ is the initial vertex map, see Section 2.6.

[14] See Section 2.6.

which finishes the proof. □

Let us give some corollaries of this proposition in the case of simplicial trees. It follows from Assertion (1) of Proposition 4.15 that if the $\mathbb{L}_2$-norms of the total mass of the Patterson densities are finite, then the Gibbs measure m_F is finite. Taking $\widetilde{F} = 0$ and $(\mu^+_x)_{x\in V\mathbb{X}} = (\mu^-_x)_{x\in V\mathbb{X}}$, so that the Gibbs measure m_F is the Bowen-Margulis measure $m_{\rm BM}$, it follows from this proposition that

$$\|m_{\rm BM}\| \leqslant \|\phi_{\mu^\pm}\|_2^{\;2} \leqslant \operatorname{Vol}(\Gamma\backslash\backslash\mathbb{X}) \sup_{x\in V\mathbb{X}} \|\mu^\pm_x\|^2 \,. \tag{4.14}$$

In particular, if Γ is a (tree) lattice[15] of $\mathbb{X}$ and if the total mass of the Patterson density is bounded, then the Bowen-Margulis measure $m_{\rm BM}$ is finite.

The following statement summarises the basic ergodic properties of the lattices of $(\mathbb{X},\lambda)$ when $F = 0$.

Proposition 4.16. *Let $(\mathbb{X},\lambda)$ be a metric or simplicial tree, with geometric realisation X. Assume that $(\mathbb{X},\lambda)$ is uniform and that Γ is a lattice in $\operatorname{Aut}(\mathbb{X},\lambda)$. Then*

1. *Γ is of divergence type, and its critical exponent δ_Γ is the Hausdorff dimension of any visual distance d_x on $\partial_\infty X = \Lambda\Gamma$;*
2. *the Patterson density $(\mu_x)_{x\in X}$ coincides, up to a scalar multiple, with the family of Hausdorff measures $(\mu^{\rm Haus}_x)_{x\in X}$ of dimension δ_Γ of the visual distances $(\partial_\infty X, d_x)$; in particular, it is $\operatorname{Aut}(\mathbb{X},\lambda)$-equivariant: for all $\gamma\in\operatorname{Aut}(\mathbb{X},\lambda)$ and $x\in X$, we have $\gamma_*\mu_x = \mu_{\gamma x}$;*
3. *the Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ of Γ on $\mathscr{G}X$ is $\operatorname{Aut}(\mathbb{X},\lambda)$-invariant, and the Bowen-Margulis measure $m_{\rm BM}$ of Γ on $\Gamma\backslash\mathscr{G}X$ is finite.*

Proof. Let Γ' be any uniform lattice of $(\mathbb{X},\lambda)$, which exists since the metric tree $(\mathbb{X},\lambda)$ is uniform. It is well-known (see for instance [Bou]) that the critical exponent $\delta_{\Gamma'}$ of Γ' is finite and equal to the Hausdorff dimension of any visual distance $(\partial_\infty X, d_x)$, and that the family $(\mu^{\rm Haus}_x)_{x\in VX}$ of Hausdorff measures of the visual distances $(\partial_\infty X, d_x)$ is a Patterson density for any discrete nonelementary subgroup of $\operatorname{Aut}(\mathbb{X},\lambda)$ with critical exponent equal to $\delta_{\Gamma'}$.

By [BuM, Coro. 6.5(2)], the lattice Γ in $\operatorname{Aut}(\mathbb{X},\lambda)$ is of divergence type and $\delta_\Gamma = \delta_{\Gamma'}$. By the uniqueness property of the Patterson densities when Γ is of divergence type (see Theorem 4.6 (3)), the family $(\mu_x)_{x\in VX}$ coincides, up to a scalar multiple, with $(\mu^{\rm Haus}_x)_{x\in VX}$.

As the graph $\Gamma'\backslash\mathbb{X}$ is compact, the total mass function of the Hausdorff measures of the visual distances is bounded, hence so is $(\|\mu_x\|)_{x\in VX}$. By Proposition 4.15, since Γ is a tree lattice of $(\mathbb{X},\lambda)$, hence of $\mathbb{X}$, this implies that the Bowen-Margulis measure $m_{\rm BM}$ of Γ is finite. □

Note that as in [DaOP], when $(\mathbb{X},\lambda)$ (or its minimal nonempty Γ-invariant subtree) is not assumed to be uniform, there are examples of Γ that are lattices (or are geometrically finite) whose Bowen-Margulis measure $m_{\rm BM}$ is infinite, see Section 15.5 for more details.

Assume till the end of this Section that $(\mathbb{X},\lambda)$ is simplicial, that is, that $\lambda \equiv 1$. Let us now discuss the mixing properties of the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ for the Gibbs measure m_F.

[15] See Section 2.6.

Let L_Γ be the *length spectrum* of Γ, which is, in the present simplicial case, the subgroup of $\mathbb{Z}$ generated by the translation lengths in $\mathbb{X}$ of the elements of Γ.

Recall that $x_0 \in V\mathbb{X}$ is a fixed basepoint. Let $V_{\rm even}\mathbb{X} = \{x \in V\mathbb{X} \;:\; d(x,x_0) = 0 \mod 2\}$ be the set of vertices of $\mathbb{X}$ at an even distance from the basepoint x_0, and let $V_{\rm odd}\mathbb{X} = V\mathbb{X} - V_{\rm even}\mathbb{X}$. Let $\mathscr{G}_{\rm even}\mathbb{X}$ (respectively $\check{\mathscr{G}}_{\rm even}\mathbb{X}$) be the subset of $\mathscr{G}\mathbb{X}$ (respectively $\check{\mathscr{G}}\mathbb{X}$) that consists of the geodesic lines (respectively generalised geodesic lines) in $\mathbb{X}$ whose origin is in $V_{\rm even}\mathbb{X}$.

Recall[16] that a discrete time one-parameter group $(h^n)_{n\in\mathbb{Z}}$ of homeomorphisms of a topological space Z is *topologically mixing* if for all nonempty open subsets U, V of Z, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $U \cap h^n(V) \neq \varnothing$.

Recall that given a measured space (Z, m), with m nonzero and finite, endowed with a discrete time one-parameter group $(h^n)_{n\in\mathbb{Z}}$ of measure-preserving transformations, the measure m is *mixing* under $(h^n)_{n\in\mathbb{Z}}$ if for all $f, g \in \mathbb{L}^2(Z, m)$, we have

$$\lim_{n\to+\infty} \int_Z f\; g\circ h^n\; dm = \frac{1}{\|m\|}\int_Z f\; dm \int_Z g\; dm\;,$$

or equivalently if for every $g \in \mathbb{L}^2(Z, m)$, the functions $g \circ h^n$ weakly converge in the Hilbert space $\mathbb{L}^2(Z, m)$ to the constant function $\frac{1}{\|m\|}\int_Z g\; dm$ as $n \to +\infty$.

Theorem 4.17. *Assume that the smallest nonempty Γ-invariant simplicial subtree of $\mathbb{X}$ is uniform, without vertices of degree 2, and that m_F is finite. Then the following assertions are equivalent:*

- *the length spectrum of Γ satisfies $L_\Gamma = \mathbb{Z}$;*
- *the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ is topologically mixing on its nonwandering set;*
- *the quotient graph $\Gamma\backslash\mathbb{X}$ is not bipartite;*
- *the Gibbs measure m_F is mixing under the discrete time geodesic flow $(\mathtt{g}^t)_{t\in\mathbb{Z}}$ on $\Gamma\backslash\mathscr{G}\mathbb{X}$.*

Otherwise $L_\Gamma = 2\mathbb{Z}$, and the square of the discrete time geodesic flow $(\mathtt{g}^{2t})_{t\in\mathbb{Z}}$ is topologically mixing on the nonwandering subset of $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ and the restriction of the Gibbs measure m_F to $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ is mixing under $(\mathtt{g}^{2t})_{t\in\mathbb{Z}}$.

Proof. The nonwandering set of $(\mathtt{g}^t)_{t\in\mathbb{Z}}$ on $\Gamma\backslash\mathscr{G}\mathbb{X}$ is $\Gamma\backslash\{\ell \in \mathscr{G}\mathbb{X} \;:\; \ell_\pm \in \Lambda\Gamma\}$, and the nonwandering set of $(\mathtt{g}^{2t})_{t\in\mathbb{Z}}$ on $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ is

$$\Omega_{\rm even} = \Gamma\backslash\{\ell \in \mathscr{G}_{\rm even}\mathbb{X} \;:\; \ell_\pm \in \Lambda\Gamma\}\;.$$

Since the translation axis of any loxodromic element of Γ is contained in the convex hull of the limit set, we may hence assume that the geometric realisation of $\mathbb{X}$ is equal to $\mathscr{C}\Lambda\Gamma$.

Lemma 4.18. *If $\mathbb{X}$ is a locally finite tree without vertices of degree 2, if Γ is a nonelementary discrete sugbroup of $\operatorname{Aut}(\mathbb{X})$ such that $\mathbb{X}$ is* tree-minimal *(that is, does not contain a Γ-invariant proper nonempty subtree), then the length spectrum L_Γ of Γ is equal either to $\mathbb{Z}$ or to $2\mathbb{Z}$, and equal to $2\mathbb{Z}$ if and only if the quotient graph $\Gamma\backslash\mathbb{X}$ is bipartite.*

Proof. This lemma is essentially due to [GaL]. By for instance [Pau1, Lem. 4.3], since $\mathbb{X}$ is tree-minimal, every geodesic segment (and in particular any two consecutive edges) is contained in the translation axis of a loxodromic element of Γ. Hence if x and y are the two endpoints of any edge e of $\mathbb{X}$, since they have degree at least 3, there exist at least two loxodromic elements α and β of Γ such that the translation axes Ax_α and Ax_β contain x and

[16] See above Theorem 4.9 for the continuous-time version

y respectively, but do not meet the interior of the edge e, so that $d(x, y) = d(\mathrm{Ax}_\alpha, \mathrm{Ax}_\beta)$. In particular Ax_α and Ax_β are disjoint, which implies by for instance [Pau1, Prop. 1.6] that the translation lengths of α, β and $\alpha\beta$ satisfy

$$\lambda(\alpha\beta) = \lambda(\alpha) + \lambda(\beta) + 2\, d(\mathrm{Ax}_\alpha, \mathrm{Ax}_\beta)\ .$$

Hence $2 = 2\, d(x, y) = \lambda(\alpha\beta) - \lambda(\alpha) - \lambda(\beta) \in L_\Gamma$. Therefore $2\mathbb{Z} \subset L_\Gamma \subset \mathbb{Z}$, and either $L_\Gamma = \mathbb{Z}$ or $L_\Gamma = 2\mathbb{Z}$.

Note that for all vertices x, y, z in a simplicial tree, if $d(x, y)$ and $d(y, z)$ are both even or both odd, then

$$d(x, z) = d(x, y) + d(y, z) - 2\, d(y, [x, z]) = 0 \mod 2\ . \tag{4.15}$$

Note that for all $x \in V\mathbb{X}$ and $\gamma \in \Gamma$, we have

$$d(x, \gamma x) = \lambda(\gamma) \mod 2\ . \tag{4.16}$$

Indeed, if γ is loxodromic, then $d(x, \gamma x) = \lambda(\gamma) + 2\ d(x, \mathrm{Ax}_\gamma)$ and otherwise, $d(x, \gamma x) = 2\ d(x, \mathrm{Fix}(\gamma))$ where $\mathrm{Fix}(\gamma)$ is the set of fixed points of the elliptic element γ. For future use, this proves that the following assertions are equivalent :

$$\begin{aligned} &(1) \quad L_\Gamma \subset 2\mathbb{Z} \\ &(2) \quad \forall\, x \in X,\ \ \forall\, \gamma \in \Gamma, \quad d(x, \gamma x) \in 2\mathbb{Z}\ . \end{aligned} \tag{4.17}$$

Assume that $L_\Gamma = 2\mathbb{Z}$. Then $V_{\mathrm{even}}\mathbb{X}$ (hence $V_{\mathrm{odd}}\mathbb{X}$) is Γ-invariant, since for all $x \in V_{\mathrm{even}}\mathbb{X}$, the distance $d(x, \gamma x)$ is even by Equation (4.16), and $d(\gamma x, \gamma x_0) = d(x, x_0)$ is even, so that $d(\gamma x, x_0)$ is even by Equation (4.15). Since no edge of $\mathbb{X}$ has both endpoints in $V_{\mathrm{even}}\mathbb{X}$, this proves that $\Gamma\backslash\mathbb{X}$ is bipartite, with partition of its set of vertices $(\Gamma\backslash V_{\mathrm{even}}\mathbb{X}) \sqcup (\Gamma\backslash V_{\mathrm{odd}}\mathbb{X})$.

Assume conversely that $\Gamma\backslash\mathbb{X}$ is bipartite. The set $V_{\mathrm{even}}\mathbb{X}$, which is the lift of one of the two elements of the partition of its vertices by the canonical projection $V\mathbb{X} \to \Gamma\backslash V\mathbb{X}$, is Γ-invariant. By Equation (4.16), this proves that $L_\Gamma \subset 2\mathbb{Z}$, hence that $L_\Gamma = 2\mathbb{Z}$. □

The equivalence of the first, second and fourth claims in the statement of Theorem 4.17 follows from a discrete time version with potential of [Rob2, Theo. 3.1] or a discrete time version of [Bab1, Theo. 1] (which can be extended to CAT(−1) spaces by the remark in [Bab1, page 70]). It can also be recovered from the following arguments when $L_\Gamma = 2\mathbb{Z}$, and we prefer to concentrate on this case, since it requires a lot of modifications and is stated with almost no proof in [BrP2, Prop. 3.3], and only when $\widetilde{F} = 0$.

Assume from now on that $L_\Gamma = 2\mathbb{Z}$. Since $V_{\mathrm{even}}\mathbb{X}$ is Γ-invariant as seen above, and since $d(\ell(0), \mathtt{g}^{2s}\ell(0)) = 2|s|$ is even for all $\ell \in \mathscr{G}\mathbb{X}$ and $s \in \mathbb{Z}$, it follows from the definition of $\mathscr{G}_{\mathrm{even}}\mathbb{X} = \{\ell \in \mathscr{G}\mathbb{X}\ :\ \pi(\ell) \in V_{\mathrm{even}}\mathbb{X}\}$ and from Equation (4.15) that $\mathscr{G}_{\mathrm{even}}\mathbb{X}$ is invariant under the even discrete time geodesic flow $(\mathtt{g}^{2s})_{s\in\mathbb{Z}}$ and under Γ. Note that the discrete Hopf parametrisation of $\mathscr{G}\mathbb{X}$ gives a homeomorphism from $\mathscr{G}_{\mathrm{even}}\mathbb{X}$ to

$$\big\{(\xi, \eta, t) \in \partial^2_\infty\mathbb{X} \times \mathbb{Z}\ :\ t = d(x_0,]\xi, \eta[\,) \mod 2\big\}\ .$$

The restriction of the Gibbs measure $\widetilde{m}_F$ to $\mathscr{G}_{\mathrm{even}}\mathbb{X}$, that we will again denote by $\widetilde{m}_F$, disintegrates by the projection on the first factor $\partial^2_\infty\mathbb{X} \times \mathbb{Z} \to \partial^2_\infty\mathbb{X}$ over the geodesic current $\widehat{m}_F$ where, for every $(\xi, \eta) \in \partial^2_\infty\mathbb{X}$ and (any) $x \in\,]\xi, \eta[$,

$$d\widehat{m}_F(\xi, \eta) = e^{C^-_\xi(x_0, x) + C^+_\eta(x_0, x)} d\mu^-_{x_0}(\xi)\, d\mu^+_{x_0}(\eta)\ , \tag{4.18}$$

with conditional measure on the fiber over (ξ,η) the counting measure on the discrete set $\{t \in \mathbb{Z} \;:\; t = d(x_0,]\xi,\eta[\,) \mod 2\}$. Since m_F is finite and invariant under the discrete time geodesic flow, it is conservative by Poincaré's recurrence theorem. Hence the measure quasi-preserving action of Γ on the measured space $(\partial^2_\infty \mathbb{X}, \widehat{m}_F)$ is ergodic by (the discrete time version of) Theorem 4.6.

Since the distance between two points in a horosphere of a simplicial tree is even, and again by Equation (4.15), every horosphere of $\mathbb{X}$ is either entirely contained in $V_{\rm even}\mathbb{X}$ or entirely contained in $V_{\rm odd}\mathbb{X}$. For every $\ell \in \mathscr{G}_{\rm even}\mathbb{X}$, its strong stable/unstable leaf

$$W^{\pm}(\ell) = \{\ell' \in \mathscr{G}\mathbb{X} \;:\; \lim_{t\to\pm\infty} d(\ell(t),\ell'(t)) = 0\}$$

is contained in $\mathscr{G}_{\rm even}\mathbb{X}$, since the image by the footpoint projection of a strong stable/instable leaf is a horosphere (see Equation (2.14)). Thus $\mathscr{G}_{\rm even}\mathbb{X}$ is saturated by the partition into strong stable/instable leaves of $\mathscr{G}\mathbb{X}$.

We now follow rather closely the arguments of [Bab1, Theo. 1] in order to prove the last claims of Theorem 4.17, the main point being that the geodesic current $\widehat{m}_F$ is a quasi-product measure.

The following lemmas are particular cases of respectively Lemma 1 and Fact page 64 of [Bab1], valid for general finite measure preserving dynamical systems, applied, with the notation of loc. cit, to $(T_t)_{t\in A} = (\mathsf{g}^{2t})_{t\in\mathbb{Z}}$.

Lemma 4.19. *Let $f \in \mathbb{L}^2(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}, m_F)$ be such that $\int f\, dm_F = 0$. If there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ such that $f\circ\mathsf{g}^{2t_n}$ does not converge to 0 for the weak topology on $\mathbb{L}^2(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}, m_F)$, then there exist an increasing sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ and a nonconstant[17] element $f^* \in \mathbb{L}^2(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}, m_F)$ such that $f\circ\mathsf{g}^{2s_n}$ and $f\circ\mathsf{g}^{-2s_n}$ both converge to f^* for the weak topology on $\mathbb{L}^2(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}, m_F)$.* □

Lemma 4.20. *If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{L}^2(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}, m_F)$ weakly converging to f^* in $\mathbb{L}^2(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}, m_F)$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that the Cesàro averages $\frac{1}{N^2}\sum_{k=0}^{N^2-1} f_{n_k}$ converge pointwise almost everywhere to f^* as $N\to+\infty$.* □

Recall that the support of m_F is the nonwandering set $\Omega_{\rm even}$. Assume for a contradiction that the restriction of m_F to $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ is not mixing under the even discrete time geodesic flow. Then there exists a continuous function f with compact support on $\Omega_{\rm even}$ such that $\int f\, dm_F = 0$ and $(f\circ\mathsf{g}^{2n})_{n\in\mathbb{N}}$ does not weakly converge to 0 in $\mathbb{L}^2(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}, m_F)$. By Lemmas 4.19 and 4.20, there exist a nonconstant element $f^* \in \mathbb{L}^2(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}, m_F)$ and increasing sequences $(n_k^{\pm})_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\frac{1}{N^2}\sum_{k=0}^{N^2-1} f\circ\mathsf{g}^{\pm 2n_k^{\pm}}$ pointwise almost everywhere converges to f^* as $N\to+\infty$.

Let $\widetilde{f}^* = f^*\circ p_{\rm even}$, where $p_{\rm even} : \mathscr{G}_{\rm even}\mathbb{X} \to \Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ is the canonical projection, be the lift of f^* to $\mathscr{G}_{\rm even}\mathbb{X}$. Since the conditional measures for the disintegration of $\widetilde{m}_F$ over $\widehat{m}_F$ are counting measures on countable sets, there exists a full $\widehat{m}_F$-measure subset E_0 of $\partial^2_\infty\mathbb{X}$ such that, for every $\ell \in \mathscr{G}_{\rm even}\mathbb{X}$ with $(\ell_-,\ell_+) \in E_0$, the above convergences hold after lifting to $\mathscr{G}_{\rm even}\mathbb{X}$ at the points $\mathsf{g}^{2n}\ell$ for all $n\in\mathbb{Z}$.

For every $\ell \in \mathscr{G}_{\rm even}\mathbb{X}$, the subgroup A_ℓ of $2\mathbb{Z}$ given by the periods of the map $2n \mapsto \widetilde{f}^*(\mathsf{g}^{2n}\ell)$ only depends on (ℓ_-,ℓ_+). Thus, we have a measurable map from E_0 into the (discrete) set

[17] Recall that an element of $\mathbb{L}^2(Z,m)$ is nonconstant if any representative function is not almost everywhere constant.

of subgroups of $2\mathbb{Z}$, which is Γ-invariant, hence is constant $\widehat{m}_F$-almost everywhere by the ergodicity of $\widehat{m}_F$ under Γ.

Assume for a contradiction that this almost everywhere constant subgroup is $2\mathbb{Z}$, that is, that the values of $\widetilde{f}^*$ almost everywhere do not depend on the time parameter in the discrete Hopf parametrisation of $\mathscr{G}_{\rm even}\mathbb{X}$. Then $\widetilde{f}^*$ defines a Γ-invariant measurable function on $\partial^2_\infty\mathbb{X}$. Again by ergodicity, this function is almost everywhere constant, contradicting the fact that f^* is not almost everywhere constant.

Hence there exist a full $\widehat{m}_F$-measure subset E_1 of E_0 and $\kappa\in\mathbb{N}-\{0,1\}$ such that $A_\ell = 2\,\kappa\,\mathbb{Z}$ for every $\ell\in\mathscr{G}_{\rm even}\mathbb{X}$ with $(\ell_-,\ell_+)\in E_1$. Let us finally prove that L_Γ is contained in $2\,\kappa\,\mathbb{Z}$, which contradicts the original assumption that $L_\Gamma = 2\mathbb{Z}$.

Let $\widetilde{f}^\pm = \limsup_{N\rightarrow+\infty}\ \frac{1}{N^2}\sum_{k=0}^{N^2} f\circ \mathsf{g}^{\pm 2n_k^\pm}\circ p_{\rm even}$, so that the set

$$E = \{(\xi,\eta)\in E_1\ :\ \forall\ \ell\in\mathscr{G}_{\rm even}\mathbb{X},\ \text{if}\ \ell_- = \xi\ \text{and}\ \ell_+ = \eta,\ \text{then}\ \widetilde{f}^+(\ell) = \widetilde{f}^-(\ell) = \widetilde{f}^*(\ell)\}$$

has full $\widehat{m}_F$-measure. By the hyperbolicity of the geodesic flow (see Equation (2.10)) and the uniform continuity of f, the map $\widetilde{f}^+$ is constant along any strong stable leaf of $\mathscr{G}_{\rm even}\mathbb{X}$ and $\widetilde{f}^-$ is constant along any strong unstable leaf. Let

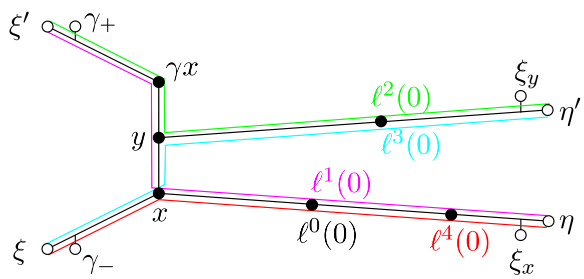
$$E^- = \{\xi\in\Lambda\Gamma\ :\ (\xi,\eta')\in E\ \text{for}\ \mu^+_{x_0}\text{-almost every}\ \eta'\in\Lambda\Gamma\}$$

and

$$E^+ = \{\eta\in\Lambda\Gamma\ :\ (\xi',\eta)\in E\ \text{for}\ \mu^-_{x_0}\text{-almost every}\ \xi'\in\Lambda\Gamma\}\ .$$

Since $\widehat{m}_F$ is in the same measure class as the product measure $\mu^-_{x_0}\otimes\mu^+_{x_0}$ (see Equation (4.18)), and by Fubini's theorem, we have $\mu^-_{x_0}(\,{}^cE^-) = 0$ and $\mu^+_{x_0}(\,{}^cE^+) = 0$, and the set $E^-\times E^+$ has full $\widehat{m}_F$-measure.

Let γ be a loxodromic element of Γ, and let x be any vertex of $\mathbb{X}$ on the translation axis of γ. Since $d(x,\gamma x)$ is even (see Equation (4.17)), the midpoint y of the geodesic segment $[x,\gamma x]$ is a vertex of $\mathbb{X}$. Since x and y have degree at least 3, there exist ξ_x and ξ_y in $\partial_\infty X$ whose closest points on the translation axis of γ are respectively x and y. Note that $\xi_x,\xi_y\in\Lambda\Gamma$ as $X = \mathscr{C}\Lambda\Gamma$. Since y and x are the closest points to γ_+ and γ_- on the geodesic line $]\xi_x,\xi_y[$, we have[18] $[\![\xi_x,\gamma_+,\xi_y,\gamma_-]\!] = d(x,y)$.



Since $E\cap(E^-\times E^+)$ has full $\widehat{m}_F$-measure, there exists $(\xi,\eta)\in E\cap(E^-\times E^+)$ arbitrarily close to (γ_-,ξ_x). Since the set

$$\{(\xi',\eta')\in\partial^2_\infty\Gamma\ :\ (\xi,\eta'),(\xi',\eta),(\xi',\eta')\in E\}$$

[18] See Section 2.6 for the definition of the crossratio of an ordered quadruple of pairwise distinct points in $\partial_\infty X$.

has full $\widehat{m}_F$-measure, there exists such a (ξ', η') arbitrarily close to (γ_+, ξ_y). Let $\ell^0 \in \mathscr{G}_{\rm even}\mathbb{X}$ be such that $\ell^0_- = \xi$ and $\ell^0_+ = \eta$. Let $\ell^1 \in W^+(\ell^0)$ be such that $\ell^1_- = \xi'$. Let $\ell^2 \in W^-(\ell^1)$ be such that $\ell^2_+ = \eta'$. Let $\ell^3 \in W^+(\ell^2)$ be such that $\ell^3_- = \xi$. Finally, let $\ell^4 \in W^-(\ell^3)$ be such that $\ell^4_+ = \eta$. Then $\ell^4 = \mathsf{g}^{2s}\ell^0$ for some $s \in \mathbb{Z}$ with $2|s| = d(\ell^0(0), \ell^4(0))$.

By the definition of E and since $\widetilde{f}^+$ (resp. $\widetilde{f}^-$) is constant along the strong stable (resp. unstable) leaves, we have

$$\begin{aligned}\widetilde{f}^*(\ell^0) = \widetilde{f}^+(\ell^0) = \widetilde{f}^+(\ell^1) = \widetilde{f}^-(\ell^1) = \widetilde{f}^-(\ell^2) = \widetilde{f}^+(\ell^2) = \widetilde{f}^+(\ell^3)\\ = \widetilde{f}^-(\ell^3) = \widetilde{f}^-(\ell^4) = \widetilde{f}^*(\ell^4) = \widetilde{f}^*(\mathsf{g}^{2\,s}\ell^0)\;.\end{aligned}$$

Hence $2\,|s|$ is a period in A_{ℓ^0}, thus is contained in $2\,\kappa\,\mathbb{Z}$.

If $t > 0$ is large enough, we have

$$\ell^0(t) = \ell^1(t), \quad \ell^1(-t) = \ell^2(-t), \quad \ell^2(t) = \ell^3(t), \quad \ell^3(-t) = \ell^4(-t)\;,$$

which respectively tend to η, ξ', η', ξ as $t \to +\infty$. Since the crossratio is locally constant and by its properties, in particular its definition in Equation (2.21), we have

$$\begin{aligned}\ell(\gamma) &= d(x, \gamma x) = 2\,d(x, y) = 2\,[\![\xi_x, \gamma_+, \xi_y, \gamma_-]\!] = 2\,[\![\eta, \xi', \eta', \xi]\!]\\ &= \lim_{t\to+\infty} d(\ell^0(t), \ell^3(-t)) - d(\ell^3(-t), \ell^2(t)) + d(\ell^2(t), \ell^1(-t)) - d(\ell^1(-t), \ell^0(t))\\ &= \lim_{t\to+\infty} d(\ell^0(t), \ell^3(-t)) - d(\ell^1(-t), \ell^1(t)) = \lim_{t\to+\infty} d(\ell^0(t), \ell^4(-t)) - 2\,t\\ &= d(\ell^0(0), \ell^4(0)) = 2\,|s| \in 2\,\kappa\,\mathbb{Z}\;.\end{aligned}$$

Thus $L_\Gamma \subset 2\,\kappa\,\mathbb{Z}$, which contradicts the fact that $L_\Gamma = 2\mathbb{Z}$. □

By Proposition 4.16, the general assumptions of Theorem 4.17 are satisfied if $\mathbb{X}$ is uniform, without vertices of degree 2, Γ is a lattice of $\mathbb{X}$ and $\widetilde{F} = 0$. Thus, if we assume furthermore that $\Gamma\backslash\mathbb{X}$ is not bipartite, then the Bowen-Margulis measure $m_{\rm BM}$ of Γ is mixing under the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$.

Chapter 5

Symbolic dynamics of geodesic flows on trees

In this Chapter, we give a coding of the discrete time geodesic flow on the nonwandering sets of quotients of locally finite simplicial trees $\mathbb{X}$ without terminal vertices by nonelementary discrete subgroups of $\operatorname{Aut}(\mathbb{X})$ by a subshift of finite type on a countable alphabet. Similarly we give a coding of the continuous time geodesic flow on the nonwandering sets of quotients of locally finite metric trees $(\mathbb{X}, \lambda)$ without terminal vertices by nonelementary discrete subgroups of $\operatorname{Aut}(\mathbb{X}, \lambda)$ by suspensions of such subshifts. These codings are used in Section 5.4 to prove the variational principle in both contexts, and in Sections 9.2 and 9.3 to obtain rates of mixing of the flows.

5.1 Two-sided topological Markov shifts

In this short and independent Section, that will be used in Sections 5.2, 5.3, 5.4, 9.2 and 9.3, we recall some definitions concerning symbolic dynamics on countable alphabets.[1]

A (two-sided) topological *Markov shift*[2] is a topological dynamical system (Σ, σ) constructed from a countable discrete *alphabet* $\mathscr{A}$ and a *transition matrix* $A = (A_{i,j})_{i,j \in \mathscr{A}} \in \{0,1\}^{\mathscr{A} \times \mathscr{A}}$, where Σ is the closed subset of the topological product space $\mathscr{A}^{\mathbb{Z}}$ defined by

$$\Sigma = \big\{x = (x_n)_{n \in \mathbb{Z}} \in \mathscr{A}^{\mathbb{Z}} \;:\; \forall\, n \in \mathbb{Z}, \quad A_{x_n, x_{n+1}} = 1\big\}\,,$$

and $\sigma : \Sigma \to \Sigma$ is the (two-sided) *shift* defined by

$$(\sigma(x))_n = x_{n+1}$$

for all $x \in \Sigma$ and $n \in \mathbb{Z}$. Note that to be given $(\mathscr{A}, A)$ is equivalent to be given an oriented graph with countable set of vertices $\mathscr{A}$ (and set of oriented edges a subset of $\mathscr{A} \times \mathscr{A}$) and with incidence matrix A such that $A_{i,j} = 1$ if there is an oriented edge from the vertex i to the vertex j and $A_{i,j} = 0$ otherwise.

[1] See for instance [Kit, Sar2].

[2] Note that the terminology could be misleading, a topological Markov shift comes a prori without a measure, and many probability measures invariant under the shift do not satisfy the Markov chain property that the probability to pass from one state to another depends only on the previous state, not of all past states.

For all $p \leqslant q$ in $\mathbb{Z}$, a finite sequence $(a_n)_{p\leqslant n\leqslant q} \in \mathscr{A}^{\{p,\dots,q\}}$ is *admissible* (or A-*admissible* when we need to make A precise) if $A_{a_n,\,a_{n+1}} = 1$ for all $n \in \{p, \dots, q-1\}$. A topological Markov shift is *transitive* if for all $x, y \in \mathscr{A}$, there exists an admissible finite sequence $(a_n)_{p\leqslant n\leqslant q}$ with $a_p = x$ and $a_q = y$. This is equivalent to requiring the dynamical system (Σ, σ) to be *topologically transitive*: for all nonempty open subsets U, V in Σ, there exists $n \in \mathbb{Z}$ such that $U \cap \sigma^n(V) \neq \varnothing$.

Note that the product space $\mathscr{A}^{\mathbb{Z}}$ is not locally compact when $\mathscr{A}$ is infinite. When the matrix A has only finitely many nonzero entries on each line and each colum, then (Σ, σ) is also called a *subshift of finite type (on a countable alphabet)*. The topological space Σ is then locally compact: By diagonal extraction, for all $p \leqslant q$ in $\mathbb{Z}$ and $a_p, a_{p+1}, \dots, a_{q-1}, a_q$ in $\mathscr{A}$, every *cylinder*

$$[a_p, a_{p+1}, \dots, a_{q-1}, a_q] = \big\{(x_n)_{n\in\mathbb{Z}} \in \Sigma \;:\; \forall\, n \in \{p, \dots, q\},\;\; x_n = a_n\big\}$$

is a compact open subset of Σ.

Given a continuous map $F_{\rm symb} : \Sigma \to \mathbb{R}$ and a constant $c_{F_{\rm symb}} \in \mathbb{R}$, we say that a measure $\mathbb{P}$ on Σ, invariant under the shift σ, *satisfies the Gibbs property*[3] *with Gibbs constant* $c_{F_{\rm symb}}$ *for the potential* $F_{\rm symb}$ if for every finite subset E of the alphabet $\mathscr{A}$, there exists $C_E \geqslant 1$ such that for all $p \leqslant q$ in $\mathbb{Z}$ and for every $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma$ such that $x_p, x_q \in E$, we have

$$\frac{1}{C_E} \leqslant \frac{\mathbb{P}([x_p, x_{p+1}, \dots, x_{q-1}, x_q])}{e^{-c_{F_{\rm symb}}(q-p+1)+\sum_{n=p}^{q} F_{\rm symb}(\sigma^n x)}} \leqslant C_E\,. \tag{5.1}$$

Two continuous maps $F_{\rm symb}, F'_{\rm symb} : \Sigma \to \mathbb{R}$ are *cohomologous* if there exists a continuous map $G : \Sigma \to \mathbb{R}$ such that

$$F'_{\rm symb} - F_{\rm symb} = G \circ \sigma - G\,.$$

5.2 Coding discrete time geodesic flows on simplicial trees

Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, with $X = |\mathbb{X}|_1$ its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X})$, and let $\widetilde{F} : T^1X \to \mathbb{R}$ be a potential for Γ.

In this Section, we give a coding of the discrete time geodesic flow $(\mathsf{g}^t)_{t\in\mathbb{Z}}$ on the nonwandering subset of $\Gamma\backslash\mathscr{G}\mathbb{X}$ by a locally compact transitive (two-sided) topological Markov shift. This explicit construction will be useful later on in order to study the variational principle (see Section 5.4) and rates of mixing (see Section 9.2).

The main technical aspect of this construction, building on [BrP2, §6], is to allow the case when Γ has torsion. When Γ is torsion free and $\Gamma/\mathbb{X}$ is finite, the construction is well-known, we refer for instance to [CoP6] for a more general setting when the potential is 0. In order to consider for instance nonuniform tree lattices, it is important to allow torsion in Γ. Our direct approach also avoids the assumption that the discrete subgroup Γ is *full*, that is, equal to the subgroup consisting of the elements $g \in \operatorname{Aut}(\mathbb{X})$ such that $p \circ g = p$ where $p : \mathbb{X} \to \Gamma\backslash\mathbb{X}$ is the canonical projection, as in [Kwo] (building on [BuM, 7.3]).

Let $\mathbb{X}'$ be the minimal nonempty Γ-invariant simplicial subtree of $\mathbb{X}$, whose geometric realisation is $\mathscr{C}\Lambda\Gamma$. Since we are only interested in the support of the Gibbs measures,

[3] Note that some references have a stronger notion of Gibbs measure (see for instance [Sar1]), with the constant C independent of E.

we will only code the geodesic flow on the nonwandering subset $\Gamma\backslash\mathbb{X}'$ of $\Gamma\backslash\mathscr{G}\mathbb{X}$. The same construction works with the full space $\Gamma\backslash\mathscr{G}\mathbb{X}$, but the resulting Markov shift is then not necessarily transitive.

Let $(\mathbb{Y}, G_*) = \Gamma\backslash\backslash\mathbb{X}'$ be the quotient graph of groups of $\mathbb{X}'$ by Γ (see for instance Example 2.10), and let $p : \mathbb{X}' \to \mathbb{Y} = \Gamma\backslash\mathbb{X}'$ be the canonical projection. We denote by $[1] = H$ the trivial double coset in any double coset set $H\backslash G/H$ of a group G by a subgroup H.

We consider the alphabet $\mathscr{A}$ consisting of the triples (e^-, h, e^+) where

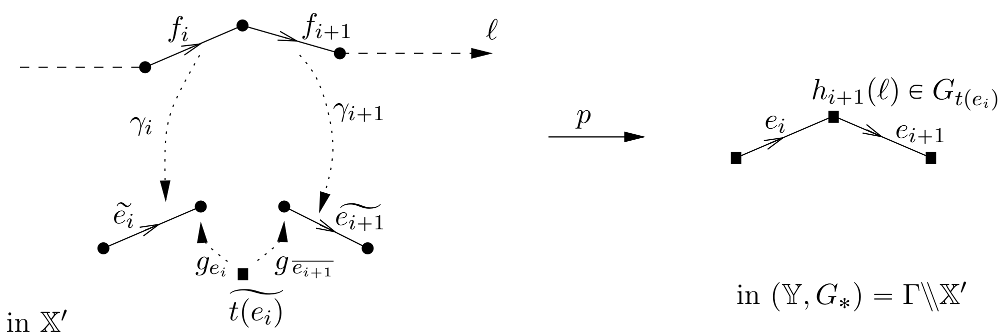
- $e^\pm \in E\mathbb{Y}$ satisfy $t(e^-) = o(e^+)$ and
- $h \in \rho_{e^-}(G_{e^-})\backslash G_{o(e^+)}/\rho_{\overline{e^+}}(G_{e^+})$ satisfy $h \neq [1]$ if $\overline{e^+} = e^-$.

This set is countable (and finite if and only if the quotient graph $\Gamma\backslash\mathbb{X}'$ is finite), we endow it with the discrete topology. We consider the (two-sided) topological Markov shift with alphabet $\mathscr{A}$ and transition matrix $A_{(e^-,\,h,\,e^+),\,(e'^-,\,h',\,e'^+)} = 1$ if $e^+ = e'^-$ and 0 otherwise. Note that this matrix $A = (A_{i,j})_{i,j\in\mathscr{A}}$ has only finitely many nonzero entries on each line and each column, since $\mathbb{X}'$ is locally finite and Γ has finite vertex stabilisers in $\mathbb{X}'$. We consider the subspace

$$\Sigma = \big\{(e_i^-, h_i, e_i^+)_{i\in\mathbb{Z}} \in \mathscr{A}^{\mathbb{Z}} \;:\; \forall\, i \in \mathbb{Z},\;\; e_{i-1}^+ = e_i^-\big\}$$

of the product space $\mathscr{A}^{\mathbb{Z}}$, and the shift $\sigma : \Sigma \to \Sigma$ defined by $(\sigma(x))_i = x_{i+1}$ for all $(x_i)_{i\in\mathbb{Z}}$ in Σ and i in $\mathbb{Z}$. As seen above, Σ is locally compact.

Let us now construct a natural coding map Θ from $\Gamma\backslash\mathscr{G}\mathbb{X}'$ to Σ, by slightly modifying the construction of [BrP2, §6].



For every discrete geodesic line $\ell \in \mathscr{G}\mathbb{X}'$, for every $i \in \mathbb{Z}$, let $f_i = f_i(\ell)$ be the edge of $\mathbb{X}'$ whose geometric realisation is $\ell([i, i+1])$ with origin $f(i)$ and endpoint $f(i+1)$, and let $e_i = p(f_i)$, which is an edge in $\mathbb{Y}$. Let us use the notation of Example 2.10: we fix lifts $\widetilde{e}$ and $\widetilde{v}$ of every edge e and vertex v of $\mathbb{Y}$ in $\mathbb{X}'$ such that $\widetilde{\overline{e}} = \overline{\widetilde{e}}$, and elements $g_e \in \Gamma$ such that $g_e\,\widetilde{t(e)} = t(\widetilde{e})$. Since $p(\widetilde{e_i}) = e_i = p(f_i)$, there exists $\gamma_i = \gamma_i(\ell) \in \Gamma$, well defined up to multiplication on the left by an element of $G_{e_i} = \Gamma_{\widetilde{e_i}}$, such that $\gamma_i f_i = \widetilde{e_i}$ for all $i \in \mathbb{Z}$.

We define $e_{i+1}^-(\ell) = e_i$, $e_{i+1}^+(\ell) = e_{i+1}$, and

$$h_{i+1}(\ell) = g_{e_{i+1}^-(\ell)}^{-1}\,\gamma_i(\ell)\,\gamma_{i+1}(\ell)^{-1}\,g_{\overline{e_{i+1}^+(\ell)}}\,. \tag{5.2}$$

Since for every edge e of $\mathbb{Y}$ the structural monomorphism

$$\rho_e : G_e = \Gamma_{\widetilde{e}} \;\longrightarrow\; G_{t(e)} = \Gamma_{\widetilde{t(e)}}$$

is the map $g \mapsto g_e^{-1} g g_e$, the double coset of $h_i(\ell)$ in $\rho_{e_i^-(\ell)}(G_{e_i^-(\ell)})\backslash G_{o(e_i^+(\ell))}/\rho_{\overline{e_i^+(\ell)}}(G_{e_i^+(\ell)})$ does not depend on the choice of the γ_i's, and we again denote it by $h_i(\ell)$.

The next result shows that, assuming only that Γ is discrete and nonelementary, the time-one discrete geodesic flow $\mathtt{g}^1$ on its nonwandering subset of $\Gamma\backslash\mathscr{G}\mathbb{X}$ is topologically conjugate to a locally compact transitive (two-sided) topological Markov shift.

Theorem 5.1. *With $\mathbb{X}' = \mathscr{C}\Lambda\Gamma$, the map $\Theta : \Gamma\backslash\mathscr{G}\mathbb{X}' \to \Sigma$ defined by*

$$\Gamma\ell \mapsto (e_i^-(\ell), h_i(\ell), e_i^+(\ell))_{i\in\mathbb{Z}}$$

is a homeomorphism which conjugates the time-one discrete geodesic flow $\mathtt{g}^1$ and the shift σ, that is, the following diagram commutes

$$\begin{array}{ccc}
\Gamma\backslash\mathscr{G}\mathbb{X}' & \xrightarrow{\ \mathtt{g}^1\ } & \Gamma\backslash\mathscr{G}\mathbb{X}' \\
\Theta\downarrow & & \downarrow\Theta \\
\Sigma & \xrightarrow{\ \sigma\ } & \Sigma\,,
\end{array}$$

and the topological Markov shift (Σ, σ) is locally compact and transitive.

Furthermore, if we endow $\Gamma\backslash\mathscr{G}\mathbb{X}'$ with the quotient distance of

$$d(\ell,\ell') = e^{-\sup\left\{n\in\mathbb{N}\,:\ \ell|_{[-n,n]} = \ell'|_{[-n,n]}\right\}}$$

on $\mathscr{G}\mathbb{X}'$ and Σ with the distance

$$d(x,x') = e^{-\sup\left\{n\in\mathbb{N}\,:\ \forall\, i\in\{-n,\dots,n\},\ x_i = x'_i\right\}}\ ,$$

then Θ is a bilipschitz homeomorphism.

Finally, if $\mathbb{X}'$ is a uniform tree without vertices of degree at most 2, if the Gibbs measure m_F of Γ is finite, and if the length spectrum L_Γ of Γ is equal to $\mathbb{Z}$, then the topological Markov shift (Σ, σ) is topologically mixing.

Note that when $\mathbb{Y}$ is finite (or equivalently when Γ is cocompact), the alphabet $\mathscr{A}$ is finite (hence (Σ, σ) is a standard subshift of finite type). When furthermore the vertex groups of $(\mathbb{Y}, G_*)$ are trivial (or equivalently when Γ acts freely, and in particular is a finitely generated free group), this result is well-known, but it is new if the vertex groups are not trivial. Compare with the construction of [CoP6], whose techniques might be applied since Γ is word-hyperbolic if $\mathbb{Y}$ is finite, up to replacing Gromov's (continuous time) geodesic flow of Γ by the (discrete time) geodesic flow on $\mathscr{G}\mathbb{X}'$, thus avoiding the suspension part (see also the end of op. cit. when Γ is a free group).

Proof. For all $\ell \in \mathscr{G}\mathbb{X}'$ and $\gamma \in \Gamma$, we can take $\gamma_i(\gamma\ell) = \gamma_i(\ell)\gamma^{-1}$, and since $p(\gamma f_i) = p(f_i)$, we have $e_i^\pm(\gamma\ell) = e_i^\pm(\ell)$ and $h_i(\gamma\ell) = h_i(\ell)$, hence the map Θ is well defined. By construction, the map Θ is equivariant for the actions of $\mathtt{g}^1$ on $\Gamma\backslash\mathscr{G}\mathbb{X}'$ and σ.

With the distances indicated in the statement of Theorem 5.1, if $\ell, \ell' \in \mathscr{G}\mathbb{X}'$ satisfy $\ell_{|[-n,n]} = \ell'_{|[-n,n]}$ for some $n \in \mathbb{N}$, then we have $e_i^\pm(\ell) = e_i^\pm(\ell')$ for $-n \leqslant i \leqslant n-1$, and we may take $\gamma_i(\ell) = \gamma_i(\ell')$ for $-n \leqslant i \leqslant n-1$, so that $h_i(\ell) = h_i(\ell')$ for $-n \leqslant i \leqslant n-1$. Therefore, we have

$$d(\,\Theta(\Gamma\ell), \Theta(\Gamma\ell')\,) \leqslant e\; d(\Gamma\ell, \Gamma\ell')\ ,$$

and Θ is Lipschitz (hence continuous).

Let us construct an inverse $\Psi : \Sigma \to \Gamma\backslash\mathscr{G}\mathbb{X}'$ of Θ, by a more general construction that will be useful later on. Let I be a nonempty interval of consecutive integers in $\mathbb{Z}$, either finite or equal to $\mathbb{Z}$ (the definition of the inverse of Θ only requires the second case $I = \mathbb{Z}$). For all $e^-, e^+ \in E\mathbb{Y}$ such that $t(e^-) = o(e^+)$, we fix once and for all a representative of every double coset in $\rho_{e^-}(G_{e^-})\backslash G_{o(e^+)}/\rho_{\overline{e^+}}(G_{e^+})$, and we will denote this double coset by its representative.

Let $w = (e_i^-, h_i, e_i^+)_{i\in I}$ be a sequence indexed by I in the alphabet $\mathscr{A}$ such that for all $i \in I$ such that $i-1 \in I$, we have $e_{i-1}^+ = e_i^-$ (when I is finite, this means that w is an A-admissible sequence in $\mathscr{A}$, and when $I = \mathbb{Z}$, this means that $w \in \Sigma$). In particular, the element $h_i \in G_{o(e_i^+)} = \Gamma_{\widetilde{o(e_i^+)}}$ is the chosen representative of its double coset $\rho_{e_i^-}(G_{e_i^-})\; h_i\; \rho_{\overline{e_i^+}}(G_{e_i^+})$.

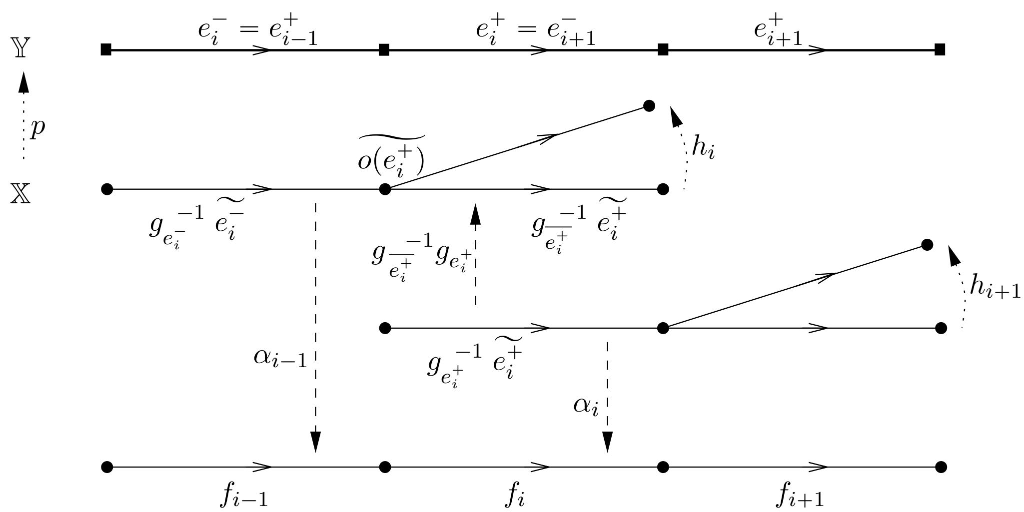
For every $i \in I$, note that

$$o(\,h_i\, g_{\overline{e_i^+}}^{-1}\, \widetilde{e_i^+}\,) = o(\,g_{\overline{e_i^+}}^{-1}\, \widetilde{e_i^+}\,) = \widetilde{o(e_i^+)} = \widetilde{t(e_i^-)} = t(\,g_{e_i^-}^{-1}\, \widetilde{e_i^-}\,)\;.$$

But $h_i\, g_{\overline{e_i^+}}^{-1}\, \widetilde{e_i^+}$ is not the opposite edge of the edge $g_{e_i^-}^{-1}\, \widetilde{e_i^-}$, since the double coset of h_i is not the trivial one [1] when $e_i^+ = \overline{e_i^-}$, hence h_i does not fix $g_{e_i^-}^{-1}\, \widetilde{e_i^-}$. Therefore the length 2 edge path (see the picture below)

$$(\,g_{e_i^-}^{-1}\, \widetilde{e_i^-},\; h_i\, g_{\overline{e_i^+}}^{-1}\, \widetilde{e_i^+}\,)$$

is geodesic.



Let us construct by induction a geodesic segment $\widetilde{w}$ in $\mathbb{X}'$ (which will be a discrete geodesic line if $I = \mathbb{Z}$), well defined up to the action of Γ, as follows.

We fix $i_0 \in I$ (for instance $i_0 = 0$ if $I = \mathbb{Z}$ or $i_0 = \min I$ if I is finite), and $\alpha_{i_0} \in \Gamma$. Let us define

$$f_{i_0} = f_{i_0}(w) = \alpha_{i_0}\, g_{e_{i_0}^+}^{-1}\, \widetilde{e_{i_0}^+}\;.$$

Let us then define

$$\alpha_{i_0-1} = \alpha_{i_0-1}(w) = \alpha_{i_0}\, g_{e_{i_0}^+}^{-1}\, g_{\overline{e_{i_0}^+}}\, h_{i_0}^{-1} \quad \text{and} \quad f_{i_0-1} = f_{i_0-1}(w) = \alpha_{i_0-1}\, g_{e_{i_0}^-}^{-1}\, \widetilde{e_{i_0}^-}\;.$$

We have $\alpha_{i_0-1}\, h_{i_0}\, g_{\overline{e_{i_0}^+}}^{-1}\, \widetilde{e_{i_0}^+} = f_{i_0}$ and (f_{i_0-1}, f_{i_0}) is a geodesic edge path of length 2 (as the image by α_{i_0-1} of such a path).

Let $i-1, i' \in I$ be such that $i' \leqslant i_0 \leqslant i-1$. Assume by increasing induction on i and decreasing induction on i' that a geodesic edge path $(f_{i'-1} = f_{i'-1}(w), \dots, f_{i-1} = f_{i-1}(w))$ in $\mathbb{X}'$ and a sequence $(\alpha_{i'-1} = \alpha_{i'-1}(w), \dots, \alpha_{i-1} = \alpha_{i-1}(w))$ in Γ have been constructed such that

$$f_j = \alpha_j\, g_{e_j^+}^{-1}\, \widetilde{e_j^+} \quad \text{and} \quad \alpha_j = \alpha_{j-1}\, h_j\, g_{\overline{e_j^+}}^{-1}\, g_{e_j^+}$$

for every $j \in \mathbb{N}$ such that $i'-1 \leqslant j \leqslant i-1$, with besides $j \geqslant i'$ for the equality on the right.

If i does not belong to I, we stop the construction on the right hand side at $i-1$. If on the contrary $i \in I$, let us define (see the above picture)

$$\alpha_i = \alpha_{i-1}\, h_i\, g_{\overline{e_i^+}}^{-1}\, g_{e_i^+} \quad \text{and} \quad f_i = f_i(w) = \alpha_i\, g_{e_i^+}^{-1}\, \widetilde{e_i^+}\ .$$

Then

$$(f_{i-1}, f_i) = \big(\alpha_{i-1}\, g_{e_i^-}^{-1}\, \widetilde{e_i^-},\ \alpha_{i-1}\, h_i\, g_{\overline{e_i^+}}^{-1}\, \widetilde{e_i^+}\big)\ ,$$

is a geodesic edge path of length 2 (as the image by α_{i-1} of such a path). As an edge path is geodesic if and only if it has no back-and-forth, $(f_{i'}, \dots, f_i)$ is a geodesic edge path in $\mathbb{X}'$. Thus the construction holds at rank i on the right.

If $i'-1$ does not belong to I, we stop the construction on the left side at i'. Otherwise we proceed as for the construction of α_{i_0-1} and f_{i_0-1} in order to construct $\alpha_{i'-2}$ and $f_{i'-2}$ with the required properties.

If $I = [p, q] \cap \mathbb{Z}$ with $p \leqslant q$ in $\mathbb{Z}$, let $I' = [p-1, q] \cap \mathbb{Z}$. If $I = \mathbb{Z}$, let $I' = \mathbb{Z}$. We have thus constructed a geodesic edge path

$$(f_i)_{i \in I'} = (f_i(w))_{i \in I'} \tag{5.3}$$

in $\mathbb{X}'$. We denote by $\widetilde{w}$ its parametrisation by $\mathbb{R}$ if $I = \mathbb{Z}$ and by $[p-1, q+1]$ if $I = [p, q] \cap \mathbb{Z}$, in such a way that $\widetilde{w}(i) = o(f_i)$ for all $i \in I$. In particular, $f_i = \widetilde{w}([i, i+1])$ for all $i \in I'$. When $I = [p, q] \cap \mathbb{Z}$, we consider $\widetilde{w}$ as a generalised discrete geodesic line, by extending it to a constant on $]-\infty, p-1]$ and on $[q+1, +\infty[$.

The orbit $\Gamma\widetilde{w}$ of $\widetilde{w}$ does not depend on the choice of α_{i_0}, since replacing α_{i_0} by α'_{i_0} replaces f_i by $\alpha'_{i_0}\alpha_{i_0}^{-1} f_i$ for all $i \in I'$, hence replaces $\widetilde{w}$ by $\alpha'_{i_0}\alpha_{i_0}^{-1}\widetilde{w}$. This also implies that $\Gamma\widetilde{w}$ does not depend on the choice of $i_0 \in I$.

Assume from now on that $I = \mathbb{Z}$, and define $\Psi : \Sigma \to \Gamma\backslash\mathscr{G}\mathbb{X}'$ by

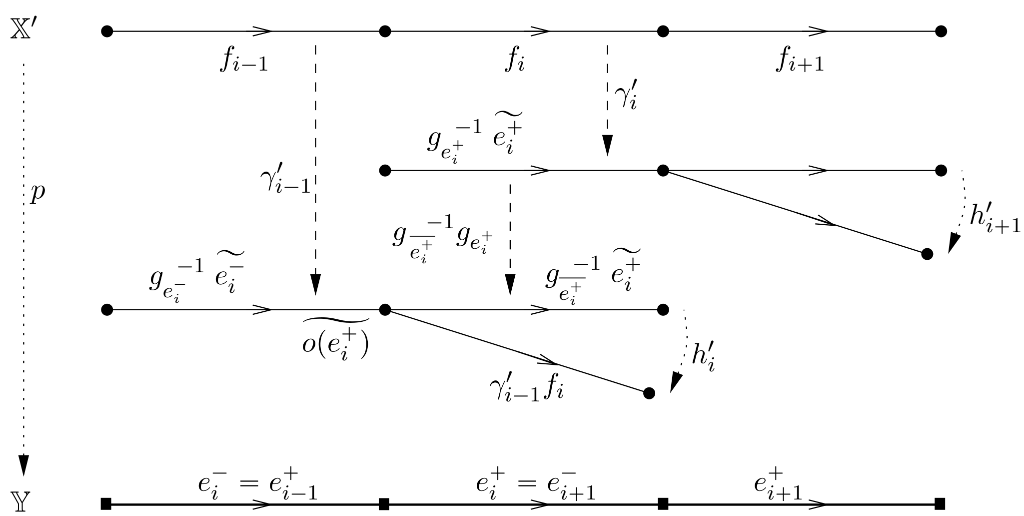
$$\Psi(w) = \Gamma\widetilde{w}\ .$$

Let $w = (e_i^-, h_i, e_i^+)_{i \in \mathbb{Z}}$ and $w' = (e_i'^-, h'_i, e_i'^+)_{i \in I}$ in Σ satisfy $e_i^\pm = e_i'^\pm$ and $h_i = h'_i$ for all $i \in \{-n, \dots, n\}$ for some $n \in \mathbb{N}$. Then we may take the same $i_0 = 0$ and α_{i_0} in the construction of $\widetilde{w}$ and $\widetilde{w'}$. We thus have $\alpha_i(w) = \alpha_i(w')$ and $f_i(w) = f_i(w')$ for $-n \leqslant i \leqslant n$. Therefore, with the distances indicated in the statement of Theorem 5.1, we have

$$d(\,\Psi(w), \Psi(w')\,) \leqslant d(w, w')\ ,$$

and Ψ is Lipschitz.

Let us prove that Ψ is indeed the inverse of Θ. As in the construction of Θ, for all $\ell \in \mathscr{G}\mathbb{X}'$ and $i \in \mathbb{Z}$, we define $f_i = \ell([i, i+1])$, $e_i^+ = p(f_i)$ and $e_i^- = e_{i-1}^+$. We denote by $\gamma'_i \in \Gamma$ an element sending f_i to $g_{e_i^+}^{-1}\,\widetilde{e_i^+}$ for all $i \in \mathbb{Z}$ (see the picture below): with the notation above the statement of Theorem 5.1, we have $\gamma'_i = g_{e_i^+}^{-1}\,\gamma_i(\ell)$.



Then γ'_i is well defined up to multiplication on the left by an element of $\Gamma_{g_{e_i^+}^{-1}\widetilde{e_i^+}} = \rho_{e_i^+}(G_{e_i^+})$. Let h'_i be an element in $G_{o(e_i^+)}$ sending $g_{\overline{e_i^+}}^{-1}\,\widetilde{e_i^+}$ to $\gamma'_{i-1} f_i$. It exists since these two edges have the same origin $\widetilde{o(e_i^+)}$, and same image by p:

$$p(\gamma'_{i-1} f_i) = p(f_i) = e_i^+ = p(\widetilde{e_i^+}) = p(\,g_{\overline{e_i^+}}^{-1}\,\widetilde{e_i^+}\,)\ .$$

Furthermore, it is well defined up to multiplication on the right by an element of $\Gamma_{g_{\overline{e_i^+}}^{-1}\widetilde{e_i^+}} = \rho_{\overline{e_i^+}}(G_{e_i^+})$, and we have (see the above picture)

$$\gamma'_{i-1}\,\gamma_i'^{-1}\;g_{e_i^-}^{-1}\,g_{\overline{e_i^+}}\;\in\;h'_i\;\rho_{\overline{e_i^+}}(G_{e_i^+})$$

Using $\gamma_j(\ell) = g_{e_j^+}\gamma'_j$ for $j = i, i-1$ in Equation (5.2) gives $h_i(\ell) = \gamma'_{i-1}\gamma_i'^{-1} g_{e_i^+}^{-1} g_{\overline{e_i^+}}$. Hence by the construction of Θ (see with $\gamma'_i = g_{e_i^-}^{-1}\,\gamma_i(\ell)$ for all $i \in \mathbb{Z}$), we have

$$\Theta(\ell) = (e_i^-,\ \rho_{e_i^-}(G_{e_i^-})\;h'_i\;\rho_{\overline{e_i^+}}(G_{e_i^+}),\ e_i^+)_{i\in\mathbb{Z}}\ .$$

Let h_i be the chosen representative of the double coset $\rho_{e_i^-}(G_{e_i^-})\;h'_i\;\rho_{\overline{e_i^+}}(G_{e_i^+})$: there exist $\alpha \in \rho_{e_i^-}(G_{e_i^-}) = \rho_{e_{i-1}^+}(G_{e_{i-1}^+})$ and $\beta \in \rho_{\overline{e_i^+}}(G_{e_i^+})$ such that $h_i = \alpha h'_i \beta$. Up to replacing γ'_{i-1} by $\alpha^{-1}\gamma'_{i-1}$ and h'_i by $h'_i\beta$, we then may have $h'_i = h_i$. By taking $\alpha_{i_0} = {\gamma'_{i_0}}^{-1}$, we have $\alpha_i = \gamma_i'^{-1}$ for all $i \in \mathbb{Z}$, and an inspection of the above two constructions gives that $\Theta \circ \Psi = \mathrm{id}$ and $\Psi \circ \Theta = \mathrm{id}$.

Since the discrete time geodesic flow is topologically transitive on its nonwandering subset and by conjugation, the topological Markov shift (Σ, σ) is topologically transitive.

If $\mathbb{X}'$ is a uniform tree without vertices of degree at most 2, if the length spectrum of Γ is equal to $\mathbb{Z}$ and if the Gibbs measure m_F for Γ is finite, then by Theorem 4.17, the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}'$ is topologically mixing, hence by conjugation by Θ, the topological Markov shift (Σ, σ) is topologically mixing. This concludes the proof of Theorem 5.1. $\square$

When the length spectrum L_Γ of Γ is different from $\mathbb{Z}$, the topological Markov shift (Σ, σ) constructed above is not always topologically mixing. We now modify the above construction in order to take care of this problem.

Recall that $\mathbb{X}' = \mathscr{C}\Lambda\Gamma$ and that $\mathscr{G}_{\rm even}\mathbb{X}'$ is the space of geodesic lines $\ell \in \mathscr{G}\mathbb{X}'$ whose origin $\ell(0)$ is at even distance from the basepoint x_0 (we assume that $x_0 \in \mathbb{X}'$), which is invariant under the time-two discrete geodesic flow g^2 and, when $L_\Gamma = 2\mathbb{Z}$, under Γ, as seen in the proof of Theorem 4.17.

Consider $\mathscr{A}_{\rm even}$ the alphabet consisting of the quintuples $(f^-, h^-, f^0, h^+, f^+)$ where the triples (f^-, h^-, f^0) and (f^0, h^+, f^+) belong to $\mathscr{A}$ and $o(f^0)$ is at even distance from the image in $\mathbb{Y} = \Gamma\backslash\mathbb{X}'$ of the basepoint x_0. Let $A_{\rm even} = (A_{{\rm even},\,i,j})_{i,j\in\mathscr{A}_{\rm even}}$ be the transition matrix with line and column indices in $\mathscr{A}_{\rm even}$ such that for all $i = (f^-, h^-, f^0, h^+, f^+)$ and $j = (f^-_*, h^-_*, f^0_*, h^+_*, f^+_*)$, we have $A_{{\rm even},\,i,j} = 1$ if and only if $f^+ = f^-_*$. We denote by $(\Sigma_{\rm even}, \sigma_{\rm even})$ the associated topological Markov shift. We endow $\Sigma_{\rm even}$ with the slightly modified distance

$$d_{\rm even}(x, x') = e^{-2\sup\big\{n\in\mathbb{N}\;:\;\forall\, k\in\{-n,\dots,n\},\;x_k = x'_k\big\}}\;,$$

where $x = (x_k)_{k\in\mathbb{Z}}$ and $x' = (x'_k)_{k\in\mathbb{Z}}$ are in $\Sigma_{\rm even}$.

We have a canonical injection ${\rm inj} : \Sigma_{\rm even} \to \Sigma$ sending the sequence $(f^-_n, h^-_n, f^0_n, h^+_n, f^+_n)_{n\in\mathbb{Z}}$ to $(e^-_n, h_n, e^+_n)_{n\in\mathbb{Z}}$ with, for every $n \in \mathbb{Z}$,

$$e^-_{2n} = f^-_n\,,\;\; h_{2n} = h^-_n\,,\;\; e^+_{2n} = f^0_n,\;\; e^-_{2n+1} = f^0_n,\;\; h_{2n+1} = h^+_n,\;\; e^+_{2n+1} = f^+_n\,,\;\; .$$

By construction, inj is clearly a homeomorphism onto its image, and

$$\Theta(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}') = {\rm inj}(\Sigma_{\rm even})\;.$$

If two sequences in $\Sigma_{\rm even}$ coincide between $-n$ and n, then their images by inj coincide between $-2n$ and $2n$. Conversely, if the images by inj of two sequences in $\Sigma_{\rm even}$ coincide between $-2n-1$ and $2n+1$, then these sequences coincide between $-n$ and n. Hence inj is bilipschitz, for the above distances.

Let us define $\Theta_{\rm even} = {\rm inj}^{-1} \circ \Theta|_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}'} : \Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}' \to \Sigma_{\rm even}$. The following diagram hence commutes

$$\begin{array}{ccc}
\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}' & \xrightarrow{\;\Theta_{\rm even}\;} & \Sigma_{\rm even} \\
\big\downarrow & & \big\downarrow {\rm inj} \\
\Gamma\backslash\mathscr{G}\mathbb{X}' & \xrightarrow{\;\;\Theta\;\;} & \Sigma\,,
\end{array}$$

where the vertical map on the left hand side is the inclusion map.

Theorem 5.2. *Assume that $\mathbb{X}' = \mathscr{C}\Lambda\Gamma$ is a uniform tree without vertices of degree at most 2, that the Gibbs measure m_F of Γ is finite, and that the length spectrum L_Γ of Γ is equal to $2\mathbb{Z}$. Then the map $\Theta_{\rm even} : \Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}' \to \Sigma_{\rm even}$ is a bilipschitz homeomorphism which conjugates the time-two discrete geodesic flow g^2 and the shift $\sigma_{\rm even}$, and the topological Markov shift $(\Sigma_{\rm even}, \sigma_{\rm even})$ is locally compact and topologically mixing.*

Proof. The only claims that remains to be proven is the last one, which follows from Theorem 4.17, by conjugation. □

Let us now study the properties of the image by the coding map Θ of finite Gibbs measures on $\Gamma\backslash\mathscr{G}\mathbb{X}$.

Let $\delta = \delta_{\Gamma, F^{\pm}}$ be the critical exponent of $(\Gamma, F^{\pm})$. Let $(\mu^{\pm}_x)_{x\in V\mathbb{X}}$ be two (normalised) Patterson densities on $\partial_\infty X$ for the pairs $(\Gamma, F^{\pm})$, where as previously $\widetilde{F}^+ = \widetilde{F}$, $\widetilde{F}^- = \widetilde{F}\circ\iota$. Assume that the associated Gibbs measure m_F on $\Gamma\backslash\mathscr{G}\mathbb{X}$ (using the convention for discrete time of Section 4.3) is finite.

Let us define

$$\mathbb{P} = \frac{1}{\|m_F\|}\,\Theta_* m_F \tag{5.4}$$

as the image of the Gibbs measure m_F (whose support is $\Gamma\backslash\mathscr{G}\mathbb{X}'$) by the homeomorphism Θ, normalised to be a probability measure. It is a probability measure on Σ, invariant under the shift σ.

Let $(Z_n)_{n\in\mathbb{Z}}$ be the random process classically associated with the full shift σ on Σ: it is the random process on the Borel space Σ indexed by $\mathbb{Z}$ with values in the discrete alphabet $\mathscr{A}$, where $Z_n : \Sigma \to \mathscr{A}$ is the (continuous hence measurable) n-th projection $(x_k)_{k\in\mathbb{N}} \mapsto x_n$ for all $n\in\mathbb{Z}$.

The following Proposition 5.5 summarises the properties of the probability measure $\mathbb{P}$. We start by recalling and giving some notation used in this proposition.

For every admissible finite sequence $w = (a_p, \dots, a_q)$ in $\mathscr{A}$, where $p \leqslant q$ in $\mathbb{Z}$, we denote

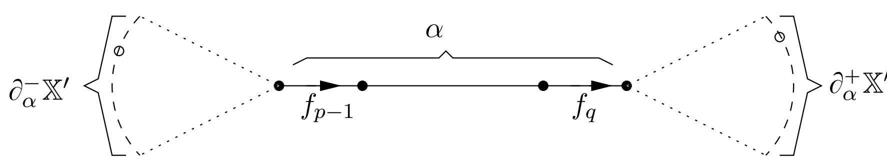
- by $[w] = [a_p, \dots, a_q] = \{(x_n)_{n\in\mathbb{Z}} \in \Sigma \;:\; \forall\, n\in\{p,\dots,q\},\;\; x_n = a_n\}$ the associated cylinder in Σ,
- by $\widehat{w}$ the associated geodesic edge path in $\mathbb{X}'$ with length $q-p+2$ constructed in the proof of Theorem 5.1 (see Equation (5.3)), with origin $\widehat{w}_-$ and endpoint $\widehat{w}_+$.

For every geodesic edge path $\alpha = (f_{p-1}, \dots, f_q)$ in $\mathbb{X}'$, we define (See Section 2.6 for the notation, and the picture below)

$$\partial^+_\alpha\mathbb{X}' = \partial_{f_q}\mathbb{X}' \quad\text{and}\quad \partial^-_\alpha\mathbb{X}' = \partial_{\overline{f_{p-1}}}\mathbb{X}'\,,$$

and

$$\mathscr{G}_\alpha\mathbb{X} = \{\ell\in\mathscr{G}\mathbb{X} \;:\; \ell(p-1) = o(f_{p-1}) \quad\text{and}\quad \ell(q+1) = t(f_q)\}\,.$$



We define a map $F_{\rm symb} : \Sigma \to \mathbb{R}$ by

$$F_{\rm symb}(x) = \int_{o(e_0^+)}^{t(e_0^+)} F \tag{5.5}$$

if $x = (x_i)_{i\in\mathbb{Z}}$ with $x_0 = (e_0^-, h_0, e_0^+)$. Note that for all $(x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}} \in \Sigma$, if $x_0 = y_0$, then $F_{\rm symb}(x) = F_{\rm symb}(y)$, so that $F_{\rm symb}$ is locally constant (constant on each cylinder of length 1 at time 0), hence continuous.

For instance, if $F = F_c$ is the potential associated with a system of conductances $c : \Gamma\backslash E\mathbb{X}' \to \mathbb{R}$ (see Section 3.5), then

$$F_{\rm symb}(x) = c(e_0^+)\ .$$

Note that if $c, c' : \Gamma\backslash E\mathbb{X}' \to \mathbb{R}$ are cohomologous systems of conductances on $\Gamma\backslash E\mathbb{X}'$, then the corresponding maps $F_{\rm symb}, F'_{\rm symb} : \Sigma \to \mathbb{R}$ are cohomologous. Indeed if $f : \Gamma\backslash V\mathbb{X} \to \mathbb{R}$ is a map such that $c'(e) - c(e) = f(t(e)) - f(o(e))$ for every $e \in \Gamma\backslash E\mathbb{X}$, with $G : \Sigma \to \mathbb{R}$ the map defined by $G(x) = f(o(e_0^+))$ if $x = (x_i)_{i\in\mathbb{Z}}$ with $x_0 = (e_0^-, h_0, e_0^+)$, then G is locally constant, hence continuous, and since $t(e_0^+) = o(e_1^+)$, we have, for every $x \in \Sigma$,

$$F'_{\rm symb}(x) - F_{\rm symb}(x) = G(\sigma x) - G(x)\ .$$

Definition 5.3. Let $\mathbb{X}''$ be a locally finite simplicial tree. A nonelementary discrete subgroup Γ' of $\mathrm{Aut}(\mathbb{X}'')$ is *Markov-good* if for every $n \in \mathbb{N} - \{0\}$ and every geodesic edge path $(e_0, \dots, e_{n+1})$ in $\mathscr{C}\Lambda\Gamma'$, we have

$$|\Gamma'_{e_0} \cap \cdots \cap \Gamma'_{e_n}|\,|\Gamma'_{e_{n-1}} \cap \Gamma'_{e_n} \cap \Gamma'_{e_{n+1}}| = |\Gamma'_{e_0} \cap \cdots \cap \Gamma'_{e_{n+1}}|\,|\Gamma'_{e_{n-1}} \cap \Gamma'_{e_n}|\ . \tag{5.6}$$

Remark 5.4. (1) Note that Equation (5.6) is automatically satisfied if $n = 1$ and that Γ' is Markov-good if Γ' acts freely on $\mathbb{X}''$.

(2) A group action on a simplicial tree is 2-*acylindrical* [4] if the stabiliser of any geodesic edge path of length 2 is trivial. If Γ' is 2-acylindrical on $\mathbb{X}$, then Γ' is Markov-good, since all groups appearing in Equation (5.6) are trivial.

(3) If $\mathbb{X}''$ has degrees at least 3 and if Γ' is a noncocompact geometrically finite lattice of $\mathbb{X}''$ with abelian edge stabilisers, then Γ' is not Markov-good.

Proof. (3) Since the quotient graph $\Gamma'\backslash\mathbb{X}''$ is infinite, the graph of groups $\Gamma'\backslash\backslash\mathbb{X}''$ contains at least one cuspidal ray. Consider a geodesic ray in $\mathbb{X}''$ with consecutive edges $(f_n)_{n\in\mathbb{N}}$ mapping injectively onto this cuspiday ray, pointing towards its end. The stabilisers of the edges f_n in Γ' are hence nondecreasing in n: we have $\Gamma'_{f_n} \subset \Gamma'_{f_{n+1}}$ for all $n \in \mathbb{N}$. By the finiteness of the volume, there exists $n \geqslant 3$ such that $\Gamma'_{f_{n-2}}$ is strictly contained in $\Gamma'_{f_{n-1}}$. Since $\mathbb{X}''$ has degrees at least 3, there exists $\gamma \in \Gamma'$ fixing $t(f_{n-1})$ but not fixing f_{n-1}. Let $e_0 = f_0, \dots, e_{n-1} = f_{n-1}$, $e_n = \gamma\ \overline{f_{n-1}}$ and $e_{n+1} = \gamma\ \overline{f_{n-2}}$. Since the stabilisers of f_{n-1} and e_n are conjugated by γ within the abelian stabiliser of f_n, they are equal. Then $(e_0, \dots, e_{n+1})$ is a geodesic edge path in the simpliciak tree $\mathbb{X}''$ (whose geometric realisation is equal to $\mathscr{C}\Lambda\Gamma'$ since Γ' is a lattice). Since $\Gamma'_{e_0} \cap \cdots \cap \Gamma'_{e_n} = \Gamma'_{f_0}$, $\Gamma'_{e_{n-1}} \cap \Gamma'_{e_n} \cap \Gamma'_{e_{n+1}} = \Gamma'_{f_{n-2}}$, $\Gamma'_{e_0} \cap \cdots \cap \Gamma'_{e_{n+1}} = \Gamma'_{f_0}$, $\Gamma'_{e_{n-1}} \cap \Gamma'_{e_n} = \Gamma'_{f_{n-1}}$ and $|\Gamma'_{f_{n-2}}| \neq |\Gamma'_{f_{n-1}}|$, the subgroup Γ' is not Markov-good. □

Recall that a random process $(Z'_n)_{n\in\mathbb{Z}}$ on $(\Sigma, \mathbb{P})$ is a *Markov chain* if and only if for all $p \leqslant q$ in $\mathbb{Z}$ and $a_p, \dots, a_q, a_{q+1}$ in $\mathscr{A}$, we have, when defined,

$$\mathbb{P}(Z'_{q+1} = a_{q+1}\,|\,Z'_q = a_q, \dots, Z'_p = a_p) = \mathbb{P}(Z'_{q+1} = a_{q+1}\,|\,Z'_q = a_q)\ . \tag{5.7}$$

Proposition 5.5. *(1) For every admissible finite sequence w in $\mathscr{A}$, we have*

$$\mathbb{P}([w]) = \frac{\mu^-_{\widehat{w}_-}(\partial^-_{\widehat{w}}\mathbb{X}')\ \mu^+_{\widehat{w}_+}(\partial^+_{\widehat{w}}\mathbb{X}')\ e^{\int_{\widehat{w}_-}^{\widehat{w}_+}(\widetilde{F}-\delta)}}{|\Gamma_{\widehat{w}}|\ \|m_F\|}\ .$$

[4] See for instance [Sel, GuL], which require other minor hypotheses that are not relevant here.

(2) The random process $(Z_n)_{n\in\mathbb{Z}}$ on $(\Sigma, \mathbb{P})$ is a Markov chain if and only if Γ is Markov-good.

(3) The measure $\mathbb{P}$ on the topological Markov shift Σ satisfies the Gibbs property with Gibbs constant δ for the potential $F_{\rm symb}$.

It follows from the above Assertion (2) and from Remark 5.4 that when $\mathbb{X}$ has degrees at least 3 and Γ is a noncocompact geometrically finite lattice of $\mathbb{X}$ with abelian edge stabilisers (and more generally, this is not a necessary assumption), then $(Z_n)_{n\in\mathbb{Z}}$ is not a Markov chain. The fact that codings of discrete time geodesic flows on trees might not satisfy the Markov chain property had been noticed by Burger and Mozes around the time the paper [BuM] was published.[5] When proving the variational principle in Section 5.4 and the exponential decay of correlations in Section 9.2, we will hence have to use tools that are not using the Markov chain property.

Proof. (1) Let $w = (a_p, \dots, a_q)$, with $p \leqslant q$ in $\mathbb{Z}$, be an admissible finite sequence in $\mathscr{A}$. Recall that $[w] = \{x \in \Sigma \;:\; \forall\, i \in \{p, \dots, q\},\;\; x_i = a_i\}$. By the construction of Θ, the preimage $\Theta^{-1}([w])$ is equal to the image $\Gamma\mathscr{G}_{\widehat{w}}\mathbb{X}'$ of $\mathscr{G}_{\widehat{w}}\mathbb{X}'$ in $\Gamma\backslash\mathscr{G}\mathbb{X}'$. Hence, since $\Gamma_{\widehat{w}}$ is the stabiliser of $\mathscr{G}_{\widehat{w}}\mathbb{X}'$ in Γ,

$$\mathbb{P}([w]) = \frac{1}{\|m_F\|}\, m_F(\Gamma\mathscr{G}_{\widehat{w}}\mathbb{X}') = \frac{1}{|\Gamma_{\widehat{w}}|\;\|m_F\|}\, \widetilde{m}_F(\mathscr{G}_{\widehat{w}}\mathbb{X}')\;.$$

In the expression of $\widetilde{m}_F$ given by Equation (4.12), let us use as basepoint the origin $\widehat{w}_-$ of the edge path $\widehat{w}$, and note that all elements of $\mathscr{G}_{\widehat{w}}\mathbb{X}'$ pass through $\widehat{w}_-$ at time $t = p - 1$, so that by the invariance of $\widetilde{m}_F$ under the discrete time geodesic flow, we have

$$\begin{aligned}\widetilde{m}_F(\mathscr{G}_{\widehat{w}}\mathbb{X}') &= \int_{\ell\in\mathscr{G}_{\widehat{w}}\mathbb{X}'} d\widetilde{m}_F(\mathsf{g}^{1-p}\ell) = \int_{\ell_-\in\partial^-_{\widehat{w}}\mathbb{X}'}\int_{\ell_+\in\partial^+_{\widehat{w}}\mathbb{X}'} d\mu^-_{\widehat{w}_-}(\ell_-)d\mu^+_{\widehat{w}_-}(\ell_+)\\ &= \mu^-_{\widehat{w}_-}(\partial^-_{\widehat{w}}\mathbb{X}')\;\mu^+_{\widehat{w}_-}(\partial^+_{\widehat{w}}\mathbb{X}') = \mu^-_{\widehat{w}_-}(\partial^-_{\widehat{w}}\mathbb{X}')\;\mu^+_{\widehat{w}_+}(\partial^+_{\widehat{w}}\mathbb{X}')\;e^{\int_{\widehat{w}_-}^{\widehat{w}_+}(\widetilde{F}-\delta)}\;,\end{aligned}$$

where this last equality follows by Equations (4.2) and (3.20) with $x = \widehat{w}_-$ and $y = \widehat{w}_+$, since for every $\ell_+ \in \partial^+_{\widehat{w}}\mathbb{X}'$, we have $\widehat{w}_+ \in [\widehat{w}_+, \ell_+[\,$.

(2) Let us fix $p \leqslant q$ in $\mathbb{Z}$ and $a_p, \dots, a_q, a_{q+1}$ in $\mathscr{A}$, and let us try to verify Equation (5.7) for $(Z'_n)_{n\in\mathbb{N}} = (Z_n)_{n\in\mathbb{N}}$. Let $\alpha_* = (a_p, \dots, a_q)$, which is an admissible sequence, since we assumed the conditional probability $\mathbb{P}(Z_{q+1} = a_{q+1}\,|\,Z_q = a_q, \dots, Z_p = a_p)$ to be well defined. We may assume that $\alpha = (a_p, \dots, a_q, a_{q+1})$ is an admissible sequence, otherwise both sides of Equation (5.7) are 0. Let us consider

$$Q_\alpha = \frac{\mathbb{P}(Z_{q+1} = a_{q+1}\,|\,Z_q = a_q, \dots, Z_p = a_p)}{\mathbb{P}(Z_{q+1} = a_{q+1}\,|\,Z_q = a_q)} = \frac{\mathbb{P}([a_p, \dots, a_{q+1}])\;\mathbb{P}([a_q])}{\mathbb{P}([a_p, \dots, a_q])\;\mathbb{P}([a_q, a_{q+1}])}\;.$$

Let us replace each one of the four terms in this ratio by its value given by Assertion (1). Since $\partial^-_{\widehat{\alpha}}\mathbb{X}' = \partial^-_{\widehat{\alpha_*}}\mathbb{X}'$, $\partial^+_{\widehat{\alpha}}\mathbb{X}' = \partial^+_{\widehat{a_q, a_{q+1}}}\mathbb{X}'$, $\partial^+_{\widehat{\alpha_*}}\mathbb{X}' = \partial^+_{\widehat{a_q}}\mathbb{X}'$ and $\partial^-_{\widehat{a_q}}\mathbb{X}' = \partial^-_{\widehat{a_q, a_{q+1}}}\mathbb{X}'$, all Patterson measure terms cancel. Denoting by y_1 the common origin of $\widehat{\alpha}$ and $\widehat{\alpha_*}$, by y_2 the common origin of $\widehat{a}_q$ and $\widehat{a_q, a_{q+1}}$, by y_3 the common terminal point of $\widehat{a}_q$ and $\widehat{\alpha_*}$, and by y_4 the common terminal point of $\widehat{a_q, a_{q+1}}$ and $\widehat{\alpha}$, we thus have by Assertion (1)

$$Q_\alpha = \frac{|\Gamma_{\widehat{\alpha_*}}|\;|\Gamma_{\widehat{a_q, a_{q+1}}}|}{|\Gamma_{\widehat{\alpha}}|\;|\Gamma_{\widehat{a_q}}|}\;\frac{e^{\int_{y_1}^{y_4}(\widetilde{F}-\delta)}\;e^{\int_{y_2}^{y_3}(\widetilde{F}-\delta)}}{e^{\int_{y_1}^{y_3}(\widetilde{F}-\delta)}\;e^{\int_{y_2}^{y_4}(\widetilde{F}-\delta)}}\;.$$

[5] Personal communication.

Since y_1, y_2, y_3, y_4 are in this order on $[y_1, y_4]$, we have

$$Q_\alpha = \frac{|\Gamma_{\widehat{\alpha_*}}|\ |\Gamma_{\widehat{a_q, a_{q+1}}}|}{|\Gamma_{\widehat{\alpha}}|\ |\Gamma_{\widehat{a_q}}|}\ .$$

Since every geodesic edge path of length $n+1$ at least 3 in $\mathbb{X}'$ defines an admissible sequence of length n at least 2 in $\mathscr{A}$, by Equation (5.6), we have $Q_\alpha = 1$ for every admissible sequence α in $\mathscr{A}$ if and only if Γ is Markov-good.

(3) Let E be a finite subset of the alphabet $\mathscr{A}$, and let $w = (a_p, \ldots, a_q)$ with $p \leqslant q$ in $\mathbb{Z}$ be an admissible sequence in $\mathscr{A}$ such that $a_p, a_q \in E$. By Assertion (1), we have

$$\mathbb{P}([w]) = \frac{\mu^-_{\widehat{w}_-}(\partial^-_{\widehat{w}}\mathbb{X}')\ \mu^+_{\widehat{w}_+}(\partial^+_{\widehat{w}}\mathbb{X}')\ e^{\int_{\widehat{w}_-}^{\widehat{w}_+}(\widetilde{F}-\delta)}}{|\Gamma_{\widehat{w}}|\ \|m_F\|}\ .$$

Since a_p, a_q are varying in the finite subset E of $\mathscr{A}$, the first and last edges of $\widehat{w}$ vary amongst the images under elements of Γ of finitely many edges of $\mathbb{X}'$. Since w is admissible, the sets $\partial^\pm_{\widehat{w}}\mathbb{X}'$ are nonempty open subsets of $\Lambda\Gamma$, hence they have positive Patterson measures. Furthermore, the quantities $\mu^\pm_{\widehat{w}_\pm}(\partial^\pm_{\widehat{w}}\mathbb{X}')$ are invariant under the action of Γ on the first/last edge of $\widehat{w}$. Hence there exists $c_1 \geqslant 1$ depending only on E such that $1 \leqslant |\Gamma_{\widehat{w}}| \leqslant |\Gamma_{\widehat{w}_-}| \leqslant c_1$ and $\frac{1}{c_1} \leqslant \mu^\pm_{\widehat{w}_\pm}(\partial^\pm_{\widehat{w}}\mathbb{X}') \leqslant c_1$.

Note that the length of $\widehat{w}$ is equal to $q-p+2$. Therefore

$$\frac{e^{-\delta}}{c_1^3\ \|m_F\|}\ e^{-\delta(q-p+1)+\int_{\widehat{w}_-}^{\widehat{w}_+}\widetilde{F}} \leqslant\ \mathbb{P}([w])\ \leqslant \frac{e^{-\delta}\ c_1^2}{\|m_F\|}\ e^{-\delta(q-p+1)+\int_{\widehat{w}_-}^{\widehat{w}_+}\widetilde{F}}\ .$$

If $\widehat{w} = (f_{p-1}, f_p, \ldots, f_q)$ and $x \in [w]$, we have by the definition of $F_{\rm symb}$

$$\int_{\widehat{w}_-}^{\widehat{w}_+}\widetilde{F} = \sum_{i=p-1}^{q}\int_{o(f_i)}^{t(f_i)}\widetilde{F} = \int_{o(f_{p-1})}^{t(f_{p-1})}\widetilde{F} + \sum_{i=p}^{q} F_{\rm symb}(\sigma^i(x))\ .$$

Since $\widetilde{F}$ is continuous and Γ-invariant, and since $o(f_{p-1})$ remains in the image under Γ of a finite subset of $V\mathbb{X}'$, there exists $c_2 > 0$ depending only on E such that $|\widetilde{F}(v)| \leqslant c_2$ for every $v \in T^1X$ with $\pi(v) \in [o(f_{p-1}), t(f_{p-1})]$. Hence $|\int_{o(f_{p-1})}^{t(f_{p-1})}\widetilde{F}| \leqslant c_2$, and Assertion (3) of Proposition 5.5 follows (see Equation (5.1) for the definition of the Gibbs property). $\Box$

Again in order to consider the case when the length spectrum L_Γ of Γ is $2\mathbb{Z}$, we define

$$\mathbb{P}_{\rm even} = \frac{1}{\|(m_F)_{|\,\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}'}\|}\ (\Theta_{\rm even})_*\big((m_F)_{|\,\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}'}\big)\ ,$$

and $(Z_{{\rm even},\,n})_{n\in\mathbb{Z}}$ the random process associated with the full shift $\sigma_{\rm even}$ on $\Sigma_{\rm even}$, with $Z_{{\rm even},\,n} : \Sigma_{\rm even} \to \mathscr{A}_{\rm even}$ the n-th projection for every $n \in \mathbb{Z}$.

By a proof similar to the one of Proposition 5.5, we have the following result. We define a map $F_{\text{symb, even}} : \Sigma_{\rm even} \to \mathbb{R}$ by

$$F_{\text{symb, even}}(x) = \int_{o(f_0^0)}^{t(f_0^+)} F \qquad (5.8)$$

if $x = (x_i)_{i\in\mathbb{Z}}$ with $x_0 = (f_0^-, h_0^-, f_0^0, h_0^+, f_0^+)$. As previously, $F_{\text{symb, even}}$ is locally constant, hence continuous.

Proposition 5.6. *The measure $\mathbb{P}_{\rm even}$ on the topological Markov shift $\Sigma_{\rm even}$ satisfies the Gibbs property with Gibbs constant δ for the potential $F_{\rm symb,\ even}$.* □

Again, if Γ is a noncocompact geometrically finite lattice of $\mathbb{X}$ with abelian edge stabilisers and $\mathbb{X}'$ has degrees at least 3, then $(Z_{{\rm even},\,n})_{n\in\mathbb{Z}}$ is not a Markov chain.

5.3 Coding continuous time geodesic flows on metric trees

Let $(\mathbb{X},\lambda)$ be a locally finite metric tree without terminal vertices, with $X=|\mathbb{X}|_\lambda$ its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X},\lambda)$, and let $\widetilde{F}:T^1X\to\mathbb{R}$ be a potential for Γ. Let $X'=\mathscr{C}\Lambda\Gamma$, which is the geometric realisation $|\mathbb{X}'|_\lambda$ of a metric subtree $(\mathbb{X}',\lambda)$. Let $\delta=\delta_{\Gamma,F^\pm}$ be the critical exponent of $(\Gamma,F^\pm)$. Let $(\mu^\pm_x)_{x\in V\mathbb{X}}$ be the (normalised) Patterson densities on $\partial_\infty X$ for the pairs $(\Gamma,F^\pm)$, and assume that the associated Gibbs measure m_F is finite. We also assume in this Section that the lengths of the edges of $(\mathbb{X}',\lambda)$ have a finite upper bound (which is in particular the case if $(\mathbb{X}',\lambda)$ is uniform). They have a positive lower bound by definition (see Section 2.6).

In this Section, we prove that the continuous time geodesic flow on $\Gamma\backslash\mathscr{G}X'$ is isomorphic to a suspension of a transitive (two-sided) topological Markov shift on a countable alphabet, by an explicit construction that will be useful later on in order to study the variational principle (see Section 5.4) and rates of mixing (see Section 9.3). Since we are only interested in the support of the Gibbs measures, we will only give such a description for the geodesic flow on the nonwandering subset $\Gamma\backslash\mathscr{G}X'$ of $\Gamma\backslash\mathscr{G}X$. The same construction works with the full space $\Gamma\backslash\mathscr{G}X$, but the resulting Markov shift is then not necessarily transitive.

We start by recalling (see for instance [BrinS, §1.11]) the definitions of the suspension of an invertible discrete time dynamical system and of the first return map on a cross-section of a continuous time dynamical system, which allow to pass from transformations to flows and back, respectively.

Let (Z,μ,T) be a metric space Z endowed with a homeomorphism T and a T-invariant (positive Borel) measure μ. Let $r:Z\to\,]0,+\infty[$ be a continuous map, such that for all $z\in Z$, the subset $\{r(T^nz)\ :\ n\in\mathbb{N}\}\cup\{-r(T^{-(n+1)}z)\ :\ n\in\mathbb{N}\}$ is discrete in $\mathbb{R}$. Then the *suspension* (or also *special flow*) *over* (Z,μ,T) *with roof function* r is the following continuous time dynamical system $(Z_r,\mu_r,(T_r^t)_{t\in\mathbb{R}})$:

- The space Z_r is the quotient topological space $(Z\times\mathbb{R})/\sim$ where $\sim$ is the equivalence relation on $Z\times\mathbb{R}$ generated by $(z,s+r(z))\sim(Tz,s)$ for all $(z,s)\in Z\times\mathbb{R}$. We denote by $[z,s]$ the equivalence class of (z,s). Note that

$$\mathscr{F}=\{(z,s)\ :\ z\in Z,\ \ 0\leqslant s<r(z)\}$$

is a measurable strict fundamental domain for this equivalence relation. We endow Z_r with the *Bowen-Walters distance*, see [BowW] and particularly the appendix in [BarS].

- For every $t\in\mathbb{R}$, the map $T_r^t:Z_r\to Z_r$ is the map $[z,s]\mapsto[z,s+t]$. Equivalently, when $(z,s)\in\mathscr{F}$ and $t\geqslant 0$, then $T_r^t([z,s])=[T^nz,s']$ where $n\in\mathbb{N}$ and $s'\in\mathbb{R}$ are such that

$$t+s=\sum_{i=0}^{n-1}r(T^iz)+s'\quad\text{and}\quad 0\leqslant s'<r(T^nz)\ .$$

- Denoting by ds the Lebesgue measure on $\mathbb{R}$, the measure μ_r is the pushforward of the restriction to $\mathscr{F}$ of the product measure $d\mu\, ds$ by the restriction to $\mathscr{F}$ of the canonical projection $(Z \times \mathbb{R}) \to Z_r$.

Note that $(T_r^t)_{t\in\mathbb{R}}$ is indeed a continuous one-parameter group of homeomorphisms of Z_r, preserving the measure μ_r. The measure μ_r is finite if and only if $\int_Z r\, d\mu$ is finite, since

$$\|\mu_r\| \;=\; \int_{\mathscr{F}} d\mu ds = \int_Z r\, d\mu \;.$$

We will denote by $(Z, \mu, T)_r$ the continuous time dynamical system $(Z_r, \mu_r, (T_r^t)_{t\in\mathbb{R}})$ thus constructed.

Conversely, let $(Z, \mu, (\phi_t)_{t\in\mathbb{R}})$ be a metric space Z endowed with a continuous one-parameter group of homeomorphisms $(\phi_t)_{t\in\mathbb{R}}$, preserving a (positive Borel) measure μ. Let Y be a *cross-section* of $(\phi_t)_{t\in\mathbb{R}}$, that is a closed subspace of Z such that for every $z \in Z$, the set $\{t \in \mathbb{R} \;:\; \phi_t(z) \in Y\}$ is infinite and discrete. Let $\tau : Y \to \,]0, +\infty[$ be the (continuous) *first return time* on the cross-section Y: for every $y \in Y$,

$$\tau(y) = \min\{t > 0 \;:\; \phi_t(y) \in Y\} \;.$$

Let $\phi_Y : Y \to Y$ be the (homeomorphic) *first return map* to (or *Poincaré map* of) the cross-section Y, defined by

$$\phi_Y : y \mapsto \phi_{\tau(y)}(y) \;.$$

By the invariance of μ under the flow $(\phi_t)_{t\in\mathbb{R}}$, the restriction of μ to

$$\{\phi_t(y) \;:\; y \in Y,\; 0 \leqslant t < \tau(y)\}$$

disintegrates[6] by the (well-defined) map $\phi_t(y) \mapsto y$ over a measure μ_Y on Y, which is invariant under the first return map ϕ_Y:

$$d\mu(\phi_t(y)) \;=\; dt\; d\mu_Y(y) \;.$$

Note that if τ has a positive lower bound and if μ is finite, then μ_Y is finite, since

$$\|\mu\| \geqslant \|\mu_Y\| \;\inf \tau \;,$$

and (Y, μ_Y, ϕ_Y) is a discrete time dynamical system.

Recall that an *isomorphism* from a continuous time dynamical system $(Z, \mu, (\phi_t)_{t\in\mathbb{R}})$ to another one $(Z', \mu', (\phi'_t)_{t\in\mathbb{R}})$ is a homeomorphism between the underlying spaces preserving the underlying measures and commuting with the underlying flows.

Example 5.7. If (Z, μ, T) and (Z', μ', T') are (invertible) discrete time dynamical systems, endowed with roof functions $r : Z \to \,]0, +\infty[$ and $r' : Z' \to \,]0, +\infty[$ respectively, if $\theta : Z \to Z'$ is a measure preserving homeomorphism commuting with the transformations T and T' (that is, $\theta_*\mu = \mu'$, $\theta \circ T = T' \circ \theta$) and such that

$$r' \circ \theta = r \;,$$

then the map $\breve{\theta} : Z_r \to Z'_{r'}$ defined by $[z, s] \mapsto [\theta(z), s]$ is an isomorphim between the suspensions $(Z, \mu, T)_r$ and $(Z', \mu', T')_{r'}$.

[6] with conditional measure on the fiber $\{\phi_t(y) \;:\; 0 \leqslant t < \tau(y)\}$ over $y \in Y$ the image of the Lebesgue measure on $[0, \tau(y)[$ by $t \mapsto \phi_t(y)$

It is well known (see for instance [BrinS, §1.11]) that the above two constructions are inverses one to another, up to isomorphism. In particular, we have the following result.

Proposition 5.8. *With the general notation above Example 5.7, the suspension* $(Y, \mu_Y, \phi_Y)_\tau$ *over* (Y, μ_Y, ϕ_Y) *with roof function* τ *is isomorphic to* $(Z, \mu, (\phi_t)_{t\in\mathbb{R}})$ *by the map* $f_Y : [y, s] \mapsto \phi_s y$. $\square$

In order to describe the continuous time dynamical system $\big(\Gamma\backslash\mathscr{G}X', \frac{m_F}{\|m_F\|}, (\mathtt{g}^t)_{t\in\mathbb{R}}\big)$ as a suspension over a topological Markov shift, we will start by describing it as a suspension of the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}'$. Note that the Patterson densities and Gibbs measures depend not only on the potential, but also on the lengths of the edges.[7] We hence need to relate precisely the continuous time and discrete time situations, and we will use in this Section the left exponent $\sharp$ to indicate a discrete time object whenever needed.

For instance, we set[8] ${}^\sharp X' = |\mathbb{X}'|_1$ and we denote by $({}^\sharp\mathtt{g}^t)_{t\in\mathbb{Z}}$ the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}'$. Note that X' and ${}^\sharp X'$ are equal as topological spaces (but not as metric spaces). The boundaries at infinity of X' and ${}^\sharp X'$, which coincide with their spaces of ends as topological spaces (by the assumption on the lengths of the edges), are hence equal and denoted by $\partial_\infty\mathbb{X}$.

We may assume by Section 3.5 that the potential $\widetilde F : T^1X \to \mathbb{R}$ is the potential $\widetilde F_c$ associated with a system of conductances $\widetilde c$ on the metric tree $(\mathbb{X}, \lambda)$ for Γ. Let $\delta_c = \delta_{F_c}$. We denote by ${}^\sharp\widetilde c : E\mathbb{X} \to \mathbb{R}$ the Γ-invariant system of conductances

$$ {}^\sharp\widetilde c : e \mapsto (\widetilde c(e) - \delta_c)\lambda(e) \tag{5.9}$$

on the simplicial tree $\mathbb{X}$ for Γ, by $\widetilde F_{\sharp c} : T^1({}^\sharp X) \to \mathbb{R}$ its associated potential, and by ${}^\sharp c : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$ and $F_{\sharp c} : \Gamma\backslash T^1({}^\sharp X) \to \mathbb{R}$ their quotient maps.

Note that the inclusion morphism $\operatorname{Aut}(\mathbb{X}, \lambda) \to \operatorname{Aut}(\mathbb{X})$ is a homeomorphism onto its image (for the compact-open topologies), by the assumption of a positive lower bound on the lengths of the edges, hence that Γ is also a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X})$.

Now, let $(\Sigma, \sigma, \mathbb{P})$ be the (two-sided) topological Markov shift conjugated to the discrete time geodesic flow $\big(\Gamma\backslash\mathscr{G}\mathbb{X}',\ {}^\sharp\mathtt{g}^1, \frac{m_{F_{\sharp c}}}{\|m_{F_{\sharp c}}\|}\big)$ by the bilipschitz homeomorphism $\Theta : \Gamma\backslash\mathscr{G}\mathbb{X}' \to \Sigma$ of Theorem 5.1 (where the potential F is replaced by $F_{\sharp c}$). Let $r : \Sigma \to \,]0, +\infty[$ be the map

$$ r : x \mapsto \lambda(e_0^+) \tag{5.10}$$

if $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma$ and $x_0 = (e_0^-, h_0, e_0^+) \in \mathscr{A}$. This map is locally constant, hence continuous on Σ, and has a positive lower bound, since the lengths of the edges of $(\mathbb{X}', \lambda)$ have a positive lower bound.

[7] The fact that the Patterson densities could be singular one with respect to another when the metric varies is a well known phenomenon, even when the potential vanishes. For instance, let $\Sigma = \Gamma\backslash\mathbb{H}^2_{\mathbb{R}}$ and $\Sigma' = \Gamma'\backslash\mathbb{H}^2_{\mathbb{R}}$ be two closed connected hyperbolic surfaces, uniformised by the real hyperbolic plane $(\mathbb{H}^2_{\mathbb{R}}, ds^2_{\rm hyp})$ endowed with torsion free cocompact Fuchsian groups Γ and Γ'. Let $\phi : \Sigma \to \Sigma'$ be a diffeomorphism, with lift $\widetilde\phi : \mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^2_{\mathbb{R}}$. Then Γ is a discrete group of isometries for the two CAT(-1)-metrics $ds^2_{\rm hyp}$ and $\widetilde\phi^* ds^2_{\rm hyp}$. Kuusalo's theorem [Kuu] says that the corresponding two Patterson densities of Γ are absolutely continuous one with respect to the other if and only if ϕ is isotopic to the identity. See an extension of this result in [HeP1]. See also the result of [KaN] which parametrises the Culler-Vogtmann space using Patterson densities for cocompact and free actions of free groups on metric trees.

[8] See Section 2.6 for the definition of the geometric realisation $|\mathbb{Y}|_1$ of a simplicial tree $\mathbb{Y}$.

Theorem 5.9. *Assume that the lengths of the edges of $(\mathbb{X}', \lambda)$ have a finite upper bound, and that the Gibbs measure m_F is finite. Then there exists $a > 0$ such that the continuous time dynamical system $\big(\Gamma\backslash\mathscr{G}X', \frac{m_F}{\|m_F\|}, (\mathtt{g}^t)_{t\in\mathbb{R}}\big)$ is isomorphic to the suspension $(\Sigma, \sigma, a\,\mathbb{P})_r$ over $(\Sigma, \sigma, a\,\mathbb{P})$ with roof function r, by a bilipschitz homeomorphism $\Theta_r : \Gamma\backslash\mathscr{G}X' \to \Sigma_r$.*

Proof. Let

$$Y = \{\ell \in \Gamma\backslash\mathscr{G}X' \;:\; \ell(0) \in \Gamma\backslash V\mathbb{X}\}\,.$$

Then the (closed) subset Y of $\Gamma\backslash\mathscr{G}X'$ is a cross-section of the continuous time geodesic flow $(\mathtt{g}^t)_{t\in\mathbb{R}}$, since every orbit meets Y infinitely many times in a discrete set of times and since the lengths of the edges of $(\mathbb{X}', \lambda)$ have a positive lower bound. Let $\tau : Y \to \,]0, +\infty[$ be the first return time, let μ_Y be the measure on Y (obtained by disintegrating $\frac{m_F}{\|m_F\|}$), and let $\mathtt{g}_Y : Y \to Y$ be the first return map associated with this cross-section Y.

We have a natural reparametrisation map $R : Y \to \Gamma\backslash\mathscr{G}\mathbb{X}'$, defined by $\ell \mapsto {}^\sharp\ell$, where ${}^\sharp\ell(n) = (\mathtt{g}_Y^n\ell)(0)$ is the n-th passage of ℓ in $V\mathbb{X}$, for every $n \in \mathbb{Z}$. Since there exist $m, M > 0$ such that $\lambda(E\mathbb{X}) \subset [m, M]$, the map R is a bilipschitz homeomorphism. It conjugates the first return map $\mathtt{g}_Y$ and the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}'$:

$$R \circ \mathtt{g}_Y = {}^\sharp\mathtt{g}^1 \circ R\,.$$

The main point of this proof is the following result relating the measures μ_Y and $m_{F_{\sharp c}}$.

Lemma 5.10.

(1) The family $(\mu_x^\pm)_{x\in V\mathbb{X}}$ is a Patterson density for $(\Gamma, F_{\sharp c})$ on the boundary at infinity of the simplicial tree $\mathbb{X}'$, and the critical exponent $\delta_{\sharp c}$ of ${}^\sharp c$ is equal to 0.

(2) We have $R_* \frac{\mu_Y}{\|\mu_Y\|} = \frac{m_{F_{\sharp c}}}{\|m_{F_{\sharp c}}\|}$.

Proof. (1) By the definition of the potential associated with a system of conductances,[9] for every $x, y \in V\mathbb{X}'$, if $(e_1, \dots, e_n)$ is the edge path in $\mathbb{X}$ with $o(e_1) = x$ and $t(e_n) = y$, then (noting that the integrals along paths depend on the lengths of the edges, the first one below being in X', the second one in ${}^\sharp X'$)

$$\int_x^y (\widetilde{F_c} - \delta_c) = \sum_{i=1}^n (\widetilde{c}(e_i)\lambda(e_i) - \delta_c\lambda(e_i)) = \int_x^y \widetilde{F_{\sharp c}}\,. \tag{5.11}$$

Let us denote[10] by

$${}^\sharp Q(s) = Q_{\Gamma,\, F_{\sharp c},\, x,\, y}(s) = \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}_{\sharp c} - s)}$$

and

$$Q(s) = Q_{\Gamma,\, F_c - \delta_c,\, x,\, y}(s) = \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}_c - \delta_c - s)}$$

the Poincaré series for the simplicial tree with potential $\widetilde{F}_{\sharp c}$ and for the metric tree with normalised potential $\widetilde{F}_c - \delta_c$, respectively. With M an upper bound on the lengths of the edges of $(\mathbb{X}, \lambda)$, the distances $d_{\sharp X'}$ on ${}^\sharp X'$ and $d_{X'}$ on X' satisfy $d_{\sharp X'} \geqslant \frac{1}{M}\, d_{X'}$ on the pairs

[9] See Section 3.5.
[10] See Section 3.3.

of vertices of $\mathbb{X}'$. We hence have ${}^\sharp Q(s) \leqslant Q(\frac{s}{M}) < +\infty$ if $s > 0$ and ${}^\sharp Q(s) \geqslant Q(\frac{s}{M}) = +\infty$ if $s < 0$. Thus the critical exponent $\delta_{\sharp c}$ of $(\Gamma, F_{\sharp c})$ for the simplicial tree $\mathbb{X}'$ is equal to 0, hence $F_{\sharp c}$ is a normalised potential.

By the definition[11] of the Gibbs cocycles (which uses the normalised potential), Equation (5.11) also implies that the Gibbs cocycles $C^\pm$ and ${}^\sharp C^\pm$ for (Γ, F_c) and $(\Gamma, F_{\sharp c})$ respectively coincide on $\partial_\infty \mathbb{X} \times V\mathbb{X} \times V\mathbb{X}$. Thus by Equations (4.1) and (4.2), the family $(\mu_x^\pm)_{x\in V\mathbb{X}}$ is indeed a Patterson density for $(\Gamma, F_{\sharp c})$: for all $\gamma \in \Gamma$ and $x, y \in V\mathbb{X}$, and for (almost) all $\xi \in \partial_\infty \mathbb{X}$,

$$\gamma_*\mu_x^\pm = \mu_{\gamma x}^\pm \quad \text{and} \quad \frac{d\mu_x^\pm}{d\mu_y^\pm}(\xi) = e^{-\,{}^\sharp C_\xi^\pm(x,y)}\,.$$

(2) We may hence choose these families $(\mu_x^\pm)_{x\in V\mathbb{X}}$ in order to define the Gibbs measure $m_{F_{\sharp c}}$ associated with the potential $F_{\sharp c}$ on $\Gamma\backslash\mathscr{G}\mathbb{X}$. Note that since we will prove that $m_{F_{\sharp c}}$ is finite, the normalised measure $\frac{m_{F_{\sharp c}}}{\|m_{F_{\sharp c}}\|}$ is independent of this choice (see Corollary 4.7).

Let $\widetilde{Y} = \{\ell \in \mathscr{G}X' \;:\; \ell(0) \in V\mathbb{X}'\}$ be the (Γ-invariant) lift of the cross-section Y to $\mathscr{G}X'$, let $\widetilde{R} : \widetilde{Y} \to \mathscr{G}\mathbb{X}'$ be the lift of R, mapping a geodesic line $\ell \in \widetilde{Y}$ to a discrete geodesic line ${}^\sharp\ell$ with same footpoint obtained by reparametrisation, and let $\widetilde{\mu_Y}$ be the measure on $\widetilde{Y}$ whose induced measure on $Y = \Gamma\backslash\widetilde{Y}$ is μ_Y. We have a partition of $\widetilde{Y}$ into the closed-open subsets $\widetilde{Y}_x = \{\ell \in \mathscr{G}X' \;:\; \ell(0) = x\}$ as x varies in $V\mathbb{X}'$.

Let us fix $x \in V\mathbb{X}$. By the definition of μ_Y as a disintegration of $\frac{m_{F_c}}{\|m_{F_c}\|}$ with respect to the continuous time, by lifting to $\mathscr{G}X'$, by using Hopf's parametrisation with respect to x and Equation (4.4) with $x_0 = x$, we have for every $\ell \in \widetilde{Y}_x$,

$$d\widetilde{\mu_Y}(\ell) = \frac{1}{\|m_{F_c}\|}\, d\mu_x^-(\ell_-)\, d\mu_x^+(\ell_+)\,.$$

Note that $\ell(0) = {}^\sharp\ell(0)$, $\ell_- = {}^\sharp\ell_-$, $\ell_+ = {}^\sharp\ell_+$ since the reparametrisation does not change the origin or the two points at infinity. Hence by Assertion (1), we have

$$\widetilde{R}_*(\widetilde{\mu_Y}) = \frac{1}{\|m_{F_c}\|}\, \widetilde{m}_{F_{\sharp c}}\,.$$

As μ_Y is a finite measure since τ has a positive lower bound, this implies that $m_{F_{\sharp c}}$ is finite. By renormalizing as probability measures, this proves Assertion (2). □

Let $a = \|\mu_Y\| > 0$, so that by Lemma 5.10 (2) we have $R_*\mu_Y = \frac{a\, m_{F_{\sharp c}}}{\|m_{F_{\sharp c}}\|}$. Let $\breve{r} : \Gamma\backslash\mathscr{G}\mathbb{X}' \to$ $]0, +\infty]$ be the map

$$\breve{r} : \Gamma\ell \mapsto \lambda\big(\ell([0,1])\big)$$

given by the length for λ of the first edge followed by a discrete geodesic line $\ell \in \mathscr{G}\mathbb{X}'$. Note that $\breve{r}$ is locally constant, hence continuous, and that $\breve{r}$ is a roof function for the discrete time dynamical system $(\Gamma\backslash\mathscr{G}\mathbb{X}',\ {}^\sharp\mathtt{g}^1)$. Also note that

$$\breve{r} \circ R = \tau \quad \text{and} \quad r \circ \Theta = \breve{r}$$

by the definitions of τ and r.

[11] See Section 3.4.

Let us finally define $\Theta_r : \Gamma\backslash\mathscr{G}X' \to \Sigma_r$ as the compositions of the following three maps

$$(\Gamma\backslash\mathscr{G}X', \frac{m_F}{\|m_F\|}, (\mathtt{g}^t)_{t\in\mathbb{R}}) \xrightarrow{f_Y^{-1}} (Y, \mu_Y, g_Y)_\tau \xrightarrow{\check{R}} (\Gamma\backslash\mathscr{G}\mathbb{X}', \frac{a\, m_{F_{\sharp_c}}}{\|m_{F_{\sharp_c}}\|}, {}^\sharp\mathtt{g}^1)_{\check{r}} \xrightarrow{\check{\Theta}} (\Sigma, a\,\mathbb{P}, \sigma)_r\ , \quad (5.12)$$

where the first one is the inverse of the tautological isomorphism given by Proposition 5.8 and the last two ones, given by Example 5.7, are the isomorphisms $\check{R}$ and $\check{\Theta}$ of continuous time dynamical systems obtained by suspensions of the isomorphisms R and Θ of discrete time dynamical systems. It is easy to check that Θ_r is a bilipschitz homeomorphism, using the following description of the Bowen-Walters distance, see for instance [BarS, Appendix].

Proposition 5.11. *Let $(Z, \mu, T)_r$ be the suspension over an invertible dynamical system such that T is a bilipschitz homeomorphism, with roof function r having a positive lower bound and a finite upper bound. Let $d_{\rm BW} : Z_r \times Z_r \to \mathbb{R}$ be the map*[12] *defined (using the canonical representatives) by*

$$d_{\rm BW}([x,s],[x',s']) = \min\{d(x,x') + |s-s'|,\ \ d(Tx,x') + r(x) - s + s',\ \ d(x,Tx') + r(x') + s - s'\}$$

Then there exists a constant $C_{\rm BW} > 0$ such that the Bowen-Walters distance d on Σ_r satisfies

$$\frac{1}{C_{\rm BW}}\, d_{\rm BW} \leqslant d \leqslant C_{\rm BW}\, d_{\rm BW}\ . \quad \square$$

This concludes the proof of Theorem 5.9. □

5.4 The variational principle for metric and simplicial trees

In this Section, we assume that X is the geometric realisation of a locally finite metric tree without terminal vertices $(\mathbb{X}, \lambda)$ (respectively of a locally finite simplicial tree $\mathbb{X}$ without terminal vertices). Let Γ be a nonelementary discrete subgroup of $\mathrm{Aut}(\mathbb{X}, \lambda)$ (respectively $\mathrm{Aut}(\mathbb{X})$).

We relate in this Section the Gibbs measures[13] to the equilibrium states[14] for the continuous time geodesic flow on $\Gamma\backslash\mathscr{G}X$ (respectively for the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$).

When X is a Riemannian manifold with pinched negative curvature such that the derivatives of the sectional curvature are uniformly bounded, and when the potential is Hölder-continuous, the analogs of the results of this Section are due to [PauPS, Theo. 6.1]. Their proofs generalise the proofs of Theorems 1 and 2 of [OtaP], with ideas and techniques going back to [LedS]. When Y is a compact locally CAT(−1)-space, a complete statement about existence, uniqueness and Gibbs property of equilibrium states for any Hölder-continuous potential is given in [ConLT].

The proof of the metric tree case will rely strongly (via the suspension process described in Section 5.3) upon the proof of the simplicial tree case, hence we start by the latter.

[12] The map $d_{\rm BW}$ is actually not a distance, but this proposition says that it may replace the Bowen-Walters true distance when working up to multiplicative constants or bilipschitz homeomorphisms.

[13] See the definition in Sections 4.2 and 4.3.

[14] See the definitions below.

The simplicial tree case.

Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, with geometric realisation $X = |X|_1$. Let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X})$. Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a system of conductances for Γ on $\mathbb{X}$ and $c : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$ its quotient map. Let $\widetilde{F}_c : T^1X \to \mathbb{R}$ be its associated potential, with quotient map $F_c : \Gamma\backslash T^1X \to \mathbb{R}$, and let δ_c be the critical exponent of c.

We define a map $\widetilde{\mathbb{F}}_c : \mathscr{G}\mathbb{X} \to \mathbb{R}$ by

$$\widetilde{\mathbb{F}}_c(\ell) = \widetilde{c}\,(e_0^+(\ell)) = \int_{o(e_0^+(\ell))}^{t(e_0^+(\ell))} \widetilde{F}_c$$

for all $\ell \in \mathscr{G}\mathbb{X}$, where $e_0^+(\ell)$ is the edge of $\mathbb{X}$ in which ℓ enters at time $t = 0$. This map is locally constant, hence continuous, and it is Γ-invariant, hence it induces a continuous map $\mathbb{F}_c : \Gamma\backslash\mathscr{G}\mathbb{X} \to \mathbb{R}$ which is also called a *potential.*[15]

The following result proves that the Gibbs measure of (Γ, F_c) for the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ is an equilibrium state for the potential $\mathbb{F}_c$. We start by recalling the definition of an equilibrium state,[16] see also [Bowe2, Rue3].

Let Z be a locally compact topological space, let $T : Z \to Z$ be a homeomorphism, and let $\phi : Z \to \mathbb{R}$ be a continuous map. Let $\mathscr{M}_\phi$ be the set of Borel probability measures m on Z, invariant under the transformation T, such that the negative part $\phi^- = \max\{0, -\phi\}$ of ϕ is m-integrable. Let $h_m(T)$ be the (metric) entropy of the transformation T with respect to $m \in \mathscr{M}_\phi$ (see for instance [BrinS]). The *metric pressure* for the potential ϕ of a measure m in $\mathscr{M}_\phi$ is

$$P_\phi(m) = h_m(T) + \int_Z \phi\, dm\ .$$

The fact that the negative part of ϕ is m-integrable, which is in particular satisfied if ϕ is bounded, implies that $P_\phi(m)$ is well defined in $\mathbb{R} \cup \{+\infty\}$. The *pressure* of the potential ϕ is the element of $\mathbb{R} \cup \{+\infty\}$ defined by

$$P_\phi = \sup_{m \in \mathscr{M}_\phi} P_\phi(m)\ .$$

A measure m_0 in $\mathscr{M}_\phi$ is an *equilibrium state* for the potential ϕ if $P_\phi(m_0) = P_\phi$.

Theorem 5.12 (The variational principle for simplicial trees)**.** *Let $\mathbb{X}, \Gamma, \widetilde{c}$ be as above. Assume that $\delta_c < +\infty$ and that there exists a finite Gibbs measure m_c for F_c such that the negative part of the potential $\mathbb{F}_c$ is m_c-integrable. Then $\frac{m_c}{\|m_c\|}$ is the unique equilibrium state for the potential $\mathbb{F}_c$ under the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$, and the pressure of $\mathbb{F}_c$ coincides with the critical exponent δ_c of c :*

$$P_{\mathbb{F}_c} = \delta_c\ .$$

In order to prove this result, using the coding of the discrete time geodesic flow given in Section 5.2, the main tool is the following result of J. Buzzi in symbolic dynamics, building on works of Sarig and Buzzi-Sarig, whose proof is given in Appendix A.

[15] See after the proof of Theorem 5.12 for a comment on cohomology classes.

[16] This definition is given for transformations and not for flows, and for possibly unbounded potentials, contrarily to the one the Introduction.

Let $\sigma : \Sigma \to \Sigma$ be a two-sided topological Markov shift[17] with (countable) alphabet $\mathscr{A}$ and transition matrix A, and let $\phi : \Sigma \to \mathbb{R}$ be a continuous map.

For every $n \in \mathbb{N}$, we denote[18] by

$$\operatorname{var}_n \phi = \sup_{\substack{x,\, y \in \Sigma \\ \forall\, i \in \{-n, \dots, n\},\, x_i = y_i}} |\phi(x) - \phi(y)|$$

the *n-variation* of ϕ. For instance, if $\phi(x)$ depends only on x_0 where $x = (x_i)_{i \in \mathbb{Z}}$, then $\operatorname{var}_n \phi = 0$ for every $n \in \mathbb{N}$ (and hence $\sum_{n \in \mathbb{N}} (n+1) \operatorname{var}_n \phi = 0$ converges).

A *weak Gibbs measure* for ϕ with Gibbs constant $C(m) \in \mathbb{R}$ is a σ-invariant (positive Borel) measure m on Σ such that for every $a \in \mathscr{A}$, there exists $c_a \geqslant 1$ such that for all $n \in \mathbb{N} - \{0\}$ and $x \in [a]$ such that $\sigma^n(x) = x$, we have

$$\frac{1}{c_a} \leqslant \frac{m([x_0, \dots, x_{n-1}])}{e^{-C(m)\, n}\; e^{\sum_{i=0}^{n-1} \phi(\sigma^i x)}} \leqslant c_a\ . \tag{5.13}$$

Theorem 5.13 (J. Buzzi, see Corollary A.5)**.** *Let (Σ, σ) be a two-sided transitive topological Markov shift on a countable alphabet and let $\phi : \Sigma \to \mathbb{R}$ be a continuous map such that $\sum_{n \in \mathbb{N}} (n+1) \operatorname{var}_n \phi$ converges. Let m be a weak Gibbs measure for ϕ on Σ with Gibbs constant $C(m)$, such that $\int \phi^-\, dm < +\infty$. Then the pressure of ϕ is finite, equal to $C(m)$, and m is the unique equilibrium state.* □

Proof of Theorem 5.12. In Section 5.2, we constructed a transitive topological Markov shift (Σ, σ) on a countable alphabet $\mathscr{A}$ and a homeomorphism $\Theta : \Gamma \backslash \mathscr{G}\mathbb{X}' \to \Sigma$ which conjugates the time-one discrete geodesic flow g^1 on the nonwandering subset $\Gamma \backslash \mathscr{G}\mathbb{X}'$ of $\Gamma \backslash \mathscr{G}\mathbb{X}$ and the shift σ on Σ (see Theorem 5.1). Let us define a potential $\mathbb{F}_{c,\,\mathrm{symb}} : \Sigma \to \mathbb{R}$ by

$$\mathbb{F}_{c,\,\mathrm{symb}}(x) = c(e_0^+) \tag{5.14}$$

if $x = (x_i)_{i \in \mathbb{Z}} \in \Sigma$ with $x_0 = (e_0^-, h_0, e_0^+)$. Note that this potential is the one denoted by F_{symb} in Equation (5.5), when the potential F on $T^1 X$ is replaced by F_c. By the construction of Θ and the definition of $\mathbb{F}_c$, we have

$$\mathbb{F}_{c,\,\mathrm{symb}} \circ \Theta = \mathbb{F}_c\ . \tag{5.15}$$

Note that all probability measures on $\Gamma \backslash \mathscr{G}\mathbb{X}$ invariant under the discrete time geodesic flow are supported on the nonwandering set $\Gamma \backslash \mathscr{G}\mathbb{X}'$. The pushforward of measures Θ_* hence gives a bijection from the space $\mathscr{M}_{\mathbb{F}_c}$ of g^1-invariant probability measures on $\Gamma \backslash \mathscr{G}\mathbb{X}$ for which the negative part of $\mathbb{F}_c$ is integrable to the space $\mathscr{M}_{\mathbb{F}_{c,\,\mathrm{symb}}}$ of σ-invariant probability measures on Σ for which the negative part of $\mathbb{F}_{c,\,\mathrm{symb}}$ is integrable. This bijection induces a bijection between the subsets of equilibrium states.

Note that $\mathbb{F}_{c,\,\mathrm{symb}}(x)$ depends only on x_0 for every $x = (x_i)_{i \in \mathbb{Z}} \in \Sigma$. Hence as seen above, the series $\sum_{n \in \mathbb{N}} (n+1) \operatorname{var}_n \mathbb{F}_{c,\,\mathrm{symb}}$ converges.

By definition,[19] the measure $\mathbb{P}$ is the pushforward of $\frac{m_c}{\|m_c\|}$ by Θ. Hence, $\mathbb{P}$ is a σ-invariant probability measure on Σ for which the negative part of $\mathbb{F}_{c,\,\mathrm{symb}}$ is integrable, by the assumption of Theorem 5.12. By Proposition 5.5 (3), the measure $\mathbb{P}$ on Σ satisfies the Gibbs property

[17] See Section 5.1 for definitions.
[18] with a shift of indices compared with the notation of Appendix A
[19] See Equation (5.4).

with Gibbs constant δ_c for the potential $\mathbb{F}_{c,\,\rm symb}$, hence[20] satisfies the weak Gibbs property with Gibbs constant δ_c. Theorem 5.12 then follows from Theorem 5.13. $\Box$

Remark. It follows from Equation (5.15), from the remark above Definition 5.3 and from the fact that $\Theta \circ \mathtt{g}^1 = \sigma \circ \Theta$, that if $c, c' : \Gamma\backslash E\mathbb{X}' \to \mathbb{R}$ are cohomologous systems of conductances on $\Gamma\backslash E\mathbb{X}'$, then the corresponding maps $\mathbb{F}_c, \mathbb{F}_{c'} : \Gamma\backslash\mathscr{G}\mathbb{X}' \to \mathbb{R}$ are cohomologous: there exists a continuous map $\mathbb{G} : \Gamma\backslash\mathscr{G}\mathbb{X}' \to \mathbb{R}$ such that for every $\ell \in \Gamma\backslash\mathscr{G}\mathbb{X}'$,

$$\mathbb{F}_{c'}(\ell) - \mathbb{F}_c(\ell) = \mathbb{G}(\mathtt{g}^1\ell) - \mathbb{G}(\ell)\ .$$

Note that given two bounded cohomologous continuous potentials on a topological dynamical system (Z, T) as above, one has finite pressure (resp. admits an equilibrium state) if and only if the other one does, and they have the same pressure and same set of equilibrium states.

The metric tree case.

Let $(\mathbb{X}, \lambda)$ be a locally finite metric tree without terminal vertices with geometric realisation $X = |\mathbb{X}|_\lambda$, let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X}, \lambda)$ and let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a system of conductances for Γ on $\mathbb{X}$. Let $\widetilde{F}_c : T^1X \to \mathbb{R}$ be its associated potential (see Section 3.5), and let $\delta_c = \delta_{F_c}$ be the critical exponent of c.

Recall[21] that we have a canonical projection $\mathscr{G}X \to T^1X$ which associates to a geodesic line ℓ its germ v_ℓ at its footpoint $\ell(0)$. Let $\widetilde{\mathbb{F}}_c^\natural : \mathscr{G}X \to \mathbb{R}$ be the Γ-invariant map obtained by precomposing the potential $\widetilde{F}_c : T^1X \to \mathbb{R}$ with this canonical projection:

$$\widetilde{\mathbb{F}}_c^\natural : \ell \mapsto \ \widetilde{F}_c(v_\ell)\ .$$

Let $\mathbb{F}_c^\natural : \Gamma\backslash\mathscr{G}X \to \mathbb{R}$ be its quotient map, which is continuous as a composition of continuous maps.

The following result proves that the Gibbs measure of (Γ, F_c) for the continuous time geodesic flow on $\Gamma\backslash\mathscr{G}X$, once renormalised to be a probability measure, is an equilibrium state for the potential $\mathbb{F}_c^\natural$. We start by recalling the definition of an equilibrium state for a possibly unbounded potential under a flow.[22]

Given $(Z, (\phi_t)_{t\in\mathbb{R}})$ a topological space endowed with a continuous one-parameter group of homeomorphisms and $\psi : Z \to \mathbb{R}$ a continuous map (called a *potential*), let $\mathscr{M}_\psi$ be the set of Borel probability measures m on Z invariant under the flow $(\phi_t)_{t\in\mathbb{R}}$, such that the negative part $\psi^- = \max\{0, -\psi\}$ of ψ is m-integrable. Let $h_m(\phi_1)$ be the (metric) entropy of the geodesic flow with respect to $m \in \mathscr{M}_\psi$ (see for instance [BrinS]). The *metric pressure* for ψ of a measure $m \in \mathscr{M}_\psi$ is

$$P_\psi(m) = h_m(\phi_1) + \int_Z \psi\ dm\ .$$

The fact that the negative part of ψ is m-integrable, which is in particular satisfied if ψ is bounded, implies that $P_\psi(m)$ is well defined in $\mathbb{R} \cup \{+\infty\}$. The *pressure* of the potential ψ is the element of $\mathbb{R} \cup \{+\infty\}$ defined by

$$P_\psi = \sup_{m\in\mathscr{M}_\psi} P_\psi(m)\ .$$

[20]For every $a \in \mathscr{A}$, for the constant c_a required by the definition of the weak Gibbs property in Equation (5.13), take the constant C_E given by the definition (see Equation (5.1)) of the Gibbs property with $E = \{a\}$.

[21]See Section 2.3.

[22]This requires only minor modifications to the definition given in the Introduction for bounded potentials.

Note that $P_{\psi+c} = P_\psi + c$ for every constant $c \in \mathbb{R}$. An element $m \in \mathscr{M}_\psi$ is an *equilibrium state* for ψ if the least upper bound defining P_ψ is attained on m.

Note that if ψ' is another potential *cohomologous* to ψ, that is, if there exists a continuous map $G : Z \to \mathbb{R}$, differentiable along every orbit of the flow, such that $\psi'(x) - \psi(x) = \frac{d}{dt}_{|t=0} G(\phi_t(x))$, if ψ and ψ' are bounded (so that $\mathscr{M}_{\psi'} = \mathscr{M}_\psi$), then, for every $m \in \mathscr{M}_\psi$, we have $P_{\psi'}(m) = P_\psi(m)$, $P_{\psi'} = P_\psi$ and the equilibrium states for ψ' are exactly the equilibrium states for ψ.

Theorem 5.14 (The variational principle for metric trees)**.** *Let $(\mathbb{X}, \lambda), \Gamma, \widetilde{c}$ be as above. Assume that the lengths of the edges of $(\mathbb{X}, \lambda)$ have a finite upper bound.*[23] *Assume that $\delta_c < +\infty$ and that there exists a finite Gibbs measure m_c for F_c such that the negative part of the potential $\mathbb{F}_c^\natural$ is m_c-integrable. Then $\frac{m_c}{\|m_c\|}$ is the unique equilibrium state for the potential $\mathbb{F}_c^\natural$ under the continuous time geodesic flow on $\Gamma\backslash\mathscr{G}X$, and the pressure of $\mathbb{F}_c^\natural$ coincides with the critical exponent δ_c of c :*

$$P_{\mathbb{F}_c^\natural} = \delta_c \;.$$

Using the description of the continuous time dynamical system $\big(\Gamma\backslash\mathscr{G}X', \frac{m_c}{\|m_c\|}, (\mathtt{g}^t)_{t\in\mathbb{R}}\big)$ as a suspension over a topological Markov shift (see Theorem 5.9), this statement reduces to well-known techniques in the thermodynamic formalism of suspension flows, see for instance [IJT], as well as [BarI, Kemp, IJ, JKL]. Our situation is greatly simplified by the fact that our roof function has a positive lower bound and a finite upper bound, and that our symbolic potential is constant on the 1-cylinders $\{x = (x_i)_{i\in\mathbb{Z}} \in \Sigma \;:\; x_0 = a\}$ for a in the alphabet.

Proof. Since finite measures invariant under the geodesic flow on $\Gamma\backslash\mathscr{G}X$ are supported on its nonwandering set, up to replacing X by $X' = \mathscr{C}\Lambda\Gamma$, we assume that $X = X'$.

Since equilibrium states are unchanged up to adding a constant to the potential, under the assumptions of Theorem 5.14, let us prove that $\frac{m_c}{\|m_c\|}$ is the unique equilibrium state for the potential $\mathbb{F}_c^\natural - \delta_c$ under the continuous time geodesic flow on $\Gamma\backslash\mathscr{G}X$, and that the pressure of $\mathbb{F}_c^\natural - \delta_c$ vanishes. The last claim of Theorem 5.14 follows, since

$$P_{\mathbb{F}_c^\natural} - \delta_c = P_{\mathbb{F}_c^\natural - \delta_c} \;.$$

We refer to the paragraphs before the statement of Theorem 5.9 for the definitions of

- the system of conductances ${}^\sharp\widetilde{c}$ for Γ on the simplicial tree $\mathbb{X}$,
- the (two-sided) topological Markov shift $(\Sigma, \sigma, \mathbb{P})$ on the alphabet $\mathscr{A}$, conjugated to the discrete time geodesic flow $\big(\Gamma\backslash\mathscr{G}\mathbb{X},\ {}^\sharp\mathtt{g}^1, \frac{m_{F_{\sharp_c}}}{\|m_{F_{\sharp_c}}\|}\big)$ by the homeomorphism $\Theta : \Gamma\backslash\mathscr{G}\mathbb{X} \to \Sigma$,
- the roof function $r : \Sigma \to]0, +\infty[$, and
- the suspension $(\Sigma, \sigma, a\,\mathbb{P})_r = (\Sigma_r, (\sigma_r^t)_{t\in\mathbb{R}}, a\,\mathbb{P}_r)$ over $(\Sigma, \sigma, a\,\mathbb{P})$ with roof function r, conjugated to the continuous time geodesic flow $\big(\Gamma\backslash\mathscr{G}X, \frac{m_c}{\|m_c\|}, (\mathtt{g}^t)_{t\in\mathbb{R}}\big)$ by the homeomorphism $\Theta_r : \Gamma\backslash\mathscr{G}X \to \Sigma_r$ defined at the end of the proof of Theorem 5.9. We will always (uniquely) represent the elements of Σ_r as $[x, s]$ with $x \in \Sigma$ and $0 \leqslant s < r(x)$.

We denote by $\mathbb{F}^\natural_{c,\,\mathrm{symb}} : \Sigma_r \to \mathbb{R}$ the potential defined by

$$\mathbb{F}^\natural_{c,\,\mathrm{symb}} = \mathbb{F}_c^\natural \circ \Theta_r^{-1} \;, \tag{5.16}$$

which is continuous as a composition of continuous maps. The key technical observation in this proof is the following one.

[23] They have a positive lower bound by definition, see Section 2.6.

Lemma 5.15. *For every $x \in \Sigma$, we have $\mathbb{F}_{\sharp c,\,\mathrm{symb}}(x) = \int_0^{r(x)} (\mathbb{F}^\natural_{c,\,\mathrm{symb}} - \delta_c)([x,s])\;ds$. For every $x \in \Sigma$, the sign of $\mathbb{F}^\natural_{c,\,\mathrm{symb}}([x,s])$ is constant on $s \in [0, r(x)[$.*

Proof. Let $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma$ and $x_0 = (e_0^-, h_0, e_0^+) \in \mathscr{A}$. By the definition of the first return time r in Equation (5.10), we have in particular

$$r(x) = \lambda(e_0^+)\;.$$

By Equation (5.14) and by the definition of $^\sharp\widetilde{c}$ in Equation (5.9), we have

$$\mathbb{F}_{\sharp c,\,\mathrm{symb}}(x) = {}^\sharp c(e_0^+) = (c(e_0^+) - \delta_c)\lambda(e_0^+)\;.$$

Using in the following sequence of equalities respectively

- the two definitions of the potential $\mathbb{F}^\natural_{c,\,\mathrm{symb}}$ in Equation (5.16) and of the suspension flow $(\sigma_r^t)_{t\in\mathbb{R}}$,
- the fact that the suspension flow $(\sigma_r^t)_{t\in\mathbb{R}}$ is conjugated to the continuous time geodesic flow by Θ_r,
- the definition of Θ_r using the reparametrisation map[24] R of continuous time geodesic lines with origin on vertices to discrete time geodesic lines,[25]
- the definition of the potential $\mathbb{F}^\natural_c$,
- the fact that e_0^+ is the first edge followed by the discrete time geodesic line $\Theta^{-1}x$, hence by the continuous time geodesic line $R^{-1}\Theta^{-1}x$, and the relation between c and the potential F_c associated with c (see Proposition 3.21),

we have

$$\begin{aligned}\int_0^{r(x)} \mathbb{F}^\natural_{c,\,\mathrm{symb}}([x,s])\;ds &= \int_0^{r(x)} \mathbb{F}^\natural_c\big(\Theta_r^{-1}\sigma_r^s[x,0]\big)\;ds = \int_0^{r(x)} \mathbb{F}^\natural_c\big(\mathtt{g}^s\Theta_r^{-1}[x,0]\big)\;ds\\ &= \int_0^{r(x)} \mathbb{F}^\natural_c\big(\mathtt{g}^s R^{-1}\Theta^{-1}x\big)\;ds = \int_0^{\lambda(e_0^+)} F_c\big(v_{\mathtt{g}^s R^{-1}\Theta^{-1}x}\big)\;ds\\ &= c(e_0^+)\lambda(e_0^+)\;.\end{aligned}$$

Since $\int_0^{r(x)} \delta_c\;ds = \delta_c\;\lambda(e_0^+)$, the first claim of Lemma 5.15 follows. The second claim follows by the definition of the potential F_c associated with c, see Equation (3.23). □

By Equation (5.16), the pushforwards of measures by the homeomorphism Θ_r, which conjugates the flows $(\mathtt{g}^t)_{t\in\mathbb{R}}$ and $(\sigma_r^t)_{t\in\mathbb{R}}$, is a bijection from $\mathscr{M}_{\mathbb{F}^\natural_c}$ to $\mathscr{M}_{\mathbb{F}^\natural_{c,\,\mathrm{symb}}}$, such that

$$P_{\mathbb{F}^\natural_{c,\,\mathrm{symb}}}((\Theta_r)_*m) = P_{\mathbb{F}^\natural_c}(m)$$

for every $m \in \mathscr{M}_{\mathbb{F}^\natural_c}$. In particular, we only have to prove that $(\Theta_r)_*\frac{m_c}{\|m_c\|} = a\,\mathbb{P}_r$ is the unique equilibrium state for the potential $\mathbb{F}^\natural_{c,\,\mathrm{symb}} - \delta_c$ under the suspension flow $(\sigma_r^t)_{t\in\mathbb{R}}$, and that the pressure of $\mathbb{F}^\natural_{c,\,\mathrm{symb}} - \delta_c$ vanishes.

The uniqueness follows for instance from [IJT, Theo. 3.5],[26] since the roof function r is locally constant and the potential $g = \mathbb{F}^\natural_{c,\,\mathrm{symb}}$ is such that the map[27] from Σ to $\mathbb{R}$ defined by

[24]See its definition above Lemma 5.10.
[25]See Equation (5.12), with the notation of Example (5.7) and Proposition 5.8.
[26]Note that a topological Markov shift which is (incorrectly) called topologically mixing in [IJT, page 551] is actually (topologically) transitive with the definition in this book, Section 5.1.
[27]denoted by Δ_g in loc. cit.

$x \mapsto \int_0^{r(x)} g([x,s])\,ds$ is locally Hölder-continuous by Lemma 5.15 and since $\mathbb{F}_{\sharp c,\,\mathrm{symb}}$ is locally constant.

Let us now relate the σ-invariant measures on Σ with the $(\sigma_r^t)_{t\in\mathbb{R}}$-invariant measures on Σ_r. Recall that we denote the Lebesgue measure on $\mathbb{R}$ by ds and the points in Σ_r by $[x,s]$ with $x \in \Sigma$ and $0 \leqslant s < r(x)$.

Lemma 5.16. *The map $S : \mathscr{M}_{\mathbb{F}_{\sharp c,\,\mathrm{symb}}} \to \mathscr{M}_{\mathbb{F}^{\natural}_{c,\,\mathrm{symb}}}$, which associates to any measure m in $\mathscr{M}_{\mathbb{F}_{\sharp c,\,\mathrm{symb}}}$ on Σ the measure*

$$d\,S(m)([x,s]) = \frac{1}{\int_\Sigma r\;dm}\,d\mu(x)\,ds$$

on Σ_r, is a bijection, such that, for every $m \in \mathscr{M}_{\mathbb{F}_{\sharp c,\,\mathrm{symb}}}$,

$$P_{\mathbb{F}^{\natural}_{c,\,\mathrm{symb}}-\delta_c}(S(m)) = \frac{P_{\mathbb{F}_{\sharp c,\,\mathrm{symb}}}(m)}{\int_\Sigma r\;dm}\;.$$

Proof. Note that $\int_\Sigma r\;dm$ is the total mass of the measure $d\mu_r([x,s]) = d\mu(x)\,ds$ on Σ_r. In particular, $S(m)$ is indeed a probability measure on Σ_r.

Since r has a positive lower bound and a finite upper bound, it is well known since [AmK], see also [IJT, §2.4], that the map S defined above[28] is a bijection from the set of σ-invariant probability measures m on Σ to the set of $(\sigma_r^t)_{t\in\mathbb{R}}$-invariant probability measures on Σ_r.

Furthermore, for every σ-invariant probability measure m on Σ, we have the following *Kac formula*, by the definition of the probability measure $S(m)$ and by Lemma 5.15,

$$\begin{aligned}\int_{\Sigma_r}\mathbb{F}^{\natural}_{c,\,\mathrm{symb}}\;d\,S(m) - \delta_c &= \int_{\Sigma_r}(\mathbb{F}^{\natural}_{c,\,\mathrm{symb}} - \delta_c)\;d\,S(m)\\ &= \frac{1}{\int_\Sigma r\;dm}\int_{x\in\Sigma}\int_0^{r(x)}(\mathbb{F}^{\natural}_{c,\,\mathrm{symb}} - \delta_c)([x,s])\;dm(x)\,ds\\ &= \frac{1}{\int_\Sigma r\;dm}\int_\Sigma \mathbb{F}_{\sharp c,\,\mathrm{symb}}\;dm\;. \end{aligned}\tag{5.17}$$

By the comment on the signs at the end of Lemma 5.15, this computation also proves that the negative part of $\mathbb{F}^{\natural}_{c,\,\mathrm{symb}}$ is integrable for $S(m)$ if and only if the negative part of $\mathbb{F}_{\sharp c,\,\mathrm{symb}}$ is integrable for m. Hence S is indeed a bijection from $\mathscr{M}_{\mathbb{F}_{\sharp c,\,\mathrm{symb}}}$ to $\mathscr{M}_{\mathbb{F}^{\natural}_{c,\,\mathrm{symb}}}$.

By Abramov's formula [Abr], see also [IJT, Prop. 2.14], we have

$$h_{S(m)}(\sigma_r^1) = \frac{h_m(\sigma)}{\int_\Sigma r\;dm}\;.\tag{5.18}$$

The last claim of Lemma 5.16 follows by summation from Equations (5.17) and (5.18). □

By the proof of Theorem 5.12 (replacing the potential c by ${}^\sharp c$), the pressure of the potential $\mathbb{F}_{\sharp c,\,\mathrm{symb}}$ is equal to the critical exponent $\delta_{\sharp c}$ of the potential ${}^\sharp c$, and by Lemma 5.10 (1), we have $\delta_{\sharp c} = 0$. Hence for every $m \in \mathscr{M}_{\mathbb{F}_{\sharp c,\,\mathrm{symb}}}$, we have

$$P_{\mathbb{F}^{\natural}_{c,\,\mathrm{symb}}-\delta_c}(S(m)) = \frac{P_{\mathbb{F}_{\sharp c,\,\mathrm{symb}}}(m)}{\int_\Sigma r\;dm} \leqslant \frac{P_{\mathbb{F}_{\sharp c,\,\mathrm{symb}}}}{\int_\Sigma r\;dm} = \frac{\delta_{\sharp c}}{\int_\Sigma r\;dm} = 0\;.$$

[28] and denoted by R in [IJT, §2.4]

In particular, the pressure of the potential $\mathbb{F}^\natural_{c,\,\mathrm{symb}} - \delta_c$ is at most 0, since S is a bijection. By the proof of Theorem 5.12 (replacing the potential c by ${}^\sharp c$), we know that $\mathbb{P}$ is an equilibrium state for the potential $\mathbb{F}_{{}^\sharp c,\,\mathrm{symb}}$. Hence

$$P_{\mathbb{F}^\natural_{c,\,\mathrm{symb}} - \delta_c}(S(\mathbb{P})) = \frac{P_{\mathbb{F}_{{}^\sharp c,\,\mathrm{symb}}}(\mathbb{P})}{\int_\Sigma r \; d\mathbb{P}} = 0\ .$$

Therefore, $S(\mathbb{P})$ is an equilibrium state of the potential $\mathbb{F}^\natural_{c,\,\mathrm{symb}} - \delta_c$, with pressure 0. But $a\mathbb{P}_r$, which is equal to $\frac{\mathbb{P}_r}{\|\mathbb{P}_r\|}$ since $a\mathbb{P}_r$ is a probability measure, is by construction equal to $S(\mathbb{P})$. The result follows. □

With slightly different notation, this result implies Theorem 1.1 in the Introduction.

Proof of Theorem 1.1. Any bounded potential $\widetilde{F}$ for Γ on T^1X is cohomologous to a bounded potential $\widetilde{F}_c$ associated with a system of conductances (see Proposition 3.22). If two potentials $\widetilde{F}$ and $\widetilde{F}'$ for Γ on T^1X are cohomologous[29] then the potentials $\ell \mapsto \widetilde{F}(v_\ell)$ and $\ell \mapsto \widetilde{F}'(v_\ell)$ for Γ on $\mathscr{G}X$ are cohomologous for the definition given before the statement of Theorem 5.14. Since the existence and uniqueness of an equilibrium state depends only on the cohomology class of the bounded[30] potentials on $\mathscr{G}X$, the result follows. □

[29]See the definition at the end of Section 3.2.
[30]hence with integrable negative part for any probability measure

Chapter 6

Random walks on weighted graphs of groups

Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, and let $X = |\mathbb{X}|_1$ be its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X})$.

In Section 6.1, given a (logarithmic) system of conductances $c : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$, we define an operator $\boldsymbol{\Delta}_c$ on the functions defined on the set of vertices of the quotient graph of groups $\Gamma\backslash\backslash\mathbb{X}$. This operator is the infinitesimal generator[1] of the random walk on $\Gamma\backslash\backslash\mathbb{X}$ associated with the (normalised) exponential of this system of conductances. When Γ is torsion free and the system of conductances vanishes, the construction recovers the standard Laplace operator on the graph $\Gamma\backslash\mathbb{X}$.

Under appropriate antireversibility assumptions on the system of conductances c, using techniques of Sullivan and Coornaert-Papadopoulos, we prove that the total mass of the Patterson densities is a positive eigenvector for the operator $\boldsymbol{\Delta}_c$ associated with c.

In Section 6.2, we study the nonsymmetric nearest neighbour random walks on $V\mathbb{X}$ associated with antireversible systems of conductivities, and we show that the Patterson densities are the harmonic measures of these random walks.

6.1 Laplacian operators on weighted graphs of groups

Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, and let $X = |\mathbb{X}|_1$ be its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X})$. Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a (Γ-invariant) system of conductances for Γ.

We define $\widetilde{c}^{\,+} = \widetilde{c}$ and $\widetilde{c}^{\,-} : e \mapsto \widetilde{c}(\overline{e})$, which is another system of conductances for Γ. Recall (see Section 3.5) that $\widetilde{c}$ is reversible (respectively antireversible) if $\widetilde{c}^{\,-} = \widetilde{c}^{\,+}$ (respectively $\widetilde{c}^{\,-} = -\,\widetilde{c}^{\,+}$). For every $x \in V\mathbb{X}$, we define

$$\deg_{\widetilde{c}^{\,\pm}}(x) = \sum_{e\in E\mathbb{X},\ o(e)=x} e^{\widetilde{c}^{\,\pm}(e)}\ .$$

The quotient graph of groups $\Gamma\backslash\backslash\mathbb{X}$ is endowed with the quotient maps $c^{\pm} : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$ of $\widetilde{c}^{\,\pm}$.

[1] More precisely, it is the infinitesimal generator of the continuous time random process on the graph $\Gamma\backslash\mathbb{X}$ whose co-called "discrete skeleton" or "jump chain" is the aforementioned random walk. The process waits an exponentially distributed time with parameter 1 at a vertex x, then instantaneously jumps along an edge e starting from x with probability $i(e)e^{c(e)}/\deg_c(x)$. See for instance [AlF, §2.1.2].

Also note that the quantity $\deg_{\widetilde{c}^\pm}(x)$ is constant on the Γ-orbit of x. Hence, it defines a map $\deg_{c^\pm} : \Gamma\backslash V\mathbb{X} \to\ ]0,+\infty[\,$.

On the vector space $\mathbb{C}^{V\mathbb{X}}$ of maps from $V\mathbb{X}$ to $\mathbb{C}$, we consider the operator $\boldsymbol{\Delta}_{\widetilde{c}^\pm}$, called the *(weighted) Laplace operator* of $(\mathbb{X}, c^\pm)$,[2] defined by setting, for all $f \in \mathbb{C}^{V\mathbb{X}}$ and $x \in V\mathbb{X}$,

$$\boldsymbol{\Delta}_{\widetilde{c}^\pm} f(x) = \frac{1}{\deg_{\widetilde{c}^\pm}(x)} \sum_{e\in E\mathbb{X},\ o(e)=x} e^{\widetilde{c}^\pm(e)}\big(f(x) - f(t(e))\big)\ . \tag{6.1}$$

This is the standard Laplace operator[3] of a weighted graph for the weight function $e \mapsto e^{\widetilde{c}^\pm(e)}$, except that usually one requires that $\widetilde{c}(e) = \widetilde{c}(\overline{e})$. Note that $p^\pm(e) = \frac{e^{\widetilde{c}^\pm(e)}}{\deg_{\widetilde{c}^\pm}(o(e))}$ is a Markov transition kernel on the tree $\mathbb{X}$, see the following Section 6.2.

The weighted Laplace operator $\boldsymbol{\Delta}_{\widetilde{c}^\pm}$ is invariant under Γ: for all $f \in \mathbb{C}^{V\mathbb{X}}$ and $\gamma\in\Gamma$, we have

$$\boldsymbol{\Delta}_{\widetilde{c}^\pm}(f\circ\gamma) = (\boldsymbol{\Delta}_{\widetilde{c}^\pm} f)\circ\gamma\ .$$

In particular, this operator induces an operator on functions defined on the quotient graph $\Gamma\backslash\mathbb{X}$, as follows.

Let $(\mathbb{Y}, G_*)$ be a graph of finite groups. We denote[4] by $\mathbb{L}^2(V\mathbb{Y}, \operatorname{vol}_{(\mathbb{Y},G_*)})$ the Hilbert space of maps $f : V\mathbb{Y} \to \mathbb{C}$ with finite norm $\|f\|_{\rm vol}$ for the following scalar product:

$$\langle f, g\rangle_{\rm vol} = \sum_{x\in V\mathbb{Y}} \frac{1}{|G_x|}\ f(x)\ \overline{g(x)}\ .$$

We denote[5] by $\mathbb{L}^2(E\mathbb{Y}, \operatorname{Tvol}_{(\mathbb{Y},G_*)})$ the Hilbert space of maps $\phi : E\mathbb{Y} \to \mathbb{C}$ with finite norm $\|\phi\|_{\rm Tvol}$ for the following scalar product:

$$\langle \phi, \psi\rangle_{\rm Tvol} = \frac{1}{2}\sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}\ \phi(e)\ \overline{\psi(e)}\ .$$

Let $i : E\mathbb{Y} \to \mathbb{N}-\{0\}$ be the index map $i(e) = [G_{o(e)} : G_e]$. For every function $c : E\mathbb{Y} \to \mathbb{R}$, let $\deg_c : V\mathbb{Y} \to \mathbb{R}$ be the positive function defined by

$$\deg_c(x) = \sum_{e\in E\mathbb{Y},\ o(e)=x} i(e)\ e^{c(e)}\ .$$

The *Laplace operator*[6] of $(\mathbb{Y}, G_*, c)$ is the operator $\boldsymbol{\Delta}_c = \boldsymbol{\Delta}_{\mathbb{Y},G_*,c}$ on $\mathbb{L}^2(V\mathbb{Y}, \operatorname{vol}_{\mathbb{Y},G_*})$ defined by

$$\boldsymbol{\Delta}_c f : x \mapsto \frac{1}{\deg_c(x)} \sum_{e\in E\mathbb{Y},\ o(e)=x} i(e)\, e^{c(e)}\big(f(x) - f(t(e))\big)\ .$$

Remark 6.1. (1) Let $(\mathbb{Y}, G_*) = \Gamma\backslash\backslash\mathbb{X}$ be a graph of finite groups with $p : V\mathbb{X} \to V\mathbb{Y} = \Gamma\backslash V\mathbb{X}$ the canonical projection. Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a potential for Γ and let $c : E\mathbb{Y} = \Gamma\backslash E\mathbb{X} \to \mathbb{R}$ be the map induced by $\widetilde{c}$. An easy computation shows that for all $f \in \mathbb{C}^{V\mathbb{Y}}$ and $x \in V\mathbb{Y}$, we have

$$\boldsymbol{\Delta}_c f(x) = \boldsymbol{\Delta}_c \widetilde{f}(\widetilde{x})$$

[2] or on $\mathbb{X}$ associated with the system of conductances $\widetilde{c}^\pm$

[3] See for example [Car] with the opposite choice of the sign, or [ChGY].

[4] See Section 2.6 for the definition of the measure $\operatorname{vol}_{(\mathbb{Y},G_*)}$ on the discrete set $V\mathbb{Y}$.

[5] See Section 2.6 for the definition of the measure $\operatorname{Tvol}_{(\mathbb{Y},G_*)}$ on the discrete set $E\mathbb{Y}$.

[6] See for instance [Mor] when $c = 0$.

if $\widetilde{f} = f \circ p : V\mathbb{X} \to \mathbb{C}$ and $\widetilde{x} \in V\mathbb{X}$ satisfies $p(\widetilde{x}) = x$.

(2) For every $x \in V\mathbb{Y}$, let

$$i(x) = \sum_{e \in E\mathbb{Y},\ o(e)=x} i(e)\ .$$

Then $i(x)$ is the degree of any vertex of any universal cover of $(\mathbb{Y}, G_*)$ above x. In particular, the map $i : V\mathbb{Y} \to \mathbb{R}$ is bounded if and only if the universal cover of $(\mathbb{Y}, G_*)$ has uniformly bounded degrees. When $c = 0$, we denote the Laplace operator by $\mathbf{\Delta} = \mathbf{\Delta}_{\mathbb{Y}, G_*}$ and for every $x \in V\mathbb{Y}$, we have

$$\mathbf{\Delta} f(x) = \frac{1}{i(x)} \sum_{e \in E\mathbb{Y},\ o(e)=x} i(e)\big(f(x) - f(t(e))\big)\ .$$

We thus recover the Laplace operator of [Mor] on the edge-indexed graph $(\mathbb{Y}, i)$.

Proposition 6.2. *Let $(\mathbb{Y}, G_*)$ be a graph of finite groups, whose map $i : V\mathbb{Y} \to \mathbb{R}$ is bounded. Let $c : E\mathbb{Y} \to \mathbb{R}$ be a system of conductances on $\mathbb{Y}$, and let*

$$p(e) = \frac{e^{c(e)}}{\deg_c(o(e))}$$

for every $e \in E\mathbb{Y}$. The following properties hold.

(1) *The Laplace operator $\mathbf{\Delta}_c : \mathbb{L}^2(V\mathbb{Y}, \mathrm{vol}_{(\mathbb{Y},G_*)}) \to \mathbb{L}^2(V\mathbb{Y}, \mathrm{vol}_{(\mathbb{Y},G_*)})$ is linear and bounded.*

(2) *The map $d_c : \mathbb{L}^2(V\mathbb{Y}, \mathrm{vol}_{(\mathbb{Y},G_*)}) \to \mathbb{L}^2(E\mathbb{Y}, \mathrm{Tvol}_{(\mathbb{Y},G_*)})$ defined by*

$$d_c(f) : e \mapsto \sqrt{p(e)}\,\big(f(t(e)) - f(o(e))\big)$$

is linear and bounded, and its dual operator

$$d_c^* : \mathbb{L}^2(E\mathbb{Y}, \mathrm{Tvol}_{(\mathbb{Y},G_*)}) \to \mathbb{L}^2(V\mathbb{Y}, \mathrm{vol}_{(\mathbb{Y},G_*)})$$

is given by

$$d_c^*(\phi) : x \mapsto \sum_{e \in E\mathbb{Y},\ o(e)=x} \frac{i(e)}{2}\Big(\sqrt{p(\overline{e})}\ \phi(\overline{e}) - \sqrt{p(e)}\ \phi(e)\Big)\ .$$

(3) *Assume that c is reversible and that the map $\deg_c : V\mathbb{Y} \to \mathbb{R}$ is constant. Then*

$$\mathbf{\Delta}_c = d_c^* d_c\ .$$

In particular, $\mathbf{\Delta}_c$ is self-adjoint and nonnegative.

Proof. By the assumptions, there exists $M \in \mathbb{N}$ such that $i(x) \leqslant M$ for every $x \in V\mathbb{Y}$, and hence $i(e) \leqslant M$ for every $e \in E\mathbb{Y}$. Note that $i(e) = \frac{|G_{o(e)}|}{|G_e|}$, that $G_e = G_{\overline{e}}$ and that $p(e) \leqslant 1$.

(1) For every $f \in \mathbb{L}^2(V\mathbb{Y}, \mathrm{vol}_{(\mathbb{Y},G_*)})$, using in the following computations

- the Cauchy-Schwarz inequality for the first inequality,
- the fact that for every $x \in V\mathbb{Y}$, we have

$$\sum_{e \in E\mathbb{Y},\ o(e)=x} i(e)^2\, p(e)^2 \leqslant M^2 \sum_{e \in E\mathbb{Y},\ o(e)=x} p(e) \leqslant M^2 \sum_{e \in E\mathbb{Y},\ o(e)=x} i(e)p(e) = M^2$$

for the second inequality,

- the fact that $|G_{o(e)}| \geqslant |G_e|$ for every $e \in E\mathbb{Y}$ for the third inequality, and
- the change of variable $e \mapsto \overline{e}$ in $\sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}|f(t(e))|^2$ (since $G_{\overline{e}} = G_e$) for the first equality on the fifth line of the computations,

we have

$$\begin{aligned}
\|\boldsymbol{\Delta}_c f\|^2_{\rm vol} &= \sum_{x\in V\mathbb{Y}} \frac{1}{|G_x|}\Big|\sum_{e\in E\mathbb{Y},\, o(e)=x} i(e)\,p(e)\big(f(x)-f(t(e))\big)\Big|^2\\
&\leqslant \sum_{x\in V\mathbb{Y}} \frac{1}{|G_x|}\Big(\sum_{e\in E\mathbb{Y},\, o(e)=x} i(e)^2\,p(e)^2\Big)\Big(\sum_{e\in E\mathbb{Y},\, o(e)=x} \big|f(x)-f(t(e))\big|^2\Big)\\
&\leqslant 2\,M^2 \sum_{x\in V\mathbb{Y}} \frac{1}{|G_x|}\Big(\sum_{e\in E\mathbb{Y},\, o(e)=x} \big(\big|f(x)\big|^2+\big|f(t(e))\big|^2\big)\Big)\\
&\leqslant 2\,M^2 \sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}\Big(\big|f(o(e))\big|^2+\big|f(t(e))\big|^2\Big)\\
&= 4\,M^2 \sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}\big|f(o(e))\big|^2 = 4\,M^2 \sum_{x\in V\mathbb{Y}} \frac{1}{|G_x|}\sum_{e\in E\mathbb{Y},\, o(e)=x} i(e)\,|f(x)|^2\\
&= 4\,M^2 \sum_{x\in V\mathbb{Y}} \frac{i(x)}{|G_x|}\,|f(x)|^2 \leqslant 4\,M^3 \sum_{x\in V\mathbb{Y}} \frac{1}{|G_x|}\,|f(x)|^2 = 4\,M^3\,\|f\|^2_{\rm vol}\,.
\end{aligned}$$

Hence the linear operator $\boldsymbol{\Delta}_c$ is bounded.

(2) For every $f \in \mathbb{L}^2(V\mathbb{Y}, {\rm vol}_{(\mathbb{Y},G_*)})$, we have

$$\begin{aligned}
\|d_c f\|^2_{\rm Tvol} &= \frac{1}{2}\sum_{e\in E\mathbb{Y}} \frac{p(e)}{|G_e|}\,\big|f(t(e))-f(o(e))\big|^2\\
&\leqslant \sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}\,\Big(\big|f(t(e))\big|^2+\big|f(o(e))\big|^2\Big) = 2\sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}\,\big|f(o(e))\big|^2\\
&= 2\sum_{e\in E\mathbb{Y}} \frac{i(e)}{|G_{o(e)}|}\,\big|f(o(e))\big|^2 = 2\sum_{x\in V\mathbb{Y}} \frac{i(x)}{|G_x|}\,|f(x)|^2 \leqslant 2\,M\|f\|^2_{\rm vol}\,.
\end{aligned}$$

Hence the linear operator d_c is bounded.

For all $f \in \mathbb{L}^2(V\mathbb{Y}, {\rm vol}_{(\mathbb{Y},G_*)})$ and $\phi \in \mathbb{L}^2(E\mathbb{Y}, {\rm Tvol}_{(\mathbb{Y},G_*)})$, using again the change of variable $e \mapsto \overline{e}$, we have

$$\begin{aligned}
\langle \phi, d_c f\rangle_{\rm Tvol} &= \frac{1}{2}\sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}\,\sqrt{p(e)}\,\phi(e)\,\overline{\big(f(t(e))-f(o(e))\big)}\\
&= \frac{1}{2}\Big(\sum_{e\in E\mathbb{Y}} \frac{\sqrt{p(\overline{e})}}{|G_e|}\,\phi(\overline{e})\,\overline{f(o(e))} - \sum_{e\in E\mathbb{Y}} \frac{\sqrt{p(e)}}{|G_e|}\,\phi(e)\,\overline{f(o(e))}\Big)\\
&= \sum_{x\in V\mathbb{Y}} \frac{1}{|G_x|}\sum_{e\in E\mathbb{Y},\, o(e)=x} \frac{i(e)}{2}\Big(\sqrt{p(\overline{e})}\,\phi(\overline{e})-\sqrt{p(e)}\,\phi(e)\Big)\,\overline{f(x)}\,.
\end{aligned}$$

This gives the formula for d_c^*.

(3) Let $f, g \in \mathbb{L}^2(V\mathbb{Y}, {\rm vol}_{(\mathbb{Y},G_*)})$. Note that $p(e) = p(\overline{e})$[7] by the reversibility of c and the fact that ${\rm deg}_c$ is constant. Hence, by developping the products in the first line and by making the

[7]This is the usual reversibility requirement for the corresponding Markov chain.

change of variable $e \mapsto \overline{e}$ in half the values, we have

$$\begin{aligned}\langle d_c f, d_c g\rangle_{\rm Tvol} &= \frac{1}{2}\sum_{e\in E\mathbb{Y}} \frac{1}{|G_e|}\; p(e)\big(f(t(e))-f(o(e))\big)\big(\,\overline{g(t(e))-g(o(e))}\,\big)\\ &= \sum_{e\in E\mathbb{Y}} \frac{i(e)}{|G_{o(e)}|}\; p(e)\big(f(o(e))\;\overline{g(o(e))}-f(t(e))\;\overline{g(o(e))}\,\big)\\ &= \sum_{x\in V\mathbb{Y}} \frac{1}{|G_x|}\sum_{e\in E\mathbb{Y},\;o(e)=x} i(e)\;p(e)\big(f(x)-f(t(e))\big)\;\overline{g(x)}\\ &= \langle \boldsymbol{\Delta}_c f, g\rangle_{\rm vol}\;.\end{aligned}$$

This proves the last claim in Proposition 6.2. □

The following result is an extension to antireversible systems of conductances of [CoP2, Prop. 3.3] (who treated the case of zero conductances), which is a discrete version of Sullivan's analogous result for hyperbolic manifolds (see [Sul1]). Let $\widetilde{F}_c : T^1X \to \mathbb{R}$ be the potential for Γ associated with $\widetilde{c}$,[8] so that $(\widetilde{F}_c)^\pm = \widetilde{F}_{c^\pm}$, and let δ_c be their common critical exponent. Let $C^\pm : \partial_\infty X \times V\mathbb{X}\times V\mathbb{X} \to \mathbb{R}$ be the associated Gibbs cocycles. Let $(\mu^\pm_x)_{x\in V\mathbb{X}}$ be two Patterson densities on $\partial_\infty X$ for the pairs $(\Gamma, F_{c^\pm})$.

Proposition 6.3. *Assume that $\mathbb{X}$ is $(q+1)$-regular, that the system of conductances $\widetilde{c}$ is antireversible and that the map $\deg_{\widetilde{c}^\pm} : V\mathbb{X}\to\mathbb{R}$ is constant with value $\kappa^\pm$. Then the total mass $\phi_{\mu^\pm} : x \mapsto \|\mu^\pm_x\|$ of the Patterson density is a positive eigenvector associated with the eigenvalue*

$$1-\frac{e^{\delta_c}+q e^{-\delta_c}}{\kappa^\pm}\,.$$

for the Laplace operator $\boldsymbol{\Delta}_{\widetilde{c}^\pm}$ on $\mathbb{C}^{V\mathbb{X}}$.

Proof. Note that the function $\widetilde{c}^\pm : E\mathbb{X}\to\mathbb{R}$ is bounded, since $e^{\widetilde{c}^\pm(e)} \leqslant \deg_{\widetilde{c}^\pm}(o(e)) = \kappa^\pm$ for every $e\in E\mathbb{X}$. Hence $(\widetilde{F}_c)^\pm = \widetilde{F}_{c^\pm}$ is bounded by its definition in Section 3.5. Since $\mathbb{X}$ is $(q+1)$-regular, the critical exponent δ_Γ is finite and hence the critical exponent $\delta_c = \delta_{\Gamma,\,F_{c^\pm}}$ is finite by Lemma 3.17 (6). Since

$$\phi_{\mu^\pm}(x) = \int_{\partial_\infty X} d\mu^\pm_x = \int_{\partial_\infty X} e^{-C^\pm_\xi(x,x_0)}\;d\mu^\pm_{x_0}\,,$$

by Equation (4.2) and by linearity, we only have to prove that for every fixed $\xi\in\partial_\infty X$ the map

$$f : x \mapsto e^{-C^\pm_\xi(x,x_0)}$$

is an eigenvector with eigenvalue $1-\frac{e^{\delta_c}+qe^{-\delta_c}}{\kappa^\pm}$ for $\boldsymbol{\Delta}_{\widetilde{c}^\pm}$.

For every $e\in E\mathbb{X}$, recall[9] that $\partial_e\mathbb{X}$ is the set of points at infinity of the geodesic rays in $\mathbb{X}$ whose initial edge is e. By Equation (3.20) and by the definition of the potential associated with a system of conductances[10], for all $e\in E\mathbb{X}$ and $\eta\in\partial_e\mathbb{X}$, since $t(e)\in[o(e),\eta[$ (independently of the choice of sign $\pm$), we have

$$C^\pm_\eta(t(e),o(e)) = \int_{o(e)}^{t(e)} (\widetilde{F}_{c^\pm}-\delta_c) = \widetilde{c}^\pm(e)-\delta_c\,.$$

[8]See Section 3.5.

[9]See Section 2.6.

[10]See Proposition 3.21 with the edge length map λ constant equal to 1.

Thus if $\xi \in \partial_e \mathbb{X}$, we have

$$f(t(e)) = e^{-C^{\pm}_{\xi}(t(e),\, o(e)) - C^{\pm}_{\xi}(o(e),\, x_0)} = e^{-\widetilde{c}^{\,\pm}(e)+\delta_c}\, f(o(e))\ ,$$

and otherwise

$$f(t(e)) = e^{C^{\pm}_{\xi}(t(\overline{e}),\, o(\overline{e})) - C^{\pm}_{\xi}(o(e),\, x_0)} = e^{\widetilde{c}^{\,\pm}(\overline{e})-\delta_c}\, f(o(e))\ .$$

For every $x \in V\mathbb{X}$, let e_ξ be the unique edge of $\mathbb{X}$ with origin x such that $\xi \in \partial_{e_\xi}\mathbb{X}$. Then,

$$\begin{aligned}
\mathbf{\Delta}_{\widetilde{c}^{\pm}} f(x) &= f(x) - \frac{1}{\deg_{\widetilde{c}^{\pm}}(x)} \sum_{o(e)=x} e^{\widetilde{c}^{\,\pm}(e)} f(t(e)) \\
&= f(x) - \frac{1}{\kappa^{\pm}} e^{\widetilde{c}^{\,\pm}(e_\xi)} f(t(e_\xi)) - \frac{1}{\kappa^{\pm}} \sum_{e \neq e_\xi,\, o(e)=x} e^{\widetilde{c}^{\,\pm}} f(t(e)) \\
&= \Big(1 - \frac{e^{\delta_c}}{\kappa^{\pm}} - \frac{q\, e^{-\delta_c}}{\kappa^{\pm}}\Big) f(x)\ .
\end{aligned}$$

This proves the result. □

Note that the antireversibility of the potential is used in an essential way in order to get the last equation in the proof of Proposition 6.3.

6.2 Patterson densities as harmonic measures for simplicial trees

In this Section, we define and study a Markov chain on the set of vertices of a simplicial tree endowed with a discrete group of automorphisms and with an appropriate system of conductances, such that the associated (nonsymmetric, nearest neighbour) random walk converges almost surely to points in the boundary of the tree, and we prove that the Patterson densities, once normalised, are the corresponding harmonic measures. We thereby generalise the zero potential case treated in [CoP2], which is also a special case of [CoM] when X is a tree under the additional restriction that the discrete group is cocompact. For other connections between harmonic measures and Patterson measures, we refer for instance to [CoM, BlHM, Tan, GouMM] and their references.

Let $\mathbb{X}$ be a $(q+1)$-regular simplicial tree, with $q \geqslant 2$. Let Γ be a nonelementary discrete subgroup of $\mathrm{Aut}(\mathbb{X})$. Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be an antireversible system of conductances for Γ, such that the associated map $\deg_{\widetilde{c}} : V\mathbb{X} \to \mathbb{R}$ on the vertices of $\mathbb{X}$ is constant. Let $(\mu_x)_{x\in V\mathbb{X}}$ be a Patterson density for (Γ, F_c), where F_c is the potential associated with c. We denote by $\phi_\mu : x \mapsto \|\mu_x\|$ the associated total mass function on $V\mathbb{X}$.

We start this Section by recalling a few facts about discrete Markov chains, for which we refer for instance to [Rev, Woe1]. A *state space* is a discrete and countable set I. A *transition kernel* on I is a map $p : I \times I \to [0,1]$ (considered as a square matrix with coefficients in $[0,1]$, and with row and column indices in I) such that for every $x \in I$,

$$\sum_{y\in I} p(x,y) = 1\ .$$

Let λ be a probability measure on I. A (discrete) *Markov chain* on a state space I with initial distribution λ and transition kernel p is a sequence $(Z_n)_{n\in\mathbb{N}}$ of random variables with values in I such that for all $n \in \mathbb{N}$ and $x_0, \dots, x_{n+1} \in I$, the probability of events $\mathbb{P}$ satisfies

(1) $\mathbb{P}[Z_0 = x_0] = \lambda(\{x_0\})$,

(2) $\mathbb{P}[Z_{n+1} = x_{n+1} \mid Z_0 = x_0, Z_1 = x_1, \dots, Z_n = x_n] =$
$\mathbb{P}[Z_{n+1} = x_{n+1} \mid Z_n = x_n] = p(x_n, x_{n+1})$.

The associated *random walk* consists in choosing a point x_0 in I with law λ, and by induction, once x_n is constructed, in choosing x_{n+1} in I with probability $p(x_n, x_{n+1})$. Note that

$$\mathbb{P}[Z_0 = x_0, Z_1 = x_1, \dots, Z_n = x_n] = \lambda(\{x_0\})\, p(x_0, x_1) \dots p(x_{n-1}, x_n) \;.$$

When the initial distribution λ is the unit Dirac mass Δ_x at $x \in I$, the Markov chain is then uniquely determined by its transition kernel p and by x, and is denoted by $(Z^x_n)_{n\in\mathbb{N}}$.

For every $n \in \mathbb{N}$, we denote by $p^{(n)}$ the iterated matrix product of the transition kernel p: we define $p^{(0)}(x, y)$ to be the Kronecker symbol $\delta_{x,y}$ for all $x, y \in I$, and by induction $p^{(n+1)} = p\, p^{(n)}$, that is, for all $x, z \in I$,

$$p^{(n+1)}(x, z) = \sum_{y\in I} p(x, y)p^{(n)}(y, z) \;.$$

Note that

$$p^{(n)}(x, y) = \mathbb{P}[Z^x_n = y]$$

is the probability for the random walk starting at time 0 from x of being at time n at the point y. The *Green kernel* of p is the map G_p from $I \times I$ to $[0, +\infty]$ defined by

$$(x, y) \mapsto G_p(x, y) = \sum_{n\in\mathbb{N}} p^{(n)}(x, y) \;,$$

and its *Green function* is the following power series in the complex variable z :

$$G_p(x, y \mid z) = \sum_{n\in\mathbb{N}} p^{(n)}(x, y)\; z^n \;.$$

Recall that if $G_p(x, y) \neq 0$ for all $x, y \in I$, then the random walk is *recurrent*[11] if $G_p(x, y) = \infty$ for any (hence all) $(x, y) \in I \times I$, and *transient* otherwise. Note that, using again matrix products of $I \times I$ matrices,

$$G_p = \mathrm{Id} + p\, G_p \;. \tag{6.2}$$

We will from now on consider as state space the set $V\mathbb{X}$ of vertices of $\mathbb{X}$. If a Markov chain $(Z^x_n)_{n\in\mathbb{N}}$ starting at time 0 from x converges almost surely in $V\mathbb{X} \cup \partial_\infty X$ to a random variable Z^x_∞, the law of Z^x_∞ is called the *harmonic measure* (or *hitting measure* on the boundary) associated with this Markov chain, and is denoted by

$$\nu_x = (Z^x_\infty)_*(\mathbb{P}) \;.$$

Note that ν_x is a probability measure on $\partial_\infty X$.

For instance, the transition kernel of the simple nearest neighbour random walk on $\mathbb{X}$ is defined by taking as transition kernel the map $\underline{p}$ where

$$\underline{p}(x, y) = \frac{1}{q+1}\; A(x, y)$$

[11] that is, $\mathrm{Card}\{n \in \mathbb{N} \;:\; Z^x_n = y\} = \infty$ for every $y \in I$ (or equivalently, there exists $y \in I$ such that $\mathrm{Card}\{n \in \mathbb{N} \;:\; Z^x_n = y\} = \infty$)

for all $x, y \in V\mathbb{X}$, with $A : V\mathbb{X} \times V\mathbb{X} \to \{0,1\}$ the *adjacency matrix* of the tree $\mathbb{X}$, defined by $A(x,y) = 1$ for any two vertices x, y of $\mathbb{X}$ that are joined by an edge in $\mathbb{X}$ and $A(x,y) = 0$ otherwise. We denote by

$$\underline{\mathfrak{G}}(x, y \mid z) = \sum_{k \in \mathbb{N}} \underline{p}^{(n)}(x,y)\, z^n \,.$$

the Green function of $\underline{p}$, whose radius of convergence is $\underline{r} = \frac{q+1}{2\sqrt{q}}$ and which diverges at $z = \underline{r}$, see for example [Woe1], [Woe2, Ex. 9.82], [LyP2, §6.3].

The antireversible system of conductances $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ defines a cocycle on the set of vertices of $\mathbb{X}$, as follows. For all $u, v \in V\mathbb{X}$, let $c(u,v) = 0$ if $u = v$ and otherwise let

$$c(x,y) = \sum_{i=1}^{n} \widetilde{c}(e_i)\,,$$

where $(e_1, e_2, \dots, e_m)$ is the geodesic edge path in $\mathbb{X}$ from $u = o(e_1)$ to $v = t(e_n)$.

Lemma 6.4.

(1) *For every edge path* $(e'_1, e'_2, \dots, e'_{n'})$ *from* u *to* v*, we have*

$$c(u,v) = \sum_{i=1}^{n'} \widetilde{c}(e'_i)\ .$$

(2) *The map* $c : V\mathbb{X} \times V\mathbb{X} \to \mathbb{R}$ *has the following cocycle property: for all* $u, v, w \in V\mathbb{X}$,

$$c(u,v) + c(v,w) = c(u,w) \quad \text{and hence} \quad c(v,u) = -c(u,v)\ .$$

(3) *We have* $c(u,v) = \int_u^v \widetilde{F}_c$.

(4) *For all* $\xi \in \partial_\infty X$ *and* $u, v \in V\mathbb{X}$*, if* $C^c_\cdot(\cdot,\cdot)$ *is the Gibbs cocycle associated with* $\widetilde{F}_c$*, we have*

$$C^c_\xi(u,v) = c(v,u) + \delta_c \beta_\xi(u,v)\ .$$

Proof. (1) Since $\mathbb{X}$ is a simplicial tree, any nongeodesic edge path from u to v has a back-and-forth on some edge, which contributes to 0 in the sum defining $c(x,y)$ by the antireversibility assumption on the system of conductances. Therefore, by induction, the sum in Assertion (1) indeed does not depend on the choice of the edge path from u to v.

Assertion (2) is immediate from Assertion (1). Assertion (3) follows from the definition of $c(\cdot,\cdot)$ by Proposition 3.21.

(4) For every $\xi \in \partial_\infty X$, if $p \in V\mathbb{X}$ is such that $[u,\xi[\, \cap [v,\xi[\, = [p,\xi[$, then using Equation (3.18) and Assertions (3) and (2), we have

$$\begin{aligned} C^c_\xi(u,v) &= \int_v^p (\widetilde{F}_c - \delta_c) - \int_u^p (\widetilde{F}_c - \delta_c) = c(v,p) - c(u,p) + \delta_c\, \beta_\xi(u,v) \\ &= c(v,u) + \delta_c\, \beta_\xi(u,v)\ . \quad \square \end{aligned}$$

Let

$$\kappa_c = \frac{q+1}{e^{\delta_c} + q\, e^{-\delta_c}}\ ,$$

which belongs to $]0, \frac{q+1}{2\sqrt{q}}]$, with $\kappa_c = \frac{q+1}{2\sqrt{q}}$ if and only if $e^{\delta_c} = \sqrt{q}$. Note that this constant κ_c is less than the radius of convergence $\underline{r} = \frac{q+1}{2\sqrt{q}}$ of the Green function $\underline{\mathfrak{G}}(x, y \mid z)$ if and only if $\delta_c \neq \frac{1}{2}\ln q$. The computation (due to Kesten) of the Green function of $\underline{p}$ is well known, and gives the following formula, see for instance [CoP2, Prop. 3.1]: If $\delta_c \neq \frac{1}{2}\ln q$, then there exists $\alpha > 0$ such that[12] for all $x, y \in V\mathbb{X}$

$$\underline{\mathfrak{G}}(x, y \mid \kappa_c) = \alpha\, e^{-\delta_c\, d(x,y)}\ . \tag{6.3}$$

We now define the transition kernel p_c *associated with*[13] *the (logarithmic) system of conductances* c by, for all $x, y \in V\mathbb{X}$,

$$p_c(x, y) = \kappa_c\, \frac{\phi_\mu(y)}{\phi_\mu(x)}\, e^{c(x,y)}\, \underline{p}(x, y)\ .$$

From now on, we denote by $(Z_n^x)_{n\in\mathbb{N}}$ the Markov chain with initial distribution Δ_x and transition kernel p_c.

Lemma 6.5.

(1) *The map p_c is a transition kernel on $V\mathbb{X}$.*

(2) *The Green kernel $\mathfrak{G}_c = G_{p_c}$ of p_c is*

$$\mathfrak{G}_c(x, y) = e^{c(x,y)}\, \frac{\phi_\mu(y)}{\phi_\mu(x)}\ \underline{\mathfrak{G}}(x, y \mid \kappa_c)\ . \tag{6.4}$$

In particular, the Green kernel of p_c is finite if $\delta_c \neq \frac{1}{2}\ln q$.

(3) *Assume that $\delta_c \neq \frac{1}{2}\ln q$. For all $x, y, z \in V\mathbb{X}$, we have*

$$\frac{\phi_\mu(y)\ \mathfrak{G}_c(y, z)}{\phi_\mu(x)\ \mathfrak{G}_c(x, z)} = e^{c(y,x)+\delta_c(d(x,z)-d(y,z))}\ .$$

If furthermore $z \notin [x, y[$, then, for every $\xi \in \mathscr{O}_x(z)$,[14]

$$\frac{\phi_\mu(y)\ \mathfrak{G}_c(y, z)}{\phi_\mu(x)\ \mathfrak{G}_c(x, z)} = e^{C^c_\xi(x,y)}\ .$$

Proof. (1) By the proof of Proposition 6.3, the positive function ϕ_μ is an eigenvector with eigenvalue $e^{\delta_c} + q\, e^{-\delta_c}$ for the operator

$$f \mapsto \{x \mapsto \sum_{e\in E\mathbb{X},\ o(e)=x} e^{\widetilde{c}(e)}\, f(t(e))\}\ .$$

[12] We actually have $\alpha = \frac{q+1}{e^{\delta_c} + (q-1)\, e^{-\delta_c}}$.

[13] The transition kernel also depends on the choice of the Patterson density if Γ is not of divergence type.

[14] Recall that given $x, z \in V\mathbb{X}$, the shadow $\mathscr{O}_x(z)$ of z seen from x is the set of points at infinity of the geodesic rays from x through z.

Since $\underline{p}(o(e), t(e)) = \frac{1}{q+1}$ for every $e \in E\mathbb{X}$, we hence have

$$\sum_{y \in V\mathbb{X}} p_c(x,y) = \sum_{e \in E\mathbb{X},\ o(e)=x} p_c(x, t(e))$$
$$= \frac{1+q}{(e^{\delta_c} + q\, e^{-\delta_c})\ \phi_\mu(x)} \sum_{e \in E\mathbb{X},\ o(e)=x} e^{\widetilde{c}(e)}\ \phi_\mu(t(e))\ \underline{p}(x, t(e)) = 1 \ .$$

(2) Let us first prove that for all $x, y \in V\mathbb{X}$ and $n \in \mathbb{N}$, we have

$$p_c^{(n)}(x,y) = (\kappa_c)^n\ \frac{\phi_\mu(y)}{\phi_\mu(x)}\ e^{c(x,\,y)}\ \underline{p}^{(n)}(x,y) \ . \tag{6.5}$$

Indeed, by the cocycle property of $c(\cdot,\cdot)$ and by a telescopic cancellation argument, we have

$$\begin{aligned}
&p_c^{(n)}(x,y)\\
&= \sum_{x_1,\dots,\,x_{n-1} \in V\mathbb{X}} p_c(x,x_1)\ p_c(x_1,x_2)\dots p_c(x_{n-2},x_{n-1})\ p_c(x_{n-1},y)\\
&= (\kappa_c)^n\ \frac{\phi_\mu(y)}{\phi_\mu(x)}\ e^{c(x,\,y)} \sum_{x_1,\dots,\,x_{n-1} \in V\mathbb{X}} \underline{p}(x,x_1)\ \underline{p}(x_1,x_2)\dots\ \underline{p}(x_{n-2},x_{n-1})\ \underline{p}(x_{n-1},y)\\
&= (\kappa_c)^n\ \frac{\phi_\mu(y)}{\phi_\mu(x)}\ e^{c(x,\,y)}\ \underline{p}^{(n)}(x,y) \ .
\end{aligned}$$

Equation (6.4) follows from Equation (6.5) by summation on n. As we have already seen, $\kappa_c < \underline{r}$ if and only if $\delta_c \neq \frac{1}{2}\ln q$. The last claim of Assertion (2) follows.

(3) Let $x, y, z \in V\mathbb{X}$. Using (twice) Assertion (2), the cocycle property of c and (twice) Equation (6.3), we have

$$\begin{aligned}
\frac{\mathfrak{G}_c(y,z)}{\mathfrak{G}_c(x,z)} &= \frac{e^{c(y,\,z)}\ \phi_\mu(z)\ \phi_\mu(x)\ \ \underline{\mathfrak{G}}(y,z \mid \kappa_c)}{e^{c(x,\,z)}\ \phi_\mu(y)\ \phi_\mu(z)\ \ \underline{\mathfrak{G}}(x,z \mid \kappa_c)} = e^{c(y,\,x)}\ \frac{\phi_\mu(x)}{\phi_\mu(y)}\ \frac{\alpha\ e^{-\delta_c\, d(y,\,z)}}{\alpha\ e^{-\delta_c\, d(x,\,z)}}\\
&= \frac{\phi_\mu(x)}{\phi_\mu(y)}\ e^{c(y,\,x)+\delta_c(d(x,\,z)-d(y,\,z))} \ .
\end{aligned}$$

This proves the first claim of Assertion (3). Under the additional assumptions on x, y, z, ξ, we have

$$\beta_\xi(x,y) = d(x,z) - d(y,z) \ .$$

The last claim of Assertion (3) hence follows from Lemma 6.4 (4). □

Using the criterion that the random walk starting from a given vertex of $\mathbb{X}$ with transition probabilities p_c is transient if and only if the Green kernel $\mathfrak{G}_c(x,y)$ of p_c is finite (for any, hence for all, $x, y \in V\mathbb{X}$), Lemma 6.5 (2) implies that if $\delta_c \neq \frac{1}{2}\ln q$, then $(Z_n^x)_{n\in\mathbb{N}}$ almost surely leaves every finite subset of $V\mathbb{X}$. The following result strengthens this remark.

Proposition 6.6. *If $\delta_c \neq \frac{1}{2}\ln q$, then for every $x \in V\mathbb{X}$, the Markov chain $(Z_n^x)_{n\in\mathbb{N}}$ (with initial distribution Δ_x and transition kernel p_c) converges almost surely in $V\mathbb{X} \cup \partial_\infty X$ to a random variable with values in $\partial_\infty X$. In particular the harmonic measure ν_x of $(Z_n^x)_{n\in\mathbb{N}}$ is well defined if $\delta_c \neq \frac{1}{2}\ln q$.*

Proof. Since $\mathbb{X}$ is a tree, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $V\mathbb{X}$ such that $d(x_n, x_{n+1}) = 1$ for every $n \in \mathbb{N}$ and which does not converge to a point in $\partial_\infty X$, then there exists a point y such that this sequence passes infinitely often through y, that is, $\{n \in \mathbb{N} \;:\; x_n = y\}$ is infinite. The result then follows from the fact that the Markov chain $(Z_n^x)_{n\in\mathbb{N}}$ is transient since $\delta_c \neq \frac{1}{2}\ln q$. $\square$

The following result, generalising [CoP2, Theo. 4.5] when $\widetilde{c} = 0$, says that the Patterson measures associated with the system of conductances $\widetilde{c}$, once renormalised to probability measures, are exactly the harmonic measures for the random walk with transition probabilities p_c.

Theorem 6.7. *Let $(\mathbb{X}, \Gamma, \widetilde{c}, (\mu_x)_{x\in V\mathbb{X}})$ be as in the beginning of Section 6.2. If $\delta_c \neq \frac{1}{2}\ln q$, then for every $x \in V\mathbb{X}$, the harmonic measure of the Markov chain $(Z_n^x)_{n\in\mathbb{N}}$ is*

$$\nu_x = \frac{\mu_x}{\|\mu_x\|}\;.$$

Proof. We fix $x \in V\mathbb{X}$. For every $n \in \mathbb{N}$, we denote by $S(x,n)$ and $B(x,n)$ the sphere and (closed) ball of centre x and radius n in $V\mathbb{X}$, and we define two maps $f_1, f_2 : V\mathbb{X} \to \mathbb{R}$ with finite support by

$$f_1(z) = \frac{\mu_x(\mathscr{O}_x(z))}{\|\mu_x\|\;\mathfrak{G}_c(x,z)} \quad\text{and}\quad f_2(z) = \frac{\nu_x(\mathscr{O}_x(z))}{\mathfrak{G}_c(x,z)}$$

if $z \in S(x,n)$, and $f_1(z) = f_2(z) = 0$ otherwise. Let us prove that $f_1 = f_2$ for every $n \in \mathbb{N}$. Since $\{\mathscr{O}_x(z) \;:\; z \in V\mathbb{X}\}$ generates the Borel σ-algebra of $\partial_\infty X$, this proves that the Borel measures ν_x and $\frac{\mu_x}{\|\mu_x\|}$ coincide.

We will use the following criterion. For all maps $G : V\mathbb{X} \times V\mathbb{X} \to \mathbb{R}$ and $f : V\mathbb{X} \to \mathbb{R}$ such that f has finite support, let us again denote by $G\;f : V\mathbb{X} \to \mathbb{R}$ the matrix product of the square matrix G and the column matrix f, defined by, for every $y \in V\mathbb{X}$,

$$G\;f(y) = \sum_{z\in V\mathbb{X}} G(y,z) f(z)\;.$$

Lemma 6.8. *For all $f, f' : V\mathbb{X} \to \mathbb{R}$ with finite support, if $\mathfrak{G}_c\;f = \mathfrak{G}_c\;f'$, then $f = f'$.*

Proof. By Equation (6.2), we have

$$f' = \mathfrak{G}_c\;f' - p_c\;\mathfrak{G}_c\;f' = \mathfrak{G}_c\;f - p_c \mathfrak{G}_c\;f = f\;. \quad \square$$

Let us hence fix $n \in \mathbb{N}$ and prove that $\mathfrak{G}_c\;f_1 = \mathfrak{G}_c\;f_2$. Theorem 6.7 then follows.

Step 1: For every $y \in B(x,n)$, since $\{\mathscr{O}_x(z) \;:\; z \in S(x,n)\}$ is a Borel partition of $\partial_\infty X$, by Equation (4.2), since $z \notin [x,y[$ if $z \in S(x,n)$ and $y \in B(x,n)$, and by the second claim of Lemma 6.5 (3), we have

$$\begin{aligned} 1 &= \frac{1}{\phi_\mu(y)} \int_{\partial_\infty X} d\mu_y = \frac{1}{\phi_\mu(y)} \sum_{z\in S(x,n)} \int_{\mathscr{O}_x(z)} e^{-C^c_\xi(y,x)}\; d\mu_x(\xi) \\ &= \frac{1}{\phi_\mu(y)} \sum_{z\in S(x,n)} \int_{\mathscr{O}_x(z)} \frac{\phi_\mu(y)\;\mathfrak{G}_c(y,z)}{\phi_\mu(x)\;\mathfrak{G}_c(x,z)}\; d\mu_x(\xi) \\ &= \sum_{z\in S(x,n)} \mathfrak{G}_c(y,z)\;\frac{\mu_x(\mathscr{O}_x(z))}{\|\mu_x\|\;\mathfrak{G}_c(x,z)} = (\mathfrak{G}_c\;f_1)(y)\;. \end{aligned} \tag{6.6}$$

Step 2: For all $y, z \in V\mathbb{X}$ such that $z \notin [x, y[$, any random walk starting at time 0 from y and converging to a point in $\mathscr{O}_x(z)$ goes through z. Let us denote by $C_x(z)$ the set of vertices different from z on the geodesic rays from z to the points in $\mathscr{O}_x(z)$. Partioning by the last time the random walk passes through z, using the Markov property saying that what happens before the random walk arrives at z and after it leaves z are independent, we have

$$\nu_y(\mathscr{O}_x(z)) = \mathbb{P}[Z^y_\infty \in \mathscr{O}_x(z)] = \mathfrak{G}_c(y, z)\ \mathbb{P}[\forall\, n > 0,\ Z^z_n \in C_x(z)]\ ,$$

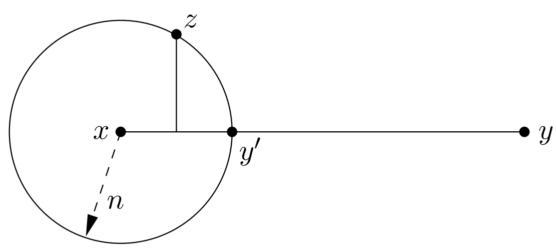
so that

$$\frac{\nu_y(\mathscr{O}_x(z))}{\nu_x(\mathscr{O}_x(z))} = \frac{\mathfrak{G}_c(y, z)}{\mathfrak{G}_c(x, z)}\ . \tag{6.7}$$

Step 3: For every $y \in B(x, n)$, again since $\{\mathscr{O}_x(z)\ :\ z \in S(x, n)\}$ is a Borel partition of $\partial_\infty X$, and by Equation (6.7), we have

$$1 = \|\nu_y\| = \sum_{z\in S(x,n)} \nu_y(\mathscr{O}_x(z)) = \sum_{z\in S(x,n)} \mathfrak{G}_c(y, z)\ \frac{\nu_x(\mathscr{O}_x(z))}{\mathfrak{G}_c(x, z)} = (\mathfrak{G}_c\ f_2)(y)\ . \tag{6.8}$$

Step 4: By Steps 1 and 3, we have $(\mathfrak{G}_c\ f_1)(y) = (\mathfrak{G}_c\ f_2)(y)$ for every $y \in B(x, n)$. Let now $y \in V\mathbb{X}-B(x, n)$. Define $y' \in S(x, n)$ as the point at distance n from x on the geodesic segment $[x, y]$. For every $z \in S(x, n)$, we have $d(y', z) - d(y, z) = -d(y, y')$, which is independent of z.



Since $y' \in B(x, n)$, we have, as just said, $(\mathfrak{G}_c\ f_1)(y') = (\mathfrak{G}_c\ f_2)(y')$. Hence by the first claim of Lemma 6.5 (3), we have

$$\begin{aligned}(\mathfrak{G}_c\ f_1)(y) &= \sum_{z\in S(x,n)} \mathfrak{G}_c(y, z)\ f_1(z)\\ &= \sum_{z\in S(x,n)} e^{c(y,\,y')+\delta_c(d(y',\,z)-d(y,\,z))}\ \frac{\phi_\mu(y')}{\phi_\mu(y)}\ \mathfrak{G}_c(y', z)\ f_1(z)\\ &= e^{c(y,\,y')-\delta_c d(y,\,y')}\ \frac{\phi_\mu(y')}{\phi_\mu(y)}\ (\mathfrak{G}_c\ f_1)(y')\\ &= e^{c(y,\,y')-\delta_c d(y,\,y')}\ \frac{\phi_\mu(y')}{\phi_\mu(y)}\ (\mathfrak{G}_c\ f_2)(y') = (\mathfrak{G}_c\ f_2)(y)\ .\end{aligned}$$

This proves that $\mathfrak{G}_c\ f_1 = \mathfrak{G}_c\ f_2$, thereby concluding the proof of Theorem 6.7. □

Chapter 7

Skinning measures with potential on CAT(−1) spaces

In this Chapter, we introduce skinning measures as weighted pushforwards of the Patterson-Sullivan densities associated with a potential to the unit normal bundles of convex subsets of a CAT(−1) space. The development follows [PaP14a] with modifications to fit the present context.

Let $X, x_0, \Gamma, \widetilde{F}$ be as in the beginning of Chapter 4, and $\widetilde{F}^{\pm}$, $F^{\pm}$, $\delta = \delta_{\Gamma,\,F^{\pm}} < +\infty$ the associated notation. Let $(\mu^{\pm}_x)_{x\in X}$ be (normalised) Patterson densities on $\partial_\infty X$ for the pairs $(\Gamma, F^{\pm})$.[1]

7.1 Skinning measures

Let D be a nonempty proper closed convex subset of X. The *outer skinning measure* $\widetilde{\sigma}^{+}_{D}$ on the outer normal bundle $\partial^1_+ D$ of D and the *inner skinning measure* $\widetilde{\sigma}^{-}_{D}$ on the inner normal bundle $\partial^1_- D$ of D associated with the Patterson densities $(\mu^{\pm}_x)_{x\in X}$ for $(\Gamma, \widetilde{F}^{\pm})$ are the measures $\widetilde{\sigma}^{\pm}_{D} = \widetilde{\sigma}^{\pm}_{D,\,F^{\pm}}$ on $\partial^1_\pm D$ defined by

$$d\widetilde{\sigma}^{\pm}_{D}(\rho) = e^{C^{\pm}_{\rho_\pm}(x_0,\,\rho(0))}\; d\mu^{\pm}_{x_0}(\rho_\pm)\;, \tag{7.1}$$

where $\rho \in \partial^1_\pm D$, using the endpoint homeomorphisms $\rho \mapsto \rho_\pm$ from $\partial^1_\pm D$ to $\partial_\infty X - \partial_\infty D$, and noting that $\rho(0) = P_D(\rho_\pm)$ depends continuously on $\rho_\pm$.

When $\widetilde{F} = 0$, the skinning measure has been defined by Oh and Shah [OhS2] for the outer unit normal bundles of spheres, horospheres and totally geodesic subspaces in real hyperbolic spaces. The definition was generalised in [PaP14a] to the outer unit normal bundles of nonempty proper closed convex sets in Riemannian manifolds with variable negative curvature.

Note that the Gibbs measure is defined on the space $\mathscr{G}X$ of geodesic lines, the potential is defined on the space T^1X of germs at time $t = 0$ of geodesic lines, and since $\partial^1_\pm D$ is contained in $\mathscr{G}_{\pm,\,0}X$ (see Section 2.4), the skinning measures are defined on the spaces $\mathscr{G}_{\pm,\,0}X$ of (generalised) geodesic rays. In the manifold case, all the above spaces are canonically identified with the standard unit tangent bundle, but in general, the natural restriction maps $\mathscr{G}X \to T^1X$ and $\mathscr{G}X \to \mathscr{G}_{\pm,\,0}X$ have infinite (though compact) fibers.

[1] See Section 4.1.

Remark 7.1. (1) If $D=\{x\}$ is a singleton, then

$$d\widetilde{\sigma}^{\pm}_{D}(\rho) = \ d\mu^{\pm}_{x}(\rho_{\pm}) \tag{7.2}$$

where ρ is a geodesic ray starting (at time $t=0$) from x.

(2) When the potential $\widetilde{F}$ is equal to $\widetilde{F}\circ\iota$ (in particular when $F=0$), we have $C^{-}=C^{+}$, and we may (and we will) take $\mu^{-}_{x}=\mu^{+}_{x}$ for all $x\in X$, hence $\iota_{*}\widetilde{m}_F=\widetilde{m}_F$ and

$$\widetilde{\sigma}^{-}_{D}=\iota_{*}\widetilde{\sigma}^{+}_{D}\ .$$

More generally, if $\widetilde{F}$ is reversible, let $\widetilde{G}: T^1X\rightarrow\mathbb{R}$ be a continuous Γ-invariant function such that, for every $\ell\in\mathscr{G}X$, the map $t\mapsto\widetilde{G}(v_{\mathtt{g}^t\ell})$ is differentiable and $\widetilde{F}^{*}(v_\ell)-\widetilde{F}(v_\ell)=\frac{d}{dt}_{|t=0}\widetilde{G}(v_{\mathtt{g}^t\ell})$. Furthermore assume that X is an $\mathbb{R}$-tree or that G is uniformly continuous (for instance Hölder-continuous). Then we have, for every $\rho\in\partial^1_+D$, denoting by $\widehat{\rho}\in\mathscr{G}X$ any extension of ρ to a geodesic line in X,

$$d\iota_{*}\widetilde{\sigma}^{-}_{D}(\rho)=e^{-\widetilde{G}(v_{\widehat{\rho}})}\;d\widetilde{\sigma}^{+}_{D}(\rho)\ .$$

Indeed, for all $x,y\in X$ and $\xi\in\partial_\infty X$, let $\ell_{x,\,\xi}$ be any geodesic line with footpoint $\ell_{x,\,\xi}(0)=x$ and positive endpoint $(\ell_{x,\,\xi})_+=\xi$. Then by Remark 4.11, we have

$$C^{-}_{\xi}(x,y)-C^{+}_{\xi}(x,y)=\widetilde{G}(v_{\ell_{x,\,\xi}})-\widetilde{G}(v_{\ell_{y,\,\xi}})\ ,$$

and we may (and we will) take

$$d\mu^{-}_{x}(\xi)=e^{-\widetilde{G}(v_{\ell_{x,\,\xi}})}\;d\mu^{+}_{x}(\xi)\ .$$

Hence for every $\rho\in\partial^1_+D$, we have

$$\begin{aligned} d\widetilde{\sigma}^{-}_{D}(\iota\rho) &= e^{C^{-}_{(\iota\rho)_-}(x_0,\,(\iota\rho)(0))}\;d\mu^{-}_{x_0}((\iota\rho)_-)=e^{C^{-}_{\rho_+}(x_0,\,\rho(0))}\;d\mu^{-}_{x_0}(\rho_+)\\ &= e^{C^{+}_{\rho_+}(x_0,\,\rho(0))-\widetilde{G}(v_{\ell_{\ell(0),\,\rho_+}})}\;d\mu^{+}_{x_0}(\rho_+)=e^{-\widetilde{G}(v_{\ell_{\rho(0),\,\rho_+}})}\;d\widetilde{\sigma}^{+}_{D}(\rho)\ . \end{aligned}$$

(3) The (normalised) Gibbs cocycle being unchanged when the potential F is replaced by the potential $F+\sigma$ for any constant σ, we may (and will) take the Patterson densities, hence the Gibbs measure and the skinning measures, to be unchanged by such a replacement.

When D is a horoball in X, let us now relate the skinning measures of D with previously known measures on $\partial_\infty X$, constructed using techniques due to Hamenstädt.

Let $\mathscr{H}$ be a horoball centred at a point $\xi\in\partial_\infty X$. Recall that $P_{\mathscr{H}}:\partial_\infty X-\{\xi\}\rightarrow\partial\mathscr{H}$ is the closest point map on $\mathscr{H}$, mapping $\eta\neq\xi$ to the intersection with the boundary of $\mathscr{H}$ of the geodesic line from η to ξ. The following result is proved in [HeP3, §2.3] when $F=0$.

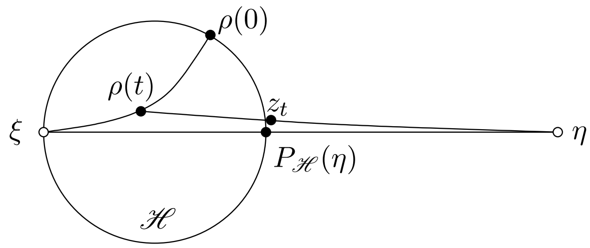
Proposition 7.2. *Let $\rho:[0,+\infty[\,\rightarrow X$ be the geodesic ray starting from any point of the boundary of $\mathscr{H}$ and converging to ξ. The following weak-star limit of measures on $\partial_\infty X-\{\xi\}$*

$$d\mu^{\pm}_{\mathscr{H}}(\eta)=\lim_{t\rightarrow+\infty}\;e^{-\int_{\rho(t)}^{P_{\mathscr{H}}(\eta)}(\widetilde{F}^{\pm}-\delta)}\;d\mu^{\pm}_{\rho(t)}(\eta)$$

exists, and it does not depend on the choice of ρ. The measure $\mu^{\pm}_{\mathscr{H}}$ is invariant under the elements of Γ preserving $\mathscr{H}$, and it satisfies, for every $x\in X$ and (almost) every $\eta\in\partial_\infty X-\{\xi\}$,

$$\frac{d\mu^{\pm}_{\mathscr{H}}}{d\mu^{\pm}_{x}}(\eta)=e^{-C^{\pm}_{\eta}(P_{\mathscr{H}}(\eta),\,x)}\ .$$

Proof. We prove all three assertions simultaneously. Let us fix $x \in X$. For all $t \geqslant 0$ and $\eta \in \partial_\infty X - \{\xi\}$, let z_t be the closest point to $P_{\mathscr{H}}(\eta)$ on the geodesic ray from $\rho(t)$ to η.



Using Equation (4.2) with x replaced by $\rho(t)$ and y by the present x, by the cocycle equation (3.19) and by Equation (3.20) as $z_t \in [\rho(t), \eta[$, we have

$$\begin{aligned} e^{-\int_{\rho(t)}^{P_{\mathscr{H}}(\eta)}(\widetilde{F}^{\pm}-\delta)}\, d\mu^{\pm}_{\rho(t)}(\eta) &= e^{-\int_{\rho(t)}^{P_{\mathscr{H}}(\eta)}(\widetilde{F}^{\pm}-\delta)}\, e^{-C^{\pm}_{\eta}(\rho(t),x)}\, d\mu^{\pm}_{x}(\eta) \\ &= e^{-\int_{\rho(t)}^{P_{\mathscr{H}}(\eta)}(\widetilde{F}^{\pm}-\delta)}\, e^{-C^{\pm}_{\eta}(\rho(t),z_t)}\, e^{-C^{\pm}_{\eta}(z_t,x)}\, d\mu^{\pm}_{x}(\eta) \\ &= e^{-\int_{\rho(t)}^{P_{\mathscr{H}}(\eta)}(\widetilde{F}^{\pm}-\delta)+\int_{\rho(t)}^{z_t}(\widetilde{F}^{\pm}-\delta)}\, e^{-C^{\pm}_{\eta}(z_t,x)}\, d\mu^{\pm}_{x}(\eta)\ . \end{aligned}$$

As $t \to +\infty$, note that z_t converges to $P_{\mathscr{H}}(\eta)$ and that by the HC-property (and since $\widetilde{F}$ is bounded on any compact neighbourhood of $P_{\mathscr{H}}(\eta)$), we have

$$\Big|\int_{\rho(t)}^{P_{\mathscr{H}}(\eta)}(\widetilde{F}^{\pm}-\delta) - \int_{\rho(t)}^{z_t}(\widetilde{F}^{\pm}-\delta)\Big| \to 0\ .$$

The result then follows by the continuity of the Gibbs cocycle (see Proposition 3.20 (3)). □

Using this proposition and the cocycle property of $C^{\pm}$ in the definition (4.4) of the Gibbs measure, we obtain, for every $\ell \in \mathscr{G}X$ such that $\ell_{\pm} \neq \xi$,

$$d\widetilde{m}_F(\ell) = e^{C^-_{\ell_-}(P_{\mathscr{H}}(\ell_-),\ell(0)) + C^+_{\ell_+}(P_{\mathscr{H}}(\ell_+),\ell(0))}\, d\mu^-_{\mathscr{H}}(\ell_-)\, d\mu^+_{\mathscr{H}}(\ell_+)\, dt\ . \tag{7.3}$$

Note that it is easy to see that for every $\rho \in \partial^1_{\pm}\mathscr{H}$, we have

$$d\widetilde{\sigma}^{\pm}_{\mathscr{H}}(\rho) = d\mu^{\pm}_{\mathscr{H}}(\rho_{\pm})\ . \tag{7.4}$$

When $F = 0$, we obtain *Hamenstädt's measure*

$$\mu_{\mathscr{H}} = \lim_{t\to+\infty} e^{\delta_\Gamma t}\mu_{\rho(t)} \tag{7.5}$$

on $\partial_\infty X - \{\xi\}$ associated with the horoball $\mathscr{H}$, which is independent of the choice of the geodesic ray ρ starting from a point of the horosphere $\partial\mathscr{H}$ and converging to ξ. Note that for every $t \geqslant 0$, if $\mathscr{H}[t]$ is the horoball contained in $\mathscr{H}$ whose boundary is at distance t from the boundary of $\mathscr{H}$, we then have

$$\mu_{\mathscr{H}[t]} = e^{-\delta_\Gamma t}\,\mu_{\mathscr{H}}\ . \tag{7.6}$$

Assume till the end of Proposition 7.3 that the potential $\widetilde{F}$ is zero. The next result gathers computations done in [PaP16, PaP17a] of the skinning measures of horoballs and some totally geodesic subspaces, when X is a real or complex hyperbolic space and Γ is a lattice. We consider the notation $\mathbb{H}^n_{\mathbb{R}}$, $\mathbb{H}^n_{\mathbb{C}}$, $\mathscr{H}_\infty$, $\operatorname{Heis}_{2n-1}$, λ_{2n-1} introduced in Section 4.2, and we again endow $T^1\mathbb{H}^n_{\mathbb{R}}$ and $T^1\mathbb{H}^n_{\mathbb{C}}$ with their Sasaki's Riemannian metric. Recall that a *complex hyperbolic line* in $\mathbb{H}^n_{\mathbb{C}}$ is a totally geodesic plane with constant sectional curvature -4.

As the arguments of the following result are purely computational and rather long, we do not copy them in this book, but we refer respectively to the proofs of [PaP16, Prop. 11 (1), (2)] and [PaP17a, Lem. 12 (iv), (v), (vi)]. Analogous computations can be done when X is the quaternionic hyperbolic n-space $\mathbb{H}^n_{\mathbb{H}}$.

Proposition 7.3. *(1) Let Γ be a lattice in $\operatorname{Isom}(\mathbb{H}^n_{\mathbb{R}})$, with Patterson density $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ normalised as in Section 4.2.*

(i) If D is a horoball in $\mathbb{H}^n_{\mathbb{R}}$, if $\operatorname{vol}_{\partial^1_\pm D}$ is the Riemannian measure of the submanifold $\partial^1_\pm D$ in $T^1\mathbb{H}^n_{\mathbb{R}}$, then

$$\widetilde{\sigma}^\pm_D = 2^{n-1}\operatorname{vol}_{\partial^1_\pm D}\ .$$

If the point at infinity of D is a parabolic fixed point of Γ, with stabiliser Γ_D in Γ, then[2]

$$\|\sigma^\pm_D\| = 2^{n-1}\operatorname{Vol}(\Gamma_D\backslash\partial^1_\pm D) = 2^{n-1}\operatorname{Vol}(\Gamma_D\backslash\partial D) = 2^{n-1}(n-1)\operatorname{Vol}(\Gamma_D\backslash D)\ .$$

(ii) If D is a totally geodesic hyperbolic subspace of dimension $k\in\{1,\dots,n-1\}$ in $\mathbb{H}^n_{\mathbb{R}}$, if $\operatorname{vol}_{\partial^1_\pm D}$ is the Riemannian measure of the submanifold $\partial^1_\pm D$ in $T^1\mathbb{H}^n_{\mathbb{R}}$, then

$$\widetilde{\sigma}^\pm_D = \operatorname{vol}_{\partial^1_\pm D}\ .$$

With Γ_D the stabiliser of D in Γ and m the order of the pointwise stabiliser of D in Γ, if $\Gamma_D\backslash D$ has finite volume, then

$$\|\sigma^\pm_D\| = \frac{\operatorname{Vol}(\mathbb{S}^{n-k-1})}{m}\ \operatorname{Vol}(\Gamma_D\backslash D)\ .$$

(2) Let Γ be a lattice in $\operatorname{Isom}(\mathbb{H}^n_{\mathbb{C}})$, with Patterson density $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{C}}}$ normalised as in Section 4.2.

(i) Using the homeomorphism $v\mapsto v_\pm$ from $\partial^1_\pm\mathscr{H}_\infty$ to $\partial_\infty\mathbb{H}^n_{\mathbb{C}}-\{\infty\}=\operatorname{Heis}_{2n-1}$, we have

$$d\widetilde{\sigma}^\pm_{\mathscr{H}_\infty}(v) = d\lambda_{2n-1}(v_\pm)\,.$$

For every horoball D in $\mathbb{H}^n_{\mathbb{C}}$, if $\operatorname{vol}_{\partial D}$ is the Riemannian measure of the hypersurface ∂D in $\mathbb{H}^n_{\mathbb{C}}$, then

$$\pi_*\widetilde{\sigma}^\pm_D = 2\ \operatorname{vol}_{\partial D}\ .$$

If the point at infinity of D is a parabolic fixed point of Γ, with stabiliser Γ_D in Γ, then

$$\|\sigma^\pm_D\| = 4n\ \operatorname{Vol}(\Gamma_D\backslash D)\,.$$

(ii) For every geodesic line D in $\mathbb{H}^n_{\mathbb{C}}$, if $\operatorname{vol}_{\partial^1_\pm D}$ is the Riemannian measure of the submanifold $\partial^1_\pm D$ in $T^1\mathbb{H}^n_{\mathbb{C}}$, we have

$$d\pi_*\widetilde{\sigma}^\pm_D = \frac{n}{4^{n-1}\,(2n-1)}\ d\pi_*\operatorname{vol}_{\partial^1_\pm D}\ .$$

[2] See for instance [Hers, p. 473] for the last equality.

With Γ_D the stabiliser of D in Γ and m the order of the pointwise stabiliser of D in Γ, if $\Gamma_D\backslash D$ has finite length, then

$$\|\sigma^{\pm}_D\| = \frac{2\,\pi^{n-1}\,n!}{m\,(2n-1)!}\;\operatorname{Vol}(\Gamma_D\backslash D)\,.$$

(iii) For every complex geodesic line D in $\mathbb{H}^n_{\mathbb{C}}$, if $\operatorname{vol}_{\partial^1_+ D}$ is the Riemannian measure of the submanifold $\partial^1_+ D$ in $T^1\mathbb{H}^n_{\mathbb{C}}$, we have

$$d\pi_*\widetilde{\sigma}^+_D = \frac{1}{2^{2n-3}}\;d\pi_*\operatorname{vol}_{\partial^1_+ D}\;.$$

With Γ_D the stabiliser of D in Γ and m the order of the pointwise stabiliser of D in Γ, if $\Gamma_D\backslash D$ has finite area, then

$$\|\sigma^{+}_D\| = \frac{\pi^{n-1}}{m\,4^{n-2}\,(n-2)!}\;\operatorname{Vol}(\Gamma_D\backslash D)\,.\quad\square$$

The following results give the basic properties of the skinning measures analogous to those in [PaP14a, Sect. 3] when the potential is zero.

Proposition 7.4. *Let D be a nonempty proper closed convex subset of X, and let $\widetilde{\sigma}^{\pm}_D$ be the skinning measures on $\partial^1_{\pm}D$ for the potential $\widetilde{F}$.*
(i) *The skinning measures $\widetilde{\sigma}^{\pm}_D$ are independent of x_0.*
(ii) *For all $\gamma\in\Gamma$, we have $\gamma_*\widetilde{\sigma}^{\pm}_D=\widetilde{\sigma}^{\pm}_{\gamma D}$. In particular, the measures $\widetilde{\sigma}^{\pm}_D$ are invariant under the stabiliser of D in Γ.*
(iii) *For all $s\geqslant 0$ and $w\in\partial^1_{\pm}D$, we have*[3]

$$\begin{aligned} d\,\widetilde{\sigma}^{\pm}_{\mathscr{N}_s D}\big((\mathsf{g}^{\pm s}w)\big|_{\pm[0,+\infty[}\big) &= e^{C^{\pm}_{w_\pm}(\pi(w),\,\pi(\mathsf{g}^{\pm s}w))}\;d\widetilde{\sigma}^{\pm}_D(w)\\ &= e^{-\int_{\pi(w)}^{\pi(\mathsf{g}^{\pm s}w)}(\widetilde{F}^{\pm}-\delta)}\;d\widetilde{\sigma}^{\pm}_D(w)\;. \end{aligned}$$

(iv) *The support of $\widetilde{\sigma}^{\pm}_D$ is*

$$\{v\in\partial^1_{\pm}D : v_{\pm}\in\Lambda\Gamma\} = P^{\pm}_D(\Lambda\Gamma-(\Lambda\Gamma\cap\partial_\infty D))\;.$$

In particular, $\widetilde{\sigma}^{\pm}_D$ is the zero measure if and only if $\Lambda\Gamma$ is contained in $\partial_\infty D$.

For future use, the version[4] of Assertion (iii) when $F=0$ is

$$\frac{d(\mathsf{g}^{\pm s})_*\widetilde{\sigma}^{\pm}_D}{d\,\widetilde{\sigma}^{\pm}_{\mathscr{N}_s D}}(\mathsf{g}^{\pm s}w) = e^{-\delta_\Gamma\, s}\;, \tag{7.7}$$

where $w\in\partial^1_{\pm}D$ and we again denote by $\mathsf{g}^{\pm s}$ the map from $\partial^1_{\pm}D$ to $\partial^1_{\pm}\mathscr{N}_1 D$ defined by $w\mapsto(\mathsf{g}^{\pm s}w)\big|_{\pm[0,+\infty[}$.

As another particular case of Assertion (iii) for future use, consider the case when $X=|\mathbb{X}|_\lambda$ is the geometric realisation of a metric tree $(\mathbb{X},\lambda)$ and when $\widetilde{F}=\widetilde{F}_c$ is the potential associated

[3] denoting by $(\mathsf{g}^{\pm s}w)_{|\pm[0,+\infty[}$ the element of $\mathscr{G}_{\pm,\,0}$ which coincides with $\mathsf{g}^{\pm s}w$ on $\pm[0,+\infty[$
[4] contained in [PaP14a, Prop. 4]

with a system of conductances $\widetilde{c}$ on $\mathbb{X}$ for a subgroup Γ of $\operatorname{Aut}(\mathbb{X})$ (see Equation (3.23) and Proposition 3.22). Then for all $w \in \partial^1_+ D$ (respectively $w \in \partial^1_- D$), if e_w is the first (respectively the last) edge followed by w, with length $\lambda(e_w)$, then

$$\int_{\pi(w)}^{\pi(\mathtt{g}^{\pm\lambda(e_w)}w)} \widetilde{F}^{\pm} = \widetilde{c}(e_w)\ \lambda(e_w)$$

by Proposition 3.21, so that

$$d\,\widetilde{\sigma}^{\pm}_{\mathscr{N}_s D}((\mathtt{g}^{\pm\lambda(e_w)}w)_{|\pm[0,+\infty[}) = e^{-(\widetilde{c}(e_w)+\delta)\lambda(e_w)}\ d\widetilde{\sigma}^{\pm}_D(w)\ . \tag{7.8}$$

Proof. The proofs of the claims are straightforward modifications of those for zero potential in [PaP14a, Prop. 4]. We give details of the proofs for the measure $\widetilde{\sigma}^{+}_D$, the case of $\widetilde{\sigma}^{-}_D$ being similar.

(i) The claim follows from Equation (4.2) and the cocycle property (3.19).

(ii) The claim follows from Equation (4.1), the first part of Equation (3.19) and Claim (i).

(iii) Since $\big((\mathtt{g}^s w)_{|[0,+\infty[}\big)_+ = w_+$ and since $w \in \partial^1_+ D$ if and only if $(\mathtt{g}^s w)_{|[0,+\infty[} \in \partial^1_+ \mathscr{N}_s D$, we have, using the definition (7.1) of the skinning measure and the cocycle property (3.19), for all $s \geqslant 0$,

$$d\,\widetilde{\sigma}^{+}_{\mathscr{N}_s D}((\mathtt{g}^s w)_{|[0,+\infty[}) = e^{C^+_{w_+}(x_0,\,\pi(\mathtt{g}^s w))}\ d\mu^+_{x_0}(w_+) = e^{C^+_{w_+}(\pi(w),\,\pi(\mathtt{g}^s w))}\ d\,\widetilde{\sigma}^{+}_D(w)\,.$$

This proves Claim (iii) for $\widetilde{\sigma}^{+}_D$, using Equation (3.20).

(iv) The claims follow from the fact that the support of any Patterson measure is $\Lambda\Gamma$, see Subsection 4.1. □

Given two nonempty closed convex subsets D and D' of X, let

$$A_{D,D'} = \partial_\infty X - (\partial_\infty D \cup \partial_\infty D')$$

and let $h^{\pm}_{D,D'} : P^{\pm}_D(A_{D,D'}) \to P^{\pm}_{D'}(A_{D,D'})$ be the restriction of $P^{\pm}_{D'} \circ (P^{\pm}_D)^{-1}$ to $P^{\pm}_D(A_{D,D'})$. It is a homeomorphism between open subsets of $\partial^1_{\pm} D$ and $\partial^1_{\pm} D'$, associating to the element w in the domain the unique element w' in the range with $w'_{\pm} = w_{\pm}$.

Proposition 7.5. *Let D and D' be nonempty closed convex subsets of X and let $h^{\pm} = h^{\pm}_{D,D'}$. The measures $(h^{\pm})_*\,\widetilde{\sigma}^{\pm}_D$ and $\widetilde{\sigma}^{\pm}_{D'}$ on $P^{\pm}_{D'}(A_{D,D'})$ are absolutely continuous one with respect to the other, with*

$$\frac{d(h^{\pm})_*\,\widetilde{\sigma}^{\pm}_D}{d\widetilde{\sigma}^{\pm}_{D'}}(w') = e^{-C^{\pm}_{w_\pm}(\pi(w),\,\pi(w'))},$$

for (almost) all $w \in P^{\pm}_D(A_{D,D'})$ and $w' = h^{\pm}(w)$.

Proof. As $w'_{\pm} = w_{\pm}$, we have

$$\begin{aligned} d\widetilde{\sigma}^{\pm}_{D'}(w') &= e^{C^{\pm}_{w'_\pm}(x_0,\,\pi(w'))}\ d\mu^{\pm}_{x_0}(w'_\pm) = e^{C^{\pm}_{w'_\pm}(x_0,\,\pi(w))}\ e^{C^{\pm}_{w'_\pm}(\pi(w),\,\pi(w'))}\ d\mu^{\pm}_{x_0}(w'_\pm) \\ &= e^{C^{\pm}_{w_\pm}(\pi(w),\,\pi(w'))}\ d\widetilde{\sigma}^{\pm}_D(w) \end{aligned}$$

using the definition (7.1) of the skinning measure and the cocycle property (3.19). □

Let $w \in \mathscr{G}_{\pm} X$. With $N^{\pm}_{w} : W^{\pm}(w) \to \partial^1_{\mp} HB_{\pm}(w)$ the canonical homeomorphism defined in Section 2.4, we define the *skinning measures* $\mu_{W^{\pm}(w)}$ on the strong stable or strong unstable leaves $W^{\pm}(w)$ by

$$\mu_{W^{\pm}(w)} = ((N^{\pm}_{w})^{-1})_{*} \widetilde{\sigma}^{\mp}_{HB_{\pm}(w)} ,$$

so that

$$d\mu_{W^{\pm}(w)}(\ell) = e^{C^{\mp}_{\ell_{\mp}}(x_0,\, \ell(0))}\, d\mu^{\mp}_{x_0}(\ell_{\mp}) \tag{7.9}$$

for every $\ell \in W^{\pm}(w)$. By Proposition 7.4 (ii) and the naturality of $N^{\pm}_{w}$, for every $\gamma \in \Gamma$, we have

$$\gamma_{*} \mu_{W^{\pm}(w)} = \mu_{W^{\pm}(\gamma w)} \,. \tag{7.10}$$

By Proposition 7.4 (iv), the support of $\mu_{W^{\pm}(w)}$ is $\{\ell \in W^{\pm}(w) \;:\; \ell_{\mp} \in \Lambda\Gamma\}$. For all $t \in \mathbb{R}$ and $\ell \in W^{\pm}(w)$, we have, using Equations (7.9), (3.19) and (3.20), and since $\ell_{\pm} = w_{\pm}$,

$$\frac{d(\mathtt{g}^{-t})_{*}\mu_{W^{\pm}(w)}}{d\,\mu_{W^{\pm}(\mathtt{g}^{t} w)}}(\mathtt{g}^{t}\ell) = e^{C^{\mp}_{\ell_{\mp}}(\ell(t),\, \ell(0))} = e^{C^{\pm}_{w_{\pm}}(\ell(0),\, \ell(t))} \,. \tag{7.11}$$

Let $w \in \mathscr{G}_{\pm} X$. The homeomorphisms $W^{\pm}(w) \times \mathbb{R} \to W^{0\pm}(w)$, defined by

$$(\ell, s) \mapsto \ell' = \mathtt{g}^{s} \ell \,,$$

conjugate the actions of $\mathbb{R}$ by translation on the second factor of the domain and by the geodesic flow on the range, and the actions of Γ (trivial on the second factor of the domain). Let us consider the measures $\nu^{\mp}_{w}$ on $W^{0\pm}(w)$ given, using the above homeomorphism, by

$$d\nu^{\mp}_{w}(\ell') = e^{C^{\pm}_{w_{\pm}}(w(0),\, \ell(0))}\, d\mu_{W^{\pm}(w)}(\ell)\, ds \,. \tag{7.12}$$

They satisfy $(\mathtt{g}^{t})_{*}\nu^{\pm}_{w} = \nu^{\pm}_{w}$ for all $t \in \mathbb{R}$ (since if $\ell' = \mathtt{g}^{s}\ell$, then $\mathtt{g}^{-t}\ell' = \mathtt{g}^{s-t}\ell$, and by invariance under translations of the Lebesgue measure on $\mathbb{R}$). Furthermore, $\gamma_{*}\nu^{\pm}_{w} = \nu^{\pm}_{\gamma w}$ for all $\gamma \in \Gamma$. In general, they depend on w, not only on $W^{\pm}(w)$. Furthermore, the support of $\nu^{\pm}_{w}$ is $\{\ell' \in W^{0\pm}(w) \;:\; \ell'_{\mp} \in \Lambda\Gamma\}$. These properties follow easily from the properties of the skinning measures on the strong stable or strong unstable leaves.

Lemma 7.6. *(i) For every nonempty proper closed convex subset D' in X, there exists $R_0 > 0$ such that for all $R \geqslant R_0$, $\eta > 0$, and $w \in \partial^1_{\pm} D'$, we have $\nu^{\mp}_{w}(V^{\pm}_{w,\,\eta,\,R}) > 0$.*[5]

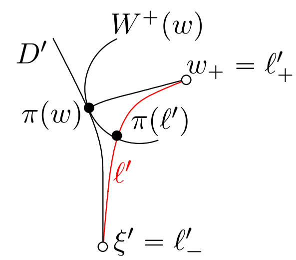
(ii) For all $w \in \mathscr{G}_{\pm} X$ and $t \in \mathbb{R}$, the measures $\nu^{\mp}_{\mathtt{g}^{t} w}$ and $\nu^{\mp}_{w}$ are proportional:

$$\nu^{\mp}_{\mathtt{g}^{t} w} = e^{C^{\pm}_{w_{\pm}}(w(t),\, w(0))}\, \nu^{\mp}_{w} \,.$$

Proof. (i) Let us show, as in [PaP14a, Lem. 7], that there exists $R_0 > 0$ (depending only on D' and on the Patterson densities) such that for all $R \geqslant R_0$, $w \in \partial^1_{+} D'$ and $w' \in \partial^1_{-} D'$, we have $\mu_{W^{+}(w)}(B^{+}(w, R)) > 0$ and $\mu_{W^{-}(w')}(B^{-}(w', R)) > 0$. The result follows from this by the definitions of $\nu^{\mp}_{w}$ and $V^{\pm}_{w,\,\eta,\,R}$.

[5] See Section 2.4 for the definition of $V^{\pm}_{w,\,\eta,\,R}$.

We give the proof of the claim on $B^+(w,R)$, the proof of the claim on $B^-(w,R)$ is similar. For all $w\in\partial^1_+D'$ and $\xi'\in D'\cup\partial_\infty D'$, by a standard comparison and convexity argument applied to the geodesic triangle with vertices $\pi(w), w_+, \xi'$, the point $\pi(w)$ is at distance at most $2\ln(\frac{1+\sqrt5}{2})$ from the intersection between the stable horosphere $H_+(w)$ and the geodesic ray or line between ξ' and w_+.



The triangle inequality and the definition of Hamenstädt distances imply that, for all $\ell,\ell'\in W^+(w)$,

$$d_{W^+(w)}(\ell,\ell')\leqslant e^{\frac12 d(\pi(\ell),\,\pi(\ell'))}\;. \tag{7.13}$$

Hence, for every $\xi'\in\partial_\infty D'$, for every extension $\widehat{w}\in\mathscr{G}X$ of w, if ℓ' is the element of $W^+(w)$ such that $\ell'_-=\xi'$, we have

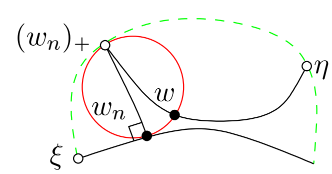
$$d_{W^+(w)}(\widehat{w},\ell')\leqslant\frac{1+\sqrt5}{2}\;.$$

Thus, if $\partial_\infty D'\cap\Lambda\Gamma\neq\varnothing$, then we may take $R_0=2>\frac{1+\sqrt5}{2}$, since by Proposition 7.4 (iv), the support of $\mu_{W^+(w)}$ is $\{\ell\in W^+(w)\;:\;\ell_-\in\Lambda\Gamma\}$.

Assume now that $\partial_\infty D'\cap\Lambda\Gamma=\varnothing$. For a contradiction, assume that, for all $n\in\mathbb N$, there exists $w_n\in\partial^1_+D'$ such that $\mu_{W^+(w_n)}(B^+(w_n,n))=0$. Assume first that $(w_n)_{n\in\mathbb N}$ has a convergent subsequence with limit $w\in\partial^1_+D'$. Since the measure $\mu_{W^+(w')}$ depends continuously on $w'\in\mathscr{G}_+X$, for every compact subset K of $W^+(w)$, we have $\mu_{W^+(w)}(K)=0$. By Proposition 7.4 (iv) and by Equation (7.9), this implies that the support of the Patterson measure $\mu^-_{x_0}$, which is the limit set of Γ, is contained in $\{w_+\}$. This is impossible, since Γ is nonelementary.

In the remaining case, the points $\pi(w_n)$ in D' converge, up to extracting a subsequence, to a point ξ in $\partial_\infty D'$. By definition of the map $P^+_{D'}$ and of ∂^1_+D', the points at infinity $(w_n)_+$ converge to ξ. For every η in $\partial_\infty X$ different from ξ, the geodesic lines from η to $(w_n)_+$ converge to the geodesic line from η to ξ.

By convexity, if n is large enough, the geodesic line $]\eta,(w_n)_+[$ meets $\mathscr{N}_1D'$, hence passes at distance at most 2 from $\pi(w_n)$. This implies by Equation (7.13) that if n is large enough, then there exists $\ell\in B^+(w_n,\,n)$ such that $\eta=\ell_-$.



Since we assumed that $\mu_{W^+(w_n)}(B^+(w_n,n))=0$ for all $n\in\mathbb N$, Proposition 7.4 (iv) implies that we have $\eta\notin\Lambda\Gamma$. Hence $\Lambda\Gamma$ is contained in $\{\xi\}$, a contradiction since Γ is nonelementary.

(ii) For all $w\in\mathscr{G}_\pm X$, $s,t\in\mathbb R$ and $\ell\in W^\pm(w)$, we have by Equations (7.12) and (7.11), and by the cocycle property (3.19) of $C^\pm$,

$$\begin{aligned}
d\nu^{\mp}_{\mathtt g^t w}(\mathtt g^s\ell)&=d\nu^{\mp}_{\mathtt g^t w}(\mathtt g^{s-t}\mathtt g^t\ell)=e^{C^{\pm}_{(\mathtt g^t w)_\pm}(\mathtt g^t w(0),\,\mathtt g^t\ell(0))}\;d\mu_{W^\pm(\mathtt g^t w)}(\mathtt g^t\ell)\;d(s-t)\\
&=e^{C^{\pm}_{w_\pm}(w(t),\,\ell(t))}\;e^{-C^{\pm}_{w_\pm}(\ell(0),\,\ell(t))}\;d\mu_{W^\pm(w)}(\ell)\;ds\\
&=e^{C^{\pm}_{w_\pm}(w(t),\,\ell(0))}\;e^{-C^{\pm}_{w_\pm}(w(0),\,\ell(0))}\;d\nu^{\mp}_{w}(\mathtt g^s\ell)\\
&=e^{C^{\pm}_{w_\pm}(w(t),\,w(0))}d\nu^{\mp}_{w}(\mathtt g^s\ell)\;.\quad\square
\end{aligned}$$

The following disintegration result of the Gibbs measure over the skinning measures of any closed convex subset is a crucial tool for our equidistribution and counting results. Recall the definition in Equation (2.15) of the flow-invariant open sets $\mathscr{U}_D^{\pm}$ and the definition of the fibrations $f_D^{\pm}:\mathscr{U}_D^{\pm}\to\partial_{\pm}^1 D$ from Section 2.4.

Proposition 7.7. *Let D be a nonempty proper closed convex subset of X. The restriction to $\mathscr{U}_D^{\pm}$ of the Gibbs measure $\widetilde{m}_F$ disintegrates by the fibration $f_D^{\pm}:\mathscr{U}_D^{\pm}\to\partial_{\pm}^1 D$ over the skinning measure $\widetilde{\sigma}_D^{\pm}$ of D, with conditional measure $\nu_\rho^{\mp}$ on the fiber $(f_D^{\pm})^{-1}(\rho)=W^{0\pm}(\rho)$ of $\rho\in\partial_{\pm}^1 D$: when ℓ ranges over $\mathscr{U}_D^{\pm}$, we have*

$$d\widetilde{m}_{F\,|\mathscr{U}_D^{\pm}}(\ell)=\int_{\rho\in\partial_{\pm}^1 D} d\nu_\rho^{\mp}(\ell)\,d\widetilde{\sigma}_D^{\pm}(\rho)\;.$$

Proof. In order to prove the claim for the fibration f_D^+, let $\phi\in\mathscr{C}_{\rm c}(\mathscr{U}_D^+)$. Using in the various steps below:

- Hopf's parametrisation with time parameter t and the definitions of $\widetilde{m}_F$ (see Equation (4.4)) and of $\mathscr{U}_D^+$ (see Equation (2.15)),
- the positive endpoint homeomorphism $w\mapsto w_+$ from $\partial_+^1 D$ to $\partial_\infty X-\partial_\infty D$, and the negative endpoint homeomorphism $\ell'\mapsto\ell'_-$ from $W^+(w)$ to $\partial_\infty X-\{w_+\}$, with $s\in\mathbb{R}$ the real parameter such that $\ell'=\mathtt{g}^{-s}\ell\in W^+(w)$ where $\ell\in W^{0+}(w)$, noting that $t-s$ depends only on $\ell_+=w_+$ and $\ell_-=\ell'_-$,
- the definitions Equation (7.9) and (7.1) of the measures $\mu_{W^+(w)}$ and $\widetilde{\sigma}_D^+$, and the cocycle property (3.19) of $C^{\pm}$,
- Equation (3.20) and the cocycle property (3.19) of C^+,

we have

$$\begin{aligned}
&\int_{\ell\in\mathscr{U}_D^+}\phi(\ell)\,d\widetilde{m}_F(\ell)\\
&=\int_{\ell_+\in\partial_\infty X-\partial_\infty D}\int_{\ell_-\in\partial_\infty X-\{\ell_+\}}\int_{t\in\mathbb{R}}\phi(\ell)\,e^{C^-_{\ell_-}(x_0,\,\pi(\ell))+C^+_{\ell_+}(x_0,\,\pi(\ell))}\,dt\,d\mu^-_{x_0}(\ell_-)\,d\mu^+_{x_0}(\ell_+)\\
&=\int_{w\in\partial_+^1 D}\int_{\ell'\in W^+(w)}\int_{s\in\mathbb{R}}\phi(\mathtt{g}^s\ell')\,e^{C^-_{\ell'_-}(x_0,\,\pi(\mathtt{g}^s\ell'))+C^+_{w_+}(x_0,\,\pi(\mathtt{g}^s\ell'))}\,ds\,d\mu^-_{x_0}(\ell'_-)\,d\mu^+_{x_0}(w_+)\\
&=\int_{w\in\partial_+^1 D}\int_{\ell'\in W^+(w)}\int_{s\in\mathbb{R}}\phi(\mathtt{g}^s\ell')\,e^{C^-_{\ell'_-}(\pi(\ell'),\,\pi(\mathtt{g}^s\ell'))+C^+_{w_+}(\pi(w),\,\pi(\mathtt{g}^s\ell'))}\,ds\,d\mu_{W^+(w)}(\ell')\,d\widetilde{\sigma}_D^+(w)\\
&=\int_{w\in\partial_+^1 D}\int_{\ell'\in W^+(w)}\int_{s\in\mathbb{R}}\phi(\mathtt{g}^s\ell')\,e^{C^+_{w_+}(\pi(w),\,\pi(\ell'))}\,ds\,d\mu_{W^+(w)}(\ell')\,d\widetilde{\sigma}_D^+(w)\,,
\end{aligned}$$

which implies the claim for the fibration f_D^+, by the definition (7.12) of the measure ν_w^-. The proof for the fibration f_D^- is similar. □

For every $u\in\mathscr{G}_-X$, if $D=HB_-(u)$, we have $\partial_+^1 D=N_u^-(W^-(u))$ and

$$\mathscr{U}_D^+=\mathscr{G}X-W^{0+}(\iota u)=\bigcup_{w\in W^-(u)}W^{0+}(w)\;.$$

Applying the above proposition and a change of variable, the restriction to $\mathscr{G}X-W^{0+}(\iota u)$ of the Gibbs measure $\widetilde{m}_F$ disintegrates over the strong unstable measure $\mu_{W^-(u)}=((N_u^-)^{-1})_*\widetilde{\sigma}_D^+$, with conditional measure on the fiber $W^{0+}(w)$ of $w \in W^-(u)$ the measure $\nu^-_w = \nu^-_{N^-_u(w)}$: for every $\phi \in \mathscr{C}_c(\mathscr{G}X - W^{0+}(\iota u))$, we have

$$\int_{\ell \in \mathscr{G}X - W^{0+}(\iota u)} \phi(\ell)\, d\widetilde{m}_F(\ell) = \int_{w \in W^-(u)} \int_{\ell' \in W^+(w)} \int_{s \in \mathbb{R}} \phi(\mathtt{g}^s \ell')\, e^{C^+_{w_+}(\pi(w),\, \pi(\ell'))}\, ds\, d\mu_{W^+(w)}(\ell')\, d\mu_{W^-(u)}(w)\,. \tag{7.14}$$

Note that if the Patterson densities have no atoms, then the stable and unstable leaves have measure zero for the associated Gibbs measure. This happens for instance if the Gibbs measure m_F is finite, see Corollary 4.7 and Theorem 4.6.

7.2 Equivariant families of convex subsets and their skinning measures

Let I be an index set endowed with a left action of Γ. A family $\mathscr{D} = (D_i)_{i \in I}$ of subsets of X or of $\breve{\mathscr{G}}X$ indexed by I is Γ-*equivariant* if $\gamma D_i = D_{\gamma i}$ for all $\gamma \in \Gamma$ and $i \in I$. We will denote by

$$\sim\; = \;\sim_{\mathscr{D}}$$

the equivalence relation on I defined by $i \sim j$ if and only if $D_i = D_j$ and there exists $\gamma \in \Gamma$ such that $j = \gamma i$. This equivalence relation is Γ-equivariant: for all $i, j \in I$ and $\gamma \in \Gamma$, we have $\gamma i \sim \gamma j$ if and only if $i \sim j$. We say that $\mathscr{D}$ is *locally finite* if for every compact subset K in X or in $\breve{\mathscr{G}}X$, the quotient set $\{i \in I : D_i \cap K \neq \varnothing\}/_\sim$ is finite.

Examples. (1) Fixing a nonempty proper closed convex subset D of X, taking $I = \Gamma$ with the left action by translations $(\gamma, i) \mapsto \gamma i$, and setting $D_i = iD$ for every $i \in \Gamma$ gives a Γ-equivariant family $\mathscr{D} = (D_i)_{i \in I}$. In this case, we have $i \sim j$ if and only if $i^{-1}j$ belongs to the stabiliser Γ_D of D in Γ, and $I/_\sim\; = \Gamma/\Gamma_D$. Note that γD depends only on the class $[\gamma]$ of γ in Γ/Γ_D. We could also take $I' = \Gamma/\Gamma_D$ with the left action by translations $(\gamma, [\gamma']) \mapsto [\gamma\gamma']$, and $\mathscr{D}' = (\gamma D)_{[\gamma] \in I'}$, so that for all $i, j \in I'$, we have $i \sim_{\mathscr{D}'} j$ if and only if $i = j$, and besides, $\mathscr{D}'$ is locally finite if and only if $\mathscr{D}$ is locally finite. The following choices of D yield equivariant families with different characteristics:

(a) Let $\gamma_0 \in \Gamma$ be a loxodromic element with translation axis $D = \mathrm{Ax}_{\gamma_0}$. The family $(\gamma D)_{\gamma \in \Gamma}$ is locally finite and Γ-equivariant. Indeed, by Lemma 2.1, only finitely many elements of the family $(\gamma D)_{\gamma \in \Gamma/\Gamma_D}$ meet any given bounded subset of X.

(b) Let $\ell \in \mathscr{G}X$ be a geodesic line whose image under the canonical map $\mathscr{G}X \to \Gamma\backslash\mathscr{G}X$ has a dense orbit in $\Gamma\backslash\mathscr{G}X$ under the geodesic flow, and let $D = \ell(\mathbb{R})$ be its image. Then the Γ-equivariant family $(\gamma D)_{\gamma \in \Gamma}$ is not locally finite.

(c) More generally, let D be a convex subset such that $\Gamma_D\backslash D$ is compact. Then the family $(\gamma D)_{\gamma \in \Gamma}$ is a locally finite Γ-equivariant family.

(d) Let $\xi \in \partial_\infty X$ be a bounded parabolic limit point of Γ, and let $\mathscr{H}$ be any horoball in X centred at ξ. Then the family $(\gamma\mathscr{H})_{\gamma \in \Gamma}$ is a locally finite Γ-equivariant family.

(2) More generally, let $(D^\alpha)_{\alpha\in A}$ be a finite family of nonempty proper closed convex subsets of X, and for every $\alpha \in A$, let F_α be a finite set. Define $I = \bigcup_{\alpha\in A} \Gamma \times \{\alpha\} \times F_\alpha$ with the action of Γ by left translation on the first factor, and for every $i = (\gamma, \alpha, x) \in I$, let $D_i = \gamma D^\alpha$. Then $I/_\sim \; = \bigcup_{\alpha\in A} \Gamma/\Gamma_{D^\alpha} \times \{\alpha\} \times F_\alpha$ and the Γ-equivariant family $\mathscr{D} = (D_i)_{i\in I}$ is locally finite if and only if the family $(\gamma D^\alpha)_{\gamma\in\Gamma}$ is locally finite for every $\alpha \in A$. The cardinalities of F_α for $\alpha \in A$ contribute to the multiplicities (see Section 12.2).

Let $\mathscr{D} = (D_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of X. Let $\Omega = (\Omega_i)_{i\in I}$ be a Γ-equivariant family of subsets of $\breve{\mathscr{G}}X$, where Ω_i is a measurable subset of $\partial^1_\pm D_i$ for all $i \in I$ (the sign $\pm$ being constant), such that $\Omega_i = \Omega_j$ if $i \sim_{\mathscr{D}} j$. Then

$$\widetilde{\sigma}^\pm_\Omega = \sum_{i\in I/_\sim} \widetilde{\sigma}^\pm_{D_i}|_{\Omega_i} \;,$$

is a well-defined Γ-invariant locally finite measure on $\breve{\mathscr{G}}X$, whose support is contained in $\mathscr{G}_{\pm,0}X$. Hence, the measure $\widetilde{\sigma}^\pm_\Omega$ induces[6] a locally finite measure on $\Gamma\backslash\breve{\mathscr{G}}X$, denoted by $\sigma^\pm_\Omega$. When $\Omega = \partial^1_\pm\mathscr{D} = (\partial^1_\pm D_i)_{i\in I}$, the measure $\widetilde{\sigma}^\pm_\Omega$ is denoted by

$$\widetilde{\sigma}^\pm_{\mathscr{D}} = \sum_{i\in I/_\sim} \widetilde{\sigma}^\pm_{D_i} \,.$$

The measures $\widetilde{\sigma}^+_{\mathscr{D}}$ and $\widetilde{\sigma}^-_{\mathscr{D}}$ are respectively called the *outer and inner skinning measures* of $\mathscr{D}$ on $\breve{\mathscr{G}}X$, and their induced measures $\sigma^+_{\mathscr{D}}$ and $\sigma^-_{\mathscr{D}}$ on $\Gamma\backslash\breve{\mathscr{G}}X$ are the *outer and inner skinning measures* of $\mathscr{D}$ on $\Gamma\backslash\breve{\mathscr{G}}X$.

Example. Consider the Γ-equivariant family $\mathscr{D} = (\gamma D)_{\gamma\in\Gamma/\Gamma_x}$ with $D = \{x\}$ a singleton in X. With $\pi^\pm = (P^\pm_D)^{-1} : \partial^1_\pm D \to \partial_\infty X$ the homeomorphism $\rho \mapsto \rho_\pm$, we have $(\pi^\pm)_*\widetilde{\sigma}^\pm_D = \mu^\pm_x$ by Remark 7.1 (1), and[7]

$$\|\sigma^\pm_{\mathscr{D}}\| \;=\; \frac{\|\mu^\pm_x\|}{|\Gamma_x|} \;. \tag{7.15}$$

[6] See for instance [PauPS, §2.6] and the beginning of Chapter 12 for details on the definition of the induced measure when Γ may have torsion, hence does not necessarily acts freely on $\breve{\mathscr{G}}X$.

[7] See also the beginning of Chapter 12.

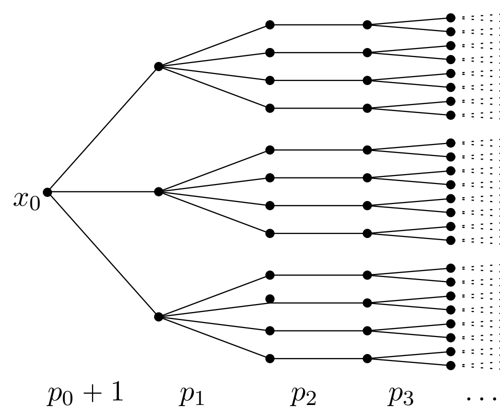
Chapter 8

Explicit measure computations for simplicial trees and graphs of groups

In this Chapter, we compute skinning measures and Bowen-Margulis measures for some highly symmetric simplicial trees $\mathbb{X}$ endowed with a nonelementary discrete subgroup Γ of $\operatorname{Aut}(\mathbb{X})$. These computations are parallel to the ones given in Section 4.2 when X is a rank one symmetric space. The potentials F are supposed to be 0 in this Chapter, and we assume that the Patterson densities $(\mu^+_x)_{x\in V\mathbb{X}}$ and $(\mu^-_x)_{x\in V\mathbb{X}}$ of Γ are equal, denoted by $(\mu_x)_{x\in V\mathbb{X}}$. As the study of geometrically finite discrete subgroups of $\operatorname{Aut}(\mathbb{X})$ mostly reduces to the study of particular (tree) lattices (see Remark 2.12), we will assume that Γ is a lattice in this Chapter.

The results of these computations will be useful when we state special cases of the equidistribution and counting results in regular and biregular trees and, in particular, in the arithmetic applications in Part III. The reader only interested in the continuous time case may skip directly to Chapter 9.

A rooted simplicial tree $(\mathbb{X}, x_0)$ is *spherically symmetric* if $\mathbb{X}$ is not reduced to x_0 and has no terminal vertex, and if the stabiliser of x_0 in $\operatorname{Aut}(\mathbb{X})$ acts transitively on each sphere of centre x_0. The set of isomorphism classes of spherically symmetric rooted simplicial trees $(\mathbb{X}, x_0)$ is in bijection with the set of sequences $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N} - \{0\}$, where $p_n + 1$ is the degree of any vertex of $\mathbb{X}$ at distance n from x_0.



If $(\mathbb{X}, x_0)$ is spherically symmetric, it is easy to check that the simplicial tree $\mathbb{X}$ is uniform[1] if and only if the sequence $(p_n)_{n\in\mathbb{N}}$ is periodic with palindromic period in the sense that there

[1]See Section 2.6 for the terminology concerning simplicial trees.

exists $N \in \mathbb{N} - \{0\}$ such that $p_{n+N} = p_n$ for every $n \in \mathbb{N}$ and $p_{N-n} = p_n$ for every $n \in \mathbb{N}$ such that $n \leqslant N$. If $N = 1$, then $\mathbb{X} = \mathbb{X}_{p_0}$ is the regular tree of degree $p_0 + 1$, and if $N = 2$, then $\mathbb{X} = \mathbb{X}_{p_0, p_1}$ is the biregular tree of degrees $p_0 + 1$ and $p_1 + 1$.

We denote by $X = |\mathbb{X}|_1$ the geometric realisation of $\mathbb{X}$. The Hausdorff dimension $h_{\mathbb{X}}$ of $\partial_\infty X$ for any visual distance is then

$$h_{\mathbb{X}} = \frac{1}{N} \ln(p_0 \dots p_{N-1}) \,, \tag{8.1}$$

see for example [Lyo, p. 935].

8.1 Computations of Bowen-Margulis measures for simplicial trees

The next result gives examples of computations of the total mass of Bowen-Margulis measures for lattices of simplicial trees having some regularity properties.

Analogous computations can be performed for Riemannian manifolds having appropriate regularity properties. We refer for instance to [PaP16, Prop. 10] and [PaP17b, Prop. 20 (1)] for computations of Bowen-Margulis measures for lattices in the isometry group of the real hyperbolic spaces, and to [PaP17a, Lem. 12 (iii)] for the computation in the complex hyperbolic case. In both cases, the main point is the computation of the proportionality constant between the Bowen-Margulis measure and Sasaki's Riemannian volume of the unit tangent bundle. When dealing now with simplicial trees, similar consequences of homogeneity properties will appear below.

We refer to Section 2.6 for the definitions of vol, Vol, $T\pi$, Tvol, TVol appearing in the following result.

Proposition 8.1. *Let $(\mathbb{X}, x_0)$ be a spherically symmetric rooted simplicial tree, with associated sequence $(p_n)_{n\in\mathbb{N}}$, such that $\mathbb{X}$ is uniform, and let Γ be a lattice of $\mathbb{X}$.*

(1) For every $x \in V\mathbb{X}$, let $r_x = d(x, \operatorname{Aut}(\mathbb{X})x_0)$, and let

$$c_x = \frac{(p_{r_x} - 1)e^{2\,r_x\,h_{\mathbb{X}}}}{(p_0+1)^2 p_1^2 \dots p_{r_x-1}^2 p_{r_x}} + \frac{2p_0}{(p_0+1)^2}$$

if $r_x \neq 0$ and $c_x = \frac{p_0}{p_0+1}$ if $r_x = 0$. Then

$$\begin{aligned} \|m_{\rm BM}\| &= \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{1}{|\Gamma_x|}\Big(\|\mu_x\|^2 - \sum_{e\in E\mathbb{X}\,:\,o(e)=x} \mu_x(\partial_e\mathbb{X})^2\Big) \\ &= \|\mu_{x_0}\|^2 \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{c_x}{|\Gamma_x|} \,. \end{aligned} \tag{8.2}$$

(2) If $\mathbb{X} = \mathbb{X}_{p,q}$ is the biregular tree of degrees $p+1$ and $q+1$, with $V\mathbb{X} = V_p\mathbb{X} \sqcup V_q\mathbb{X}$ the corresponding partition of the set of vertices of $\mathbb{X}$, if the Patterson density $(\mu_x)_{x\in V\mathbb{X}}$ of Γ is normalised so that $\|\mu_x\| = \frac{p+1}{\sqrt{p}}$ for all $x \in V_p\mathbb{X}$, then

$$(T\pi)_* m_{\rm BM} = {\rm Tvol}_{\Gamma\backslash\backslash\mathbb{X}}$$

and

$$\|m_{\rm BM}\| = {\rm TVol}(\Gamma\backslash\backslash\mathbb{X}) = \sum_{[x]\in\Gamma\backslash V_p\mathbb{X}} \frac{p+1}{|\Gamma_x|} \;+\; \sum_{[x]\in\Gamma\backslash V_q\mathbb{X}} \frac{q+1}{|\Gamma_x|}\;. \tag{8.3}$$

(3) If $\mathbb{X} = \mathbb{X}_q$ is the regular tree of degree $q+1$, if the Patterson density $(\mu_x)_{x\in V\mathbb{X}}$ of Γ is normalised to be a family of probability measures, then

$$\pi_* m_{\rm BM} = \frac{q}{q+1}\;{\rm vol}_{\Gamma\backslash\backslash\mathbb{X}}$$

and in particular

$$\|m_{\rm BM}\| = \frac{q}{q+1}\,{\rm Vol}(\Gamma\backslash\backslash\mathbb{X})\;. \tag{8.4}$$

Proof. We start by proving the first equality of Assertion (1). For every $x \in V\mathbb{X}$, we may partition the set of geodesic lines $\ell \in \mathscr{G}\mathbb{X}$ with $\ell(0) = x$ according to the two edges starting from x contained in the image of ℓ. The only restriction for the edges is that they are required to be distinct.

For every $e \in E\mathbb{X}$, recall from Section 2.6 that $\partial_e\mathbb{X}$ is the set of points at infinity of the geodesic rays whose initial edge is e. For all $e \in E\mathbb{X}$ and $x \in V\mathbb{X}$, say that e *points away* from x if $o(e) \in [x, t(e)]$, and that e *points towards* x otherwise. In particular, all edges with origin x point away from x. Hence by Equation (4.13), and since $\mu_x = \mu_x^- = \mu_x^+$, we have

$$\pi_* m_{\rm BM} = \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{1}{|\Gamma_x|} \sum_{e,\,e'\in E\mathbb{X}\,:\,o(e)=o(e')=x,\;e\neq e'} \mu_x^-(\partial_e\mathbb{X})\;\mu_x^+(\partial_{e'}\mathbb{X})\;\Delta_{[x]} \tag{8.5}$$

$$= \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{1}{|\Gamma_x|} \Big(\Big(\sum_{e\in E\mathbb{X}\,:\,o(e)=x} \mu_x(\partial_e\mathbb{X})\;\Big)^2 - \sum_{e\in E\mathbb{X}\,:\,o(e)=x} \mu_x(\partial_e\mathbb{X})^2\Big)\;\Delta_{[x]}\;. \tag{8.6}$$

This gives the first equality of Assertion (1).

Let us prove the second equality of Assertion (1). By homogeneity, we assume that $\|\mu_{x_0}\| = 1$ and we will prove that

$$\|m_{\rm BM}\| = \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{c_x}{|\Gamma_x|}\;.$$

Let $N \in \mathbb{N} - \{0\}$ be such that $p_{n+N} = p_n$ for every $n \in \mathbb{N}$ and $p_{N-n} = p_n$ for every $n \in \{0, \dots, N\}$, which exists since $\mathbb{X}$ is assumed to be uniform. Then the automorphism group ${\rm Aut}(\mathbb{X})$ of the simplicial tree $\mathbb{X}$ acts transitively on the set of vertices at distance a multiple of N from x_0. Hence for every $x \in V\mathbb{X}$, the distance $r_x = d(x, {\rm Aut}(\mathbb{X})x_0)$ belongs to $\big\{0, 1, \dots, \lfloor\frac{N}{2}\rfloor\big\}$, and there exist $\gamma_x, \gamma'_x \in {\rm Aut}(\mathbb{X})$ such that

$$d(x, \gamma_x x_0) = r_x, \quad x \in [\gamma_x x_0, \gamma'_x x_0] \quad\text{and}\quad d(\gamma_x x_0, \gamma'_x x_0) = N\;.$$

The map $x \mapsto r_x$ is constant on the orbits of Γ in $V\mathbb{X}$,[2] hence so is the map $x \mapsto c_x$, and thus the right hand side of Equation (8.2) is well defined.

Since the family $(\mu_x^{\rm Haus})_{x\in V\mathbb{X}}$ of Hausdorff measures of the visual distances $(\partial_\infty X, d_x)$ is invariant under any element of ${\rm Aut}(\mathbb{X})$, since Γ is a lattice and by Proposition 4.16, we have $\delta_\Gamma = h_\mathbb{X}$ and $\gamma_*\mu_x = \mu_{\gamma x}$ for all $x \in V\mathbb{X}$ and $\gamma \in {\rm Aut}(\mathbb{X})$.

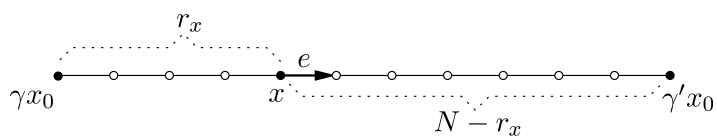
[2] In fact, it is constant on the orbits of ${\rm Aut}(\mathbb{X})$.

Since $(\mathbb{X}, x_0)$ is spherically symmetric, and since μ_{x_0} is a probability measure, we have by induction, for every $e \in E\mathbb{X}$ pointing away from x_0 with $d(x_0, o(e)) = n$,

$$\mu_{x_0}(\partial_e \mathbb{X}) = \frac{1}{(p_0+1)\, p_1 \,\dots\, p_n} \tag{8.7}$$

if $n \neq 0$, and $\mu_{x_0}(\partial_e \mathbb{X}) = \frac{1}{p_0+1}$ otherwise.

For every fixed $x \in V\mathbb{X}$, let us now compute $\mu_x(\partial_e \mathbb{X})$ for every edge e of $\mathbb{X}$ with origin x. Let $\gamma = \gamma_x, \gamma' = \gamma'_x \in \mathrm{Aut}(\mathbb{X})$ be as above. By the spherical transitivity, we may assume that e or $\overline{e}$ belongs to the edge path from γx_0 to $\gamma' x_0$.



There are two cases to consider.

Case 1: Assume first that e points away from γx_0. There are $p_0 + 1$ such edges starting from x if $r_x = 0$, and p_{r_x} otherwise. By Equation (8.7) and by invariance under $\mathrm{Aut}(\mathbb{X})$ of $(\mu_x^{\mathrm{Haus}})_{x \in V\mathbb{X}}$, we have

$$\mu_{\gamma x_0}(\partial_e \mathbb{X}) = \frac{1}{(p_0+1)\, p_1 \,\dots\, p_{r_x}}\ ,$$

with the convention that the denominator is $p_0 + 1$ if $r_x = 0$. Since the map $\xi \mapsto \beta_\xi(x, \gamma x_0)$ is constant with value $-r_x$ on $\partial_e \mathbb{X}$, and by the quasi-invariance property of the Patterson density (see Equation (4.2)), we have

$$\mu_x(\partial_e \mathbb{X}) = e^{-\delta_\Gamma(-r_x)} \mu_{\gamma x_0}(\partial_e \mathbb{X}) = \frac{e^{r_x h_{\mathbb{X}}}}{(p_0+1)\, p_1 \,\dots\, p_{r_x}}\ ,$$

with the same convention as above.

Case 2: Assume now that e points towards γx_0. This implies that $r_x \geqslant 1$, and there is one and only one such edge starting from x. Then as above we have

$$\mu_{\gamma' x_0}(\partial_e \mathbb{X}) = \frac{1}{(p_N+1)\, p_{N-1} \,\dots\, p_{r_x}}\ ,$$

and

$$\mu_x(\partial_e \mathbb{X}) = e^{-\delta_\Gamma(-(N-r_x))} \mu_{\gamma' x_0}(\partial_e \mathbb{X}) = \frac{e^{(N-r_x) h_{\mathbb{X}}}}{(p_N+1)\, p_{N-1} \,\dots\, p_{r_x}}\ .$$

For every $x \in V\mathbb{X}$, let

$$C_x = \Big(\sum_{e \in E\mathbb{X}\,:\, o(e)=x} \mu_x(\partial_e \mathbb{X}) \Big)^2 - \sum_{e \in E\mathbb{X}\,:\, o(e)=x} \mu_x(\partial_e \mathbb{X})^2\ . \tag{8.8}$$

If $r_x \neq 0$, since the stabiliser of γx_0 in $\mathrm{Aut}(\mathbb{X})$ acts transitively on the p_{r_x} edges with origin x pointing away from γx_0, since $e^{N h_{\mathbb{X}}} = p_0 p_1 \dots p_{N-1}$ and $p_N = p_0$, we have

$$\begin{aligned} C_x = &\; \Big(p_{r_x} \frac{e^{r_x h_{\mathbb{X}}}}{(p_0+1)\, p_1 \,\dots\, p_{r_x}} + \frac{e^{(N-r_x)h_{\mathbb{X}}}}{(p_N+1)\, p_{N-1} \,\dots\, p_{r_x}} \Big)^2 \\ & - \Big(p_{r_x} \big(\frac{e^{r_x h_{\mathbb{X}}}}{(p_0+1)\, p_1 \,\dots\, p_{r_x}} \big)^2 + \big(\frac{e^{(N-r_x)h_{\mathbb{X}}}}{(p_N+1)\, p_{N-1} \,\dots\, p_{r_x}} \big)^2 \Big) \\ = &\; \frac{({p_{r_x}}^2 - p_{r_x})\, e^{2\, r_x h_{\mathbb{X}}}}{(p_0+1)^2\, {p_1}^2 \,\dots\, {p_{r_x}}^2} + \frac{2\, p_{r_x} e^{N\, h_{\mathbb{X}}}}{(p_0+1)\, p_1 \,\dots\, p_{r_x} p_{r_x} \dots p_{N-1}(p_N+1)} \\ = &\; \frac{(p_{r_x} - 1)\, e^{2\, r_x h_{\mathbb{X}}}}{(p_0+1)^2\, {p_1}^2 \,\dots\, {p_{r_x - 1}}^2 p_{r_x}} + \frac{2\, p_0}{(p_0+1)^2} = c_x \;. \end{aligned}$$

If $r_x = 0$, we have

$$C_x = \Big((p_0+1) \frac{1}{p_0+1} \Big)^2 - (p_0+1) \Big(\frac{1}{p_0+1} \Big)^2 = \frac{p_0}{p_0+1} = c_x \;.$$

Assertion (1) now follows from Equation (8.6).

Let us prove Assertion (2). Note that $\mathbb{X} = \mathbb{X}_{p,\,q}$ is spherically symmetric with respect to any vertex of $\mathbb{X}$, and that, by Equation (8.1),

$$h_{\mathbb{X}} = \frac{1}{2} \ln(pq) \;.$$

Let e be an edge of $\mathbb{X}$, with $x = o(e) \in V_p\mathbb{X}$ and $y = t(e) \in V_q\mathbb{X}$. For every $z \in V\mathbb{X}$, we define C_z as in Equation (8.8).

Note that by homogeneity, we have $C_z = C_x$ and $\|\mu_z\| = \|\mu_x\|$ for all $z \in V_p\mathbb{X}$, as well as $C_z = C_y$ and $\|\mu_z\| = \|\mu_y\|$ for all $z \in V_q\mathbb{X}$. Hence the normalisation of the Patterson density as in the statement of Assertion (2) is possible. By the spherical symmetry at x, and the normalisation of the measure, we have $\mu_x(\partial_e \mathbb{X}) = \frac{1}{\sqrt{p}}$ and $\mu_x(\partial_{\overline{e}} \mathbb{X}) = \sqrt{p}$. Therefore

$$\begin{aligned} \|\mu_y\| &= \mu_y(\partial_e \mathbb{X}) + \mu_y(\partial_{\overline{e}} \mathbb{X}) = e^{h_{\mathbb{X}}} \mu_x(\partial_e \mathbb{X}) + e^{-h_{\mathbb{X}}} \mu_x(\partial_{\overline{e}} \mathbb{X}) \\ &= \sqrt{pq}\, \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{pq}} \sqrt{p} = \frac{q+1}{\sqrt{q}} \;. \end{aligned}$$

This symmetry in the values of $\|\mu_y\|$ and $\|\mu_x\|$ explains the choice of our normalisation of the Patterson density. We have

$$C_x = \|\mu_x\|^2 - (p+1) \Big(\frac{\|\mu_x\|}{p+1} \Big)^2 = \frac{p}{p+1} \|\mu_x\|^2 = p+1$$

and similarly $C_y = \frac{q}{q+1} \|\mu_y\|^2 = q+1$. This proves the second equality in Equation (8.3), by the first equation of Assertion (1).

In order to prove that $(T\pi)_* m_{\rm BM} = \mathrm{Tvol}_{\Gamma \backslash\backslash \mathbb{X}}$, we now partition $\Gamma \backslash \mathscr{G}\mathbb{X}$ as

$$\bigcup_{[e] \in \Gamma \backslash E\mathbb{X}} \Gamma \backslash \big\{ \ell \in \mathscr{G}\mathbb{X} \;:\; \ell(0) = \pi(o(e)),\; \ell(1) = \pi(t(e)) \big\} \;.$$

Using on every element of this partition Hopf's decomposition with respect to the basepoint $o(e)$, we have, by a ramified covering argument already used in the proof of the second part of Proposition 4.15,

$$(T\pi)_* m_{\rm BM} = \sum_{[e]\in\Gamma\backslash E\mathbb{X}} \frac{1}{|\Gamma_e|}\, \mu_{o(e)}(\partial_\infty X - \partial_e\mathbb{X})\, \mu_{o(e)}(\partial_e\mathbb{X})\, \Delta_{[e]}\,.$$

Since $\mu_{o(e)}(\partial_e\mathbb{X}) = e^{-h_{\mathbb{X}}}\mu_{t(e)}(\partial_e\mathbb{X})$ and by homogeneity, we have

$$\begin{aligned}(T\pi)_* m_{\rm BM} &= \sum_{[e]\in\Gamma\backslash E\mathbb{X}} \frac{1}{|\Gamma_e|}\, \frac{\deg o(e)-1}{\deg o(e)}\, \|\mu_{o(e)}\|\, \frac{\deg t(e)-1}{\deg t(e)}\, \|\mu_{t(e)}\|\, e^{-h_{\mathbb{X}}}\, \Delta_{[e]}\\ &= \sum_{[e]\in\Gamma\backslash E\mathbb{X}} \frac{1}{|\Gamma_e|}\, \frac{\|\mu_{o(e)}\|\,\|\mu_{t(e)}\|\,\sqrt{pq}}{(p+1)(q+1)}\, \Delta_{[e]} = {\rm Tvol}_{\Gamma\backslash\backslash\mathbb{X}}\,.\end{aligned}$$

The first equality of Equation (8.3) follows, since pushforwards of measures preserve the total mass.

Finally, the last claim of Assertion (3) of Proposition 8.1 follows from Equation (8.2), since $c_x = \frac{q}{q+1}$ for every $x\in V\mathbb{X}_q$ (or by taking $q=p$ in Equation (8.3) and by renormalising). The first claim of Assertion (3) follows from the first claim of Assertion (2), by using Equation (2.23) and renormalising. □

Remark 8.2. (1) In particular, if $\mathbb{X} = \mathbb{X}_q$ is regular, if the Patterson density is normalised to be a family of probability measures and if Γ is torsion free, then $\pi_* m_{\rm BM}$ is $\frac{q}{q+1}$ times the counting measure on $\Gamma\backslash V\mathbb{X}$. In this case, Equation (8.4) is given by [CoP4, Rem. 2].

(2) If $\mathbb{X} = \mathbb{X}_{p,q}$ is biregular with $p\neq q$, then $\pi_* m_{\rm BM}$ is not proportional to ${\rm vol}_{\Gamma\backslash\backslash\mathbb{X}}$. In particular, if Γ is torsion free and if the Patterson density is normalised to be a family of probability measures, then $\pi_* m_{\rm BM}$ is the sum of $\frac{p}{p+1}$ times the counting measure on $\Gamma\backslash V_p\mathbb{X}$ and $\frac{q}{q+1}$ times the counting measure on $\Gamma\backslash V_q\mathbb{X}$.

This statement is coherent with the well-known fact that in pinched but variable curvature, the Bowen-Margulis measure is generally not absolutely continuous with respect to Sasaki's Riemannian measure on the unit tangent bundle (they would then be proportional by ergodicity of the geodesic flow in the lattice case).

8.2 Computations of skinning measures for simplicial trees

We now give examples of computations of the total mass of skinning measures (for zero potentials), after introducing some notation. Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, and let Γ be a discrete subgroup of ${\rm Aut}(\mathbb{X})$.

For every simplicial subtree $\mathbb{D}$ of $\mathbb{X}$, we define the *boundary* $\partial V\mathbb{D}$ of $V\mathbb{D}$ in $\mathbb{X}$ as

$$\partial V\mathbb{D} = \{x\in V\mathbb{D}\;:\;\exists\, e\in E\mathbb{X},\;\; o(e)=x,\; t(e)\notin V\mathbb{D}\}\,.$$

The *boundary* $\partial\mathbb{D}$ of $\mathbb{D}$ is the maximal subgraph (which might be non connected) of $\mathbb{X}$ with set of vertices $\partial V\mathbb{D}$. It is contained in $\mathbb{D}$. The stabiliser $\Gamma_{\mathbb{D}}$ of $\mathbb{D}$ in Γ acts discretely on $\partial\mathbb{D}$.

For every $x\in V\mathbb{X}$, we define the *codegree* of x in $\mathbb{D}$ as ${\rm codeg}_{\mathbb{D}}(x)=0$ if $x\notin\mathbb{D}$ and otherwise

$${\rm codeg}_{\mathbb{D}}(x) = \deg_{\mathbb{X}}(x) - \deg_{\mathbb{D}}(x)\,.$$

Note that $\operatorname{codeg}_{\mathbb{D}}(x) = 0$ if $x \notin \partial V\mathbb{D}$, and that the codegree $\operatorname{codeg}_{\mathscr{N}_1\mathbb{D}}(x)$ of $x \in V\mathbb{X}$ in the 1-neighbourhood $\mathscr{N}_1\mathbb{D}$ of $\mathbb{D}$ is equal to 0 unless x lies in the boundary of $\mathscr{N}_1\mathbb{D}$, in which case it is equal to $\deg_{\mathbb{X}}(x) - 1$.

Let $\mathscr{D} = (\mathbb{D}_i)_{i\in I}$ be a locally finite Γ-equivariant family of simplicial subtrees of $\mathbb{X}$, and let $x \in V\mathbb{X}$. We define the *multiplicity*[3] of x in (the boundary of) $\mathscr{D}$ as (see Section 7.2 for the definition of $\sim_{\mathscr{D}}$)

$$m_{\mathscr{D}}(x) = \frac{\operatorname{Card}\{i \in I/_{\sim_{\mathscr{D}}} \;:\; x \in \partial V\mathbb{D}_i\}}{|\Gamma_x|} \;.$$

The numerator and the denominator are finite by the local finiteness of the family $\mathscr{D}$ and the discreteness of Γ, and they depend only on the orbit of x under Γ. Note that if $\mathbb{D}$ is a simplicial subtree of $\mathbb{X}$ which is *precisely invariant* under Γ (that is, whenever $\gamma \in \Gamma$ is such that $\mathbb{D} \cap \gamma\mathbb{D}$ is nonempty, then γ belongs to the stabiliser $\Gamma_{\mathbb{D}}$ of $\mathbb{D}$ in Γ), if $\mathscr{D} = (\gamma\mathbb{D})_{\gamma\in\Gamma/\Gamma_{\mathbb{D}}}$, and if $x \in \partial V\mathbb{D}$, then

$$m_{\mathscr{D}}(x) = \frac{1}{|\Gamma_x|} \;.$$

In particular, if furthermore Γ is torsion free, then $m_{\mathscr{D}}(x) = 1$ if $x \in \partial V\mathbb{D}$, and $m_{\mathscr{D}}(x) = 0$ otherwise.

Example 8.3. Let $\mathscr{G}$ be a connected graph without vertices of degree $\leqslant 2$ and let $\mathbb{X}$ be its universal cover, with covering group Γ. If C is a cycle in $\mathscr{G} = \Gamma\backslash\mathbb{X}$ and if $\mathscr{D}$ is the family of geodesic lines in $\mathbb{X}$ lifting C, then $m_{\mathscr{D}}(x) = 1$ for all $x \in V\mathbb{X}$ whose image in $\mathscr{G} = \Gamma\backslash\mathbb{X}$ belongs to C if C is a *simple* cycle (that is, if C passes through no vertex twice).

We define the *codegree* of x in $\mathscr{D}$ as

$$\operatorname{codeg}_{\mathscr{D}}(x) = \sum_{i\in I/\sim_{\mathscr{D}}} \operatorname{codeg}_{\mathbb{D}_i}(x) \;,$$

which is well defined as $\operatorname{codeg}_{\mathbb{D}_i}(x)$ depends only on the class of $i \in I$ modulo $\sim_{\mathscr{D}}$. Note that

$$\operatorname{codeg}_{\mathscr{D}}(x) = (\deg_{\mathbb{X}} x - k)\,|\Gamma_x|\, m_{\mathscr{D}}(x) \tag{8.9}$$

if $\deg_{\mathbb{D}_i}(x) = k$ for every $x \in \partial V\mathbb{D}_i$ and $i \in I$. If every vertex of $\mathbb{X}$ has degree at least 3, this is in particular the case with $k = 2$ if $\mathbb{D}_i$ is a line for all $i \in I$ and with $k = 1$ if $\mathbb{D}_i$ is a horoball for all $i \in I$.

We will say that a simplicial subtree $\mathbb{D}$ of $\mathbb{X}$, with stabiliser $\Gamma_{\mathbb{D}}$ in Γ, is *almost precisely invariant* if there exists $N \in \mathbb{N}$ such that for every $x \in V\mathbb{D}$, the number of $\gamma \in \Gamma/\Gamma_{\mathbb{D}}$ such that $x \in \gamma V\mathbb{D}$ is at most N. It follows from this property that if $\mathscr{D} = (\gamma\mathbb{D})_{\gamma\in\Gamma}$, then $\mathscr{D}$ is locally finite and $\operatorname{codeg}_{\mathscr{D}}(x) \leqslant N \operatorname{codeg}_{\mathbb{D}}(x)$ for every $x \in V\mathbb{X}$.

Proposition 8.4. *Assume that $\mathbb{X}$ is a regular or biregular simplicial tree with degrees at least 3, and that Γ is a lattice of $\mathbb{X}$.*

(1) For every simplicial subtree $\mathbb{D}$ of $\mathbb{X}$, we have

$$\pi_*\widetilde{\sigma}^{\pm}_{\mathbb{D}} = \sum_{x\in V\mathbb{X}} \frac{\|\mu_x\| \;\operatorname{codeg}_{\mathbb{D}}(x)}{\deg_{\mathbb{X}}(x)}\;\Delta_x \;.$$

[3] See Section 12.2 for explanations on the terminology.

(2) If $\mathscr{D} = (\mathbb{D}_i)_{i\in I}$ is a locally finite Γ-equivariant family of simplicial subtrees of $\mathbb{X}$, then

$$\pi_*\sigma^{\pm}_{\mathscr{D}} = \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{\|\mu_x\|\ \operatorname{codeg}_{\mathscr{D}}(x)}{|\Gamma_x|\ \deg_{\mathbb{X}}(x)}\ \Delta_{[x]}\ .$$

(3) Let $k\in\mathbb{N}$ and let $\mathbb{D}$ be a simplicial subtree of $\mathbb{X}$ such that $\deg_{\mathbb{D}}(x) = k$ for every $x\in\partial V\mathbb{D}$ and the Γ-equivariant family $\mathscr{D} = (\gamma\mathbb{D})_{\Gamma/\Gamma_{\mathbb{D}}}$ is locally finite. Then

$$\pi_*\sigma^{\pm}_{\mathscr{D}} = \sum_{\Gamma_{\mathbb{D}} y\,\in\,\Gamma_{\mathbb{D}}\backslash\partial V\mathbb{D}} \frac{\|\mu_y\|\ (\deg_{\mathbb{X}}(y)-k)}{|(\Gamma_{\mathbb{D}})_y|\ \deg_{\mathbb{X}}(y)}\ \Delta_{\Gamma y}\ .$$

(4) If $\mathbb{D}$ is a simplicial subtree of $\mathbb{X}$ such that the Γ-equivariant family $\mathscr{D} = (\gamma\mathbb{D})_{\gamma\in\Gamma}$ is locally finite, then the skinning measure $\sigma^{\pm}_{\mathscr{D}}$ is finite if and only if the graph of groups $\Gamma_{\mathbb{D}}\backslash\backslash\partial\mathbb{D}$ has finite volume.

Before proving Proposition 8.4, let us give some immediate consequences of its Assertion (3). If $\mathbb{X} = \mathbb{X}_{p,q}$ is biregular of degrees $p+1$ and $q+1$, let $V\mathbb{X} = V_p\mathbb{X} \sqcup V_q\mathbb{X}$ be the corresponding partition of the set of vertices of $\mathbb{X}$ and, for $r\in\{p,q\}$, let $\partial_r\mathbb{D}$ be the edgeless graph with set of vertices $\partial V\mathbb{D}\cap V_r\mathbb{X}$.

Corollary 8.5. *Assume that $(\mathbb{X},\Gamma)$ is as in Proposition 8.4. Let $\mathbb{D}$ be a simplicial subtree of $\mathbb{X}$ such that the Γ-equivariant family $\mathscr{D} = (\gamma\mathbb{D})_{\gamma\in\Gamma/\Gamma_{\mathbb{D}}}$ is locally finite.*

(1) If $\mathbb{X} = \mathbb{X}_{p,q}$ is biregular of degrees $p+1$ and $q+1$ and if the Patterson density $(\mu_x)_{x\in V\mathbb{X}}$ of Γ is normalised so that $\|\mu_x\| = \frac{\deg_{\mathbb{X}}(x)}{\sqrt{\deg_{\mathbb{X}}(x)-1}}$ for all $x\in V\mathbb{X}$, then

- *if $\mathbb{D}$ is a horoball,*

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \sqrt{p}\ \operatorname{Vol}(\Gamma_{\mathbb{D}}\backslash\backslash\partial_p\mathbb{D}) + \sqrt{q}\ \operatorname{Vol}(\Gamma_{\mathbb{D}}\backslash\backslash\partial_q\mathbb{D})\ ,$$

- *if $\mathbb{D}$ is a line,*

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \frac{p-1}{\sqrt{p}}\ \operatorname{Vol}(\Gamma_{\mathbb{D}}\backslash\backslash\partial_p\mathbb{D}) + \frac{q-1}{\sqrt{q}}\ \operatorname{Vol}(\Gamma_{\mathbb{D}}\backslash\backslash\partial_q\mathbb{D})\ . \tag{8.10}$$

(2) If $\mathbb{X} = \mathbb{X}_q$ is the regular tree of degree $q+1$ and if the Patterson measures $(\mu_x)_{x\in V\mathbb{X}}$ are normalised to be probability measures, then

- *if $\mathbb{D}$ is a horoball,*

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \frac{q}{q+1}\operatorname{Vol}(\Gamma_{\mathbb{D}}\backslash\backslash\partial\mathbb{D}) \tag{8.11}$$

- *if $\mathbb{D}$ is a line,*

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \frac{q-1}{q+1}\operatorname{Vol}(\Gamma_{\mathbb{D}}\backslash\backslash\mathbb{D})\ .\ \square \tag{8.12}$$

Proof of Proposition 8.4. (1) We may partition the outer/inner unit normal bundle $\partial^1_{\pm}\mathbb{D}$ of $\mathbb{D}$ according to the first/last edge of the elements in $\partial^1_{\pm}\mathbb{D}$. On each of the elements of this partition, for the computation of the skinning measures using its definition and its independence of the basepoint (see Section 7.1), we take as basepoint the initial/terminal point of the corresponding edge. Since $\mathbb{D}$ is a simplicial tree, note that for every $e \in E\mathbb{X}$ such that $o(e) \in V\mathbb{D}$, we have $e \in E\mathbb{D}$ if and only if $t(e) \in V\mathbb{D}$. Thus, we have

$$\begin{aligned}\pi_*\widetilde{\sigma}^+_{\mathbb{D}} &= \sum_{e\in E\mathbb{X}\,:\, o(e)\in V\mathbb{D},\ t(e)\notin V\mathbb{D}} \mu_{o(e)}(\partial_e\mathbb{X})\ \Delta_{o(e)} \\ &= \sum_{x\in\partial V\mathbb{D}} \Big(\sum_{e\in E\mathbb{X}\,:\, o(e)=x,\ t(e)\notin V\mathbb{D}} \mu_x(\partial_e\mathbb{X})\Big)\ \Delta_x\ .\end{aligned}$$

and similarly

$$\begin{aligned}\pi_*\widetilde{\sigma}^-_{\mathbb{D}} &= \sum_{e\in E\mathbb{X}\,:\, t(e)\in V\mathbb{D},\ o(e)\notin V\mathbb{D}} \mu_{t(e)}(\partial_{\overline{e}}\mathbb{X})\ \Delta_{t(e)} \\ &= \sum_{x\in\partial V\mathbb{D}} \Big(\sum_{e\in E\mathbb{X}\,:\, o(e)=x,\ t(e)\notin V\mathbb{D}} \mu_x(\partial_e\mathbb{X})\Big)\ \Delta_x\ .\end{aligned}$$

As in the proof of Proposition 8.1 (2), since $\mathbb{X}$ is spherically homogeneous around each point and since Γ is a lattice (so that the Patterson density is $\operatorname{Aut}(\mathbb{X})$-equivariant, see Proposition 4.16), we have $\mu_x(\partial_e\mathbb{X}) = \frac{\|\mu_x\|}{\deg_{\mathbb{X}}(x)}$ for all $x \in V\mathbb{X}$ and $e \in E\mathbb{X}$ with $o(e) = x$. Assertion (1) follows, since $\sum_{e\in E\mathbb{X}\,:\, o(e)=x,\ t(e)\notin V\mathbb{D}} 1 = \operatorname{codeg}_{\mathbb{D}}(x)$ if $x \in \partial V\mathbb{D}$ and $\operatorname{codeg}_{\mathbb{D}}(x) = 0$ otherwise.

(2) By the definition[4] of the skinning measures associated with Γ-equivariant families, we have $\widetilde{\sigma}^{\pm}_{\mathscr{D}} = \sum_{i\in I/\sim} \widetilde{\sigma}^{\pm}_{\mathbb{D}_i}$, where $\sim\ =\ \sim_{\mathscr{D}}$. Hence by Assertion (1)

$$\begin{aligned}\pi_*\widetilde{\sigma}^{\pm}_{\mathscr{D}} &= \sum_{i\in I/\sim} \widetilde{\pi}_*\sigma^{\pm}_{\mathbb{D}_i} = \sum_{i\in I/\sim}\ \sum_{x\in V\mathbb{X}} \frac{\|\mu_x\|\ \operatorname{codeg}_{\mathbb{D}_i}(x)}{\deg_{\mathbb{X}}(x)}\ \Delta_x \\ &= \sum_{x\in V\mathbb{X}} \Big(\sum_{i\in I/\sim} \operatorname{codeg}_{\mathbb{D}_i}(x)\Big) \frac{\|\mu_x\|}{\deg_{\mathbb{X}}(x)}\ \Delta_x = \sum_{x\in V\mathbb{X}} \frac{\|\mu_x\|\ \operatorname{codeg}_{\mathscr{D}}(x)}{\deg_{\mathbb{X}}(x)}\ \Delta_x\ .\end{aligned}$$

By the definition of the measure induced in $\Gamma\backslash V\mathbb{X}$ when Γ may have torsion (see for instance [PauPS, §2.6] and the beginning of Chapter 12), Assertion (2) follows.

(3) It follows from Assertion (2) and from Equation (8.9) that

$$\pi_*\sigma^{\pm}_{\mathscr{D}} = \sum_{[x]\in\Gamma\backslash V\mathbb{X}} \frac{\deg_{\mathbb{X}}(x) - k}{\deg_{\mathbb{X}}(x)}\ \|\mu_x\|\ m_{\mathscr{D}}(x)\ \ \Delta_{[x]}\ .$$

For every $x \in V\mathbb{X}$, by the definition of $m_{\mathscr{D}}(x)$, we have, by partitioning $\partial V\mathbb{D}$ into its orbits under $\Gamma_{\mathbb{D}}$,

$$\begin{aligned}m_{\mathscr{D}}(x) &= \frac{1}{|\Gamma_x|}\operatorname{Card}\{\gamma\in\Gamma_{\mathbb{D}}\backslash\Gamma\ :\ \gamma x\in\partial V\mathbb{D}\} \\ &= \frac{1}{|\Gamma_x|}\sum_{\Gamma_{\mathbb{D}}y\,\in\,\Gamma_{\mathbb{D}}\backslash\partial V\mathbb{D}} \operatorname{Card}\{\gamma\in\Gamma_{\mathbb{D}}\backslash\Gamma\ :\ \Gamma_{\mathbb{D}}\gamma x = \Gamma_{\mathbb{D}}y\} \\ &= \frac{1}{|\Gamma_x|}\sum_{\Gamma_{\mathbb{D}}y\,\in\,\Gamma_{\mathbb{D}}\backslash\partial V\mathbb{D},\ \Gamma x=\Gamma y} \operatorname{Card}\{\gamma\in\Gamma_{\mathbb{D}}\backslash\Gamma\ :\ \Gamma_{\mathbb{D}}\gamma y = \Gamma_{\mathbb{D}}y\} \\ &= \frac{1}{|\Gamma_x|}\sum_{\Gamma_{\mathbb{D}}y\,\in\,\Gamma_{\mathbb{D}}\backslash\partial V\mathbb{D},\ \Gamma x=\Gamma y} [\Gamma_y : (\Gamma_{\mathbb{D}})_y] = \sum_{\Gamma_{\mathbb{D}}y\,\in\,\Gamma_{\mathbb{D}}\backslash\partial V\mathbb{D},\ \Gamma x=\Gamma y} \frac{1}{|(\Gamma_{\mathbb{D}})_y|}\ .\end{aligned}$$

[4] See Section 7.2.

This proves Assertion (3), since $\sum_{[x]\in\Gamma\backslash V\mathbb{X},\ \Gamma x=\Gamma y}\Delta_{[x]}=\Delta_{\Gamma y}$.

(4) It follows from Assertion (2) that

$$\|\sigma^{\pm}_{\mathscr{D}}\|=\sum_{[x]\in\Gamma\backslash V\mathbb{X}}\frac{\|\mu_x\|\ \operatorname{codeg}_{\mathscr{D}}(x)}{|\Gamma_x|\ \deg_{\mathbb{X}}(x)}\ .$$

Note that for every $x\in\partial V\mathbb{D}$, we have

$$|\Gamma_x|\,m_{\mathscr{D}}(x)\leqslant\operatorname{codeg}_{\mathscr{D}}(x)\leqslant\deg_{\mathbb{X}}(x)\ |\Gamma_x|\,m_{\mathscr{D}}(x)\ .$$

Let $m=\min_{x\in V\mathbb{X}}\|\mu_x\|$ and $M=\max_{x\in V\mathbb{X}}\|\mu_x\|$, which are positive and finite, as the total mass of the Patterson measures takes at most two values, since Γ is a lattice and $\mathbb{X}$ is biregular. By arguments similar to those in the proof of Assertion (3), we hence have

$$\frac{m}{\min_{x\in V\mathbb{X}}\deg_{\mathbb{X}}(x)}\ \operatorname{Vol}(\Gamma_{\mathbb{D}}\backslash\!\backslash\partial\mathbb{D})\leqslant\ \|\sigma^{\pm}_{\mathscr{D}}\|\ \leqslant M\ \operatorname{Vol}(\Gamma_{\mathbb{D}}\backslash\!\backslash\partial\mathbb{D})$$

The result follows. □

We now give a formula for the skinning measure (with zero potential) of a geodesic line in the simplicial tree $\mathbb{X}$, using[5] Hamenstädt's distance $d_{\mathscr{H}}$ and measure $\mu_{\mathscr{H}}$ associated with a fixed horoball $\mathscr{H}$ in $\mathbb{X}$. This expression for the skinning measure will be useful in Part III.

Lemma 8.6. *Let $\mathscr{H}$ be a horoball in $\mathbb{X}$ centred at a point $\xi\in\partial_\infty X$. Let L be a geodesic line in $\mathbb{X}$ with endpoints $L_\pm\in\partial_\infty X-\{\xi\}$. Then for all $\rho\in\partial^1_+L$ such that $\rho_+\neq\xi$,*

$$d\widetilde{\sigma}^+_L(\rho)=\frac{d_{\mathscr{H}}(L_+,L_-)^{\delta_\Gamma}}{d_{\mathscr{H}}(\rho_+,L_-)^{\delta_\Gamma}\,d_{\mathscr{H}}(\rho_+,L_+)^{\delta_\Gamma}}\,d\mu_{\mathscr{H}}(\rho_+)\,.$$

Proof. By Equations (2.13) and (7.6), the power ${d_{\mathscr{H}}}^{\delta_\Gamma}$ of the distance and the measure $\mu_{\mathscr{H}}$ scale by the same factor when the horoball is replaced by another one centred at the same point. Thus, we can assume in the proof that L does not intersect the interior of $\mathscr{H}$.

Fix $\rho\in\partial^1_+L$ such that $\rho_+\neq\xi$. Let y be the closest point to ξ on L, let x_0 be the closest point to L on $\mathscr{H}$, and let z be the closest point to ξ on $\rho([0,+\infty[)$. Let $t\mapsto x_t$ be the geodesic ray starting from x_0 at time $t=0$ and converging to ξ. When t is large enough, the points ρ_+, z, x_t and ξ are in this order on the geodesic line $]\rho_+,\xi[$.

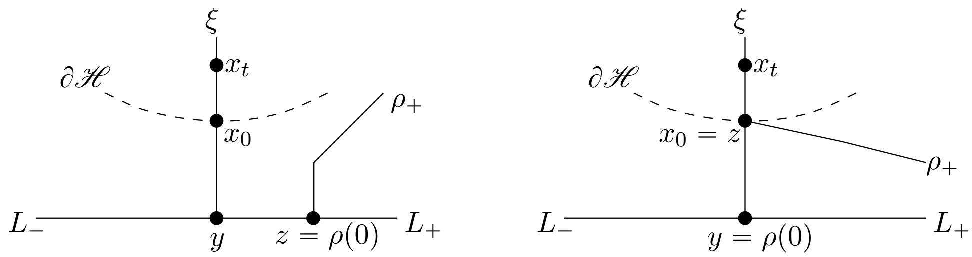
We have, by the definition in Equation (7.1) of the skinning measure, the cocycle property of the Busemann function, Equation (3.20) and the definition of z,

$$\begin{aligned}d\widetilde{\sigma}^+_L(\rho)&=e^{\delta_\Gamma\,\beta_{\rho_+}(x_t,\,\rho(0))}d\mu_{x_t}(\rho_+)=e^{\delta_\Gamma\,\beta_{\rho_+}(x_t,\,z)+\delta_\Gamma\,\beta_{\rho_+}(z,\,\rho(0))}d\mu_{x_t}(\rho_+)\\&=e^{-\delta_\Gamma\,\beta_\xi(x_t,\,z)-\delta_\Gamma\,d(z,\,\rho(0))}d\mu_{x_t}(\rho_+)\\&=e^{\delta_\Gamma\,t-\delta_\Gamma\,\beta_\xi(x_0,\,z)-\delta_\Gamma\,d(z,\,\rho(0))}d\mu_{x_t}(\rho_+)\ ,\end{aligned}$$

and, by the definition of Hamenstädt's measure $\mu_{\mathscr{H}}$ (see Equation (7.5)),

$$d\mu_{\mathscr{H}}(\rho_+)=e^{\delta_\Gamma\,t}d\mu_{x_t}(\rho_+)\,.$$

[5] See the definitions of Hamenstädt's distance and measure in Sections 2.2 and 7.1 respectively.



Case 1: Assume first that $\rho(0) \neq y$. We may assume that $\rho(0) \in [y, L_+[$. Then $z = \rho(0)$ and z is the closest point to $\mathscr{H}$ on the geodesic line $]L_+, \rho_+[$. Thus $d_{\mathscr{H}}(L_-, L_+) = d_{\mathscr{H}}(L_-, \rho_+)$ and $d_{\mathscr{H}}(L_+, \rho_+) = e^{-d(z,\, x_0)} = e^{\beta_\xi(x_0,\, z)}$, and the claim follows.

Case 2: Assume now that $y = \rho(0)$. Then $[y, z] = [y, \xi[\, \cap\, [y, \rho_+[$, and we may assume that $x_0 = z$ up to adjusting the horoball $\mathscr{H}$ while keeping its point at infinity. Thus $d_{\mathscr{H}}(L_-, L_+) = e^{-d(y,\, x_0)} = e^{-d(z,\, \rho(0))}$ and $d_{\mathscr{H}}(L_-, \rho_+) = d_{\mathscr{H}}(L_+, \rho_+) = 1$, and the claim follows. □

Chapter 9

Rate of mixing for the geodesic flow

Let X, x_0, Γ, $\widetilde{F}$, $(\mu^{\pm}_x)_{x\in X}$ be as in the beginning of Chapter 7, and $\widetilde{F}^{\pm}$, $F^{\pm}$, $\delta = \delta_{\Gamma,\, F^{\pm}} < \infty$, $\widetilde{m}_F$, m_F the associated notation. In this Chapter, we start by collecting in Section 9.1 known results on the rate of mixing of the geodesic flow for manifolds. The main part of the Chapter then consists in proving analogous bounds for the discrete time and continuous time geodesic flow for quotient spaces of simplicial and metric trees respectively.

We define $\overline{m_F} = \frac{m_F}{\|m_F\|}$ when the Gibbs measure is finite. Recall that this measure is nonzero since Γ is nonelementary.

Let $\alpha \in \,]0,1]$.[1] We will say that the (continuous time) geodesic flow on $\Gamma\backslash\mathscr{G}X$ is *exponentially mixing for the α-Hölder regularity* or that it has *exponential decay of α-Hölder correlations* for the potential F if there exist $C, \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}^{\alpha}_{\rm b}(\Gamma\backslash\mathscr{G}X)$ and $t \in \mathbb{R}$, we have

$$\Big|\int_{\Gamma\backslash\mathscr{G}X} \phi\circ \mathtt{g}^{-t}\,\psi\; d\overline{m_F} - \int_{\Gamma\backslash\mathscr{G}X}\phi\; d\overline{m_F}\int_{\Gamma\backslash\mathscr{G}X}\psi\; d\overline{m_F}\,\Big| \leqslant C\, e^{-\kappa|t|}\;\|\phi\|_\alpha\;\|\psi\|_\alpha\,, \tag{9.1}$$

and that it is *polynomially mixing* for the α-Hölder regularity or has *polynomial decay of α-Hölder correlations* if there exist $C > 0$ and $n \in \mathbb{N} - \{0\}$ such that for all $\phi, \psi \in \mathscr{C}^{\alpha}_{\rm b}(\Gamma\backslash\mathscr{G}X)$ and $t \in \mathbb{R}$, we have

$$\Big|\int_{\Gamma\backslash\mathscr{G}X} \phi\circ \mathtt{g}^{-t}\,\psi\; d\overline{m_F} - \int_{\Gamma\backslash\mathscr{G}X}\phi\; d\overline{m_F}\int_{\Gamma\backslash\mathscr{G}X}\psi\; d\overline{m_F}\,\Big| \leqslant C\,(1+|t|)^{-n}\;\|\phi\|_\alpha\;\|\psi\|_\alpha\,.$$

9.1 Rate of mixing for Riemannian manifolds

When $X = \widetilde{M}$ is a complete simply connected Riemannian manifold with pinched negative sectional curvature with bounded derivatives, then the boundary at infinity of $\widetilde{M}$, the strong unstable, unstable, stable, and strong stable foliations of $T^1\widetilde{M}$ are Hölder-smooth and only Hölder-smooth in general.[2] Hence the assumption of Hölder regularity on functions on $T^1\widetilde{M}$ is appropriate for these manifolds.

The geodesic flow is known to have exponential decay of Hölder correlations for compact manifolds $M = \Gamma\backslash\widetilde{M}$ when

[1] We refer to Section 3.1 for the definition of the Banach space $\mathscr{C}^{\alpha}_{\rm b}(Z)$ of bounded α-Hölder-continuous functions on a metric space Z.

[2] See for instance [Brin] when $\widetilde{M}$ has a compact quotient (a result first proved by Anosov), and [PauPS, Theo. 7.3].

- M is two-dimensional and F is any Hölder-continous potential by [Dol1],
- M is 1/9-pinched and $F = 0$ by [GLP, Coro. 2.7],
- m_F is the Liouville measure by [Live], see also [Tsu], [NZ, Coro. 5] who give more precise estimates,
- M is locally symmetric and F is any Hölder-continuous potential by [Sto], see also [MO].

When $\widetilde{M}$ is a symmetric space, then the boundary at infinity of $\widetilde{M}$, the strong unstable, unstable, stable, and strong stable foliations of $T^1\widetilde{M}$ are smooth. Hence talking about leafwise $\mathscr{C}^\ell$-smooth functions on $T^1\widetilde{M}$ makes sense. For every $\ell \in \mathbb{N}$, we will denote by $\mathscr{C}_c^\ell(T^1M)$ the vector space of real-valued $\mathscr{C}^\ell$-smooth functions on the orbifold $T^1M = \Gamma\backslash T^1\widetilde{M}$ with compact support in T^1M, and by $\|\psi\|_\ell$ the Sobolev $W^{\ell,2}$-norm of any $\psi \in \mathscr{C}_c^\ell(T^1M)$. This space consists of functions induced on T^1M by $\mathscr{C}^\ell$-smooth Γ-invariant functions with compact support on $T^1\widetilde{M}$.

Given $\ell \in \mathbb{N}$, we will say that the geodesic flow on T^1M is *exponentially mixing for the ℓ-Sobolev regularity* (or that it has *exponential decay of ℓ-Sobolev correlations*) for the potential F if there exist $c, \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}_c^\ell(T^1M)$ and all $t \in \mathbb{R}$, we have

$$\Big|\int_{T^1M} \phi \circ \mathtt{g}^{-t}\, \psi\, d\overline{m_F} - \int_{T^1M} \phi\, d\overline{m_F} \int_{T^1M} \psi\, d\overline{m_F}\,\Big| \leqslant c\, e^{-\kappa|t|}\ \|\psi\|_\ell\ \|\phi\|_\ell\ .$$

When $F = 0$ and Γ is an arithmetic lattice in the isometry group of $\widetilde{M}$ (the Gibbs measure then coincides, up to a multiplicative constant, with the Liouville measure), this property, for some $\ell \in \mathbb{N}$, follows from [KM1, Theorem 2.4.5], with the help of [Clo, Theorem 3.1] to check its spectral gap property, and of [KM2, Lemma 3.1] to deal with finite cover problems.

9.2 Rate of mixing for simplicial trees

Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, with geometric realisation $X = |\mathbb{X}|_1$, and $x_0 \in V\mathbb{X}$. Let Γ be a nonelementary discrete subgroup of $\mathrm{Aut}(\mathbb{X})$ and let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a system of conductances for Γ on $\mathbb{X}$. Let $\widetilde{F}_c : T^1X \to \mathbb{R}$ be the associated potential of $\widetilde{c}$, and $c : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$, $F_c : \Gamma\backslash T^1X \to \mathbb{R}$ the quotient functions. Let $\delta_c = \delta_{\Gamma,\,F_c}$ be the critical exponent of c, assumed to be finite.[3] Let $(\mu_x^\pm)_{x\in V\mathbb{X}}$ be (normalised) Patterson densities on $\partial_\infty X$ for the pairs $(\Gamma, F_c^\pm)$, and let $\widetilde{m}_c = \widetilde{m}_{F_c}$ and $m_c = m_{F_c}$ be the associated Gibbs measures on $\mathscr{G}\mathbb{X}$ and $\Gamma\backslash\mathscr{G}\mathbb{X}$.

In this Section, building on the end of Section 4.4 concerning the mixing properties themselves,[4] we now study the rate of mixing of the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ for the Gibbs measure $m_c = m_{F_c}$, when it is mixing.

Let (Z, m, T) be a dynamical system with (Z, m) a probability space and $T : Z \to Z$ a (not necessarily invertible) measure preserving map. For all $n \in \mathbb{N}$ and $\phi, \psi \in \mathbb{L}^2(m)$, the (well-defined) n-th *correlation coefficient* of ϕ, ψ is

$$\mathrm{cov}_{m,\,n}(\phi,\psi) = \int_Z \phi \circ T^n\, \psi\, dm - \int_Z \phi\, dm \int_Z \psi\, dm\ .$$

Let $\alpha \in\ ]0,1]$ and assume that Z is a metric space (endowed with its Borel σ-algebra). Similarly as for the case of flows in the beginning of Chapter 9, we will say that the dynamical

[3] That is, to be $< +\infty$, since The critical exponentas it is $> -\infty$ by Lemma 3.17 (7).
[4] See Theorem 4.17.

system (Z, m, T) is *exponentially mixing for the α-Hölder regularity* or that it has *exponential decay of α-Hölder correlations* if there exist $C, \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}_{\rm b}^{\alpha}(Z)$ and $n \in \mathbb{N}$, we have

$$|\operatorname{cov}_{m,\,n}(\phi, \psi)| \leqslant C\, e^{-\kappa\, n}\, \|\phi\|_\alpha\, \|\psi\|_\alpha\,.$$

Note that this property is invariant under measure preserving conjugations of dynamical systems by bilipschitz homeomorphisms.

The main result of this Section is a simple criterion for the exponential decay of correlation of the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$.

We define $\overline{m_c} = \frac{m_c}{\|m_c\|}$ when the Gibbs measure m_c on $\Gamma\backslash\mathscr{G}\mathbb{X}$ is finite, and we use the dynamical system $(\Gamma\backslash\mathscr{G}\mathbb{X}, \overline{m_c}, \mathsf{g}^1)$ in the definition of the correlation coefficients.

Given a finite subset E of $\Gamma\backslash V\mathbb{X}$, we denote by $\tau_E : \Gamma\backslash\mathscr{G}\mathbb{X} \to \mathbb{N} \cup \{+\infty\}$ the first return[5] time to E of the discrete time geodesic flow:

$$\tau_E(\ell) = \inf\{n \in \mathbb{N} - \{0\} \;:\; \mathsf{g}^n \ell(0) \in E\}\,,$$

with the usual convention that $\inf \varnothing = +\infty$.

Theorem 9.1. *Let $\mathbb{X}, \Gamma, \widetilde{c}$ be as above, with δ_c finite. Assume that the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$. Assume moreover that there exist a finite subset E of $\Gamma\backslash V\mathbb{X}$ and $C', \kappa' > 0$ such that for all $n \in \mathbb{N}$, we have*

$$m_c\big(\{\ell \in \Gamma\backslash\mathscr{G}\mathbb{X} \;:\; \ell(0) \in E \text{ and } \tau_E(\ell) \geqslant n\}\big) \leqslant C'\, e^{-\kappa' n}\,. \tag{9.2}$$

Then the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ has exponential decay of α-Hölder correlations for the system of conductances c.

A similar statement holds for the square of the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ when m_c is finite, $\mathscr{C}\Lambda\Gamma$ is a uniform simplicial tree with degrees at least 3 and $L_\Gamma = 2\mathbb{Z}$.

Note that the crucial Hypothesis (9.2) of Theorem 9.1 is in particular satisfied if $\Gamma\backslash\mathbb{X}$ is finite, by taking $E = \Gamma\backslash V\mathbb{X}$. In this case, the result is quite well-known: when Γ is torsion free, it follows from Bowen's result [Bowe2, 1.26] that a mixing subshift of finite type is exponentially mixing.

Proof. Let $\mathbb{X}' = \mathscr{C}\Lambda\Gamma$. Using the coding introduced in Section 5.2, we first reduce this statement to a statement in symbolic dynamics.

Step 1 : Reduction to two-sided symbolic dynamics. Let (Σ, σ) be the (two-sided) topological Markov shift with alphabet $\mathscr{A}$ and transition matrix A constructed in Section 5.2, which is conjugated to $(\Gamma\backslash\mathscr{G}\mathbb{X}', \mathsf{g}^1)$ by the homeomorphism $\Theta : \Gamma\backslash\mathscr{G}\mathbb{X}' \to \Sigma$ (see Theorem 5.1). Let $\mathbb{P} = \Theta_* \frac{m_c}{\|m_c\|}$, which is a mixing σ-invariant probability measure on Σ with full support, since $\Gamma\backslash\mathscr{G}\mathbb{X}'$ is the support of m_c. Let

$$\mathscr{E} = \{(e^-, h, e^+) \in \mathscr{A} : t(e^-) = o(e^+) \in E\}\,.$$

The set $\mathscr{E}$ is finite since the degrees and the vertex stabilisers of $\mathbb{X}$ are finite. For all $x \in \Sigma$ and $k \in \mathbb{Z}$, we denote by x_k the k-th component of $x = (x_n)_{n\in\mathbb{Z}}$. Let

$$\tau_{\mathscr{E}}(x) = \inf\{n \in \mathbb{N} - \{0\} \;:\; x_n \in \mathscr{E}\}$$

[5] Actually, a more precise terminology is "first positive passage time", but we use the shorter one. If $\ell \in \Gamma\backslash\mathscr{G}\mathbb{X}$ is such that $\ell(0) \in E$, then "return" is appropriate.

be the first return[6] time to $\mathscr{E}$ of x under iteration of the shift σ.

Let $\pi_+ : \Sigma \to \mathscr{A}^{\mathbb{N}}$ be the *natural extension*[7] $(x_n)_{n\in\mathbb{Z}} \mapsto (x_n)_{n\in\mathbb{N}}$. Theorem 9.1 will follow from the following two-sided symbolic dynamics result.[8]

Theorem 9.2. *Let (Σ, σ) be a locally compact transitive two-sided topological Markov shift with alphabet $\mathscr{A}$ and transition matrix A, and let $\mathbb{P}$ be a mixing σ-invariant probability measure with full support on Σ. Assume that*

(1) for every A-admissible finite sequence $w = (w_0, \dots, w_n)$ in $\mathscr{A}$, the Jacobian of the map f_w from $\{(x_k)_{k\in\mathbb{N}} \in \pi_+(\Sigma) \;:\; x_0 = w_n\}$ to $\{(y_k)_{k\in\mathbb{N}} \in \pi_+(\Sigma) \;:\; y_0 = w_0, \dots, y_n = w_n\}$ defined by $(x_0, x_1, x_2, \dots) \mapsto (w_0, \dots, w_n, x_1, x_2, \dots)$, with respect to the restrictions of the pushforward measure $(\pi_+)_*\mathbb{P}$, is constant;

(2) there exist a finite subset $\mathscr{E}$ of $\mathscr{A}$ and $C', \kappa' > 0$ such that for all $n \in \mathbb{N}$, we have

$$\mathbb{P}\big(\{x \in \Sigma \;:\; x_0 \in \mathscr{E} \text{ and } \tau_{\mathscr{E}}(x) \geqslant n\}\big) \leqslant C' \, e^{-\kappa' n} \,. \tag{9.3}$$

Then $(\Sigma, \mathbb{P}, \sigma)$ has exponential decay of α-Hölder correlations.

Proof that Theorem 9.2 implies Theorem 9.1. Since $\widetilde{m}_c$ is supported on $\mathscr{G}\mathbb{X}'$, up to replacing $\mathbb{X}$ by $\mathbb{X}'$, we may assume that $\partial_\infty X = \Lambda\Gamma$.

By the construction of Θ just before the statement of Theorem 5.1, for every $\ell = \Gamma\widetilde{\ell} \in \Gamma\backslash\mathscr{G}\mathbb{X}$, we have $(\Theta\ell)_0 = (e_0^-(\widetilde{\ell}), h_0(\widetilde{\ell}), e_0^+(\widetilde{\ell}))$ with $e_0^+(\widetilde{\ell}) = p(\widetilde{\ell}([0,1]))$ where $p : \mathbb{X} \to \Gamma\backslash\mathbb{X}$ is the canonical projection, so that $o\big(e_0^+(\widetilde{\ell})\big) = \ell(0)$. Since Θ conjugates g^1 to σ, for every $n \in \mathbb{N}$, we have

$$(\Theta\ell)_n = (\sigma^n(\Theta\ell))_0 = (\Theta(\mathsf{g}^n\ell))_0 \,.$$

Thus $(\Theta\ell)_n \in \mathscr{E}$ if and only if $\mathsf{g}^n\ell(0) \in E$, and

$$\tau_{\mathscr{E}}(\Theta\ell) = \tau_E(\ell) \,.$$

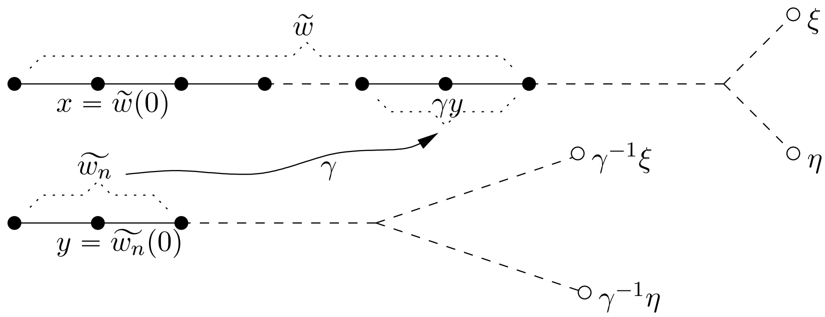
Therefore Theorem 9.1 will follow from Theorem 9.2 by conjugation since Θ is bilipschitz, once we have proved that Hypothesis (1) of Theorem 9.2 is satisfied for the two-sided topological Markov shift (Σ, σ) conjugated by Θ to $(\Gamma\backslash\mathscr{G}\mathbb{X}, \mathsf{g}^1)$, which is the main point in this proof.

We hence fix an A-admissible finite sequence $w = (w_0, \dots, w_n)$ in $\mathscr{A}$. We consider the (one-sided) *cylinders*

$$[w_n] = \{(x_k)_{k\in\mathbb{N}} \in \pi_+(\Sigma) \;:\; x_0 = w_n\} \,,$$
$$[w] = \{(y_k)_{k\in\mathbb{N}} \in \pi_+(\Sigma) \;:\; y_0 = w_0, \dots, y_n = w_n\}$$

and the map $f_w : [w_n] \to [w]$ with $(x_0, x_1, x_2, \dots) \mapsto (w_0, \dots, w_n, x_1, x_2, \dots)$ that appear in Hypothesis (1). We denote by $\widetilde{w}$ and $\widetilde{w_n}$ the discrete generalised geodesic lines in $\mathbb{X}$ associated with w and w_n (see the proof of Theorem 5.1 just after Equation (5.3)). Since w ends with w_n, by the construction of Θ, there exists $\gamma \in \Gamma$ sending the two consecutive edges of $\widetilde{w_n}$ to the last two consecutive edges of w. We denote by $x = \widetilde{w}(0)$ and $y = \widetilde{w_n}(0)$ the footpoints of $\widetilde{w}$ and $\widetilde{w_n}$ respectively.

[6] See the previous footnote.
[7] The authors are not responsible for this questionable terminology, rather standard in symbolic dynamics.
[8] Assumption (1) of Theorem 9.2 is far from being optimal, but will be sufficient for our purpose.



For every discrete generalised geodesic line $\omega \in \check{\mathscr{G}}\mathbb{X}$ which is isometric exactly on an interval I containing 0 in its interior (as for $\omega = \widetilde{w}, \widetilde{w_n}$), let

$$\mathscr{G}_\omega \mathbb{X} = \{\ell \in \mathscr{G}\mathbb{X} \;:\; \ell_{|I} = \omega_{|I}\}$$

be the space of extensions of $\omega_{|I}$ to geodesic lines. With $\partial^\pm_\omega \mathbb{X} = \{\ell_\pm \;:\; \ell \in \mathscr{G}_\omega\mathbb{X}\}$ its set of points at $\pm\infty$, we have a homeomorphism $\mathscr{G}_\omega\mathbb{X} \to (\partial^-_\omega\mathbb{X} \times \partial^+_\omega\mathbb{X})$ defined by $\ell \mapsto (\ell_-, \ell_+)$, using Hopf's parametrisation with respect to the point $\omega(0)$, since all the geodesic lines in $\mathscr{G}_\omega\mathbb{X}$ are at the point $\omega(0)$ at time $t = 0$. Using as basepoint $x_0 = \omega(0)$ in the definition of the Gibbs measure (see Equation (4.12)), this homeomorphism sends the restriction to $\mathscr{G}_\omega\mathbb{X}$ of the Gibbs measure $d\,\widetilde{m}_c(\ell)$ to the product measure $d\mu^-_{\omega(0)}(\ell_-)\; d\mu^+_{\omega(0)}(\ell_+)$. Hence the pushforward of $\widetilde{m}_{c|\mathscr{G}_\omega\mathbb{X}}$ by the positive endpoint map $e_+ : \ell \mapsto \ell_+$ is $\mu^-_{\omega(0)}(\partial^-_\omega\mathbb{X})\; d\mu^+_{\omega(0)}(\ell_+)$, and note that $\mu^-_{\omega(0)}(\partial^-_\omega\mathbb{X})$ is a positive constant.

Let $\widetilde{p} : \mathscr{G}\mathbb{X} \to \Gamma\backslash\mathscr{G}\mathbb{X}$ be the canonical projection. Since $\pi_+ : \Sigma \to \Sigma_+$ is the map which forgets about the past, there exist measurable maps $u_w : \partial^+_{\widetilde{w}}\mathbb{X} \to [w]$ and $u_{w_n} : \partial^+_{\widetilde{w_n}}\mathbb{X} \to [w_n]$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathscr{G}_{\widetilde{w}}\mathbb{X} & \xrightarrow{\pi_+\circ\Theta\circ\widetilde{p}} & [w] \\ & {\scriptstyle e_+}\searrow \quad \nearrow{\scriptstyle u_w} & \\ & \partial^+_{\widetilde{w}}\mathbb{X} & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathscr{G}_{\widetilde{w_n}}\mathbb{X} & \xrightarrow{\pi_+\circ\Theta\circ\widetilde{p}} & [w_n] \\ & {\scriptstyle e_+}\searrow \quad \nearrow{\scriptstyle u_{w_n}} & \\ & \partial^+_{\widetilde{w}}\mathbb{X} & \end{array}\;.$$

Furthermore, the map u_w (respectively u_{w_n}) is surjective, and has constant finite order fibers given by the orbits of the finite stabiliser $\Gamma_{\widetilde{w}}$ (respectively $\Gamma_{\widetilde{w_n}}$). Since $\mathbb{P} = \Theta_*\frac{m_c}{\|m_c\|}$, the pushforward by the map u_w (respectively u_{w_n}) of the measure μ^+_x (respectively μ^+_y) is a constant time the restriction of $(\pi_+)_*\mathbb{P}$ to $[w]$ (respectively $[w_n]$). Finally, by the construction of the (inverse of the) coding in the proof of Theorem 5.1, the following diagram is commutative:

$$\begin{array}{ccc} \partial^+_{\widetilde{w_n}}\mathbb{X} & \xrightarrow{u_{w_n}} & [w_n] \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle f_w} \\ \partial^+_{\widetilde{w}}\mathbb{X} & \xrightarrow{u_w} & [w]\,. \end{array}$$

Recall that the pushforwards of measures μ, ν, which are absolutely continuous one with respect to the other, by a measurable map f are again absolutely continuous one with respect to the other, and satisfy (almost everywhere)

$$\frac{d\,f_*\mu}{d\,f_*\nu}\circ f \;=\; \frac{d\,\mu}{d\,\nu}\;.$$

Hence in order to prove that Hypothesis (1) in the statement of Theorem 9.2 is satisfied, we only have to prove that the map $\gamma : \partial^+_{\widetilde{w_n}}\mathbb{X} \to \partial^+_{\widetilde{w}}\mathbb{X}$ has a constant Jacobian for the measures μ^+_y on $\partial^+_{\widetilde{w_n}}\mathbb{X}$ and μ^+_x on $\partial^+_{\widetilde{w}}\mathbb{X}$ respectively.

For all $\xi, \eta \in \partial^+_{\widetilde{w}}\mathbb{X}$, by the properties of the Patterson densities (see Equations (4.1) and (4.2)), since γy belongs to the geodesic ray from x to ξ and η (see the above picture), and by Equation (3.20), we have

$$\frac{\frac{d\,\gamma_*\mu^+_y}{d\,\mu^+_x}(\xi)}{\frac{d\,\gamma_*\mu^+_y}{d\,\mu^+_x}(\eta)} = \frac{\frac{d\,\mu^+_{\gamma y}}{d\,\mu^+_x}(\xi)}{\frac{d\,\mu^+_{\gamma y}}{d\,\mu^+_x}(\eta)} = \frac{e^{-C^+_\xi(\gamma y,\, x)}}{e^{-C^+_\eta(\gamma y,\, x)}} = \frac{e^{-\int_x^{\gamma y}(\widetilde{F}^+_c-\delta_c)}}{e^{-\int_x^{\gamma y}(\widetilde{F}^+_c-\delta_c)}} = 1\ .$$

This proves that Hypothesis (1) in Theorem 9.2 is satisfied, and concludes the proof of Theorem 9.1. □

We now indicate how to pass from a one-sided version of Theorem 9.2 to the two-sided one, as was communicated to us by J. Buzzi.

Step 2 : Reduction to one-sided symbolic dynamics. Let (Σ_+, σ_+) be the *one-sided topological Markov shift* with alphabet $\mathscr{A}$ and transition matrix A, that is, Σ_+ is the closed subset of the topological product space $\mathscr{A}^{\mathbb{N}}$ defined by

$$\Sigma_+ = \big\{x = (x_n)_{n\in\mathbb{N}} \in \mathscr{A}^{\mathbb{N}}\ :\ \forall\, n\in\mathbb{N},\quad A_{x_n, x_{n+1}} = 1\big\}\ ,$$

and $\sigma_+ : \Sigma_+ \to \Sigma_+$ is the (one-sided) *shift*[9] defined by

$$(\sigma_+(x))_n = x_{n+1}$$

for all $x \in \Sigma_+$ and $n \in \mathbb{N}$. We endow Σ_+ with the distance

$$d(x, x') = e^{-\max\big\{n\in\mathbb{N}\,:\ \forall\, i\in\{0,\dots,n\},\ x_i = x'_i\big\}}\ .$$

Note that the distances on Σ and Σ_+ are bounded by 1.

The natural extension $\pi_+ : (x_n)_{n\in\mathbb{Z}} \mapsto (x_n)_{n\in\mathbb{N}}$ maps Σ to Σ_+. It satisfies $\pi_+\circ\sigma = \sigma_+\circ\pi_+$ and is 1-Lipschitz. Note that Σ is transitive (respectively locally compact) if and only if Σ_+ is transitive (respectively locally compact).

In the one-sided case, we always assume that the cylinders start at time $t = 0$: Given an admissible sequence $w = (w_0, w_1, \dots, w_{n-1})$, we will say that the cylinder

$$[w] = [w_0, \dots, w_{n-1}] = \{(x_n)_{n\in\mathbb{N}} \in \Sigma_+\ :\ \forall\, i\ \in\ \{0, \dots, n-1\},\ \ x_i\ =\ w_i\}$$

defined by w has length $|w| = n$.

We first explain how to relate the decay of correlations for the two-sided and one-sided systems. This is well-known since the works of Sinai [Sin, §3] and Bowen [Bowe2, Lem. 1.6], see for instance [You1, §4], and the following proof has been communicated to us by J. Buzzi. We

[9] Although it is standard to denote the one-sided shift by σ in the same way as the two-sided shift, we use σ_+ for readability.

fix $\alpha \in]0,1]$. For all metric spaces Z and bounded α-Hölder-continuous functions $f : Z \to \mathbb{R}$, recall[10] that

$$\|f\|'_\alpha = \sup_{\substack{x,\,y\,\in Z\\ 0<d(x,\,y)\leqslant 1}} \frac{|f(x)-f(y)|}{d(x,y)^\alpha} \quad\text{and}\quad \|f\|_\alpha = \|f\|_\infty + \|f\|'_\alpha\ .$$

For every $a \in \mathscr{A}$, let us fix $z^a \in \Sigma$ such that $(z^a)_0 = a$.

Lemma 9.3. *Let $\phi : \Sigma \to \mathbb{R}$ be a bounded α-Hölder-continuous map and $N \in \mathbb{N}$. For all $x \in \Sigma_+$, let $y^{(N)}(x) = (y_i)_{i\in\mathbb{N}} \in \Sigma$, where $y_i = x_{i+N}$ if $i \geqslant -N$ and $y_i = (z^{x_0})_{i+N}$ otherwise. Define $\phi^{(N)} : \Sigma_+ \to \mathbb{R}$ by $\phi^{(N)}(x) = \phi(y^{(N)}(x))$. Then $\phi^{(N)}$ is bounded and α-Hölder-continuous on Σ_+, with*

$$|\phi\circ\sigma^N - \phi^{(N)}\circ\pi_+| \leqslant \|\phi\|'_\alpha\, e^{-\alpha\,N}\ .$$

Moreover,

$$\|\phi^{(N)}\|'_\alpha \leqslant e^{\alpha\,N}\,\|\phi\|'_\alpha \quad\textit{and}\quad \|\phi^{(N)}\|_\infty \leqslant \|\phi\|_\infty\ .$$

Proof. For every $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma$, if $y = y^{(N)}(\pi_+(x))$, we have $\big(\sigma^N(x)\big)_n = x_{n+N} = y_n$ if $|n| \leqslant N$. Hence,

$$\begin{aligned}|\phi\circ\sigma^N(x) - \phi^{(N)}(\pi_+(x))| &= |\phi(\sigma^N(x)) - \phi(y)|\\ &\leqslant \|\phi\|'_\alpha\; d(\sigma^N(x),y)^\alpha \leqslant \|\phi\|'_\alpha\; e^{-\alpha\,N}\ .\end{aligned}$$

Moreover, for all $x = (x_n)_{n\in\mathbb{N}}$, $x' = (x'_n)_{n\in\mathbb{N}}$ in Σ_+, if $y = y^{(N)}(x)$ and $y' = y^{(N)}(x')$, then $d(y,y') = e^N\, d(x,x')$ if $d(x,x') < e^{-N}$ and otherwise $d(y,y') \leqslant 1 \leqslant e^N d(x,x')$, so that

$$|\phi^{(N)}(x) - \phi^{(N)}(x')| = |\phi(y)-\phi(y')| \leqslant \|\phi\|'_\alpha\; d(y,y')^\alpha \leqslant \|\phi\|'_\alpha\; e^{\alpha\,N}\, d(x,x')^\alpha\ . \quad \square$$

Proposition 9.4. *Let μ be a σ-invariant probability measure on Σ. Assume that the dynamical system $(\Sigma_+,\sigma_+,(\pi_+)_*\mu)$ has exponential decay of α-Hölder correlations. Then (Σ,σ,μ) has exponential decay of α-Hölder correlations.*

Proof. Let $C,\kappa > 0$ be such that for all bounded α-Hölder-continuous maps $\phi',\psi' : \Sigma_+ \to \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$|\operatorname{cov}_{(\pi_+)_*\mu,\,n}(\phi',\psi')| \leqslant C\;\|\phi'\|_\alpha\;\|\psi'\|_\alpha\; e^{-\kappa n}\ .$$

Let $\phi,\psi : \Sigma \to \mathbb{R}$ be bounded α-Hölder-continuous maps and $n \in \mathbb{N}$. Denoting by $\pm\, t$ any value in $[-t,t]$ for every $t \geqslant 0$, we have, by the first part of Lemma 9.3 and for every $N \in \mathbb{N}$,[11]

$$\begin{aligned}\int_\Sigma \phi\circ\sigma^n\;\;\psi\;d\mu &= \int_\Sigma \phi\circ\sigma^{n+N}\;\;\psi\circ\sigma^N\;d\mu\\ &= \int_\Sigma (\phi^{(N)}\circ\pi_+ \pm \|\phi\|'_\alpha\;e^{-\alpha\,N})\circ\sigma^n\;\;(\psi^{(N)}\circ\pi_+ \pm \|\psi\|'_\alpha\;e^{-\alpha\,N})\;d\mu\\ &= \int_{\Sigma_+} \phi^{(N)}\circ\sigma_+^n\;\;\psi^{(N)}\;d(\pi_+)_*\mu\;\pm\;\|\phi\|_\alpha\;\|\psi\|_\alpha\;e^{-\alpha\,N}\ .\end{aligned}$$

[10] See Section 3.1 for the definition of the Hölder norm $\|\cdot\|_\alpha$.
[11] to be chosen appropriately below

A similar estimate holds for the second term in the definition of the correlation coefficients. Hence, by the second part of Lemma 9.3,

$$\begin{aligned}
&|\operatorname{cov}_{\mu,n}(\phi,\psi)| \\
&\leqslant |\operatorname{cov}_{(\pi_+)_*\mu,n}(\phi^{(N)},\psi^{(N)})| + 2\,\|\phi\|_\alpha\,\|\psi\|_\alpha\,e^{-\alpha N} \\
&\leqslant C\,(\|\phi\|_\infty + \|\phi\|'_\alpha\,e^{\alpha N})\,(\|\psi\|_\infty + \|\psi\|'_\alpha\,e^{\alpha N})\,e^{-\kappa n} + 2\,\|\phi\|_\alpha\,\|\psi\|_\alpha\,e^{-\alpha N} \\
&\leqslant \|\phi\|_\alpha\,\|\psi\|_\alpha (C\,e^{2\alpha N-\kappa n} + 2\,e^{-\alpha N})\,.
\end{aligned}$$

Taking $N = \lfloor\frac{\kappa n}{4\alpha}\rfloor$, $C' = C + 2\,e^{\alpha}$ and $\kappa' = \frac{\kappa}{4}$, we have

$$|\operatorname{cov}_{\mu,n}(\phi,\psi)| \leqslant C'\,\|\phi\|_\alpha\,\|\psi\|_\alpha\,e^{-\kappa' n}\,,$$

and the result follows. □

In order to conclude Step 2, we now state the one-sided version of Theorem 9.2 and prove how it implies Theorem 9.2.[12] For every finite subset $\mathscr{E}$ of $\mathscr{A}$, let

$$\tau_{\xi}(x) = \inf\{n \in \mathbb{N} - \{0\} \;:\; x_n \in \mathscr{E}\}$$

be the first return time[13] of $x \in \Sigma_+$ under iteration of the one-sided shift.

Theorem 9.5. *Let (Σ_+, σ_+) be a locally compact transitive one-sided topological Markov shift with alphabet $\mathscr{A}$ and transition matrix A, and let $\mathbb{P}_+$ be a mixing σ_+-invariant probability measure with full support on Σ_+. Assume that*

(1) for every A-admissible finite sequence $w = (w_0, \dots, w_n)$ in $\mathscr{A}$, the Jacobian of the map from $[w_n]$ to $[w]$ defined by $(w_n, x_1, x_2, \dots) \mapsto (w_0, \dots, w_n, x_1, x_2, \dots)$ with respect to the restrictions of the measure $\mathbb{P}_+$ is constant;

(2) there exist a finite subset $\mathscr{E}$ of $\mathscr{A}$ and $C', \kappa' > 0$ such that for all $n \in \mathbb{N}$, we have

$$\mathbb{P}_+\big(\{x \in \Sigma_+ \;:\; x_0 \in \mathscr{E} \text{ and } \tau_{\mathscr{E}}(x) \geqslant n\}\big) \leqslant C'\,e^{-\kappa' n}\,. \tag{9.4}$$

Then $(\Sigma_+, \mathbb{P}_+, \sigma_+)$ has exponential decay of α-Hölder correlations.

Proof that Theorem 9.5 implies Theorem 9.2. Let $(\Sigma, \sigma, \mathbb{P}, \mathscr{E})$ be as in the statement of Theorem 9.2. Let $\mathbb{P}_+ = (\pi_+)_*\mathbb{P}$, which is a mixing σ_+-invariant probability measure on Σ_+ with full support. Note that Hypothesis (1) in Theorem 9.5 follows from Hypothesis (1) of Theorem 9.2. Similarly, Equation (9.4) follows from Equation (9.3). Hence Theorem 9.2 follows from Theorem 9.5 and Proposition 9.4. □

Let us now consider Theorem 9.5. The scheme of its proof, using inducing and Young tower arguments, was communicated to us by O. Sarig.

[12] Assumption (1) of Theorem 9.5 is far from being optimal, but will be sufficient for our purpose.
[13] or rather the first positive passage time

Step 3 : Proof of Theorem 9.5. In this final Step, using inducing of the dynamical system (Σ_+, σ_+) on the subspace $\{x \in \Sigma_+ \;:\; x_0 \in \mathscr{E}\} = \bigcup_{a\in\mathscr{E}}[a]$ (a finite union of 1-cylinders), we present (Σ_+, σ_+) as a Young tower to which we will apply the results of [You2].

Note that since σ_+ is mixing and $\mathbb{P}_+$ has full support, there exists a σ_+-invariant measurable subset of full measure Δ of Σ_+ such that the orbit under σ_+ of every element of Δ passes infinitely many times inside the nonempty open subset $\bigcup_{a\in\mathscr{E}}[a]$. We again denote by $\tau_{\mathscr{E}} : \Delta \to \mathbb{N} - \{0\}$ the restriction to Δ of the first return time in $\bigcup_{a\in\mathscr{E}}[a]$, so that if

$$\Delta_0 = \{x \in \Delta \;:\; x_0 \in \mathscr{E}\} = \bigcup_{a\in\mathscr{E}} \Delta \cap [a] \;,$$

then $\tau_{\mathscr{E}}(x) = \min\{n \in \mathbb{N} - \{0\} \;:\; \sigma_+^n(x) \in \Delta_0\}$ for all $x \in \Delta$. We denote by $F : \Delta \to \Delta_0$ the first return map to Δ_0 under iteration of the one-sided shift, that is

$$F : x \mapsto \sigma_+^{\tau_{\mathscr{E}}(x)}(x) \;.$$

Let W be the set of admissible sequences w of length $|w|$ at least 2 such that if $w = (w_0, \dots, w_n)$ with $n = |w| - 1$ then

$$w_0, w_n \in \mathscr{E} \quad \text{and} \quad w_1, \dots, w_{n-1} \notin \mathscr{E} \;.$$

We have the following properties:

- the sets $\Delta_a = \Delta \cap [a]$ for $a \in \mathscr{E}$ form a finite measurable partition of Δ_0 and for every $a \in \mathscr{E}$, the sets $\Delta_w = \Delta \cap [w]$ for $w \in W$ and $w_0 = a$ form a countable measurable partition of Δ_a;
- for every $w \in W$, the first return time $\tau_{\mathscr{E}}$ is constant (equal to $|w| - 1$) on Δ_w, and if $w_{|w|-1} = b$, then the first return map F is a bijection from Δ_w to Δ_b;
- for all $w \in W$ and $x, y \in \Delta_w$, since x, y have the same $|w|$ first components, we have

$$d(F(x), F(y)) = d(\sigma_+^{|w|-1}x, \sigma_+^{|w|-1}y) = e^{|w|-1}\, d(x, y) \geqslant e\, d(x, y) \;;$$

- for all $w \in W$, $n \in \{0, \dots, |w| - 2\}$ and $x, y \in \Delta_w$, we have

$$d(\sigma_+^n x, \sigma_+^n y) = e^n\, d(x, y) \leqslant e^{|w|-2}\, d(x, y) < d(F(x), F(y)) \;;$$

- for every $w \in W$, the Jacobian of the first return map $F : \Delta_w \to \Delta_{w_{|w|-1}}$ for the restrictions to Δ_w and $\Delta_{w_{|w|-1}}$ of $\mathbb{P}_+$ is constant.[14]

By an easy adaptation of [You2, Theo. 3] (see also [Mel1, §2.1]) which considers the case when $\mathscr{E}$ is a singleton, we have the following noneffective[15] exponential decay of correlation: there exists $\kappa > 0$ such that for every $\phi, \psi \in \mathscr{C}_{\mathrm{b}}^{\alpha}(\Sigma_+)$, there exists a constant $C_{\phi,\psi} > 0$ such that

$$|\,\mathrm{cov}_{\mathbb{P}_+,\,n}(\phi, \psi)| \leqslant C_{\phi,\psi}\, e^{-\kappa n}$$

By an elegant argument using the Principle of Uniform Boundedness, it is proved in [ChCS, Appendix B] that this implies that there exist $C, \kappa > 0$ such that for every $\phi, \psi \in \mathscr{C}_{\mathrm{b}}^{\alpha}(\Sigma_+)$, we have

$$|\,\mathrm{cov}_{\mathbb{P}_+,\,n}(\phi, \psi)| \leqslant C\, \|\phi\|_\alpha\, \|\psi\|_\alpha\, e^{-\kappa\, n} \;.$$

[14] Actually, only a much weaker assumption is required, such as a Hölder-continuity property of this Jacobian, see [You2].

[15] Actually, there is in [You2] (see also [CyS]) a control on the constant in terms of some norms of the test functions, but these norms are not the ones we are interested in.

This concludes the proof of Theorem 9.5, hence the proof of Theorem 9.1. □ □

The next result gives examples of applications of Theorem 9.1 when $\Gamma\backslash\mathbb{X}$ is infinite. It strengthens [AtGP, Theo. 2.1] that applies only to arithmetic lattices and only for the locally constant regularity (see Section 15.4), see also [BekL] for an approach using spectral gaps. It was claimed in [Kwo], but was retracted by the author.

Corollary 9.6. *Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices. Let Γ be a geometrically finite subgroup of $\operatorname{Aut}(\mathbb{X})$ such that the smallest nonempty Γ-invariant subtree of $\mathbb{X}$ is uniform without vertices of degree 2. Let $\alpha \in \,]0,1]$.*

(1) If $L_\Gamma = \mathbb{Z}$, then the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ has exponential decay of α-Hölder correlations for the zero system of conductances.

(2) If $L_\Gamma = 2\mathbb{Z}$, then the square of the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ has exponential decay of α-Hölder correlations for the zero system of conductances, that is, there exist $C, \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}^\alpha_{\rm b}(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X})$ and $n \in \mathbb{Z}$, we have

$$\Big|\int_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}} \phi\circ \mathsf{g}^{-2n}\,\psi\,dm_{\rm BM} - \frac{1}{m_{\rm BM}(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X})}\int_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}}\phi\,dm_{\rm BM}\int_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}}\psi\,dm_{\rm BM}\Big| \leqslant C\,e^{-\kappa|n|}\,\|\phi\|_\alpha\,\|\psi\|_\alpha\,.$$

The main point in order to obtain this corollary is to prove the exponential decay of volumes of geodesic lines going high in the cuspidal rays of $\Gamma\backslash\mathbb{X}$, stated as Assumption (9.2) in Theorem 9.1. There is a long history of similar results, starting from the exponential decay of volumes of small cusp neighbourhoods in noncompact finite volume hyperbolic manifolds (based on the description of their ends) used by Sullivan to deduce Diophantine approximation results (see [Sul3, §9]).[16] These results were extended to the case of locally symmetric Riemannian manifolds by Kleinbock-Margulis [KM2] (based on the description of their ends using Siegel sets). Note that the geometrically finite lattice assumption on Γ is here in order to obtain similar descriptions of the ends of $\Gamma\backslash\mathbb{X}$.

Proof. Up to replacing $\mathbb{X}$ by $\mathscr{C}\Lambda\Gamma$, we assume that $\mathbb{X}$ is a uniform simplicial tree with degrees at least 3 and that Γ is a geometrically finite lattice of $\mathbb{X}$. We use the zero system of conductances.

(1) By [Pau4],[17] the graph $\Gamma\backslash\mathbb{X}$ is the union of a finite graph $\mathbb{Y}$ and finitely many cuspidal rays R_i for $i \in \{1,\dots,k\}$. If $(x_{i,n})_{n\in\mathbb{N}}$ is the sequence of vertices in increasing order along R_i for $i = 1,\dots,k$, then the vertex group $G_{x_{i,n}}$ of $x_{i,n}$ in the quotient graph of groups $\Gamma\backslash\backslash\mathbb{X}$ satisfies $G_{x_{i,n}} \subset G_{x_{i,n+1}}$ for every $n \in \mathbb{N}$, and the edge group of the edge $e_{i,n}$ with origin $x_{i,n}$ and endpoint $x_{i,n+1}$ is equal to $G_{x_{i,n}}$.[18] Note that since the degrees of the vertices of $\mathbb{X}$ are at least 3, we have $[G_{x_{i,n+1}} : G_{x_{i,n}}] \geqslant 2$ and $|G_{x_{i,0}}| \geqslant 1$, so that, for every $n \in \mathbb{N}$,

$$|G_{x_{i,n}}| \geqslant 2^n\,. \tag{9.5}$$

Let E be the (finite) set of vertices $V\mathbb{Y}$ of $\mathbb{Y}$. Note that for all $n \in \mathbb{N}-\{0\}$ and $\ell \in \Gamma\backslash\mathscr{G}\mathbb{X}$, if $\ell(0) \in E$ and $\tau_E(\ell) \geqslant 2n$, then ℓ leaves $\mathbb{Y}$ after time 0 and it travels (geodesically) inside some

[16] and by probabilists in order to study the statistics of cusp excursions (see for instance [EF])
[17] See also Section 2.6.
[18] identifying the edge group of an edge e with its image by the structural map $G_e \to G_{o(e)}$

cuspidal ray for a time at least n, so that there exists $i \in \{1, \ldots, k\}$ such that $\ell(n) = x_{i,n}$. Hence for all $n \in \mathbb{N}$, using

- the invariance of $m_{\rm BM}$ under the discrete time geodesic flow in order to get the third term,
- Equation (8.5) where $\widetilde{x}_{i,n}$ is a fixed lift of $x_{i,n}$ in $V\mathbb{X}$ for the fifth term, and
- Equation (9.5) since $|\Gamma_{\widetilde{x}_{i,n}}| = |G_{x_{i,n}}|$, and the facts that the degrees of the uniform simplicial tree $\mathbb{X}$ are uniformly bounded and that the total mass of the Patterson measures of the lattice Γ are uniformly bounded (see Proposition 4.16) for the last term,

we have

$$\begin{aligned}
& m_{\rm BM}\big(\{\ell \in \Gamma\backslash\mathscr{G}\mathbb{X} \;:\; \ell(0) \in E \text{ and } \tau_E(\ell) \geqslant 2n\}\big) \\
\leqslant\; & \sum_{i=1}^{k} m_{\rm BM}\big(\{\ell \in \Gamma\backslash\mathscr{G}\mathbb{X} \;:\; \ell(n) = x_{i,n}\}\big) \\
=\; & \sum_{i=1}^{k} m_{\rm BM}\big(\{\ell \in \Gamma\backslash\mathscr{G}\mathbb{X} \;:\; \ell(0) = x_{i,n}\}\big) = \sum_{i=1}^{k} \pi_* m_{\rm BM}(\{x_{i,n}\}) \\
=\; & \sum_{i=1}^{k} \frac{1}{|\Gamma_{\widetilde{x}_{i,n}}|} \sum_{e,\,e' \in E\mathbb{X}\,:\, o(e)=o(e')=\widetilde{x}_{i,n},\; e\neq e'} \mu_{\widetilde{x}_{i,n}}(\partial_e \mathbb{X})\, \mu_{\widetilde{x}_{i,n}}(\partial_{e'} \mathbb{X}) \\
\leqslant\; & k\, \frac{1}{2^n} \max_{x \in V\mathbb{X}} \deg(x)^2 \max_{x \in V\mathbb{X}} \|\mu_x\|^2 \,.
\end{aligned}$$

The result then follows from Theorem 9.1 using the above finite set E which satisfies Assumption (9.2) as we just proved, and using Proposition 4.16 and Theorem 4.17 in order to check that under the assumption that $L_\Gamma = \mathbb{Z}$, the Bowen-Margulis measure $m_{\rm BM}$ of Γ is finite and mixing under the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$.

(2) The proof of Assertion (2) is similar to the one of Assertion (1). □

Remark. The techniques introduced in the above proof in order to check the main hypothesis of Theorem 9.1 may be applied to numerous other examples. For instance, let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices. Let Γ be a nonelementary discrete subgroup of $\mathrm{Aut}(\mathbb{X})$ such that the smallest nonempty Γ-invariant subtree of $\mathbb{X}$ is uniform without vertices of degree 2, and such that $L_\Gamma = \mathbb{Z}$. Let $\alpha \in \,]0,1]$. Assume that $\Gamma\backslash\mathbb{X}$ is the union of a finite graph A and finitely many trees $\mathbb{T}_1, \ldots, \mathbb{T}_k$ meeting A in one and exactly one vertex $*_1, \ldots, *_n$ such that for every edge e in $\mathbb{T}_i$ pointing away from the root $*_i$ of $\mathbb{T}_i$, the canonical morphism $G_e \to G_{o(e)}$ between edge and vertex groups of the quotient graph of groups $\Gamma\backslash\backslash\mathbb{X} = (\Gamma\backslash\mathbb{X}, G_*)$ is an isomorphism. Assume that there exist $C, \kappa > 0$ such that for all $n \in \mathbb{N}$,

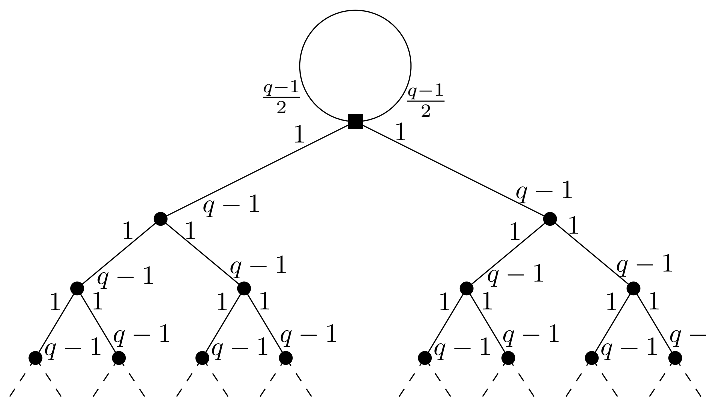
$$\sum_{i=1,\ldots,k,\; x \in V\mathbb{T}_i \,:\, d(x,*_i)=n} \frac{1}{|G_x|} \leqslant C e^{-\kappa\, n} \,.$$

Then the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ has exponential decay of α-Hölder correlations for the zero system of conductances.

This is in particular the case for every $k, q \in \mathbb{N}$ such that $k \geqslant 2$, $q > 2k+1$ and $q-k$ is odd, when the quotient graph of groups $\Gamma\backslash\backslash\mathbb{X}$ has underlying edge-indexed graph[19] a loop-edge with both indices equal to $\frac{q-k+1}{2}$ glued to the root of a regular k-ary rooted tree, with indices

[19]See definition in Section 2.6.

1 for the edges pointing towards the root and $q-k+1$ for the edges pointing away from the root (see the picture below with $k=2$). Note that $\mathbb{X}$ is then the $(q+1)$-regular tree, and that the loop edge is here in order to ensure that $L_\Gamma = \mathbb{Z}$. For instance, the vertex group of a point at distance n from the root may be chosen to be $\mathbb{Z}/(\frac{q-k+1}{2})\mathbb{Z} \times \big(\mathbb{Z}/(q-k+1)\mathbb{Z}\big)^n$.



9.3 Rate of mixing for metric trees

Let $(\mathbb{X},\lambda)$, X, Γ, $\widetilde{F}$, $\widetilde{F}^\pm$, $\delta=\delta_{\Gamma,\,F^\pm}<+\infty$ and $(\mu^\pm_x)_{x\in VX}$ be as in the beginning of Section 4.4. Let $\widetilde{m}_F$ and m_F be the associated Gibbs measures on $\mathscr{G}X$ and $\Gamma\backslash\mathscr{G}X$. The aim of this Section is to study the problem of finding conditions on these data under which the (continuous time) geodesic flow on $\Gamma\backslash\mathscr{G}X$ is polynomially mixing for the Gibbs measure m_F.

We will actually prove a stronger property, though it applies only to observables which are smooth enough along the flow. Let us fix $\alpha\in\,]0,1]$. Let $(Z,\mu,(\phi_t)_{t\in\mathbb{R}})$ be a topological space Z endowed with a continuous one-parameter group $(\phi_t)_{t\in\mathbb{R}}$ of homeomorphisms preserving a (Borel) probability measure μ on Z. For all $k\in\mathbb{N}$, let $\mathscr{C}^{k,\,\alpha}_{\rm b}(Z)$ be the real vector space of maps $f:Z\rightarrow\mathbb{R}$ such that for all $z\in Z$, the map $t\mapsto f(\phi_t z)$ is $\mathscr{C}^k$-smooth, and such that the maps $\partial^i_t f:Z\rightarrow\mathbb{R}$ defined by $z\mapsto \frac{d^i}{dt^i}\big|_{t=0} f(\phi_t z)$ for $0\leqslant i\leqslant k$ are bounded and α-Hölder-continuous. It is a Banach space when endowed with the norm

$$\|f\|_{k,\,\alpha}=\sum_{i=0}^{k}\;\|\partial^i_t f\|_\alpha\;,$$

and it is contained in $\mathbb{L}^2(Z,\mu)$ by the finiteness of μ. We denote by $\mathscr{C}^{k,\,\alpha}_{\rm c}(Z)$ the vector subspace of elements of $\mathscr{C}^{k,\,\alpha}_{\rm b}(Z)$ with compact support.

For all $\psi,\psi'\in\mathbb{L}^2(Z,\mu)$ and $t\in\mathbb{R}$, let

$$\operatorname{cov}_{\mu,\,t}(\psi,\psi')=\int_Z \psi\circ\phi_t\;\psi'\;d\mu-\int_Z\psi\;d\mu\;\int_Z\psi'\;d\mu$$

be the (well-defined) *correlation coefficient* of the observables ψ,ψ' at time t under the flow $(\phi_t)_{t\in\mathbb{R}}$ for the measure μ. We say[20] that the (continuous time) dynamical system

[20] See [Dol2], and more precisely [Mel1, Def. 2.2] whose definition is slightly different but implies the one given in this paper by the Principle of Uniform Boundedness argument of [ChCS, Appendix B] already used in Section 9.2.

$(Z, \mu, (\phi_t)_{t\in\mathbb{R}})$ has *superpolynomial decay of α-Hölder correlations* if for every $n \in \mathbb{N}$ there exist $C = C_n > 0$ and $k = k_n \in \mathbb{N}$ such that for all $\psi, \psi' \in \mathscr{C}_{\rm b}^{k,\alpha}(Z)$ and $t \in \mathbb{R}$, we have

$$|\operatorname{cov}_{\mu,t}(\psi, \psi')| \leqslant C\,(1 + |t|)^{-n}\,\|\psi\|_{k,\alpha}\,\|\psi'\|_{k,\alpha}\,.$$

Following Dolgopyat, we say that the dynamical system $(Z, \mu, (\phi_t)_{t\in\mathbb{R}})$ is *rapidly mixing* if there exists $\alpha > 0$ such that $(Z, \mu, (\phi_t)_{t\in\mathbb{R}})$ has superpolynomial decay of α-Hölder correlations

We will use the following two assumptions on our data, introduced respectively in [Dol2] and [Mel1]. Recall that the Gibbs measure m_F, when finite, is mixing if and only the length spectrum L_Γ is dense in $\mathbb{R}$ (see Theorem 4.9). The rapidly mixing property will require stronger assumptions on L_Γ.

We say that the length spectrum L_Γ of Γ is 2-*Diophantine* if there exists a ratio of two translation lengths of elements of Γ which is Diophantine. Recall that a real number x is *Diophantine* if there exist $\alpha, \beta > 0$ such that

$$\big|x - \frac{p}{q}\big| \geqslant \alpha\, q^{-\beta}$$

for all $p, q \in \mathbb{Z}$ with $q > 0$.

Let E be a finite subset of vertices of $\Gamma\backslash\mathbb{X}$, and let $\widetilde{E}$ be the set of vertices of $\mathbb{X}$ mapping to E. We denote by T_E the set of triples $(\lambda(\gamma), d(\gamma), q(\gamma, p))$ where $\gamma \in \Gamma$ has translation length $\lambda(\gamma) > 0$, has $d(\gamma)$ vertices on its translation axis $\operatorname{Ax}(\gamma)$ modulo $\gamma^{\mathbb{Z}}$ and if the first return time of a vertex p in $\widetilde{E} \cap \operatorname{Ax}(\gamma)$ under the discrete time geodesic flow along the translation axis has period $q(\gamma, p)$. We say that the length spectrum L_Γ of Γ is 4-*Diophantine* with respect to E if for all sequences $(b_k)_{k\in\mathbb{N}}$ in $[1 + \infty[$ converging to $+\infty$ and $(\omega_k)_{k\in\mathbb{N}}$, $(\varphi_k)_{k\in\mathbb{N}}$ in $[0, 2\pi[$, there exists $N \in \mathbb{N}$ such that for all $a \geqslant N$ and $C, \beta \geqslant 1$, there exist $k \geqslant 1$ and $(\tau, d, q) \in T_E$ such that

$$d\big((b_k\tau + \omega_k d)\lfloor\beta \ln b_k\rfloor + q\varphi_k, 2\pi\mathbb{Z}\big) \geqslant C\, q\, b_k^{-a}\,.$$

We define the first return time after time ϵ on a finite subset E of vertices of $\Gamma\backslash\mathbb{X}$ as the map $\tau_E^{>\epsilon} : \Gamma\backslash\mathscr{G}X \to [0, +\infty]$ defined by $\tau_E^{>\epsilon}(\ell) = \inf\{t > \epsilon\ :\ \ell(t) \in E\}$.

Theorem 9.7. *Assume that the Gibbs measure m_F is finite and mixing for the (continuous time) geodesic flow, and that the lengths of the edges of $(\mathbb{X}, \lambda)$ have a finite upper bound.*[21] *Furthermore assume that*

(a) either $\Gamma\backslash X$ is compact and the length spectrum of Γ is 2-Diophantine,

(b) or there exists a finite subset E of vertices of $\Gamma\backslash\mathbb{X}$ satisfying the following properties:

(1) there exist $C, \kappa > 0$ and $\epsilon \in\,]0, \min \lambda[$ such that for all $t \geqslant 0$,

$$m_F(\{\ell \in \Gamma\backslash\mathscr{G}X\ :\ d(\ell(0), E) \leqslant \epsilon \quad\text{and}\quad \tau_E^{>\epsilon}(\ell) \geqslant t\}) \leqslant C\, e^{-\kappa t}\,,$$

(2) the length spectrum of Γ is 4-Diophantine with respect to E.

Then the (continuous time) geodesic flow on $\Gamma\backslash\mathscr{G}X$ has superpolynomial decay of α-Hölder correlations for the normalised Gibbs measure $\frac{m_F}{\|m_F\|}$.

[21] They have a positive lower bound by definition, see Section 2.6.

Note that the existence of E satisfying the exponentially small tail Hypothesis (1) is in particular satisfied if Γ is geometrically finite with E the set of vertices of a finite subgraph of $\Gamma\backslash\mathbb{X}$ whose complement in $\Gamma\backslash\mathbb{X}$ is the underlying graph of a union of cuspidal rays in $\Gamma\backslash\backslash\mathbb{X}$: see the proof of Corollary 9.6 and use the hypothesis on the lengths of edges.

Note that the exponentially small tail Hypothesis (1) might be weakened to a superpolynomially small tail hypothesis while keeping the same conclusion, see [Mel2]. Since the former is easier to check than the latter, we prefer to state Theorem 9.7 as it is.

We will follow a scheme of proof analogous to the one in Section 9.2 for simplicial trees, by reducing the study to a problem of suspensions of Young towers, and then apply results of [Dol2] and [Mel1] for the rapid mixing property of suspensions of hyperbolic and nonuniformly hyperbolic dynamical systems.

Proof. Since the Gibbs measure normalised to be a probability measure depends only on the cohomology class of the potential (see Equation (4.10)), we may assume by Proposition 3.22 that $F = F_c$ is the potential on $\Gamma\backslash T^1X$ associated with a system of conductances $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ for Γ. We denote by δ_c the critical exponent of (Γ, F_c), and by m_c the Gibbs measure m_{F_c}. Up to replacing $\mathbb{X}$ by its minimal nonempty Γ-invariant subtree, we assume that $X = \mathscr{C}\Lambda\Gamma$.

Step 1 : Reduction to a suspension of a two-sided symbolic dynamics. We refer to the paragraphs before the statement of Theorem 5.9 and at the beginning of Section 5.3 for the definitions of

- the system of conductances ${}^\sharp c$ for Γ on the simplicial tree $\mathbb{X}$,
- the (two-sided) topological Markov shift $(\Sigma, \sigma, \mathbb{P})$ on the alphabet $\mathscr{A}$, conjugated to the discrete time geodesic flow $\big(\Gamma\backslash\mathscr{G}\mathbb{X},\ {}^\sharp\mathrm{g}^1,\ \frac{m_{\sharp_c}}{\|m_{\sharp_c}\|}\big)$ by the homeomorphism $\Theta : \Gamma\backslash\mathscr{G}\mathbb{X} \to \Sigma$,
- the roof function $r : \Sigma \to \,]0, +\infty[\,$,
- the suspension $(\Sigma, \sigma, a\,\mathbb{P})_r = (\Sigma_r, (\sigma_r^t)_{t\in\mathbb{R}}, a\,\mathbb{P}_r)$ over $(\Sigma, \sigma, a\,\mathbb{P})$ with roof function r, where $a = \frac{1}{\|\mathbb{P}_r\|}$.

The suspension $(\Sigma, \sigma, a\,\mathbb{P})_r$ is conjugated to the geodesic flow $\big(\Gamma\backslash\mathscr{G}X, \frac{m_c}{\|m_c\|}, (\mathrm{g}^t)_{t\in\mathbb{R}}\big)$ by the bilipschitz homeomorphism $\Theta_r : \Gamma\backslash\mathscr{G}X \to \Sigma_r$ defined at the end of the proof of Theorem 5.9. We will always (uniquely) represent the elements of Σ_r as $[x, s]$ with $x \in \Sigma$ and $0 \leqslant s < r(x)$.

Note that since Θ_r^{-1} conjugates $(\sigma_r^t)_{t\in\mathbb{R}}$ and $(\mathrm{g}^t)_{t\in\mathbb{R}}$, we have for all $f : \Gamma\backslash\mathscr{G}X \to \mathbb{R}$ and $x \in \Sigma_r$, when defined,

$$\partial_t^i(f\circ\Theta_r^{-1})(x) = \frac{d^i}{dt^i}\Big|_{t=0} f\circ\Theta_r^{-1}(\sigma_r^t x) = \frac{d^i}{dt^i}\Big|_{t=0} f(\mathrm{g}^t\Theta_r^{-1}(x)) = (\partial_t^i f)\circ\Theta_r^{-1}(x)\ .$$

Hence if $f : \Gamma\backslash\mathscr{G}X \to \mathbb{R}$ is C^k-smooth along the orbits of $(\mathrm{g}^t)_{t\in\mathbb{R}}$, then $f\circ\Theta_r^{-1}$ is C^k-smooth along the orbits of $(\sigma_r^t)_{t\in\mathbb{R}}$. Furthermore, since Θ_r is bilipschitz, the precomposition map by Θ_r^{-1} is a continuous linear isomorphism from $\mathscr{C}_{\rm b}^{k,\alpha}(\Gamma\backslash\mathscr{G}X)$ to $\mathscr{C}_{\rm b}^{k,\alpha}(\Sigma_r)$.

Note that since Θ_r conjugates $(\mathrm{g}^t)_{t\in\mathbb{R}}$ and $(\sigma_r^t)_{t\in\mathbb{R}}$, and sends $\frac{m_c}{\|m_c\|}$ to $\frac{\mathbb{P}_r}{\|\mathbb{P}_r\|}$, we have, for all $\psi, \psi' \in \mathbb{L}^2(\Gamma\backslash\mathscr{G}X)$ and $t \in \mathbb{R}$,

$$\mathrm{cov}_{\frac{m_c}{\|m_c\|},\,t}(\psi, \psi') = \mathrm{cov}_{\frac{\mathbb{P}_r}{\|\mathbb{P}_r\|},\,t}(\psi\circ\Theta_r^{-1}, \psi'\circ\Theta_r^{-1})\ .$$

Therefore we only have to prove that the suspension $(\Sigma_r, (\sigma_r^t)_{t\in\mathbb{R}}, \frac{\mathbb{P}_r}{\|\mathbb{P}_r\|})$ is rapidly mixing under one of the assumptions (a) and (b).

Step 2 : Reduction to a suspension of a one-sided symbolic dynamics. In this Step, we explain the rather standard reduction concerning mixing rates from suspensions of two-sided topological Markov shifts to suspensions of one-sided topological Markov shifts. We use the obvious modifications of the notation and constructions concerning the suspension of a noninvertible transformation to a semiflow, given for invertible transformations at the beginning of Section 5.3.

We consider the one-sided topological Markov shift $(\Sigma_+, \sigma_+, \mathbb{P}_+)$ over the alphabet $\mathscr{A}$ constructed at the beginning of Step 2 of the proof of Theorem 9.1, where the system of conductances c is now replaced by $^\sharp c$, so that $\mathbb{P} = \Theta_*(m_{^\sharp c}/\|m_{^\sharp c}\|)$. Let $\pi_+ : \Sigma \to \Sigma_+$ be the natural extension so that $\mathbb{P}_+ = (\pi_+)_*\mathbb{P}$ and $\pi_+ \circ \sigma = \sigma_+ \circ \pi_+$.

We are going to construct in Step 2, as the suspension of $(\Sigma_+, \sigma_+, \mathbb{P}_+)$ with an appropriate roof function r_+, a semiflow $((\Sigma_+)_{r_+}, \big((\sigma_+)^t_{r_+}\big)_{t\geqslant 0}, (\mathbb{P}_+)_{r_+})$, and prove that the flow $\big(\Sigma_r, (\sigma^t_r)_{t\in\mathbb{R}}, \frac{\mathbb{P}_r}{\|\mathbb{P}_r\|}\big)$ is rapidly mixing if the semiflow $((\Sigma_+)_{r_+}, \big((\sigma_+)^t_{r_+}\big)_{t\geqslant 0}, \frac{(\mathbb{P}_+)_{r_+}}{\|(\mathbb{P}_+)_{r_+}\|})$ is rapidly mixing.

Let $r_+ : \Sigma_+ \to \,]0, +\infty[$ be the map defined by

$$r_+ : x \mapsto \lambda(e_0^+) \;, \tag{9.6}$$

where if $x = (x_n)_{n\in\mathbb{N}}$, the edge e_0^+ is such that $x_0 = (e_0^-, h_0, e_0^+)$. Note that this map has a positive lower bound, and a finite upper bound, and that it is locally constant (and even constant on the 1-cylinders of Σ_+). By Equation (5.10), we have

$$r_+ \circ \pi_+ = r \;. \tag{9.7}$$

We denote by $((\Sigma_+)_{r_+}, \big((\sigma_+)^t_{r_+}\big)_{t\geqslant 0}, (\mathbb{P}_+)_{r_+})$ the suspension semiflow over $(\Sigma_+, \sigma_+, \mathbb{P}_+)$ with roof function r_+. We (uniquely) represent the points of the suspension space $(\Sigma_+)_{r_+}$ as $[x, s]$ for $x \in \Sigma_+$ and $0 \leqslant s < r_+(x)$. For all $t \geqslant 0$, we have $(\sigma_+)^t_{r_+}([x, s]) = [\sigma_+^n x, s']$ where $n \in \mathbb{N}$ and $s' \in \mathbb{R}$ are such that $t + s = \sum_{i=0}^{n-1} r_+(\sigma_+^i x) + s'$ and $0 \leqslant s' < r_+(\sigma_+^n x)$.

We define the *suspended natural extension* as the map $\pi_+^r : \Sigma_r \to (\Sigma_+)_{r_+}$ by

$$\pi_+^r : [x, s] \mapsto [\pi_+(x), s] \;,$$

which is well defined by Equation (9.7). Note that π_+^r is 1-Lipschitz for the Bowen-Walters distances on Σ_r and $(\Sigma_+)_{r_+}$ (see Proposition 5.11).[22]

For all $\psi : \Sigma_r \to \mathbb{R}$ and $T \geqslant 0$, let us construct a function $\psi^{(T)} : (\Sigma_+)_{r_+} \to \mathbb{R}$ as follows. For every $[x, s] \in (\Sigma_+)_{r_+}$, let $N \in \mathbb{N}$ and $s' \geqslant 0$ be such that $(\sigma_+)^T_{r_+}[x, s] = [\sigma_+^N x, s']$, with

$$0 \leqslant s' < r_+(\sigma_+^N x) \quad \text{and} \quad s + T = \sum_{i=0}^{N-1} r_+(\sigma_+^i x) \;+ s' \;.$$

Let

$$\psi^{(T)}([x, s]) = \psi([y, s'])$$

where $y = (y_n)_{n\in\mathbb{Z}}$ is such that $y_i = x_{i+N}$ if $i \geqslant -N$ and $y_i = (z^{x_0})_{i+N}$ otherwise. Note that $y_0 = x_N$, hence $r(y) = r_+(\sigma_+^N(x))$ and $0 \leqslant s' < r(y)$, so that the above map is well defined.

[22]Note that Proposition 5.11 is stated for suspensions of invertible maps, but the roof function is constant on 1-cylinders, and the branches of the inverse of σ_+ on these 1-cylinders are uniformely Lipschitz, hence the proof of [BarS, Appendix] extends.

Finally, for every $\psi \in \mathscr{C}_{\rm b}^{k,\,\alpha}(\Sigma_r)$ or $\psi \in \mathscr{C}_{\rm b}^{k,\,\alpha}((\Sigma_+)_{r_+})$, let

$$\|\psi\|_{k,\,\infty} = \sum_{i=0}^{k} \|\partial_t^i \psi\|_\infty \quad \text{and} \quad \|\psi\|'_{k,\,\alpha} = \sum_{i=0}^{k} \|\partial_t^i \psi\|'_\alpha \;,$$

so that

$$\|\psi\|_{k,\,\alpha} = \|\psi\|_{k,\,\infty} + \|\psi\|'_{k,\,\alpha} \;.$$

Lemma 9.8. *Let $T \geqslant 0$ and $\psi \in \mathscr{C}_{\rm b}^{k,\,\alpha}(\Sigma_r)$.*

(1) For all $t \geqslant 0$, we have $(\sigma_+)^t_{r_+} \circ \pi_+^{\,r} = \pi_+^{\,r} \circ \sigma_r^t$.

(2) With $\alpha' = \frac{\alpha}{\sup \lambda}$, there exists a constant $C_1 \geqslant 1$ (independent of k, T and ψ) such that $|\psi \circ \sigma_r^T - \psi^{(T)} \circ \pi_+^{\,r}| \leqslant C_1 \,\|\psi\|'_\alpha \, e^{-\alpha' T}$.

(3) We have $\psi^{(T)} \in \mathscr{C}_{\rm b}^{k,\,\alpha}((\Sigma_+)_{r_+})$ and $\|\psi^{(T)}\|_{k,\,\infty} \leqslant \|\psi\|_{k,\,\infty}$. With $\alpha'' = \frac{\alpha}{\inf \lambda}$, there exists a constant $C_2 \geqslant 1$ (independent of k, T and ψ) such that

$$\|\psi^{(T)}\|'_{k,\,\alpha} \leqslant C_2 \; e^{\alpha'' T} \; \|\psi\|'_{k,\,\alpha} \;. \tag{9.8}$$

Proof. (1) For all $t \geqslant 0$ and $[x, s] \in \Sigma_r$, let $n \in \mathbb{N}$ and $s' \geqslant 0$ be such that

$$t + s = \sum_{i=0}^{n-1} r_+(\sigma_+^i \pi_+(x)) + s' \quad \text{and} \quad 0 \leqslant s' < r_+(\sigma_+^n \pi_+(x)) \;.$$

Since $r_+ \circ \sigma_+^i \circ \pi_+ = r \circ \sigma^i$ for every $i \in \mathbb{N}$, these two conditions are equivalent to

$$t + s = \sum_{i=0}^{n-1} r(\sigma^i x) + s' \quad \text{and} \quad 0 \leqslant s' < r(\sigma^n x) \,.$$

Hence

$$(\sigma_+)^t_{r_+} \circ \pi_+^{\,r}([x, s]) = (\sigma_+)^t_{r_+}([\pi_+(x), s]) = [\sigma_+^n \pi_+(x), s']$$

and

$$\pi_+^{\,r} \circ \sigma_r^t([x, s]) = \pi_+^{\,r}([\sigma^n x, s']) = [\pi_+(\sigma^n x), s'] \;.$$

This proves Assertion (1) since $\pi_+ \circ \sigma = \sigma_+ \circ \pi_+$.

(2) By Proposition 5.11, we may assume that, in the formula of the Hölder norms, the Bowen-Walters distance is replaced by the function $d_{\rm BW}$, as this will only change C_1 by $C_{\rm BW}^\alpha \, C_1$.

For every $[x, s] \in \Sigma_r$, with $[y, s']$ and N associated with $\pi_+^{\,r}([x, s]) = [\pi_+(x), s]$ as in the definition of $\psi^{(T)}([\pi_+(x), s])$, we have

$$d_{\rm BW}(\sigma_r^T[x, s], [y, s']) = d_{\rm BW}([\sigma^N x, s'], [y, s']) \leqslant d(\sigma^N x, y) \leqslant e^{-N} \;.$$

Since the positive roof function r is bounded from above by the least upper bound $\sup \lambda$ of the lengths of the edges, we have

$$N \geqslant \sum_{i=0}^{N-1} \frac{r(\sigma^i x)}{\sup \lambda} = \frac{1}{\sup \lambda} \,(s + T - s') \geqslant \frac{T}{\sup \lambda} - 1 \;.$$

Hence

$$\begin{aligned}|\psi\circ\sigma_r^T([x,s]) - \psi^{(T)}(\pi_+^{\,r}(x))| &= |\psi(\sigma_r^T[x,s]) - \psi([y,s'])|\\ &\leqslant \|\psi\|'_\alpha\; d_{\rm BW}(\sigma_r^T[x,s],[y,s'])^\alpha \leqslant \|\psi\|'_\alpha\; e^{-\frac{\alpha}{\sup\lambda}T+\alpha}\;.\end{aligned}$$

(3) Let us prove that $\psi^{(T)}$ is $\mathscr{C}^k$ along semiflow lines. Fix $[x,s]\in(\Sigma_+)_{r_+}$. With $[y,s']$ and N as in the construction of $\psi^{(T)}([x,s])$, let us consider $\epsilon>0$ small enough, so that $\epsilon<r_+(x)-s$ and $\epsilon<r_+(\sigma_+^N(x))-s'=r(y)-s'$. Then

$$\psi^{(T)}\circ(\sigma_+)^\epsilon_{r_+}([x,s]) = \psi^{(T)}([x,s+\epsilon]) = \psi([y,s'+\epsilon]) = \psi\circ\sigma_r^\epsilon([y,s'])\;.$$

Therefore by taking derivatives with respect to ϵ in this formula, $\psi^{(T)}$ is indeed $\mathscr{C}^k$ along semiflow lines, and, for $i=0,\dots,k$, we have

$$\partial_t^i\big(\psi^{(T)}\big) = \big(\partial_t^i\psi\big)^{(T)}\;. \tag{9.9}$$

The inequality $\|\psi^{(T)}\|_\infty\leqslant\|\psi\|_\infty$ is immediate by construction. Using the above centred equation, we have $\|\psi^{(T)}\|_{k,\infty}\leqslant\|\psi\|_{k,\infty}$.

Let us prove that there exists a constant $C_2\geqslant 1$ (independent of T and ψ) such that

$$\|\psi^{(T)}\|'_\alpha \leqslant C_2\, e^{\alpha'' T}\;\|\psi\|'_\alpha\;. \tag{9.10}$$

Let $[x,s]\in(\Sigma_+)_{r_+}$, take $[y,s']$ and N as in the definition of $\psi^{(T)}([x,s])$. Let $[\underline{x},\underline{s}\,]\in(\Sigma_+)_{r_+}$, take $[\underline{y},\underline{s'}]$ and $\underline{N}$ as in the definition of $\psi^{(T)}([\underline{x},\underline{s}\,])$. Up to exchanging $[x,s]$ and $[\underline{x},\underline{s}\,]$, we assume that $\underline{N}\geqslant N$.

By Proposition 5.11,[23] we may assume that, in the Hölder norms formulas, the Bowen-Walters distance is replaced by the function $d_{\rm BW}$, as this will only change C_2 by $C_{\rm BW}^{2\,\alpha}\,C_2$. Let

$$C_3=\min\{e^{-1},\ \inf\lambda\}\;.$$

We have

$$\big|\psi^{(T)}([x,s])-\psi^{(T)}([\underline{x},\underline{s}\,])\big| = \big|\psi([y,s'])-\psi([\underline{y},\underline{s'}])\big| \leqslant \|\psi\|'_\alpha\; d_{\rm BW}([y,s'],[\underline{y},\underline{s'}])^\alpha\;. \tag{9.11}$$

Note that the map $d_{\rm BW}$ on $\Sigma_r\times\Sigma_r$ is bounded from above by $1+\sup\lambda$, since the distance on Σ is at most 1 and since the roof function r is bounded from above by $\sup\lambda$. If $d_{\rm BW}([x,s],[\underline{x},\underline{s}\,])\geqslant C_3\,e^{-\frac{T}{\inf\lambda}}$, then

$$d_{\rm BW}([y,s'],[\underline{y},\underline{s'}])\leqslant 1+\sup\lambda\leqslant\frac{1+\sup\lambda}{C_3}\;e^{\frac{T}{\inf\lambda}}\;d_{\rm BW}([x,s],[\underline{x},\underline{s}\,])\;.$$

Therefore Equation (9.10) follows from Equation (9.11) whenever $C_2\geqslant\frac{(1+\sup\lambda)^\alpha}{C_3^\alpha}$.

Conversely, suppose that $d_{\rm BW}([x,s],[\underline{x},\underline{s}\,])<C_3\,e^{-\frac{T}{\inf\lambda}}$. Assume that

$$d_{\rm BW}([x,s],[\underline{x},\underline{s}\,]) = d(x,\underline{x})+|s-\underline{s}|\;,$$

the other possibilities are treated similarly. Since

$$s+T=\sum_{i=0}^{N-1} r_+(\sigma_+^i x)\;+s'$$

[23]See the previous footnote.

and since the roof function r_+ is bounded from below by $\inf\lambda$, we have $T \geqslant N\inf\lambda - \inf\lambda$, or equivalently $N \leqslant \frac{T}{\inf\lambda} + 1$. Hence $d(x,\underline{x}) < C_3\ e^{-\frac{T}{\inf\lambda}} \leqslant e^{-N}$ by the definition of C_3. In particular the sequences x and $\underline{x}$ indexed by $\mathbb{N}$ have the same $N+1$ first coefficients. Since $r_+(z)$ depends only on z_0 for all $z \in \Sigma_+$, we thus have $r_+(\sigma_+^i x) = r_+(\sigma_+^i\,\underline{x})$ for $i = 0,\dots,N$. Note that we have

$$\underline{s} + T = \sum_{i=0}^{\underline{N}-1} r_+(\sigma_+^i\,\underline{x})\ + \underline{s'}\ .$$

If $\underline{N} = N$, then by taking the difference of the last two centred equations, we have $\underline{s} - s = \underline{s'} - s'$, and by construction, the sequences y and $\underline{y}$ indexed by $\mathbb{Z}$ satisfy $y_i = (\underline{y})_i$ if $i \leqslant 0$ and if $0 \leqslant i \leqslant -\ln d(x,\underline{x}) - N$. Therefore $d(y,\underline{y}) \leqslant e^N\ d(x,\underline{x})$ and

$$\begin{aligned} d_{\rm BW}([y,s'],[\underline{y},\underline{s'}]) &\leqslant d(y,\underline{y}) + |s' - \underline{s'}| \leqslant e^N\ d(x,\underline{x}) + |\underline{s} - s| \\ &\leqslant e^N\ d_{\rm BW}([x,s],[\underline{x},\underline{s}\,]) \leqslant e^{\frac{T}{\inf\lambda}+1}\ d_{\rm BW}([x,s],[\underline{x},\underline{s}\,])\ . \end{aligned}$$

Therefore Equation (9.10) follows from Equation (9.11) whenever $C_2 \geqslant e^{\alpha}$.

If $\underline{N} > N$, then again by difference

$$\underline{s} - s = \sum_{i=N+1}^{\underline{N}-1} r_+(\sigma_+^i\,\underline{x}) + r_+(\sigma_+^N x) - s' + \underline{s'}.$$

Note that $\underline{s'} \geqslant 0$, that $r_+(\sigma_+^N x) - s' \geqslant 0$, and that $|\underline{s} - s| < C_3\ e^{-\frac{T}{\inf\lambda}} \leqslant \inf\lambda$ by the definition of C_3. Hence we have $\underline{N} = N+1$ and $\underline{s} - s = r_+(\sigma_+^N x) - s' + \underline{s'}$. By construction, the sequences σy and $\underline{y}$ indexed by $\mathbb{Z}$ satisfy $(\sigma y)_i = (\underline{y})_i$ if $i \leqslant 0$ and if $0 \leqslant i \leqslant -\ln d(x,\underline{x}) - N - 1$. Hence by the definition of $d_{\rm BW}$ and since $r(y) = r_+(\sigma_+^N x)$, we have

$$\begin{aligned} d_{\rm BW}([y,s'],[\underline{y},\underline{s'}]) &\leqslant d(\sigma y,\underline{y}) + r(y) - s' + \underline{s'} \leqslant e^{N+1}\ d(x,\underline{x}) + |\underline{s} - s| \\ &\leqslant e^{N+1}\ d_{\rm BW}([x,s],[\underline{x},\underline{s}\,]) \leqslant e^{\frac{T}{\inf\lambda}+2}\ d_{\rm BW}([x,s],[\underline{x},\underline{s}\,])\ . \end{aligned}$$

Therefore Equation (9.10) follows from Equation (9.11) whenever $C_2 \geqslant e^{2\alpha}$. This ends the proof of Equation (9.10).

Now note that Equations (9.9) and (9.10) imply Equation (9.8) by summation (using the independence of C_2 on ψ), thus concluding the proof of Lemma 9.8. □

Proposition 9.9. *Let μ be a $(\sigma_r^t)_{t\in\mathbb{R}}$-invariant probability measure on Σ_r. Assume that the dynamical system $\big((\Sigma_+)_{r_+},((\sigma_+)_{r_+}^t)_{t\in\mathbb{R}},(\pi_+^r)_*\mu\big)$ has superpolynomial decay of α-Hölder correlations. Then $(\Sigma_r,(\sigma_r^t)_{t\in\mathbb{R}},\mu)$ has superpolynomial decay of α-Hölder correlations.*

Proof. We fix $n \in \mathbb{N}$. Let $N = 1 + 2\lceil\frac{\sup\lambda}{\inf\lambda}\rceil$. Let $k \in \mathbb{N}$ and $C_4 > 0$ (depending on n) be such that for all $\psi,\psi' \in \mathscr{C}_{\rm b}^{k,\alpha}((\Sigma_+)_{r_+})$, we have for all $t \geqslant 1$

$$\big|\operatorname{cov}_{(\pi_+^r)_*\mu,\,t}(\psi,\psi')\big| \leqslant C_4\ \|\psi\|_{k,\alpha}\ \|\psi'\|_{k,\alpha}\ t^{N\,n}\ . \tag{9.12}$$

Now let $\psi,\psi' \in \mathscr{C}_{\rm b}^{k,\alpha}(\Sigma_r)$. We again denote by $\pm\,a$ any value in $[-a,a]$ for every $a \geqslant 0$. By invariance of μ under $(\sigma_r^t)_{t\in\mathbb{R}}$, by Lemma 9.8 (2) and by Lemma 9.8 (1), we have, for every

$T \geqslant 0$ (to be chosen appropriately later on),

$$\begin{aligned}
&\int_{\Sigma_r} \psi\circ\sigma_r^t\;\;\psi'\;d\mu = \int_{\Sigma_r} \psi\circ\sigma_r^{T+t}\;\;\psi'\circ\sigma_r^{T}\;d\mu\\
&= \int_{\Sigma_r} (\psi^{(T)}\circ\pi_+^{\,r} \pm C_1\,\|\psi\|'_\alpha\,e^{-\alpha' T})\circ\sigma_r^t\;\;(\psi'^{(T)}\circ\pi_+^{\,r} \pm C_1\,\|\psi'\|'_\alpha\,e^{-\alpha' T})\,d\mu\\
&= \int_{(\Sigma_+)_{r_+}} \psi^{(T)}\circ(\sigma_+)^t_{r_+}\;\;\psi'^{(T)}\;d(\pi_+^{\,r})_*\mu\;\;\pm\;C_1^2\,\|\psi\|_\alpha\,\|\psi'\|_\alpha\,e^{-\alpha' T}\;.
\end{aligned}$$

A similar estimate holds for the second term in the definition of the correlation coefficients. Hence, applying Equation (9.12) to the observables $\psi^{(T)}$ and $\psi'^{(T)}$, by Lemma 9.8 (3), we have since $C_2\,e^{\alpha'' T} \geqslant 1$,

$$\begin{aligned}
&|\operatorname{cov}_{\mu,\,t}(\psi,\psi')| \leqslant |\operatorname{cov}_{(\pi_+^{\,r})_*\mu,\,t}(\psi^{(T)},\psi'^{(T)})| + 2\,C_1^2\,\|\psi\|_\alpha\,\|\psi'\|_\alpha\,e^{-\alpha' T}\\
&\leqslant C_4\,(\|\psi\|_{k,\,\infty} + C_2\,\|\psi\|'_{k,\,\alpha}\,e^{\alpha'' T})\,(\|\psi'\|_{k,\,\infty} + C_2\,\|\psi'\|'_{k,\,\alpha}\,e^{\alpha'' T})\;t^{-N\,n}\\
&\quad + 2\,C_1^2\,\|\psi\|_\alpha\,\|\psi'\|_\alpha\,e^{-\alpha' T}\\
&\leqslant \|\psi\|_{k,\,\alpha}\,\|\psi'\|_{k,\,\alpha}\,(C_4\,C_2^2\,e^{2\alpha'' T}\,t^{-N\,n} + 2\,C_1^2\,e^{-\alpha' T})\;.
\end{aligned}$$

Take $T = \frac{n}{\alpha'}\,\ln t \geqslant 0$. Since $N = 1 + 2\lceil\frac{\alpha''}{\alpha'}\rceil$, we have $2\alpha''\,\frac{n}{\alpha'} - N\,n \leqslant -n$. Hence with $C_5 = C_4\,C_2^2 + 2\,C_1^2$, we have for all $t \geqslant 1$

$$|\operatorname{cov}_{\mu,\,t}(\psi,\psi')| \leqslant C_5\,\|\psi\|_{k,\,\alpha}\,\|\psi'\|_{k,\,\alpha}\,t^{-n}\;.$$

This concludes the proof of Proposition 9.9. □

Step 3 : Conclusion of the proof of Theorem 9.7. In this Step, we prove that the semi-flow $\big((\Sigma_+)_{r_+},\big((\sigma_+)^t_{r_+}\big)_{t\geqslant 0},\frac{(\mathbb{P}_+)_{r_+}}{\|(\mathbb{P}_+)_{r_+}\|}\big)$ is rapidly mixing, which concludes the proof of Theorem 9.7, using Proposition 9.9 with $\mu = \frac{\mathbb{P}_r}{\|\mathbb{P}_r\|}$.

Recall[24] that $Y = \{\ell \in \Gamma\backslash\mathscr{G}X \;:\; \ell(0) \in V\mathbb{X}\}$ is a cross-section of the geodesic flow on $\Gamma\backslash\mathscr{G}X$, and that if $R : Y \to \Gamma\backslash\mathscr{G}\mathbb{X}$ is the reparametrisation map of $\ell \in Y$ to a discrete geodesic line $^\sharp\ell \in \Gamma\backslash\mathscr{G}\mathbb{X}$ with the same origin, then the measure μ_Y, induced by the Gibbs measure m_c on the cross-section Y by disintegration along the flow, maps by R_* to a constant multiple of $m_{\sharp_c} = m_{F_{\sharp_c}}$ (see Lemma 5.10 (2)). Hence for all $n \in \mathbb{N} - \{0\}$ and $\epsilon \in \,]0, \frac{1}{2}\inf\lambda[\,$, by

[24] See the proof of Theorem 5.9.

Assumption (b) (1) in the statement of Theorem 9.7, we have

$$m_{\sharp c}\Big(\Big\{\,{}^\sharp\ell\in\Gamma\backslash\mathscr{G}\mathbb{X}\;:\;\begin{array}{c}{}^\sharp\ell(0)\in E\\ \forall\, k\in\{1,\dots,n-1\},\;\;{}^\sharp\ell(k)\notin E\end{array}\Big\}\Big)$$

$$\leqslant\frac{\|m_{\sharp c}\|}{\|\mu_Y\|}\,\mu_Y\Big(\Big\{\,R^{-1}({}^\sharp\ell)\in Y\;:\;\begin{array}{c}R^{-1}({}^\sharp\ell)(0)\in E\\ \forall\, t\in\,]0,n\inf\lambda[\,,\;\;R^{-1}({}^\sharp\ell)(t)\notin E\end{array}\Big\}\Big)$$

$$\leqslant\frac{\|m_{\sharp c}\|}{\epsilon\,\|\mu_Y\|}\,m_c\Big(\Big\{\,\mathsf{g}^sR^{-1}({}^\sharp\ell)\in\Gamma\backslash\mathscr{G}X\;:\;\begin{array}{c}0\leqslant s\leqslant\epsilon,\;\;d(\mathsf{g}^sR^{-1}({}^\sharp\ell)(0),E)\leqslant\epsilon\\ \forall\, t\in\,]\epsilon,n\inf\lambda-\epsilon[\,,\;\;\mathsf{g}^sR^{-1}({}^\sharp\ell)(t)\notin E\end{array}\Big\}\Big)$$

$$\leqslant\frac{\|m_{\sharp c}\|}{\epsilon\,\|\mu_Y\|}\,C\,e^{-\kappa\,(\inf\lambda)\,n+\kappa\,\epsilon}\;.$$

Therefore Equation (9.2) (where c is replaced by $\sharp c$) is satisfied, with $C'=\frac{\|m_{\sharp c}\|\,C\,e^{\kappa\,\epsilon}}{\epsilon\,\|\mu_Y\|}$ and $\kappa'=\kappa\,\inf\lambda$. As seen in the proof of Theorem 9.1, this implies that there exists a finite subset $\mathscr{E}$ of the alphabet $\mathscr{A}$ such that Equation (9.4) is satisfied.

We now apply [Mel1, Theo. 2.3] with the dynamical system $(X,m_0,T)=(\Sigma_+,\mathbb{P}_+,\sigma_+)$ (using the system of conductances ${}^\sharp c$) and the roof function $h=r_+$. This dynamical system is presented as a Young tower in Step 3 of the proof of Theorem 9.1. Equation (9.4) for the first return map $\tau_{\mathscr{E}}$ and the 4-Diophantine hypothesis are exactly the hypothesis needed in order to apply [Mel1, Theo. 2.3]. Thus the semiflow $\big((\Sigma_+)_{r_+},\big((\sigma_+)^t_{r_+}\big)_{t\geqslant 0},\frac{(\mathbb{P}_+)_{r_+}}{\|(\mathbb{P}_+)_{r_+}\|}\big)$ has superpolynomial decay of α-Hölder correlations.

When $\Gamma\backslash X$ is compact, the alphabet $\mathscr{A}$ is finite and $(\Sigma_+,\sigma_+,\mathbb{P}_+)$ is a (one-sided) subshift of finite type, hence we do not need the exponentially small tail assumption, but only the 2-Diophantine hypothesis, and we may apply [Dol2]. □

Corollary 9.10. *Assume that the Gibbs measure m_F is finite and mixing for the (continuous time) geodesic flow, that the lengths of the edges of $(\mathbb{X},\lambda)$ have a finite upper bound, and that Γ is geometrically finite. There exists a full measure subset A of $\mathbb{R}^4$ (for the Lebesgue measure) such that if Γ has a quadruple of translation lengths in A, or if the length spectrum is 4-Diophantine, then the (continuous time) geodesic flow on $\Gamma\backslash\mathscr{G}X$ has superpolynomial decay of α-Hölder correlations for the Bowen-Margulis measure $m_{\rm BM}$.*

Proof. The exponentially small tail Assumption (b) (1) is checked as in the proof of Corollary 9.6. The deduction of Corollary 9.10 from Theorem 9.7 then proceeds, by an argument going back in part to Dolgopyat, as for the deduction of Corollary 2.4 from Theorem 2.3 in [Mel1]. □

Note that under the general assumptions of Theorem 4.9, the geodesic flow on $\Gamma\backslash\mathscr{G}X$ might not be exponentially mixing, see for instance [Pol1, page 162] or [Rue2] for analogous behaviour.

Part II

Geometric equidistribution and counting

Chapter 10

Equidistribution of equidistant level sets to Gibbs measures

Let X be a geodesically complete proper CAT(-1) space, let Γ be a nonelementary discrete group of isometries of X, let $\widetilde{F}$ be a continuous Γ-invariant map on T^1X such that $\delta = \delta_{\Gamma,\, F^\pm}$ is finite and positive and that the triple $(X, \Gamma, \widetilde{F})$ satisfies the HC-property,[1] and let $(\mu^\pm_x)_{x\in X}$ be Patterson densities for the pairs $(\Gamma, F^\pm)$.

In this Chapter, we prove that the skinning measure on (any nontrivial piece of) the outer unit normal bundle of any properly immersed nonempty proper closed convex subset of X, pushed a long time by the geodesic flow, equidistributes towards the Gibbs measure, under finiteness and mixing assumptions. This result gives four important extensions of [PaP14a, Theo. 1], one for general CAT(-1) spaces with constant potentials, one for Riemannian manifolds with pinched negative curvature and Hölder-continuous potentials, one for $\mathbb{R}$-trees with general potentials, and one for simplicial trees.

10.1 A general equidistribution result

Before stating this equidistribution result, we start by a technical construction which will also be useful in the following Chapter 11. We refer to Section 2.4 for the notation concerning the dynamical neighbourhoods (including $V^\mp_{w,\,\eta',\,R}$) and to Chapter 7 for the notation concerning the skinning measures (including $\nu^\pm_w$).

Technical construction of bump functions. Let $D^\pm$ be nonempty proper closed convex subsets of X, and let $R > 0$ be such that $\nu^\pm_w(V^\mp_{w,\,\eta'',\,R}) > 0$ for all $\eta'' > 0$ and $w \in \partial^1_\mp D^\pm$. Let $\eta > 0$ and let $\Omega^\pm$ be measurable subsets of $\partial^1_\mp D^\pm$. We now construct functions

$$\phi^\mp_{\eta,\,R,\,\Omega^\pm} : \mathscr{G}X \to [0, +\infty[$$

whose supports are contained in dynamical neigbourhoods of $\Omega^\pm$. If $X = \widetilde{M}$ is a Riemanian manifold and $\widetilde{F} = 0$, we recover the same bump functions as in [PaP14a] after the standard identifications.

[1] See Definition 3.13.

For all $\eta' > 0$, let $h^{\pm}_{\eta,\,\eta'} : \mathscr{G}_{\mp}X \to [0, +\infty[$ be the Γ-invariant measurable maps defined by

$$h^{\pm}_{\eta,\,\eta'}(w) = \frac{1}{\nu^{\pm}_{w}(V^{\mp}_{w,\,\eta,\,\eta'})} \tag{10.1}$$

if $\nu^{\pm}_{w}(V^{\mp}_{w,\,\eta,\,\eta'}) > 0$ (which is for instance satisfied if $w_{\pm} \in \Lambda\Gamma$ and for every $w \in \partial^1_{\mp} D^{\pm}$ if $\eta' = R$ by the choice of R) and $h^{\pm}_{\eta,\,\eta'}(w) = 0$ otherwise.

These functions $h^{\pm}_{\eta,\,\eta'}$ have the following behaviour under precomposition by the geodesic flow. By Lemma 7.6 (2), by Equation (2.18), and by the invariance of $\nu^{\pm}_{w}$ under the geodesic flow, we have, for all $t \in \mathbb{R}$ and $w \in \mathscr{G}_{\pm}X$,

$$h^{\mp}_{\eta,\,\eta'}(\mathtt{g}^{\mp t}w) = e^{C^{\pm}_{w_{\pm}}(w(0),\,w(\mp t))}\; h^{\mp}_{\eta,\,e^{-t}\eta'}(w)\,. \tag{10.2}$$

Let us also describe the behaviour of $h^{\pm}_{\eta,\,\eta'}$ when η' is small. Let $w \in \mathscr{G}_{\pm}X$ be such that w is isometric at least on $\pm[0, +\infty[$, which is for instance the case if $w \in \partial^1_{\pm} D^{\mp}$. For all $\eta' > 0$ and $\ell \in B^{\pm}(w, \eta')$, let $\widehat{w}$ be an extension of w such that $d_{W^{\pm}(w)}(\ell, \widehat{w}) < \eta'$. Then $\widehat{w}(0) = w(0)$ by the assumption on w, and using Lemma 2.4, we have

$$d(\ell(0), w(0)) = d(\ell(0), \widehat{w}(0)) \leqslant d_{W^{\pm}(w)}(\ell, \widehat{w}) < \eta'\;.$$

Hence, with κ_1 and κ_2 the constants in Definition 3.13, if $\eta' \leqslant 1$ and

$$c_1 = \kappa_1 + 2\,\delta + 2 \sup_{\pi^{-1}(B(w(0),\,2))} |\widetilde{F}|\;,$$

we have, by Proposition 3.20 (2),

$$|\; C^{\mp}_{w_{\mp}}(w(0), \ell(0))\;| \leqslant c_1\,(\eta')^{\kappa_2}\;.$$

Using the defining Equation (7.12) of $\nu^{\mp}_{w}$, for all $s \in \mathbb{R}$, $\eta' \in\,]0, 1]$ and $\ell \in B^{\pm}(w, \eta')$, we have

$$e^{-c_1(\eta')^{\kappa_2}}\; ds\, d\mu_{W^{\pm}(w)}(\ell) \leqslant d\nu^{\mp}_{w}(\mathtt{g}^{s}\ell) \leqslant e^{c_1(\eta')^{\kappa_2}}\; ds\, d\mu_{W^{\pm}(w)}(\ell)\,.$$

It follows that for all $\eta' \in\,]0, 1]$ and $w \in \partial^1_{\pm} D^{\mp}$ such that $w_{\pm} \in \Lambda\Gamma$, we have the following control of $h^{\mp}_{\eta,\,\eta'}(w)$:

$$\frac{e^{-c_1(\eta')^{\kappa_2}}}{2\eta\;\mu_{W^{\pm}(w)}(B^{\pm}(w, \eta'))} \leqslant h^{\mp}_{\eta,\,\eta'}(w) \leqslant \frac{e^{c_1(\eta')^{\kappa_2}}}{2\eta\;\mu_{W^{\pm}(w)}(B^{\pm}(w, \eta'))}\,. \tag{10.3}$$

Note that when X is an $\mathbb{R}$-tree, we may take $\kappa_2 = 1$ and $c_1 = \sup_{\pi^{-1}(B(w(0),1))} |\widetilde{F} - \delta|$ in this equation, as said in the last claim of Proposition 3.20 (2). Note that c_1 is bounded when w ranges over any compact subset of $\mathscr{G}_{\pm}X$, and is uniformly bounded when $\widetilde{F}$ is bounded.

Recall that $\mathbb{1}_A$ denotes the characteristic function of a subset A. We now define the test functions $\phi^{\mp}_{\eta,\,R,\,\Omega^{\pm}} : \mathscr{G}X \to [0, +\infty[$ with support in a dynamical neighbourhood of $\Omega^{\mp}$ by

$$\phi^{\mp}_{\eta,\,R,\,\Omega^{\mp}} \;=\; h^{\mp}_{\eta,\,R} \circ f^{\pm}_{D^{\mp}}\;\; \mathbb{1}_{\mathscr{V}^{\pm}_{\eta,\,R}(\Omega^{\mp})}\,, \tag{10.4}$$

where $\mathscr{V}^{\pm}_{\eta,\,R}(\Omega^{\mp})$ and $f^{\pm}_{D^{\mp}}$ are as in Section 2.4. Note that if $\ell \in \mathscr{V}^{\pm}_{\eta,\,R}(\Omega^{\mp})$, then $\ell_{\pm} \notin \partial_{\infty} D^{\mp}$ by convexity. Thus, ℓ belongs to the domain of definition $\mathscr{U}^{\pm}_{D^{\mp}}$ of $f^{\pm}_{D^{\mp}}$. Hence $\phi^{\mp}_{\eta,\,R,\,\Omega^{\mp}}(\ell) = h^{\mp}_{\eta,\,R} \circ f^{\pm}_{D^{\mp}}(\ell)$ is well defined. By convention, $\phi^{\mp}_{\eta,\,R,\,\Omega^{\mp}}(\ell) = 0$ if $\ell \notin \mathscr{V}^{\pm}_{\eta,\,R}(\Omega^{\mp})$.

We now globalise these test functions in order to apply them to equivariant families of supports.

Let $\eta > 0$. Let $\mathscr{D} = (D_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of X with $\Gamma\backslash I$ finite, and $\sim=\sim_{\mathscr{D}}$. Let $R > 0$ be such that $\nu_w^{\pm}(V^{\mp}_{w,\eta'',R}) > 0$ for all $\eta'' > 0$, $i \in I$ and $w \in \partial^1_{\pm} D_i$. Let $\Omega = (\Omega_i)_{i\in I}$ be a locally finite Γ-equivariant family of measurable subsets of $\breve{\mathscr{G}}X$, with $\Omega_i \subset \partial^1_{\pm} D_i$ for all $i \in I$ and $\Omega_i = \Omega_j$ if $i \sim j$. We define the global test functions $\widetilde{\Phi}^{\mp}_{\eta} : \mathscr{G}X \to [0, +\infty[$ by

$$\widetilde{\Phi}^{\mp}_{\eta} = \sum_{i\in I/\sim} \phi^{\mp}_{\eta,R,\Omega_i} = \sum_{i\in I/\sim} h^{\mp}_{\eta,R} \circ f^{\pm}_{D^{\mp}}\ \mathbb{1}_{\mathscr{V}^{\pm}_{\eta,R}(\Omega^{\mp})}\,. \tag{10.5}$$

A subset Δ_Γ of $\breve{\mathscr{G}}X$ is a *fundamental domain* for the action of Γ if the interiors of its translates are disjoint and any compact subset of $\breve{\mathscr{G}}X$ meets only finitely many translates of Δ_Γ: If m_F is finite, a fundamental domain with boundary of zero measure exists by [Rob2, p. 13], using the fact that $\widetilde{m}_F$ has no atoms according to Corollary 4.7 (1) and to Theorem 4.6.

The following properties of the bump functions are proved as in [PaP14a, Prop. 18].

Lemma 10.1. *(1) For every $\eta > 0$, the functions $\phi^{\mp}_{\eta,R,\Omega^{\pm}}$ are measurable, nonnegative and satisfy*

$$\int_{\mathscr{G}X} \phi^{\mp}_{\eta,R,\Omega^{\pm}}\ d\widetilde{m}_F = \widetilde{\sigma}^{\pm}_{D^{\mp}}(\Omega^{\mp})\,.$$

(2) For every $\eta > 0$, the function $\widetilde{\Phi}^{\mp}_{\eta}$ is well defined, measurable and Γ-invariant. It defines, by passing to the quotient, a measurable function $\Phi^{\mp}_{\eta} : \Gamma\backslash\mathscr{G}X \to [0, +\infty[$ such that

$$\int_{\Gamma\backslash\mathscr{G}X} \Phi^{\mp}_{\eta}\ dm_F = \|\sigma^{\pm}_{\Omega}\|\,. \tag{10.6}$$

Proof. (1) Recall that the fiber of the restriction of $f^{\pm}_{D^{\mp}}$ to $\mathscr{V}^{\pm}_{\eta,R}(\Omega^{\mp})$ over $w \in \Omega^{\mp}$ is the open subset $V^{\pm}_{w,\eta,\eta'}$ of $W^{0\pm}(w)$. By the disintegration result of Proposition 7.7, by the definition of $h^{\mp}_{\eta,R}$ and by the choice of R, we have

$$\begin{aligned}\int_{\mathscr{G}X} \phi^{\mp}_{\eta,R,\Omega^{\pm}}\ d\widetilde{m}_F &= \int_{\ell\in\mathscr{V}^{\pm}_{\eta,R}(\Omega^{\mp})} h^{\mp}_{\eta,R} \circ f^{\pm}_{D^{\mp}}(\ell)\ d\widetilde{m}_F(\ell)\\ &= \int_{w\in\Omega^{\mp}} h^{\mp}_{\eta,R}(w) \int_{\ell\in V^{\pm}_{w,\eta,R}} d\nu^{\mp}_w(\ell)\ d\widetilde{\sigma}^{\pm}_{D^{\mp}}(w) = \widetilde{\sigma}^{\pm}_{D^{\mp}}(\Omega^{\mp})\,.\end{aligned}$$

(2) The function $\widetilde{\Phi}^{\mp}_{\eta}$ is well defined, since $\Omega_i = \Omega_j$ and thus $\mathscr{V}^{\pm}_{\eta,R}(\Omega_i) = \mathscr{V}^{\pm}_{\eta,R}(\Omega_j)$ if $i \sim j$, since $h^{\mp}_{\eta,R} \circ f^{\pm}_{D_i}(\ell)$ is finite if $\ell \in \mathscr{V}^{\pm}_{\eta,R}(\Omega_i)$ (by the definition of R), and since the sum defining $\widetilde{\Phi}^{\mp}_{\eta}$ has only finitely many nonzero terms, by the local finiteness of the family Ω (given $\ell \in \mathscr{G}X$, the summation over $I/\sim$ giving $\widetilde{\Phi}^{\mp}_{\eta}(\ell)$ may be replaced by a summation over the finite set $\{i \in I : \ell \in \mathscr{V}^{\pm}_{\eta,R}(\Omega_i)\}/\sim$).

The function $\widetilde{\Phi}^{\mp}_{\eta}$ is Γ-invariant since

$$\mathbb{1}_{\mathscr{V}^{\pm}_{\eta,R}(\Omega_i)} \circ \gamma = \mathbb{1}_{\gamma^{-1}\mathscr{V}^{\pm}_{\eta,R}(\Omega_i)} = \mathbb{1}_{\mathscr{V}^{\pm}_{\eta,R}(\Omega_{\gamma^{-1}i})}$$

and[2]

$$h^{\mp}_{\eta,\,R}\circ f^{\pm}_{D_i}\circ\gamma = h^{\mp}_{\eta,\,R}\circ\gamma\circ f^{\pm}_{\gamma^{-1}D_i} = h^{\mp}_{\eta,\,R}\circ f^{\pm}_{D_{\gamma^{-1}i}}$$

and by a change of index in Equation (10.5). If Δ_Γ is a fundamental domain for the action of Γ, we have by Assertion (1)

$$\int_{\Gamma\backslash\mathscr{G}X}\Phi^{\mp}_{\eta}\,dm_F=\int_{\Delta_\Gamma}\widetilde{\Phi}^{\mp}_{\eta}\,d\widetilde{m}_F=\sum_{i\in I/\sim}\int_{\mathscr{G}X}\phi^{\mp}_{\eta,\,R,\,\Omega_i\cap\Delta_\Gamma}\,d\widetilde{m}_F$$
$$=\sum_{i\in I/\sim}\widetilde{\sigma}^{\pm}_{D_i}(\Delta_\Gamma\cap\Omega_i)=\widetilde{\sigma}^{\pm}_{\Omega}(\Delta_\Gamma)=\|\sigma^{\pm}_{\Omega}\|\,.\quad\square$$

We now state and prove the aforementioned equidistribution result. Note that as the elements of the outer unit normal bundles are only geodesic rays on $[0,+\infty[$, their pushforwards by the geodesic flow at time t are geodesic rays on $[-t,+\infty[$ and the convergence towards geodesic lines (defined on $]-\infty,+\infty[$) does take place in the full space of generalised geodesic lines $\breve{\mathscr{G}}X$. This explains why it is important not to forget to consider the negative times in order for the skinning measures, supported on geodesic rays, when pushed by the geodesic flow, to have a chance to weak-star converge to Gibbs measures, supported on geodesic lines, up to renormalisation.

The proof of the following result has similarities with that of [PaP14a, Theo. 1], but the computations do not apply in the present context because the proof in loc. cit. does not keep track of the past: here we can no longer reduce our study to the outer unit normal bundle of the t-neighbourhood of the elements of $\mathscr{D}$.

Theorem 10.2. *Let $(X,\Gamma,\widetilde{F})$ be as in the beginning of Chapter 10. Assume that the Gibbs measure m_F on $\Gamma\backslash\mathscr{G}X$ is finite and mixing for the geodesic flow. Let $\mathscr{D}=(D_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of X. Let $\Omega=(\Omega_i)_{i\in I}$ be a locally finite Γ-equivariant family of measurable subsets of $\breve{\mathscr{G}}X$, with $\Omega_i\subset\partial^1_{\pm}D_i$ for all $i\in I$ and $\Omega_i=\Omega_j$ if $i\sim_{\mathscr{D}} j$. Assume that $\sigma^{\pm}_{\Omega}$ is finite and nonzero. Then, as $t\to+\infty$, for the weak-star convergence of measures on $\Gamma\backslash\breve{\mathscr{G}}X$,*

$$\frac{1}{\|(\mathsf{g}^{\pm t})_*\sigma^{\pm}_{\Omega}\|}\,(\mathsf{g}^{\pm t})_*\sigma^{\pm}_{\Omega}\;\overset{*}{\rightharpoonup}\;\frac{1}{\|m_F\|}\,m_F\,.$$

Proof. We only give the proof when $\pm=+$, the other case is treated similarly. Given three numbers a,b,c (depending on some parameters), we write $a=b\pm c$ if $|a-b|\leqslant c$.

Let $\eta\in\,]0,1]$. We may assume that $\Gamma\backslash I$ is finite, since for every $\epsilon>0$, there exists a Γ-invariant partition $I=I'\cup I''$ with $\Gamma\backslash I'$ finite such that if $\Omega'=(\Omega_i)_{i\in I'}$ and $\Omega''=(\Omega_i)_{i\in I''}$, then $\sigma^+_{\Omega}=\sigma^+_{\Omega'}+\sigma^+_{\Omega''}$ with $\|(\mathsf{g}^{-t})_*\sigma^+_{\Omega''}\|=\|\sigma^+_{\Omega''}\|<\epsilon$. Hence, using Lemma 7.6 (i), we may fix $R>0$ such that $\nu^-_w(V^+_{w,\,\eta,\,R})>0$ for all $i\in I$ and $w\in\partial^1_+D_i$.

Fix $\psi\in\mathscr{C}_{\rm c}(\Gamma\backslash\breve{\mathscr{G}}X)$, a continuous function with compact support on $\Gamma\backslash\breve{\mathscr{G}}X$. Let us prove that

$$\lim_{t\to+\infty}\;\frac{1}{\|(\mathsf{g}^t)_*\sigma^+_{\Omega}\|}\int_{\Gamma\backslash\mathscr{G}X}\psi\;d(\mathsf{g}^t)_*\sigma^+_{\Omega}=\frac{1}{\|m_F\|}\int_{\Gamma\backslash\mathscr{G}X}\psi\;dm_F\,.$$

Let Δ_Γ be a fundamental domain for the action of Γ on $\breve{\mathscr{G}}X$, such that the boundary of Δ_Γ has zero measure. By a standard argument of finite partition of unity and up to modifying

[2] See Equation (2.16).

Δ_Γ, we may assume that there exists a function $\widetilde{\psi} : \breve{\mathscr{G}}X \to \mathbb{R}$ whose support has a small neighbourhood contained in Δ_Γ such that $\widetilde{\psi} = \psi \circ p$ on this neighbourhood, where $p : \breve{\mathscr{G}}X \to \Gamma\backslash\breve{\mathscr{G}}X$ is the canonical projection (which is Lipschitz). Fix $\epsilon > 0$. Since $\widetilde{\psi}$ is uniformly continuous, for every $\eta > 0$ small enough and for every $t \geqslant 0$ large enough, for all $w \in \mathscr{G}_+X$ isometric on $[-t, +\infty[$ and $\ell \in V^+_{w,\,\eta,\,e^{-t}R}$, we have

$$\widetilde{\psi}(\ell) = \widetilde{\psi}(w) \pm \frac{\epsilon}{2}\ . \tag{10.7}$$

If t is large enough and η small enough, we have, using respectively

- the definition of the global test function $\widetilde{\Phi}_\eta = \widetilde{\Phi}^-_\eta$, since the support of $\widetilde{\psi}$ is contained in Δ_Γ and the support of $\phi^-_{\eta,\,R,\,\Omega_i}$ is contained in $\mathscr{U}^+_{D_i}$, for the second equality,
- the disintegration property of $f^+_{D_i}$ in Proposition 7.7 for the third equality,
- the fact that if ℓ is in the support of ν^-_ρ, then $f^+_{D_i}(\mathsf{g}^{-t}\ell) = f^+_{D_i}(\ell) = \rho$ and the change of variables by the geodesic flow $w = \mathsf{g}^t\rho$ for the fourth equality,
- the fact that the support of $\nu^-_{\mathsf{g}^{-t}w}$ is contained in $W^{0+}(\mathsf{g}^{-t}w)$, and that

  $$W^{0+}(\mathsf{g}^{-t}w) \cap \mathsf{g}^t V^+_{\eta,\,R}(\Omega_i) = \mathsf{g}^t\big(W^{0+}(\mathsf{g}^{-t}w) \cap V^+_{\eta,\,R}(\Omega_i)\big) = \mathsf{g}^t V^+_{\mathsf{g}^{-t}w,\,\eta,\,R}$$

  for the fifth equality,
- Equation (10.7) for the sixth equality, and
- the definition of h^-, the invariance of the measure $\nu^-_{\mathsf{g}^{-t}w}$ and the Gibbs measure $\widetilde{m}_F$ under the geodesic flow, and the definition of the measure σ^+_Ω for the last two equalities:

$$\begin{aligned}
&\int_{\Gamma\backslash\mathscr{G}X} \psi\ \ \Phi_\eta\circ\mathsf{g}^{-t}\ dm_F = \int_{\Delta_\Gamma\cap\mathscr{G}X} \widetilde{\psi}\ \ \widetilde{\Phi}_\eta\circ\mathsf{g}^{-t}\ d\widetilde{m}_F\\
&= \sum_{i\in I/\sim}\ \int_{\ell\in\mathscr{U}^+_{D_i}} \widetilde{\psi}(\ell)\ \phi^-_{\eta,\,R,\,\Omega_i}(\mathsf{g}^{-t}\ell)\ d\widetilde{m}_F(\ell)\\
&= \sum_{i\in I/\sim}\ \int_{\rho\in\partial^1_+D_i}\int_{\ell\in\mathscr{U}^+_{D_i}} \widetilde{\psi}(\ell)\ h^-_{\eta,\,R}(f^+_{D_i}(\mathsf{g}^{-t}\ell))\,\mathbb{1}_{\mathscr{V}^+_{\eta,R}(\Omega_i)}(\mathsf{g}^{-t}\ell)\ d\nu^-_\rho(\ell)\ d\widetilde{\sigma}^+_{D_i}(\rho)\\
&= \sum_{i\in I/\sim}\ \int_{w\in\mathsf{g}^t\partial^1_+D_i}\int_{\ell\in\mathsf{g}^t\mathscr{V}^+_{\eta,R}(\Omega_i)} \widetilde{\psi}(\ell)\ h^-_{\eta,\,R}(\mathsf{g}^{-t}w))\ d\nu^-_{\mathsf{g}^{-t}w}(\ell)\ d(\mathsf{g}^t)_*\widetilde{\sigma}^+_{D_i}(w)\\
&= \sum_{i\in I/\sim}\ \int_{w\in\mathsf{g}^t\partial^1_+D_i}\int_{\ell\in\mathsf{g}^tV^+_{\mathsf{g}^{-t}w,\,\eta,\,R}} \widetilde{\psi}(\ell)\ h^-_{\eta,\,R}(\mathsf{g}^{-t}w)\ d\nu^-_{\mathsf{g}^{-t}w}(\ell)\ d(\mathsf{g}^t)_*\widetilde{\sigma}^+_{D_i}(w)\\
&= \sum_{i\in I/\sim}\ \int_{w\in\mathsf{g}^t\partial^1_+D_i} \widetilde{\psi}(w)\ h^-_{\eta,\,R}(\mathsf{g}^{-t}w)\ \nu^-_{\mathsf{g}^{-t}w}(\mathsf{g}^tV^+_{\mathsf{g}^{-t}w,\,\eta,\,R})\ d(\mathsf{g}^t)_*\widetilde{\sigma}^+_{D_i}(w)\\
&\qquad \pm\frac{\epsilon}{2}\int_{\Delta_\Gamma\cap\mathscr{G}X} \widetilde{\Phi}_\eta\circ\mathsf{g}^{-t}\ d\widetilde{m}_F\\
&= \sum_{i\in I/\sim}\ \int_{\breve{\mathscr{G}}X} \widetilde{\psi}\ d(\mathsf{g}^t)_*\widetilde{\sigma}^+_{D_i} \pm \frac{\epsilon}{2}\int_{\Gamma\backslash\mathscr{G}X} \Phi_\eta\circ\mathsf{g}^{-t}\ dm_F\\
&= \int_{\Gamma\backslash\breve{\mathscr{G}}X} \psi\ d(\mathsf{g}^t)_*\sigma^+_\Omega \pm \frac{\epsilon}{2}\int_{\Gamma\backslash\mathscr{G}X} \Phi_\eta\ dm_F\ .
\end{aligned} \tag{10.8}$$

We then conclude as in the end of the proof of [PaP14a, Theo. 19]. By Equation (10.6), we have $\|(\mathsf{g}^t)_*\sigma^+_\Omega\| = \|\sigma^+_\Omega\| = \int_{\Gamma\backslash\mathscr{G}X}\Phi_\eta\ dm_F$. By the mixing property of the geodesic flow on

$\Gamma\backslash\mathscr{G}X$ for the Gibbs measure m_F, for $t \geqslant 0$ large enough (while η is small but fixed), we hence have

$$\frac{\int_{\Gamma\backslash\breve{\mathscr{G}}X}\psi\; d(\mathtt{g}^t)_*\sigma^+_\Omega}{\|(\mathtt{g}^t)_*\sigma^+_\Omega\|} = \frac{\int_{\Gamma\backslash\mathscr{G}X}\Phi_\eta\circ\mathtt{g}^{-t}\;\psi\; dm_F}{\int_{\Gamma\backslash\mathscr{G}X}\Phi_\eta\; dm_F} \pm\frac{\epsilon}{2} = \frac{\int_{\Gamma\backslash\mathscr{G}X}\psi\; dm_F}{\|m_F\|}\pm\epsilon\,.$$

This proves the result. □

Recall that by Proposition 3.14, Theorem 10.2 applies to Riemannian manifolds with pinched negative curvature and for $\mathbb{R}$-trees for which the geodesic flow is mixing and which satisfy the finiteness requirements of the Theorem.

Since pushforwards of measures are weak-star continuous and preserve total mass, we have, under the assumptions of Theorem 10.2, the following equidistribution result in X of the immersed t-neighbourhood of a properly immersed nonempty proper closed convex subset of X: as $t\to+\infty$,

$$\frac{1}{\|\sigma^+_\Omega\|}\,\pi_*(\mathtt{g}^t)_*\sigma^+_\Omega \;\overset{*}{\rightharpoonup}\; \frac{1}{\|m_F\|}\,\pi_* m_F\,. \tag{10.9}$$

10.2 Rate of equidistribution of equidistant level sets for manifolds

If $X=\widetilde{M}$ is a Riemannian manifold and if the geodesic flow of $\Gamma\backslash\widetilde{M}$ is mixing with exponentially decaying correlations, we get a version of Theorem 10.2 with error bounds. See Section 9.1 for conditions on Γ and $\widetilde{F}$ that imply the exponential mixing.

Theorem 10.3. *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with negative sectional curvature. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F}: T^1\widetilde{M}\to\mathbb{R}$ be a bounded Γ-invariant Hölder-continuous function with critical exponent $\delta=\delta_{\Gamma,\,F}$. Let $\mathscr{D}=(D_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of $\widetilde{M}$, with finite nonzero skinning measure $\sigma_{\mathscr{D}}$. Let $M=\Gamma\backslash\widetilde{M}$ and let $F:T^1M\to\mathbb{R}$ be the potential induced by $\widetilde{F}$.*

(i) If M is compact and if the geodesic flow on T^1M is mixing with exponential speed for the Hölder regularity for the potential F, then there exist $\alpha\in\,]0,1]$ and $\kappa''>0$ such that for all $\psi\in\mathscr{C}^\alpha_{\rm c}(T^1M)$, we have, as $t\to+\infty$,

$$\frac{1}{\|\sigma_{\mathscr{D}}\|}\int\psi\; d(\mathtt{g}^t)_*\sigma_{\mathscr{D}} = \frac{1}{\|m_F\|}\int\psi\; dm_F + \operatorname{O}(e^{-\kappa'' t}\,\|\psi\|_\alpha)\,.$$

(ii) If $\widetilde{M}$ is a symmetric space, if D_i has smooth boundary for every $i\in I$, if m_F is finite and smooth, and if the geodesic flow on T^1M is mixing with exponential speed for the Sobolev regularity for the potential F, then there exist $\ell\in\mathbb{N}$ and $\kappa''>0$ such that for all $\psi\in\mathscr{C}^\ell_{\rm c}(T^1M)$, we have, as $t\to+\infty$,

$$\frac{1}{\|\sigma_{\mathscr{D}}\|}\int\psi\; d(\mathtt{g}^t)_*\sigma_{\mathscr{D}} = \frac{1}{\|m_F\|}\int\psi\; dm_F + \operatorname{O}(e^{-\kappa'' t}\,\|\psi\|_\ell)\,.$$

Note that if $\widetilde{M}$ is a symmetric space and M has finite volume, then M is geometrically finite. Theorem 4.8 implies that m_F is finite if F is small enough. The maps $\operatorname{O}(\cdot)$ depend on $\widetilde{M},\Gamma,F,\mathscr{D}$, and the speeds of mixing.

Proof. Up to rescaling, we may assume under the assumptions of Claim (i) or (ii) that the sectional curvature is bounded from above by -1. The critical exponent δ and the Gibbs measure m_F are finite under the assumptions of the theorem.

Let us consider Claim (i). Under its assumptions, there exists $\alpha \in \,]0,1[$ such that the geodesic flow on T^1M is exponentially mixing for the Hölder regularity α and such that the strong stable foliation of $T^1\widetilde{M}$ is α-Hölder.[3]

First assume that $\Gamma\backslash I$ is finite. Fix $R > 0$ large enough and, for every $\eta > 0$, let us consider the test function Φ_η as in the proof of Theorem 10.2. Up to replacing D_i by $\mathscr{N}_1 D_i$, we may assume that the boundary of D_i is $\mathscr{C}^{1,1}$-smooth, for every $i \in I$, see for instance [Walt].

Fix $\psi \in \mathscr{C}_{\rm c}^{\alpha}(T^1M)$. We may assume as in the proof of Theorem 10.2 that there exists a lift $\widetilde{\psi} : T^1\widetilde{M} \to \mathbb{R}$ of ψ whose support is contained in a given fundamental domain Δ_Γ for the action of Γ on $T^1\widetilde{M}$. There exist $\eta_0 > 0$ and $t_0 \geqslant 0$ such that for every $\eta \in \,]0,\eta_0]$, and for every $t \in [t_0,+\infty[$, for every $w \in T^1\widetilde{M}$ and $v \in V^+_{w,\,\eta,\,e^{-t}R}$, we have

$$\widetilde{\psi}(v) = \widetilde{\psi}(w) + \mathrm{O}\left((\eta + e^{-t})^{\alpha}\|\psi\|_{\alpha}\right) , \tag{10.10}$$

since $d(v,w) = \mathrm{O}(\eta + e^{-t})$ by Equation (2.8) and Lemma 2.4.

Let $\overline{m}_F = \frac{m_F}{\|m_F\|}$ be the normalisation of the Gibbs measure m_F to a probability measure. As in the proof of Theorem 10.2 using Equation (10.10) instead of Equation (10.7), we have

$$\frac{\int_{T^1M} \psi \; d(\mathtt{g}^t)_*\sigma^+_\Omega}{\|(\mathtt{g}^t)_*\sigma^+_\Omega\|} = \frac{\int_{T^1M} \Phi_\eta \; \psi\circ \mathtt{g}^t \; d\overline{m}_F}{\int_{T^1M} \Phi_\eta \; d\overline{m}_F} + \mathrm{O}\left((\eta + e^{-t})^{\alpha}\|\psi\|_{\alpha}\right) .$$

As M is compact, the Patterson densities and the Bowen-Margulis measure are doubling measures by Lemma 4.3 (4).[4] Using discrete convolution approximation,[5] there exist $\kappa' > 0$ and, for every $\eta > 0$, a nonnegative function $\mathtt{R}\Phi_\eta \in \mathrm{C}_{\rm c}^{\alpha}(T^1M)$ such that

- $\int_{T^1M} \mathtt{R}\Phi_\eta \; d\overline{m}_F = \int_{T^1M} \Phi_\eta \; d\overline{m}_F$,
- $\int_{T^1M} |\,\mathtt{R}\Phi_\eta - \Phi_\eta| \; d\overline{m}_F = \mathrm{O}(\eta \int_{T^1M} \Phi_\eta \; d\overline{m}_F)$,
- $\|\,\mathtt{R}\Phi_\eta\|_{\alpha} = \mathrm{O}(\eta^{-\kappa'} \int_{T^1M} \Phi_\eta \; d\overline{m}_F)$.

Hence, applying the exponential mixing of the geodesic flow, with $\kappa > 0$ as in its definition (9.1), we have, for $\eta \in \,]0,\eta_0]$ and $t \in [t_0,+\infty[$,

$$\begin{aligned}
&\frac{\int_{T^1M} \psi \; d(\mathtt{g}^t)_*\sigma^+_\Omega}{\|(\mathtt{g}^t)_*\sigma^+_\Omega\|} = \\
&= \frac{\int_{T^1M} \mathtt{R}\Phi_\eta \; \psi\circ \mathtt{g}^t \; d\overline{m}_F}{\int_{T^1M} \Phi_\eta \; d\overline{m}_F} + \mathrm{O}\left(\eta\;\|\psi\|_\infty + (\eta + e^{-t})^{\alpha}\|\psi\|_{\alpha}\right) \\
&= \frac{\int_{T^1M} \mathtt{R}\Phi_\eta \; d\overline{m}_F}{\int_{T^1M} \Phi_\eta \; d\overline{m}_F} \int_{T^1M} \psi \; d\overline{m}_F + \mathrm{O}\left(e^{-\kappa t}\|\,\mathtt{R}\Phi_\eta\|_{\alpha}\|\psi\|_{\alpha} + \eta\;\|\psi\|_\infty + (\eta + e^{-t})^{\alpha}\|\psi\|_{\alpha}\right) \\
&= \int_{T^1M} \psi \; d\overline{m}_F + \mathrm{O}\left((e^{-\kappa t}\eta^{-\kappa'} + \eta + (\eta + e^{-t})^{\alpha})\|\psi\|_{\alpha}\right) .
\end{aligned}$$

[3] See Section 9.1.
[4] See also [PauPS, Prop. 3.12].
[5] See for instance [Sem, p. 290-292] or [KinKST].

Taking $\eta = e^{-t\lambda}$ for λ small enough (for instance $\lambda = \kappa/(2\kappa')$), the result follows (for instance with $\kappa'' = \min\{\kappa/2,\ \kappa/(2\kappa'),\ \alpha\min\{1, \kappa/(2\kappa')\}\}$), when $\Gamma\backslash I$ is finite. As the implied constants do not depend on the family $\mathscr{D}$, the result holds in general.

For Claim (ii), the required smoothness of m_F (that is, the fact that m_F is absolutely continuous with respect to the Lebesgue measure with smooth Radon-Nikodym derivative) allows to use the standard convolution approximation described for instance in [Zie, §1.6], instead of the operator $\mathtt{R}$ as above, and the proof proceeds similarly. □

10.3 Equidistribution of equidistant level sets on simplicial graphs and random walks on graphs of groups

Let $\mathbb{X}$, X, Γ, $\widetilde{c}$, c, $\widetilde{F}_c$, F_c, $\delta_c < +\infty$, $(\mu^\pm_x)_{x\in V\mathbb{X}}$, $\widetilde{m}_c = \widetilde{m}_{F_c}$, $m_c = m_{F_c}$ be as in the beginning of Section 9.2.

In this Section, we state an equidistribution result analogous to Theorem 10.2, which now holds in the space of generalised discrete geodesic lines $\Gamma\backslash\check{\mathscr{G}}\mathbb{X}$, but whose proof is completely analogous.

Theorem 10.4. *Let $\mathbb{X}, \Gamma, \widetilde{c}, (\mu^\pm_x)_{x\in V\mathbb{X}}$ be as above. Assume that the Gibbs measure m_c on $\Gamma\backslash\mathscr{G}\mathbb{X}$ is finite and mixing for the discrete time geodesic flow. Let $\mathscr{D} = (\mathbb{D}_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper simplicial subtrees of $\mathbb{X}$, and $D_i = |\mathbb{D}_i|_1$. Let $\Omega = (\Omega_i)_{i\in I}$ be a locally finite Γ-equivariant family of measurable subsets of $\check{\mathscr{G}}\mathbb{X}$, with $\Omega_i \subset \partial^1_+ D_i$ for all $i \in I$ and $\Omega_i = \Omega_j$ if $i \sim_{\mathscr{D}} j$. Assume that σ^+_Ω is finite and nonzero. Then, as $n \to +\infty$, for the weak-star convergence of measures on $\Gamma\backslash\check{\mathscr{G}}\mathbb{X}$,*

$$\frac{1}{\|(\mathtt{g}^n)_*\sigma^+_\Omega\|}\,(\mathtt{g}^n)_*\sigma^+_\Omega \;\overset{*}{\rightharpoonup}\; \frac{1}{\|m_F\|}\,m_F\,. \quad \square$$

We leave to the reader the analog of this result when the restriction to $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ of the Gibbs measure is finite and mixing for the square of the discrete time geodesic flow.

Using Proposition 4.16 and Theorem 4.17 in order to check that the Bowen-Margulis measure $m_{\rm BM}$ on $\Gamma\backslash\mathscr{G}\mathbb{X}$ is finite and mixing, we have the following consequence of Theorem 10.4, using the system of conductances $\widetilde{c} = 0$.

Corollary 10.5. *Let $\mathbb{X}$ be a uniform simplicial tree. Let Γ be a lattice of $\mathbb{X}$ such that the graph $\Gamma\backslash\mathbb{X}$ is not bipartite. Let $\mathscr{D} = (\mathbb{D}_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper simplicial subtrees of $\mathbb{X}$ and $D_i = |\mathbb{D}_i|_1$. Let $\Omega = (\Omega_i)_{i\in I}$ be a locally finite Γ-equivariant family of measurable subsets of $\check{\mathscr{G}}\mathbb{X}$, with $\Omega_i \subset \partial^1_+ D_i$ for all $i \in I$ and $\Omega_i = \Omega_j$ if $i \sim_{\mathscr{D}} j$. Assume that the skinning measure σ^+_Ω (with vanishing potential) is finite and nonzero. Then, as $n \to +\infty$, for the weak-star convergence of measures on $\Gamma\backslash\check{\mathscr{G}}\mathbb{X}$,*

$$\frac{1}{\|(\mathtt{g}^n)_*\sigma^+_\Omega\|}\,(\mathtt{g}^n)_*\sigma^+_\Omega \;\overset{*}{\rightharpoonup}\; \frac{1}{\|m_{\rm BM}\|}\,m_{\rm BM}\,. \quad \square$$

When furthermore $\mathbb{X}$ is regular, we have the following corollary, using Proposition 8.1 (3).

Corollary 10.6. *Let $\mathbb{X}$ be a regular simplicial tree of degree at least 3. Let Γ be a lattice of $\mathbb{X}$ such that the graph $\Gamma\backslash\mathbb{X}$ is not bipartite. Let $\mathscr{D} = (\mathbb{D}_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper simplicial subtrees of $\mathbb{X}$ and $D_i = |\mathbb{D}_i|_1$. Let $\Omega = (\Omega_i)_{i\in I}$ be a* *locally finite Γ-equivariant family of measurable subsets of $\check{\mathscr{G}}\mathbb{X}$, with $\Omega_i \subset \partial^1_+ D_i$ for all $i \in I$ and $\Omega_i = \Omega_j$ if $i \sim_{\mathscr{D}} j$. Assume that the skinning measure σ^+_Ω (with vanishing potential) is finite and nonzero. Then, as $n \to +\infty$, for the weak-star convergence of measures on $\Gamma\backslash V\mathbb{X}$,*

$$\frac{1}{\|(\mathrm{g}^n)_*\sigma^+_\Omega\|}\,\pi_*(\mathrm{g}^n)_*\sigma^+_\Omega \;\overset{*}{\rightharpoonup}\; \frac{1}{\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X})}\,\mathrm{vol}_{\Gamma\backslash\backslash\mathbb{X}}\;. \quad \Box$$

Let us give an application of Corollary 10.6 in terms of random walks on graphs of groups, which might also be deduced from general result on random walks, as indicated to the third author by M. Burger and S. Mozes.

Let $(\mathbb{Y}, G_*)$ be a graph of finite groups with finite volume, and let $(\mathbb{Y}', G'_*)$ be a connected subgraph of subgroups.[6] Note that $(\mathbb{Y}', G'_*)$ also has finite volume, less than or equal to the volume of $(\mathbb{Y}, G_*)$. We say that $(\mathbb{Y}, G_*)$ is *locally homogeneous* if $\sum_{e\in E\mathbb{Y},\ o(e)=x} \frac{|G_x|}{|G_e|}$ is constant at least 3 for all $x \in V\mathbb{Y}$. We say that a graph of groups is 2-*acylindrical* if the action of its fundamental group on its Bass-Serre tree is 2-acylindrical (see Remark 5.4). In particular, this action is faithful if the graph has at least two edges.

The *non-backtracking simple random walk on $(\mathbb{Y}, G_*)$ starting transversally to $(\mathbb{Y}', G'_*)$* is the following Markovian random process $(X_n = (f_n, \gamma_n))_{n\in\mathbb{N}}$ where $f_n \in E\mathbb{Y}$ and γ_n is a double coset or right coset of $G_{o(f_n)}$ for all $n \in \mathbb{N}$. Choose at random a vertex y_0 of $\mathbb{Y}'$ for the probability measure $\frac{1}{\mathrm{Vol}(\mathbb{Y}', G'_*)}\ \mathrm{vol}_{\mathbb{Y}', G'_*}$ (we will call y_0 the *origin* of the random path). Then choose uniformly at random $X_0 = (f_0, \gamma_0)$ where $f_0 \in E\mathbb{Y}$ is such that $o(f_0) = y_0$ and γ_0 is a double coset in $G'_{y_0}\backslash G_{y_0}/\rho_{\overline{f_0}}(G_{f_0})$ such that if $f_0 \in E\mathbb{Y}'$ then $\gamma_0 \notin G'_{y_0}\,\rho_{\overline{f_0}}(G_{f_0})$.[7] Assuming $X_n = (f_n, \gamma_n)$ constructed, choose uniformly at random $X_{n+1} = (f_{n+1}, \gamma_{n+1})$ where $f_{n+1} \in E\mathbb{Y}$ is such that $o(f_{n+1}) = t(f_n)$ and $\gamma_{n+1} \in G_{o(f_{n+1})}/\rho_{\overline{f_{n+1}}}(G_{f_{n+1}})$ is such that if $f_{n+1} = \overline{f_n}$ then $\gamma_{n+1} \notin \rho_{\overline{f_{n+1}}}(G_{f_{n+1}})$. The *$n$-th vertex* of $(X_n = (f_n, \gamma_n))_{n\in\mathbb{N}}$ is $o(f_n)$.

Corollary 10.7. *Let $(\mathbb{Y}, G_*)$ be a locally homogeneous 2-acylindrical nonbipartite graph of finite groups with finite volume, and let $(\mathbb{Y}', G'_*)$ be a locally homogeneous nonempty proper connected subgraph of subgroups. Then the n-th vertex of the non-backtracking simple random walk on $(\mathbb{Y}, G_*)$ starting transversally to $(\mathbb{Y}', G'_*)$ converges in distribution to $\frac{1}{\mathrm{Vol}(\mathbb{Y}, G_*)}\ \mathrm{vol}_{\mathbb{Y}, G_*}$ as $n \to +\infty$.*

Proof. Let Γ be the fundamental group of $(\mathbb{Y}, G_*)$ (with respect to a choice of basepoint in $V\mathbb{Y}'$), which is a lattice of the Bass-Serre tree $\mathbb{X}$ of $(\mathbb{Y}, G_*)$, since Γ acts faithfully on $\mathbb{X}$ and $(\mathbb{Y}, G_*)$ has finite volume. Note that $\mathbb{X}$ is regular since $(\mathbb{Y}, G_*)$ is locally homogeneous. Let $p : \mathbb{X} \to \mathbb{Y} = \Gamma\backslash\mathbb{X}$ be the canonical projection.

Let Γ' be the fundamental group of $(\mathbb{Y}', G'_*)$ (with respect to the same choice of basepoint). As seen in Section 2.6, there exists a simplicial subtree $\mathbb{X}'$ whose stabiliser in Γ is Γ', such that the quotient graph of groups $\Gamma'\backslash\backslash\mathbb{X}'$ identifies with $(\mathbb{Y}', G'_*)$ and the map $(\Gamma'\backslash\mathbb{X}') \to (\Gamma\backslash\mathbb{X})$ is injective. Similarly, $\mathbb{X}'$ is regular since $(\mathbb{Y}', G'_*)$ is locally homogeneous. Let $\mathscr{D} = (\gamma\mathbb{X}')_{\gamma\in\Gamma}$, which is a locally finite Γ-equivariant family of nonempty proper simplicial subtrees of $\mathbb{X}$.

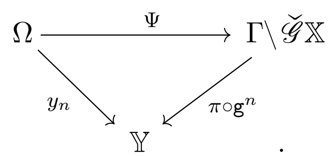
Using the notation of Example 2.10 for the graph of groups $\Gamma\backslash\backslash\mathbb{X}$ (which identifies with $(\mathbb{Y}, G_*)$), we fix lifts $\widetilde{f}$ and $\widetilde{y}$ in $\mathbb{X}$ by p of every edge f and vertex y of $\mathbb{Y}$ such that $\widetilde{\overline{f}} = \overline{\widetilde{f}}$, and elements $g_f \in \Gamma$ such that $g_f\,\widetilde{t(f)} = t(\widetilde{f})$. We may assume that $\widetilde{f} \in E\mathbb{X}'$ if $f \in E\mathbb{Y}'$, that $\widetilde{y} \in V\mathbb{X}'$ if $y \in V\mathbb{Y}'$, and that $g_f \in \Gamma'$ if $f \in E\mathbb{Y}'$, which is possible by Equation (2.22).

[6]See Section 2.6 for definitions and background.

[7]This last condition says that γ_0 is not the double coset of the trivial element.

Let $(\Omega, \mathbb{P})$ be the (canonically constructed) probability space of the random walk $(X_n = (f_n, \gamma_n))_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, let $y_n = o(f_n)$ be the random variable (with values in the discrete space $\mathbb{Y} = \Gamma\backslash\mathbb{X}$) of the n-th vertex of the random walk $(X_n)_{n\in\mathbb{N}}$.

Let us define a measurable map $\Psi : \Omega \to \Gamma\backslash\breve{\mathscr{G}}\mathbb{X}$, with image contained in the image of $\partial^1_+\mathbb{X}'$ by the canonical projection $\breve{\mathscr{G}}\mathbb{X} \to \Gamma\backslash\breve{\mathscr{G}}\mathbb{X}$, such that $\Psi_*\mathbb{P}$ is the normalised skinning measure $\frac{\sigma^+_{\mathscr{D}}}{\|\sigma^+_{\mathscr{D}}\|}$ and that the following diagram commutes for all $n \in \mathbb{N}$:

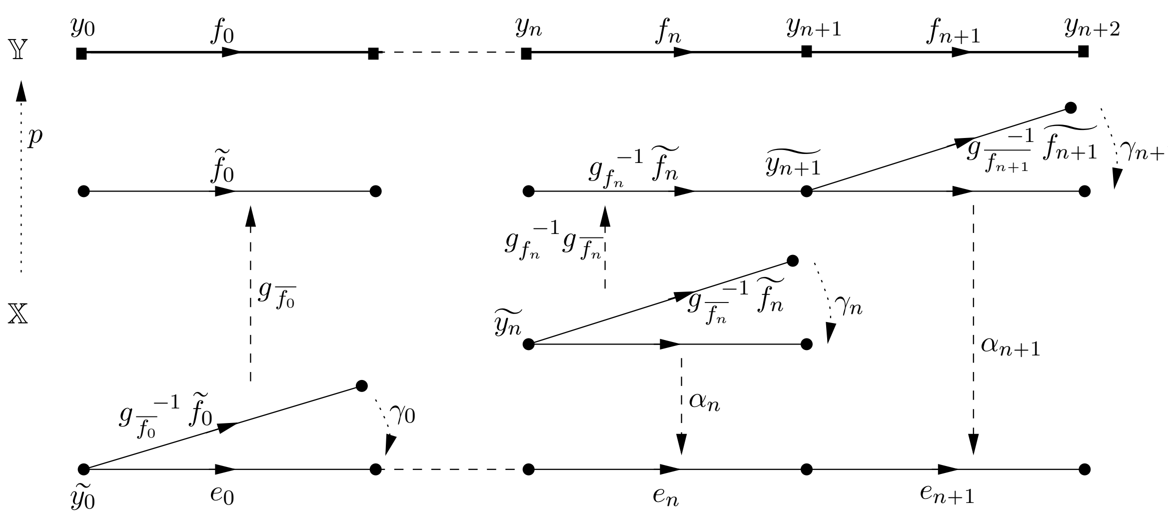


(10.11)

Assuming that we have such a map, we have

$$(y_n)_*\mathbb{P} \;=\; (\pi_* \circ (\mathsf{g}^n)_* \circ \Psi_*)\mathbb{P} \;=\; \pi_*(\mathsf{g}^n)_*\,\frac{\sigma^+_{\mathscr{D}}}{\|\sigma^+_{\mathscr{D}}\|} \;=\; \frac{1}{\|(\mathsf{g}^n)_*\sigma^+_{\mathscr{D}}\|}\,\pi_*(\mathsf{g}^n)_*\sigma^+_{\mathscr{D}}$$

so that the convergence of the law of y_n to $\frac{1}{\operatorname{Vol}(\mathbb{Y},G_*)}\ \operatorname{vol}_{\mathbb{Y},G_*}$ follows from Corollary 10.6 applied to $\Omega = (\partial^1_+ D_i)_{i\in I}$.



Let $(X_n = (f_n, \gamma_n))_{n\in\mathbb{N}}$ be a random path with origin $y_0 \in \mathbb{Y}'$, corresponding to $\omega \in \Omega$. Fix a representative of γ_n in its right class for every $n \geqslant 1$, and a representative of γ_0 in its double class, that we still denote by γ_n and γ_0 respectively. Using ideas introduced for the coding in Section 5.2, let us construct by induction an infinite geodesic edge path $(e_n)_{n\in\mathbb{N}}$ with origin $o(e_0) = \widetilde{y_0}$ and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in Γ such that

$$e_n = \alpha_n\gamma_n g_{\overline{f_n}}{}^{-1}\,\widetilde{f_n}\ . \tag{10.12}$$

Let $\alpha_0 = \mathrm{id}$ and $e_0 = \gamma_0\, g_{\overline{f_0}}{}^{-1}\ \widetilde{f_0}$. Since $o(\widetilde{f_0}) = g_{\overline{f_0}}\ \widetilde{o(\overline{f_0})} = g_{\overline{f_0}}\ \widetilde{y_0}$ by the construction of the lifts and since $\gamma_0 \in G_{y_0} = \Gamma_{\widetilde{y_0}}$, we have $o(e_0) = \widetilde{y_0}$. Since the stabiliser of $g_{\overline{f_0}}{}^{-1}\ \widetilde{f_0}$ is

$\rho_{\overline{f_0}}(G_{f_0})$, the edge e_0 does not depend on the choice of the representative γ_0 modulo $\rho_{\overline{f_0}}(G_{f_0})$ on the right, but depends on the choice of the representative γ_0 modulo $G'_{y_0} = \Gamma'_{\widetilde{y_0}}$ on the left.

The hypothesis that if $f_0 \in E\mathbb{Y}'$ then $\gamma_0 \notin G'_{y_0}\, \rho_{\overline{f_0}}(G_{f_0})$ ensures that the edge e_0 does not belong to $E\mathbb{X}'$. Indeed, assume otherwise that e_0 belongs to $E\mathbb{X}'$. Then $f_0 = p(e_0) \in E\mathbb{Y}'$, and by the assumptions on the choice of lifts, the edges $g_{\overline{f_0}}^{\;-1}\, \widetilde{f}_0$ and e_0 both belong to $E\mathbb{X}'$. Since they are both mapped to f_0 by the map $\mathbb{X}' \to \mathbb{Y}' = \Gamma' \backslash \mathbb{X}'$, and by Equation (2.22), they are mapped one to the other by an element of $\Gamma'_{\widetilde{y_0}} = G'_{y_0}$. Let $\gamma'_0 \in \Gamma'_{\widetilde{y_0}}$ be such that $\gamma'_0\, e_0 = g_{\overline{f_0}}^{\;-1}\, \widetilde{f}_0$. Then ${\gamma'_0}^{-1}\gamma_0$ belongs to the stabiliser in Γ of the edge $g_{\overline{f_0}}^{\;-1}\, \widetilde{f}_0$, which is equal to $g_{\overline{f_0}}^{\;-1}\, \Gamma_{\widetilde{f_0}} g_{\overline{f_0}} = \rho_{\overline{f_0}}(G_{f_0})$. Therefore $\gamma_0 \in G'_{y_0}\, \rho_{\overline{f_0}}(G_{f_0})$, a contradiction.

Assume by induction that e_n and α_n are constructed. Define

$$\alpha_{n+1} = \alpha_n\, \gamma_n\, g_{\overline{f_n}}^{\;-1}\, g_{f_n}$$

and

$$e_{n+1} = \alpha_{n+1}\, \gamma_{n+1}\, g_{\overline{f_{n+1}}}^{\;-1}\, \widetilde{f_{n+1}}\ ,$$

so that the induction formula (10.12) at rank $n+1$ is satisfied. By the construction of the lifts, since $y_{n+1} = t(f_n) = o(f_{n+1})$, we have

$$\widetilde{y_{n+1}} = g_{f_n}^{\;-1}\, t(\widetilde{f_n}) = g_{\overline{f_{n+1}}}^{\;-1}\, o(\widetilde{f_{n+1}})\ .$$

Hence, since $\gamma_{n+1} \in G_{y_{n+1}} = \Gamma_{\widetilde{y_{n+1}}}$ fixes $\widetilde{y_{n+1}}$, using the induction formula (10.12) at rank n for the last equality,

$$\begin{aligned} o(e_{n+1}) &= \alpha_{n+1}\, \gamma_{n+1}\, g_{\overline{f_{n+1}}}^{\;-1}\, o(\widetilde{f_{n+1}}) = \alpha_{n+1}\, \gamma_{n+1}\, \widetilde{y_{n+1}} = \alpha_{n+1}\, \widetilde{y_{n+1}} \\ &= \alpha_{n+1}\, g_{f_n}^{\;-1}\, t(\widetilde{f_n}) = \alpha_n\, \gamma_n\, g_{\overline{f_n}}^{\;-1}\, t(\widetilde{f_n}) = t(e_n)\ . \end{aligned}$$

In particular, the sequence $(e_n)_{n\in\mathbb{N}}$ is an edge path in $\mathbb{X}$.

Since the stabiliser of $g_{\overline{f_{n+1}}}^{\;-1}\, \widetilde{f_{n+1}}$ is

$$g_{\overline{f_{n+1}}}^{\;-1}\, \Gamma_{\widetilde{f_{n+1}}}\, g_{\overline{f_{n+1}}} = \rho_{\overline{f_{n+1}}}(G_{f_{n+1}})\ ,$$

the edge $\gamma_{n+1}\, g_{\overline{f_{n+1}}}^{\;-1}\, \widetilde{f_{n+1}}$ does not depend on the choice of the representative of the right coset γ_{n+1}. Let us prove that the length 2 edge path $(g_{f_n}^{\;-1}\, \widetilde{f_n},\ \gamma_{n+1}\, g_{\overline{f_{n+1}}}^{\;-1}\, \widetilde{f_{n+1}})$ is geodesic. Otherwise, the two edges of this path are opposite one to another, hence $f_{n+1} = \overline{f_n}$ by using the projection $p : \mathbb{X} \to \mathbb{Y}$, therefore $g_{\overline{f_{n+1}}} = g_{f_n}$. Thus γ_{n+1} maps $g_{f_n}^{\;-1}\, \widetilde{f_n}$ to $g_{f_n}^{\;-1}\, \overline{\widetilde{f_n}}$, hence belongs to $\rho_{f_n}(G_{f_n}) = \rho_{\overline{f_{n+1}}}(G_{f_{n+1}})$, a contradiction by the assumptions on the random walk.

By construction, the element α_{n+1} of Γ sends the above length 2 geodesic edge path $(g_{f_n}^{\;-1}\, \widetilde{f_n},\ \gamma_{n+1}\, g_{\overline{f_{n+1}}}^{\;-1}\, \widetilde{f_{n+1}})$ to (e_n, e_{n+1}). This implies on the one hand that the edge path (e_n, e_{n+1}) is geodesic, and on the other hand that α_{n+1} is uniquely defined, since the action of Γ on $\mathbb{X}$ is 2-acylindrical.

In particular, $(e_n)_{n\in\mathbb{N}}$ is the sequence of edges followed by a (discrete) geodesic ray in $\mathbb{X}$, starting from a point of $\mathbb{X}'$ but not by an edge of $\mathbb{X}'$, that is, an element of $\partial^1_+\mathbb{X}'$. Furthermore, this ray is well defined up to the action of $\Gamma'_{\widetilde{y_0}}$, hence its image, that we denote by $\Psi(\omega)$, is well defined in $\Gamma\backslash\breve{\mathscr{G}}\mathbb{X}$. Since $p(o(e_n)) = p(\widetilde{y_n}) = y_n$ for all $n \in \mathbb{N}$, the commutativity of the diagram (10.11) is immediate.

For every $x \in V\mathbb{X}'$, let $\partial^1_+\mathbb{X}'(x)$ be the subset of $\partial^1_+\mathbb{X}'$ consisting of the elements w with $w(0) = x$. By construction, the above map from the subset of random paths in Ω starting from y_0 to $\Gamma'_{\widetilde{y_0}}\backslash\partial^1_+\mathbb{X}'(\widetilde{y_0})$, which associates to $(X_n)_{n\in\mathbb{N}}$ the $\Gamma'_{\widetilde{y_0}}$-orbit of the geodesic ray with consecutive edges $(e_n)_{n\in\mathbb{N}}$, is clearly a bijection. This bijection maps the measure $\mathbb{P}$ to the normalised skinning measure $\frac{\sigma^+_{\mathscr{D}}}{\|\sigma^+_{\mathscr{D}}\|}$, since by homogeneity, the restriction to $\partial^1_+\mathbb{X}'(\widetilde{y_0})$ of $\widetilde{\sigma}^+_{\mathbb{X}'}$ to $\partial^1_+\mathbb{X}'(\widetilde{y_0})$, normalised to be a probability measure, is the restriction to $\partial^1_+\mathbb{X}'(\widetilde{y_0})$ of the $\operatorname{Aut}(\mathbb{X})_x$-homogeneous probability measure on the space of geodesic rays with origin x in the regular tree $\mathbb{X}$. This proves the result. □

When $\mathbb{Y}$ is finite, all the groups G_y for $y \in V\mathbb{Y}$ are trivial and $\mathbb{Y}'$ is reduced to a vertex,[8] the above random walk is the non-backtracking simple random walk on the nonbipartite regular finite graph $\mathbb{Y}$, and $\frac{1}{\operatorname{Vol}(\mathbb{Y},G_*)}\operatorname{vol}_{\mathbb{Y},G_*}$ is the uniform distribution on $V\mathbb{Y}$. Hence this result (stated as Corollary 1.3 in the Introduction) is classical. See for instance [OW, Theo. 1.2] and [AloBLS], which under further assumptions on the spectral properties of $\mathbb{Y}$ gives precise rates of convergence, and also the book [LyP2], including its Section 6.3 and its references.

10.4 Rate of equidistribution for metric and simplicial trees

In this Section, we give error terms for the equidistribution results stated in Theorem 10.2 for metric trees, and in Theorem 10.4 for simplicial trees, under additional assumptions required in order to get the error terms for the mixing property discussed in Chapter 9.

We first consider the simplicial case, for the discrete time geodesic flow. Let $\mathbb{X}$, X, Γ, $\widetilde{c}$, c, $\widetilde{F}_c$, F_c, $\delta_c < +\infty$, $(\mu^\pm_x)_{x\in V\mathbb{X}}$, $\widetilde{m}_c$, m_c be as in the beginning of Section 9.2.

Theorem 10.8. *Assume that δ_c is finite and that the Gibbs measure m_c on $\Gamma\backslash\mathscr{G}\mathbb{X}$ is finite. Assume furthermore that*

(1) *the families $(\Lambda\Gamma, \mu^-_x, d_x)_{x\in V\mathscr{C}\Lambda\Gamma}$ and $(\Lambda\Gamma, \mu^+_x, d_x)_{x\in V\mathscr{C}\Lambda\Gamma}$ of metric measure spaces are uniformly doubling,*[9]

(2) *there exists $\alpha \in \,]0,1]$ such that the discrete time geodesic flow on $(\Gamma\backslash\mathscr{G}\mathbb{X}, m_c)$ is exponentially mixing for the α-Hölder regularity.*

Let $\mathscr{D} = (D_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper simplicial subtrees of $\mathbb{X}$ with $\Gamma\backslash I$ finite. Let $\Omega = (\Omega_i)_{i\in I}$ be a locally finite Γ-equivariant family of measurable subsets of $\breve{\mathscr{G}}\mathbb{X}$, with $\Omega_i \subset \partial^1_\pm D_i$ for all $i \in I$ and $\Omega_i = \Omega_j$ if $i \sim_{\mathscr{D}} j$. Assume that $\sigma^\pm_\Omega$ is finite and positive. Then there exists $\kappa' > 0$ such that for all $\psi \in \mathscr{C}^\alpha_{\rm c}(\Gamma\backslash\breve{\mathscr{G}}\mathbb{X})$, we have, as $n \to +\infty$,

$$\frac{1}{\|(\mathsf{g}^{\pm n})_*\sigma^\pm_\Omega\|}\int \psi\, d\,\pi_*(\mathsf{g}^{\pm n})_*\sigma^\pm_\Omega \;=\; \frac{1}{\|m_c\|}\int \psi\, d\,m_c + \operatorname{O}(\|\psi\|_\alpha\, e^{-\kappa' n})\,.$$

Remarks. (1) If $\widetilde{c} = 0$, if the simplicial subtree $\mathbb{X}'$ of $\mathbb{X}$ satisfying $|\mathbb{X}'|_1 = \mathscr{C}\Lambda\Gamma$ is uniform, if $L_\Gamma = \mathbb{Z}$ and if Γ is a lattice of $\mathbb{X}'$, then we claim that $\delta_c = \delta_\Gamma$ is finite, $m_c = m_{\rm BM}$ is finite and mixing, and $(\Lambda\Gamma, \mu_x = \mu^\pm_x, d_x)_{x\in V\mathscr{C}\Lambda\Gamma}$ is uniformly doubling.

[8] The result for general $\mathbb{Y}'$ follows by averaging.
[9] See Section 4.1 for definitions.

Indeed, the above finiteness and mixing properties follow from the results of Section 4.4. Since $\mathbb{X}'$ is uniform, it has a cocompact discrete group of isometries Γ' whose Patterson density (for the vanishing potential) is uniformly doubling on $\Lambda\Gamma' = \Lambda\Gamma$, by Lemma 4.3 (4). Since $\widetilde{c} = 0$ and Γ is a lattice, the Patterson densities of Γ and of Γ' coincide (up a scalar multiple) by Proposition 4.16 (2).

(2) Assume that $\widetilde{c} = 0$, that the simplicial subtree $\mathbb{X}'$ of $\mathbb{X}$ satisfying $|\mathbb{X}'|_1 = \mathscr{C}\Lambda\Gamma$ is uniform without vertices of degree 2, that $L_\Gamma = \mathbb{Z}$ and that Γ is a geometrically finite lattice of $\mathbb{X}$. Then all assumptions of Theorem 10.8 are satisfied by the first remark and by Corollary 9.6. Therefore we have an exponentially small error term in the equidistribution of the equidistant levels sets.

Proof. We only give the proof when $\pm = +$, the other case is treated similarly. We follow the proof of Theorem 10.2, concentrating on the new features. We now have $\Gamma\backslash I$ finite by assumption. Let $\eta \in \,]0, 1]$ and $\psi \in \mathscr{C}_c^\alpha(\Gamma\backslash\breve{\mathscr{G}}X)$. We consider the constant $R > 0$, the test function Φ_η, the fundamental domain Δ_Γ and the lift $\widetilde{\psi}$ of ψ as in the proof of Theorem 10.2.

For all $n \in \mathbb{N}$, all $w \in \mathscr{G}_+X$ isometric on $[-n, +\infty[$ and all $\ell \in V^+_{w,\,\eta,\,e^{-n}R} = B^+(w, e^{-n}R)$,[10] by Lemma 2.7 where we can take $\eta = 0$, we have $d(\ell, w) = \operatorname{O}(e^{-n})$. Since p is Lipschitz, the map $\widetilde{\psi}$ is α-Hölder-continuous with α-Hölder norm at most $\|\psi\|_\alpha$. Hence for all $n \in \mathbb{N}$, all $w \in \mathscr{G}_+X$ isometric on $[-n, +\infty[$ and all $\ell \in V^+_{w,\,\eta,\,e^{-n}R}$, we have

$$\widetilde{\psi}(\ell) = \widetilde{\psi}(w) + \operatorname{O}(e^{-n\,\alpha}\,\|\psi\|_\alpha)\;. \tag{10.13}$$

As in the proof of Theorem 10.2 with t replaced by n, using Equation (10.13) instead of Equation (10.7) in the series of equations (10.8), since the symbols w that appear in them are indeed generalised geodesic lines isometric on $[-n, +\infty[$, we have

$$\frac{\int_{\Gamma\backslash\breve{\mathscr{G}}\mathbb{X}}\;\psi\;d(\mathtt{g}^n)_*\sigma^+_\Omega}{\|(\mathtt{g}^n)_*\sigma^+_\Omega\|} = \frac{\int_{\Gamma\backslash\mathscr{G}\mathbb{X}}\;\psi\;\;\Phi_\eta\circ\mathtt{g}^{-n}\;dm_c}{\int_{\Gamma\backslash\mathscr{G}\mathbb{X}}\;\Phi_\eta\;dm_c} + \operatorname{O}(e^{-n\,\alpha}\;\|\psi\|_\alpha)\;. \tag{10.14}$$

Let us now apply the assumption on the decay of correlations. In order to do that, we need to regularise our test functions Φ_η.

By the definition of the Gibbs measures,[11] Lemma 3.3 implies that, for all $\epsilon > 0$ small enough and $\ell \in \mathscr{G}\mathbb{X}$,

$$\begin{aligned}
\mu^-_{\ell(0)}\big(B_{d_{\ell(0)}}(\ell(-\infty), \tfrac{1}{c_0}\,\sqrt{\epsilon}\,)\big)\;\mu^+_{\ell(0)}\big(B_{d_{\ell(0)}}(\ell(+\infty), \tfrac{1}{c_0}\,\sqrt{\epsilon}\,)\big)&\\
\leqslant \widetilde{m}_c(B_d(\ell, \epsilon))&\\
\leqslant \mu^-_{\ell(0)}\big(B_{d_{\ell(0)}}(\ell(-\infty), c_0\,\sqrt{\epsilon}\,)\big)\;\mu^+_{\ell(0)}\big(B_{d_{\ell(0)}}(\ell(+\infty), c_0\,\sqrt{\epsilon}\,)\big)\;.&
\end{aligned}$$

Since the Patterson densities are uniformly doubling for basepoints in $\mathscr{C}\Lambda\Gamma$, since the footpoints of the geodesic lines in the support of $\widetilde{m}_c$ belong to $\mathscr{C}\Lambda\Gamma$, the Gibbs measure m_c is hence doubling on its support. Let $\overline{m}_c = \frac{m_c}{\|m_c\|}$. As in the proof of Theorem 10.3, using discrete convolution approximation, there exists $\kappa'' > 0$ and, for every $\eta > 0$, a nonnegative function $\mathtt{R}\Phi_\eta \in \mathrm{C}^\alpha_c(\Gamma\backslash\mathscr{G}\mathbb{X})$ such that

[10] As said in Section 2.6, the subsets $V^+_{w,\,\eta,\,s}$ and $B^+(w, s)$ of the space of discrete geodesic lines $\mathscr{G}\mathbb{X}$ are equal for every $s > 0$ since $\mathbb{X}$ is simplicial and $\eta < 1$.

[11] See Equation (4.4), using $x_0 = \ell(0)$ as the basepoint for the Hopf parametrisation, and the fact that if $\epsilon > 0$ is small enough and $\ell' \in B_d(\ell, \epsilon)$, then $\ell'(0) = \ell(0)$ as seen in the proof of Lemma 3.3.

(1) $\int_{\Gamma\backslash\mathscr{G}\mathbb{X}} \mathtt{R}\Phi_\eta \, d\,\overline{m}_c = \int_{\Gamma\backslash\mathscr{G}\mathbb{X}} \Phi_\eta \, d\,\overline{m}_c$,

(2) $\int_{\Gamma\backslash\mathscr{G}\mathbb{X}} |\,\mathtt{R}\Phi_\eta - \Phi_\eta| \, d\,\overline{m}_c = \operatorname{O}\big(\eta \int_{\Gamma\backslash\mathscr{G}\mathbb{X}} \Phi_\eta \, d\,\overline{m}_c\big)$,

(3) $\|\,\mathtt{R}\Phi_\eta\|_\alpha = \operatorname{O}\big(\eta^{-\kappa''} \int_{\Gamma\backslash\mathscr{G}\mathbb{X}} \Phi_\eta \, d\,\overline{m}_c\big)$.

By Equation (10.6), the integral $\int_{\Gamma\backslash\mathscr{G}\mathbb{X}} \Phi_\eta \, dm_c = \|\sigma^+_\Omega\|$ is constant (in particular independent of η). All integrals below besides the first one being over $\Gamma\backslash\mathscr{G}\mathbb{X}$, and using

- Equation (10.14) and the above property (2) of the regularised map $\mathtt{R}\Phi_\eta$ for the first equality,
- the assumption of exponential decay of correlations for the second one, involving some constant $\kappa > 0$, for the second equality,
- the above properties (1) and (3) of the regularised map $\mathtt{R}\Phi_\eta$ for the last equality,

we hence have

$$\begin{aligned}
\frac{\int_{\Gamma\backslash\breve{\mathscr{G}}\mathbb{X}} \psi \, d(\mathtt{g}^n)_*\sigma^+_\Omega}{\|(\mathtt{g}^n)_*\sigma^+_\Omega\|} &= \frac{\int \psi \;\mathtt{R}\Phi_\eta\circ\mathtt{g}^{-n}\, d\,\overline{m}_c}{\int \Phi_\eta \, d\,\overline{m}_c} + \operatorname{O}(e^{-n\,\alpha}\,\|\psi\|_\alpha + \eta\,\|\psi\|_\infty) \\
&= \frac{\int \mathtt{R}\Phi_\eta \, d\,\overline{m}_c \int \psi \, d\,\overline{m}_c}{\int \Phi_\eta \, d\,\overline{m}_c} + \operatorname{O}(e^{-n\,\alpha}\,\|\psi\|_\alpha + \eta\,\|\psi\|_\infty + \\
&\qquad \frac{1}{\int \Phi_\eta \, d\,\overline{m}_c}\, e^{-\kappa\, n}\|\,\mathtt{R}\Phi_\eta\|_\alpha\|\psi\|_\alpha) \\
&= \int \psi \, d\,\overline{m}_c + \operatorname{O}\big((e^{-n\,\alpha} + \eta + e^{-\kappa\, n}\,\eta^{-\kappa''})\|\psi\|_\alpha\big)\;.
\end{aligned}$$

Taking $\eta = e^{-\lambda\, n}$ with $\lambda = \frac{\kappa}{2\kappa''}$, the result follows with $\kappa' = \min\{\alpha, \frac{\kappa}{2\kappa''}, \frac{\kappa}{2}\}$. □

Let us now consider the metric tree case, for the continuous time geodesic flow, where the main change is to assume a superpolynomial decay of correlations and hence get a superpolynomial error term, for observables which are smooth enough along the flow lines. Let $(\mathbb{X}, \lambda)$, X, Γ, $\widetilde{F}$, $\delta = \delta_{\Gamma,\,F^\pm} < \infty$, $(\mu^\pm_x)_{x\in X}$, $\widetilde{m}_F$ and m_F be as in the beginning of Section 9.3.

Theorem 10.9. *Assume that the Gibbs measure m_F on $\Gamma\backslash\mathscr{G}X$ is finite. Assume furthermore that*

(1) *the families $(\Lambda\Gamma, \mu^-_x, d_x)_{x\in\mathscr{C}\Lambda\Gamma}$ and $(\Lambda\Gamma, \mu^+_x, d_x)_{x\in\mathscr{C}\Lambda\Gamma}$ of metric measure spaces are uniformly doubling,[12] and $\widetilde{F}$ is bounded on $T^1\mathscr{C}\Lambda\Gamma$,*

(2) *there exists $\alpha \in\,]0,1]$ such that the (continuous time) geodesic flow on $(\Gamma\backslash\mathscr{G}X, m_F)$ has superpolynomial decay of α-Hölder correlations.*

Let $\mathscr{D} = (D_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of X with $\Gamma\backslash I$ finite. Let $\Omega = (\Omega_i)_{i\in I}$ be a locally finite Γ-equivariant family of measurable subsets of $\breve{\mathscr{G}}X$, with $\Omega_i \subset \partial^1_\pm D_i$ for all $i \in I$ and $\Omega_i = \Omega_j$ if $i \sim_{\mathscr{D}} j$. Assume that $\sigma^\pm_\Omega$ is finite and nonzero. Then for every $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that for all $\psi \in \mathscr{C}^{k,\,\alpha}_{\rm c}(\Gamma\backslash\breve{\mathscr{G}}X)$, we have, as $t \to +\infty$,

$$\frac{1}{\|(\mathtt{g}^{\pm t})_*\sigma^\pm_\Omega\|}\int \psi \, d\,\pi_*(\mathtt{g}^{\pm t})_*\sigma^\pm_\Omega \;=\; \frac{1}{\|m_F\|}\int \psi \, d\,m_F + \operatorname{O}(\|\psi\|_{k,\,\alpha}\, t^{-n})\;.$$

[12] See Section 4.1 for definitions.

Remarks. (1) If $F=0$, if the metric subtree $X'=\mathscr{C}\Lambda\Gamma$ of X is uniform, if the length spectrum of Γ on X is not contained in a discrete subgroup of $\mathbb{R}$ and if Γ is a lattice of X', then we claim that $\delta=\delta_\Gamma$ is finite, $m_F=m_{\rm BM}$ is finite and mixing, and $(\Lambda\Gamma,\mu_x=\mu^\pm_x,d_x)_{x\in X'}$ is uniformly doubling.

Indeed, the above finiteness and mixing properties follow from Proposition 4.16 and Theorem 4.9. Since X' is uniform, it has a cocompact discrete sugbroup of isometries Γ' whose Patterson density (for the vanishing potential) is uniformly doubling on $\Lambda\Gamma'=\Lambda\Gamma$, by Lemma 4.3 (4). Since $F=0$ and Γ is a lattice, the Patterson densities of Γ and of Γ' coincide (up to a scalar multiple) by Proposition 4.16.

(2) Assume that $F=0$, that the metric subtree $X'=\mathscr{C}\Lambda\Gamma$ of X is uniform, that the length spectrum of Γ on X is 4-Diophantine and that Γ is a geometrically finite lattice of X'. Then all assumptions of Theorem 10.9 are satisfied by the first remark and by Corollary 9.10. Therefore we have a superpolynomially small error term in the equidistribution of the equidistant levels sets.

Proof. The proof is similar to the one of Theorem 10.8, except for the doubling property of the Gibbs measure on its support and the conclusion of the proof. Let $X'=\mathscr{C}\Lambda\Gamma$. The modification of Lemma 3.3 used in the previous proof is now the third assertion of Lemma 3.4.

If the footpoints of $\ell,\ell'\in\mathscr{G}X'$ are at distance bounded by $c_0\,\epsilon_0$, then by Proposition 3.20 (2), since $|\widetilde{F}|$ is bounded on T^1X' by assumption, the quantities $|C^\pm_\xi(\ell(0),\ell'(0))|$ for $\xi\in\Lambda\Gamma$ are bounded by the constant $c'_0=c_0\,\epsilon_0\,(\max_{T^1X'}|\widetilde{F}-\delta|)$. By the definition of the Gibbs measures (see Equation (4.4)), Assertion (3) of Lemma 3.4 hence implies that if $\epsilon\leqslant\epsilon_0$ then for every $\ell\in\mathscr{G}\mathbb{X}$,

$$\begin{aligned}e^{-2c'_0}\,\epsilon\;&\mu^-_{\ell(0)}\big(B_{d_{\ell(0)}}(\ell(-\infty),\frac{1}{c_0}\sqrt{\epsilon}\,)\big)\;\mu^+_{\ell(0)}\big(B_{d_{\ell(0)}}(\ell(+\infty),\frac{1}{c_0}\sqrt{\epsilon}\,)\big)\\ &\leqslant\widetilde{m}_c(B_d(\ell,\epsilon))\\ &\leqslant e^{2c'_0}\,\epsilon\;\mu^-_{\ell(0)}\big(B_{d_{\ell(0)}}(\ell(-\infty),c_0\sqrt{\epsilon}\,)\big)\;\mu^+_{\ell(0)}\big(B_{d_{\ell(0)}}(\ell(+\infty),c_0\sqrt{\epsilon}\,)\big)\;.\end{aligned}$$

As in the simplicial case, since the Patterson densities are uniformly doubling for basepoints in X', the Gibbs measure m_c is hence doubling on its support.

Fix $n\in\mathbb{N}$. As in the end of the proof of Theorem 10.8, using the assumption of superpolynomial decay of correlations, involving some degree of regularity k in order to have polynomial decay in t^{-Nn} where $N=\lceil\kappa''\rceil+1$, instead of the exponential one, we have, for all $t\geqslant1$ and $\psi\in\mathscr{C}^{k,\alpha}_c(\Gamma\backslash\breve{\mathscr{G}}X)$,

$$\frac{\int_{\Gamma\backslash\breve{\mathscr{G}}\mathbb{X}}\psi\;d(\mathrm{g}^t)_*\sigma^+_\Omega}{\|(\mathrm{g}^t)_*\sigma^+_\Omega\|}=\int\psi\;d\,\overline{m}_c+\mathrm{O}\big((e^{-t\,\alpha}+\eta+t^{-Nn}\;\eta^{-\kappa''})\|\psi\|_{k,\,\alpha}\big)\;.$$

Taking $\eta=t^{-n}$, by the definition of N, we hence have

$$\frac{\int_{\Gamma\backslash\breve{\mathscr{G}}\mathbb{X}}\psi\;d(\mathrm{g}^t)_*\sigma^+_\Omega}{\|(\mathrm{g}^t)_*\sigma^+_\Omega\|}=\int\psi\;d\,\overline{m}_c+\mathrm{O}\big(t^{-n}\;\|\psi\|_{k,\,\alpha}\big)\;.$$

This proves Theorem 10.9. □

Chapter 11

Equidistribution of common perpendicular arcs

In this Chapter, we prove the equidistribution of the initial and terminal vectors of common perpendiculars of convex subsets, at the universal covering space level, for Riemannian manifolds and for metric and simplicial trees. The results generalise [PaP17b, Theo. 8].

From now untill the end of Section 11.3, we consider the continuous time situation where X is a proper CAT(-1)-space which is either an $\mathbb{R}$-tree without terminal point or a complete Riemannian manifold with pinched negative curvature at most -1. In Section 11.4, X will be the geometric realisation of a simplicial tree $\mathbb{X}$, and we will consider the discrete time geodesic flow.

Let Γ be a nonelementary discrete group of isometries of X. Let x_0 be any basepoint in X. Let $\widetilde{F}$ be a continuous Γ-invariant potential on T^1X, which is Hölder-continuous if X is a manifold. Assume that $\delta = \delta_{\Gamma,\, F^\pm}$ is finite and positive and let $(\mu^\pm_x)_{x\in X}$ be (normalised) Patterson densities for the pairs $(\Gamma, F^\pm)$, with associated Gibbs measure m_F. Let $\mathscr{D}^- = (D^-_i)_{i\in I^-}$ and $\mathscr{D}^+ = (D^+_j)_{j\in I^+}$ be locally finite Γ-equivariant families of nonempty proper closed convex subsets of X.

For every (i,j) in $I^-\times I^+$ such that the closures $\overline{D^-_i}$ and $\overline{D^+_j}$ of D^-_i and D^+_j in $X\cup\partial_\infty X$ have empty intersection, let $\lambda_{i,j} = d(D^-_i, D^+_j)$ be the length of the common perpendicular from D^-_i to D^+_j, and let $\alpha^-_{i,\,j}\in\breve{\mathscr{G}}X$ be its parametrisation: it is the unique map from $\mathbb{R}$ to X such that

- $\alpha^-_{i,\,j}(t) = \alpha^-_{i,\,j}(0)\in D^-_i$ if $t\leqslant 0$,
- $\alpha^-_{i,\,j}(t) = \alpha^-_{i,\,j}(\lambda_{i,\,j})\in D^+_j$ if $t\geqslant \lambda_{i,\,j}$, and
- $\alpha^-_{i,j}|_{[0,\,\lambda_{i,j}]} = \alpha_{i,\,j}$ is the shortest geodesic arc starting from a point of D^-_i and ending at a point of D^+_j.

Let $\alpha^+_{i,j} = \mathsf{g}^{\lambda_{i,j}}\alpha^-_{i,j}$. In particular, we have $\mathsf{g}^{\frac{\lambda_{i,j}}{2}}\alpha^-_{i,j} = \mathsf{g}^{\frac{-\lambda_{i,j}}{2}}\alpha^+_{i,j}$.

We now state our main equidistribution result of common perpendiculars between convex subsets in the continuous time and upstairs settings. We will give the discrete time version in Section 11.4, and the downstairs version in Chapter 12.

Theorem 11.1. *Let X be either a proper $\mathbb{R}$-tree without terminal points or a complete simply connected Riemannian manifold with pinched negative curvature at most -1. Let Γ be a nonelementary discrete group of isometries of X and let $\widetilde{F}$ be a bounded Γ-invariant potential* *on X which is Hölder-continuous if X is a manifold. Let $\mathscr{D}^\pm = (D_k^\pm)_{k\in I^\pm}$ be locally finite Γ-equivariant families of nonempty proper closed convex subsets of X. Assume that the critical exponent δ is positive,*[1] *and that the Gibbs measure m_F is finite and mixing for the geodesic flow on $\Gamma\backslash\mathscr{G}X$. Then*

$$\lim_{t\to+\infty}\ \delta\,\|m_F\|\,e^{-\delta\,t}\sum_{\substack{i\in I^-/_\sim,\ j\in I^+/_\sim,\ \gamma\in\Gamma\\ \overline{D}_i^-\cap\overline{D}_{\gamma j}^+=\varnothing,\ \lambda_{i,\,\gamma j}\leqslant t}} e^{\int_{\alpha_{i,\gamma j}}\widetilde{F}}\ \Delta_{\alpha^-_{i,\,\gamma j}}\otimes\Delta_{\alpha^+_{\gamma^{-1}i,\,j}}\ =\ \widetilde{\sigma}^+_{\mathscr{D}^-}\otimes\widetilde{\sigma}^-_{\mathscr{D}^+}$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}X\times\breve{\mathscr{G}}X$.

The proof of Theorem 11.1 follows that of [PaP17b, Theo. 8], which proves this result when X is a manifold and $F=0$. The first two and a half steps work for both trees and manifolds and are given in Section 11.1. The differences begin in Step 3T. After this, the steps for trees are called $3T$ and $4T$ and are given in Section 11.2 and the corresponding steps for manifolds are $3M$ and $4M$, given in Section 11.3.

In the special case of $\mathscr{D}^-=(\gamma x)_{\gamma\in\Gamma}$ and $\mathscr{D}^+=(\gamma y)_{\gamma\in\Gamma}$ for some $x,y\in X$, this statement,[2] with Equation (7.2), gives the following version with potentials of Roblin's double equidistribution theorem [Rob2, Theo. 4.1.1] when $F=0$, see [PauPS, Theo. 9.1] for general F when X is a Riemannian manifold with pinched sectional curvature at most -1.

Corollary 11.2. *Let X, Γ, $\widetilde{F}$, δ, m_F be as in Theorem 11.1, and let $x,y\in X$. We have*

$$\lim_{t\to+\infty}\ \delta\,\|m_F\|\,e^{-\delta\,t}\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\leqslant t} e^{\int_x^{\gamma y}\widetilde{F}}\ \Delta_{\gamma y}\otimes\Delta_{\gamma^{-1}x}\ =\ \mu_x^+\otimes\mu_y^-$$

for the weak-star convergence of measures on the compact space $(X\cup\partial_\infty X)\times(X\cup\partial_\infty X)$. □

Let us give a version of Theorem 11.1 without the assumption $\delta>0$.

Theorem 11.3. *Let X, Γ, $\widetilde{F}$, δ, m_F be as in Theorem 11.1, except that the critical exponent δ is not assumed to be positive. Then for every $\tau>0$, we have*

$$\lim_{t\to+\infty}\ \frac{\delta\,\|m_F\|}{1-e^{-\tau\,\delta}}\,e^{-\delta\,t}\sum_{\substack{i\in I^-/_\sim,\ j\in I^+/_\sim,\ \gamma\in\Gamma\\ \overline{D}_i^-\cap\overline{D}_{\gamma j}^+=\varnothing,\ t-\tau<\lambda_{i,\,\gamma j}\leqslant t}} e^{\int_{\alpha_{i,\gamma j}}\widetilde{F}}\ \Delta_{\alpha^-_{i,\,\gamma j}}\otimes\Delta_{\alpha^+_{\gamma^{-1}i,\,j}}\ =\ \widetilde{\sigma}^+_{\mathscr{D}^-}\otimes\widetilde{\sigma}^-_{\mathscr{D}^+}$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}X\times\breve{\mathscr{G}}X$.

Proof. The key ingredient in order to deduce Theorem 11.3 from Theorem 11.1 is the following classical lemma (see [PauPS, Lem. 9.5] for a proof).

Lemma 11.4. *Let I be a discrete set and let $f,g:I\to[0,+\infty[$ be maps with f proper. If $\delta+\kappa>0$ and, as $t\to+\infty$,*

$$\sum_{i\in I,\ f(i)\leqslant t} e^{\kappa\,f(i)}g(i)\ \sim\ \frac{e^{(\delta+\kappa)t}}{\delta+\kappa}\,,$$

[1] It is finite since $\widetilde{F}$ is bounded, see Lemma 3.17 (6).

[2] or rather the following Equation (11.1)

then for every $c > 0$, as $t \to +\infty$,

$$\sum_{i \in I,\, t-c < f(i) \leqslant t} g(i) \;\sim\; \frac{1-e^{-c\,\delta}}{\delta}\, e^{\delta t}\,. \quad \Box$$

Let $\kappa > 0$ be such that $\delta_{\Gamma,\,F+\kappa} = \delta_{\Gamma,\,F} + \kappa > 0$. As the definition of the Gibbs measure only involves the normalised potential, we have $\|m_{F+\kappa}\| = \|m_F\|$. Thus, the statement of Theorem 11.1 for the potential $F+\kappa$ is equivalent to the claim that, as $t \to +\infty$,

$$\sum_{\substack{i \in I^-/_\sim,\; j \in I^+/_\sim,\; \gamma \in \Gamma \\ \overline{D_i^-} \cap \overline{D_{\gamma j}^+} = \varnothing,\; \lambda_{i,\,\gamma j} \leqslant t}} e^{\kappa \lambda_{i,\,\gamma j}}\, e^{\int_{\alpha_{i,\gamma j}} \widetilde F}\, \psi(\alpha^-_{i,\,\gamma j}, \alpha^+_{\gamma^{-1} i,\, j}) \;\sim\; \frac{e^{(\delta+\kappa)\,t}}{(\delta+\kappa)\,\|m_F\|} \int \psi\; d(\widetilde\sigma^+_{\mathscr{D}^-} \otimes \widetilde\sigma^-_{\mathscr{D}^+})$$

for all positive functions $\psi \in \mathscr{C}_{\rm c}(\partial_{\!\!\!\smile}\mathscr{G}X \times \partial_{\!\!\!\smile}\mathscr{G}X)$. Since the function $(i,j,\gamma) \mapsto \lambda_{i,\gamma j}$ is proper by the local finiteness of the families $\mathscr{D}^-$ and $\mathscr{D}^+$, Lemma 11.4 implies that for every $\tau > 0$, as $t \to +\infty$,

$$\sum_{\substack{i \in I^-/_\sim,\; j \in I^+/_\sim,\; \gamma \in \Gamma \\ \overline{D_i^-} \cap \overline{D_{\gamma j}^+} = \varnothing,\; t-\tau < \lambda_{i,\,\gamma j} \leqslant t}} e^{\int_{\alpha_{i,\gamma j}} \widetilde F}\, \psi(\alpha^-_{i,\,\gamma j}, \alpha^+_{\gamma^{-1} i,\, j}) \;\sim\; e^{\delta\,t}\,\frac{1-e^{\tau\,\delta}}{\delta\,\|m_F\|} \int \psi\; d(\widetilde\sigma^+_{\mathscr{D}^-} \otimes \widetilde\sigma^-_{\mathscr{D}^+})\,,$$

which yields Theorem 11.3. $\Box$

11.1 Part I of the proof of Theorem 11.1: the common part

Step 1: Reduction. By additivity, by the local finiteness of the families $\mathscr{D}^\pm$, and by the definition of $\widetilde\sigma^\pm_{\mathscr{D}^\mp} = \sum_{k \in I^\mp/_\sim} \widetilde\sigma^\pm_{D_k^\mp}$, we only have to prove, for all fixed $i \in I^-$ and $j \in I^+$, that

$$\lim_{t \to +\infty} \delta\,\|m_F\|\, e^{-\delta\,t} \sum_{\gamma \in \Gamma\,:\, 0 < \lambda_{i,\,\gamma j} \leqslant t} e^{\int_{\alpha_{i,\gamma j}} \widetilde F}\, \Delta_{\alpha^-_{i,\,\gamma j}} \otimes \Delta_{\alpha^+_{\gamma^{-1} i,\, j}} \;=\; \widetilde\sigma^+_{D_i^-} \otimes \widetilde\sigma^-_{D_j^+} \tag{11.1}$$

for the weak-star convergence of measures on $\partial_{\!\!\!\smile}\mathscr{G}X \times \partial_{\!\!\!\smile}\mathscr{G}X$.

Let Ω^- be a Borel subset of $\partial^1_+ D_i^-$ and let Ω^+ be a Borel subset of $\partial^1_- D_j^+$. In order to simplify the notation, let

$$D^- = D_i^-,\;\; D^+ = D_j^+,\;\; \alpha_\gamma = \alpha_{i,\gamma j},\;\; \alpha^-_\gamma = \alpha^-_{i,\,\gamma j},\;\; \alpha^+_\gamma = \alpha^+_{\gamma^{-1} i,\, j},$$
$$\lambda_\gamma = \lambda_{i,\,\gamma j},\;\; \widetilde\sigma^\pm = \widetilde\sigma^\pm_{D^\mp}\,. \tag{11.2}$$

Assume that Ω^- and Ω^+ have positive finite skinning measures and that their boundaries in $\partial^1_+ D^-$ and $\partial^1_- D^+$ have zero skinning measures (for $\widetilde\sigma^+$ and $\widetilde\sigma^-$ respectively). Let

$$I_{\Omega^-,\,\Omega^+}(t) = \delta\,\|m_F\|\, e^{-\delta\,t} \sum_{\substack{\gamma \in \Gamma\,:\, 0 < \lambda_\gamma \leqslant t \\ \alpha^-_\gamma|_{]0,\lambda_\gamma]} \in \Omega^-|_{]0,\lambda_\gamma]},\;\; \alpha^+_\gamma|_{]-\lambda_\gamma,0]} \in \Omega^+|_{]-\lambda_\gamma,0]}}} e^{\int_{\alpha_\gamma} \widetilde F}\,. \tag{11.3}$$

We will prove the stronger statement, implying Equation (11.1) by restricting to $\Omega^\pm$ compact, and useful in this generality for Chapter 12, that, for every such $\Omega^\pm$, we have

$$\lim_{t \to +\infty} I_{\Omega^-,\,\Omega^+}(t) \;=\; \widetilde\sigma^+(\Omega^-)\,\widetilde\sigma^-(\Omega^+)\,. \tag{11.4}$$

Step 2: First upper and lower bounds. Using Lemma 7.6 (1), we may fix $R > e^2$ such that $\nu_w^{\pm}(V^{\mp}_{w,\,\eta,\,R}) > 0$ for all $\eta \in \,]0,1]$ and $w \in \partial^1_{\mp} D^{\pm}$. Let $\phi^{\mp}_\eta = \phi^{\mp}_{\eta,\,R,\,\Omega^{\pm}}$ be the test functions defined in Equation (10.4).

For all $t \geqslant 0$, let

$$a_\eta(t) = \sum_{\gamma\in\Gamma} \int_{\ell\in\mathscr{G}X} \phi^-_\eta(\mathsf{g}^{-t/2}\ell)\ \phi^+_\eta(\mathsf{g}^{t/2}\gamma^{-1}\ell)\ d\widetilde{m}_F(\ell)\,. \tag{11.5}$$

As in [PaP17b], the heart of the proof is to give two pairs of upper and lower bounds, as $T \geqslant 0$ is large enough and $\eta \in \,]0,1]$ is small enough, of the (Cesàro-type) quantity

$$i_\eta(T) = \int_0^T e^{\delta\, t}\ a_\eta(t)\ dt\,. \tag{11.6}$$

By passing to the universal cover, the mixing property of the geodesic flow on $\Gamma\backslash\mathscr{G}X$ for the Gibbs measure m_F gives that, for every $\epsilon > 0$, there exists $T_\epsilon = T_{\epsilon,\eta} \geqslant 0$ such that for all $t \geqslant T_\epsilon$, we have

$$\frac{e^{-\epsilon}}{\|m_F\|}\int_{\mathscr{G}X}\phi^-_\eta\ d\widetilde{m}_F\int_{\mathscr{G}X}\phi^+_\eta\ d\widetilde{m}_F \leqslant a_\eta(t) \leqslant \frac{e^{\epsilon}}{\|m_F\|}\int_{\mathscr{G}X}\phi^-_\eta\ d\widetilde{m}_F\int_{\mathscr{G}X}\phi^+_\eta\ d\widetilde{m}_F\,.$$

Hence by Lemma 10.1 (1), for all $\epsilon > 0$ and $\eta \in \,]0,1]$, there exists $c_\epsilon = c_{\epsilon,\eta} > 0$ such that for every $T \geqslant 0$, we have

$$e^{-\epsilon}\,\frac{e^{\delta\,T}}{\delta\,\|m_F\|}\,\widetilde{\sigma}^+(\Omega^-)\ \widetilde{\sigma}^-(\Omega^+) - c_\epsilon \leqslant i_\eta(T) \leqslant e^{\epsilon}\,\frac{e^{\delta\,T}}{\delta\,\|m_F\|}\,\widetilde{\sigma}^+(\Omega^-)\ \widetilde{\sigma}^-(\Omega^+) + c_\epsilon\,. \tag{11.7}$$

Step 3: Second upper and lower bounds. Let $T \geqslant 0$ and $\eta \in \,]0,1]$. By Fubini's theorem for nonnegative measurable maps, the definition[3] of the test functions $\phi^{\pm}_\eta$ and the flow-invariance[4] of the fibrations $f^{\pm}_{D^{\mp}}$, we have

$$\begin{aligned} i_\eta(T) = \sum_{\gamma\in\Gamma}\int_0^T e^{\delta\,t}\int_{\ell\in\mathscr{G}X}\ & h^-_{\eta,\,R}\circ f^+_{D^-}(\ell)\ h^+_{\eta,\,R}\circ f^-_{D^+}(\gamma^{-1}\ell) \\ & \mathbb{1}_{\mathsf{g}^{t/2}\mathscr{V}^+_{\eta,\,R}(\Omega^-)}(\ell)\ \mathbb{1}_{\mathsf{g}^{-t/2}\mathscr{V}^-_{\eta,\,R}(\gamma\Omega^+)}(\ell)\ d\widetilde{m}_F\ dt\,. \end{aligned} \tag{11.8}$$

We start the computations by rewriting the product term involving the functions $h^{\pm}_{\eta,\,R}$. For all $\gamma\in\Gamma$ and $\ell \in \mathscr{U}^+_{D^-}\cap\mathscr{U}^-_{\gamma D^+}$, define (using Equation (2.16))

$$w^- = f^+_{D^-}(\ell) \in \mathscr{G}_{+,\,0}X \quad\text{and}\quad w^+ = f^-_{\gamma D^+}(\ell) = \gamma f^-_{D^+}(\gamma^{-1}\ell) \in \mathscr{G}_{-,\,0}X\,. \tag{11.9}$$

This notation is ambiguous (w^- depends on ℓ, and w^+ depends on ℓ and γ), but it makes the computations less heavy. By Equations (10.2) and (3.20), we have, for every $t \geqslant 0$,

$$h^-_{\eta,\,R}(w^-) = h^-_{\eta,\,R}\circ\mathsf{g}^{-t/2}(\mathsf{g}^{t/2}w^-) = e^{\int_{w^-(0)}^{w^-(t/2)}(\widetilde{F}-\delta)}\ h^-_{\eta,\,e^{-t/2}R}(\mathsf{g}^{t/2}w^-)\,.$$

[3] See Equation (10.4).
[4] See Equation (2.16).

Similarly,

$$h^+_{\eta,\,R}(\gamma^{-1}w^+) = e^{\int_{w^+(-t/2)}^{w^+(0)}(\widetilde{F}-\delta)}\; h^+_{\eta,\,e^{-t/2}R}(\mathtt{g}^{-t/2}w^+)\,.$$

Hence,

$$\begin{aligned} h^-_{\eta,\,R}\circ f^+_{D^-}(\ell)\; h^+_{\eta,\,R}\circ f^-_{D^+}(\gamma^{-1}\ell) \\ = e^{-\delta\, t}\; e^{\int_{w^-(0)}^{w^-(t/2)}\widetilde{F}+\int_{w^+(-t/2)}^{w^+(0)}\widetilde{F}}\; h^-_{\eta,\,e^{-t/2}R}(\mathtt{g}^{t/2}w^-)\; h^+_{\eta,\,e^{-t/2}R}(\mathtt{g}^{-t/2}w^+)\,. \end{aligned} \tag{11.10}$$

11.2 Part II of the proof of Theorem 11.1: the metric tree case

In this Section, we assume that X is an $\mathbb{R}$-tree and we will consider the manifold case separately in Section 11.3.

Step 3T. Consider the product term in Equation (11.8) involving the characteristic functions. By Lemma 2.8 (applied by replacing D^+ by γD^+), there exists $t_0 \geqslant 2\ln R + 4$ such that for all $\eta \in \,]0,1]$, $t \geqslant t_0$, $\gamma \in \Gamma$ and $\ell \in \mathscr{G}X$, if $\mathbb{1}_{\mathtt{g}^{t/2}\mathscr{V}^+_{\eta,\,R}(\Omega^-)}(\ell)\; \mathbb{1}_{\mathtt{g}^{-t/2}\mathscr{V}^-_{\eta,\,R}(\gamma\Omega^+)}(\ell) \neq 0$, or equivalently by Equation (2.20) if

$$\ell \in \mathscr{V}^+_{\eta,\,e^{-t/2}R}(\mathtt{g}^{t/2}\Omega^-) \cap \mathscr{V}^-_{\eta,\,e^{-t/2}R}(\gamma\mathtt{g}^{-t/2}\Omega^+)\;,$$

then the following facts hold.

(i) By the convexity of $D^\pm$, we have $\ell \in \mathscr{U}^+_{D^-} \cap \mathscr{U}^-_{\gamma D^+}$.

(ii) By the definition[5] of $w^\pm$, we have $w^- \in \Omega^-$ and $w^+ \in \gamma\Omega^+$. The notation (w^-, w^+) here coincides with the notation (w^-, w^+) in Lemma 2.8.

(iii) There exists a common perpendicular α_γ from D^- to γD^+, whose length λ_γ satisfies

$$|\,\lambda_\gamma - t\,| \leqslant 2\eta\,, \tag{11.11}$$

whose origin is $\alpha^-_\gamma(0) = w^-(0)$, whose endpoint is $\gamma\,\alpha^+_\gamma(0) = w^+(0)$, such that the points $w^-(\frac{t}{2})$ and $w^+(-\frac{t}{2})$ are at distance at most η from $\ell(0) \in \alpha_\gamma$.

See the picture following Lemma 2.8 (replacing D^+ by γD^+). Hence, by Lemma 3.12 and since $\widetilde{F}$ is bounded,

$$e^{-2\eta\,\|F\|_\infty}\; e^{\int_{\alpha_\gamma}\widetilde{F}} \leqslant e^{\int_{w^-(0)}^{w^-(t/2)}\widetilde{F}+\int_{w^+(-t/2)}^{w^+(0)}\widetilde{F}} \leqslant e^{2\eta\,\|F\|_\infty}\; e^{\int_{\alpha_\gamma}\widetilde{F}}\,. \tag{11.12}$$

For all $\eta \in \,]0,1]$, $\gamma \in \Gamma$ and $T \geqslant t_0$, let

$$\mathscr{A}_{\eta,\gamma}(T) = \big\{(t,\ell) \in [t_0, T] \times \mathscr{G}X : \ell \in \mathscr{V}^+_{\eta,\,e^{-t/2}R}(\mathtt{g}^{t/2}\Omega^-) \cap \mathscr{V}^-_{\eta,\,e^{-t/2}R}(\gamma\mathtt{g}^{-t/2}\Omega^+)\big\}$$

and

$$j_{\eta,\,\gamma}(T) = \iint_{(t,\,\ell)\in\mathscr{A}_{\eta,\gamma}(T)} h^-_{\eta,\,e^{-t/2}R}(\mathtt{g}^{t/2}w^-)\; h^+_{\eta,\,e^{-t/2}R}(\mathtt{g}^{-t/2}w^+)\; dt\; d\widetilde{m}_F(\ell)\,.$$

[5] See Equation (11.9).

By the above, since the integral of a function is equal to the integral on any Borel set containing its support, and since the integral of a nonnegative function is nondecreasing in the integration domain, there hence exists $c_4 > 0$ such that for all $T \geqslant 0$ and $\eta \in]0,1]$, we have

$$i_\eta(T) \geqslant -c_4 + e^{-2\eta\|F\|_\infty} \sum_{\substack{\gamma\in\Gamma\,:\, t_0+2\leqslant\lambda_\gamma\leqslant T-2\eta \\ \alpha_\gamma^-|_{[0,\,\lambda_\gamma]}\in\Omega^-|_{[0,\,\lambda_\gamma]},\ \alpha_\gamma^+|_{[-\lambda_\gamma,\,0]}\in\Omega^+|_{[-\lambda_\gamma,\,0]}}} e^{\int_{\alpha_\gamma}\widetilde{F}}\ j_{\eta,\gamma}(T)\ ,$$

and similarly, for every $T' \geqslant T$ (later on, we will take T' to be $T + 4\eta$),

$$i_\eta(T) \leqslant c_4 + e^{2\eta\,\|F\|_\infty} \sum_{\substack{\gamma\in\Gamma\,:\, t_0+2\leqslant\lambda_\gamma\leqslant T+2\eta \\ \alpha_\gamma^-|_{[0,\,\lambda_\gamma]}\in\Omega^-|_{[0,\,\lambda_\gamma]},\ \alpha_\gamma^+|_{[-\lambda_\gamma,\,0]}\in\Omega^+|_{[-\lambda_\gamma,\,0]}}} e^{\int_{\alpha_\gamma}\widetilde{F}}\ j_{\eta,\gamma}(T')\ .$$

Step 4T: Conclusion. Let $\epsilon > 0$. Let $\gamma \in \Gamma$ be such that D^- and γD^+ do not intersect and the length of their common perpendicular satisfies $\lambda_\gamma \geqslant t_0 + 2$. Let us prove that if η is small enough and λ_γ is large enough,[6] then for every $T \geqslant \lambda_\gamma + 2\eta$, we have

$$1 - \epsilon \leqslant j_{\eta,\gamma}(T) \leqslant 1 + \epsilon\,. \tag{11.13}$$

This estimate proves the claim (11.4), as follows. For every $\epsilon > 0$, if $\eta > 0$ is small enough, we have

$$i_\eta(T + 2\eta) \geqslant -c_4 + e^{-2\eta\|F\|_\infty}(1-\epsilon)\Big(\frac{I_{\Omega^-,\Omega^+}(T)}{\delta\,\|m_F\|e^{-\delta\,T}} - \frac{I_{\Omega^-,\Omega^+}(t_0+2)}{\delta\,\|m_F\|e^{-\delta(t_0+2)}}\Big)$$

and by Equation (11.7)

$$i_\eta(T + 2\eta) \leqslant c_\epsilon + \frac{e^\epsilon\,\widetilde{\sigma}^+(\Omega^-)\,\widetilde{\sigma}^-(\Omega^+)}{\delta\,\|m_F\|\,e^{-\delta(T+2\eta)}}\,.$$

Thus, for η small enough,

$$\widetilde{\sigma}^+(\Omega^-)\,\widetilde{\sigma}^-(\Omega^+) \geqslant \frac{1-\epsilon}{e^\epsilon}\,I_{\Omega^-,\Omega^+}(T) + \mathrm{o}(1)$$

as $T \to +\infty$, which gives

$$\limsup_{T\to+\infty} I_{\Omega^-,\Omega^+}(T) \leqslant \widetilde{\sigma}^+(\Omega^-)\,\widetilde{\sigma}^-(\Omega^+).$$

The similar estimate for the lower limit proves the claim (11.4).

In order to prove the claim (11.13), let $\eta \in]0,1]$ and $T \geqslant \lambda_\gamma + 2\eta$. In order to simplify the notation, let

$$r_t = e^{-t/2}R,\quad w_t^- = \mathsf{g}^{t/2}w^- \quad\text{and}\quad w_t^+ = \mathsf{g}^{-t/2}w^+\,.$$

By the definition of $j_{\eta,\gamma}$, using the inequalities (10.3) where the constant c_1 is uniform since $\widetilde{F}$ is bounded (with the comment following them) and the fact that $r_t = \mathrm{O}(e^{-\lambda_\gamma/2})$ by Equation

[6] with the enough's independent of γ

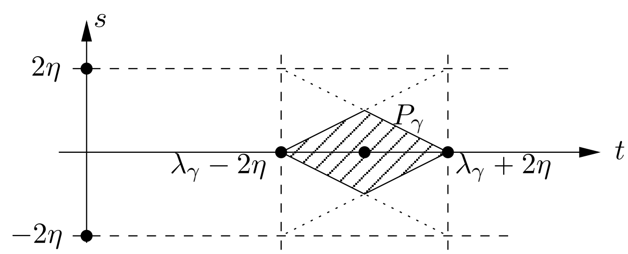
(11.11), we hence have

$$j_{\eta,\gamma}(T) = \iint_{(t,\ell)\in\mathscr{A}_{\eta,\gamma}(T)} h^-_{\eta,r_t}(w^-_t)\, h^+_{\eta,r_t}(w^+_t)\, dt\, d\widetilde{m}_F(\ell)$$
$$= \frac{e^{\mathrm{O}(e^{-\lambda_\gamma/2})}}{(2\eta)^2} \iint_{(t,\ell)\in\mathscr{A}_{\eta,\gamma}(T)} \frac{dt\, d\widetilde{m}_F(\ell)}{\mu_{W^+(w^-_t)}(B^+(w^-_t, r_t))\, \mu_{W^-(w^+_t)}(B^-(w^+_t, r_t))}\,. \tag{11.14}$$

Let x_γ be the midpoint of the common perpendicular α_γ, so that $d(x_\gamma, \ell(0)) = \mathrm{O}(\eta)$ for every $(t,\ell) \in \mathscr{A}_{\eta,\gamma}(T)$ by the above Claim (iii). Let us use the Hopf parametrisation of $\mathscr{G}X$ with basepoint x_γ, denoting by s its time parameter. When $(t,\ell) \in \mathscr{A}_{\eta,\gamma}(T)$, by Definition (4.4) of the Gibbs measure $\widetilde{m}_F$, by the $\mathbb{R}$-tree case of Proposition 3.20 (2), and since $\widetilde{F}$ is bounded, we have

$$d\widetilde{m}_F(\ell) = e^{C^-_{\ell_-}(x_\gamma,\ell(0)) + C^+_{\ell_+}(x_\gamma,\ell(0))}\, d\mu^-_{x_\gamma}(\ell_-)\, d\mu^+_{x_\gamma}(\ell_+)\, ds$$
$$= e^{\mathrm{O}(\eta)} d\mu^-_{x_\gamma}(\ell_-)\, d\mu^+_{x_\gamma}(\ell_+)\, ds\,. \tag{11.15}$$

Let P_γ be the plane domain of the $(t,s) \in \mathbb{R}^2$ such that $|\lambda_\gamma - t| \leqslant 2\eta$ and there exist $s^\pm \in\,]-\eta,\eta[$ with $s^\mp = \frac{\lambda_\gamma - t}{2} \pm s$. It is easy to see that P_γ is a rhombus centred at $(\lambda_\gamma, 0)$ whose area is $(2\eta)^2$.



Let $\xi^\pm_\gamma$ be the point at infinity of any fixed geodesic ray from x_γ through $\alpha^\pm_\gamma(0)$. If A is a subset of $\mathscr{G}X$, we denote by $A_\pm$ the subset $\{\ell_\pm \,:\, \ell \in A\}$ of $\partial_\infty X$.

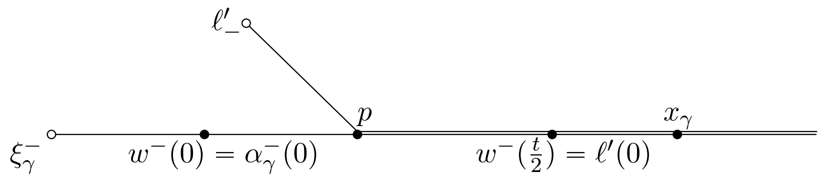
Lemma 11.5. *For every $t \geqslant t_0$ such that $|\lambda_\gamma - t| \leqslant 2\eta$, we have*

$$(B^\pm(w^\mp_t, r_t))_\mp = B_{d_{x_\gamma}}(\xi^\mp_\gamma, R\, e^{-\frac{\lambda_\gamma}{2}})\,.$$

Proof. We prove the statement for the negative endpoints, the proof of the claim for positive endpoints is similar. Since $R \geqslant 1$ and $d(x_\gamma, \alpha^\mp_\gamma(0)) = \frac{\lambda_\gamma}{2}$, the term on the right hand side does not not depend on the choice of $\xi^\mp_\gamma$.

Let us first prove the inclusion on the set on left hand side into the one on the right hand side. Let $\ell' \in B^+(w^-_t, r_t)$, so that[7] there exists a geodesic line $\widehat{w}^-_t \in \mathscr{G}X$, extension of the geodesic ray $w^-_t : [-\frac{t}{2}, +\infty[\, \to X$, with $d_{W^+(w^-_t)}(\widehat{w}^-_t, \ell') < r_t$. We may assume that $(\widehat{w}^-_t)_- = \xi^-_\gamma$ and $\ell'_- \neq \xi^-_\gamma$. Let $p \in X$ be such that $[\ell'(0), \xi^-_\gamma[\, \cap\, [\ell'(0), \ell'_-[\, = [\ell'(0), p]$.

[7] See the definition of the strong stable ball $B^+(w^-_t, r_t)$ in Section 2.3.



Since $t \geqslant t_0 > 2\ln R$, we have $r_t < 1$, hence[8] $\ell'(0) = w_t^-(0) = w^-(t/2)$ and $p \in \,]\ell'(0), \xi_\gamma^-[\,$. Since

$$d(p, w^-(t/2)) = -\ln d_{W^-+w_t^-)}(\widehat{w}_t^-, \ell') > -\ln r_t = \frac{t}{2} - \ln R \geqslant \frac{t_0}{2} - \ln R \geqslant 1 \geqslant \eta$$

and $d(x_\gamma, w^-(t/2)) = \big|\frac{\lambda_\gamma}{2} - \frac{t}{2}\big| \leqslant \eta$, we have $p \in [x_\gamma, \xi_\gamma^-[\,$. Hence

$$d_{x_\gamma}(\ell'_-, \xi_\gamma^-) = e^{-d(p,x_\gamma)} \leqslant \begin{cases} e^{-d(p,\,w^-(\frac{t}{2}))-d(w^-(t/2),\,x_\gamma)} & \text{if } \frac{t}{2} \leqslant \lambda_\gamma \\ e^{-d(p,\,w^-(\frac{t}{2}))+d(w^-(t/2),\,x_\gamma)} & \text{otherwise.} \end{cases}$$

In both cases,

$$d_{x_\gamma}(\ell'_-, \xi_\gamma^-) = d_{W^+(w_t^-)}(\widehat{w}_t^-, \ell')\, e^{-\frac{\lambda_\gamma}{2}+\frac{t}{2}} < r_t\, e^{-\frac{\lambda_\gamma}{2}+\frac{t}{2}} = R\, e^{-\frac{\lambda_\gamma}{2}}\ .$$

Conversely, if $\xi \in B_{d_{x_\gamma}}(\xi_\gamma^-, R\, e^{-\frac{\lambda_\gamma}{2}})$, let $\ell' \in \mathscr{G}X$ be such that $\ell'(0) = w^-(t/2)$ and $\ell'_- = \xi$. We may assume that $\xi \neq \xi_\gamma^-$. Let $\widehat{w}_t^-$ be the extension of w_t^- such that $(\widehat{w}_t^-)_- = \xi_\gamma^-$. Let $p \in X$ be such that $[\ell'(0), \xi_\gamma^-[\, \cap [\ell'(0), \ell'_-[\, = [\ell'(0), p]$. Then as above, we have $R\, e^{-\frac{\lambda_\gamma}{2}} < 1$, hence

$$d_{W^+(w_t^-)}(\widehat{w}_t^-, \ell') = e^{-d(p,\,\ell'(0))} = d_{x_\gamma}(\ell'_-, \xi_\gamma^-)\; e^{\frac{\lambda_\gamma}{2}-\frac{t}{2}} < r_t\ ,$$

thus $\ell' \in B^+(w_t^-, r_t)$. □

It follows from this lemma that, for all $t \geqslant t_0$, $s^\pm \in\,]-\eta, \eta[$ and $\ell \in \mathscr{G}X$, we have $\mathsf{g}^{\mp s^\mp}\ell \in B^\pm(w_t^\mp, r_t)$ if and only if $d(\ell(0), \alpha_\gamma^\pm(0)) = s^\pm + \frac{t}{2}$ (or equivalently, by the definition of the time parameter s of ℓ in Hopf's parametrisation with basepoint x_γ, when $s^\pm + \frac{t}{2} = \frac{\lambda_\gamma}{2} \pm s$), and $\ell_\pm \in B_{d_{x_\gamma}}(\xi_\gamma^\pm, R\, e^{\frac{-\lambda_\gamma}{2}})$. Thus,

$$\mathscr{A}_{\eta,\gamma}(T) = P_\gamma \times B_{d_{x_\gamma}}(\xi_\gamma^-, R\, e^{-\frac{\lambda_\gamma}{2}}) \times B_{d_{x_\gamma}}(\xi_\gamma^+, R\, e^{-\frac{\lambda_\gamma}{2}})\,.$$

To finish Step 4T and the proof of the theorem for $\mathbb{R}$-trees, note that by the definition of the skinning measure (using again the Hopf parametrisation with basepoint x_γ), by the above Lemma 11.5, by the claim for $\mathbb{R}$-trees of Assertion (2) of Proposition 3.20 and the boundedness of $\widetilde{F}$, we have

$$\mu_{W^\pm(w_t^\mp)}(B^\pm(w_t^\mp, r_t)) = e^{\mathrm{O}(\eta)}\mu_{x_\gamma}^\mp(B_{d_{x_\gamma}}(\xi_\gamma^\mp, R\, e^{-\frac{\lambda_\gamma}{2}}))\,. \tag{11.16}$$

Thus, by the above and by Equations (11.14) and (11.15), (and noting that $\mathrm{O}(\eta) \pm \mathrm{O}(\eta) = \mathrm{O}(\eta)$)

$$\begin{aligned} j_{\eta,\gamma}(T) &= \frac{e^{\mathrm{O}(e^{-\frac{\lambda_\gamma}{2}})}}{4\eta^2} e^{\mathrm{O}(\eta)}(2\eta)^2 \frac{\mu_{x_\gamma}^-(B_{d_{x_\gamma}}(\xi_\gamma^-, Re^{-\frac{\lambda_\gamma}{2}}))\mu_{x_\gamma}^+(B_{d_{x_\gamma}}(\xi_\gamma^+, Re^{-\frac{\lambda_\gamma}{2}}))}{\mu_{x_\gamma}^-(B_{d_{x_\gamma}}(\xi_\gamma^-, Re^{-\frac{\lambda_\gamma}{2}}))\mu_{x_\gamma}^+(B_{d_{x_\gamma}}(\xi_\gamma^+, Re^{-\frac{\lambda_\gamma}{2}}))} \\ &= e^{\mathrm{O}(\eta + e^{-\frac{\lambda_\gamma}{2}})}\,, \end{aligned} \tag{11.17}$$

[8] See the definition of the Hamenstädt distance $d_{W^+(w_t^-)}$ in Section 2.3.

which gives the inequalities (11.13). □

The effective control on $j_{\eta,\gamma}(T)$ given by Equation (11.17) is stronger than what is needed in order to prove Equation (11.13) in Step 4T. We will use it in Section 12.6 in order to obtain error terms.

11.3 Part III of the proof of Theorem 11.1: the manifold case

The proof of Theorem 11.1 for manifolds is the same one as for trees until Equation (11.10). The remaining part of the proof that we give below is more technical than for trees but the structure of the proof is similar. In this Section, $X = \widetilde{M}$ is a Riemannian manifold, and we identify $\mathscr{G}X$ and T^1X with the standard unit tangent bundle of $\widetilde{M}$, as explained in Section 2.3.

Step 3M. Consider the product term in Equation (11.8) involving the characteristic functions. The quantity $\mathbb{1}_{\mathscr{V}^+_{\eta,\,R}(\Omega^-)}(\mathsf{g}^{-t/2}v)\;\mathbb{1}_{\mathscr{V}^-_{\eta,\,R}(\Omega^+)}(\gamma^{-1}\mathsf{g}^{t/2}v)$ is different from 0 (hence equal to 1) if and only if

$$v \in \mathsf{g}^{t/2}\mathscr{V}^+_{\eta,\,R}(\Omega^-) \cap \gamma\mathsf{g}^{-t/2}\mathscr{V}^-_{\eta,\,R}(\Omega^+) = \mathscr{V}^+_{\eta,\,e^{-t/2}R}(\mathsf{g}^{t/2}\Omega^-) \cap \mathscr{V}^-_{\eta,\,e^{-t/2}R}(\gamma\mathsf{g}^{-t/2}\Omega^+)\,,$$

see Section 2.4 and in particular Equation (2.20). By Lemma 2.9 (applied by replacing D^+ by γD^+ and w by v), there exist $t_0, c_0 > 0$ such that for all $\eta \in \,]0,1]$ and $t \geqslant t_0$, for all $v \in T^1\widetilde{M}$, if $\mathbb{1}_{\mathscr{V}^+_{\eta,\,R}(\Omega^-)}(\mathsf{g}^{-t/2}v)\;\mathbb{1}_{\mathscr{V}^-_{\eta,\,R}(\Omega^+)}(\gamma^{-1}\mathsf{g}^{t/2}v) \neq 0$, then the following facts hold:

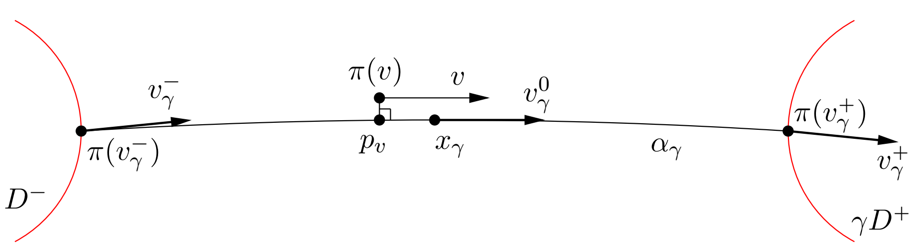
(i) by the convexity of $D^\pm$, we have $v \in \mathscr{U}^+_{D^-} \cap \mathscr{U}^-_{\gamma D^+}$,

(ii) by the definition of $w^\pm$ (see Equation (11.9)), we have $w^- \in \Omega^-$ and $w^+ \in \gamma\Omega^+$ (the notation (w^-, w^+) here coincides with the notation (w^-, w^+) in Lemma 2.9),

(iii) there exists a common perpendicular α_γ from D^- to γD^+,[9] whose length λ_γ satisfies

$$|\,\lambda_\gamma - t\,| \leqslant 2\eta + c_0\,e^{-t/2}\,, \tag{11.18}$$

whose origin $\pi(v^-_\gamma)$ is at distance at most $c_0\,e^{-t/2}$ from $\pi(w^-)$, whose endpoint $\pi(v^+_\gamma)$ is at distance at most $c_0\,e^{-t/2}$ from $\pi(w^+)$, such that both points $\pi(\mathsf{g}^{t/2}w^-)$ and $\pi(\mathsf{g}^{-t/2}w^+)$ are at distance at most $\eta + c_0\,e^{-t/2}$ from $\pi(v)$, which is at distance at most $c_0\,e^{-t/2}$ from some point p_v of α_γ.



[9] and we then denote as previously by v^-_γ its tangent vector at its origin, by v^+_γ its tangent vector at its terminal point, and by v^0_γ its tangent vector at its midpoint

Using (iii) and the (HC)-property[10] which introduces a constant $\kappa_2 \in \,]0,1]$, and since $\widetilde{F}$ is bounded, for all $\eta \in \,]0,1]$, $t \geqslant t_0$ and $v \in T^1\widetilde{M}$ for which $\mathbb{1}_{\mathscr{V}^+_{\eta,R}(\Omega^-)}(\mathsf{g}^{-t/2}v)\,\mathbb{1}_{\mathscr{V}^-_{\eta,R}(\Omega^+)}(\gamma^{-1}\mathsf{g}^{t/2}v) \neq 0$, we have

$$e^{\int_{\pi(w^-)}^{\pi(\mathsf{g}^{t/2}w^-)}\widetilde{F}+\int_{\pi(\mathsf{g}^{-t/2}w^+)}^{\pi(w^+)}\widetilde{F}} = e^{\int_{\pi(v^-_\gamma)}^{pv}\widetilde{F}+\int_{pv}^{\pi(v^+_\gamma)}\widetilde{F}+\mathrm{O}((\eta+e^{-t/2})^{\kappa_2})}$$
$$= e^{\int_{\alpha_\gamma}\widetilde{F}}\; e^{\mathrm{O}((\eta+e^{-\lambda_\gamma/2})^{\kappa_2})}\,. \tag{11.19}$$

For all $\eta \in \,]0,1]$, $\gamma \in \Gamma$ and $T \geqslant t_0$, define

$$\mathscr{A}_{\eta,\gamma}(T) = \big\{(t,v) \in [t_0,T] \times T^1\widetilde{M} \;:\; v \in \mathscr{V}^+_{\eta,\,e^{-t/2}R}(\mathsf{g}^{t/2}\Omega^-) \cap \mathscr{V}^-_{\eta,\,e^{-t/2}R}(\gamma\mathsf{g}^{-t/2}\Omega^+)\big\}\,,$$

and

$$j_{\eta,\gamma}(T) = \iint_{(t,v)\in\mathscr{A}_{\eta,\gamma}(T)} h^-_{\eta,\,e^{-t/2}R}(\mathsf{g}^{t/2}w^-)\; h^+_{\eta,\,e^{-t/2}R}(\mathsf{g}^{-t/2}w^+)\;dt\;d\widetilde{m}_F(v)\,.$$

By the above, since the integral of a function is equal to the integral on any Borel set containing its support, and since the integral of a nonnegative function is nondecreasing in the integration domain, there hence exists $c_4 > 0$ such that for all $T \geqslant 0$ and $\eta \in \,]0,1]$, we have

$$i_\eta(T) \geqslant -\,c_4 + \sum_{\substack{\gamma\in\Gamma\,:\,t_0+2+c_0\leqslant\lambda_\gamma\leqslant T-\mathrm{O}(\eta+e^{-\lambda_\gamma/2})\\ v^-_\gamma\in\mathscr{N}_{-\,\mathrm{O}(\eta+e^{-\lambda_\gamma/2})}\Omega^-,\; v^+_\gamma\in\gamma\mathscr{N}_{-\,\mathrm{O}(\eta+e^{-\lambda_\gamma/2})}\Omega^+}} e^{\int_{\alpha_\gamma}\widetilde{F}}\; j_{\eta,\gamma}(T)\; e^{-\,\mathrm{O}((\eta+e^{-\lambda_\gamma/2})^{\kappa_2})}\,,$$

and similarly, for every $T' \geqslant T$,

$$i_\eta(T) \leqslant c_4 + \sum_{\substack{\gamma\in\Gamma\,:\,t_0+2+c_0\leqslant\lambda_\gamma\leqslant T+\mathrm{O}(\eta+e^{-\lambda_\gamma/2})\\ v^-_\gamma\in\mathscr{N}_{\mathrm{O}(\eta+e^{-\lambda_\gamma/2})}\Omega^-,\; v^+_\gamma\in\gamma\mathscr{N}_{\mathrm{O}(\eta+e^{-\lambda_\gamma/2})}\Omega^+}} e^{\int_{\alpha_\gamma}\widetilde{F}}\; j_{\eta,\gamma}(T')\; e^{\mathrm{O}((\eta+e^{-\lambda_\gamma/2})^{\kappa_2})}\,.$$

We will take T' to be of the form $T + \mathrm{O}(\eta + e^{-\lambda_\gamma/2})$, for a bigger $\mathrm{O}(\cdot)$ than the one appearing in the index of the above summation.

Step 4M: Conclusion. Let $\gamma \in \Gamma$ be such that D^- and γD^+ have a common perpendicular with length $\lambda_\gamma \geqslant t_0 + 2 + c_0$. Let us prove that for all $\epsilon > 0$, if η is small enough and λ_γ is large enough, then for every $T \geqslant \lambda_\gamma + \mathrm{O}(\eta + e^{-\lambda_\gamma/2})$ (with the enough's and $\mathrm{O}(\cdot)$ independent of γ), we have

$$1 - \epsilon \leqslant j_{\eta,\gamma}(T) \leqslant 1 + \epsilon\,. \tag{11.20}$$

Note that $\widetilde{\sigma}^\pm(\mathscr{N}_\varepsilon(\Omega^\mp))$ and $\widetilde{\sigma}^\pm(\mathscr{N}_{-\varepsilon}(\Omega^\mp))$ tend to $\widetilde{\sigma}^\pm(\Omega^\mp)$ as $\varepsilon \to 0$ (since $\widetilde{\sigma}^\pm(\partial\Omega^\mp) = 0$ as required in Step 1). Using Steps 2, 3M and 4M, this will prove Equation (11.4), hence will complete the proof of Theorem 11.1.

We say that $(\widetilde{M}, \Gamma, \widetilde{F})$ *has radius-continuous strong stable/unstable ball masses* if for every $\epsilon > 0$, if $r > 1$ is close enough to 1, then for every $v \in T^1\widetilde{M}$, if $B^-(v,1)$ meets the support of $\mu^+_{W^-(v)}$, then

$$\mu^+_{W^-(v)}(B^-(v,r)) \leqslant e^\epsilon \mu^+_{W^-(v)}(B^-(v,1))$$

[10] See Definition 3.13.

and if $B^+(v,1)$ meets the support of $\mu^-_{W^+(v)}$, then

$$\mu^-_{W^+(v)}(B^+(v,r)) \leqslant e^{\epsilon}\mu^-_{W^+(v)}(B^+(v,1))\,.$$

We say that $(\widetilde{M},\Gamma,\widetilde{F})$ *has radius-Hölder-continuous strong stable/unstable ball masses* if there exist $c\in\,]0,1]$ and $c'>0$ such that for every $\epsilon\in\,]0,1]$, if $r>1$ is close enough to 1, then for every $v\in T^1\widetilde{M}$, if $B^-(v,1)$ meets the support of $\mu^+_{W^-(v)}$, then

$$\mu^+_{W^-(v)}(B^-(v,r)) \leqslant e^{c'\epsilon^c}\mu^+_{W^-(v)}(B^-(v,1))$$

and if $B^+(v,1)$ meets the support of $\mu^-_{W^+(v)}$, then

$$\mu^-_{W^+(v)}(B^+(v,r)) \leqslant e^{c'\epsilon^c}\mu^-_{W^+(v)}(B^+(v,1))\,.$$

Note that when $F=0$ and M is locally symmetric with finite volume, the conditional measures on the strong stable/unstable leaves are homogeneous. Hence $(\widetilde{M},\Gamma,\widetilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses.

When the sectional curvature of $\widetilde{M}$ has bounded derivatives and when $(\widetilde{M},\Gamma,\widetilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses, we will prove a stronger statement: With a constant $c_7>0$ and functions $\operatorname{O}(\cdot)$ independent of γ, for all $\eta\in\,]0,1]$ and $T\geqslant\lambda_\gamma+\operatorname{O}(\eta+e^{-\lambda_\gamma/2})$, we have

$$j_{\eta,\gamma}(T) = \Big(1+\operatorname{O}\Big(\frac{e^{-\lambda_\gamma/2}}{2\eta}\Big)\Big)^2 e^{\operatorname{O}((\eta+e^{-\lambda_\gamma/2})^{c_7})}\,. \tag{11.21}$$

This stronger version will be needed for the error term estimate in Section 12.3. In order to obtain Theorem 11.1, only the fact that $j_{\eta,\gamma}(T)$ tends to 1 as firstly λ_γ tends to $+\infty$, secondly η tends to 0 is needed. A reader not interested in the error term may skip many technical details below.

Given $a,b>0$ and a point x in a metric space X (with a,b,x depending on parameters), we will denote by $B(x,a\,e^{\operatorname{O}(b)})$ any subset Y of X such that there exists a constant $c>0$ (independent of the parameters) with

$$B(x,a\,e^{-c\,b})\subset Y\subset B(x,a\,e^{c\,b})\,. \tag{11.22}$$

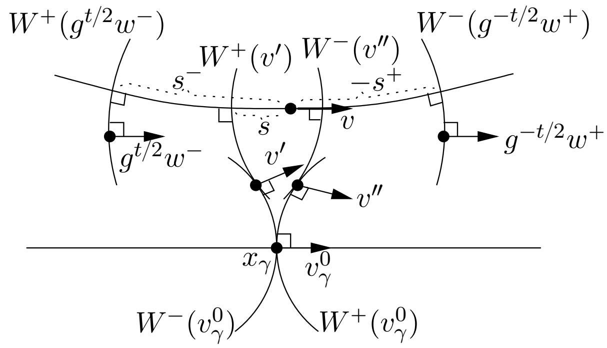
Let $\eta\in\,]0,1]$ and $T\geqslant\lambda_\gamma+\operatorname{O}(\eta+e^{-\lambda_\gamma/2})$. In order to simplify the notation, let

$$r_t=e^{-t/2}R,\;\; w_t^-=\mathsf{g}^{t/2}w^- \quad\text{and}\quad w_t^+=\mathsf{g}^{-t/2}w^+\,.$$

By the definition of $j_{\eta,\gamma}$, using the inequalities (10.3) where the constant c_1 is uniform since $\widetilde{F}$ is bounded, and the fact that $r_t=\operatorname{O}(e^{-\lambda_\gamma/2})$ by Equation (11.18), we hence have

$$\begin{aligned} j_{\eta,\gamma}(T) &= \iint_{(t,v)\in\mathscr{A}_{\eta,\gamma}(T)} h^-_{\eta,r_t}(w_t^-)\;h^+_{\eta,r_t}(w_t^+)\;dt\;d\widetilde{m}_F(v)\\ &= \frac{e^{\operatorname{O}(e^{-\kappa_2\lambda_\gamma/2})}}{(2\eta)^2}\iint_{(t,v)\in\mathscr{A}_{\eta,\gamma}(T)}\frac{dt\;d\widetilde{m}_F(v)}{\mu_{W^+(w_t^-)}(B^+(w_t^-,r_t))\;\mu_{W^-(w_t^+)}(B^-(w_t^+,r_t))}\,. \end{aligned} \tag{11.23}$$

We start the proof of Equation (11.20) by defining parameters s^+,s^-,s,v',v'' associated with $(t,v)\in\mathscr{A}_{\eta,\gamma}(T)$.



We have $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$ if and only if there exist $s^\pm \in \,]-\eta, \eta[$ such that

$$\mathsf{g}^{\mp s^\mp} v \in B^\pm(\mathsf{g}^{\pm t/2} w^\mp, e^{-t/2} R) \,.$$

In order to define the parameters s, v', v'', we use the well known local product structure of the unit tangent bundle in negative curvature. If $v \in T^1M$ is close enough to v_γ^0 (in particular, $v_- \neq (v_\gamma^0)_+$ and $v_+ \neq (v_\gamma^0)_-$), then let $v' = f^+_{HB_-(v_\gamma^0)}(v)$ be the unique element of $W^-(v_\gamma^0)$ such that $v'_+ = v_+$, let $v'' = f^-_{HB_+(v_\gamma^0)}(v)$ be the unique element of $W^+(v_\gamma^0)$ such that $v''_- = v_-$, and let s be the unique element of $\mathbb{R}$ such that $\mathsf{g}^{-s} v \in W^+(v')$. The map $v \mapsto (s, v', v'')$ is a homeomorphism from a neighbourhood of v_γ^0 in $T^1\widetilde{M}$ to a neighbourhood of $(0, v_\gamma^0, v_\gamma^0)$ in $\mathbb{R} \times W^-(v_\gamma^0) \times W^+(v_\gamma^0)$. Note that if $v = \mathsf{g}^r v_\gamma^0$ for some $r \in \mathbb{R}$ close to 0, then

$$w^- = v_\gamma^-,\ w^+ = v_\gamma^+,\ s = r,\ v' = v'' = v_\gamma^0\,,$$

$$s^- = \frac{\lambda_\gamma - t}{2} + s,\ s^+ = \frac{\lambda_\gamma - t}{2} - s\,.$$

Up to increasing t_0 (which does not change Step 3M, up to increasing c_4), we may assume that for every $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$, the vector v belongs to the domain of this local product structure of $T^1\widetilde{M}$ at v_γ^0.

The vectors v, v', v'' are close to v_γ^0 if t is large and η small, as the following result shows. We denote (also) by d the Riemannian distance induced by Sasaki's metric on $T^1\widetilde{M}$.

Lemma 11.6. *For every $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$, we have $d(v, v_\gamma^0), d(v', v_\gamma^0), d(v'', v_\gamma^0) = \mathrm{O}(\eta + e^{-t/2})$.*

Proof. Consider the distance d' on $T^1\widetilde{M}$, defined by

$$\forall\ v_1, v_2 \in T^1\widetilde{M}, \quad d'(v_1, v_2) = \max_{r \in [-1,0]} d\big(\pi(\mathsf{g}^r v_1), \pi(\mathsf{g}^r v_2)\big)\ .$$

As seen in Claim (iii) of Step 3M, we have $d(\pi(w^\pm), \pi(v_\gamma^\pm))$, $d(\pi(v), \alpha_\gamma) = \mathrm{O}(e^{-t/2})$, and furthermore, $d(\pi(\mathsf{g}^{\pm t/2} w^\mp), \pi(v))$, $\frac{\lambda_\gamma}{2} - \frac{t}{2} = \mathrm{O}(\eta + e^{-t/2})$. Hence $d(\pi(v), \pi(v_\gamma^0)) = \mathrm{O}(\eta + e^{-t/2})$. By Lemma 2.4, we have

$$d(\pi(\mathsf{g}^{-\frac{t}{2} - s^-} v), \pi(v_\gamma^-)) \leqslant d(\pi(\mathsf{g}^{-\frac{t}{2} - s^-} v), \pi(w^-)) + d(\pi(w^-), \pi(v_\gamma^-)) \leqslant R + c_0\, e^{-t/2}\ .$$

By an exponential pinching argument, we hence have $d'(v, v_\gamma^0) = \mathrm{O}(\eta + e^{-\lambda_\gamma/2})$. Since d and d' are equivalent by Proposition 3.5,[11] we therefore have $d(v, v_\gamma^0) = \mathrm{O}(\eta + e^{-\lambda_\gamma/2})$.

[11] In fact, Proposition 3.5 considers the distance $\delta_2(v, v') = \sup_{r \in [0,1]} d(\pi(\mathsf{g}^r v), \pi(\mathsf{g}^r v'))$ instead of d', but the argument is similar.

For all $w \in T^1\widetilde{M}$ and $V \in T_w T^1\widetilde{M}$, we may uniquely write $V = V^- + V^0 + V^+$ with $V^- \in T_w W^-(w)$, $V^0 \in \mathbb{R}\,\frac{d}{dt}\big|_{t_0} \mathtt{g}^t w$ and $V^+ \in T_w W^+(w)$. By [PauPS, Lem. 7.4],[12] Sasaki's metric (with norm $\|\cdot\|$) is equivalent to the Riemannian metric with (product) norm

$$\|V\|' = \sqrt{\|\,V^-\,\|^2 + \|\,V^0\,\|^2 + \|\,V^+\,\|^2}\,.$$

By the dynamical local product structure of $T^1\widetilde{M}$ in the neighbourhood of v^0_γ and by the definition of v', v'', the result follows, since the exponential map of $T^1\widetilde{M}$ at v^0_γ is almost isometric close to 0 and the projection to a factor of a product norm is Lipschitz. □

We now use the local product structure of the Gibbs measure to prove the following result.

Lemma 11.7. *For every $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$, we have*

$$dt\, d\widetilde{m}_F(v) = e^{\mathrm{O}((\eta + e^{-\lambda_\gamma/2})^{\kappa_2})}\, dt\, ds\, d\mu_{W^-(v^0_\gamma)}(v')\, d\mu_{W^+(v^0_\gamma)}(v'')\,.$$

Proof. By the definition of the measures (see Equations (4.4) and (7.9)), since the above parameter s differs, when v_-, v_+ are fixed, only up to a constant from the time parameter in Hopf's parametrisation with respect to the basepoint $x_\gamma = \pi(v^0_\gamma)$, we have

$$d\widetilde{m}_F(v) = e^{C^-_{v_-}(x_\gamma,\, \pi(v)) \,+\, C^+_{v_+}(x_\gamma,\, \pi(v))}\, d\mu^-_{x_\gamma}(v_-)\, d\mu^+_{x_\gamma}(v_+)\, ds$$

$$d\mu_{W^-(v^0_\gamma)}(v') = e^{C^+_{v'_+}(x_\gamma,\, \pi(v'))}\, d\mu^+_{x_\gamma}(v'_+)\,,$$

$$d\mu_{W^+(v^0_\gamma)}(v'') = e^{C^-_{v''_-}(x_\gamma,\, \pi(v''))}\, d\mu^-_{x_\gamma}(v''_-)\,.$$

By Proposition 3.20 (2) since F is bounded, we have $|\,C^\pm_\xi(z, z')\,| = \mathrm{O}(d(z, z')^{\kappa_2})$ for all $\xi \in \partial_\infty\widetilde{M}$ and $z, z' \in \widetilde{M}$ with $d(z, z')$ bounded. Since the map $\pi : T^1\widetilde{M} \to \widetilde{M}$ is Lipschitz, and since $v_+ = v'_+$ and $v_- = v''_-$, the result follows from Lemma 11.6. □

When λ_γ is large, the submanifold $\mathtt{g}^{\lambda_\gamma/2}\Omega^-$ has a second order contact at v^0_γ with $W^-(v^0_\gamma)$ and similarly, $\mathtt{g}^{-\lambda_\gamma/2}\Omega^+$ has a second order contact at v^0_γ with $W^+(v^0_\gamma)$. Let P_γ be the plane domain of $(t, s) \in \mathbb{R}^2$ such that $|\lambda_\gamma - t| \leqslant 2\eta + c_0 e^{-t/2}$ and there exist $s^\pm \in\,]-\eta, \eta[$ with $s^\mp = \frac{\lambda_\gamma - t}{2} \pm s + \mathrm{O}(e^{-\lambda_\gamma/2})$. Note that its area is $(2\eta + \mathrm{O}(e^{-\lambda_\gamma/2}))^2$. By the above, we have[13]

$$\mathscr{A}_{\eta,\gamma}(T) = P_\gamma \times B^-(v^0_\gamma, r_{\lambda_\gamma}\, e^{\mathrm{O}(\eta + e^{-\lambda_\gamma/2})}) \times B^+(v^0_\gamma, r_{\lambda_\gamma}\, e^{\mathrm{O}(\eta + e^{-\lambda_\gamma/2})})\,.$$

By Lemma 11.7, we hence have

$$\int_{\mathscr{A}_{\eta,\gamma}(T)} dt\, d\widetilde{m}_F(v) = e^{\mathrm{O}((\eta + e^{-\lambda_\gamma/2})^{\kappa_2})}\, (2\eta + \mathrm{O}(e^{-\lambda_\gamma/2}))^2 \times \mu_{W^-(v^0_\gamma)}(B^-(v^0_\gamma, r_{\lambda_\gamma}\, e^{\mathrm{O}(\eta + e^{-\lambda_\gamma/2})}))\, \mu_{W^+(v^0_\gamma)}(B^+(v^0_\gamma, r_{\lambda_\gamma}\, e^{\mathrm{O}(\eta + e^{-\lambda_\gamma/2})}))\,. \tag{11.24}$$

The last ingredient of the proof of Step 4M is the following continuity property of the masses of balls in the strong stable and strong unstable manifolds as their centre varies. This result generalises [PaP17b, Lem. 11]. The precise control for the error term is used in Section 12.3.

[12] building on [Brin] whose compactness assumption on M and torsion free assumption on Γ are not necessary for this, the pinched negative curvature assumption is sufficient

[13] with the obvious meaning of a double inclusion by Equation (11.22)

Lemma 11.8. *Assume that $(\widetilde{M}, \Gamma, \widetilde{F})$ has radius-continuous strong stable/unstable ball masses. There exists $c_5 > 0$ such that for every $\epsilon > 0$, if η is small enough and λ_γ large enough, then for every $(t, v) \in \mathscr{A}_{\eta,\gamma}(T)$, we have*

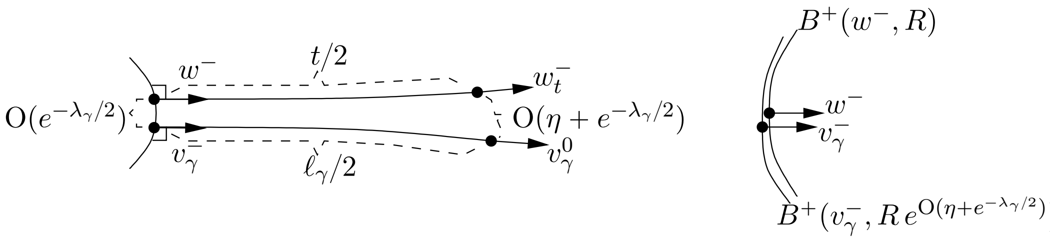
$$\mu_{W^-(w_t^+)}(B^-(w_t^+, r_t)) = e^{\mathrm{O}(\epsilon^{c_5})}\ \mu_{W^-(v_\gamma^0)}(B^-(v_\gamma^0, r_{\lambda_\gamma}))$$

and

$$\mu_{W^+(w_t^-)}(B^+(w_t^-, r_t)) = e^{\mathrm{O}(\epsilon^{c_5})}\ \mu_{W^+(v_\gamma^0)}(B^+(v_\gamma^0, r_{\lambda_\gamma}))\,.$$

If we furthermore assume that the sectional curvature of $\widetilde{M}$ has bounded derivatives and that $(\widetilde{M}, \Gamma, \widetilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses, then we may replace ϵ by $(\eta + e^{-\lambda_\gamma/2})^{c_6}$ for some constant $c_6 > 0$.

Proof. We prove the (second) claim for W^+, the (first) one for W^- follows similarly. The final statement is only used for the error estimates in Section 12.3.



Using respectively Equation (2.18) since $w_t^- = \mathsf{g}^{t/2} w^-$ and $r_t = e^{-t/2} R$, Equation (7.11) where (ℓ, t, w) is replaced by $(v, t/2, w^-)$, and Equation (3.20), we have

$$\begin{aligned}\mu_{W^+(w_t^-)}(B^+(w_t^-, r_t)) &= \int_{v \in B^+(w^-, R)} d\mu_{W^+(\mathsf{g}^{t/2} w^-)}(\mathsf{g}^{t/2} v)\\ &= \int_{v \in B^+(w^-, R)} e^{C^-_{v_-}(\pi(v),\, \pi(\mathsf{g}^{t/2} v))}\ d\mu_{W^+(w^-)}(v)\\ &= \int_{v \in B^+(w^-, R)} e^{\int_{\pi(v)}^{\pi(\mathsf{g}^{t/2} v)} (\widetilde{F} - \delta)}\ d\mu_{W^+(w^-)}(v)\,. \end{aligned} \tag{11.25}$$

Similarly, for every $a > 0$, we have

$$\mu_{W^+(v_\gamma^0)}(B^+(v_\gamma^0, a r_t)) = \int_{v \in B^+(v_\gamma^-, aR)} e^{\int_{\pi(v)}^{\pi(\mathsf{g}^{t/2} v)} (\widetilde{F} - \delta)}\ d\mu_{W^+(v_\gamma^-)}(v)\ . \tag{11.26}$$

Let $h^- : B^+(w^-, R) \to W^+(v_\gamma^-)$ be the map such that $(h^-(v))_- = v_-$, which is well defined and a homeomorphism onto its image if λ_γ is large enough (since R is fixed). By Proposition 7.5 applied with $D = HB_+(w^-)$ and $D' = HB_+(v_\gamma^-)$, we have, for every $v \in B^+(w^-, R)$,

$$d\mu_{W^+(w^-)}(v) = e^{-C^-_{v_-}(\pi(v),\, \pi(h^-(v)))}\ d\mu_{W^+(v_\gamma^-)}(h^-(v))\,.$$

Let us fix $\epsilon > 0$. The strong stable balls of radius R centred at w^- and v_γ^- are very close (see the above picture). More precisely, recall that R is fixed, and that

$$d(\pi(w^-), \pi(v_\gamma^-)) = \mathrm{O}(e^{-\lambda_\gamma/2}) \quad \text{and} \quad d(\pi(\mathsf{g}^{t/2} w^-), \pi(\mathsf{g}^{\lambda_\gamma/2} v_\gamma^-)) = \mathrm{O}(\eta + e^{-\lambda_\gamma/2})\ .$$

Therefore we have $d(\pi(v), \pi(h^-(v))) \leqslant \epsilon$ for every $v \in B^+(w^-, R)$ if η is small enough and λ_γ is large enough. If we furthermore assume that the sectional curvature has bounded derivatives, then by Anosov's arguments, the strong stable foliation is Hölder-continuous, see for instance [PauPS, Theo. 7.3]. Hence we have $d(\pi(v), \pi(h^-(v))) = \mathrm{O}((\eta + e^{-\lambda_\gamma/2})^{c_6})$ for every $v \in B^+(w^-, R)$, for some constant $c_6 > 0$, under the additional regularity assumption on the curvature. We also have $h^-(B^+(w^-, R)) = B^+(v_\gamma^-, R\, e^{\mathrm{O}(\epsilon)})$ and, under the additional hypothesis on the curvature, $h^-(B^+(w^-, R)) = B^+(v_\gamma^-, R\, e^{\mathrm{O}((\eta + e^{-\lambda_\gamma/2})^{c_6})})$.

In what follows, we assume that $\epsilon = (\eta + e^{-\lambda_\gamma/2})^{c_6}$ under the additional assumption on the curvature. By Proposition 3.20 (2) since $\widetilde{F}$ is bounded, we hence have, for every $v \in B^+(w^-, R)$,

$$d\mu_{W^+(w^-)}(v) = e^{\mathrm{O}(\epsilon^{\kappa_2})}\, d\mu_{W^+(v_\gamma^-)}(h^-(v))$$

and, using the (HC)-property and the boundedness of $\widetilde{F}$,

$$\int_{\pi(v)}^{\pi(\mathrm{g}^{t/2} v)} (\widetilde{F} - \delta) - \int_{\pi(h^-(v))}^{\pi(\mathrm{g}^{t/2} h^-(v))} (\widetilde{F} - \delta) = \mathrm{O}(\epsilon^{\kappa_2})\ .$$

The result follows, by Equation (11.25) and (11.26) and the continuity properties in the radius of the strong stable/unstable ball masses. $\square$

Now Lemma 11.8 (with ϵ as in its statement, and when its hypotheses are satisfied) implies that

$$\iint_{(t,v) \in \mathscr{A}_{\eta,\gamma}(T)} \frac{dt\; d\widetilde{m}_F(v)}{\mu^-_{W^+(w_t^-)}(B^+(w_t^-,\, r_t))\; \mu^+_{W^-(w_t^+)}(B^-(w_t^+,\, r_t))}$$
$$= \frac{e^{\mathrm{O}(\epsilon^{c_5})} \iint_{(t,v) \in \mathscr{A}_{\eta,\gamma}(T)} dt\; d\widetilde{m}_F(v)}{\mu^-_{W^+(v_\gamma^0)}(B^+(v_\gamma^0,\, r_t))\; \mu^+_{W^-(v_\gamma^0)}(B^-(v_\gamma^0,\, r_t))}\ .$$

By Equation (11.23) and Equation (11.24), we hence have

$$j_{\eta,\gamma}(T) = e^{\mathrm{O}((\eta + e^{-\lambda_\gamma/2})^{\kappa_2})}\; e^{\mathrm{O}(\epsilon^{c_5})}\; \frac{(2\eta + \mathrm{O}(e^{-\lambda_\gamma/2}))^2}{(2\eta)^2}$$

under the technical assumptions of Lemma 11.8. The assumption on radius-continuity of strong stable/unstable ball masses can be bypassed using bump functions, as explained in [Rob2, page 81]. This completes the proof of Equation (11.20), hence the proof of Theorem 11.1. $\square$

11.4 Equidistribution of common perpendiculars in simplicial trees

In this Section, we prove a version of Theorem 11.1 for the discrete time geodesic flow on simplicial trees (and we leave to the reader the version without the assumption that the critical exponent of the system of conductances is positive).

Let $\mathbb{X}$, X, x_0, Γ, $\widetilde{c}$, c, $\widetilde{F}_c$, F_c, $\delta_c < +\infty$, $(\mu_x^\pm)_{x \in V\mathbb{X}}$, $\widetilde{m}_c$, m_c be as in the beginning of Section 9.2. Let $\mathscr{D}^- = (\mathbb{D}_i^-)_{i \in I^-}$ and $\mathscr{D}^+ = (\mathbb{D}_j^+)_{j \in I^+}$ be locally finite Γ-equivariant families of nonempty proper simplicial subtrees of $\mathbb{X}$. We denote by $D_k^\pm = |\mathbb{D}_k^\pm|_1$ the geometric realisation of $\mathbb{D}_k^\pm$ for $k \in I^\pm$.

For every edge path $\alpha = (e_1, \dots, e_n)$ in $\mathbb{X}$, we set

$$\widetilde{c}(\alpha) = \sum_{i=1}^{n} \widetilde{c}(e_i)\ .$$

Theorem 11.9. *Assume that the critical exponent δ_c of $\widetilde{c}$ is positive and that the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$. Then*

$$\lim_{t\to+\infty} \frac{e^{\delta_c}-1}{e^{\delta_c}}\,\|m_c\|\,e^{-\delta_c\,t} \sum_{\substack{i\in I^-/_\sim,\ j\in I^+/_\sim,\ \gamma\in\Gamma\\ D_i^-\cap D_{\gamma j}^+=\varnothing,\ \lambda_{i,\,\gamma j}\leqslant t}} e^{\widetilde{c}(\alpha_{i,\gamma j})}\ \Delta_{\alpha^-_{i,\,\gamma j}} \otimes \Delta_{\alpha^+_{\gamma^{-1}i,\,j}} \;=\; \widetilde{\sigma}^+_{\mathscr{D}^-} \otimes \widetilde{\sigma}^-_{\mathscr{D}^+}$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}\mathbb{X} \times \breve{\mathscr{G}}\mathbb{X}$.

Proof. The proof is a modification of the continuous time proof for metric trees in Sections 11.1 and 11.2. Here, we indicate the changes to adapt the proof to the discrete time. We use the conventions for the discrete time geodesic flow described in Section 2.6.

Note that for all $i \in I^-$, $j \in I^+$, $\gamma \in \Gamma$, the common perpendicular $\alpha_{i,\gamma j}$ is now an edge path from D_i^- to $D_{\gamma j}^+$, and that by Proposition 3.21, we have

$$\int_{\alpha_{i,\gamma j}} \widetilde{F}_c = \widetilde{c}(\alpha_{i,\gamma j})\ .$$

In the definition of the bump functions in Section 10.1, we assume (as we may) that $\eta < 1$, so that for all $\eta' \in\,]0,1[$ and $w \in \partial^1_\mp D^\pm$ such that $w_\mp \in \Lambda\Gamma$, we have

$$V^\pm_{w,\eta,\eta'} = B^\pm(w,\eta')\,,$$

see Equation (2.17) and recall that we are only considering discrete geodesic lines. As $\ell(0) = w(0)$ for every $\ell \in B^\pm(w,\eta')$ since $\eta' < 1$, and as the time is now discrete, Equations (10.1) and (7.12) give

$$h^\mp_{\eta,\,\eta'}(w) = \frac{1}{\mu_{W^\pm(w)}(B^\pm(w,\eta'))}\ . \tag{11.27}$$

This is a considerable simplification compared with the inequalities of Equation (10.3).

In the whole proof, we restrict to $t = n \in \mathbb{N}$, $T = N \in \mathbb{N}$. We keep the notation of Equation (11.2), as well as the only assumptions on the Borel sets $\Omega^\pm \subset \partial^1_\mp D^\pm$ to have finite positive skinning measure, with boundary of zero skinning measure. In Steps 1 and 2, we define instead of Equation (11.3)

$$I_{\Omega^-,\,\Omega^+}(N) = \ (e^{\delta_c}-1)\,\|m_c\|\,e^{-\delta_c(N+1)} \sum_{\substack{\gamma\in\Gamma\,:\,0<\lambda_\gamma\leqslant N\\ \alpha^-_\gamma|_{]0,\lambda_\gamma]}\in\Omega^-|_{]0,\lambda_\gamma]},\ \alpha^+_\gamma|_{]-\lambda_\gamma,0]}\in\Omega^+|_{]-\lambda_\gamma,0]}}} e^{\int_{\alpha_\gamma}\widetilde{F}_c}\,,$$

and instead of Equation (11.5)

$$a_\eta(n) = \sum_{\gamma\in\Gamma} \int_{\ell\in\mathscr{G}\mathbb{X}} \phi^-_\eta(\mathsf{g}^{-\lfloor n/2\rfloor}\ell)\ \phi^+_\eta(\mathsf{g}^{\lceil n/2\rceil}\gamma^{-1}\ell)\ d\widetilde{m}_c(\ell)\,.$$

Equation (11.6) is replaced by

$$i_\eta(N)=\sum_{n=0}^{N} e^{\delta_c\, n}\; a_\eta(n)\,,$$

so that by a geometric sum argument, the pair of inequalities (11.7) becomes

$$e^{-\epsilon}\;\frac{e^{\delta_c\,(N+1)}\,\widetilde{\sigma}^+(\Omega^-)\;\widetilde{\sigma}^-(\Omega^+)}{(e^{\delta_c}-1)\,\|m_c\|}\;-c_\epsilon\leqslant\; i_\eta(N)\leqslant\; e^{\epsilon}\;\frac{e^{\delta_c\,(N+1)}\,\widetilde{\sigma}^+(\Omega^-)\;\widetilde{\sigma}^-(\Omega^+)}{(e^{\delta_c}-1)\,\|m_c\|}\;+c_\epsilon\,.$$

Step 3 is unchanged up to replacing $\int_0^T$ by $\sum_{n=0}^N$, $\widetilde{F}$ by $\widetilde{F}_c$, δ by δ_c and $t/2$ by either $\lfloor n/2\rfloor$ or $\lceil n/2\rceil$, so that Equation (11.10) becomes, since $\lfloor n/2\rfloor+\lceil n/2\rceil=n$,

$$\begin{aligned}&h^-_{\eta,\,R}\circ f^+_{D^-}(\ell)\;h^+_{\eta,\,R}\circ f^-_{D^+}(\gamma^{-1}\ell)\\&\quad=e^{-\delta_c\,n}\;e^{\int_{w^-(0)}^{w^-(\lfloor n/2\rfloor)}\widetilde{F}_c+\int_{w^+(-\lceil n/2\rceil)}^{w^+(0)}\widetilde{F}_c}\;h^-_{\eta,\,e^{-\lfloor n/2\rfloor}R}(\mathsf{g}^{\lfloor n/2\rfloor}w^-)\;h^+_{\eta,\,e^{-\lceil n/2\rceil}R}(\mathsf{g}^{-\lceil n/2\rceil}w^+)\,.\end{aligned}$$

The proof then follows in the same way as in Section 11.2, with the simplifications in the point (iii) that, taking $\eta<1/2$, we have λ_γ equal to $t=n$, and the points $w^-(\lfloor\frac{n}{2}\rfloor)$, $w^+(-\lceil\frac{n}{2}\rceil)$ and $\ell(0)$ are equal. In particular, Equation (11.12) simplifies as

$$e^{\int_{w^-(0)}^{w^-(\lfloor\frac{t}{2}\rfloor)}\widetilde{F}_c+\int_{w^+(-\lceil\frac{t}{2}\rceil)}^{w^+(0)}\widetilde{F}_c}=e^{\int_{\alpha_\gamma}\widetilde{F}_c}\;,$$

thus avoiding the assumption that F_c (or equivalently c, see Section 3.2) is bounded. We now define

$$\mathscr{A}_{\eta,\gamma}(N)=\big\{(n,\ell)\in[t_0,N]\times\mathscr{G}\mathbb{X}:\ell\in\mathscr{V}^+_{\eta,\,e^{-\lfloor\frac{n}{2}\rfloor}R}(\mathsf{g}^{\lfloor\frac{n}{2}\rfloor}\Omega^-)\cap\mathscr{V}^-_{\eta,\,e^{-\lceil\frac{n}{2}\rceil}R}(\gamma\mathsf{g}^{-\lceil\frac{n}{2}\rceil}\Omega^+)\big\}\;.$$

The end of Step 3T simplifies as

$$-\,c_4\leqslant\; i_\eta(N)-\sum_{\substack{\gamma\in\Gamma\,:\;t_0+2\leqslant\lambda_\gamma\leqslant N\\ \alpha^-_\gamma|_{[0,\,\lambda_\gamma]}\in\Omega^-|_{[0,\,\lambda_\gamma]},\;\;\alpha^+_\gamma|_{[-\lambda_\gamma,\,0]}\in\Omega^+|_{[-\lambda_\gamma,\,0]}}} e^{\int_{\alpha_\gamma}\widetilde{F}_c}\;j_{\eta,\,\gamma}(N)\;\leqslant\; c_4\;.$$

The statement of Step 4T now simplifies as

$$j_{\eta,\gamma}(N)=1\;,$$

if $\eta<\frac{1}{2}$, and if $\gamma\in\Gamma$ is such that D^- and γD^+ do not intersect and λ_γ is large enough. We introduce in its proof the slightly modified notation

$$r^-_n=e^{-\lfloor\frac{n}{2}\rfloor}R,\;\; r^+_n=e^{-\lceil\frac{n}{2}\rceil}R,\;\; w^-_n=\mathsf{g}^{\lfloor\frac{n}{2}\rfloor}w^-\quad\text{and}\quad w^+_n=\mathsf{g}^{-\lceil\frac{n}{2}\rceil}w^+\,.$$

and we now take as x_γ the point at distance $\lfloor\frac{n}{2}\rfloor$ from its origin on the common perpendicular α_γ. Equation (11.14) becomes (using Equation (11.27) instead of Equation (10.3))

$$j_{\eta,\,\gamma}(N)=\iint_{(n,\,\ell)\in\mathscr{A}_{\eta,\,\gamma}(N)}\frac{dn\;d\widetilde{m}_c(\ell)}{\mu_{W^+(w^-_n)}(B^+(w^-_n,\,r^-_n))\;\mu_{W^-(w^+_n)}(B^-(w^+_n,\,r^+_n))}\,.$$

Since $\ell(0) = x_\gamma$ if $(n,\ell) \in \mathscr{A}_{\eta,\gamma}(N)$, Equation (11.15) simplifies as

$$d\widetilde{m}_c(\ell) = d\mu^-_{x_\gamma}(\ell_-)\, d\mu^+_{x_\gamma}(\ell_+)\, ds\ ,$$

with ds the counting measure on the Hopf parameter $s \in \mathbb{Z}$ of ℓ (with basepoint x_γ). If $\eta < \frac{1}{2}$, replacing P_γ with its intersection with $\mathbb{Z}^2$ reduces it to one point $(\lambda_\gamma, 0)$, and now $s = s^\pm = 0$. Lemma 11.5 becomes

$$(B^+(w^-_n, r^-_n))_- = B_{d_{x_\gamma}}(\xi^-_\gamma, R\, e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor}), \quad (B^-(w^+_n, r^+_n))_+ = B_{d_{x_\gamma}}(\xi^+_\gamma, R\, e^{-\lceil \frac{\lambda_\gamma}{2} \rceil})\,,$$

so that

$$\mathscr{A}_{\eta,\gamma}(N) = \{(\lambda_\gamma, 0)\} \times B_{d_{x_\gamma}}(\xi^-_\gamma, R\, e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor}) \times B_{d_{x_\gamma}}(\xi^+_\gamma, R\, e^{-\lceil \frac{\lambda_\gamma}{2} \rceil})\,.$$

Finally, since $\ell(0) = x_\gamma$ if $(n,\ell) \in \mathscr{A}_{\eta,\gamma}(N)$, Equation (11.16) becomes

$$\mu_{W^+(w^-_n)}(B^+(w^-_n, r^-_n)) = \mu^-_{x_\gamma}(B_{d_{x_\gamma}}(\xi^-_\gamma, R\, e^{-\lfloor \frac{\lambda_\gamma}{2} \rfloor})),$$
$$\mu_{W^-(w^+_t)}(B^-(w^+_n, r^+_n)) = \mu^+_{x_\gamma}(B_{d_{x_\gamma}}(\xi^+_\gamma, R\, e^{-\lceil \frac{\lambda_\gamma}{2} \rceil}))\,.$$

The last centred equation in Step 4T now reduces to $j_{\eta,\gamma}(T) = 1$. □

For lattices in regular trees, we get more explicit expressions.

Corollary 11.10. *Let $\mathbb{X}$ be a $(q+1)$-regular simplicial tree (with $q \geqslant 2$) and let Γ be a lattice of $\mathbb{X}$ such that $\Gamma\backslash\mathbb{X}$ is not bipartite. Assume that the Patterson density is normalised to be a family of probability measures. Let $\mathbb{D}^\pm$ be nonempty proper simplicial subtrees of $\mathbb{X}$ with stabilisers $\Gamma_{\mathbb{D}^\pm}$ in Γ, such that $\mathscr{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^\pm}}$ is locally finite. Let $\sigma^\mp_{\mathscr{D}^\pm}$ be their skinning measures for the sero system of conductances. Then*

$$\lim_{t\to+\infty} \frac{q-1}{q+1}\ \mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X})\ q^{-t} \sum_{\substack{(\alpha,\beta,\gamma)\in\Gamma/\Gamma_{\mathbb{D}^-}\times\Gamma/\Gamma_{\mathbb{D}^+}\times\Gamma \\ 0<d(\alpha\mathbb{D}^-,\,\gamma\beta\mathbb{D}^+)\leqslant t}} \Delta_{\alpha^-_{\alpha,\,\gamma\beta}} \otimes \Delta_{\alpha^+_{\gamma^{-1}\alpha,\,\beta}} \;=\; \widetilde{\sigma}^+_{\mathscr{D}^-} \otimes \widetilde{\sigma}^-_{\mathscr{D}^+}\ ,$$

for the weak-star convergence of measures on the locally compact space $\check{\mathscr{G}}\mathbb{X} \times \check{\mathscr{G}}\mathbb{X}$.

If the measure $\sigma^-_{\mathscr{D}^+}$ is nonzero and finite, then

$$\lim_{t\to+\infty} \frac{q-1}{q+1}\ \frac{\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X})}{\|\sigma^-_{\mathscr{D}^+}\|}\ q^{-t} \sum_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+},\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant t} \Delta_{\alpha^-_{e,\,\gamma}} \;=\; \widetilde{\sigma}^+_{\mathbb{D}^-}\ ,$$

for the weak-star convergence of measures on the locally compact space $\check{\mathscr{G}}\mathbb{X}$.

Proof. In order to prove the first claim, we apply Theorem 11.9 with $\widetilde{c} \equiv 0$, so that by Proposition 4.16, Theorem 4.17, and Proposition 8.1 (3), we have $\delta_c = \ln q > 0$, $m_c = m_{\rm BM}$ is finite and mixing, and $\|m_{\rm BM}\| = \frac{q}{q+1}\,\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X})$.

The second claim follows by restricting to $\alpha = \beta = e$ and integrating on an appropriate fundamental domain (note that Equation (11.4) does not require Ω^+ to be relatively compact, just to have finite measure for $\widetilde{\sigma}^-$). □

The mixing assumption in Theorem 11.9 implies that the length spectrum L_Γ of Γ is equal to $\mathbb{Z}$. The next result considers the other case, when only the square of the geodesic flow is mixing, while appropriately restricted. Note that the smallest nonempty Γ-invariant simplicial subtree of $\mathbb{X}$ is uniform, without vertices of degree 2, for instance in the case when $\mathbb{X}$ is $(p+1, q+1)$-biregular with $p, q \geqslant 2$ and Γ is a lattice of $\mathbb{X}$.

Theorem 11.11. *Assume that the smallest nonempty Γ-invariant simplicial subtree of $\mathbb{X}$ is uniform, without vertices of degree 2, and that the length spectrum L_Γ of Γ is $2\mathbb{Z}$. Assume that the critical exponent δ_c of $\widetilde{c}$ is positive, that the Gibbs measure m_c is finite and that its restriction to $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ is mixing for the square of the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$. Then*

$$\lim_{t\to+\infty}\frac{e^{2\,\delta_c}-1}{2\,e^{2\,\delta_c}}\,\|m_c\|\,e^{-\delta_c\,t}\sum_{\substack{i\in I^-/_\sim,\,j\in I^+/_\sim,\,\gamma\in\Gamma\\ D_i^-\cap D_{\gamma j}^+=\varnothing,\,\lambda_{i,\,\gamma j}\leqslant t}} e^{\widetilde{c}(\alpha_{i,\gamma j})}\,\Delta_{\alpha^-_{i,\,\gamma j}}\otimes\Delta_{\alpha^+_{\gamma^{-1}i,\,j}} = \widetilde{\sigma}^+_{\mathscr{D}^-}\otimes\widetilde{\sigma}^-_{\mathscr{D}^+}$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}\mathbb{X}\times\breve{\mathscr{G}}\mathbb{X}$.

Proof. We denote by $\widetilde{\sigma}^\pm_{\mathscr{D}^\mp,\,{\rm even}}$ the restriction of $\widetilde{\sigma}^\pm_{\mathscr{D}^\mp}$ to $\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$, and by $\widetilde{\sigma}^\pm_{\mathscr{D}^\mp,\,{\rm odd}}$ the restriction of $\widetilde{\sigma}^\pm_{\mathscr{D}^\mp}$ to $\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}=\breve{\mathscr{G}}\mathbb{X}-\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$. We denote by $V_{\rm even}\mathbb{X}$ the subset of $V\mathbb{X}$ consisting of the vertices at even distance from x_0, and by $V_{\rm odd}\mathbb{X}=V\mathbb{X}-V_{\rm even}\mathbb{X}$ its complement. The subsets $V_{\rm even}\mathbb{X}$ and $V_{\rm odd}\mathbb{X}$ are Γ-invariant if $L_\Gamma=2\mathbb{Z}$ by Equation (4.17).

Let us first prove that

$$\lim_{t\to+\infty}\frac{e^{2\,\delta_c}-1}{2\,e^{2\,\delta_c}}\,\|m_c\|\,e^{-\delta_c\,t}\sum_{\substack{i\in I^-/_\sim,\,j\in I^+/_\sim,\,\gamma\in\Gamma\\ \pi(\alpha^-_{i,\gamma j}),\,\pi(\alpha^+_{\gamma^{-1}i,j})\,\in\,V_{\rm even}\mathbb{X}\\ D_i^-\cap D_{\gamma j}^+=\varnothing,\,\lambda_{i,\,\gamma j}\leqslant t}} e^{\widetilde{c}(\alpha_{i,\gamma j})}\,\Delta_{\alpha^-_{i,\,\gamma j}}\otimes\Delta_{\alpha^+_{\gamma^{-1}i,\,j}} = \widetilde{\sigma}^+_{\mathscr{D}^-,\,{\rm even}}\otimes\widetilde{\sigma}^-_{\mathscr{D}^+,\,{\rm even}} \tag{11.28}$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}\times\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$.

The proof of this Equation (11.28) is a modification of the proof of the previous Theorem 11.9. We now restrict to $t=2n\in\mathbb{N}$, $T=2N\in\mathbb{N}$, and we replace $\widetilde{m}_c$ by $(\widetilde{m}_c)|_{\mathscr{G}_{\rm even}\mathbb{X}}$ and $(\mathsf{g}^t)_{t\in\mathbb{Z}}$ by $(\mathsf{g}^{2t})_{t\in\mathbb{Z}}$. Note that since $\widetilde{m}_c$ is invariant under the time 1 of the geodesic flow, which maps $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ to $\Gamma\backslash\mathscr{G}\mathbb{X}-\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$, we have

$$\big\|(m_c)|_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}}\big\| = \frac{1}{2}\,\|m_c\|\,. \tag{11.29}$$

Note that for all $i\in I^-$, $j\in I^+$ and $\gamma\in\Gamma$, if $\pi(\alpha^-_{i,\gamma j})$ and $\pi(\alpha^+_{\gamma^{-1}i,j})$ belong to $V_{\rm even}\mathbb{X}$, then the distance between D_i^- and γD_j^+ is even.[14]

In Steps 1 and 2, we now consider $\Omega^\pm$ two Borel subsets of $\partial^1_\mp D^\pm\cap\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$, and we define instead of Equation (11.3)

$$I_{\Omega^-,\,\Omega^+}(2N)=(e^{2\,\delta_c}-1)\,\frac{\|m_c\|}{2}\,e^{-2\,\delta_c(N+1)}\times\sum_{\substack{\gamma\in\Gamma\,:\,0<\lambda_\gamma\leqslant 2N,\;\;\pi(\alpha^-_\gamma),\,\pi(\alpha^+_\gamma)\,\in\,V_{\rm even}\mathbb{X}\\ \alpha^-_\gamma|_{]0,\lambda_\gamma]}\in\Omega^-|_{]0,\lambda_\gamma]},\;\;\alpha^+_\gamma|_{]-\lambda_\gamma,0]}\in\Omega^+|_{]-\lambda_\gamma,0]}}} e^{\int_{\alpha_\gamma}\widetilde{F}_c}\,,$$

[14]Indeed, for all x,y,z in a simplicial tree, if p is the closest point to x on $[y,z]$, then $d(y,z)=d(y,x)+d(x,z)-2\,d(x,p)$.

and instead of Equation (11.5)

$$a_\eta(2n) = \sum_{\gamma\in\Gamma} \int_{\ell\in\mathscr{G}_{\rm even}\mathbb{X}} \phi^-_\eta(\mathsf{g}^{-2\lfloor n/2\rfloor}\ell)\ \phi^+_\eta(\mathsf{g}^{2\lceil n/2\rceil}\gamma^{-1}\ell)\ d\widetilde{m}_c(\ell)\,.$$

Equation (11.6) is replaced by

$$i_\eta(2N) = \sum_{n=0}^{N} e^{\delta_c\,2n}\ a_\eta(2n)\,.$$

The mixing property of the square of the geodesic flow on $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ for the restriction of the Gibbs measure m_c gives that, for every $\epsilon > 0$, there exists $T_\epsilon = T_{\epsilon,\eta} \geqslant 0$ such that for all $n \geqslant T_\epsilon$, we have

$$\frac{e^{-\epsilon}\int_{\mathscr{G}_{\rm even}\mathbb{X}}\phi^-_\eta\ d\widetilde{m}_c\int_{\mathscr{G}_{\rm even}\mathbb{X}}\phi^+_\eta\ d\widetilde{m}_c}{\|(m_c)|_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}}\|} \leqslant\ a_\eta(2n) \leqslant \frac{e^{\epsilon}\int_{\mathscr{G}_{\rm even}\mathbb{X}}\phi^-_\eta\ d\widetilde{m}_c\int_{\mathscr{G}_{\rm even}\mathbb{X}}\phi^+_\eta\ d\widetilde{m}_c}{\|(m_c)|_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}}\|}\,.$$

Note that $\mathscr{G}_{\rm even}\mathbb{X}$ is saturated by the strong stable and strong unstable leaves, since two points x, y on a given horosphere of centre $\xi\in\partial_\infty X$ are at even distance one from another (equal to $2d(x,p)$ where $[x,\xi[\ \cap\ [y,\xi[\ = [p,\xi[\,$). By the disintegration statement in Proposition 7.7, when ℓ ranges over $\mathscr{U}^\pm_D\cap\mathscr{G}_{\rm even}\mathbb{X}$, we have

$$d\widetilde{m}_c|_{\mathscr{U}^\pm_D\cap\mathscr{G}_{\rm even}\mathbb{X}}(\ell) = \int_{\rho\in\partial^1_\pm D\cap\check{\mathscr{G}}_{\rm even}\,\mathbb{X}} d\nu^\mp_\rho(\ell)\, d\widetilde{\sigma}^\pm_D(\rho)\,.$$

Hence the proof of Lemma 10.1 extends to give

$$\int_{\mathscr{G}_{\rm even}\mathbb{X}}\phi^\mp_\eta\ d\widetilde{m}_c = \widetilde{\sigma}^\pm_{\rm even}(\Omega^\mp)\,, \tag{11.30}$$

where in order to simplify notation $\widetilde{\sigma}^\pm_{\rm even} = \widetilde{\sigma}^\pm_{D^\mp,\,{\rm even}}$.

Therefore, by Equations (11.29) and (11.30), and by a geometric sum argument, the pair of inequalities (11.7) becomes

$$\frac{2\ e^{-\epsilon}e^{2\delta_c\,(N+1)}\,\widetilde{\sigma}^+_{\rm even}(\Omega^-)\,\widetilde{\sigma}^-_{\rm even}(\Omega^+)}{(e^{2\delta_c}-1)\,\|m_c\|} - c_\epsilon \leqslant i_\eta(2N) \leqslant \frac{2\ e^{\epsilon}e^{2\delta_c\,(N+1)}\,\widetilde{\sigma}^+_{\rm even}(\Omega^-)\,\widetilde{\sigma}^-_{\rm even}(\Omega^+)}{(e^{2\delta_c}-1)\,\|m_c\|} + c_\epsilon\,.$$

Up to replacing the summations from $n = 0$ to N to summations on even numbers between 0 to $2N$, and replacing $\lfloor n/2\rfloor$ by $2\lfloor n/2\rfloor$ as well as $\lceil n/2\rceil$ by $2\lceil n/2\rceil$, the remaining part of the proof applies and gives the result, noting that in Claim (iii) of Step 3T, we furthermore have that the origin and endpoint of the constructed common perpendicular α_γ are in $V_{\rm even}\mathbb{X}$. This concludes the proof of Equation (11.28).

The remainder of the proof of Theorem 11.11 consists in proving versions of the equidistribution result Equation (11.28) in $\check{\mathscr{G}}_{\rm odd}\,\mathbb{X}\times\check{\mathscr{G}}_{\rm odd}\,\mathbb{X}$, $\check{\mathscr{G}}_{\rm even}\,\mathbb{X}\times\check{\mathscr{G}}_{\rm odd}\,\mathbb{X}$, $\check{\mathscr{G}}_{\rm odd}\,\mathbb{X}\times\check{\mathscr{G}}_{\rm even}\,\mathbb{X}$ respectively , and in summing these four contributions.

By applying Equation (11.28) by replacing x_0 by a vertex x_0' in $V_{\rm odd}\mathbb{X}$, which exchanges $V_{\rm even}\mathbb{X}$ and $V_{\rm odd}\mathbb{X}$, $\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$ and $\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}$, as well as $\widetilde{\sigma}^{\pm}_{\mathscr{D}^{\mp},\,{\rm even}}$ and $\widetilde{\sigma}^{\pm}_{\mathscr{D}^{\mp},\,{\rm odd}}$, we have

$$\lim_{t\to+\infty}\frac{e^{2\,\delta_c}-1}{2\,e^{2\,\delta_c}}\,\|m_c\|\;e^{-\delta_c\,t}\sum_{\substack{i\in I^-/_\sim,\;j\in I^+/_\sim,\;\gamma\in\Gamma\\ \pi(\alpha^-_{i,\gamma j}),\,\pi(\alpha^+_{\gamma^{-1}i,j})\in V_{\rm odd}\mathbb{X}\\ D^-_i\cap D^+_{\gamma j}=\varnothing,\;\lambda_{i,\,\gamma j}\leqslant t}} e^{\widetilde{c}\,(\alpha_{i,\gamma j})}\;\Delta_{\alpha^-_{i,\,\gamma j}}\otimes\Delta_{\alpha^+_{\gamma^{-1}i,\,j}}$$

$$=\;\widetilde{\sigma}^{+}_{\mathscr{D}^-,\,{\rm odd}}\otimes\widetilde{\sigma}^{-}_{\mathscr{D}^+,\,{\rm odd}}\qquad(11.31)$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}\times\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}$.

Let us now apply Equation (11.28) by replacing $\mathscr{D}^-=(D^-_i)_{i\in I^-}$ by $\mathscr{N}_1\mathscr{D}^-=(\mathscr{N}_1D^-_i)_{i\in I^-}$. Let us consider the map $\varphi_+:\breve{\mathscr{G}}\mathbb{X}\to\breve{\mathscr{G}}\mathbb{X}$, which maps a generalised geodesic line ℓ to the generalised geodesic line which coincides with $\mathtt{g}^{+1}\ell$ on $[0,+\infty[$ and is constant (with value $\ell(1)$) on $]-\infty,0[$. Note that this map is continuous and Γ-equivariant, and that it maps $\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$ in $\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}$ and $\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}$ in $\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$.

Furthermore, by convexity, φ_+ induces for every $i\in I^-$ a homeomorphism from $\partial^1_+D^-_i$ to $\partial^1_+\mathscr{N}_1D^-_i$, which sends $\partial^1_+D^-_i\cap\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}$ to $\partial^1_+\mathscr{N}_1D^-_i\cap\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$, such that, by Equation (7.8), for all $w\in\partial^1_+D^-_i\cap\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}$, if e_w is the first edge followed by w

$$d\widetilde{\sigma}^+_{D^-_i,\,{\rm odd}}(w)=e^{\widetilde{c}\,(e_w)-\delta_c}\;d\widetilde{\sigma}^+_{\mathscr{N}_1D^-_i,\,{\rm even}}(\varphi_+(w))\;.$$

Note that for all $\ell>0$, there is a one-to-one correspondence between the set of common perpendiculars of length ℓ, with origin and endpoint both in $V_{\rm even}$, between $\mathscr{N}_1D^-_i$ and γD^+_j for all $i\in I^-$, $j\in I^+$ and $\gamma\in\Gamma$, and the set of common perpendiculars of length $\ell+1$, with origin in $V_{\rm odd}$ and endpoint in $V_{\rm even}$, between D^-_i and γD^+_j for all $i\in I^-$, $j\in I^+$ and $\gamma\in\Gamma$. In particular, $\varphi_+(\alpha^-_{i,\gamma j})$ is the common perpendicular between $\mathscr{N}_1D^-_i$ and γD^+_j, starting at time $t=0$ from $\mathscr{N}_1D^-_i$.

Therefore Equation (11.28) applied by replacing $\mathscr{D}^-=(D^-_i)_{i\in I^-}$ by $\mathscr{N}_1\mathscr{D}^-=(\mathscr{N}_1D^-_i)_{i\in I^-}$ gives

$$\lim_{t\to+\infty}\frac{e^{2\,\delta_c}-1}{2\,e^{2\,\delta_c}}\,\|m_c\|\;e^{-\delta_c\,t}$$
$$\sum_{\substack{i\in I^-/_\sim,\;j\in I^+/_\sim,\;\gamma\in\Gamma\\ \pi(\alpha^-_{i,\gamma j})\in V_{\rm odd}\mathbb{X},\,\pi(\alpha^+_{\gamma^{-1}i,j})\in V_{\rm even}\mathbb{X}\\ D^-_i\cap D^+_{\gamma j}=\varnothing,\;\lambda_{i,\,\gamma j}\leqslant t+1}} e^{\widetilde{c}\,(e_{\alpha^-_{i,\,\gamma j}})+\widetilde{c}\,(\varphi_+(\alpha^-_{i,\gamma j}))}\;\Delta_{\alpha^-_{i,\,\gamma j}}\otimes\Delta_{\alpha^+_{\gamma^{-1}i,\,j}}$$
$$=\;e^{\delta_c}\;\widetilde{\sigma}^{+}_{\mathscr{D}^-,\,{\rm odd}}\otimes\widetilde{\sigma}^{-}_{\mathscr{D}^+,\,{\rm even}}$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}\times\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$.

Since $\widetilde{c}(e_{\alpha^-_{i,\gamma j}}) + \widetilde{c}(\varphi_+(\alpha^-_{i,\gamma j})) = \widetilde{c}(\alpha^-_{i,\gamma j})$, replacing t by $t-1$ and simplifying by e^{δ_c}, we get

$$\lim_{t\to+\infty} \frac{e^{2\,\delta_c}-1}{2\,e^{2\,\delta_c}}\,\|m_c\|\,e^{-\delta_c\,t} \sum_{\substack{i\in I^-/\sim,\ j\in I^+/\sim,\ \gamma\in\Gamma\\ \pi(\alpha^-_{i,\gamma j})\in V_{\rm odd}\mathbb{X},\ \pi(\alpha^+_{\gamma^{-1}i,j})\in V_{\rm even}\mathbb{X}\\ D^-_i\cap D^+_{\gamma j}=\varnothing,\ \lambda_{i,\gamma j}\leqslant t}} e^{\widetilde{c}(\alpha^-_{i,\gamma j})}\ \Delta_{\alpha^-_{i,\gamma j}}\otimes\Delta_{\alpha^+_{\gamma^{-1}i,j}}$$
$$= \ \widetilde{\sigma}^+_{\mathscr{D}^-,\,\rm odd}\otimes\widetilde{\sigma}^-_{\mathscr{D}^+,\,\rm even} \tag{11.32}$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}_{\rm odd}\,\mathbb{X}\times\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$.

Now Theorem 11.11 follows by summing Equation (11.28), Equation (11.31), Equation (11.32) and the formula, proven similarly, obtained from Equation (11.28) by replacing $\mathscr{D}^+ = (D^+_j)_{j\in I^+}$ by $(\mathscr{N}_1 D^+_j)_{j\in I^+}$. □

The following result for bipartite graphs (of groups) is used in the arithmetic applications in Part III (see Section 15.4).

Corollary 11.12. *Let $\mathbb{X}$ be a $(p+1,q+1)$-biregular simplicial tree (with $p,q\geqslant 2$, possibly with $p=q$), with corresponding partition $V\mathbb{X} = V_p\mathbb{X}\sqcup V_q\mathbb{X}$. Let Γ be a lattice of $\mathbb{X}$ such that this partition is Γ-invariant. Assume that the Patterson density is normalised so that $\|\mu_x\| = \frac{p+1}{\sqrt{p}}$ for every $x\in V_p\mathbb{X}$. Let $\mathbb{D}^\pm$ be nonempty proper simplicial subtrees of $\mathbb{X}$ with stabilisers $\Gamma_{\mathbb{D}^\pm}$ in Γ, such that the families $\mathscr{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite. Let $\sigma^\mp_{\mathscr{D}^\pm}$ be their skinning measures for the sero system of conductances. Then*

$$\lim_{t\to+\infty}\frac{pq-1}{2}\,{\rm TVol}(\Gamma\backslash\backslash\mathbb{X})\,\sqrt{pq}^{\,-t-2}\sum_{\substack{(\alpha,\beta,\gamma)\in\Gamma/\Gamma_{\mathbb{D}^-}\times\Gamma/\Gamma_{\mathbb{D}^+}\times\Gamma\\ 0<d(\alpha\mathbb{D}^-,\gamma\beta\mathbb{D}^+)\leqslant t}}\Delta_{\alpha^-_{\alpha,\gamma\beta}}\otimes\Delta_{\alpha^+_{\gamma^{-1}\alpha,\beta}}$$
$$= \ \widetilde{\sigma}^+_{\mathscr{D}^-}\otimes\widetilde{\sigma}^-_{\mathscr{D}^+}$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}\mathbb{X}\times\breve{\mathscr{G}}\mathbb{X}$.

If the measure $\sigma^-_{\mathscr{D}^+}$ is nonzero and finite, then

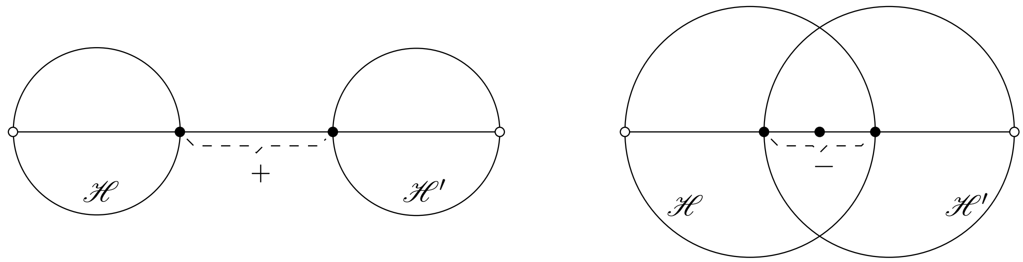
$$\lim_{t\to+\infty}\frac{pq-1}{2}\,\frac{{\rm TVol}(\Gamma\backslash\backslash\mathbb{X})}{\|\sigma^-_{\mathscr{D}^+}\|}\,\sqrt{pq}^{\,-t-2}\sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\gamma\mathbb{D}^+)\leqslant t}}\Delta_{\alpha^-_{e,\gamma}} \ = \ \widetilde{\sigma}^+_{\mathbb{D}^-}\ ,$$

for the weak-star convergence of measures on the locally compact space $\breve{\mathscr{G}}\mathbb{X}$.

Proof. In order to prove the first result, we apply Theorem 11.11 with $\widetilde{c}\equiv 0$, so that by Equation (8.1), Proposition 4.16, Theorem 4.17, and Proposition 8.1 (2), we have $\delta_c = \frac{1}{2}\ln(pq) > 0$, $m_c = m_{\rm BM}$ is finite and its restriction to $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ is mixing under the square of the geodesic flow, and $\|m_{\rm BM}\| = {\rm TVol}(\Gamma\backslash\backslash\mathbb{X})$.

The second claim follows as in the proof of Corollary 11.10. □

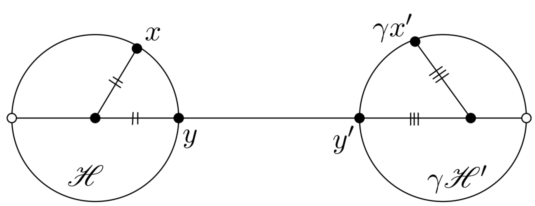
Remark. In some special occasions, the measures involved in the statements of Theorem 11.11 and Corollary 11.12 (whether skinning measures or Dirac masses) are actually all supported on $\breve{\mathscr{G}}_{\rm even}\,\mathbb{X}$ (up to choosing appropriately x_0). This is in particular the case if $\mathscr{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^\pm}}$ with $\mathbb{D}^-,\mathbb{D}^+$ horoballs at even signed distance (see below), as the following lemma shows.



The *signed distance* between horoballs $\mathscr{H}$ and $\mathscr{H}'$ in an $\mathbb{R}$-tree that are not centred at the same point at infinity is the distance between them (that is, the length of their common perpendicular) if they are disjoint, or the opposite of the diameter of their intersection otherwise. Note that if nonempty, the intersection of $\mathscr{H}$ and $\mathscr{H}'$ is a ball centred at the midpoint of the segment contained in the geodesic line between the two points at infinity of the horoballs, which lies in both horoballs.

Lemma 11.13. *Let $\mathbb{X}$ be a simplicial tree, Γ a subgroup of $\operatorname{Aut}(\mathbb{X})$ and $\mathscr{H}, \mathscr{H}'$ two horoballs in $\mathbb{X}$ (whose boundaries are contained in $V\mathbb{X}$), which either are equal or have distinct points at infinity. If $\Lambda\Gamma \subset 2\mathbb{Z}$ and $\mathscr{H}, \mathscr{H}'$ are at even signed distance, then the signed distance between $\mathscr{H}$ and $\gamma\mathscr{H}'$ is even for every $\gamma \in \Gamma$ such that $\mathscr{H}$ and $\gamma\mathscr{H}'$ do not have the same point at infinity.*

Proof. Fix such a γ. For every horoball $\mathscr{H}''$ and for all $s \in \mathbb{N}$, let $\mathscr{H}''[s]$ be the horoball contained in $\mathscr{H}$, whose boundary is at distance s from the boundary of $\mathscr{H}$. Shrinking the horoballs $\mathscr{H}$ and $\mathscr{H}'$, by replacing them by the horoballs $\mathscr{H}[s]$ and $\mathscr{H}'[s]$ for any $s \in \mathbb{N}$, only changes by $\pm 2s$ the considered signed distances. Hence, taking s large enough, we may assume that $\mathscr{H}$ and $\gamma\mathscr{H}'$ are disjoint, and that $\mathscr{H}$ and $\mathscr{H}'$ are disjoint or equal. Let $[x, x']$ be the common perpendicular between $\mathscr{H}$ and $\mathscr{H}'$ with $x \in \partial\mathscr{H}$, $x' \in \partial\mathscr{H}'$ if $\mathscr{H}$ and $\mathscr{H}'$ are disjoint, and otherwise, let $x = x'$ be any point in $\partial\mathscr{H} = \partial\mathscr{H}'$. Let $[y, y']$ be the common perpendicular between $\mathscr{H}$ and $\gamma\mathscr{H}'$, with $y \in \partial\mathscr{H}$, $y' \in \partial(\gamma\mathscr{H}')$. Note that $\gamma x' \in \partial(\gamma\mathscr{H}')$.



The distance between two points x, y of a horosphere is always even (equal to twice the distance from x to the geodesic ray from y to the point at infinity of the horosphere). Since geodesic triangles in trees are tripods, for all a, b, c in a simplicial tree, since

$$d(a, c) = d(a, b) + d(b, c) - 2d(b, [a, c]) ,$$

if $d(a, b)$ and $d(b, c)$ are even, so is $d(a, c)$.

Since $\Lambda\Gamma \subset 2\mathbb{Z}$, the distance between x' and $\gamma x'$ is even by Equation (4.17). Since $d(x, x')$ is even by assumption, we hence have that $d(x, \gamma x')$ is even. Therefore

$$d(y, y') = d(x, \gamma x') - d(x, y) - d(y', \gamma x')$$

is even. □

Chapter 12

Equidistribution and counting of common perpendiculars in quotient spaces

In this Chapter, we use the results of Chapter 11 to prove equidistribution and counting results in Riemannian manifolds (or good orbifolds) and in metric and simplicial graphs (of groups).

Let X, x_0, Γ and $\widetilde{F}$ be as in the beginning of Chapter 11. We will need the following two notions in this chapter.

Recall that the *narrow topology*[1] on the set $\mathscr{M}_{\rm f}(Y)$ of finite measures on a Polish space Y is the smallest topology such that the map from $\mathscr{M}_{\rm f}(Y)$ to $\mathbb{R}$ defined by $\mu \mapsto \mu(g)$ is continuous for every bounded continuous function $g : Y \to \mathbb{R}$. Since continous functions with compact support are bounded, the narrow convergence of finite measures implies their weak-star convergence.

Recall that given a discrete group G acting properly (but not necessarily freely) on a locally compact space Z, the *induced measure*[2] on $G\backslash Z$ of a (positive, Radon) measure μ on Z is a measure $\overline{\mu}$ which depends linearly and continuously for the weak-star topology on μ, and satisfies $\overline{\Delta_z} = \frac{1}{|G_z|}\,\Delta_{Gz}$ for every $z \in Z$. The following observation on the behaviour of induced measures under quotients by properly discontinuous group actions will be used in the proofs of Corollary 12.3 and its analogues in Section 12.4. Let G be a discrete group that acts properly on a Polish space $\widetilde{Y}$ and let $Y = G\backslash\widetilde{Y}$. Let $\widetilde{\mu}_k$ for $k \in \mathbb{N}$ and $\widetilde{\mu}$ be G-invariant locally finite measures on $\widetilde{Y}$, with finite induced measures μ_k for $k \in \mathbb{N}$ and μ on Y. If for every Borel subset B of $\widetilde{Y}$ with $\widetilde{\mu}(B)$ finite and $\widetilde{\mu}(\partial B) = 0$ we have $\lim_{k\to\infty} \widetilde{\mu}_k(B) = \widetilde{\mu}(B)$, then the sequence $(\mu_k)_{k\in\mathbb{N}}$ narrowly converges to μ.

12.1 Multiplicities and counting functions in Riemannian orbifolds

In this Section, we assume that $X = \widetilde{M}$ is a Riemannian manifold. We denote its quotient Riemannian orbifold under Γ by $M = \Gamma\backslash\widetilde{M}$, and the quotient Riemannian orbifold under Γ

[1] also called *weak topology* see for instance [DM, p. 71-III] or [Bil, Part]

[2] See for instance [PauPS, §2.6] for details.

of its unit tangent bundle by $T^1M = \Gamma\backslash T^1\widetilde{M}$. We use the identifications $\mathscr{G}X = \mathscr{G}_{\pm,0}X = T^1X = T^1\widetilde{M}$ explained in Chapter 2.

Let $\mathscr{D} = (D_i)_{i\in I}$ be a locally finite Γ-equivariant family of nonempty proper closed convex subsets of $\widetilde{M}$. Let $\Omega = (\Omega_i)_{i\in I}$ be a Γ-equivariant family of subsets of $T^1\widetilde{M}$, where Ω_i is a measurable subset of $\partial^1_\pm D_i$ for all $i \in I$ (the sign $\pm$ being constant) and $\Omega_i = \Omega_j$ if $i \sim_{\mathscr{D}} j$. The *multiplicity* of an element $v \in T^1M$ with respect to Ω is

$$m_\Omega(v) = \frac{\operatorname{Card}\{i \in I/_\sim \;:\; \widetilde{v} \in \Omega_i\}}{\operatorname{Card}(\operatorname{Stab}_\Gamma \widetilde{v})}\;,$$

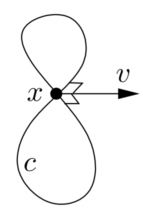
for any preimage $\widetilde{v}$ of v in $T^1\widetilde{M}$. The numerator and the denominator are finite by the local finiteness of the family $\mathscr{D}$ and the discreteness of Γ, and they depend only on the orbit of $\widetilde{v}$ under Γ.

The numerator takes into account the multiplicities of the images of the elements of Ω in T^1M. The denominator of this multiplicity is also natural, as any counting problem of objects possibly having symmetries, the appropriate counting function consists in taking as the multiplicity of an object the inverse of the cardinality of its symmetry group.

Examples 12.1. The following examples illustrate the behaviour of the multiplicity when Γ is torsion-free and $\Omega = \partial^1_\pm\mathscr{D}$.

(1) If for every $i \in I$, the quotient $\Gamma_{D_i}\backslash D_i$ of D_i by its stabiliser Γ_{D_i} maps injectively in M by the map induced by the inclusion of D_i in $\widetilde{M}$, and if for every $i, j \in I$ such that $j \notin \Gamma i$, the intersection $D_i \cap D_j$ is empty, then the nonzero multiplicities $m_\Omega(\ell)$ are all equal to 1.

(2) Here is a simple example of a multiplicity different from 0 or 1. Let c be a closed geodesic in the Riemannian manifold M, let $\widetilde{c}$ be a geodesic line in $\widetilde{M}$ mapping to c in M, let $\mathscr{D} = (\gamma\,\widetilde{c})_{\gamma\in\Gamma}$, let x be a double point of c, let $v \in T^1_xM$ be orthogonal to the two tangent lines to c at x (this requires the dimension of $\widetilde{M}$ to be at least 3, if x is a transverse self-intersection point). Then $m_{\partial^1_\pm\mathscr{D}}(v) = 2$.



Given $t > 0$ and two unit tangent vectors $v, w \in T^1M$, we define the number $n_t(v, w)$ of locally geodesic paths having v and w as initial and terminal tangent vectors respectively, weighted by the potential F, with length at most t, by

$$n_t(v, w) = \sum_\alpha \operatorname{Card}(\Gamma_\alpha)\; e^{\int_\alpha F}\;,$$

where the sum ranges over the locally geodesic paths $\alpha : [0, s] \to M$ in the Riemannian orbifold M such that $\dot\alpha(0) = v$, $\dot\alpha(s) = w$ and $s \in\,]0, t]$, and Γ_α is the stabiliser in Γ of any geodesic path $\widetilde{\alpha}$ in $\widetilde{M}$ mapping to α by the quotient map $\widetilde{M} \to M$. If $F = 0$ and Γ is torsion free, then $n_t(v, w)$ is precisely the number of locally geodesic paths having v and w as initial and terminal tangent vectors respectively, with length at most t.

Let $\Omega^- = (\Omega^-_i)_{i\in I^-}$ and $\Omega^+ = (\Omega^+_j)_{j\in I^+}$ be Γ-equivariant families of subsets of $T^1\widetilde{M}$, where $\Omega^\mp_k$ is a measurable subset of $\partial^1_\pm D^\mp_k$ for all $k \in I^\mp$ and $\Omega^\pm_k = \Omega^\pm_{k'}$ if $k \sim_{\mathscr{D}^\pm} k'$. We will denote by $\mathscr{N}_{\Omega^-,\Omega^+,F} : \,]0, +\infty[\, \to \mathbb{R}$ the following *counting function*: for every $t > 0$, let $\mathscr{N}_{\Omega^-,\Omega^+,F}(t)$ be the number of common perpendiculars whose initial vectors belong to the images in T^1M of the elements of Ω^- and terminal vectors to the images in T^1M of the elements of Ω^+, counted with multiplicities and weighted by the potential F, that is:

$$\mathscr{N}_{\Omega^-,\,\Omega^+,\,F}(t)=\sum_{v,\,w\,\in\,T^1M} m_{\Omega^-}(v)\; m_{\Omega^+}(w)\; n_t(v,w)\;.$$

When $\Omega^\pm=\partial^1_\mp\mathscr{D}^\pm$, we denote $\mathscr{N}_{\Omega^-,\,\Omega^+,\,F}$ by $\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}$.

Remark 12.2. Let Y be a negatively curved complete connected Riemannian manifold and let $\widetilde{Y}\to Y$ be its Riemannian universal cover. Let $D^\pm$ be a locally convex[3] geodesic metric space endowed with a continuous map $f^\pm : D^\pm \to Y$ such that if $\widetilde{D}^\pm\to D^\pm$ is a locally isometric universal cover and if $\widetilde{f}^\pm:\widetilde{D}^\pm\to\widetilde{Y}$ is a lift of $f^\pm$, then $\widetilde{f}^\pm$ is on each connected component of $\widetilde{D}^\pm$ an isometric embedding whose image is a proper nonempty closed locally convex subset of $\widetilde{Y}$, and the family of images under the covering group of $\widetilde{Y}\to Y$ of the images by $\widetilde{f}^\pm$ of the connected components of $\widetilde{D}^\pm$ is locally finite. Then $D^\pm$ (or the pair $(D^\pm,f^\pm)$) is a *proper nonempty properly immersed closed locally convex subset* of Y.

If Γ is a discrete subgroup without torsion of isometries of a CAT(-1) Riemannian manifold X, if $\mathscr{D}^\pm=(\gamma\widetilde{D}^\pm)_{\gamma\in\Gamma}$ where $\widetilde{D}^\pm$ is a nonempty proper closed convex subset of X such that the family $\mathscr{D}^\pm$ is locally finite, and if $D^\pm$ is the image of $\widetilde{D}^\pm$ by the covering map $X\to\Gamma\backslash X$, then $D^\pm$ is a proper nonempty properly immersed closed convex subset of $\Gamma\backslash X$. Under these assumptions, $\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}$ is the counting function $\mathscr{N}_{D^-,\,D^+,\,F}$ given in the introduction.

Let us continue fixing the notation used in Sections 12.2 and 12.3. For every (i,j) in $I^-\times I^+$ such that D^-_i and D^+_j have a common perpendicular[4], we denote by $\alpha_{i,\,j}$ this common perpendicular, by $\lambda_{i,\,j}$ its length, by $v^-_{i,\,j}\in\partial^1_+D^-_i$ its initial tangent vector and by $v^+_{i,\,j}\in\partial^1_-D^+_i$ its terminal tangent vector. Note that if $i'\sim i$, $j'\sim j$ and $\gamma\in\Gamma$, then

$$\gamma\,\alpha_{i',\,j'}=\alpha_{\gamma i,\,\gamma j},\quad \lambda_{i',\,j'}=\lambda_{\gamma i,\,\gamma j}\quad\text{and}\quad \gamma\, v^\pm_{i',\,j'}=v^\pm_{\gamma i,\,\gamma j}\;. \tag{12.1}$$

When Γ has no torsion, we have, for the diagonal action of Γ on $I^-\times I^+$,

$$\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t)=\sum_{(i,\,j)\in\Gamma\backslash((I^-/\sim)\times(I^+/\sim))\,:\,\overline{D^-_i}\,\cap\,\overline{D^+_j}=\varnothing,\;\lambda_{i,\,j}\leqslant t} e^{\int_{\alpha_{i,\,j}}\widetilde{F}}\;.$$

When the potential F is zero and Γ has no torsion, $\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t)$ is the number of common perpendiculars of length at most t, and the counting function $t\mapsto\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,0}(t)$ has been studied in various special cases of negatively curved manifolds since the 1950's and in a number of recent works, see the Introduction. The asymptotics of $\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,0}(t)$ as $t\to+\infty$ in the case when X is a Riemannian manifold with pinched negative curvature are described in general in [PaP17b, Theo. 1], where it is shown that if the skinning measures $\sigma^+_{\mathscr{D}^-}$ and $\sigma^-_{\mathscr{D}^+}$ are finite and nonzero, then as $s\to+\infty$,

$$\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,0}(s)\sim\frac{\|\sigma^+_{\mathscr{D}^-}\|\,\|\sigma^-_{\mathscr{D}^+}\|}{\|m_{\rm BM}\|}\,\frac{e^{\delta_\Gamma\, s}}{\delta_\Gamma}\;. \tag{12.2}$$

[3] not necessarily connected

[4] that is, whose closures $\overline{D^-_i}$ and $\overline{D^+_j}$ in $X\cup\partial_\infty X$ have empty intersection

12.2 Common perpendiculars in Riemannian orbifolds

Corollary 12.3 below is the main result of this text on the counting with weights of common perpendiculars and on the equidistribution of their initial and terminal tangent vectors in negatively curved Riemannian manifolds endowed with a Hölder-continuous potential. We use the notation of Section 12.1.

Corollary 12.3. *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with pinched negative sectional curvature at most -1. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a bounded Γ-invariant Hölder-continuous function with positive critical exponent δ. Let $\mathscr{D}^- = (D_i^-)_{i\in I^-}$ and $\mathscr{D}^+ = (D_j^+)_{j\in I^+}$ be locally finite Γ-equivariant families of nonempty proper closed convex subsets of $\widetilde{M}$. Assume that the Gibbs measure m_F is finite and mixing for the geodesic flow on T^1M. Then,*

$$\begin{aligned}\lim_{t\to+\infty} \delta\,\|m_F\|\,e^{-\delta t} \sum_{v,\,w\in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\, m_{\partial^1_-\mathscr{D}^+}(w)\, n_t(v,w)\,\Delta_v\otimes\Delta_w \\ = \sigma^+_{\mathscr{D}^-}\otimes\sigma^-_{\mathscr{D}^+} \qquad (12.3)\end{aligned}$$

for the weak-star convergence of measures on the locally compact space $T^1M\times T^1M$. If $\sigma^+_{\mathscr{D}^-}$ and $\sigma^-_{\mathscr{D}^+}$ are finite, the result also holds for the narrow convergence.

Furthermore, for all Γ-equivariant families $\Omega^\pm = (\Omega^\pm_k)_{k\in I^\pm}$ of subsets of $T^1\widetilde{M}$ with $\Omega^\pm_k$ a Borel subset of $\partial^1_\mp D^\pm_k$ for all $k\in I^\pm$ and $\Omega^\pm_k = \Omega^\pm_{k'}$ if $k\sim_{\mathscr{D}^\pm} k'$, with nonzero finite skinning measure and with boundary in $\partial^1_\pm D^\mp_k$ of zero skinning measure, we have, as $t\to+\infty$,

$$\mathscr{N}_{\Omega^-,\,\Omega^+,\,F}(t) \;\sim\; \frac{\|\sigma^+_{\Omega^-}\|\,\|\sigma^-_{\Omega^+}\|}{\delta\,\|m_F\|}\,e^{\delta t}\,.$$

Proof. Note that the sum in Equation (12.3) is locally finite, hence it defines a locally finite measure on $T^1M\times T^1M$. We are going to rewrite the sum in the statement of Theorem 11.1 in a way which makes it easier to push it down from $T^1\widetilde{M}\times T^1\widetilde{M}$ to $T^1M\times T^1M$.

For every $\widetilde{v}\in T^1\widetilde{M}$, let

$$m^\mp(\widetilde{v}) = \operatorname{Card}\{k\in I^\mp/_\sim \;:\; \widetilde{v}\in\partial^1_\pm D^\mp_k\}\,,$$

so that for every $v\in T^1M$, the multiplicity of v with respect to the family $\partial^1_\pm\mathscr{D}^\mp$ is[5]

$$m_{\partial^1_\pm\mathscr{D}^\mp}(v) = \frac{m^\mp(\widetilde{v})}{\operatorname{Card}(\operatorname{Stab}_\Gamma\widetilde{v})}\,,$$

for any preimage $\widetilde{v}$ of v in $T^1\widetilde{M}$.

For all $\gamma\in\Gamma$ and $\widetilde{v},\widetilde{w}\in T^1\widetilde{M}$, there exists $(i,j)\in(I^-/_\sim)\times(I^+/_\sim)$ such that $\widetilde{v} = v^-_{i,\gamma j}$ and $\widetilde{w} = v^+_{\gamma^{-1}i,j} = \gamma^{-1}v^+_{i,\gamma j}$ if and only if $\gamma\widetilde{w}\in \mathtt{g}^{\mathbb{R}}\,\widetilde{v}$, there exists $i'\in I^-/_\sim$ such that $\widetilde{v}\in\partial^1_+D^-_{i'}$ and there exists $j'\in I^+/_\sim$ such that $\gamma\widetilde{w}\in\partial^1_-D^+_{j'}$. Then the choice of such elements (i,j), as

[5] See Section 12.1.

well as i' and j', is free. We hence have

$$\sum_{\substack{i\in I^-/_\sim,\, j\in I^+/_\sim,\, \gamma\in\Gamma \\ 0<\lambda_{i,\,\gamma j}\leqslant t\,,\; v^-_{i,\,\gamma j}=\widetilde{v}\,,\; v^+_{\gamma^{-1}i,\,j}=\widetilde{w}}} e^{\int_{\alpha_{i,\,\gamma j}}\widetilde{F}}\;\Delta_{v^-_{i,\,\gamma j}}\otimes\Delta_{v^+_{\gamma^{-1}i,\,j}}$$

$$=\sum_{\substack{\gamma\in\Gamma,\; 0<s\leqslant t \\ \gamma\widetilde{w}=\mathtt{g}^s\widetilde{v}}} e^{\int_{\pi(\widetilde{v})}^{\gamma\pi(\widetilde{w})}\widetilde{F}}\;\operatorname{Card}\big\{(i,j)\in(I^-/_\sim)\times(I^+/_\sim)\;:\;v^-_{i,\gamma j}=\widetilde{v}\,,\;v^+_{\gamma^{-1}i,\,j}=\widetilde{w}\big\}\;\Delta_{\widetilde{v}}\otimes\Delta_{\widetilde{w}}$$

$$=\sum_{\substack{\gamma\in\Gamma,\; 0<s\leqslant t \\ \gamma\widetilde{w}=\mathtt{g}^s\widetilde{v}}} e^{\int_{\pi(\widetilde{v})}^{\gamma\pi(\widetilde{w})}\widetilde{F}}\;m^-(\widetilde{v})\;m^+(\gamma\widetilde{w})\;\Delta_{\widetilde{v}}\otimes\Delta_{\widetilde{w}}\;.$$

Therefore

$$\sum_{\substack{i\in I^-/_\sim,\, j\in I^+/_\sim,\, \gamma\in\Gamma \\ 0<\lambda_{i,\,\gamma j}\leqslant t}} e^{\int_{\alpha_{i,\,\gamma j}}\widetilde{F}}\;\Delta_{v^-_{i,\,\gamma j}}\otimes\Delta_{v^+_{\gamma^{-1}i,\,j}}$$

$$=\sum_{\widetilde{v},\,\widetilde{w}\,\in\,T^1\widetilde{M}}\Big(\sum_{\substack{\gamma\in\Gamma,\; 0<s\leqslant t \\ \gamma\widetilde{w}=\mathtt{g}^s\widetilde{v}}} e^{\int_{\pi(\widetilde{v})}^{\gamma\pi(\widetilde{w})}\widetilde{F}}\Big)\;m^-(\widetilde{v})\;m^+(\widetilde{w})\;\Delta_{\widetilde{v}}\otimes\Delta\widetilde{w}\;.$$

By definition, $\sigma^\pm_{\mathscr{D}^\mp}$ is the measure on T^1M induced by the Γ-invariant measure $\widetilde{\sigma}^\pm_{\mathscr{D}^\mp}$. Thus Corollary 12.3 follows from Theorem 11.1 after a similar reduction[6] as in Section 11.1. The narrow convergence is obtained when the skinning measures $\sigma^\pm_{\mathscr{D}^\mp}$ are finite, using the continuity properties of the induced measures recalled at the beginning of Chapter 12, since no compactness assumptions were made in Equation (11.4) on $\Omega^\pm$.

The counting statement follows from the equidistribution result by integration. □

In particular, if the skinning measures $\sigma^+_{\mathscr{D}^-}$ and $\sigma^-_{\mathscr{D}^+}$ are positive and finite, Corollary 12.3 gives, as $t\to+\infty$,

$$\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t)\;\sim\;\frac{\|\sigma^+_{\mathscr{D}^-}\|\;\|\sigma^-_{\mathscr{D}^+}\|}{\delta\;\|m_F\|}\;e^{\delta\,t}\;.$$

Remark 12.4. Under the assumptions of Corollary 12.3 with the exception that δ may now be nonpositive, we have the following asymptotic result as $t\to+\infty$ for the growth of the weighted number of common perpendiculars with lengths in $]t-\tau,t]$ for every fixed $\tau>0$:

$$\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t)-\mathscr{N}_{\mathscr{D}^-,\,\mathscr{D}^+,\,F}(t-\tau)\;\sim\;\frac{(1-e^{-\delta\,\tau})\;\|\sigma^+_{\mathscr{D}^-}\|\;\|\sigma^-_{\mathscr{D}^+}\|}{\delta\;\|m_F\|}\;e^{\delta\,t}\;.$$

This result follows by considering a large enough constant σ such that $\delta_{\Gamma,\,F+\sigma}=\delta+\sigma>0$, by applying Corollary 12.3 with the potential $F+\sigma$ (see Remark 7.1 (2)) as in the proof of Theorem 11.3.

Using the continuity of the pushforwards of measures for the weak-star and the narrow topologies, applied to the basepoint maps $\pi\times\pi$ from $T^1\widetilde{M}\times T^1\widetilde{M}$ to $\widetilde{M}\times\widetilde{M}$, and from

[6] See Step 1 of the proof of Theorem 11.1.

$T^1M \times T^1M$ to $M \times M$, we have the following result of equidistribution of the ordered pairs of endpoints of common perpendiculars between two equivariant families of convex sets in $\widetilde{M}$ or two families of locally convex sets in M. When M has constant curvature and finite volume, $F = 0$ and $\mathscr{D}^-$ is the Γ-orbit of a point and $\mathscr{D}^+$ is the Γ-orbit of a totally geodesic cocompact submanifold, this result is due to Herrmann [Herr]. When $\mathscr{D}^\pm$ are Γ-orbits of points and F is a Hölder-continuous potential, see [PauPS, Theo. 9.1, 9.3], and we refer for instance to [BoyM] for an application of this particular case.

Corollary 12.5. *Let $\widetilde{M}, \Gamma, \widetilde{F}, \mathscr{D}^-, \mathscr{D}^+$ be as in Corollary 12.3. Then*

$$\lim_{t\to+\infty} \delta\, \|m_F\|\, e^{-\delta\, t} \sum_{\substack{i\in I^-/_\sim,\, j\in I^+/_\sim,\, \gamma\in\Gamma \\ 0<\lambda_{i,\,\gamma j}\leqslant t}} e^{\int_{\alpha_{i,\,\gamma j}} \widetilde{F}}\, \Delta_{\pi(v^-_{i,\,\gamma j})} \otimes \Delta_{\pi(v^+_{\gamma^{-1} i,\, j})}$$
$$= \pi_* \widetilde{\sigma}^+_{\mathscr{D}^-} \otimes \pi_* \widetilde{\sigma}^-_{\mathscr{D}^+}\;,$$

for the weak-star convergence of measures on the locally compact space $\widetilde{M} \times \widetilde{M}$, and

$$\lim_{t\to+\infty} \delta\, \|m_F\|\, e^{-\delta\, t} \sum_{v,\,w\in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\; m_{\partial^1_-\mathscr{D}^+}(w)\; n_t(v,w)\; \Delta_{\pi(v)} \otimes \Delta_{\pi(w)}$$
$$= \pi_*\sigma^+_{\mathscr{D}^-} \otimes \pi_*\sigma^-_{\mathscr{D}^+}\;,$$

for the weak-star convergence of measures on $M \times M$. If the measures $\sigma^\pm_{\mathscr{D}^\mp}$ are finite, then the above claim holds for the narrow convergence of measures on $M \times M$. □

We will now prove Theorems 1.4 and 1.5 (1) in the Introduction for Riemannian manifolds. Recall from Remark 12.2 the definition of proper nonempty properly immersed closed locally convex subsets $D^\pm$ in a pinched negatively curved complete connected Riemannian manifold Y and the associated maps $\widetilde{f}^\pm : \widetilde{D}^\pm \to \widetilde{Y}$.

Proof of Theorems 1.4 and 1.5 (1) for Riemannian manifolds. Let $Y, F, D^\pm$ be as in these statements and assume that Y is a Riemannian manifold. Let Γ be the covering group of a universal Riemannian cover $\widetilde{Y} \to Y$. Let $I^\pm = \Gamma \times \pi_0(\widetilde{D}^\pm)$ with the action of Γ defined by $\gamma \cdot (\alpha, c) = (\gamma\alpha, c)$ for all $\gamma, \alpha \in \Gamma$ and every connected component c of $\widetilde{D}^\pm$. Consider the families $\mathscr{D}^\pm = (D^\pm_k)_{k\in I^\pm}$ where $D^\pm_k = \alpha\, \widetilde{f}^\pm(c)$ if $k = (\alpha, c)$. Then $\mathscr{D}^\pm$ are Γ-equivariant families of nonempty proper closed convex subsets of $\widetilde{Y}$, which are locally finite since $D^\pm$ are properly immersed in Y. The conclusions in Theorems 1.4 and 1.5 (1) when Y is a manifold then follow from Corollary 12.3, applied with $\widetilde{M} = \widetilde{Y}$ and with $\widetilde{F}$ the lift of F to $T^1\widetilde{M}$. □

Corollary 12.6. *Let $\widetilde{M}, \Gamma, \widetilde{F}, \mathscr{D}^-, \mathscr{D}^+$ be as in Corollary 12.3. Assume that $\sigma^\pm_{\mathscr{D}^\mp}$ are finite and nonzero. Let*

$$n_{t,\mathscr{D}^+}(v) = \sum_{w\in T^1M} m_{\partial^1_-\mathscr{D}^+}(w)\; n_t(v,w)$$

be the number (counted with multiplicities) of locally geodesic paths in M of length at most t, with initial vector v, arriving perpendicularly to $\mathscr{D}^+$. Then

$$\lim_{s\to+\infty} \lim_{t\to+\infty} \frac{\delta\, \|m_F\|^2\, e^{-\delta t}}{\|\sigma^+_{\mathscr{D}^-}\|\|\sigma^-_{\mathscr{D}^+}\|} \sum_{v\in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\; n_{t,\,\mathscr{D}^+}(v)\; \Delta_{\mathtt{g}^s v} = m_F\,,$$

for the narrow convergence on $\Gamma\backslash\mathscr{G}\widetilde{M}$.

Proof. For every $s \in \mathbb{R}$, by Corollary 12.3, using the continuity of the pushforwards of measures by the first projection $(v, w) \mapsto v$ from $T^1M \times T^1M$ to T^1M, and by the geodesic flow on T^1M at time s, since $(\mathtt{g}^s)_*\Delta_v = \Delta_{\mathtt{g}^s v}$, we have

$$\lim_{t\to+\infty} \delta\,\|m_F\|\,e^{-\delta t} \sum_{v\in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\,n_{t,\mathscr{D}^+}(v)\,\Delta_{\mathtt{g}^s v} \;=\; (\mathtt{g}^s)_*\sigma^+_{\mathscr{D}^-}\|\sigma^-_{\mathscr{D}^+}\|\,.$$

The result then follows from Theorem 10.2 with $\Omega = \partial^1_+\mathscr{D}^-$. □

12.3 Error terms for equidistribution and counting for Riemannian orbifolds

In Section 9.1, we discussed various results on the rate of mixing of the geodesic flow for Riemannian manifolds. In this Section, we apply these results to give error bounds to the statements of equidistribution and counting of common perpendicular arcs given in Section 12.2. We use again the notation of Section 12.1.

Theorem 12.7. *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with pinched negative sectional curvature at most -1. Let Γ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a bounded Γ-invariant Hölder-continuous function with positive critical exponent δ. Assume that $(\widetilde{M}, \Gamma, \widetilde{F})$ has radius-Hölder-continuous strong stable/unstable ball masses. Let $\mathscr{D}^- = (D^-_i)_{i\in I^-}$ and $\mathscr{D}^+ = (D^+_j)_{j\in I^+}$ be locally finite Γ-equivariant families of nonempty proper closed convex subsets of $\widetilde{M}$ such that $\Gamma\backslash I^\pm$ are finite, with finite nonzero skinning measure $\sigma_{\mathscr{D}^-}$ and $\sigma_{\mathscr{D}^+}$. Let $M = \Gamma\backslash\widetilde{M}$ and let $F : T^1M \to \mathbb{R}$ be the potential induced by $\widetilde{F}$.*

(1) Assume that M is compact and that the geodesic flow on T^1M is mixing with exponential speed for the Hölder regularity for the potential F. Then there exist $\alpha \in \,]0, 1]$ and $\kappa' > 0$ such that for all nonnegative $\psi^\pm \in \mathscr{C}^\alpha_{\rm c}(T^1M)$, we have, as $t \to +\infty$,

$$\frac{\delta\,\|m_F\|}{e^{\delta\,t}} \sum_{v,\,w\in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\,m_{\partial^1_-\mathscr{D}^+}(w)\,n_t(v,w)\,\psi^-(v)\,\psi^+(w)$$
$$= \int_{T^1M}\psi^- d\sigma^+_{\mathscr{D}^-}\int_{T^1M}\psi^+ d\sigma^-_{\mathscr{D}^+} + \mathrm{O}(e^{-\kappa' t}\|\psi^-\|_\alpha\,\|\psi^+\|_\alpha)\,.$$

(2) Assume that $\widetilde{M}$ is a symmetric space, that $D^\pm_k$ has smooth boundary for every $k \in I^\pm$, that m_F is finite and smooth,[7] and that the geodesic flow on T^1M is mixing with exponential speed for the Sobolev regularity for the potential F. Then there exist $\ell \in \mathbb{N}$ and $\kappa' > 0$ such that for all nonnegative maps $\psi^\pm \in \mathscr{C}^\ell_{\rm c}(T^1M)$, we have, as $t \to +\infty$,

$$\frac{\delta\,\|m_F\|}{e^{\delta\,t}} \sum_{v,\,w\in T^1M} m_{\partial^1_+\mathscr{D}^-}(v)\,m_{\partial^1_-\mathscr{D}^+}(w)\,n_t(v,w)\,\psi^-(v)\,\psi^+(w)$$
$$= \int_{T^1M}\psi^- d\sigma^+_{\mathscr{D}^-}\int_{T^1M}\psi^+ d\sigma^-_{\mathscr{D}^+} + \mathrm{O}(e^{-\kappa' t}\|\psi^-\|_\ell\,\|\psi^+\|_\ell)\,.$$

[7]Recall that a measure on a smooth manifold N is *smooth* if any local chart, it is absolutely continuous with respect to the Lebesgue measure, with smooth Radon-Nikodym derivative.

Furthermore, if $\mathscr{D}^-$ and $\mathscr{D}^+$ respectively have nonzero finite outer and inner skinning measures, and if $(\widetilde{M}, \Gamma, \widetilde{F})$ satisfies the conditions of (1) or of (2) above, then there exists $\kappa'' > 0$ such that, as $t \to +\infty$,

$$\mathscr{N}_{\mathscr{D}^-,\, \mathscr{D}^+,\, F}(t) = \frac{\|\sigma^+_{\mathscr{D}^-}\|\, \|\sigma^-_{\mathscr{D}^+}\|}{\delta\, \|m_F\|}\, e^{\delta\, t}\big(1 + \operatorname{O}(e^{-\kappa'' t})\big)\,.$$

The maps $\operatorname{O}(\cdot)$ depend on $\widetilde{M}, \Gamma, F, \mathscr{D}$, and the speeds of mixing. The proof is a generalisation to nonzero potential of [PaP17b, Theo. 15].

Proof. We will follow the proofs of Theorem 11.1 and Corollary 12.3 to prove generalisations of the assertions (1) and (2) by adding to these proofs a regularisation process of the test functions $\widetilde{\phi}^{\pm}_{\eta}$ as for the deduction of Theorem 10.3 from Theorem 10.2. We will then deduce the last statement from these generalisations, again using this regularisation process.

Let β be either $\alpha \in \,]0,1]$ small enough in the Hölder regularity case or $\ell \in \mathbb{N}$ large enough in the Sobolev regularity case. We fix $i \in I^-$, $j \in I^+$, and we use the notation $D^{\pm}, \alpha_\gamma, \lambda_\gamma$ and $\widetilde{\sigma}^{\pm}$ of Equation (11.2). Let $v^{\pm}_{\gamma} \in \partial^1_{\mp} D^{\pm}$ be the initial and terminal tangent vectors to α_γ and $\gamma^{-1}\alpha_\gamma$ respectively. Let $\widetilde{\psi}^{\pm} \in \mathscr{C}^{\beta}_{\rm c}(\partial^1_{\mp} D^{\pm})$. Under the assumptions of Assertion (1) or of Assertion (2), we first prove the following avatar of Equation (11.4), indicating only the required changes in its proof: there exists $\kappa_0 > 0$ (independent of $\widetilde{\psi}^{\pm}$) such that, as $T \to +\infty$,

$$\begin{aligned}\delta\, \|m_F\|\, e^{-\delta\, T} &\sum_{\gamma \in \Gamma,\, 0 < \lambda_\gamma \leqslant T} e^{\int_{\alpha_\gamma} \widetilde{F}}\; \widetilde{\psi}^-(v^-_\gamma)\, \widetilde{\psi}^+(v^+_\gamma) \\ &= \int_{\partial^1_+ D^-} \widetilde{\psi}^-\, d\widetilde{\sigma}^+ \int_{\partial^1_- D^+} \widetilde{\psi}^+\, d\widetilde{\sigma}^- + \operatorname{O}(e^{-\kappa_0 T} \|\widetilde{\psi}^-\|_\beta\, \|\widetilde{\psi}^+\|_\beta)\,. \end{aligned} \tag{12.4}$$

By Lemma 3.7 and the Hölder regularity of the strong stable and unstable foliations under the assumptions of Assertion (1), or by the smoothness of the boundary of $D^{\pm}$ under the assumptions of Assertion (2), the maps $f^{\pm}_{D^{\mp}} : \mathscr{V}^{\pm}_{\eta,\,R}(\partial^1_{\pm} D^{\mp}) \to \partial^1_{\pm} D^{\mp}$ are respectively Hölder-continuous or smooth fibrations, whose fiber over $w \in \partial^1_{\pm} D^{\mp}$ is exactly $V^{\pm}_{w,\,\eta,\,R}$. By applying leafwise the regularisation process described in the proof of Theorem 10.3 to characteristic functions, there exist a constant $\kappa'_1 > 0$ and $\chi^{\pm}_{\eta,\,R} \in \mathscr{C}^{\beta}(T^1\widetilde{M})$ such that

- $\|\chi^{\pm}_{\eta,\,R}\|_\beta = \operatorname{O}(\eta^{-\kappa'_1})$,
- $\mathbb{1}_{\mathscr{V}^{\mp}_{\eta\, e^{-\operatorname{O}(\eta)},\, R\, e^{-\operatorname{O}(\eta)}}(\partial^1_{\mp} D^{\pm})} \leqslant \chi^{\pm}_{\eta,\,R} \leqslant \mathbb{1}_{\mathscr{V}^{\mp}_{\eta,\,R}(\partial^1_{\mp} D^{\pm})}$,
- for every $w \in \partial^1_{\mp} D^{\pm}$, we have

$$\int_{V^{\mp}_{w,\,\eta,\,R}} \chi^{\pm}_{\eta,\,R}\, d\nu^{\pm}_{w} = \nu^{\pm}_{w}(V^{\mp}_{w,\,\eta,\,R})\, e^{-\operatorname{O}(\eta)} = \nu^{\pm}_{w}(V^{\mp}_{w,\,\eta\, e^{-\operatorname{O}(\eta)},\, R\, e^{-\operatorname{O}(\eta)}})\, e^{\operatorname{O}(\eta)}\,.$$

We now define the new test functions (compare with Section 10.1). For every $w \in \partial^1_{\mp} D^{\pm}$, let

$$H^{\pm}_{\eta,\,R}(w) = \frac{1}{\int_{V^{\mp}_{w,\,\eta,\,R}} \chi^{\pm}_{\eta,\,R}\, d\nu^{\pm}_{w}}\,.$$

Let $\Phi^{\pm}_{\eta} : T^1\widetilde{M} \to \mathbb{R}$ be the map defined by

$$\Phi^{\pm}_{\eta} = (H^{\pm}_{\eta,\,R}\, \widetilde{\psi}^{\pm}) \circ f^{\mp}_{D^{\pm}}\;\; \chi^{\pm}_{\eta,\,R}\,.$$

The support of this map is contained in $\mathscr{V}^{\mp}_{\eta,\,R}(\partial^1_{\mp} D^{\pm})$. Since M is compact in Assertion (1) and by homogeneity in Assertion (2), if R is large enough, by the definitions of the measures $\nu^{\pm}_{w}$, the denominator of $H^{\pm}_{\eta,\,R}(w)$ is at least $c\,\eta$ where $c>0$. The map $H^{\pm}_{\eta,\,R}$ is hence Hölder-continuous under the assumptions of Assertion (1), and it is smooth under the assumptions of Assertion (2). Therefore $\Phi^{\pm}_{\eta}\in\mathscr{C}^{\beta}(T^1\widetilde{M})$ and there exists a constant $\kappa'_2>0$ such that

$$\|\Phi^{\pm}_{\eta}\|_{\beta}=\mathrm{O}(\eta^{-\kappa'_2}\|\widetilde{\psi}^{\pm}\|_{\beta})\,.$$

As in Lemma 10.1, the functions $\Phi^{\pm}_{\eta}$ are measurable, nonnegative and satisfy

$$\int_{T^1\widetilde{M}}\Phi^{\pm}_{\eta}\;d\widetilde{m}_F=\int_{\partial^1_{\mp}D^{\pm}}\widetilde{\psi}^{\pm}\;d\widetilde{\sigma}^{\mp}\,.$$

As in the proof of Theorem 11.1, we will estimate in two ways the quantity

$$I_{\eta}(T)=\int_0^T e^{\delta\,t}\sum_{\gamma\in\Gamma}\int_{T^1\widetilde{M}}(\Phi^{-}_{\eta}\circ\mathsf{g}^{-t/2})\;(\Phi^{+}_{\eta}\circ\mathsf{g}^{t/2}\circ\gamma^{-1})\;d\widetilde{m}_F\;dt\,. \tag{12.5}$$

We first apply the mixing property, now with exponential decay of correlations, as in Step 2 of the proof of Theorem 11.1. For all $t\geqslant 0$, let

$$A_{\eta}(t)=\sum_{\gamma\in\Gamma}\int_{v\in T^1\widetilde{M}}\Phi^{-}_{\eta}(\mathsf{g}^{-t/2}v)\;\Phi^{+}_{\eta}(\mathsf{g}^{t/2}\gamma^{-1}v)\;d\widetilde{m}_F(v)\,.$$

Then with $\kappa>0$ as in the definitions of the exponential mixing for the Hölder or Sobolev regularity, we have

$$\begin{aligned}A_{\eta}(t)&=\frac{1}{\|m_F\|}\int_{T^1\widetilde{M}}\Phi^{-}_{\eta}\;d\widetilde{m}_F\int_{T^1\widetilde{M}}\Phi^{+}_{\eta}\;d\widetilde{m}_F\;+\;\mathrm{O}\left(e^{-\kappa\,t}\|\Phi^{-}_{\eta}\|_{\beta}\|\Phi^{+}_{\eta}\|_{\beta}\right)\\&=\frac{1}{\|m_F\|}\int_{\partial^1_{+}D^{-}}\widetilde{\psi}^{-}\;d\widetilde{\sigma}^{+}\int_{\partial^1_{-}D^{+}}\widetilde{\psi}^{+}\;d\widetilde{\sigma}^{-}\;+\;\mathrm{O}\left(e^{-\kappa\,t}\eta^{-2\kappa'_2}\|\widetilde{\psi}^{-}\|_{\beta}\|\widetilde{\psi}^{+}\|_{\beta}\right).\end{aligned}$$

Hence by integrating,

$$I_{\eta}(T)=\frac{e^{\delta\,T}}{\delta\,\|m_F\|}\Big(\int_{\partial^1_{+}D^{-}}\widetilde{\psi}^{-}\;d\widetilde{\sigma}^{+}\int_{\partial^1_{-}D^{+}}\widetilde{\psi}^{+}\;d\widetilde{\sigma}^{-}\;+\;\mathrm{O}\left(e^{-\kappa\,T}\eta^{-2\kappa'_2}\|\widetilde{\psi}^{-}\|_{\beta}\|\widetilde{\psi}^{+}\|_{\beta}\right)\Big)\,. \tag{12.6}$$

Now, as in Step 3 of the proof of Theorem 11.1, we exchange the integral over t and the summation over γ in the definition of $I_{\eta}(T)$, and we estimate the integral term independently of γ:

$$I_{\eta}(T)=\sum_{\gamma\in\Gamma}\int_0^T e^{\delta\,t}\int_{T^1\widetilde{M}}(\Phi^{-}_{\eta}\circ\mathsf{g}^{-t/2})\;(\Phi^{+}_{\eta}\circ\mathsf{g}^{t/2}\circ\gamma^{-1})\;d\widetilde{m}_F\;dt\,.$$

Let $\widehat{\Phi}^{\pm}_{\eta}=H^{\pm}_{\eta,\,R}\circ f^{\mp}_{D^{\pm}}\;\chi^{\pm}_{\eta,\,R}$, so that $\Phi^{\pm}_{\eta}=\widetilde{\psi}^{\pm}\circ f^{\mp}_{D^{\pm}}\;\widehat{\Phi}^{\pm}_{\eta}$. By the last two properties of the regularised maps $\chi^{\pm}_{\eta,\,R}$, we have, with $\phi^{\pm}_{\eta',\,\eta'',\,\Omega^{\pm}}$ defined as in Equation (10.4),

$$\phi^{\pm}_{\eta\,e^{-\mathrm{O}(\eta)},\,R\,e^{-\mathrm{O}(\eta)},\,\partial^1_{\mp}D^{\pm}}\;e^{-\,\mathrm{O}(\eta)}\leqslant\widehat{\Phi}^{\pm}_{\eta}\leqslant\phi^{\pm}_{\eta}\;e^{\mathrm{O}(\eta)}\,. \tag{12.7}$$

If $v\in T^1\widetilde{M}$ belongs to the support of $(\Phi^{-}_{\eta}\circ\mathsf{g}^{-t/2})\;(\Phi^{+}_{\eta}\circ\mathsf{g}^{t/2}\circ\gamma^{-1})$, then we have $v\in\mathsf{g}^{t/2}\mathscr{V}^{+}_{\eta,\,R}(\partial^1_{+}D^{-})\cap\mathsf{g}^{-t/2}\mathscr{V}^{-}_{\eta,\,R}(\gamma\partial^1_{-}D^{+})$. Hence the properties (i), (ii) and (iii) of Step 3M of the proof of Theorem 11.1 still hold (with $\Omega_- = \partial^1_+ D^-$ and $\Omega_+ = \partial^1_-(\gamma D^+)$). In particular, if $w^- = f^+_{D^-}(v)$ and $w^+ = f^-_{\gamma D^+}(v)$, we have, by Assertion (iii) in Step 3M of the proof of Theorem 11.1,[8] that

$$d(w^\pm, v^\pm_\gamma) = \operatorname{O}(\eta + e^{-\lambda_\gamma/2})\,.$$

Hence, with $\kappa'_3 = \alpha$ in the Hölder case and $\kappa'_3 = 1$ in the Sobolev case (we may assume that $\ell \geqslant 1$), we have

$$|\,\widetilde{\psi}^\pm(w^\pm) - \widetilde{\psi}^\pm(v^\pm_\gamma)\,| = \operatorname{O}((\eta + e^{-\lambda_\gamma/2})^{\kappa'_3}\|\widetilde{\psi}^\pm\|_\beta)\,.$$

Therefore there exists a constant $\kappa'_4 > 0$ such that

$$I_\eta(T) = \sum_{\gamma\in\Gamma} \big(\widetilde{\psi}^-(v^-_\gamma)\widetilde{\psi}^+(v^+_\gamma) + \operatorname{O}((\eta + e^{-\lambda_\gamma/2})^{\kappa'_4}\|\widetilde{\psi}^-\|_\beta\|\widetilde{\psi}^+\|_\beta)\big)\times$$
$$\int_0^T e^{\delta\, t}\int_{v\in T^1\widetilde{M}} \widehat{\Phi}^-_\eta(\mathtt{g}^{-t/2}v)\;\widehat{\Phi}^+_\eta(\gamma^{-1}\mathtt{g}^{t/2}v)\;d\widetilde{m}_F(v)\;dt\,.$$

Now, using the inequalities (12.7), Equation (12.4) follows as in Steps 3M and 4M of the proof of Theorem 11.1, by taking $\eta = e^{-\kappa'_5 T}$ for some $\kappa'_5 > 0$ and using the effective control given by Equation (11.21) in Step 4M.

In order to prove Assertions (1) and (2) of Theorem 12.7, we may assume that the supports of $\psi^\pm$ are small enough, say contained in $B(x^\pm,\epsilon)$ for some $x^\pm \in T^1M$ and ϵ small enough. Let $\widetilde{x}^\pm$ be lifts of $x^\pm$ and let $\widetilde{\psi}^\pm \in \mathscr{C}^\beta_{\rm c}(T^1\widetilde{M})$ with support in $B(\widetilde{x}^\pm,\epsilon)$ be such that $\widetilde{\psi}^\pm = \psi^\pm \circ Tp$ on $B(\widetilde{x}^\pm,\epsilon)$ where $p:\widetilde{M}\to M$ is the universal Riemanian orbifold cover. By a finite summation argument since $\Gamma\backslash I^\pm$ are finite, and by Equation (12.4), we have

$$\delta\,\|m_F\|\,e^{-\delta\,T}\sum_{\substack{i\in I^-/\sim,\; j\in I^+/\sim,\;\gamma\in\Gamma\\ 0<\lambda_{i,\,\gamma j}\leqslant T}} e^{\int_{\alpha_\gamma}\widetilde{F}}\;\widetilde{\psi}^-(v^-_\gamma)\,\widetilde{\psi}^+(v^+_\gamma)$$
$$= \int_{\partial^1_+ D^-}\widetilde{\psi}^-\,d\widetilde{\sigma}^+\int_{\partial^1_- D^+}\widetilde{\psi}^+\,d\widetilde{\sigma}^- + \operatorname{O}(e^{-\kappa_0 T}\|\widetilde{\psi}^-\|_\beta\,\|\widetilde{\psi}^+\|_\beta)\,. \qquad (12.8)$$

Assertions (1) and (2) are deduced from this equation in the same way that Corollary 12.3 is deduced from Theorem 11.1. Taking the functions $\psi^\pm_k$ to be the constant functions 1 in Assertion (1) gives the last statement of Theorem 12.7 under the assumptions of Assertion (1). An approximation argument gives the result under the assumptions of Assertion (2). □

12.4 Equidistribution and counting for quotient simplicial and metric trees

In this Section, we assume that X is the geometric realisation of a locally finite metric tree without terminal vertices $(\mathbb{X},\lambda)$, and that Γ is a (nonelementary discrete) subgroup of $\operatorname{Aut}(\mathbb{X},\lambda)$. Let $\widetilde{c}: E\mathbb{X}\to\mathbb{R}$ be a system of conductances for Γ, and let $c:\Gamma\backslash E\mathbb{X}\to\mathbb{R}$ be its quotient function. We assume in this Section that the potential $\widetilde{F}$ is the potential $\widetilde{F}_c$ associated[9] with c. Let $\delta_c = \delta_{\Gamma,\,F_c}$ be the critical exponent of (Γ, F_c) and let $\widetilde{m}_c = \widetilde{m}_{F_c}$ and

[8] See also the picture at the beginning of the proof of Lemma 11.8.
[9] See Section 3.5.

$m_c = m_{F_c}$ be the Gibbs measures of F_c for the continuous time geodesic flow on respectively $\mathscr{G}X$ and $\Gamma\backslash\mathscr{G}X$, as well as for the discrete time geodesic flow on respectively $\mathscr{G}\mathbb{X}$ and $\Gamma\backslash\mathscr{G}\mathbb{X}$ when $(\mathbb{X}, \lambda)$ is simplicial, that is, if λ is constant with value 1.

Let $\mathbb{D}^\pm$ be simplicial subtrees of $\mathbb{X}$, with the edge length map induced by λ, and $D^\pm = |\mathbb{D}^\pm|_\lambda$ its geometric realisation, such that the Γ-equivariant families $\mathscr{D}^\pm = (\gamma D^\pm)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite in X.[10]

For all γ, γ' in Γ such that γD^- and $\gamma' D^+$ are disjoint, we denote by $\alpha_{\gamma,\gamma'}$ the common perpendicular from γD^- to $\gamma' D^+$ (which is an edge path in $\mathbb{X}$), with length $\lambda_{\gamma,\gamma'} = d(\gamma D^-, \gamma' D^+) \in \mathbb{N}$, and by $\alpha^\pm_{\gamma,\gamma'} \in \check{\mathscr{G}}\mathbb{X}$ its parametrisations as in the beginning of Chapter 11: it is the unique map from $\mathbb{R}$ to X such that $\alpha^-_{\gamma,\gamma'}(t) \in \gamma V\mathbb{D}^-$ is the origin $o(\alpha_{\gamma,\gamma'})$ of the edge path $\alpha_{\gamma,\gamma'}$ if $t \leqslant 0$, $\alpha^-_{\gamma,\gamma'}(t) \in \gamma' V\mathbb{D}^+$ is the endpoint $t(\alpha_{\gamma,\gamma'})$ of the edge path $\alpha_{\gamma,\gamma'}$ if $t \geqslant \lambda_{\gamma,\gamma'}$, and $\alpha^-_{\gamma,\gamma'}|_{[0,\lambda_{\gamma,\gamma'}]}$ is the shortest geodesic arc starting from a point of γD^- and ending at a point of $\gamma' D^+$.

For all γ, γ' in Γ such that γD^- and $\gamma' D^+$ are disjoint, we define the *multiplicity* of the common perpendicular $\alpha_{\gamma,\gamma'}$ from γD^- to $\gamma' D^+$ as

$$m_{\gamma,\gamma'} = \frac{1}{\operatorname{Card}(\gamma\Gamma_{\mathbb{D}^-}\gamma^{-1} \cap \gamma'\Gamma_{\mathbb{D}^+}\gamma'^{-1})}\,. \tag{12.9}$$

Note that $m_{\gamma,\gamma'} = 1$ for all $\gamma, \gamma' \in \Gamma$ when Γ acts freely on $E\mathbb{X}$ (for instance when Γ is torsion-free). Generalising the definition for simplicial trees in Section 11.4, we set

$$\widetilde{c}(\alpha) = \sum_{i=1}^{k} \widetilde{c}\,(e_i)\,\lambda(e_i)\,,$$

for any edge path $\alpha = (e_1, \dots, e_k)$ in $\mathbb{X}$.

For $n \in \mathbb{N} - \{0\}$, let

$$\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+}(n) = \sum_{\substack{[\gamma]\in\Gamma_{\mathbb{D}^-}\backslash\Gamma/\Gamma_{\mathbb{D}^+} \\ 0<d(D^-,\gamma D^+)\leqslant n}} m_{e,\gamma}\; e^{\widetilde{c}\,(\alpha_{e,\gamma})}\,, \tag{12.10}$$

where d is the the distance on $X = |\mathbb{X}|_\lambda$. When Γ is torsion-free, $\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+}(n)$ is the number of edge paths in the graph $\Gamma\backslash\mathbb{X}$ of length at most n, starting by an outgoing edge from the image of $\mathbb{D}^-$ and ending by the opposite of an outgoing edge from the image of $\mathbb{D}^+$, with multiplicities coming from the fact that $\Gamma_{D^\pm}\backslash\mathbb{D}^\pm$ is not assumed to be embedded in $\Gamma\backslash\mathbb{X}$, and with weights coming from the conductances.

In the next results, we distinguish the continuous time case (Theorem 12.8) from the discrete time case (Theorem 12.9). We leave to the reader the versions without the assumption $\delta_c > 0$, giving for every $\tau \in \mathbb{N} - \{0\}$ an asymptotic on

$$\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+,\,\tau}(n) = \sum_{\substack{[\gamma]\in\Gamma_{\mathbb{D}^-}\backslash\Gamma/\Gamma_{\mathbb{D}^+} \\ n-\tau<d(D^-,\gamma D^+)\leqslant n}} m_{e,\gamma}\; e^{\widetilde{c}\,(\alpha_{e,\gamma})}\,.$$

When $\Gamma\backslash X$ is compact, $c = 0$ and $\mathbb{D}^\pm$ are reduced to points, the counting results in Theorems 12.8 and 12.9 are proved in [Gui]. When $\mathbb{D}^\pm$ are singletons, Theorem 12.8 is due to [Rob2] if $c = 0$. Otherwise, the result seems to be new.

[10] We leave to the reader the extension to more general locally finite families of subtrees, as for instance finite unions of those above.

Theorem 12.8. *Let $(\mathbb{X}, \lambda)$, Γ, $\mathbb{D}^{\pm}$ and c be as in the beginning of this Section. Assume that the critical exponent δ_c is finite and positive, that the skinning measures $\sigma^{\pm}_{\mathscr{D}^{\mp}}$ are finite and nonzero, and that the Gibbs measure m_c is finite and mixing for the continuous time geodesic flow. Then as $t \to +\infty$, the measures*

$$\delta_c \, \|m_c\| \, e^{-\delta_c \, t} \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \, \mathbb{D}^+) \leqslant t}} m_{e,\gamma} \, e^{\widetilde{c}\,(\alpha_{e,\gamma})} \, \Delta_{\Gamma \alpha^-_{e,\gamma}} \otimes \Delta_{\Gamma \alpha^+_{\gamma^{-1},e}}$$

narrow converge to $\sigma^+_{\mathscr{D}^-} \otimes \sigma^-_{\mathscr{D}^+}$ in $\Gamma \backslash \check{\mathscr{G}} X \times \Gamma \backslash \check{\mathscr{G}} X$, and

$$\mathscr{N}_{\mathbb{D}^-, \mathbb{D}^+}(t) \sim \frac{\|\sigma^+_{\mathscr{D}^-}\| \, \|\sigma^-_{\mathscr{D}^+}\|}{\delta_c \, \|m_c\|} \; e^{\delta_c \, t} \, .$$

Proof. By Theorem 11.1, we have

$$\lim_{t \to +\infty} \; \delta_c \, \|m_c\| \, e^{-\delta_c \, t} \sum_{\substack{(a,b,\gamma) \in \Gamma/\Gamma_{\mathbb{D}^-} \times \Gamma/\Gamma_{\mathbb{D}^+} \times \Gamma \\ 0 < d(a\mathbb{D}^-, \, \gamma b \mathbb{D}^+) \leqslant t}} e^{\int_{\alpha_{a,\gamma b}} \widetilde{F}_c} \, \Delta_{\alpha^-_{a,\gamma b}} \otimes \Delta_{\alpha^+_{\gamma^{-1} a, b}} \; = \; \widetilde{\sigma}^+_{\mathscr{D}^-} \otimes \widetilde{\sigma}^-_{\mathscr{D}^+} \, ,$$

for the weak-star convergence on $\check{\mathscr{G}} \mathbb{X} \times \check{\mathscr{G}} \mathbb{X}$.

The group $\Gamma \times \Gamma$ acts on $\Gamma/\Gamma_{\mathbb{D}^-} \times \Gamma/\Gamma_{\mathbb{D}^+} \times \Gamma$ by

$$(a', b') \cdot (a, b, \gamma) = (a' a, b' b, a' \gamma (b')^{-1}) \, .$$

and the map from the discrete set $\Gamma/\Gamma_{\mathbb{D}^-} \times \Gamma/\Gamma_{\mathbb{D}^+} \times \Gamma$ to $\check{\mathscr{G}} \mathbb{X} \times \check{\mathscr{G}} \mathbb{X}$ which sends (a, b, γ) to $(\alpha^-_{a,\gamma b} \, , \alpha^+_{\gamma^{-1} a, b})$ is $(\Gamma \times \Gamma)$-equivariant. In particular, the pushforward of measures by this map sends the unit Dirac mass at (a, b, γ) to $\Delta_{\alpha^-_{a,\gamma b}} \otimes \Delta_{\alpha^+_{\gamma^{-1} a, b}}$.

Every orbit of $\Gamma \times \Gamma$ on $\Gamma/\Gamma_{\mathbb{D}^-} \times \Gamma/\Gamma_{\mathbb{D}^+} \times \Gamma$ has a representative of the form $(\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, \gamma)$ for some $\gamma \in \Gamma$, since $(a, b) \cdot (\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, a^{-1} \gamma \, b) = (a \Gamma_{\mathbb{D}^-}, b \Gamma_{\mathbb{D}^-}, \gamma)$. Furthermore the double class in $\Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+}$ of such a γ is uniquely defined, and the stabiliser of $(\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, \gamma)$ has cardinality $|\Gamma_{\mathbb{D}^-} \cap \gamma \Gamma_{\mathbb{D}^+} \gamma^{-1}|$, since $(a, b) \cdot (\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, \gamma) = (\Gamma_{\mathbb{D}^-}, \Gamma_{\mathbb{D}^+}, \gamma')$ if and only if $a \in \Gamma_{\mathbb{D}^-}$, $b \in \Gamma_{\mathbb{D}^+}$ and $a \gamma \, b^{-1} = \gamma'$. When $\gamma' = \gamma$, this happens if and only if $b = \gamma^{-1} a \gamma$ and $a \in \Gamma_{\mathbb{D}^-} \cap \gamma \Gamma_{\mathbb{D}^+} \gamma^{-1}$.

By using the properties recalled at the beginning of Chapter 12 on the narrow convergence of induced measures, and since no compactness assumptions were made in Equation (11.4) on $\Omega^{\pm}$, the measures

$$\delta_c \, \|m_c\| \, e^{-\delta_c \, t} \sum_{\substack{[\gamma] \in \Gamma_{\mathbb{D}^-} \backslash \Gamma / \Gamma_{\mathbb{D}^+} \\ 0 < d(\mathbb{D}^-, \gamma \, \mathbb{D}^+) \leqslant t}} \frac{1}{|\Gamma_{\mathbb{D}^-} \cap \gamma \Gamma_{\mathbb{D}^+} \gamma^{-1}|} \, e^{\int_{\Gamma \alpha_{e,\gamma}} F_c} \, \Delta_{\Gamma \alpha^-_{e,\gamma}} \otimes \Delta_{\Gamma \alpha^+_{\gamma^{-1},e}}$$

hence narrow converge as $t \to +\infty$ to $\sigma^+_{\mathscr{D}^-} \otimes \sigma^-_{\mathscr{D}^+}$ in $\Gamma \backslash \check{\mathscr{G}} \mathbb{X} \times \Gamma \backslash \check{\mathscr{G}} \mathbb{X}$. By applying this convergence to the constant function 1, and by the finiteness and nonvanishing of $\sigma^+_{\mathscr{D}^-}$ and $\sigma^-_{\mathscr{D}^+}$, the result follows using the defining property of the potential F_c, see Proposition 3.21. $\square$

In the remainder of this Section, we consider simplicial trees with the discrete time geodesic flow.

Theorem 12.9. *Let $(\mathbb{X},\lambda)$, Γ, $\widetilde{c}$ and $\mathbb{D}^\pm$ be as in the beginning of this Section, with λ constant with value 1. Assume that the critical exponent δ_c is finite and positive. If the Gibbs measure m_c on the space $\Gamma\backslash\mathscr{G}\mathbb{X}$ of discrete geodesic lines modulo Γ is finite and mixing for the discrete time geodesic flow, and if the skinning measures $\sigma^\pm_{\mathscr{D}^\mp}$ are finite and nonzero, then as $n\to+\infty$, the measures*

$$\frac{e^{\delta_c}-1}{e^{\delta_c}}\,\|m_c\|\;e^{-\delta_c\,n}\sum_{\substack{[\gamma]\in\Gamma_{\mathbb{D}^-}\backslash\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\gamma\,\mathbb{D}^+)\leqslant n}} m_{e,\,\gamma}\;e^{\widetilde{c}\,(\alpha_{e,\,\gamma})}\;\Delta_{\Gamma\alpha^-_{e,\gamma}}\otimes\Delta_{\Gamma\alpha^+_{\gamma^{-1},e}}$$

narrow converge to $\sigma^+_{\mathscr{D}^-}\otimes\sigma^-_{\mathscr{D}^+}$ in $\Gamma\backslash\check{\mathscr{G}}\,\mathbb{X}\times\Gamma\backslash\check{\mathscr{G}}\,\mathbb{X}$ and

$$\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+}(n)\sim\frac{e^{\delta_c}\;\|\sigma^+_{\mathscr{D}^-}\|\;\|\sigma^-_{\mathscr{D}^+}\|}{(e^{\delta_c}-1)\;\|m_c\|}\;\;e^{\delta_c\,n}\,.$$

Proof. The claims follow as in Theorem 12.8, replacing Theorem 11.1 by Theorem 11.9. □

Remark 12.10. A common perpendicular in a simplicial tree is, in the language of graph theory, a non-backtracking walk. Among other applications,[11] Theorem 12.9 gives a complete asymptotic solution to the problem of counting non-backtracking walks from a given vertex to a given vertex of a (finite) nonbipartite graph. See Theorem 12.12 for the corresponding result in bipartite graphs, and for example [AloBLS, Th. 1.1], [AnFH, p. 4290,4302], [Fri2, L. 2.3], [Sod, Prop. 6.4] for related results. Anticipating on the error terms that we will give in Section 12.6, note that the paper [AloBLS, Th. 1.1] for instance gives a precise speed using spectral properties, more precise than the ones we obtain.

Examples 12.11. (1) Let $\mathbb{X},\Gamma,\widetilde{c}$ be as in Theorem 12.9, and let $\mathbb{D}^-=\{x\}$ and $\mathbb{D}^+=\{y\}$ for some $x,y\in V\mathbb{X}$. If the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$, then we have a discrete time version of Roblin's simultaneous equidistribution theorem with potential,[12] and the number $\mathscr{N}_{x,\,y}(n)$ of nonbacktracking edge paths of length at most n from the image of x to the image of y (counted with weights and multiplicities) satisfies, by Equation (7.15),

$$\mathscr{N}_{x,\,y}(n)\sim\frac{e^{\delta_c}\;\|\mu^+_x\|\;\|\mu^-_y\|}{(e^{\delta_c}-1)\;\|m_c\|\;|\Gamma_x|\;|\Gamma_y|}\;\;e^{\delta_c\,n}\,.$$

(2) If $\mathbb{Y}$ is a finite connected nonbipartite $(q+1)$-regular graph (with $q\geqslant 2$) and $\mathbb{Y}^\pm$ are points, then the number of nonbacktracking edge paths from $\mathbb{Y}^-$ to $\mathbb{Y}^+$ of length at most n is equivalent as $n\to+\infty$ to

$$\frac{q+1}{q-1}\,\frac{q^n}{|V\mathbb{Y}|}+\mathrm{O}(r^n) \tag{12.11}$$

for some $r<q$. Indeed, by Theorem 12.9 with $\mathbb{X}$ the universal cover of $\mathbb{Y}$, Γ its covering group and $c=0$, we have $\delta_c=\ln q$ and m_c is the Bowen-Margulis measure, so that normalising the Patterson measures to be probability measures, we have $\|m_c\|=\frac{q}{q+1}\,|V\mathbb{Y}|$ by Equation (8.4). We refer to Section 12.6 (see Remark (i) following the proof of Theorem 12.16) for the error term.

[11] when restricting to groups Γ acting freely, which is never the case if Γ is a nonuniform lattice in the tree $\mathbb{X}$, that is, when the quotient graph of groups $\Gamma\backslash\mathbb{X}$ is infinite but has finite volume

[12] See Corollary 11.2 for the continuous time version.

Let $\mathbb{Y}$ be the figure 8-graph with a single vertex and four directed edges, and let $\mathbb{Y}^\pm$ be the singleton consisting of its vertex. In this simple example, it is easy to count by hand that the number of loops of length exactly n without backtracking in $\mathbb{Y}$ is $4\,3^{n-1}$. Thus the number $\mathscr{N}(n)$ of common perpendiculars of the vertex to itself of length at most n is by a simple geometric sum $2(3^n-1)$. This agrees with Equation (12.11) that gives $\mathscr{N}(n) \sim 2\,3^n$ as $n \to +\infty$.

(3) Let $\mathbb{Y}$ be a finite connected nonbipartite $(q+1)$-regular graph (with $q \geqslant 2$). Let $\mathbb{Y}^\pm$ be regular connected subgraphs of degrees $q^\pm \geqslant 0$, with $q^\pm < q+1$. Then the number $\mathscr{N}(n)$ of nonbacktracking edge paths of length at most n starting transversally to $\mathbb{Y}^-$ and ending transversally to $\mathbb{Y}^+$ satisfies

$$\mathscr{N}(n) = \frac{(q+1-q^-)(q+1-q^+)\,|V\mathbb{Y}^-|\,|V\mathbb{Y}^+|}{(q^2-1)\,|V\mathbb{Y}|}\; q^n + \mathrm{O}(r^n)$$

for some $r<q$. This is a direct consequence of Theorem 12.9, using Proposition 8.1 (3) and Proposition 8.4 (3), again refering to Section 12.6 for the error term.

We refer for instance to Chapters 15 and 16 for examples of counting results in graphs of groups where the underlying graph is infinite.

In some applications (see the examples at the end of this Section), we encounter bipartite simplicial graphs and, consequently, their discrete time geodesic flow is not mixing. The following result applies in this context.

Until the end of this section, we assume that the simplicial tree $\mathbb{X}$ has a Γ-invariant structure of a bipartite graph, and we denote by $V\mathbb{X} = V_1\mathbb{X} \sqcup V_2\mathbb{X}$ the corresponding partition of its set of vertices. For every $i \in \{1,2\}$, we denote by $\breve{\mathscr{G}}_i\,\mathbb{X}$ the space of generalised discrete geodesic lines $\ell \in \breve{\mathscr{G}}\,\mathbb{X}$ such that $\ell(0) \in V_i\mathbb{X}$, so that we have a partition $\breve{\mathscr{G}}\,\mathbb{X} = \breve{\mathscr{G}}_1\,\mathbb{X} \sqcup \breve{\mathscr{G}}_2\,\mathbb{X}$. Note that if the basepoint x_0 lies in $V_i\mathbb{X}$, then $\mathscr{G}_{\rm even}\mathbb{X}$ is equal to $\breve{\mathscr{G}}_i\,\mathbb{X} \cap \mathscr{G}\mathbb{X}$. Let $\sigma^\pm_{\mathscr{D}^\mp,\,i} = \sigma^\pm_{\mathscr{D}^\mp}\,|_{\Gamma\backslash\breve{\mathscr{G}}_i\,\mathbb{X}}$. For all $i,j \in \{1,2\}$, we define

$$\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+,\,i,\,j}(n) = \sum_{\substack{[\gamma]\in\Gamma_{\mathbb{D}^-}\backslash\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(D^-,\gamma D^+)\leqslant n\\ o(\alpha_{e,\gamma})\in V_i\mathbb{X},\ t(\alpha_{e,\gamma})\in V_j\mathbb{X}}} m_{e,\,\gamma}\, e^{\widetilde{c}\,(\alpha_{e,\,\gamma})}\,.$$

Theorem 12.12. *Let $(\mathbb{X},\lambda)$, Γ and c be as in the beginning of this Section, with λ constant with value 1. Assume that the critical exponent δ_c is finite and positive. Assume that $\mathbb{X}$ has a Γ-invariant structure of a bipartite graph as above, and that the restriction to $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ of the Gibbs measure m_c is finite and mixing for the square of the discrete time geodesic flow. Then for all $i,j\in\{1,2\}$ such that the measures $\sigma^-_{\mathscr{D}^-,\,i}$ and $\sigma^+_{\mathscr{D}^+,\,j}$ are finite and nonzero, as n tends to $+\infty$ with $n \equiv i-j \mod 2$, the measures*

$$\frac{e^{2\,\delta_c}-1}{2\,e^{2\,\delta_c}}\,\|m_c\|\;e^{-\delta_c\,n} \sum_{\substack{[\gamma]\in\Gamma_{\mathbb{D}^-}\backslash\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(D^-,\gamma\, D^+)\leqslant n\\ o(\alpha_{e,\gamma})\in V_i\mathbb{X},\ t(\alpha_{e,\gamma})\in V_j\mathbb{X}}} m_{e,\,\gamma}\; e^{\widetilde{c}\,(\alpha_{e,\,\gamma})}\;\Delta_{\Gamma\alpha^-_{e,\gamma}} \otimes \Delta_{\Gamma\alpha^+_{\gamma^{-1},e}}$$

narrow converge to $\sigma^+_{\mathscr{D}^-,\,i} \otimes \sigma^-_{\mathscr{D}^+,\,j}$ in $\Gamma\backslash\breve{\mathscr{G}}\,\mathbb{X} \times \Gamma\backslash\breve{\mathscr{G}}\,\mathbb{X}$ and

$$\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+,\,i,\,j}(n) \sim \frac{2\,e^{2\,\delta_c}\,\|\sigma^+_{\mathscr{D}^-,\,i}\|\;\|\sigma^-_{\mathscr{D}^+,\,j}\|}{(e^{2\,\delta_c}-1)\,\|m_c\|}\;\; e^{\delta_c\,n}\,.$$

Proof. This Theorem is proved in the same way as the above Theorem 12.9 using Theorem 11.11. Note that we have a $(\Gamma\times\Gamma)$-invariant partition

$$\breve{\mathscr{G}}\,\mathbb{X}\times\breve{\mathscr{G}}\,\mathbb{X}=\bigsqcup_{(i,\,j)\in\{1,\,2\}^2}\breve{\mathscr{G}}_i\,\mathbb{X}\times\breve{\mathscr{G}}_j\,\mathbb{X}\;,$$

that $\alpha^-_{e,\,\gamma}\in\breve{\mathscr{G}}_i\,\mathbb{X}$ if and only if $o(\alpha_{e,\,\gamma})\in V_i\mathbb{X}$, and that $\alpha^+_{\gamma^{-1},\,e}\in\breve{\mathscr{G}}_j\,\mathbb{X}$ if and only if $t(\alpha_{e,\,\gamma})\in V_j\mathbb{X}$, since $\alpha^+_{\gamma^{-1},\,e}(0)=\gamma^{-1}\alpha^+_{e,\,\gamma}(0)=\gamma^{-1}t(\alpha_{e,\,\gamma})$. □

Examples 12.13. (1) Let $\mathbb{X},\Gamma,c$ be as in Theorem 12.12, and let $\mathbb{D}^-=\{x\}$ and $\mathbb{D}^+=\{y\}$ for some vertices x,y in the same $V_i\mathbb{X}$ for $i\in\{1,2\}$. If the restriction to $\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}$ of the Gibbs measure m_c is finite and mixing for the square of the discrete time geodesic flow, then as n is even and tends to $+\infty$,

$$\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+}(n)\sim\frac{2\,e^{2\,\delta_c}}{e^{2\,\delta_c}-1}\;\frac{\|\mu^+_x\|\;\|\mu^-_y\|}{\|m_c\|\,|\Gamma_x|\,|\Gamma_y|}\;e^{\delta_c\,n}\,.$$

Indeed, we have $\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+}(n)=\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+,\,i,\,i}(n)$ and $\sigma^\pm_{\mathscr{D}^\pm,\,i}=\sigma^\pm_{\mathscr{D}^\pm}$, and we conclude as in Example 12.11 (1).

(2) Let $\mathbb{Y}$ be the complete biregular graph with $q+1$ vertices of degree $p+1$ and $p+1$ vertices of degree $q+1$. Let $\mathbb{Y}^\pm=\{y\}$ be a fixed vertex of degree $p+1$. Note that $\mathbb{Y}$ being bipartite, all common perpendiculars from y to y have even length (the shortest one having length 4). Then as n is even and tends to $+\infty$, we have

$$\mathscr{N}_{\mathbb{Y}^-,\mathbb{Y}^+}(n)\sim\frac{q(p+1)}{(q+1)(pq-1)}(pq)^{n/2}\,.$$

Indeed, the biregular tree $\mathbb{X}_{p,q}$ of degrees $(p+1,q+1)$ is a universal cover of $\mathbb{Y}$ with covering group Γ acting freely and cocompactly, so that with $c=0$ we have $\delta_c=\ln\sqrt{pq}$ and the Gibbs measure m_c is the Bowen-Margulis measure $m_{\rm BM}$. If we normalise the Patterson density such that $\|\mu_y\|=\frac{p+1}{\sqrt{p}}$, then by Proposition 8.1 (2), we have $\|m_{\rm BM}\|=2(p+1)(q+1)$. Thus the result follows from the above Example (1). Note that if $p=q$, then

$$\mathscr{N}_{\mathbb{Y}^-,\mathbb{Y}^+}(n)\sim\frac{q}{q^2-1}q^n\,,$$

and the constant in front of q^n is indeed different from that in the nonbipartite case.

(3) Let $\mathbb{Y}$ be a finite biregular graph with vertices of degrees $p+1$ and $q+1$, where $p,q\geqslant 2$, and let $V\mathbb{Y}=V_p\mathbb{Y}\sqcup V_q\mathbb{Y}$ be the corresponding partition. If $\mathbb{Y}^-=\{v\}$ where $v\in V_p\mathbb{Y}$ and $\mathbb{Y}^+$ is a cycle of length $L\geqslant 2$, then as $N\to+\infty$, the number of common perpendiculars of even length at most $2N$ from $\mathbb{Y}^-$ to $\mathbb{Y}^+$ is equivalent to

$$\frac{L\;q\;(p-1)}{2\;(pq-1)\;|V_p\mathbb{Y}|}\;(pq)^N\;,$$

and the number of common perpendiculars of odd length at most $2N-1$ from v to $\mathbb{Y}^+$ is equivalent to

$$\frac{L\;(q-1)}{2\;(pq-1)\;|V_p\mathbb{Y}|}\;(pq)^N\;.$$

Proof. The cycle $\mathbb{Y}^+$ has even length L and has $\frac{L}{2}$ vertices in both $V_p\mathbb{Y}$ and $V_q\mathbb{Y}$. A common perpendicular from $\mathbb{Y}^-$ to $\mathbb{Y}^+$ has even length if and only if it ends at a vertex in $V_p\mathbb{Y}$.

Let $\mathbb{X} \to \mathbb{Y}$ be a universal cover of $\mathbb{Y}$, whose covering group Γ acts freely and cocompactly on $\mathbb{X}$. Let $\mathbb{D}^- = \{\widetilde{v}\}$ where $\widetilde{v} \in V_p\mathbb{X}$ is a lift of v, and let $\mathbb{D}^-$ be a geodesic line in $\mathbb{X}$ mapping to $\mathbb{Y}^+$. We use Theorem 12.12 with $V_1\mathbb{X}$ the (full) preimage of $V_p\mathbb{Y}$ in $\mathbb{X}$, with $V_2\mathbb{X}$ the preimage of $V_q\mathbb{Y}$ in $\mathbb{X}$ and with $c = 0$, so that $\delta_c = \ln\sqrt{pq}$ and $m_c = m_{\rm BM}$. Let us normalise the Patterson density of Γ as in Proposition 8.1 (2), so that

$$\|\sigma^+_{\mathscr{D}^-,1}\| = \|\mu_{\widetilde{v}}\| = \frac{p+1}{\sqrt{p}}\,.$$

By the proof of Equation (8.10), the mass for the skinning measure of the part of the inner unit normal bundle of $\mathbb{Y}^+$ with basepoint in $V_p\mathbb{Y}$ is $\frac{p-1}{\sqrt{p}}\frac{L}{2}$ and its complement has mass $\frac{L(q-1)}{2\sqrt{q}}$. Recall also that, by Proposition 8.1 (2) and Remark 2.11, considering the graph $\mathbb{Y}$ as a graph of groups with trivial groups,

$$\|m_{\rm BM}\| = \operatorname{TVol}(\mathbb{Y}) = |E\mathbb{Y}| = 2(p+1)|V_p\mathbb{Y}| = 2(q+1)|V_q\mathbb{Y}|\,.$$

The claim about the common perpendiculars of even length at most $2N$ follows from Theorem 12.12 with $i = j = 1$, since

$$\frac{2\,e^{2\,\delta_c}\,\|\sigma^+_{\mathscr{D}^-,\,i}\|\,\|\sigma^-_{\mathscr{D}^+,\,j}\|}{(e^{2\,\delta_c}-1)\,\|m_c\|} = \frac{2\,pq\,\frac{p+1}{\sqrt{p}}\,\frac{L(p-1)}{2\sqrt{p}}}{(pq-1)\;\,2\,(p+1)\,|V_p\mathbb{Y}|} = \frac{L\,q\,(p-1)}{2\,(pq-1)\,|V_p\mathbb{Y}|}\,.$$

The claim about the common perpendiculars of odd length at most $2N-1$ follows similarly from Theorem 12.12 with $i = 1$ and $j = 2$. □

(4) Let $\mathbb{Y}$ be a finite biregular graph with vertices of degrees $p+1$ and $q+1$, where $p, q \geqslant 2$, and let $V\mathbb{Y} = V_p\mathbb{Y} \sqcup V_q\mathbb{Y}$ be the corresponding partition. If $\mathbb{Y}^-$ and $\mathbb{Y}^+$ are cycles of length $L^- \geqslant 2$ and $L^+ \geqslant 2$ respectively, then as $N \to +\infty$, the number of common perpendiculars of even length at most $2N$ from $\mathbb{Y}^-$ to $\mathbb{Y}^+$ is equal to

$$\frac{(p+q)\,L^-\,L^+}{2\,(pq-1)\,|E\mathbb{Y}|}\,(pq)^{N+1} + \operatorname{O}(r^N) \tag{12.12}$$

for some $r < \sqrt{pq}$.

Proof. As in the above proof of Example (3), let $\mathbb{X} \to \mathbb{Y}$ be a universal cover of $\mathbb{Y}$, with covering group Γ and let $\mathbb{D}^\pm$ be a geodesic line in $\mathbb{X}$ mapping to $\mathbb{Y}^\pm$. Let $V_1\mathbb{X}$ be the preimage of $V_p\mathbb{Y}$ and $V_2\mathbb{X}$ be one of $V_q\mathbb{Y}$. We normalise the Patterson density $(\mu_x)_{x\in V\mathbb{X}}$ of Γ so that $\|\mu_x\| = \frac{\deg_{\mathbb{X}}(x)}{\sqrt{\deg_{\mathbb{X}}(x)-1}}$. By Proposition 8.4 (3) with $k = 1$ and trivial vertex stabilisers, and since a (simple) cycle of length λ in a biregular graph of different degrees $p+1$ and $q+1$ has exactly $\frac{\lambda}{2}$ vertices of degree either $p+1$ or $q+1$, we have

$$\|\sigma^\pm_{\mathscr{D}^\mp,1}\| = \sum_{\Gamma x \in V_p\mathbb{Y}^\mp} \frac{\|\mu_x\|\,(\deg_{\mathbb{X}}(x)-k)}{\deg_{\mathbb{X}}(x)} = \sum_{y\in V_p\mathbb{Y}^\mp} \sqrt{p} = \frac{L^\mp\sqrt{p}}{2}\,.$$

Similarly, $\|\sigma^\pm_{\mathscr{D}^\mp,2}\| == \frac{L^\mp\sqrt{q}}{2}$. The result without the error term then follows from Theorem 12.12, using Proposition 8.1 (2) and Remark 2.11, since the number we are looking for is $\mathscr{N}_{\mathbb{D}^-,\mathbb{D}^+,1,1}(2N) + \mathscr{N}_{\mathbb{D}^-,\mathbb{D}^+,2,2}(2N)$. We refer to Section 12.6 (see Remark (ii) following the proof of Theorem 12.16) for the error term. □

12.5 Counting for simplicial graphs of groups

In this Section, we give an intrinsic translation "a la Bass-Serre" of the counting result in Theorem 12.9 using graphs of groups (see [Ser3] and Section 2.6 for background information).

Let $(\mathbb{Y}, G_*)$ be a locally finite, connected graph of finite groups, and let $(\mathbb{Y}^\pm, G_*^\pm)$ be connected subgraphs of subgroups.[13] Let $c : E\mathbb{Y} \to \mathbb{R}$ be a system of conductances on $\mathbb{Y}$.

Let $\mathbb{X}$ be the Bass-Serre tree of the graph of groups $(\mathbb{Y}, G_*)$ (with geometric realisation $X = |\mathbb{X}|_1$) and Γ its fundamental group (for an indifferent choice of basepoint). Assume that Γ is nonelementary. We denote by $\mathscr{G}(\mathbb{Y}, G_*) = \Gamma\backslash\mathscr{G}\mathbb{X}$ and $\big(\mathsf{g}^t : \mathscr{G}(\mathbb{Y}, G_*) \to \mathscr{G}(\mathbb{Y}, G_*)\big)_{t\in\mathbb{Z}}$ the quotient of the (discrete time) geodesic flow on $\mathscr{G}\mathbb{X}$, by $\widetilde{c} : \mathbb{X} \to \mathbb{R}$ the (Γ-invariant) lift of c, with δ_c its critical exponent and $\widetilde{F}_c : T^1X \to \mathbb{R}$ its associated potential, by m_c the Gibbs measure on $\mathscr{G}(\mathbb{Y}, G_*)$ associated with a choice of Patterson densities $(\mu_x^\pm)_{x\in X}$ for the pairs $(\Gamma, F_c^\pm)$, by $\mathbb{D}^\pm$ two subtrees in $\mathbb{X}$ such that the quotient graphs of groups $\Gamma_{\mathbb{D}^\pm}\backslash\backslash\mathbb{D}^\pm$ identify with $(\mathbb{Y}^\pm, G_*^\pm)$ (see below for precisions), and by $\sigma^\pm_{(\mathbb{Y}^\mp, G_*^\mp)}$ the associated skinning measures.

The *fundamental groupoid* $\pi(\mathbb{Y}, G_*)$ of $(\mathbb{Y}, G_*)$[14] is the quotient of the free product of the groups G_v for $v \in V\mathbb{Y}$ and of the free group on $E\mathbb{Y}$ by the normal subgroup generated by the elements $e\,\overline{e}$ and $e\,\rho_e(g)\,\overline{e}\,\rho_{\overline{e}}(g)^{-1}$ for all $e \in E\mathbb{Y}$ and $g \in G_e$. We identify each G_x for $x \in V\mathbb{Y}$ with its image in $\pi(\mathbb{Y}, G_*)$.

Let $n \in \mathbb{N} - \{0\}$. A *(locally) geodesic path* of length n in the graph of groups $(\mathbb{Y}, G_*)$ is the image α in $\pi(\mathbb{Y}, G_*)$ of a word, called *reduced* in [Bass, 1.7],

$$h_0\, e_1\, h_1\, e_2\, \dots\, h_{n-1}\, e_n\, h_n$$

with

- $e_i \in E\mathbb{Y}$ and $t(e_i) = o(e_{i+1})$ for $1 \leqslant i \leqslant n-1$ (so that $(e_1, \dots, e_n)$ is an edge path in the graph $\mathbb{Y}$);
- $h_0 \in G_{o(e_1)}$ and $h_i \in G_{t(e_i)}$ for $1 \leqslant i \leqslant n$;
- if $e_{i+1} = \overline{e_i}$ then h_i does not belong to $\rho_{e_i}(G_{e_i})$, for $1 \leqslant i \leqslant n-1$.

Its *origin* is $o(\alpha) = o(e_1)$ and its *endpoint* is $t(\alpha) = t(e_n)$. They do not depend on the chosen words with image α in $\pi(\mathbb{Y}, G_*)$.

A *common perpendicular* of length n from $(\mathbb{Y}^-, G_*^-)$ to $(\mathbb{Y}^+, G_*^+)$ in the graph of groups $(\mathbb{Y}, G_*)$ is the double coset

$$[\alpha] = G^-_{o(\alpha)}\,\alpha\,G^+_{t(\alpha)}$$

of a geodesic path α of length n as above, such that:

- α *starts transversally from* $(\mathbb{Y}^-, G_*^-)$, that is, its origin $o(\alpha) = o(e_1)$ belongs to $V\mathbb{Y}^-$ and $h_0 \notin G^-_{o(e_1)}\,\rho_{\overline{e_1}}(G_{e_1})$ if $e_1 \in E\mathbb{Y}^-$,
- α *ends transversally in* $(\mathbb{Y}^+, G_*^+)$, that is, its endpoint $t(\alpha) = t(e_n)$ belongs to $\mathbb{Y}^+$ and $h_n \notin \rho_{e_n}(G_{e_n})\,G^+_{t(e_n)}$ if $e_n \in E\mathbb{Y}^+$.

Note that these two notions do not depend on the representative of the double coset $G^-_{o(\alpha)}\,\alpha\,G^+_{t(\alpha)}$, and we also say that the double coset $[\alpha]$ *starts transversally from* $(\mathbb{Y}^-, G_*^-)$ or *ends transversally in* $(\mathbb{Y}^-, G_*^-)$.

[13] See Section 2.6 for definitions and background.

[14] denoted by $F(\mathbb{Y}, G_*)$ in [Ser3, §5.1], called the *path group* in [Bass, 1.5], see also [Hig]

We denote by $\mathrm{Perp}((\mathbb{Y}^\pm, G_*^\pm), n)$ the set of common perpendiculars in $(\mathbb{Y}, G_*)$ of length at most n from $(\mathbb{Y}^-, G_*^-)$ to $(\mathbb{Y}^+, G_*^+)$. We denote by

$$c(\alpha) = \sum_{i=1}^{n} c(e_i)$$

the *conductance* of a geodesic path α as above, which depends only on the double class $[\alpha]$. We define the *multiplicity* m_α of a geodesic path α as above by

$$m_\alpha = \frac{1}{\mathrm{Card}(G^-_{o(\alpha)} \cap \alpha\, G^+_{t(\alpha)}\, \alpha^{-1})}\;.$$

It depends only on the double class $[\alpha]$ of α. We define the *counting function* of the common perpendiculars in $(\mathbb{Y}, G_*)$ of length at most n from $(\mathbb{Y}^-, G_*^-)$ to $(\mathbb{Y}^+, G_*^+)$ (counted with multiplicities and with weights given by the system of conductances c) as

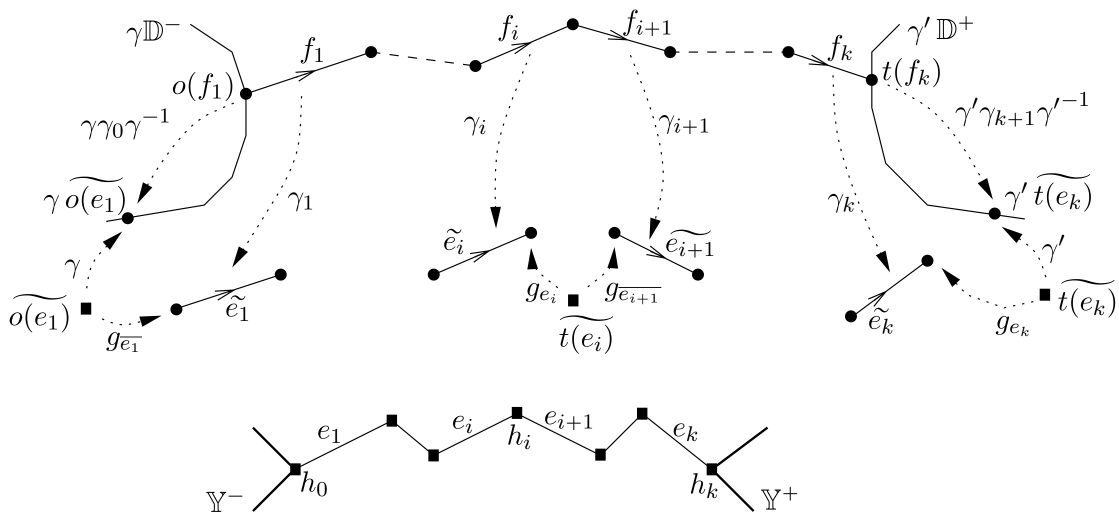
$$\mathscr{N}_{(\mathbb{Y}^-, G_*^-),\,(\mathbb{Y}^+, G_*^+)}(n) = \sum_{[\alpha]\in \mathrm{Perp}((\mathbb{Y}^\pm, G_*^\pm), n)} m_\alpha\; e^{c(\alpha)}\;.$$

Theorem 12.14. *Let $(\mathbb{Y}, G_*)$, $(\mathbb{Y}^\pm, G_*^\pm)$ and c be as in the beginning of this Section. Assume that the critical exponent δ_c of c is finite and positive and that the Gibbs measure m_c on $\mathscr{G}(\mathbb{Y}, G_*)$ is finite and mixing for the discrete time geodesic flow. Assume that the skinning measures $\sigma^\pm_{(\mathbb{Y}^\mp, G_*^\mp)}$ are finite and nonzero. Then as $n \in \mathbb{N}$ tends to ∞*

$$\mathscr{N}_{(\mathbb{Y}^-, G_*^-),\,(\mathbb{Y}^+, G_*^+)}(n) \sim \frac{e^{\delta_c}\,\|\sigma^+_{(\mathbb{Y}^-, G_*^-)}\|\,\|\sigma^-_{(\mathbb{Y}^+, G_*^+)}\|}{(e^{\delta_c}-1)\,\|m_c\|}\;\; e^{\delta_c\, n}\;.$$

Proof. Let $\mathbb{X}$ be the Bass-Serre tree of $(\mathbb{Y}, G_*)$ and Γ its fundamental group (for an indifferent choice of basepoint). As seen in Section 2.6, the Bass-Serre trees $\mathbb{D}^\pm$ of $(\mathbb{Y}^\pm, G_*^\pm)$, with fundamental groups $\Gamma^\pm$, identify with simplicial subtrees $\mathbb{D}^\pm$ of $\mathbb{X}$, such that $\Gamma^\pm$ are the stabilisers $\Gamma_{\mathbb{D}^\pm}$ of $\mathbb{D}^\pm$ in Γ, and that the maps $(\Gamma_{\mathbb{D}^\pm}\backslash\mathbb{D}^\pm) \to (\Gamma\backslash\mathbb{X})$ induced by the inclusion maps $\mathbb{D}^\pm \to \mathbb{X}$ by taking quotient, are injective.

As in Definition 2.10, for all $z \in V\mathbb{Y} \cup E\mathbb{Y}$ and $e \in E\mathbb{Y}$, we fix a lift $\widetilde{z} \in V\mathbb{X} \cup E\mathbb{X}$ of z and $g_e \in \Gamma$, such that $\widetilde{\overline{e}} = \overline{\widetilde{e}}$, $g_e\,\widetilde{t(e)} = t(\widetilde{e})$, $G_z = \Gamma_{\widetilde{z}}$, and the monomorphism $\rho_e : G_e \to G_{t(e)}$ is $\gamma \mapsto g_e^{-1}\gamma g_e$. We assume, as we may, that $\widetilde{z} \in V\mathbb{D}^\pm \cup E\mathbb{D}^\pm$ if $z \in V\mathbb{Y}^\pm \cup E\mathbb{Y}^\pm$. We assume, as we may using Equation (2.22), that if $e \in E\mathbb{Y}^\pm$, then $g_e \in \Gamma_{\mathbb{D}^\pm}$. We denote by $p : \mathbb{X} \to \mathbb{Y} = \Gamma\backslash\mathbb{X}$ the canonical projection.



For all $\gamma, \gamma' \in \Gamma$ such that $\gamma\mathbb{D}^-$ and $\gamma'\mathbb{D}^+$ are disjoint, the common perpendicular $\alpha_{\gamma\mathbb{D}^-,\,\gamma'\mathbb{D}^+}$ from $\gamma\mathbb{D}^-$ to $\gamma'\mathbb{D}^+$ is an edge path $(f_1, f_2, \dots, f_k)$ with $o(f_1) \in \gamma\mathbb{D}^-$ and $t(f_k) \in \gamma'\mathbb{D}^+$. Note that $\gamma^{-1}\, o(f_1)$ and $\widetilde{p(o(f_1))}$ are two vertices of $\mathbb{D}^-$ in the same Γ-orbit, and that ${\gamma'}^{-1}\, t(f_k)$ and $\widetilde{p(t(f_k))}$ are two vertices of $\mathbb{D}^+$ in the same Γ-orbit. Hence by Equation (2.22), we may choose $\gamma_0 \in \Gamma_{\mathbb{D}^-}$ such that $\gamma_0\gamma^{-1}\, o(f_1) = \widetilde{p(o(f_1))}$ and $\gamma_{k+1} \in \Gamma_{\mathbb{D}^-}$ such that $\gamma_{k+1}{\gamma'}^{-1}\, t(f_k) = \widetilde{p(t(f_k))}$. For $1 \leqslant i \leqslant k$, choose $\gamma_i \in \Gamma$ such that $\gamma_i f_i = \widetilde{p(f_i)}$. We define (see the above picture)

- $e_i = p(f_i)$ for $1 \leqslant i \leqslant k$,
- $h_i = g_{e_i}^{-1}\gamma_i{\gamma_{i+1}}^{-1}g_{\overline{e_{i+1}}}$, which belongs to $\Gamma_{\widetilde{t(e_i)}} = G_{t(e_i)}$ for $1 \leqslant i \leqslant k-1$,
- $h_0 = \gamma_0\gamma^{-1}\gamma_1^{-1}g_{\overline{e_1}} = \gamma^{-1}(\gamma\gamma_0\gamma^{-1})\gamma_1^{-1}g_{\overline{e_1}}$, which belongs to $\Gamma_{\widetilde{o(e_1)}} = G_{o(e_1)}$,
- $h_k = g_{e_k}^{-1}\gamma_k\gamma'{\gamma_{k+1}}^{-1} = g_{e_k}^{-1}\gamma_k(\gamma'\gamma_{k+1}{\gamma'}^{-1})^{-1}\gamma'$, which belongs to $\Gamma_{\widetilde{t(e_k)}} = G_{t(e_k)}$.

Lemma 12.15.

(1) The word $h_0e_1h_1\dots h_{k-1}e_kh_k$ is reduced. Its image α in the fundamental groupoid $\pi(\mathbb{Y}, G_*)$ does not depend on the choices of $\gamma_1, \dots, \gamma_k$, and starts transversally from $(\mathbb{Y}^-, G_*^-)$ and ends transversally in $(\mathbb{Y}^+, G_*^+)$. The double class $[\alpha]$ of α is independent of the choices of γ_0 and γ_{k+1}.

(2) The map $\widetilde{\Theta}$ from the set of common perpendiculars in $\mathbb{X}$ between disjoint images of $\mathbb{D}^-$ and $\mathbb{D}^+$ under elements of Γ, into the set of common perpendiculars in $(\mathbb{Y}, G_*)$ from $(\mathbb{Y}^-, G_*^-)$ to $(\mathbb{Y}^+, G_*^+)$, sending $\alpha_{\gamma\mathbb{D}^-,\,\gamma'\mathbb{D}^+}$ to $[\alpha]$, is invariant under the action of Γ at the source, and preserves the lengths and the multiplicities.

(3) The map Θ induced by $\widetilde{\Theta}$ from the set of Γ-orbits of common perpendiculars in $\mathbb{X}$ between disjoint images of $\mathbb{D}^-$ and $\mathbb{D}^+$ under elements of Γ into the set of common perpendiculars in $(\mathbb{Y}, G_*)$ from $(\mathbb{Y}^-, G_*^-)$ to $(\mathbb{Y}^+, G_*^+)$ is a bijection, preserving the lengths and the multiplicities.

Proof. (1) If $e_{i+1} = \overline{e_i}$, then by the definition of h_i, we have

$$\begin{aligned} h_i \in \rho_{e_i}(G_{e_i}) = g_{e_i}^{-1}\Gamma_{\widetilde{e_i}} g_{e_i} &\iff g_{e_i}\, h_i\, g_{e_i}^{-1}\, \widetilde{e_i} = \widetilde{e_i} \\ &\iff g_{e_i}\, g_{e_i}^{-1}\gamma_i{\gamma_{i+1}}^{-1} g_{\overline{e_{i+1}}}\, g_{e_i}^{-1}\, \widetilde{e_i} = \widetilde{e_i} \\ &\iff \gamma_i{\gamma_{i+1}}^{-1}\, \overline{\widetilde{e_{i+1}}} = \widetilde{e_i} \\ &\iff \overline{{\gamma_{i+1}}^{-1}\, \widetilde{e_{i+1}}} = \gamma_i^{-1}\widetilde{e_i} \iff \overline{f_{i+1}} = f_i\ . \end{aligned}$$

Hence the word $h_0 e_1 h_1 \dots h_{k-1} e_k h_k$ is reduced.

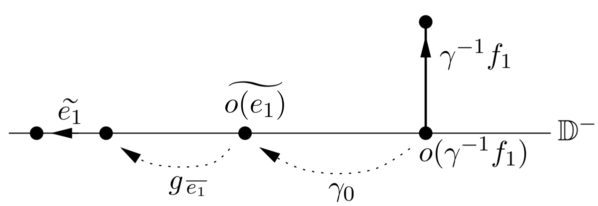
The element γ_i for $i \in \{1, \dots, k\}$ is uniquely determined up to multiplication on the left by an element of $\Gamma_{\widetilde{e_i}} = G_{e_i}$. If we fix[15] $i \in \{1, \dots, k\}$ and if we replace γ_i by $\gamma_i' = \alpha\, \gamma_i$ for some $\alpha \in G_{e_i}$, then only the elements h_{i-1} and h_i change, replaced by elements that we denote by h_{i-1}' and h_i' respectively. We have (if $2 \leqslant i \leqslant k-1$, but otherwise the argument is similar by the definitions of h_0 and h_k)

$$\begin{aligned} h_{i-1}'\, e_i\, h_i' &= g_{e_{i-1}}^{-1}\gamma_{i-1}\, {\gamma_i}^{-1}\alpha^{-1} g_{\overline{e_i}}\; e_i\; g_{e_i}^{-1}\alpha\, \gamma_i\, {\gamma_{i+1}}^{-1} g_{\overline{e_{i+1}}} \\ &= g_{e_{i-1}}^{-1}\gamma_{i-1}\, {\gamma_i}^{-1} g_{\overline{e_i}}\, \rho_{\overline{e_i}}(\alpha)^{-1}\; e_i\; \rho_{e_i}(\alpha)\, g_{e_i}^{-1}\gamma_i\, {\gamma_{i+1}}^{-1} g_{\overline{e_{i+1}}}\ . \end{aligned}$$

Since $\rho_{\overline{e_i}}(\alpha)^{-1}\, e_i\, \rho_{e_i}(\alpha)$ is equal to $\overline{e_i}^{\;-1} = e_i$ in the fundamental groupoid, the words $h_{i-1}'\, e_i\, h_i'$ and $h_{i-1}\, e_i\, h_i$ have the same image in $\pi(\mathbb{Y}, G_*)$. Therefore α is does not depend on the choices of $\gamma_1, \dots, \gamma_k$.

We have $o(\alpha) = o(e_1) \in V\mathbb{Y}^-$ and $t(\alpha) = t(e_k) \in V\mathbb{Y}^+$, hence α starts from $\mathbb{Y}^-$ and ends in $\mathbb{Y}^+$.

Assume that $e_1 \in E\mathbb{Y}^-$. Let us prove that $h_0 \in G_{o(e_1)}^-\, \rho_{\overline{e_1}}(G_{e_1})$ if and only if $f_1 \in \gamma\, E\mathbb{D}^-$.



By the definition of $\rho_{\overline{e_1}}$, we have $h_0 \in G_{o(e_1)}^-\, \rho_{\overline{e_1}}(G_{e_1})$ if and only if there exists $\alpha \in \Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-}$ such that $\alpha^{-1}\, h_0 \in g_{\overline{e_1}}^{-1}\, \Gamma_{\widetilde{e_1}}\, g_{\overline{e_1}}$. By the definition of h_0 and since γ_1 maps f_1 to $\widetilde{e_1}$, we have

$$\begin{aligned} \alpha^{-1}\, h_0 \in g_{\overline{e_1}}^{-1}\, \Gamma_{\widetilde{e_1}}\, g_{\overline{e_1}} &\iff g_{\overline{e_1}}\, \alpha^{-1}\left(\gamma_0\gamma^{-1}\gamma_1^{-1} g_{\overline{e_1}}\right) g_{\overline{e_1}}^{-1}\, \widetilde{e_1} \;=\; \widetilde{e_1} \\ &\iff f_1 = \gamma\, \gamma_0^{-1}\, \alpha\, g_{\overline{e_1}}^{-1}\, \widetilde{e_1}\ . \end{aligned}$$

Since $\widetilde{e_1} \in E\mathbb{D}^-$ and γ_0, α, $g_{\overline{e_1}}$ all belong to $\Gamma_{\mathbb{D}^-}$, this last condition implies that $f_1 \in \gamma\, E\mathbb{D}^-$. Conversely (for future use), if $f_1 \in \gamma\, E\mathbb{D}^-$, then (see the above picture) $\gamma_0\, \gamma^{-1} f_1$ is an edge of $\mathbb{D}^-$ with origin $\widetilde{o(e_1)}$, in the same Γ-orbit than the edge $g_{\overline{e_1}}^{-1}\, \widetilde{e_1}$ of $\mathbb{D}^-$, which also has origin $\widetilde{o(e_1)}$. By Equation (2.22), this implies that there exists $\alpha \in \Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-}$ such that $f_1 = \gamma\, \gamma_0^{-1}\, \alpha\, g_{\overline{e_1}}^{-1}\, \widetilde{e_1}$. By the above equivalences, we hence have that $h_0 \in G_{o(e_1)}^-\, \rho_{\overline{e_1}}(G_{e_1})$.

[15] We leave to the reader the verification that the changes induced by various i's do not overlap.

Similarly, one proves that if $e_k \in E\mathbb{Y}^+$, then $h_k \in \rho_{e_k}(G_{e_k})\, G^+_{t(e_k)}$ if and only if $f_k \in \gamma' E\mathbb{D}^+$. Since $(f_1, \dots, f_n)$ is the common perpendicular edge path from $\gamma\,\mathbb{D}^-$ to $\gamma'\mathbb{D}^+$, this proves that α starts transversally from $\mathbb{Y}^-$ and ends transversally in $\mathbb{Y}^-$.

Note that the element $\gamma_0 \in \Gamma_{\mathbb{D}^-}$ is uniquely defined up to multiplication on the left by an element of $\Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-} = G^-_{o(e_1)}$, and appears only as the first letter in the expression of h_0. Note that the element $\gamma_{k+1} \in \Gamma_{\mathbb{D}^+}$ is uniquely defined up to multiplication on the left by an element of $\Gamma_{\widetilde{t(e_k)}} \cap \Gamma_{\mathbb{D}^+} = G^+_{t(e_k)}$, hence γ_{k+1}^{-1} is uniquely defined up to multiplication on the right by an element of $G^+_{t(e_k)}$, and appears only as the last letter in the expression of h_k. Therefore α is uniquely defined in the fundamental groupoid $\pi(\mathbb{Y}, G_*)$ up to multiplication on the left by an element of $G^-_{o(e_1)}$ and multiplication on the right by an element of $G^+_{t(e_k)}$, that is, the double class $[\alpha] \subset \pi(\mathbb{Y}, G_*)$ is uniquely defined.

(2) Let β be an element in Γ and let $x = \alpha_{\gamma\mathbb{D}^-,\,\gamma'\mathbb{D}^+}$ be a common perpendicular in $\mathbb{X}$ between disjoint images of $\mathbb{D}^-$ and $\mathbb{D}^+$ under elements of Γ. Let us prove that $\widetilde{\Theta}(\beta\, x) = \widetilde{\Theta}(x)$.

Since $\beta\, x = \alpha_{\beta\,\gamma\mathbb{D}^-,\,\beta\,\gamma'\mathbb{D}^+}$, in the construction of $\widetilde{\Theta}(\beta\, x)$, we may take, instead of the elements $\gamma_0, \gamma_1, \dots, \gamma_k, \gamma_{k+1}$ used to construct $\widetilde{\Theta}(x)$, the elements

$$\gamma^\sharp_0 = \gamma_0,\;\; \gamma^\sharp_1 = \gamma_1\,\beta^{-1}, \dots,\;\; \gamma^\sharp_k = \gamma_k\,\beta^{-1},\;\; \gamma^\sharp_{k+1} = \gamma_{k+1}.$$

And instead of γ and γ', we now may use $\gamma^\sharp = \beta\,\gamma$ and ${\gamma'}^\sharp = \beta\,\gamma'$.

The only terms involving $\gamma, \gamma', \gamma_1, \dots, \gamma_k$ in the construction of $\widetilde{\Theta}(x)$ come under the form $\gamma^{-1}\gamma_1^{-1}$ in h_0, $\gamma_i\,{\gamma_{i+1}}^{-1}$ in h_i for $1 \leqslant i \leqslant k-1$, and $\gamma_k\gamma'$ in h_k. Since $(\gamma^\sharp)^{-1}(\gamma^\sharp_1)^{-1} = \gamma^{-1}\gamma_1^{-1}$, $(\gamma^\sharp_i)(\gamma^\sharp_{i+1})^{-1} = \gamma_i\,{\gamma_{i+1}}^{-1}$ for $1 \leqslant i \leqslant k-1$, and $(\gamma^\sharp_k)({\gamma'}^\sharp) = \gamma_k\gamma'$, this proves that $\widetilde{\Theta}(\beta\, x) = \widetilde{\Theta}(x)$, as wanted.

It is immediate that if the length of $\alpha_{\gamma\mathbb{D}^-,\,\gamma'\mathbb{D}^+}$ is k, then the length of $[\alpha]$ is k.

Let us prove that the multiplicity, given in Equation (12.9),

$$m_{\gamma,\,\gamma'} = \frac{1}{\operatorname{Card}(\gamma\Gamma_{\mathbb{D}^-}\gamma^{-1} \cap \gamma'\Gamma_{\mathbb{D}^+}{\gamma'}^{-1})}$$

of the common perpendicular $\alpha_{\gamma\mathbb{D}^-,\,\gamma'\mathbb{D}^+}$ in $\mathbb{X}$ between $\gamma\,\mathbb{D}^-$ and $\gamma'\,\mathbb{D}^+$ is equal to the multiplicity

$$m_\alpha = \frac{1}{\operatorname{Card}(G^-_{o(\alpha)} \cap \alpha\, G^+_{t(\alpha)}\,\alpha^{-1})}$$

of the common perpendicular α in $(\mathbb{Y}, G_*)$ from $(\mathbb{Y}^-, G^-_*)$ to $(\mathbb{Y}^+, G^+_*)$.

Since the multiplicity $m_{\gamma,\,\gamma'}$ is invariant under the diagonal action by left translations of $\gamma_0^{-1}\gamma^{-1} \in \Gamma$ on (γ, γ'), we may assume that $\gamma = \gamma_0 = \mathrm{id}$. Since the multiplicity $m_{\gamma,\,\gamma'}$ is invariant under right translation by ${\gamma_{k+1}}^{-1}$, which stabilises $\mathbb{D}^+$, on the element γ', we may assume that $\gamma_{k+1} = \mathrm{id}$. In particular, we have

$$o(f_1) = \widetilde{o(e_1)} \quad\text{and}\quad t(f_k) = \gamma'\;\widetilde{t(e_k)}\;.$$

We use the basepoint $x_0 = o(e_1)$ in the construction of the fundamental group and the Bass-Serre tree of $(\mathbb{Y}, G_*)$, so that (see in particular [Bass, Eq. (1.3)])

$$V\mathbb{X} = \coprod_{\beta\in\pi(\mathbb{Y},\,G_*)\;:\;o(\beta)=x_0} \beta\; G_{t(\beta)}$$

and

$$\Gamma = \pi_1(\mathbb{Y}, G_*) = \{\beta \in \pi(\mathbb{Y}, G_*)\ :\ o(\beta) = t(\beta) = x_0\}\ .$$

Since an element in Γ which preserves $\mathbb{D}^-$ and $\gamma'\,\mathbb{D}^+$ fixes pointwise its (unique) common perpendicular in $\mathbb{X}$, we have

$$\begin{aligned}\Gamma_{\mathbb{D}^-}\ \cap\ \gamma'\Gamma_{\mathbb{D}^+}\gamma'^{-1} &= \Gamma_{\mathbb{D}^-}\ \cap\ \Gamma_{\gamma'\,\mathbb{D}^+} = (\Gamma_{o(f_1)} \cap \Gamma_{\mathbb{D}^-})\ \cap\ (\Gamma_{t(f_k)} \cap \Gamma_{\gamma'\,\mathbb{D}^+})\\ &= (\Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-})\ \cap\ (\Gamma_{\gamma'\,\widetilde{t(e_k)}} \cap \Gamma_{\gamma'\,\mathbb{D}^+})\ .\end{aligned}$$

Note that $\Gamma_{\widetilde{o(e_1)}} \cap \Gamma_{\mathbb{D}^-} = G^-_{o(e_1)}$. By the construction of the edges in the Bass-Serre tree of a graph of groups (see [Bass, page 11]), the vertex $\alpha\, G_{t(e_k)}$ is exactly the vertex $t(f_k) = \gamma'\,\widetilde{t(e_k)}$. By [Bass, Eq. (1.4)], we hence have

$$\alpha\, G_{t(e_k)}\, \alpha^{-1} = \mathrm{Stab}_{\pi_1(\mathbb{Y}, G_*)}(\alpha\, G_{t(e_k)}) = \Gamma_{\gamma'\,\widetilde{t(e_k)}}\ .$$

Therefore $m_{\gamma,\,\gamma'} = m_\alpha$.

(3) Let $[\alpha] = G_{o(\alpha)}\,\alpha\, G_{t(\alpha)}$ be a common perpendicular in $(\mathbb{Y}, G_*)$ from $(\mathbb{Y}^-, G^-_*)$ to $(\mathbb{Y}^+, G^+_*)$, with representative $\alpha \in \pi(\mathbb{Y}, G_*)$, and let $h_0\, e_1\, h_1\, \dots\, e_k\, h_k$ be a reduced word whose image in $\pi(\mathbb{Y}, G_*)$ is α.

We define

- $\gamma_1 = g_{\overline{e_1}}\, h_0^{-1}$,
- $f_1 = \gamma_1^{-1}\, \widetilde{e_1}$,
- assuming that γ_i and f_i for some $1 \leqslant i \leqslant k-1$ are constructed, let

$$\gamma_{i+1} = g_{\overline{e_{i+1}}}\, h_i^{-1}\, g_{e_i}^{-1}\, \gamma_i \quad \text{and} \quad f_{i+1} = {\gamma_{i+1}}^{-1}\, \widetilde{e_{i+1}}\ ,$$

- with γ_k and f_k constructed by induction, finally let $\gamma' = \gamma_k^{-1}\, g_{e_k}\, h_k$.

It is easy to check, using the equivalences in the proof of Lemma 12.15 (1) with $\gamma = \gamma_0 = \gamma_{k+1} = \mathrm{id}$, that the sequence $(f_1, \dots, f_k)$ is the edge path of a common perpendicular in $\mathbb{X}$ from $\mathbb{D}^-$ to $\gamma'\mathbb{D}^+$ with origin $\widetilde{o(e_1)}$ and endpoint $\gamma'\,\widetilde{t(e_k)}$.

If h_0 is replaced by $\alpha\, h_0$ with $\alpha \in G^-_{o(e_1)}$, then by induction, $f_1, f_2 \dots, f_k$ are replaced by $\alpha f_1, \alpha f_2, \dots, \alpha f_k$ and γ' is replaced by $\alpha\gamma'$. Note that $(\alpha f_1, \alpha f_2, \dots, \alpha f_k)$ is then the common perpendicular edge path from $\mathbb{D}^- = \alpha\mathbb{D}^-$ to $\alpha\gamma'\mathbb{D}^+$. If h_k is replaced by $h_k\, \alpha$ with $\alpha \in G^+_{t(e_k)}$, then $f_1, f_2 \dots, f_k$ are unchanged, and γ' is replaced by $\gamma'\,\alpha$. Note that $\gamma'\,\alpha\,\mathbb{D}^+ = \gamma'\,\mathbb{D}^+$.

Hence the map which associates to $[\alpha]$ the Γ-orbit of the common perpendicular in $\mathbb{X}$ from $\mathbb{D}^-$ to $\gamma'\mathbb{D}^+$ with edge path $(f_1, \dots, f_k)$ is well defined. It is easy to see by construction that this map is the inverse of Θ. □

Theorem 12.14 now follows from Theorem 12.9. □

12.6 Error terms for equidistribution and counting for metric and simplicial graphs of groups

In this Section, we give error terms to the equidistribution and counting results of Section 12.4, given by Theorem 12.8 for metric trees (and their continuous time geodesic flows) and by

Theorem 12.9 for simplicial trees (and their discrete time geodesic flows), under appropriate assumptions on bounded geometry and the rate of mixing.

Let $(\mathbb{X}, \lambda)$, X, Γ, $\widetilde{c}$, c, $\widetilde{F}_c$, F_c, δ_c, $\mathbb{D}^\pm$, $D^\pm$, $\mathscr{D}^\pm$, $\lambda_{\gamma,\gamma'}$, $\alpha_{\gamma,\gamma'}$, $\alpha^\pm_{\gamma,\gamma'}$, $m_{\gamma,\gamma'}$ be as in Section 12.4. We first consider the simplicial case (when $\lambda = 1$), for the discrete time geodesic flow.

Theorem 12.16. *Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X})$, let $\widetilde{c}$ be a system of conductances on $\mathbb{X}$ for Γ and let $\mathbb{D}^\pm$ be nonempty proper simplicial subtrees of $\mathbb{X}$. Assume that the critical exponent δ_c is finite and positive, that the Gibbs measure m_c (for the discrete time geodesic flow) is finite and that the skinning measures $\sigma^\pm_{\mathscr{D}^\mp}$ are finite and nonzero. Assume furthermore that*

(1) *at least one of the following holds :*
   - *$\Gamma_{\mathbb{D}^\pm}\backslash\partial D^\pm$ is compact*
   - *$\mathscr{C}\Lambda\Gamma$ is uniform and Γ is a lattice of $\mathscr{C}\Lambda\Gamma$,*

(2) *there exists $\beta \in \,]0,1]$ such that the discrete time geodesic flow on $(\Gamma\backslash\mathscr{G}\mathbb{X}, m_c)$ is exponentially mixing for the β-Hölder regularity.*

Then there exists $\kappa' > 0$ such that for all $\psi^\pm \in \mathscr{C}_c^\beta(\Gamma\backslash\check{\mathscr{G}}\mathbb{X})$, we have, as $n \to +\infty$,

$$\frac{e^{\delta_c}-1}{e^{\delta_c}}\,\|m_c\|\,e^{-\delta_c\,n}\sum_{\substack{[\gamma]\in\Gamma_{\mathbb{D}^-}\backslash\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\gamma\,\mathbb{D}^+)\leqslant n}} m_{e,\gamma}\,e^{\widetilde{c}\,(\alpha_{e,\gamma})}\,\psi^-(\Gamma\alpha^-_{e,\gamma})\,\psi^+(\Gamma\alpha^+_{\gamma^{-1},e})$$
$$=\int\psi^-\,d\sigma^+_{\mathscr{D}^-}\int\psi^+\,d\sigma^-_{\mathscr{D}^+}+\operatorname{O}\big(e^{-\kappa'\,n}\,\|\psi^-\|_\beta\,\|\psi^+\|_\beta\big)$$

and if $\Gamma_{\mathbb{D}^\pm}\backslash\partial D^\pm$ is compact, then

$$\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+}(n)=\frac{e^{\delta_c}\,\|\sigma^+_{\mathscr{D}^-}\|\,\|\sigma^-_{\mathscr{D}^+}\|}{(e^{\delta_c}-1)\,\|m_c\|}\;e^{\delta_c\,n}+\operatorname{O}\big(e^{(\delta_c-\kappa')n}\big)\,.$$

Proof. We follow the scheme of proof of Theorem 12.7, replacing aspects of Riemannian manifolds by aspects of simplicial trees as in the proof of Theorem 11.9. Let $\widetilde{\psi}^\pm \in \mathscr{C}_c^\beta(\check{\mathscr{G}}\mathbb{X})$. In order to simplify the notation, let $\lambda_\gamma = \lambda_{e,\gamma}$, $\alpha_\gamma = \alpha_{e,\gamma}$, $\alpha^-_\gamma = \alpha^-_{e,\gamma}$, $\alpha^+_\gamma = \alpha^+_{\gamma^{-1},e}$ and $\widetilde{\sigma}^\pm = \widetilde{\sigma}^\pm_{\mathbb{D}^\mp}$.

Let us first prove the following avatar of Equation (12.4), indicating only the required changes in its proof: there exists $\kappa_0 > 0$ (independent of $\widetilde{\psi}^\pm$) such that, as $n \to +\infty$,

$$\frac{e^{\delta_c}-1}{e^{\delta_c}}\,\|m_c\|\,e^{-\delta_c\,n}\sum_{\gamma\in\Gamma,\,0<\lambda_\gamma\leqslant n} e^{\widetilde{c}\,(\alpha_\gamma)}\,\widetilde{\psi}^-(\alpha^-_\gamma)\,\widetilde{\psi}^+(\alpha^+_\gamma)$$
$$=\int_{\partial^1_+D^-}\widetilde{\psi}^-\,d\widetilde{\sigma}^+\int_{\partial^1_-D^+}\widetilde{\psi}^+\,d\widetilde{\sigma}^-+\operatorname{O}(e^{-\kappa_0 n}\,\|\widetilde{\psi}^-\|_\beta\,\|\widetilde{\psi}^+\|_\beta)\,. \qquad (12.13)$$

Most of the new work to be done in order to prove this formula concerns regularity properties of the test functions that will be introduced later on.

We fix $R > e^2$ large enough. Let $0 < \eta < 1$. We introduce the following modification of the test functions $\phi_\eta^{\pm}$:[16]

$$\Phi_\eta^{\pm} = (h_{\eta,\,R}^{\pm}\; \widetilde{\psi}^{\pm}) \circ f_{D^\pm}^{\mp} \;\; \mathbb{1}_{\mathscr{V}_{\eta,\,R}^{\mp}(\partial_{\mp}^{1} D^{\pm})}\,.$$

As in Lemma 10.1, the functions $\Phi_\eta^{\pm}$ are measurable and satisfy

$$\int_{\mathscr{G}\mathbb{X}} \Phi_\eta^{\pm}\; d\widetilde{m}_c = \int_{\partial_{\mp}^{1}\mathbb{D}^{\pm}} \widetilde{\psi}^{\pm}\; d\widetilde{\sigma}^{\mp}\,. \tag{12.14}$$

Lemma 12.17. *The maps $\Phi_\eta^{\pm}$ are β-Hölder-continuous with*

$$\|\Phi_\eta^{\pm}\|_\beta = \operatorname{O}(\|\widetilde{\psi}^{\pm}\|_\beta)\,. \tag{12.15}$$

Proof. Since $\mathbb{X}$ is a simplicial tree and $\eta < 1$, we have $V_{w,\,\eta,\,R}^{\pm} = B^{\pm}(w,R)$ for every $w \in \partial_{\pm}^{1} D^{\mp}$. By the proof of Lemma 3.3,[17] there exists $c_R > 0$ depending only on R such that if $\ell' \in \mathscr{G}\mathbb{X}$ satisfy $d(\ell,\ell') \leqslant c_R$, then ℓ' coincides with ℓ on $\pm[0, \ln R + 1]$. Therefore, if $\ell \in B^{\pm}(w,R)$ for some $w \in \partial_{\pm}^{1} D^{\mp}$ and $d(\ell,\ell') \leqslant c_R$, then ℓ' coincides with w on $\pm[\ln R, \ln R + 1]$, thus $\ell' \in B^{\pm}(w',R)$ where $w' \in \partial_{\pm}^{1} D^{\mp}$ is the geodesic ray with $w'(0) = w(0)$ and $w'_{\pm} = \ell'_{\pm}$. Hence (see Section 3.1) the characteristic function $\mathbb{1}_{\mathscr{V}_{\eta,\,R}^{\pm}(\partial_{\pm}^{1} D^{\mp})}$ is c_R-locally constant, thus β-Hölder-continuous by Remark 3.11.

By Assumption (1) in the statement of Theorem 12.16, the denominator of

$$h_{\eta,\,R}^{\mp}(w) = \frac{1}{\mu_{W^{\pm}(w)}(B^{\pm}(w,R))}$$

is at least a positive constant depending only on R, hence $h_{\eta,\,R}^{\mp}$ is bounded by a constant depending only on R. Since the map $\mathbb{1}_{B^{\pm}(w,R)}$ is c_R-locally constant, so is the map $h_{\eta,\,R}^{\mp}$. The result then follows from Lemma 3.8 and Equation (3.7). □

In order to prove Equation (12.13), as in the proofs of Theorems 12.7 and 11.9, for all $N \in \mathbb{N}$, we estimate in two ways the quantity

$$I_\eta(N) = \sum_{n=0}^{N} e^{\delta_c\, n} \sum_{\gamma\in\Gamma} \int_{\ell\in\mathscr{G}\mathbb{X}} \Phi_\eta^{-}(\mathsf{g}^{-\lfloor n/2\rfloor}\ell)\; \Phi_\eta^{+}(\mathsf{g}^{\lceil n/2\rceil}\gamma^{-1}\ell)\; d\widetilde{m}_c(\ell)\,. \tag{12.16}$$

On the one hand, as in order to obtain Equation (12.6), using now Assumption (2) in the statement of Theorem 12.16 on the exponential mixing for the discrete time geodesic flow, a geometric sum argument and Equations (12.14) and (12.15), we have

$$I_\eta(N) = \frac{e^{\delta_c(N+1)}}{(e^{\delta_c}-1)\,\|m_c\|} \Big(\int_{\partial_{+}^{1}\mathbb{D}^{-}} \widetilde{\psi}^{-}\; d\widetilde{\sigma}^{+} \int_{\partial_{-}^{1}\mathbb{D}^{+}} \widetilde{\psi}^{+}\; d\widetilde{\sigma}^{-} + \operatorname{O}(e^{-\kappa\, N}\|\widetilde{\psi}^{-}\|_\beta \|\widetilde{\psi}^{+}\|_\beta) \Big)\,. \tag{12.17}$$

[16]See Equation (10.4) for the definition of $\phi_\eta^{\pm}$ and Equation (10.1) for the definition of $h_{\eta,\,R}^{\pm}$, that simplifies as $h_{\eta,\,R}^{\mp}(w) = (\mu_{W^{\pm}(w)}(B^{\pm}(w,R)))^{-1}$ since $\mathbb{X}$ is simplicial, as seen in Equation (11.27).
[17]See also the proof of Lemmas 3.4 and 3.8.

On the other hand, exchanging the summations over γ and n in the definition of $I_\eta(N)$, we have

$$I_\eta(N)=\sum_{\gamma\in\Gamma}\;\sum_{n=0}^{N} e^{\delta_c\, n}\int_{\mathscr{G}\mathbb{X}}\Phi^-_\eta(\mathsf{g}^{-\lfloor n/2\rfloor}\ell)\;\Phi^+_\eta(\mathsf{g}^{\lceil n/2\rceil}\gamma^{-1}\ell)\;d\widetilde{m}_c(\ell)\,.$$

With the simplifications in Step 3T of the proof of Theorem 11.1 given by the proof of Theorem 11.9, if $\eta<\frac{1}{2}$, if $\ell\in\mathscr{G}\mathbb{X}$ belongs to the support of $\Phi^-_\eta\circ\mathsf{g}^{-\lfloor n/2\rfloor}\;\Phi^+_\eta\circ\mathsf{g}^{\lceil n/2\rceil}\circ\gamma^{-1}$, setting $w^-=f^+_{D^-}(\ell)$ and $w^+=f^-_{\gamma D^+}(\ell)$, we then have $\lambda_\gamma=n$, $w^-(0)=\alpha^-_\gamma(0)$, $w^+(0)=\gamma\alpha^+_\gamma(0)$ and

$$w^-(\lfloor n/2\rfloor)=w^+(-\lceil n/2\rceil)=\ell(0)=\alpha^-_\gamma(\lfloor n/2\rfloor)=\gamma\alpha^+_\gamma(-\lceil n/2\rceil)\,.$$

Hence by the triangle inequality

$$\begin{aligned}d(w^-,\alpha^-_\gamma)&=\int_{\lfloor n/2\rfloor}^{+\infty}d(w^-(s),\alpha^-_\gamma(s))\;e^{-2s}\;ds\leqslant e^{-2\lfloor n/2\rfloor}\int_0^{+\infty}2s'\,e^{-2s'}\;ds'\\&=\mathrm{O}(e^{-\lambda_\gamma})\,.\end{aligned}$$

Similarly, $d(w^+,\gamma\alpha^+_\gamma)=\mathrm{O}(e^{-\lambda_\gamma})$. Therefore, since $\widetilde{\psi}^\pm$ is β-Hölder-continuous,

$$|\,\widetilde{\psi}^-(w^-)-\widetilde{\psi}^-(v^-_\gamma)\,|,\;\;|\,\widetilde{\psi}^+(\gamma^{-1}w^+)-\widetilde{\psi}^+(v^\pm_\gamma)\,|=\mathrm{O}(e^{-\beta\lambda_\gamma}\;\|\widetilde{\psi}^\pm\|_\beta)\,.$$

Note that now $\Phi^\pm_\eta=\widetilde{\psi}^\pm\circ f^\mp_{D^\pm}\;\phi^\pm_\eta$, so that

$$\begin{aligned}I_\eta(N)=\sum_{\gamma\in\Gamma}\;&\big(\widetilde{\psi}^-(\alpha^-_\gamma)\widetilde{\psi}^+(\alpha^+_\gamma)+\mathrm{O}(e^{-2\beta\lambda_\gamma}\;\|\widetilde{\psi}^-\|_\beta\|\widetilde{\psi}^+\|_\beta)\big)\;\times\\&\sum_{n=0}^{N}e^{\delta_c\,n}\int_{\mathscr{G}\mathbb{X}}\phi^-_\eta(\mathsf{g}^{-\lfloor n/2\rfloor}\ell)\;\phi^+_\eta(\mathsf{g}^{\lceil n/2\rceil}\gamma^{-1}\ell)\;d\widetilde{m}_c(\ell)\,.\end{aligned}$$

Now if $\eta<\frac{1}{2}$, Equation (12.13) with $\kappa_0=\min\{2\beta,\kappa\}$ follows as in Steps 3T and 4T of the proof of Theorem 11.1 with the simplifications given by the proof of Theorem 11.9.

The end of the proof of the equidistribution claim of Theorem 12.16 follows from Equation (12.13) as the one of Theorem 12.7 from Equation (12.4).

The counting claim follows from the equidistribution one by taking $\psi^\pm=\mathbb{1}_{\Gamma\mathscr{V}_{\eta,R}(\partial^1_\mp\mathbb{D}^\pm)}$, which has compact support since $\Gamma_{\mathbb{D}^\pm}\backslash\partial\mathbb{D}^\pm$ is assumed to be compact, and is β-Hölder-continuous by previous arguments. □

Remarks. (i) Assume that $\widetilde{c}=0$, that the simplicial tree $\mathbb{X}'$ with $|\mathbb{X}'|_1=\mathscr{C}\Lambda\Gamma$ is uniform without vertices of degree 2, that $L_\Gamma=\mathbb{Z}$ and that Γ is a geometrically finite lattice of $\mathbb{X}'$. Then all assumptions of Theorem 12.16 are satisfied by the results of Section 4.4 and by Corollary 9.6. Therefore we have an exponentially small error term in the (joint) equidistribution of the common perpendiculars, and in their counting if $\Gamma_{\mathbb{D}^\pm}\backslash\partial\mathbb{D}^\pm$ is compact, see Examples 12.11 (2) and 12.13 (4).

(ii) Assume in this remark that Assumption (2) of the above theorem is replaced by the assumptions that $\mathscr{C}\Lambda\Gamma$ is uniform without vertices of degree 2, that $L_\Gamma=2\mathbb{Z}$, and that there exists $\beta\in\;]0,1]$ such that the square of the discrete time geodesic flow on $(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X},m_c)$ is exponentially mixing for the β-Hölder regularity, for instance by Corollary 9.6 (2) if Γ is geometrically finite. Then a similar proof (replacing the references to Theorem 11.9 by references to Theorem 11.11) shows that there exists $\kappa' > 0$ such that for all $\psi^\pm \in \mathscr{C}_{\rm c}^{\beta}(\Gamma\backslash\breve{\mathscr{G}}\mathbb{X})$, we have, as $n \to +\infty$,

$$\frac{e^{2\delta_c}-1}{2\,e^{2\delta_c}}\;\|m_c\|\;e^{-\delta_c\,n}\sum_{\substack{[\gamma]\in\Gamma_{\mathbb{D}^-}\backslash\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\gamma\,\mathbb{D}^+)\leqslant n}} m_{e,\gamma}\;e^{\widetilde{c}\,(\alpha_{e,\gamma})}\;\psi^-(\Gamma\alpha^-_{e,\gamma})\;\psi^+(\Gamma\alpha^+_{\gamma^{-1},e})$$
$$=\int\psi^-\,d\sigma^+_{\mathscr{D}^-}\;\int\psi^+\,d\sigma^-_{\mathscr{D}^+}+\operatorname{O}\big(e^{-\kappa'\,n}\,\|\psi^-\|_\beta\,\|\psi^+\|_\beta\big)$$

and if $\Gamma_{\mathbb{D}^\pm}\backslash\partial\mathbb{D}^\pm$ is compact, then

$$\mathscr{N}_{\mathbb{D}^-,\,\mathbb{D}^+}(n)=\frac{2\,e^{2\delta_c}\;\|\sigma^+_{\mathscr{D}^-}\|\;\|\sigma^-_{\mathscr{D}^+}\|}{(e^{2\delta_c}-1)\;\|m_c\|}\;\;e^{\delta_c\,n}+\operatorname{O}\big(e^{(\delta_c-\kappa')n}\big)\,.$$

Let us now consider the metric tree case, for the continuous time geodesic flow, where the main change is to assume a superpolynomial decay of correlations and hence get a superpolynomial error term. We refer to the beginning of Section 9.3 for the definitions of the function space $\mathscr{C}_{\rm c}^{k,\,\beta}(\Gamma\backslash\breve{\mathscr{G}}X)$ and the superpolynomial mixing.

Theorem 12.18. *Let $(\mathbb{X},\lambda)$, Γ, $\widetilde{c}$ and $\mathbb{D}^\pm$ be as in the beginning of this Section, and let $D^\pm = |\mathbb{D}^\pm|_\lambda$. Assume that the critical exponent δ_c is finite and positive, that the Gibbs measure m_c (for the continuous time geodesic flow) is finite and that the skinning measures $\sigma^\pm_{\mathscr{D}^\mp}$ are finite and nonzero. Assume furthermore that*

(1) *at least one of the following holds :*
- *$\Gamma_{D^\pm}\backslash\partial D^\pm$ is compact*
- *the metric subtree $\mathscr{C}\Lambda\Gamma$ is uniform and Γ is a lattice of $\mathscr{C}\Lambda\Gamma$,*

(2) *there exists $\beta\in\,]0,1]$ such that the continous time geodesic flow on $(\Gamma\backslash\mathscr{G}X, m_c)$ has superpolynomial decay of β-Hölder correlations.*

Then for every $n\in\mathbb{N}$ there exists $k\in\mathbb{N}$ such that for all $\psi^\pm\in\mathscr{C}_{\rm c}^{k,\,\beta}(\Gamma\backslash\breve{\mathscr{G}}X)$, we have, as $T\to+\infty$,

$$\delta_c\;\|m_c\|\;e^{-\delta_c\,T}\sum_{\substack{[\gamma]\in\Gamma_{D^-}\backslash\Gamma/\Gamma_{D^+}\\ 0<d(D^-,\gamma\,D^+)\leqslant T}} m_{e,\gamma}\;e^{\widetilde{c}\,(\alpha_{e,\gamma})}\;\psi^-(\Gamma\alpha^-_{e,\gamma})\;\psi^+(\Gamma\alpha^+_{\gamma^{-1},e})$$
$$=\int_{\Gamma\backslash\breve{\mathscr{G}}X}\psi^-\,d\sigma^+_{\mathscr{D}^-}\;\int_{\Gamma\backslash\breve{\mathscr{G}}X}\psi^+\,d\sigma^-_{\mathscr{D}^+}+\operatorname{O}\big(T^{-n}\,\|\psi^-\|_{k,\,\beta}\,\|\psi^+\|_{k,\,\beta}\big)$$

and if $\Gamma_{D^\pm}\backslash\partial D^\pm$ is compact, then for every $n\in\mathbb{N}$

$$\mathscr{N}_{D^-,\,D^+}(T)=\frac{\|\sigma^+_{\mathscr{D}^-}\|\;\|\sigma^-_{\mathscr{D}^+}\|}{\delta_c\;\|m_c\|}\;\;e^{\delta_c\,T}+\operatorname{O}\big(e^{\delta_c\,T}T^{-n}\big)\,.$$

Remark. Assume that $\widetilde{c}=0$, that the metric tree $\mathscr{C}\Lambda\Gamma$ is uniform, either that $\Gamma\backslash\mathbb{X}$ is finite and the length spectrum L_Γ of Γ is 2-Diophantine, or that Γ is a geometrically finite lattice of $\mathscr{C}\Lambda\Gamma$ and L_Γ is 4-Diophantine. Then all assumptions of Theorem 12.18 are satisfied by the results of Section 4.4 and by Corollary 9.10. Therefore we have a superpolynomially small error term in the (joint) equidistribution of the common perpendiculars (and in their counting if $\Gamma_{D^\pm}\backslash\partial D^\pm$ is compact).

Proof. The proof is similar to the one of Theorem 12.16, except that since the time is now continuous, we need to regularise our test functions also in the time direction in order to obtain the regularity required for the application of the assumption on the mixing rate. We again use the simplifying notation $\lambda_\gamma = \lambda_{e,\gamma}$, $\alpha_\gamma = \alpha_{e,\gamma}$, $\alpha^-_\gamma = \alpha^-_{e,\gamma}$, $\alpha^+_\gamma = \alpha^+_{\gamma^{-1},e}$ and $\widetilde{\sigma}^\pm = \widetilde{\sigma}^\pm_{D^\mp}$.

We fix $n \in \mathbb{N} - \{0\}$. Using the rapidly mixing property, there exists a regularity k such that for all $\psi, \psi' \in \mathscr{C}^{k,\beta}_{\rm b}(\Gamma\backslash\mathscr{G}X)$ we have as $t \to +\infty$

$$\operatorname{cov}_{\overline{m_c},\,t}(\psi,\psi') = \operatorname{O}(t^{-N\,n}\,\|\psi\|_{k,\beta}\,\|\psi'\|_{k,\beta})\,, \tag{12.18}$$

where $N \in \mathbb{N} - \{0\}$ is a constant which will be made precise later on.

Let us first prove that for all $\widetilde{\psi}^\pm \in \mathscr{C}^{k,\beta}_{\rm c}(\breve{\mathscr{G}}X)$, we have, as $T \to +\infty$,

$$\begin{aligned}\delta_c\,\|m_c\|\,e^{-\delta_c T}&\sum_{\gamma\in\Gamma,\,0<\lambda_\gamma\leqslant T} e^{\widetilde{c}(\alpha_\gamma)}\,\widetilde{\psi}^-(\alpha^-_\gamma)\,\widetilde{\psi}^+(\alpha^+_\gamma)\\ &= \int_{\partial^1_+D^-}\widetilde{\psi}^-\,d\widetilde{\sigma}^+\int_{\partial^1_-D^+}\widetilde{\psi}^+\,d\widetilde{\sigma}^- + \operatorname{O}(T^{-n}\,\|\widetilde{\psi}^-\|_{k,\beta}\,\|\widetilde{\psi}^+\|_{k,\beta})\,.\end{aligned} \tag{12.19}$$

In order to prove this formula, we introduce modified test functions with bounded Hölder-continuous derivatives up to order k (by a standard construction) in the time direction (the stable leaf and unstable leaf directions remain discrete). We fix $R > 0$ large enough.

For every $\eta \in \,]0,1[\,$, there exists a map $\widehat{\mathbb{1}_\eta} : \mathbb{R} \to [0,1]$ which has bounded β-Hölder-continuous derivatives up to order k, which is equal to 0 if $t \notin [-\eta,\eta]$ and to 1 if $t \in [-\eta\,e^{-\eta},\eta\,e^{-\eta}]$ (when $k = 0$, just take $\widehat{\mathbb{1}_\eta}$ to be continuous and affine on each remaining segment $[-\eta,-\eta\,e^{-\eta}]$ and $[\eta\,e^{-\eta},\eta]$), such that, for some constant $\kappa'_1 > 0$,

$$\|\widehat{\mathbb{1}_\eta}\|_{k,\beta} = \operatorname{O}(\eta^{-\kappa'_1})\,.$$

Using leafwise this regularisation process, there exists $\chi^\pm_{\eta,R} \in \mathscr{C}^{k,\beta}_{\rm b}(\mathscr{G}X)$ such that

- $\|\chi^\pm_{\eta,R}\|_{k,\beta} = \operatorname{O}(\eta^{-\kappa'_1})$,
- $\mathbb{1}_{\mathscr{V}^\mp_{\eta e^{-\eta},R}(\partial^1_\mp D^\pm)} \leqslant \chi^\pm_{\eta,R} \leqslant \mathbb{1}_{\mathscr{V}^\mp_{\eta,R}(\partial^1_\mp D^\pm)}$,
- for every $w \in \partial^1_\mp D^\pm$, we have

$$\int_{V^\mp_{w,\eta,R}} \chi^\pm_{\eta,R}\,d\nu^\pm_w = \nu^\pm_w(V^\mp_{w,\eta,R})\,e^{-\operatorname{O}(\eta)} = \nu^\pm_w(V^\mp_{w,\eta\,e^{-\eta},R})\,e^{\operatorname{O}(\eta)}\,.$$

As in the proof of Theorem 12.7 in the manifold case, the new test functions are defined, with

$$H^\pm_{\eta,R}\;:\;w \in \partial^1_\mp D^\pm \mapsto \frac{1}{\int_{V^\mp_{w,\eta,R}} \chi^\pm_{\eta,R}\,d\nu^\pm_w}\,,$$

by

$$\Phi^\pm_\eta = (H^\pm_{\eta,R}\,\widetilde{\psi}^\pm)\circ f^\mp_{D^\pm}\;\;\chi^\pm_{\eta,R}\quad:\mathscr{G}X \to \mathbb{R}\,.$$

Let $\widehat{\Phi}^\pm_\eta = H^\pm_{\eta,R}\circ f^\mp_{D^\pm}\;\chi^\pm_{\eta,R}$, so that $\Phi^\pm_\eta = \widetilde{\psi}^\pm\circ f^\mp_{D^\pm}\;\widehat{\Phi}^\pm_\eta$. By the last two properties of the regularised maps $\chi^\pm_{\eta,R}$, we have, with $\phi^\mp_\eta$ defined as in Equation (10.4),

$$\phi^\pm_{\eta\,e^{-\eta}}\,e^{-\operatorname{O}(\eta)} \leqslant \widehat{\Phi}^\pm_\eta \leqslant \phi^\pm_\eta\,e^{\operatorname{O}(\eta)}\,. \tag{12.20}$$

By Assumption (1), if R is large enough, by the definitions of the measures $\nu^{\pm}_w$, the denominator of $H^{\pm}_{\eta,\,R}(w)$ is at least $c\,\eta$ where $c>0$. As in the proof of Theorem 12.7,

$$\int_{\mathscr{G}X} \Phi^{\pm}_\eta \; d\widetilde{m}_c = \int_{\partial^1_{\mp} D^{\pm}} \widetilde{\psi}^{\pm}\; d\widetilde{\sigma}^{\mp}$$

and there exists $\kappa''>0$ such that

$$\|\Phi^{\pm}_\eta\|_{k,\,\beta} = \operatorname{O}(\eta^{-\kappa''}\|\widetilde{\psi}^{\pm}\|_{k,\,\beta})\,.$$

We again estimate in two ways as $T\rightarrow+\infty$ the quantity

$$I_\eta(T) = \int_0^T e^{\delta_c\,t}\sum_{\gamma\in\Gamma}\int_{\ell\in\mathscr{G}X}\Phi^-_\eta(\mathtt{g}^{-t/2}\ell)\;\Phi^+_\eta(\mathtt{g}^{t/2}\gamma^{-1}\ell)\;d\widetilde{m}_c(\ell)\,dt\,. \tag{12.21}$$

Note that as $T\rightarrow+\infty$,

$$\begin{aligned} e^{-\delta_c\,T}\int_1^T e^{\delta_c\,t}\,t^{-N\,n}\,dt &= e^{-\delta_c\,T}\int_1^{T/2} e^{\delta_c\,t}\,t^{-N\,n}\,dt + e^{-\delta_c\,T}\int_{T/2}^{T} e^{\delta_c\,t}\,t^{-N\,n}\,dt \\ &= \operatorname{O}(e^{-\delta_c\,T/2}) + \operatorname{O}(T^{-N\,n+1}) = \operatorname{O}(T^{-(N-1)\,n})\,. \end{aligned}$$

Using Equation (12.18), an integration argument and the above two properties of the test functions, we hence have

$$\begin{aligned} I_\eta(T) = \frac{e^{\delta_c\,T}}{\delta_c\,\|m_c\|}\Big(\int_{\partial^1_+ D^-}\widetilde{\psi}^-\,d\widetilde{\sigma}^+\int_{\partial^1_- D^+}\widetilde{\psi}^+\,d\widetilde{\sigma}^- + {} \\ \operatorname{O}(T^{-(N-1)\,n}\eta^{-2\kappa''}\|\widetilde{\psi}^-\|_{k,\,\beta}\|\widetilde{\psi}^+\|_{k,\,\beta})\Big)\,. \end{aligned} \tag{12.22}$$

As in Step 3T of the proof of Theorem 11.1, for all $\gamma\in\Gamma$ and $t>0$ large enough, if $\ell\in\mathscr{G}X$ belongs to the support of $\Phi^-_\eta\circ\mathtt{g}^{-t/2}\;\Phi^+_\eta\circ\mathtt{g}^{t/2}\circ\gamma^{-1}$ (which is contained in the support of $\phi^-_\eta\circ\mathtt{g}^{-t/2}\;\phi^+_\eta\circ\mathtt{g}^{t/2}\circ\gamma^{-1}$), then we may define $w^-=f^+_{D^-}(\ell)$ and $w^+=f^-_{\gamma D^+}(\ell)$.

By the property (iii) in Step 3T of the proof of Theorem 11.1, the generalised geodesic lines w^- and α^-_γ coincide, besides on $]-\infty,0]$, at least on $[0,\frac{t}{2}-\eta]$, and similarly, w^+ and $\gamma\alpha^+_\gamma$ coincide, besides on $[0,+\infty[$, at least on $[-\frac{t}{2}+\eta,0]$. Therefore, by an easy change of variable and since $|\frac{t}{2}-\frac{\lambda_\gamma}{2}|\leqslant\eta$,

$$\begin{aligned} d(w^-,\alpha^-_\gamma) &\leqslant \int_{\frac{t}{2}-\eta}^{+\infty} d(w^-(s),\alpha^-_\gamma(s))\,e^{-2s}\,ds \leqslant e^{-2(\frac{t}{2}-\eta)}\int_0^{+\infty} 2s\,e^{-2s}\,ds \\ &= \operatorname{O}(e^{-t}) = \operatorname{O}(e^{-\lambda_\gamma})\,. \end{aligned}$$

Similarly, $d(w^+,\gamma\alpha^+_\gamma)=\operatorname{O}(e^{-\lambda_\gamma})$. Hence since $\widetilde{\psi}^{\pm}$ is β-Hölder-continuous, we have

$$|\,\widetilde{\psi}^-(w^-)-\widetilde{\psi}^-(\alpha^-_\gamma)\,|,\;\;|\,\widetilde{\psi}^+(\gamma^{-1}w^+)-\widetilde{\psi}^+(\alpha^+_\gamma)\,| = \operatorname{O}(e^{-\beta\lambda_\gamma}\|\widetilde{\psi}^{\pm}\|_\beta)\,.$$

Therefore, as in the proof of Theorem 12.16, we have

$$\begin{aligned} I_\eta(T) = \sum_{\gamma\in\Gamma}\;\big(\widetilde{\psi}^-(\alpha^-_\gamma)\widetilde{\psi}^+(\alpha^+_\gamma) + \operatorname{O}(e^{-2\beta\lambda_\gamma}\|\widetilde{\psi}^-\|_\beta\|\widetilde{\psi}^+\|_\beta)\big)\times \\ \int_0^T e^{\delta\,t}\int_{\ell\in\mathscr{G}X}\widehat{\Phi}^-_\eta(\mathtt{g}^{-t/2}\ell)\;\widehat{\Phi}^+_\eta(\gamma^{-1}\mathtt{g}^{t/2}\ell)\;d\widetilde{m}_c(\ell)\;dt\,. \end{aligned}$$

Finally, Equation (12.19) follows as in the end of the proof of Equation (12.4), using Equations (12.20) and (11.17) instead of Equations (12.7) and (11.21), by taking $\eta = T^{-n}$ and $N = 2(\lceil \kappa'' \rceil + 1)$.

The end of the proof of the equidistribution claim of Theorem 12.18 follows from Equation (12.19) as the one of Theorem 12.7 from Equation (12.4).

The counting claim follows from the equidistribution one by taking $\psi^{\pm}$ to be β-Hölder-continuous plateau functions around $\Gamma \mathscr{V}_{\eta,R}(\partial^1_{\mp}\mathbb{D}^{\pm})$. $\square$

We are now in a position to prove one of the counting results in the introduction.

Proof of Theorem 1.9. Let $\mathbb{X}$ be the universal cover of $\mathbb{Y}$, with fundamental group Γ for an indifferent choice of basepoint, and let $\mathbb{D}^{\pm}$ be connected components of the preimages of $\mathbb{Y}^{\pm}$ in $\mathbb{X}$. Assertion (1) of Theorem 1.9 follows from Theorem 12.18 and its subsequent Remark. Assertion (2) of Theorem 1.9 follows from Theorem 12.16 and its Remarks (ii) and (i) following its proof, respectively, if $\mathbb{Y}$ is bipartite or not. $\square$

Chapter 13

Geometric applications

In this final Chapter of Part II, we apply the equidistribution and counting results obtained in the previous Chapters in order to study geometric equidistribution and counting problems for metric and simplicial trees concerning conjugacy classes in discrete isometry groups and closed orbits of the geodesic flows.

13.1 Orbit counting in conjugacy classes for groups acting on trees

In this Section, we study the orbital counting problem for groups acting on metric or simplicial trees when we consider only the images by elements in a given conjugacy class. We refer to the Introduction for motivations and previously known results for manifolds (see [Hub1] and [PaP15]) and graphs (see [Dou] and [KeS]). The main tools we use are Theorem 12.8 for the metric tree case and Theorem 12.16 for the simplicial tree case, as well as their error terms. In particular, we obtain a much more general version of Theorem 1.12 in the Introduction.

Let $(\mathbb{X}, \lambda)$ be a locally finite metric tree without terminal vertices, let $X = |\mathbb{X}|_\lambda$ be its geometric realisation, let $x_0 \in V\mathbb{X}$ and let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X}, \lambda)$.[1] Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a Γ-invariant system of conductances, let $\widetilde{F}_c$ and F_c be its associated potentials on T^1X and $\Gamma\backslash T^1X$ respectively, and let $\delta_c = \delta_{\Gamma,\, F_c^\pm}$ be its critical exponent.[2] Let $(\mu_x^\pm)_{x\in X}$ (respectively $(\mu_x^\pm)_{x\in V\mathbb{X}}$) be (normalised) Patterson densities for the pairs $(\Gamma, F_c^\pm)$, and let $\widetilde{m}_c = \widetilde{m}_{F_c}$ and $m_c = m_{F_c}$ be the associated Gibbs measures on $\mathscr{G}X$ and $\Gamma\backslash\mathscr{G}X$ (respectively $\mathscr{G}\mathbb{X}$ and $\Gamma\backslash\mathscr{G}\mathbb{X}$) for the continuous time geodesic flow (respectively the discrete time geodesic flow, when $\lambda \equiv 1$).[3]

Recall that the *virtual centre* $Z^{\rm virt}(\Gamma)$ of Γ is the finite (normal) subgroup of Γ consisting of the elements $\gamma \in \Gamma$ acting by the identity on the limit set $\Lambda\Gamma$ of Γ in $\partial_\infty X$, see for instance [Cha, §5.1]. If $\Lambda\Gamma = \partial_\infty X$ (for instance if Γ is a lattice), then $Z^{\rm virt}(\Gamma) = \{\mathrm{id}\}$.

For any nontrivial element γ in Γ with translation length $\lambda(\gamma)$ in X, let C_γ be

- the translation axis of γ if γ is loxodromic on X,
- the fixed point set of γ if γ is elliptic on X,

[1] See Section 2.6 for definitions and notation.
[2] See Section 3.5 for definitions and notation.
[3] See Sections 4.3 and 4.4 for definitions and notation.

and let Γ_{C_γ} be the stabiliser of C_γ in Γ. In the simplicial case (that is, when $\lambda \equiv 1$), C_γ is a simplicial subtree of $\mathbb{X}$. Note that $\lambda(\gamma) = \lambda(\gamma'\gamma(\gamma')^{-1})$ and $\gamma' C_\gamma = C_{\gamma'\gamma(\gamma')^{-1}}$ for all $\gamma' \in \Gamma$, and that for any $x_0 \in X$

$$d(x_0, C_\gamma) = \frac{d(x_0, \gamma x_0) - \lambda(\gamma)}{2}\,. \tag{13.1}$$

Let $\mathscr{D} = (\gamma' C_\gamma)_{\gamma' \in \Gamma/\Gamma_{C_\gamma}}$, which is a locally finite Γ-invariant family of nonempty proper (since $\gamma \neq \mathrm{id}$) closed convex subsets of X.[4] By the equivariance properties of the skinning measures, the total mass of the skinning measure[5] $\sigma^-_{\mathscr{D}}$ depends only on the conjugacy class $\mathfrak{K}$ of γ in Γ, and will be denoted by $\|\sigma^-_{\mathfrak{K}}\|$. This quantity, called the *skinning measure* of $\mathfrak{K}$, is positive unless $\partial_\infty C_\gamma = \Lambda\Gamma$, which is equivalent to $\gamma \in Z^{\rm virt}(\Gamma)$ (and implies in particular that γ is elliptic). Furthermore, $\|\sigma^-_{\mathfrak{K}}\|$ is finite if γ is loxodromic, and it is finite if γ is elliptic and $\Gamma_{C_\gamma}\backslash(C_\gamma \cap \mathscr{C}\Lambda\Gamma)$ is compact. This last condition is in particular satisfied if $C_\gamma \cap \mathscr{C}\Lambda\Gamma$ itself is compact, and this is the case for instance if, for some $k \geqslant 0$, the action of Γ on X is k-*acylindrical* (see for instance [Sel, GuL]), that is, if any element of Γ fixing a segment of length k in $\mathscr{C}\Lambda\Gamma$ is the identity.

For every $\gamma \in \Gamma - \{e\}$, we define

$$m_\gamma = \frac{1}{\mathrm{Card}(\Gamma_{x_0} \cap \Gamma_{C_\gamma})}\,,$$

which is a natural multiplicity of γ, and equals 1 if the stabiliser of x_0 in Γ is trivial (for instance if Γ is torsion-free). Note that for every $\beta \in \Gamma$, the real number $m_{\beta\gamma\beta^{-1}}$ depends only on the double coset of β in $\Gamma_{x_0}\backslash\Gamma/\Gamma_{C_\gamma}$.

The centraliser $Z_\Gamma(\gamma)$ of γ in Γ is contained in the stabiliser of C_γ in Γ. The index

$$i_{\mathfrak{K}} = [\Gamma_{C_\gamma} : Z_\Gamma(\gamma)]$$

depends only on the conjugacy class $\mathfrak{K}$ of γ; it will be called the *index* of $\mathfrak{K}$. The index $i_{\mathfrak{K}}$ is finite if γ is loxodromic (the stabiliser of its translation axis C_γ is then virtually cyclic), and also finite if C_γ is compact (as for instance if the action of Γ on X is k-acylindrical for some $k \geqslant 0$).

We define

$$c_\gamma = \sum_{i=1}^{k} \widetilde{c}\,(e_i)\,\lambda(e_i)\,,$$

where $(e_1, \dots, e_k)$ is the shortest edge path from x_0 to C_γ.

We finally define the orbital counting function in conjugacy classes, counting with multiplicities and weights coming from the system of conductances, as

$$N_{\mathfrak{K},\,x_0}(t) = \sum_{\alpha \in \mathfrak{K},\; d(x_0,\, \alpha x_0) \leqslant t} m_\alpha\, e^{c_\alpha}\,.$$

for $t \in [0, +\infty[$ (simply $t \in \mathbb{N}$ in the simplicial case). When the stabiliser of x_0 in Γ is trivial and when the system of conductances c vanishes, we recover the definition of the Introduction (above Theorem 1.12).

[4] See Section 7.2 for definitions and notations.

[5] See the previous footnote.

Theorem 13.1. *Let $\mathfrak{K}$ be the conjugacy class of a nontrivial element γ_0 of Γ, with finite index $i_{\mathfrak{K}}$, and with positive and finite skinning measure $\|\sigma^-_{\mathfrak{K}}\|$. Assume that δ_c is finite and positive.*

(1) Assume that m_c is finite and mixing for the continuous time geodesic flow on $\Gamma\backslash X$. Then, as $t \to +\infty$,

$$N_{\mathfrak{K},\,x_0}(t) \sim \frac{i_{\mathfrak{K}}\,\|\mu^+_{x_0}\|\,\|\sigma^-_{\mathfrak{K}}\|\,e^{-\frac{\lambda(\gamma_0)}{2}}}{\delta_c\,\|m_c\|}\;e^{\frac{\delta_c}{2}\,t}\,.$$

If $\Gamma_{C_\gamma}\backslash(C_\gamma \cap \mathscr{C}\Lambda\Gamma)$ is compact when $\gamma \in \mathfrak{K}$ is elliptic and if there exists $\beta \in\,]0,1]$ such that the continous time geodesic flow on $(\Gamma\backslash\mathscr{G}X, m_c)$ has superpolynomial decay of β-Hölder correlations, then the error term is $\operatorname{O}\big(t^{-n}\,e^{\frac{\delta_c}{2}\,t}\big)$ for every $n \in \mathbb{N}$.

(2) Assume that $\lambda \equiv 1$ and that m_c is finite and mixing for the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$. Then, as $n \to +\infty$,

$$N_{\mathfrak{K},\,x_0}(n) \sim \frac{e^{\delta_c}\,i_{\mathfrak{K}}\,\|\mu^+_{x_0}\|\,\|\sigma^-_{\mathfrak{K}}\|}{(e^{\delta_c}-1)\,\|m_c\|}\;e^{\delta_c\lfloor\frac{n-\lambda(\gamma_0)}{2}\rfloor}\,.$$

If $\Gamma_{C_\gamma}\backslash(C_\gamma \cap \mathscr{C}\Lambda\Gamma)$ is compact when $\gamma \in \mathfrak{K}$ is elliptic and if there exists $\beta \in\,]0,1]$ such that the discrete time geodesic flow on $(\Gamma\backslash\mathscr{G}\mathbb{X}, m_c)$ is exponentially mixing for the β-Hölder regularity, then the error term is $\operatorname{O}\big(e^{(\delta_c-\kappa)n/2}\big)$ for some $\kappa > 0$.

One can also formulate a version of the above result for groups acting on bipartite simplicial trees based on Theorem 12.12 and Remark (ii) following the proof of Theorem 12.16.

The error term in Assertion (1) holds for instance if $\widetilde{c} = 0$, X is uniform, and either $\Gamma\backslash X$ is compact and the length spectrum L_Γ is 2-Diophantine or Γ is a geometrically finite lattice of X whose length spectrum L_Γ is 4-Diophantine, by the Remark following Theorem 12.18. When $\Gamma\backslash X$ is compact and Γ has no torsion (in particular, Γ has then a very restricted group structure, as it is then a free group), we thus recover a result of [KeS].

The error term in Assertion (2) holds for instance if $\widetilde{c} = 0$, $\mathbb{X}$ is uniform with vertices of degrees at least 3, Γ is a geometrically finite lattice of $\mathbb{X}$ with length spectrum equal to $\mathbb{Z}$, by Remark (i) following the proof of Theorem 12.16.

Theorem 1.12 in the introduction follows from this theorem, using Proposition 4.16 (3) and Theorem 4.17.

Proof. We only give a full proof of Assertion (1) of this theorem, Assertion (2) follows similarly using Theorems 12.9 and 12.16 instead of Theorems 12.8 and 12.18.

The proof is similar to the proof of [PaP15, Theo. 8]. Let $D^- = \{x_0\}$ and $D^+ = C_{\gamma_0}$. Let $\mathscr{D}^- = (\gamma D^-)_{\gamma' \in \Gamma/\Gamma_{D^-}}$ and $\mathscr{D}^+ = (\gamma D^+)_{\gamma \in \Gamma/\Gamma_{D^+}}$. By Equation (7.15), we have

$$\|\sigma^+_{\mathscr{D}^-}\| = \frac{\|\mu^+_{x_0}\|}{|\Gamma_{x_0}|}\,.$$

By Equation (13.1), by the definition[6] of the counting function $\mathscr{N}_{D^-,\,D^+}$ and by the last claim

[6] See Equation (12.10) in Section 12.4.

of Theorem 12.8, we have, as $t \to +\infty$,

$$\begin{aligned}
\sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,\alpha x_0)\leqslant t} m_\alpha\ e^{c_\alpha} &= \sum_{\alpha\in\mathfrak{K},\ 0<d(x_0,\,C_\alpha)\leqslant \frac{t-\lambda(\gamma_0)}{2}} m_\alpha\ e^{c_\alpha}\\
&= \sum_{\gamma\in\Gamma/Z_\Gamma(\gamma_0),\ 0<d(x_0,\,\gamma C_{\gamma_0})\leqslant \frac{t-\lambda(\gamma_0)}{2}} m_{\gamma\gamma_0\gamma^{-1}}\ e^{c_{\gamma\gamma_0\gamma^{-1}}}\\
&= |\Gamma_{x_0}|\, i_{\mathfrak{K}} \sum_{\gamma\in\Gamma_{x_0}\backslash\Gamma/\Gamma_{C_{\gamma_0}},\ 0<d(x_0,\,\gamma C_{\gamma_0})\leqslant \frac{t-\lambda(\gamma_0)}{2}} m_{\gamma\gamma_0\gamma^{-1}}\ e^{c_{\gamma\gamma_0\gamma^{-1}}}\\
&= |\Gamma_{x_0}|\, i_{\mathfrak{K}}\, \mathscr{N}_{D^-,\,D^+}\big(\frac{t-\lambda(\gamma_0)}{2}\big)\\
&\sim |\Gamma_{x_0}|\, i_{\mathfrak{K}}\, \frac{\|\sigma^+_{\mathscr{D}^-}\|\ \|\sigma^-_{\mathscr{D}^+}\|}{\delta_c\ \|m_c\|}\ e^{\delta_c\ \frac{t-\lambda(\gamma_0)}{2}}\ .
\end{aligned}$$

Assertion (1) without the error term follows, and the error term statement follows similarly from Theorem 12.18. □

Theorem 13.1 (1) without an explicit form of the multiplicative constant in the asymptotic is due to [KeS] under the strong restriction that Γ is a free group acting freely on X and $\Gamma\backslash X$ is a finite graph. The following result is due to [Dou, Theo. 1] in the special case when $\mathbb{X}$ is a regular tree and the group Γ has no torsion and finite quotient $\Gamma\backslash\mathbb{X}$.

Corollary 13.2. *Let $\mathbb{X}$ be a regular simplicial tree with vertices of degree $q+1\geqslant 3$, let $x_0\in V\mathbb{X}$, let Γ be a lattice of $\mathbb{X}$ such that $\Gamma\backslash\mathbb{X}$ is nonbipartite, and let $\mathfrak{K}$ be the conjugacy class of a loxodromic element $\gamma_0\in\Gamma$. Then, as $n\to+\infty$,*

$$\sum_{\alpha\in\mathfrak{K},\ d(x_0,\,\alpha x_0)\leqslant n} m_\alpha \sim \frac{\lambda(\gamma_0)}{[Z_\Gamma(\gamma_0):\gamma_0^{\mathbb{Z}}]\ \operatorname{Vol}(\Gamma\backslash\backslash\mathbb{X})}\ q^{\lfloor\frac{n-\lambda(\gamma_0)}{2}\rfloor}\,.$$

If we assume furthermore that Γ has no torsion and that γ_0 is primitive, then we have as $n\to+\infty$,

$$\operatorname{Card}\{\alpha\in\mathfrak{K} : d(x_0,\,\alpha x_0)\leqslant n\} \sim \frac{\lambda(\gamma_0)}{|\Gamma\backslash V\mathbb{X}|}\ q^{\lfloor\frac{n-\lambda(\gamma_0)}{2}\rfloor}\,.$$

Proof. Under these assumptions, taking $c\equiv 0$ in Theorem 13.1 so that the Gibbs measure is the Bowen-Margulis measure, the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ is finite and mixing by Proposition 4.16 (3) and Theorem 4.17. We also have $\delta_c=\ln q$. Using the normalisation of the Patterson density $(\mu^\pm_x)_{x\in V\mathbb{X}}$ to probability measures, Proposition 8.1 (3) and Equation (8.12), the result follows, since when γ is loxodromic,

$$\operatorname{Vol}(\Gamma_{C_\gamma}\backslash\backslash C_\gamma)=\frac{\operatorname{Vol}(\gamma^{\mathbb{Z}}\backslash\backslash C_\gamma)}{[\Gamma_{C_\gamma}:\gamma^{\mathbb{Z}}]}=\frac{\lambda(\gamma)}{[\Gamma_{C_\gamma}:Z_\Gamma(\gamma)]\ [Z_\Gamma(\gamma):\gamma^{\mathbb{Z}}]}\,.\ \ \square$$

The value of C' given below Theorem 1.12 in the Introduction follows from this corollary.

We leave to the reader an extension with nonzero potential F of the results for manifolds in [PaP15], along the lines of the above proofs.

13.2 Equidistribution and counting of closed orbits on metric and simplicial graphs (of groups)

Classically, an important characterisation of the Bowen-Margulis measure on compact negatively curved Riemannian manifolds is that it coincides with the weak-star limit of properly normalised sums of Lebesgue measures supported on periodic orbits, see [Bowe1]. Under much weaker assumptions than compactness, this result was extended to CAT(−1) spaces with zero potential in [Rob2] and to Gibbs measures in the manifold case in [PauPS, Theo. 9.11]. As a corollary of the simultaneous equidistribution results from Chapter 11, we prove in this Section the equidistribution towards the Gibbs measure of weighted closed orbits in quotients of metric and simplicial graphs of groups and as a corollary of this result, in the standard manner, we obtain asymptotic counting results for weighted (primitive) closed orbits.

Let $(\mathbb{X}, \lambda)$ be a locally finite metric tree without terminal vertices, and let $X = |\mathbb{X}|_\lambda$ be its geometric realisation. Let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X}, \lambda)$. Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a Γ-invariant system of conductances, and $c : \Gamma\backslash E\mathbb{X} \to \mathbb{R}$ its induced function.

Given a periodic orbit g of the geodesic flow on $\Gamma\backslash\mathscr{G}X$, if $(e_1, \dots, e_k)$ is the sequence of edges followed by g, we denote by $\mathscr{L}_g$ the Lebesgue measure along g, by $\lambda(g)$ the length of g and by $c(g)$ its *period* for the system of conductances c:

$$\lambda(g) = \sum_{i=1}^{k} \lambda(e_i) \quad \text{and} \quad c(g) = \sum_{i=1}^{k} \lambda(e_i)\, c(e_i) \;.$$

Let $\mathbf{Per}(t)$ be the set of periodic orbits of the continuous time geodesic flow on $\Gamma\backslash\mathscr{G}X$ with length at most t and let $\mathbf{Per'}(t)$ be the subset of primitive ones.

Theorem 13.3. *Assume that the critical exponent δ_c of c is finite and positive and that the Gibbs measure m_c of c is finite and mixing for the continuous time geodesic flow. As $t \to +\infty$, the measures*

$$\delta_c\, e^{-\delta_c\, t} \sum_{g \in \mathbf{Per'}(t)} e^{c(g)}\, \mathscr{L}_g$$

and

$$\delta_c\, t\, e^{-\delta_c\, t} \sum_{g \in \mathbf{Per'}(t)} e^{c(g)}\, \frac{\mathscr{L}_g}{\lambda(g)}$$

converge to $\frac{m_c}{\|m_c\|}$ for the weak-star convergence of measures. If Γ is geometrically finite, the convergence holds for the narrow convergence.

We conjecture that if Γ is geometrically finite and if its length spectrum is 4-Diophantine[7], then for all $n \in \mathbb{N}$ and $\beta \in \,]0, 1]$, there exist $k \in \mathbb{N}$ and an error term of the form $\operatorname{O}(t^n\, \|\psi\|_{k,\,\beta})$ for these equidistribution claims evaluated on any $\psi \in \mathscr{C}_c^{k,\,\beta}(\Gamma\backslash\mathscr{G}X)$. But since we will not need this result and since the proof is likely to be very long, we do not address the problem here.

Proof. Let $\widetilde{F}_c$ and F_c be the potentials on T^1X and $\Gamma\backslash T^1X$ respectively associated[8] with c, and note that the period of a periodic orbit g for the geodesic flow on $\Gamma\backslash\mathscr{G}X$ satisfies[9]

$$c(g) = \mathscr{L}_g(F_c^\sharp) = \operatorname{Per}_{F_c}(\gamma) \;,$$

[7] See the definition in Section 9.3.

[8] See Section 3.5.

[9] See Proposition 3.21 and Section 3.2.

where $F_c^\sharp$ is the composition of the canonical map $\Gamma\backslash\mathscr{G}X \to \Gamma\backslash T^1X$ with $F_c : \Gamma\backslash T^1X \to \mathbb{R}$, and $\gamma \in \Gamma$ is the loxodromic element of Γ whose conjugacy class corresponds to g.

Let $\mathscr{H}_{\Gamma,t}$ be the subset of Γ that consists in the loxodromic elements whose translation length is at most t, and let $\mathscr{H}'_{\Gamma,t}$ be the subset of $\mathscr{H}_{\Gamma,t}$ that consists in the primitive ones. For every $\gamma \in \mathscr{H}_{\Gamma,t}$, we denote by g_γ its corresponding periodic orbit in $\Gamma\backslash\mathscr{G}X$. The first claim is equivalent to the following assertion: we have

$$\delta_c\, e^{-\delta_c\, t} \sum_{\gamma\in\mathscr{H}'_{\Gamma,t}} e^{\mathrm{Per}_{F_c}(\gamma)}\, \mathscr{L}_{g_\gamma} \overset{*}{\rightharpoonup} \frac{m_c}{\|m_c\|} \tag{13.2}$$

as $t \to +\infty$. We proceed with the proof of the convergence claimed in Equation (13.2) as in [PauPS, Theo. 9.11]. We first prove that

$$\nu''_t = \delta_c\, \|m_c\|\, e^{-\delta_c\, t} \sum_{\gamma\in\mathscr{H}_{\Gamma,t}} e^{\mathrm{Per}_{F_c}(\gamma)}\, \mathscr{L}_{g_\gamma} \overset{*}{\rightharpoonup} m_c\,. \tag{13.3}$$

We then refer to Step 2 of the proof of [PauPS, Theo. 9.11] for the fact that the contribution of the non primitive elements is negligible, so that Equation (13.3) implies Equation (13.2). Although the proof of this deduction in loc. cit. is written for manifolds, the arguments are directly applicable for any CAT(-1) space X and potential F satisfying the HC-property.[10] In particular, the use of Proposition 5.13 (i) and (ii) of op. cit. in the proof of Step 2 in loc. cit. is replaced now by the use of Theorem 4.6 (1) and (4) respectively.

Let us fix $x \in X$. Let

$$V(x) = \big\{(\xi,\eta) \in (X \cup \partial_\infty X)^2 : \xi \neq \eta,\ x \in \,]\xi,\eta[\,\big\}\,,$$

which is an open subset of $X \cup \partial_\infty X$. Note that the family $(V(y))_{y\in X}$ covers the set of pairs of distinct points of $\partial_\infty X$. For every $t > 0$, let ν_t be the measure on $(X \cup \partial_\infty X)^2$ defined by

$$\nu_t = \delta_c\, \|m_c\|\, e^{-\delta_c t} \sum_{\gamma\in\Gamma\,:\, d(x,\gamma x)\leqslant t} e^{\int_x^{\gamma x} \widetilde{F}_c}\, \Delta_{\gamma^{-1}x} \otimes \Delta_{\gamma x}\,.$$

The measures ν_t weak-star converge to $\mu^-_x \otimes \mu^+_x$ as $t \to +\infty$ by Corollary 11.2 (taking in its statement $y = x$).

Let $\gamma_\pm$ be the attracting and repelling fixed points of any loxodromic element $\gamma \in \Gamma$. Let

$$\nu'''_t = \delta_c\, \|m_c\|\, e^{-\delta_c\, t} \sum_{\gamma\in\mathscr{H}_{\Gamma,\,t}} e^{\mathrm{Per}_{F_c}(\gamma)} \Delta_{\gamma_-} \otimes \Delta_{\gamma_+}\,.$$

Since $\mathbb{X}$ is an $\mathbb{R}$-tree, every element $\gamma \in \Gamma$ such that $x \in \,]\gamma^{-1}x, \gamma x[$ is loxodromic, and for such x and γ we have

$$d(x,\gamma x) = \lambda(\gamma) \quad\text{and}\quad \int_x^{\gamma x} \widetilde{F}_c = \mathrm{Per}_{F_c}(\gamma)\,.$$

If furthermore $d(x,\gamma x)$ is large, then $\gamma^{-1}x$ and γx are respectively close to γ_- and γ_+ in $X \cup \partial_\infty X$.

Hence, for every continuous map $\psi : (X \cup \partial_\infty X)^2 \to [0,+\infty[$ with compact support contained in $V(x)$, and for every $\epsilon > 0$, if t is large enough, we have

$$e^{-\epsilon}\nu_t(\psi) \leqslant \nu'''_t(\psi) \leqslant e^{\epsilon}\nu_t(\psi)\,.$$

[10] See Definition 3.13.

Using Hopf's parametrisation with basepoint x, and by Equation (3.20), the measures $\widetilde{m}_c$ and $\mu_x^- \otimes \mu_x^+ \otimes ds$ are equal on $V(x) \times \mathbb{R}$. Hence the measures $\nu_t \otimes ds$ weak-star converge to $\widetilde{m}_c$ on $V(x) \times \mathbb{R}$ as $t \to +\infty$. For every continuous $\psi' : \mathscr{G}X \to \mathbb{R}$ which is a product of continuous functions in each variable with compact support in $V(x) \times \mathbb{R}$ and for every $\epsilon > 0$, if t is large enough, we hence have

$$e^{-\epsilon}\widetilde{m}_c(\psi') \leqslant \nu_t''' \otimes ds(\psi') \leqslant e^{\epsilon}\widetilde{m}_c(\psi') \,.$$

Note that the support of any continuous function with compact support on $\mathscr{G}X$ may be covered by finitely many open sets $V(x) \times \mathbb{R}$ where $x \in X$. The induced measure[11] of the Γ-invariant measure $\nu_t''' \otimes ds$ on $\mathscr{G}X$ is the measure ν_t'' on $\Gamma\backslash\mathscr{G}X$. Since m_c is the induced mesasure of $\widetilde{m}_c$, this proves Equation (13.3), hence gives the first claim of Theorem 13.3.

The second claim follows from the first one in the same way as in [PauPS, Theo. 9.11]: Consider the measures

$$m'_t = \delta_c e^{-\delta_c t} \sum_{g \in \mathbf{Per}'(t)} e^{\mathscr{L}_g(F)}\, \mathscr{L}_g \quad \text{and} \quad m''_t = \delta_c t e^{-\delta_c t} \sum_{g \in \mathbf{Per}'(t)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g}{\lambda(g)}$$

on $\Gamma\backslash\mathscr{G}X$. Fix a continuous map $\psi : \Gamma\backslash\mathscr{G}X \to [0, +\infty[$ with compact support. For every $\epsilon > 0$, for every $t > 0$, we have, since $\lambda(g) \geqslant e^{-\epsilon}t$ for all g in the second sum below,

$$\begin{aligned}
m''_t(\psi) \geqslant m'_t(\psi) &\geqslant \delta_c e^{-\delta_c t} \sum_{g \in \mathbf{Per}'(t) - \mathbf{Per}'(e^{-\epsilon}t)} e^{\mathscr{L}_g(F)}\, \mathscr{L}_g(\psi) \\
&\geqslant e^{-\epsilon}\delta_c t e^{-\delta_c t} \sum_{g \in \mathbf{Per}'(t) - \mathbf{Per}'(e^{-\epsilon}t)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g(\psi)}{\lambda(g)} \\
&= e^{-\epsilon} m''_t(\psi) - e^{-\epsilon}\delta_c t\, e^{-\delta_c t} \sum_{g \in \mathbf{Per}'(e^{-\epsilon}t)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g(\psi)}{\lambda(g)} \,.
\end{aligned}$$

By the local finiteness of X, the closed orbits meeting the support of ψ have a positive lower bound on their lengths. Thus, by the first claim of Theorem 13.3, there exists a constant $C > 0$ such that the second term of the above difference is at most $C\,t\,e^{-\delta_c t} e^{\delta_c\, e^{-\epsilon} t}$, which tends to 0 as t tends to $+\infty$. Hence by applying twice the first claim of Theorem 13.3, we have

$$\frac{m_c(\psi)}{\|m_c\|} = \lim_{t \to +\infty} m'_t(\psi) \leqslant \liminf_{t \to +\infty} m''_t(\psi) \leqslant \limsup_{t \to +\infty} m''_t(\psi) \leqslant \lim_{t \to +\infty} e^{\epsilon} m'_t(\psi) = e^{\epsilon}\frac{m_c(\psi)}{\|m_c\|} \,,$$

and the result follows by letting ϵ go to 0. (and writing any continuous map $\psi : T^1M \to \mathbb{R}$ with compact support into the sum of its positive and negative parts).

In order to prove the last claim of Theorem 13.3, assume that Γ is geometrically finite. The narrow convergence follows as in [PauPS, Theo. 9.16], using the fact that there exists a compact subset of $\Gamma\backslash\mathscr{G}X$ meeting every periodic orbit of the geodesic flow, and replacing Lemma 3.10, Lemma 3.2, Equation (112), Theorem 8.3 and Corollary 5.15 of op. cit. by respectively Lemma 4.3 (1), the (HC)-property, Proposition 3.20 (4), Theorem 4.8 and Corollary 4.7 (1). □

[11] See the beginning of Chapter 12.

In a similar way, replacing in the above proof Corollary 11.2 of Theorem 11.1 by the similar corollary of Theorem 11.9 with $\mathscr{D}^- = (\gamma x)_{\gamma\in\Gamma}$ and $\mathscr{D}^- = (\gamma y)_{\gamma\in\Gamma}$ for any $x, y \in V\mathbb{X}$, we get the following analogous result for simplicial trees. For every $n \in \mathbb{N}$, let now $\mathbf{Per}(n)$ be the set of periodic orbits of the discrete time geodesic flow on $\Gamma\backslash\mathscr{G}\mathbb{X}$ with length at most n and let $\mathbf{Per}'(n)$ be the subset of primitive ones.

Theorem 13.4. *Let $\mathbb{X}$ be a locally finite simplicial tree without terminal vertices, let Γ be a nonelementary discrete subgroup of $\operatorname{Aut}(\mathbb{X})$ and let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a Γ-invariant system of conductances. Assume that the critical exponent δ_c of c is finite and positive and that the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow. As $n \to +\infty$, the measures*

$$\frac{e^{\delta_c} - 1}{e^{\delta_c}}\, e^{-\delta_c n} \sum_{g \in \mathbf{Per}'(n)} e^{c(g)}\, \mathscr{L}_g$$

and

$$\frac{e^{\delta_c} - 1}{e^{\delta_c}}\, n\, e^{-\delta_c n} \sum_{g \in \mathbf{Per}'(n)} e^{c(g)}\, \frac{\mathscr{L}_g}{\lambda(g)}$$

converge to $\frac{m_c}{\|m_c\|}$ for the weak-star convergence of measures. If Γ is geometrically finite, the convergence holds for the narrow convergence. □

In the special case when $\Gamma\backslash X$ is a compact graph and $F = 0$, the following immediate corollary of Theorem 13.3 is proved in [Gui], and it follows from the results of [ParP].[12] There are also some works on non-backtracking random walks with related results. For example, for regular finite graphs, [LuPS1] and [Fri1] (see [Fri2, Lem. 2.3]) give an expression of the irreducible trace which is the number of closed walks of a given length.

Corollary 13.5. *Let $(\mathbb{X}, \lambda)$ be a locally finite metric tree without terminal vertices. Let Γ be a geometrically finite discrete subgroup of $\operatorname{Aut}(\mathbb{X}, \lambda)$. Let $c : E\mathbb{X} \to \mathbb{R}$ be a Γ-invariant system of conductances, with finite and positive critical exponent δ_c.*

(1) If the Gibbs measure m_c is finite and mixing for the continuous time geodesic flow, then

$$\sum_{g \in \mathbf{Per}'(t)} e^{c(g)} \sim \frac{e^{\delta_c t}}{\delta_c\, t}$$

as $t \to +\infty$.

(2) If $\lambda = 1$ and if the Gibbs measure m_c is finite and mixing for the discrete time geodesic flow, then

$$\sum_{g \in \mathbf{Per}'(n)} e^{c(g)} \sim \frac{e^{\delta_c}}{e^{\delta_c} - 1}\, \frac{e^{\delta_c n}}{n}$$

as $n \to +\infty$. □

[12] See the introduction of [Sha] for comments.

Part III

Arithmetic applications

Chapter 14

Fields with discrete valuations

Let $\widehat{K}$ be a non-Archimedean local field. Basic examples of such fields are the field of formal Laurent series over a finite field, and the field of p-adic numbers (see Examples 14.1 and 14.2). In Part III of this book, we apply the geometric equidistribution and counting results for simplicial trees given in Part II, in order to prove arithmetic equidistribution and counting results in such fields $\widehat{K}$. The link between the geometry and the algebra is provided by the Bruhat-Tits tree of $(\mathrm{PGL}_2, \widehat{K})$, the construction of which is recalled in Section 15.1. We will only use the system of conductances equal to 0 in this Part III.

In the present Chapter, before embarking on our arithmetic applications, we recall basic facts on local fields for the convenience of the geometer reader. For more details, we refer for instance to [Ser2, Gos]. We refer to [BrPP] for an announcement of the results of Part III, with a presentation different from the one in the Introduction.

We will only give results for the algebraic group $\underline{G} = \mathrm{PGL}_2$ over $\widehat{K}$ and special discrete subgroups Γ of $\mathrm{PGL}_2(\widehat{K})$, even though the same methods give equidistribution and counting results when $\underline{G}$ is any semisimple connected linear algebraic group over $\widehat{K}$ of $\widehat{K}$-rank 1 and Γ any lattice in $G = \underline{G}(\widehat{K})$.

14.1 Local fields and valuations

Let F be a field and let $F^\times = (F - \{0\}, \times)$ be its multiplicative group. A surjective group morphism $v : F^\times \to \mathbb{Z}$ to the additive group $\mathbb{Z}$, that satisfies

$$v(a+b) \geqslant \min\{v(a), v(b)\}$$

for all $a, b \in F^\times$, is a (normalised discrete) *valuation* v on F. We make the usual convention and extend the definition of v to F by setting $v(0) = +\infty$. Note that $v(a+b) = \min\{v(a), v(b)\}$ if $v(a) \neq v(b)$. When F is an extension of a finite field k, the valuation v vanishes on $k^\times$.

The subring

$$\mathscr{O}_v = \{x \in F \;:\; v(x) \geqslant 0\}$$

is the *valuation ring* (or local ring) of v (or of F is v is implicit).

The maximal ideal

$$\mathfrak{m}_v = \{x \in F \;:\; v(x) > 0\}$$

of $\mathscr{O}_v$ is principal and it is generated as an ideal of $\mathscr{O}_v$ by any element $\pi_v \in F$ with

$$v(\pi_v) = 1$$

which is called a *uniformiser* of F.

The *residual field* of the valuation v is

$$k_v = \mathscr{O}_v/\mathfrak{m}_v \ .$$

When k_v is finite, the valuation v defines a (normalised, non-Archimedean) *absolute value* $|\cdot|_v$ on F by

$$|x|_v = |k_v|^{-v(x)} \ ,$$

with the convention that $|k_v|^{-\infty} = 0$. This absolute value induces an ultrametric distance on F by

$$(x, y) \mapsto |x - y|_v \ .$$

Let F_v be the completion of F with respect to this distance. The valuation v of F uniquely extends to a (normalised discrete) valuation on F_v, again denoted by v.

Example 14.1. Let $K = \mathbb{F}_q(Y)$ be the field of rational functions in one variable Y with coefficients in a finite field $\mathbb{F}_q$ of order a positive power q of a positive prime p in $\mathbb{Z}$, let $\mathbb{F}_q[Y]$ be the ring of polynomials in one variable Y with coefficients in $\mathbb{F}_q$, and let $v_\infty : K^\times \to \mathbb{Z}$ be the *valuation at infinity* of K, defined on every $P/Q \in K$ with $P \in \mathbb{F}_q[Y]$ and $Q \in \mathbb{F}_q[Y] - \{0\}$ by

$$v_\infty(P/Q) = \deg Q - \deg P \ .$$

The absolute value associated with v_∞ is

$$|P/Q|_\infty = q^{\deg P - \deg Q} \ .$$

The completion of K for v_∞ is the field $K_{v_\infty} = \mathbb{F}_q((Y^{-1}))$ of formal Laurent series in one variable Y^{-1} with coefficients in $\mathbb{F}_q$. The elements x in $\mathbb{F}_q((Y^{-1}))$ are of the form

$$x = \sum_{i \in \mathbb{Z}} x_i \, Y^{-i}$$

where $x_i \in \mathbb{F}_q$ for every $i \in \mathbb{Z}$, and $x_i = 0$ for i small enough. The valuation at infinity of $\mathbb{F}_q((Y^{-1}))$ extending the valuation at infinity of $\mathbb{F}_q(Y)$ is

$$v_\infty(x) = \sup\{i \in \mathbb{Z} \ : \ \forall \ j < i, \quad x_j = 0\} \, ,$$

that is,

$$v_\infty\big(\sum_{i=i_0}^{\infty} x_i Y^{-i}\big) = i_0$$

if $x_{i_0} \neq 0$. The valuation ring of v_∞ is the ring $\mathscr{O}_{v_\infty} = \mathbb{F}_q[[Y^{-1}]]$ of formal power series in one variable Y^{-1} with coefficients in $\mathbb{F}_q$. The element

$$\pi_{v_\infty} = Y^{-1}$$

is a uniformiser of v_∞, the residual field $\mathscr{O}_{v_\infty}/\pi_{v_\infty}\mathscr{O}_{v_\infty}$ of v_∞ is $k_{v_\infty} = \mathbb{F}_q$.

Example 14.2. Given a positive prime $p \in \mathbb{Z}$, the field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$ of the *p-adic valuation* v_p defined by setting

$$v_p\big(p^n \frac{a}{b}\big) = n\,,$$

when $n \in \mathbb{Z}$, $a, b \in \mathbb{Z} - \{0\}$ are not divisible by p. Then the valuation ring $\mathscr{O}_{v_p}$, denoted by $\mathbb{Z}_p$, of $\mathbb{Q}_p$ is the closure of $\mathbb{Z}$ for the absolute value $|\cdot|_p$, the element $\pi_{v_p} = p$ is a uniformiser, and the residual field is $k_{v_p} = \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$, a finite field of order p.

A field endowed with a valuation is a *non-Archimedean local field* if it is complete with respect to its absolute value and if its residual field is finite.[1] Its valuation ring is then a compact open additive subgroup. Any non-Archimedean local field is isomorphic to a finite extension of the p-adic field $\mathbb{Q}_p$ for some prime p, or to the field $\mathbb{F}_q((Y^{-1}))$ of formal Laurent series in one variable Y^{-1} over $\mathbb{F}_q$ for some positive power q of a prime p.

These formal Laurent series fields may occur as completions of numerous (global) fonctions fields, that we now define. The basic case is described in Example 14.1 above, and the general case is detailed in Section 14.2 below. The geometer reader may skip Section 14.2 and use only Example 14.1 in the remainder of Part III (using $g = 0$ when the constant g occurs).

14.2 Global function fields

In this Section, we fix a finite field $\mathbb{F}_q$ with q elements, where q is a positive power of a positive prime $p \in \mathbb{Z}$, and we recall the definitions and basic properties of a function field K over $\mathbb{F}_q$, its genus g, its valuations v, its completion K_v for the associated absolute value $|\cdot|_v$ and the associated affine function ring R_v. See for instance [Gos, Ros] for the content of this Section.

Let K be a (global) *function field* over $\mathbb{F}_q$, which can be defined in two equivalent ways as

(1) the field of rational functions on a geometrically irreducible smooth projective curve $\mathbf{C}$ over $\mathbb{F}_q$, or

(2) an extension of $\mathbb{F}_q$ of transcendence degree 1, in which $\mathbb{F}_q$ is algebraically closed.

There is a bijection between the set of closed points of $\mathbf{C}$ and the set of (normalised discrete) valuations of its function field K, the valuation of a given element $f \in K$ being the order of the zero or the opposite of the order of the pole of f at the given closed point. We fix such an element v from now on. We denote by g the genus of the curve $\mathbf{C}$.

In the basic Example 14.1, $\mathbf{C}$ is the projective line $\mathbb{P}^1$ over $\mathbb{F}_q$, which is a curve of genus $g = 0$, and the closed point associated with the valuation at infinity v_∞ is the point at infinity $[1:0]$.

We denote by K_v the completion of K for v, and by

$$\mathscr{O}_v = \{x \in K_v \;:\; v(x) \geqslant 0\}$$

the valuation ring of (the unique extension to K_v) of v. We choose a uniformiser $\pi_v \in K$ of v. We denote by $k_v = \mathscr{O}_v/\pi_v\mathscr{O}_v$ the residual field of v, which is a finite field of order

$$q_v = |k_v|\;.$$

[1]There are also two Archimedean local fields $\mathbb{C}$ and $\mathbb{R}$, see for example [Cas].

The field k_v is from now on identified with a fixed lift in $\mathscr{O}_v$ (see for instance [Col, Théo. 1.3]), and is an extension of the field of constants $\mathbb{F}_q$. The degree of this extension is denoted by $\deg v$, so that

$$q_v = q^{\deg v} \ .$$

We denote by $|\cdot|_v$ the (normalised) absolute value associated with v: for every $x \in K_v$, we have

$$|x|_v = (q_v)^{-v(x)} = q^{-v(x)\deg v} \ .$$

Every element $x \in K_v$ is[2] a (converging) Laurent series $x = \sum_{i\in\mathbb{Z}} x_i\,(\pi_v)^i$ in the variable π_v over k_v, where $x_i \in k_v$ is zero for $i \in \mathbb{Z}$ small enough. We then have

$$|x|_v = (q_v)^{-\sup\{j\in\mathbb{Z}\,:\,\forall\, i<j,\ x_i=0\}} \ , \tag{14.1}$$

and $\mathscr{O}_v$ consists of the (converging) power series $x = \sum_{i\in\mathbb{N}} x_i\,(\pi_v)^i$ (where $x_i \in k_v$) in the variable π_v over k_v.

We denote by R_v the affine algebra of the affine curve $\mathbf{C} - \{v\}$, consisting of the elements of K whose only poles are at the closed point v of $\mathbf{C}$. Its field of fractions is equal to K, hence we will often write elements of K as x/y with $x, y \in R_v$ and $y \neq 0$. In the basic Example 14.1, we have $R_{v_\infty} = \mathbb{F}_q[Y]$. Note that

$$R_v \cap \mathscr{O}_v = \mathbb{F}_q \ , \tag{14.2}$$

since the only rational functions on $\mathbf{C}$ whose only poles are at v and whose valuation at v is nonnegative are the constant ones. We have (see for instance [Ser3, II.2 Notation], [Gos, page 63])

$$(R_v)^\times = (\mathbb{F}_q)^\times \ . \tag{14.3}$$

The following result is immediate when $\mathbf{C} = \mathbb{P}^1$, since then $R_v + \mathscr{O}_v = K_v$.

Lemma 14.3. *The dimension of the quotient vector space $K_v/(R_v + \mathscr{O}_v)$ over $\mathbb{F}_q$ is equal to the genus g of $\mathbf{C}$.*

Proof. (indicated by J.-B. Bost) We refer for instance to [Ser1] for background on sheaf cohomology. We denote in the same way the valuation v and the corresponding closed point on $\mathbf{C}$.

Let $\mathscr{O} = K \cap \mathscr{O}_v$ be the discrete valuation ring of v restricted to K. Since K is dense in K_v and $\mathscr{O}_v$ is open and contains 0, we have $K_v = K + \mathscr{O}_v$. Therefore the canonical map

$$K/(R_v + \mathscr{O}) \ \to K_v/(R_v + \mathscr{O}_v)$$

is a linear isomorphism over $\mathbb{F}_q$. Let us hence prove that $\dim_{\mathbb{F}_q} K/(R_v + \mathscr{O}) = g$.

In what follows, $\mathscr{V}$ ranges over the affine Zariski-open neighbourhoods of v in $\mathbf{C}$, ordered by inclusion. Let $\mathscr{O}_{\mathbf{C}}$ be the structural sheaf of $\mathbf{C}$. Note that by the definition of R_v, since the zeros of elements of $K^\times$ are isolated and by the relation between valuations of K and closed points of $\mathbf{C}$,

$$R_v = H^0(\mathbf{C} - \{v\}, \mathscr{O}_{\mathbf{C}}), \quad K = \varinjlim_{\mathscr{V}} H^0(\mathscr{V} - \{v\}, \mathscr{O}_{\mathbf{C}}) \quad \text{and} \quad \mathscr{O} = \varinjlim_{\mathscr{V}} H^0(\mathscr{V}, \mathscr{O}_{\mathbf{C}}) \ .$$

[2] See for instance [Col, Coro. 1.6].

Since $\mathscr{V}$ and $\mathbf{C}-\{v\}$ are affine curves, we have $H^1(\mathbf{C}-\{v\},\mathscr{O}_{\mathbf{C}}) = H^1(\mathscr{V},\mathscr{O}_{\mathbf{C}}) = 0$. By the Mayer-Vietoris exact sequence for the covering $\{\mathbf{C}-\{v\},\mathscr{V}\}$ of $\mathbf{C}$, we hence have an exact sequence

$$H^0(\mathbf{C},\mathscr{O}_{\mathbf{C}}) \to H^0(\mathbf{C}-\{v\},\mathscr{O}_{\mathbf{C}}) \times H^0(\mathscr{V},\mathscr{O}_{\mathbf{C}}) \to H^0(\mathscr{V}-\{v\},\mathscr{O}_{\mathbf{C}}) \to H^1(\mathbf{C},\mathscr{O}_{\mathbf{C}}) \;.$$

Therefore

$$\begin{aligned} K/(R_v+\mathscr{O}) &= \varinjlim_{\mathscr{V}} \; H^0(\mathscr{V}-\{v\},\mathscr{O}_{\mathbf{C}})/\big(H^0(\mathbf{C}-\{v\},\mathscr{O}_{\mathbf{C}}) + H^0(\mathscr{V},\mathscr{O}_{\mathbf{C}})\big) \\ &\simeq H^1(\mathbf{C},\mathscr{O}_{\mathbf{C}}) \;. \end{aligned}$$

Since $\dim_{\mathbb{F}_q} H^1(\mathbf{C},\mathscr{O}_{\mathbf{C}}) = g$ by one definition of the genus of $\mathbf{C}$, the result follows. □

Recall that R_v is a Dedekind ring.[3] In particular, every nonzero ideal (respectively fractional ideal) I of R_v may be written uniquely as $I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(I)}$ where $\mathfrak{p}$ ranges over the prime ideals in R_v and $v_{\mathfrak{p}}(I) \in \mathbb{N}$ (respectively $v_{\mathfrak{p}}(I) \in \mathbb{Z}$), with only finitely many of them nonzero. By convention $I = R_v$ if $v_{\mathfrak{p}}(I) = 0$ for all $\mathfrak{p}$. For all $x,y \in R_v$ (respectively $x,y \in K$), we denote by

$$\langle x,\, y\rangle = x\,R_v + y\,R_v$$

the ideal (respectively fractional ideal) of R_v generated by x,y. If I,J are nonzero fractional ideals of R_v, we have

$$I \cap J = \prod_{\mathfrak{p}} \mathfrak{p}^{\max\{v_{\mathfrak{p}}(I),\, v_{\mathfrak{p}}(J)\}} \quad \text{and} \quad I + J = \prod_{\mathfrak{p}} \mathfrak{p}^{\min\{v_{\mathfrak{p}}(I),\, v_{\mathfrak{p}}(J)\}} \;. \tag{14.4}$$

The (absolute) *norm* of a nonzero ideal $I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(I)}$ of R_v is

$$N(I) = [R_v : I] = |R_v/I| = \prod_{\mathfrak{p}} q^{v_{\mathfrak{p}}(I)\deg \mathfrak{p}} \;,$$

where $\deg \mathfrak{p}$ is the degree of the field $R_v/\mathfrak{p}$ over $\mathbb{F}_q$, so that $N(R_v) = 1$. By convention $N(0) = 0$. This norm is multiplicative:

$$N(IJ) = N(I)N(J) \;,$$

and the norm of a nonzero fractional ideal $I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(I)}$ of R_v is defined by the same formula $N(I) = \prod_{\mathfrak{p}} q^{v_{\mathfrak{p}}(I)\deg \mathfrak{p}}$. Note that if (a) is the principal fractional ideal in R_v generated by a nonzero element $a \in K$, we define $N(a) = N\big((a)\big)$. We have (see for instance [Gos, page 63])

$$N(a) = |a|_v \;. \tag{14.5}$$

Dedekind's zeta function of K is (see for instance [Gos, §7.8] or [Ros, §5])

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s}$$

[3] See for instance [Ser3, II.2 Notation]. We refer for instance to [Nar, §1.1] for background on Dedekind rings.

if Re $s > 1$, where the summation is over the nonzero ideals I of R_v. By for instance [Ros, §5], it has an analytic continuation on $\mathbb{C} - \{0, 1\}$ with simple poles at $s = 0, s = 1$. It is actually a rational function of q^{-s}. In particular, if $K = \mathbb{F}_q(Y)$, then (see [Ros, Theo. 5.9])

$$\zeta_{\mathbb{F}_q(Y)}(-1) = \frac{1}{(q-1)(q^2-1)} \ . \tag{14.6}$$

We denote by Haar_{K_v} the Haar measure of the (abelian) locally compact topological group $(K_v, +)$, normalised so that $\mathrm{Haar}_{K_v}(\mathscr{O}_\nu) = 1$.[4] The Haar measure scales as follows under multiplication: for all $\lambda, x \in K_v$, we have

$$d\,\mathrm{Haar}_{K_v}(\lambda x) = |\lambda|_v \, d\,\mathrm{Haar}_{K_v}(x) \ . \tag{14.7}$$

Note that any nonzero fractional ideal I of R_v is a discrete subgroup of $(K_v, +)$, and we will again denote by Haar_{K_v} the Haar measure on the compact group K_v/I which is induced by the above normalised Haar measure of K_v.

Lemma 14.4. *For every nonzero fractional ideal I of R_v, we have*

$$\mathrm{Haar}_{K_v}(K_v/I) = q^{\,g-1}\, N(I) \ .$$

Proof. By the scaling properties of the Haar measure, we may assume that I is an ideal in R_v. By Lemma 14.3, we have Card $K_v/(R_v + \mathscr{O}_v) = q^g$. By Equation (14.2) and by the normalisation of the Haar measure, we have

$$\mathrm{Haar}_{K_v}(R_v + \mathscr{O}_v)/R_v = \mathrm{Haar}_{K_v}\, \mathscr{O}_v/(R_v \cap \mathscr{O}_v) = \mathrm{Haar}_{K_v}\, \mathscr{O}_v/\mathbb{F}_q = \frac{1}{q} \ .$$

Hence

$$\mathrm{Haar}_{K_v}(K_v/R_v) = \mathrm{Card}\big(K_v/(R_v + \mathscr{O}_v)\big)\ \mathrm{Haar}_{K_v}(R_v + \mathscr{O}_v)/R_v = q^{g-1} \ .$$

Since $\mathrm{Haar}_{K_v}(K_v/I) = N(I)\ \mathrm{Haar}_{K_v}(K_v/R_v)$, the result follows. □

[4] Other normalisations are useful when considering Fourier transforms, see for instance Tate's thesis [Tat].

Chapter 15

Bruhat-Tits trees and modular groups

In this Chapter, we give background information and preliminary results on the main link between the geometry and the algebra used for our arithmetic applications: the (discrete time) geodesic flow on quotients of Bruhat-Tits trees by arithmetic lattices. We refer to Section 2.6 for the basic notation and terminology for trees and graphs (of groups).

We denote the image in PGL_2 of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2$.

15.1 Bruhat-Tits trees

Let K_v be a non-Archimedean local field, with valuation v, valuation ring $\mathscr{O}_v$, choice of uniformiser π_v, and residual field k_v of order q_v (see Section 14.1 for definitions).

In this Section, we recall the construction and basic properties of the Bruhat-Tits tree $\mathbb{X}_v$ of (PGL_2, K_v), see for instance [Tit]. We use its description given in [Ser3], to which we refer for proofs and further information.

An $\mathscr{O}_v$*-lattice* Λ in the K_v-vector space $K_v \times K_v$ is a rank 2 free $\mathscr{O}_v$-submodule of $K_v \times K_v$, generating $K_v \times K_v$ as a vector space. The Bruhat-Tits tree $\mathbb{X}_v$ of (PGL_2, K_v) is the graph whose set of vertices $V\mathbb{X}_v$ is the set of homothety classes (under $(K_v)^\times$) $[\Lambda]$ of $\mathscr{O}_v$-lattices Λ in $K_v \times K_v$, and whose nonoriented edges are the pairs $\{x, x'\}$ of vertices such that there exist representatives Λ of x and Λ' of x' for which $\Lambda \subset \Lambda'$ and Λ'/Λ is isomorphic to $\mathscr{O}_v/\pi_v\mathscr{O}_v$. We again denote by $\mathbb{X}_v$ the geometric realisation of $\mathbb{X}_v$.[1] Two (oriented) edges are naturally associated with each nonoriented edge. If K is any field endowed with a valuation v whose completion is K_v, then the similarly defined Bruhat-Tits tree of (PGL_2, K) coincides with $\mathbb{X}_v$, see [Ser3, p. 71].

The graph $\mathbb{X}_v$ is a regular tree of degree $|\mathbb{P}_1(k_v)| = q_v + 1$. In particular, the Bruhat-Tits tree of $(\mathrm{PGL}_2, \mathbb{Q}_p)$ is regular of degree $p + 1$, and if $K_v = \mathbb{F}_q((Y^{-1}))$ and $v = v_\infty$, then the Bruhat-Tits tree $\mathbb{X}_v$ of (PGL_2, K_v) is regular of degree $q + 1$. More generally, if K_v is the completion of a function field over $\mathbb{F}_q$ endowed with a valuation v as in Section 14.2, then the Bruhat-Tits tree of (PGL_2, K_v) is regular of degree $q_v + 1 = q^{\deg v} + 1$.

The *standard base point* $*_v$ of $\mathbb{X}$ is the homothety class $[\mathscr{O}_v \times \mathscr{O}_v]$ of the $\mathscr{O}_v$-lattice $\mathscr{O}_v \times \mathscr{O}_v$, generated by the canonical basis of $K_v \times K_v$. In particular, we have

$$d(*_v, [\mathscr{O}_v \times x\mathscr{O}_v]) = |v(x)| \tag{15.1}$$

[1] giving length 1 to (the geometric realisation of) each nonoriented edge, see Section 2.6.

for every $x \in (K_v)^\times$. The *link*

$$\mathrm{lk}(*_v) = \{y \in V\mathbb{X}_v \ : \ d(y, *_v) = 1\}$$

of $*_v$ in $\mathbb{X}_v$ identifies with the projective line $\mathbb{P}_1(k_v)$.

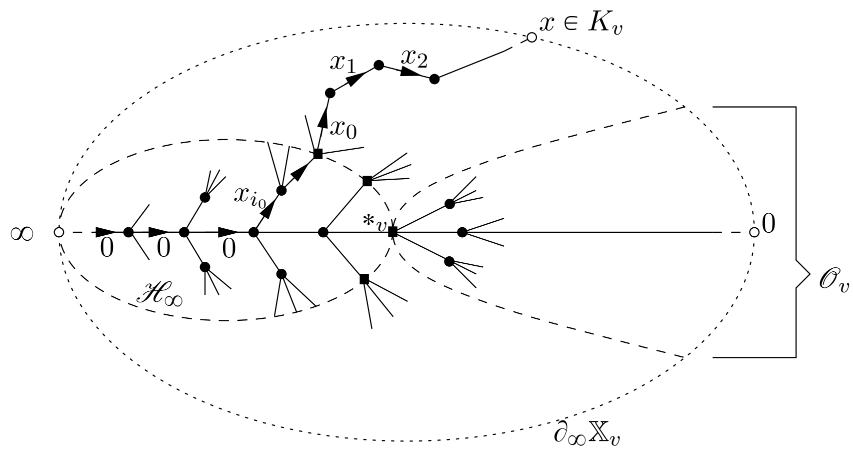
The left linear action of $\mathrm{GL}_2(K_v)$ on $K_v \times K_v$ induces a faithful, vertex-transitive left action by automorphisms of $\mathrm{PGL}_2(K_v)$ on the Bruhat-Tits tree $\mathbb{X}_v$. The stabiliser in $\mathrm{PGL}_2(K_v)$ of $*_v$ is $\mathrm{PGL}_2(\mathscr{O}_v)$, acting projectively on $\mathrm{lk}(*_v) = \mathbb{P}_1(k_v)$ by reduction modulo $\pi_v \mathscr{O}_v$ of the coefficients. We will hence identify $\mathrm{PGL}_2(K_v)/\mathrm{PGL}_2(\mathscr{O}_v)$ with $V\mathbb{X}_v$ by the map $g\,\mathrm{PGL}_2(\mathscr{O}_v) \mapsto g\,*_v$.

We identify the projective line $\mathbb{P}_1(K_v)$ with $K_v \cup \{\infty\}$ using the map $K_v(x, y) \mapsto \frac{x}{y}$, so that

$$\infty = [1 : 0]\ .$$

The projective action of $\mathrm{GL}_2(K_v)$ or $\mathrm{PGL}_2(K_v)$ on $\mathbb{P}^1(K_v)$ is the action by *homographies*[2] on $K_v \cup \{\infty\}$, given by $(g, z) \mapsto g \cdot z = \frac{a\,z+b}{c\,z+d}$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_v)$, or $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(K_v)$. As usual we define $\infty \mapsto \frac{a}{c}$ and $-\frac{d}{c} \mapsto \infty$.

There exists a unique homeomorphism between the boundary at infinity $\partial_\infty \mathbb{X}_v$ of $\mathbb{X}_v$ and $\mathbb{P}_1(K_v)$ such that the (continuous) extension to $\partial_\infty \mathbb{X}_v$ of the isometric action of $\mathrm{PGL}_2(K_v)$ on $\mathbb{X}_v$ corresponds to the projective action of $\mathrm{PGL}_2(K_v)$ on $\mathbb{P}_1(K_v)$. From now on, we identify $\partial_\infty \mathbb{X}_v$ and $\mathbb{P}_1(K_v)$ by this homeomorphism. Under this identification, $\mathscr{O}_v$ consists of the positive endpoints ℓ_+ of the geodesic lines ℓ of $\mathbb{X}_v$ with negative endpoint $\ell_- = \infty$ that pass through the vertex $*_v$ (see the picture below).



Let $\mathscr{H}_\infty$ be the horoball centred at $\infty \in \partial_\infty \mathbb{X}_v$ whose associated horosphere passes through $*_v$. There is a unique labeling of the edges of $\mathbb{X}_v$ by elements of $\mathbb{P}_1(k_v) = k_v \cup \{\infty\}$ such that (see the above picture)

- the label of any edge of $\mathbb{X}_v$ pointing towards $\infty \in \partial_\infty \mathbb{X}_v$ is ∞,
- for any $x = \sum_{i \in \mathbb{Z}} x_i\,(\pi_v)^i \in K_v$, the sequence $(x_i)_{i \in \mathbb{Z}}$ is the sequence of the labels of the (directed) edges that make up the geodesic line $]\infty, x[$ oriented from ∞ towards x,
- x_0 is the label of the edge of $]\infty, x[$ exiting the horoball $\mathscr{H}_\infty$.

[2] or linear fractional transformations

We refer to [Pau3, Sect. 5] for a detailed treatment of the case $K_v = \mathbb{F}_q((Y^{-1}))$ and $v = v_\infty$.

For all $\eta, \eta' \in K_v = \partial_\infty \mathbb{X}_v - \{\infty\}$, we have

$$|\eta - \eta'|_v = d_{\mathscr{H}_\infty}(\eta, \eta')^{\ln q_v} \tag{15.2}$$

by the definitions of the absolute value $|\cdot|_v$ and of Hamenstädt's distance, see Equation (14.1), the above geometric interpretation, and Equation (2.12). Note that in [Pau3], Hamenstädt's distance in a regular tree is defined in a different way: In that reference, the distance $|\eta - \eta'|_v$ equals Hamenstädt's distance between η and η'.

In particular, the Hölder norms[3] $\|\psi\|_{\beta,\,|\cdot - \cdot|_v}$ and $\|\psi\|_{\beta',\,d_{\mathscr{H}_\infty}}$ of a function $\psi : K_v \to \mathbb{R}$, respectively for the distances $(x, y) \mapsto |x - y|_v$ and $d_{\mathscr{H}_\infty}$ on K_v, are related by the following formula:

$$\forall\ \beta \in\ \Big]0, \frac{1}{\ln q_v}\Big], \quad \|\psi\|_{\beta,\,|\cdot - \cdot|_v} = \|\psi\|_{\beta \ln q_v,\, d_{\mathscr{H}_\infty}}\ . \tag{15.3}$$

The group $\mathrm{PGL}_2(K_v)$ acts simply transitively on the set of ordered triples of distinct points in $\partial_\infty \mathbb{X}_v = \mathbb{P}_1(K_v)$. In particular, it acts transitively on the space $\mathscr{G}\mathbb{X}_v$ of (discrete) geodesic lines in $\mathbb{X}_v$. The stabiliser under this action of the geodesic line (from $\infty = [1:0]$ to $0 = [0:1]$)

$$\ell^* : n \mapsto [\mathscr{O}_v \times (\pi_v)^{-n} \mathscr{O}_v]$$

is the maximal compact-open subgroup

$$\underline{A}(\mathscr{O}_v) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;:\; a, d \in (\mathscr{O}_v)^\times \right\}$$

of the diagonal group

$$\underline{A}(K_v) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;:\; a, d \in (K_v)^\times \right\}.$$

We will hence identify $\mathrm{PGL}_2(K_v)/\underline{A}(\mathscr{O}_v)$ with $\mathscr{G}\mathbb{X}_v$ by the mapping $\widetilde{\Xi} : g\underline{A}(\mathscr{O}_v) \mapsto g\,\ell^*$. Define

$$a_v = \begin{bmatrix} 1 & 0 \\ 0 & \pi_v^{-1} \end{bmatrix},$$

which belongs to $\underline{A}(K_v)$ and centralises $\underline{A}(\mathscr{O}_v)$. The homeomorphism $\widetilde{\Xi}$ is equivariant for the actions on the left of $\mathrm{PGL}_2(K_v)$ on $\mathrm{PGL}_2(K_v)/\underline{A}(\mathscr{O}_v)$ and $\mathscr{G}\mathbb{X}_v$. It is also equivariant for the actions on $\mathrm{PGL}_2(K_v)/\underline{A}(\mathscr{O}_v)$ under translations on the right by $(a_v)^{\mathbb{Z}}$ and on $\mathscr{G}\mathbb{X}_v$ under the discrete geodesic flow $(\mathsf{g}^n)_{n\in\mathbb{Z}}$: for all $n \in \mathbb{Z}$ and $x \in \mathrm{PGL}_2(K_v)/\underline{A}(\mathscr{O}_v)$, we have

$$\widetilde{\Xi}\,(x\; {a_v}^n) = \mathsf{g}^n\, \widetilde{\Xi}\,(x)\ . \tag{15.4}$$

Furthermore, the stabiliser in $\mathrm{PGL}_2(K_v)$ of the ordered pair of endpoints $(\ell^*_- = \infty, \ell^*_+ = 0)$ of ℓ^* in $\partial_\infty \mathbb{X}_v = \mathbb{P}_1(K_v)$ is $\underline{A}(K_v)$. Therefore any element $\gamma \in \mathrm{PGL}_2(K_v)$ which is loxodromic on $\mathbb{X}_v$ is diagonalisable over K_v. By [Ser3, page 108], the translation length on $\mathbb{X}_v$ of $\gamma_0 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is

$$\lambda(\gamma_0) = |v(a) - v(d)|\ . \tag{15.5}$$

[3]See the definition in Section 3.1.

Note that if $\widetilde{\gamma}_0 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(K_v)$ is a representative of γ_0 such that $\det \widetilde{\gamma}_0 \in (\mathscr{O}_v)^\times$, then $0 = v(\det \widetilde{\gamma}_0) = v(ad) = v(a) + v(d)$, so that $v(d) = -v(a)$ and $\lambda(\gamma_0) = 2|v(a)|$. Since $v(a) \neq v(d)$ if $\lambda(\gamma_0) \neq 0$, we have $v(\mathrm{tr}\, \widetilde{\gamma}_0) = v(a + d) = \min\{v(a), v(d)\} = -|v(a)|$. Thus,

$$\lambda(\gamma_0) = 2|v(\mathrm{tr}\, \widetilde{\gamma}_0)| \; . \tag{15.6}$$

By conjugation, this formula is valid if $\gamma_0 \in \mathrm{PGL}_2(K_v)$ is loxodromic on $\mathbb{X}_v$ and represented by $\widetilde{\gamma}_0 \in \mathrm{GL}_2(K_v)$ such that $\det \widetilde{\gamma}_0 \in (\mathscr{O}_v)^\times$.

Let $\mathscr{H}$ be a horoball in $\mathbb{X}_v$ whose boundary is contained in $V\mathbb{X}_v$ and whose point at infinity ξ is different from ∞. With $\beta_\infty : V\mathbb{X}_v \times V\mathbb{X}_v \to \mathbb{Z}$ the Busemann function at ∞ (see Equation (2.5)), the *height* of $\mathscr{H}$ is

$$\mathrm{ht}_\infty(\mathscr{H}) = \max\{\beta_\infty(x, *_v) \;:\; x \in \partial\mathscr{H}\} \in \mathbb{Z} \; ,$$

which is the signed distance between $\mathscr{H}_\infty$ and $\mathscr{H}$.[4] It is attained at the intersection point with $\partial\mathscr{H}$ of the geodesic line from ∞ to ξ, which is then called the *highest point* of $\mathscr{H}$. Note that the height of $\mathscr{H}$ is invariant under the action of the stabiliser of $\mathscr{H}_\infty$ in $\mathrm{PGL}_2(K_v)$ on the set of such horoballs $\mathscr{H}$.

The following lemma is a generalisation of [Pau3, Prop. 6.1] that only covered the particular case of $K = \mathbb{F}_q(Y)$ and $v = v_\infty$.

Lemma 15.1. *Assume that K_v is the completion of a function field K over $\mathbb{F}_q$ endowed with a valuation v, with associated affine function ring R_v. For every $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(K)$ with $a, b, c, d \in K$ such that $ad - bc \in (\mathscr{O}_v)^\times$ and $c \neq 0$, the image of $\mathscr{H}_\infty$ by γ is the horoball centred at $\frac{a}{c} \in K \subset K_v = \partial_\infty\mathbb{X}_v - \{\infty\}$ with height*

$$\mathrm{ht}_\infty(\gamma\mathscr{H}_\infty) = -2\, v(c) \; .$$

Proof. It is immediate that $\gamma\infty = \frac{a}{c}$ under the projective action. Up to multiplying γ on the left by $\begin{bmatrix} 1 & -\frac{a}{c} \\ 0 & 1 \end{bmatrix} \in \mathrm{PGL}_2(K)$, which does not change c nor the height of $\gamma\mathscr{H}_\infty$, we may assume that $a = 0$ and that b has the form $c^{-1}u$ with $u = bc - ad \in (\mathscr{O}_v)^\times$. Multiplying γ on the right by $\begin{bmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{bmatrix} \in \mathrm{PGL}_2(K)$ preserves $\gamma\mathscr{H}_\infty$ and does not change $a = 0$, $b = c^{-1}u$ or c. Hence we may assume that $d = 0$. Since γ then exchanges the points ∞ and 0 in $\partial_\infty\mathbb{X}_v$, the highest point of $\gamma\mathscr{H}_\infty$ is $\gamma *_v$. Assuming first that $0, \gamma*_v, *_v, \infty$ are in this order on the geodesic line from 0 to ∞, we have by Equation (15.1)

$$\begin{aligned} \mathrm{ht}_\infty(\gamma\mathscr{H}_\infty) &= d(*_v, \gamma*_v) = d([\mathscr{O}_v \times \mathscr{O}_v], [c^{-1}u\mathscr{O}_v \times c\mathscr{O}_v]) \\ &= d([\mathscr{O}_v \times \mathscr{O}_v], [\mathscr{O}_v \times c^2\mathscr{O}_v]) = -v(c^2) = -2\, v(c) \; . \end{aligned}$$

If $0, \gamma*_v, *_v, \infty$ are in the opposite order, then the same computation holds, up to replacing the distance d by its opposite $-d$. □

[4] See the definition of the signed distance just above Lemma 11.13.

15.2 Modular graphs of groups

Let K be a function field over $\mathbb{F}_q$, let v be a (normalised discrete) valuation of K, let K_v be the completion of K associated with v, and let R_v be the affine function ring associated with v (see Section 14.2 for definitions).

The group $\Gamma_v = \mathrm{PGL}_2(R_v)$ is a lattice in the locally compact group $\mathrm{PGL}_2(K_v)$, and a lattice[5] of the Bruhat-Tits tree $\mathbb{X}_v$ of (PGL_2, K_v), called the *modular group* at v of K. The quotient graph $\Gamma_v\backslash\mathbb{X}_v$ is called the *modular graph* at v of K, and the quotient graph of groups[6] $\Gamma_v\backslash\backslash\mathbb{X}_v$ is called the *modular graph of groups* at v of K. We refer to [Ser3] for background information on these objects, and for instance to [Pau3] for a geometric treatment when $K = \mathbb{F}_q(Y)$ and $v = v_\infty$.

By for instance [Ser3], the *set of cusps* $\Gamma_v\backslash\mathbb{P}_1(K)$ is finite, and $\Gamma_v\backslash\mathbb{X}_v$ is the disjoint union of a finite connected subgraph containing $\Gamma_v *_v$ and of maximal open geodesic rays $h_z(\,]0,+\infty[)$, for $z = \Gamma_v\tilde{z} \in \Gamma_v\backslash\mathbb{P}_1(K)$, where h_z (called a cuspidal ray, see Section 2.6) is the injective mage by the canonical projection $\mathbb{X}_v \to \Gamma_v\backslash\mathbb{X}_v$ of a geodesic ray whose point at infinity in $\mathbb{P}_1(K) \subset \partial_\infty\mathbb{X}_v$ is equal to $\tilde{z}$. Conversely, any geodesic ray whose point at infinity lies in $\mathbb{P}_1(K) \subset \partial_\infty\mathbb{X}_v$ contains a subray that maps injectively by the canonical projection $\mathbb{X}_v \to \Gamma_v\backslash\mathbb{X}_v$.

The group $\Gamma_v = \mathrm{PGL}_2(R_v)$ is a geometrically finite lattice[7] by for instance [Pau4]. The set of bounded parabolic fixed points of Γ_v is exactly $\mathbb{P}_1(K) \subset \partial_\infty\mathbb{X}_v$, and the set of conical limit points of Γ_v is $\mathbb{P}_1(K_v) - \mathbb{P}_1(K)$.

Let us denote by $\widehat{\Gamma_v\backslash\mathbb{X}_v} = (\Gamma_v\backslash\mathbb{X}_v) \sqcup \mathscr{E}_v$ Freudenthal's compactification of $\Gamma_v\backslash\mathbb{X}_v$ by its finite set of ends $\mathscr{E}_v$, see [Fre]. This set of ends is indeed finite, in bijection with $\Gamma_v\backslash\mathbb{P}_1(K)$ by the map which associates to $z \in \Gamma_v\backslash\mathbb{P}_1(K)$ the end towards which the cuspidal ray h_z converges. See for instance [Ser3] for a geometric interpretation of $\mathscr{E}_v$ in terms of the curve **C**.

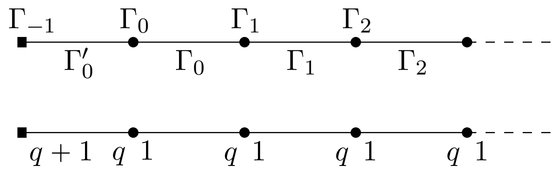
Let $\mathscr{I}_v$ be the set of classes of fractional ideals of R_v. The map which associates to an element $[x:y] \in \mathbb{P}_1(K)$ the class of the fractional ideal $xR_v + yR_v$ generated by x, y induces a bijection from the set of cusps $\Gamma\backslash\mathbb{P}_1(K)$ to $\mathscr{I}_v$.

The volume[8] of the modular graph of groups $\Gamma_v\backslash\backslash\mathbb{X}_v$ can be computed using Equation (14.3) and Exercice 2 b) in [Ser3, II.2.3]:

$$\mathrm{Vol}(\mathrm{PGL}_2(R_v)\backslash\backslash\mathbb{X}_v) = (q-1)\,\mathrm{Vol}(\mathrm{GL}_2(R_v)\backslash\backslash\mathbb{X}_v) = 2\,\zeta_K(-1)\;. \tag{15.7}$$

If $K = \mathbb{F}_q(Y)$ is the rational function field over $\mathbb{F}_q$ and if we consider the valuation at infinity $v = v_\infty$ of K, then the *Nagao lattice*[9] $\Gamma_v = \mathrm{PGL}_2(\mathbb{F}_q[Y])$ acts transitively on $\mathbb{P}^1(K)$. Its quotient graph of groups $\Gamma_v\backslash\backslash\mathbb{X}_v$ is the following *modular ray* (with associated edge-indexed graph)

[5] See Section 2.6 for a definition.
[6] See Section 2.6 for a definition.
[7] See Section 2.6 for a definition and for instance [BasL] for a profusion of geometrically infinite lattices in simplicial trees.
[8] See Section 2.6 for a definition.
[9] This lattice was studied by Nagao in [Nag], see also [Moz, BasL]. It is called the modular group in [Wei2].



where $\Gamma_{-1} = \mathrm{PGL}_2(\mathbb{F}_q)$, $\Gamma'_0 = \Gamma_0 \cap \Gamma_{-1}$ and, for every $n \in \mathbb{N}$,

$$\Gamma_n = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{F}_q[Y]) \;:\; a, d \in \mathbb{F}_q^\times, b \in \mathbb{F}_q[Y], \deg b \leqslant n+1 \right\}.$$

Note that even though $\mathrm{PGL}_2(K_v)$ has inversions on $\mathbb{X}_v$, its subgroup $\Gamma_v = \mathrm{PGL}_2(R_v)$ acts without inversion on $\mathbb{X}_v$ (see for instance [Ser3, II.1.3]). In particular, the quotient graph $\Gamma_v \backslash \mathbb{X}_v$ is then well defined.

15.3 Computations of measures for Bruhat-Tits trees

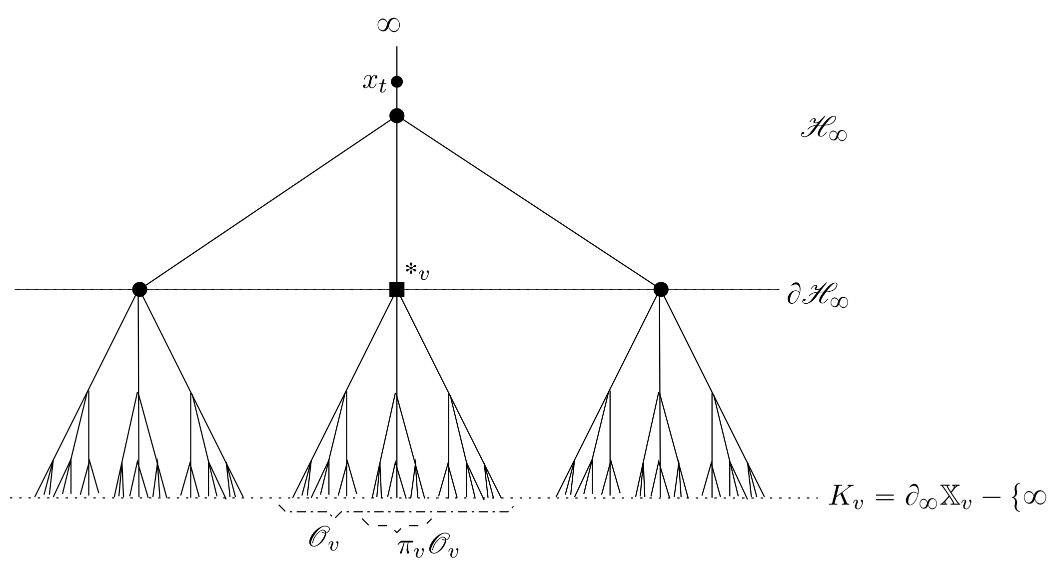
In this Section, we compute explicit expressions for the skinning measures[10] of horoballs and geodesic lines, and for the Bowen-Margulis measures,[11] when considering lattices of Bruhat-Tits trees. See [PaP16, Section 7] and [PaP17a, Section 4] for analogous computations in the real and complex hyperbolic spaces respectively, and [BrP1] for related computations in the tree case.

Let (K_v, v) be as in the beginning of Section 15.1. Let Γ be a lattice of the Bruhat-Tits tree $\mathbb{X}_v$ of (PGL_2, K_v). Since $\mathbb{X}_v$ is regular of degree $q_v + 1$, the critical exponent of Γ is

$$\delta_\Gamma = \ln q_v \tag{15.8}$$

by Proposition 4.16 and Equation (8.1).

We normalise the Patterson density $(\mu_x)_{x \in V\mathbb{X}_v}$ of Γ as follows.[12] Let $\mathscr{H}_\infty$ be the horoball in $\mathbb{X}_v$ centred at ∞ whose associated horosphere passes through $*_v$. Let $t \mapsto x_t$ be the geodesic ray in $\mathbb{X}_v$ such that $x_0 = *_v$ and which converges to ∞.



[10] See Section 7.1.
[11] See Section 4.2.
[12] Since the system of conductances is 0, note that $\mu_x^+ = \mu_x^- = \mu_x$ is the standard Patterson measure of Γ.

Hamenstädt's measure[13] associated with $\mathscr{H}_\infty$

$$\mu_{\mathscr{H}_\infty} = \lim_{t\to+\infty} e^{\delta_\Gamma t}\mu_{x_t} = \lim_{t\to+\infty} {q_v}^t \mu_{x_t}$$

is a Radon measure on $\partial_\infty \mathbb{X}_v - \{\infty\} = K_v$, invariant under all isometries of $\mathbb{X}_v$ preserving $\mathscr{H}_\infty$, since Γ is a lattice. Hence it is invariant under the translations by the elements of K_v. By the uniqueness property of Haar measures, $\mu_{\mathscr{H}_\infty}$ is a constant multiple of the chosen Haar measure[14] of K_v, and we normalise the Patterson density $(\mu_x)_{x\in V\mathbb{X}_v}$ so that

$$\mu_{\mathscr{H}_\infty} = \operatorname{Haar}_{K_v} \ . \tag{15.9}$$

We summarise the various measure computations in the following result.

Proposition 15.2. *Let Γ be a lattice of the Bruhat-Tits tree $\mathbb{X}_v$ of (PGL_2, K_v), with Patterson density normalised as above.*

(1) *The outer/inner skinning measures of the singleton $\{*_v\}$ are given by*

$$d\,\widetilde{\sigma}^{\pm}_{\{*_v\}}(\rho) = d\mu_{*_v}(\rho_\pm) = \big(\max\{1, |\rho_\pm|_v\}\big)^{-2}\, d\operatorname{Haar}_{K_v}(\rho_\pm)$$

on the set of $\rho \in \partial^1_\pm\{*_v\}$ such that $\rho_\pm \neq \infty$.

(2) *The total mass of the Patterson density is*

$$\|\mu_x\| = \frac{q_v+1}{q_v}$$

for all $x \in V\mathbb{X}_v$.

(3) *The skinning measure of the horoball $\mathscr{H}_\infty$ is the projection of the Haar measure of K_v: For all $\rho \in \partial^1_\pm \mathscr{H}_\infty$, we have*

$$d\widetilde{\sigma}^{\pm}_{\mathscr{H}_\infty}(\rho) = d\mu_{\mathscr{H}_\infty}(\rho_\pm) = d\operatorname{Haar}_{K_v}(\rho_\pm)\ .$$

(4) *If ∞ is a bounded parabolic fixed point of Γ, with Γ_∞ its stabiliser in Γ, if $\mathscr{D} = (\gamma\mathscr{H}_\infty)_{\gamma\in\Gamma/\Gamma_\infty}$, we have*

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \operatorname{Haar}_{K_v}(\Gamma_\infty\backslash K_v) = \operatorname{Vol}(\Gamma_\infty\backslash\backslash\partial\mathscr{H}_\infty)\ .$$

(5) *Let L be a geodesic line in $\mathbb{X}_v$ with endpoints $L_\pm \in K_v = \partial_\infty\mathbb{X}_v - \{\infty\}$. Then on the set of $\rho \in \partial^1_+ L$ such that $\rho_+ \in K_v = \partial_\infty \mathbb{X}_v - \{\infty\}$ and $\rho_+ \neq L_\pm$, the outer skinning measure of L is*

$$d\,\widetilde{\sigma}^{+}_{L}(\rho) = \frac{|L_+ - L_-|_v}{|\rho_+ - L_-|_v\,|\rho_+ - L_+|_v}\, d\operatorname{Haar}_{K_v}(\rho_+)\,.$$

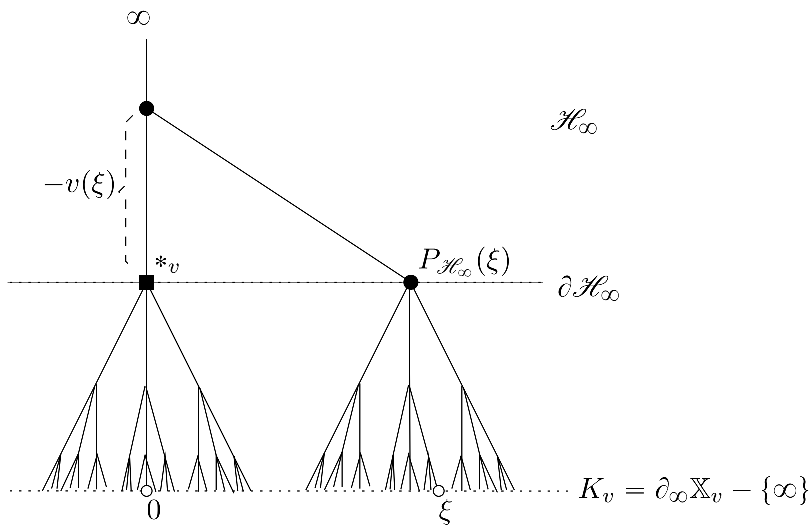
(6) *Let L be a geodesic line in $\mathbb{X}_v$, let Γ_L be the stabiliser in Γ of L, and assume that $\Gamma_L\backslash L$ has finite length. Then with $\mathscr{D} = (\gamma L)_{\gamma\in\Gamma/\Gamma_L}$, we have*

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \frac{q_v - 1}{q_v}\ \operatorname{Vol}(\Gamma_L\backslash\backslash L)\,.$$

[13]See Equation (7.5).

[14]Recall that we normalise the Haar measure of $(K_v, +)$ such that $\operatorname{Haar}_{K_v}(\mathscr{O}_v) = 1$.

Proof. (1) For every $\xi \in K_v$, by the description of the geodesic lines in the Bruhat-Tits tree $\mathbb{X}_v$ starting from ∞ given in Section 15.1, we have $\xi \in \mathscr{O}_v$ if and only if $P_{\mathscr{H}_\infty}(\xi) = *_v$.[15]



For every $\xi \in K_v - \mathscr{O}_v$, by Equations (15.2) and (2.12), we have

$$|\xi|_v = d_{\mathscr{H}_\infty}(0,\xi)^{\ln q_v} = {q_v}^{\frac{1}{2}\, d(*_v,\, P_{\mathscr{H}_\infty}(\xi))} \,. \tag{15.10}$$

On the set of geodesic rays $\rho \in \partial^1_\pm\{*_v\}$ such that $\rho_\pm \neq \infty$, by Equation (7.2), by the last claim of Proposition 7.2, by Equation (15.8),[16] since $P_{\mathscr{H}_\infty}(\rho_\pm)$ belongs to the geodesic ray $[*_v, \rho_\pm[$ (even when $\rho_\pm \in \mathscr{O}_v$), and by Equation (15.9), we have

$$\begin{aligned} d\widetilde{\sigma}^{\pm}_{\{*_v\}}(\rho) = \; & d\mu_{*_v}(\rho_\pm) = e^{C_{\rho_\pm}(P_{\mathscr{H}_\infty}(\rho_\pm),\, *_v)}\; d\mu_{\mathscr{H}_\infty}(\rho_\pm) \\ & = e^{\delta_\Gamma\, \beta_{\rho_\pm}(P_{\mathscr{H}_\infty}(\rho_\pm),\, *_v)}\; d\mu_{\mathscr{H}_\infty}(\rho_\pm) \\ & = {q_v}^{-d(P_{\mathscr{H}_\infty}(\rho_\pm),\, *_v)}\; d\,\mathrm{Haar}_{K_v}(\rho_\pm)\,. \end{aligned}$$

Therefore, if $\rho \in \partial^1_\pm\{*_v\}$ is such that

$$\rho_\pm \in \mathscr{O}_v = \{\xi \in K_v \;:\; |\xi|_v \leqslant 1\} = \{\xi \in K_v \;:\; P_{\mathscr{H}_\infty}(\xi) = *_v\}\,,$$

then $d\,\widetilde{\sigma}^{\pm}_{\{*_v\}}(\rho) = d\,\mathrm{Haar}_{K_v}(\rho_\pm)$. If $\rho_\pm \in K_v - \mathscr{O}_v$, Equation (15.10) gives the claim.

(2) This Assertion follows from Assertion (1) by a geometric series argument, but we give a direct proof.

As Γ is a lattice, the family $(\mu_x)_{x\in V\mathbb{X}_v}$ is actually equivariant under $\mathrm{Aut}(\mathbb{X}_v)$,[17] which acts transitively on the vertices of $\mathbb{X}_v$, and the stabiliser in $\mathrm{Aut}(\mathbb{X}_v)$ of the standard base point $*_v$ acts transitively on the edges starting from $*_v$.

Since $\mathbb{X}_v$ is (q_v+1)-regular, since the set of points at infinity of the geodesic rays starting from $*_v$, whose initial edge has endpoint $0 \in \mathrm{lk}(*_v) = \mathbb{P}_1(k_v)$, is equal to $\pi_v\mathscr{O}_v$, since all

[15] Recall that $P_{\mathscr{H}_\infty} : \partial_v\mathbb{X}_v - \{\infty\} \to \partial\mathscr{H}_\infty$ is the closest point map to the horoball $\mathscr{H}_\infty$, see Section 2.4.
[16] Since the potential is zero, the Gibbs cocycle is the critical exponent times the Busemann cocycle.
[17] See Proposition 4.16 (2).

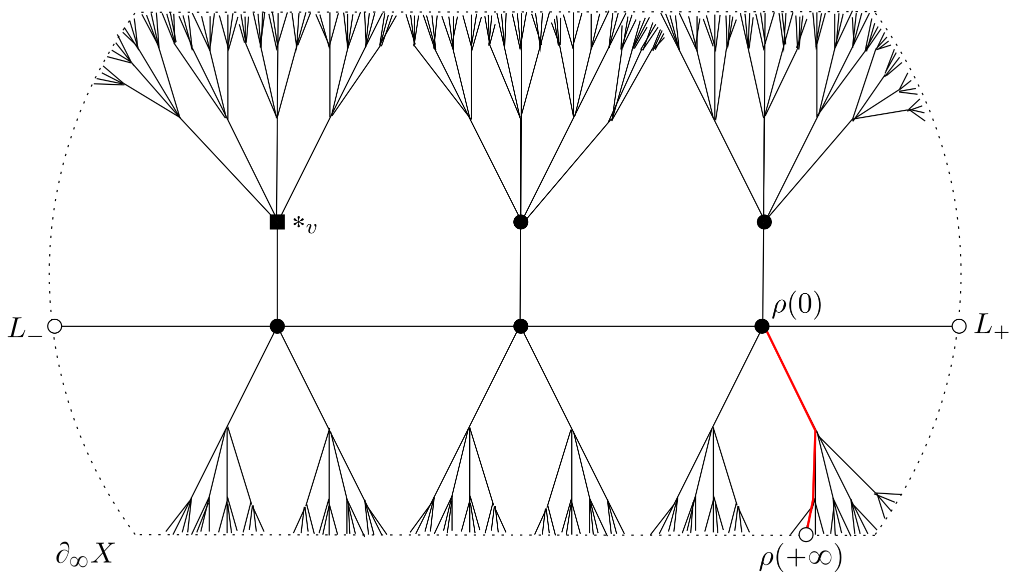
geodesic lines from $\infty \in \partial_\infty \mathbb{X}_v$ to points of $\pi_v \mathscr{O}_v \subset \partial_\infty \mathbb{X}_v$ pass through $*_v$, and by the normalisation of the Patterson density and of the Haar measure, we have

$$\|\mu_{*_v}\| = (q_v+1)\ \mu_{*_v}(\pi_v \mathscr{O}_v) = (q_v+1)\ \mu_{\mathscr{H}_\infty}(\pi_v \mathscr{O}_v) = (q_v+1)\,\mathrm{Haar}_{K_v}(\pi_v \mathscr{O}_v)$$
$$= \frac{q_v+1}{q_v}\,\mathrm{Haar}_{K_v}(\mathscr{O}_v) = \frac{q_v+1}{q_v}\ .$$

(3) This follows from Equation (7.4), and from the normalisation $\mu_{\mathscr{H}_\infty} = \mathrm{Haar}_{K_v}$ of the Patterson density.

(4) This follows from Assertion (3) and from Equation (8.11) (where the normalisation of the Patterson density was different than the one given by Assertion (2)).

(5) Let L, L_+, L_- be as in the statement of Assertion (5), see the picture below. The result follows from Lemma 8.6 applied with $\mathscr{H} = \mathscr{H}_\infty$, from Equations (15.9), (15.8) and (15.2).



(6) This follows from Equation (8.12) (where the normalisation of the Patterson density was different), since $\mathbb{X}_v$ is (q_v+1)-regular, and from Assertion (2):

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \|\mu_{*_v}\|\ \frac{q_v-1}{q_v+1}\ \mathrm{Vol}(\Gamma_L \backslash\!\backslash L) = \frac{q_v-1}{q_v}\ \mathrm{Vol}(\Gamma_L \backslash\!\backslash L)\ . \quad \square$$

We now turn to measure computations for arithmetic lattices Γ in $\mathbb{X}_v$ in the function field case. We still assume that the Patterson density of Γ is normalised so that $\mu_{\mathscr{H}_\infty}(\mathscr{O}_v) = 1$, and we denote by $m_{\rm BM}$ the Bowen-Margulis measure of Γ associated with this choice of Patterson density.

Proposition 15.3. *Let K be a function field over $\mathbb{F}_q$ of genus g and let v be a valuation of K. Let Γ be a finite index subgroup of $\Gamma_v = \mathrm{PGL}_2(R_v)$, with Patterson density normalised such that $\mu_{\mathscr{H}_\infty} = \mathrm{Haar}_{K_v}$.*

(1) *We have*

$$\|m_{\rm BM}\| = \frac{(q_v+1)\ [\Gamma_v:\Gamma]}{q_v}\ \mathrm{Vol}(\Gamma_v \backslash\!\backslash \mathbb{X}_v) = \frac{2\ (q_v+1)\ \zeta_K(-1)\ [\Gamma_v:\Gamma]}{q_v}\ ,$$

and if $K = \mathbb{F}_q(Y)$ *and* $v = v_\infty$ *is the valuation at infinity of* $\mathbf{C} = \mathbb{P}_1$*, then*

$$\|m_{\rm BM}\| = \frac{2\,[\mathrm{PGL}_2(\mathbb{F}_q[Y]):\Gamma]}{q\,(q-1)^2}\,.$$

(2) *Let* Γ_∞ *be the stabiliser in* Γ *of* $\infty \in \partial_\infty \mathbb{X}_v$*, and let* $\mathscr{D} = (\gamma\mathscr{H}_\infty)_{\gamma\in\Gamma/\Gamma_\infty}$*. We have*

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \frac{q^{\,g-1}\,[(\Gamma_v)_\infty:\Gamma_\infty]}{q-1}\,.$$

Proof. (1) Recall that $\Gamma_v = \mathrm{PGL}_2(R_v)$ acts without inversion on $\mathbb{X}_v$. By Equation (8.4) (which uses a different normalisation of the Patterson density of Γ), and by Proposition 15.2 (2), we have

$$\|m_{\rm BM}\| = \|\mu_{*_v}\|^2\,\frac{q_v}{q_v+1}\,\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v) = \frac{(q_v+1)\,[\Gamma_v:\Gamma]}{q_v}\;\mathrm{Vol}(\Gamma_v\backslash\!\backslash\mathbb{X}_v)\,.$$

The first claim of Assertion (1) hence follows from Equation (15.7).

If $K = \mathbb{F}_q(Y)$ and $v = v_\infty$, then the second claim of Assertion (1) follows either from the first claim where the value of $\zeta_K(-1)$ is given by Equation (14.6), or from the fact that $q_v = q$ and that the covolume $\mathrm{Vol}(\mathrm{PGL}_2(\mathbb{F}_q[Y])\backslash\!\backslash\mathbb{X}_{v_\infty})$ of the Nagao lattice $\mathrm{PGL}_2(\mathbb{F}_q[Y])$ is

$$\mathrm{Vol}(\mathrm{PGL}_2(\mathbb{F}_q[Y])\backslash\!\backslash\mathbb{X}_{v_\infty}) = \frac{2}{(q-1)(q^2-1)}\,, \tag{15.11}$$

as an easy geometric series computation shows using the description of the modular ray in Section 15.2 (see also [BasL, Sect. 10.2]).

(2) Let us prove that

$$\mathrm{Haar}_{K_v}((\Gamma_v)_\infty\backslash K_v) = \frac{q^{\,g-1}}{q-1}\,. \tag{15.12}$$

The result then follows by Proposition 15.2 (4) since

$$\|\sigma^{\pm}_{\mathscr{D}}\| = \mathrm{Haar}_{K_v}(\Gamma_\infty\backslash K_v) = [(\Gamma_v)_\infty:\Gamma_\infty]\;\mathrm{Haar}_{K_v}((\Gamma_v)_\infty\backslash K_v)\,.$$

The stabiliser of $\infty = [1:0]$ in Γ_v acts on K_v exactly by the set of transformations $z\mapsto az+b$ with $a\in(R_v)^\times$ and $b\in R_v$. Since[18] $(R_v)^\times = (\mathbb{F}_q)^\times$ acts freely by multiplication on the left on $(K_v - R_v)/R_v$, and by Lemma 14.4, we have

$$\mathrm{Haar}_{K_v}((\Gamma_v)_\infty\backslash K_v) = \frac{1}{q-1}\,\mathrm{Haar}_{K_v}(K_v/R_v) = \frac{q^{\,g-1}}{q-1}\,.$$

This proves Equation (15.12). □

[18] See Equation (14.3).

15.4 Exponential decay of correlation and error terms for arithmetic quotients of Bruhat-Tits trees

As in the beginning of Section 15.1, let K_v be a non-Archimedean local field, with valuation v, valuation ring $\mathscr{O}_v$, choice of uniformiser π_v, and residual field k_v of order q_v. Let Γ be a lattice of the Bruhat-Tits tree $\mathbb{X}_v$ of (PGL_2, K_v). In this Section, we discuss the error terms estimates that we will use in Part III.

Partly in order to simplify the references, we start by summarizing in the next statement the only results from the geometric Part II of this book, on geometric equidistribution and counting problems, that we will use in this algebraic Part III. We state it with the normalisation introduced in Section 15.3 which will be useful in what follows.

Theorem 15.4. *Let Γ be a lattice of $\mathbb{X}_v$ whose length spectrum L_Γ is equal to $2\mathbb{Z}$. Assume that the Patterson density of Γ is normalised so that $\|\mu_x\| = \frac{q_v+1}{q_v}$ for every $x \in V\mathbb{X}_v$. Let $\mathbb{D}^\pm$ be nonempty proper simplicial subtrees of $\mathbb{X}_v$ with stabilisers $\Gamma_{\mathbb{D}^\pm}$ in Γ, such that the families $\mathscr{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite in $\mathbb{X}_v$. For every $\gamma \in \Gamma$ such that $\mathbb{D}^-$ and $\gamma\mathbb{D}^+$ are disjoint, let $\alpha^-_{e,\gamma}$ be the generalised geodesic line, isometric exactly on $[0, d(\mathbb{D}^-, \gamma\mathbb{D}^+)]$, whose image is the common perpendicular between $\mathbb{D}^-$ and $\gamma\mathbb{D}^+$. If the measure $\sigma^-_{\mathscr{D}^+}$ is nonzero and finite, then*

$$\lim_{n\to+\infty} \frac{({q_v}^2-1)(q_v+1)}{2\,{q_v}^3}\;\frac{\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;{q_v}^{-n} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\gamma\mathbb{D}^+)\leqslant n}} \Delta_{\alpha^-_{e,\gamma}} \;=\; \widetilde{\sigma}^+_{\mathbb{D}^-}\;,$$

for the weak-star convergence of measures on the locally compact space $\check{\mathscr{G}}\mathbb{X}_v$.

Furthermore, if Γ is geometrically finite, then for every $\beta \in\,]0,1]$, there exists an error term for this equidistribution claim when evaluated on $\widetilde{\psi} \in \mathscr{C}^\beta_c(\check{\mathscr{G}}\mathbb{X})$ of the form $\mathrm{O}(\,\|\widetilde{\psi}\,\|_\beta\, e^{-\kappa\, n})$ for some $\kappa > 0$.

As recalled at the end of Section 2.6, lattices in $\mathrm{PGL}_2(\mathbb{K}_v)$ are geometrically finite, see [Lub1]. We will hence be able to use the error term in Theorem 15.4 in particular when

- K_v is the completion of a function field K over $\mathbb{F}_q$ with respect to a (normalised discrete) valuation v of K and Γ is a finite index subgroup of $\mathrm{PGL}_2(R_v)$ with R_v the affine function ring associated with v,[19] as in Chapters 16 and 19, and in Sections 17.2 and 18.2;
- when $K_v = \mathbb{Q}_p$ and Γ is an arithmetic lattice in $\mathrm{PGL}_2(K_v)$ derived from a quaternion algebra, see Sections 17.3 and 18.2.

Proof. In order to prove the first claim, we apply Corollary 11.12 with $\mathbb{X} = \mathbb{X}_v$ and $p = q = q_v$. Since $L_\Gamma = 2\mathbb{Z}$, the lattice Γ leaves invariant the partition of $V\mathbb{X}_v$ into vertices at even distance from a basepoint $x_0 \in V\mathbb{X}_v$ and vertices at odd distance from x_0. Since the Patterson density is now normalised so that $\|\mu_{x_0}\| = \frac{q_v+1}{q_v}$ (instead of $\|\mu_{x_0}\| = \frac{q_v+1}{\sqrt{q_v}}$ in Corollary 11.12), the skinning measures $\widetilde{\sigma}^\mp_{\mathscr{D}^\pm}$ are now $\frac{1}{\sqrt{q_v}}$ times the ones in the statement of Corollary 11.12. Hence the second assertion of Corollary 11.12 gives

$$\lim_{n\to+\infty} \frac{{q_v}^2-1}{2\,{q_v}^2}\;\frac{\mathrm{TVol}(\Gamma\backslash\backslash\mathbb{X}_v)}{\sqrt{q_v}\;\|\sigma^-_{\mathscr{D}^+}\|}\;{q_v}^{-n} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\gamma\mathbb{D}^+)\leqslant n}} \Delta_{\alpha^-_{e,\gamma}} \;=\; \sqrt{q_v}\;\widetilde{\sigma}^+_{\mathbb{D}^-}\;.$$

[19] See Section 14.2 for definitions.

By Equation (2.23), we have

$$\mathrm{TVol}(\Gamma\backslash\backslash\mathbb{X}_v) = (q_v + 1)\ \mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)\ .$$

The first claim follows.

The last claim concerning error terms follows from Remark (ii) following the proof of Theorem 12.16. □

In the last four Chapters 16, 17, 18 and 19 of this book, we will need to push to infinity the measures appearing in the statement of Theorem 15.4. We regroup in the following two remarks the necessary control tools for such a pushing.

Remark 15.5. With the notation of Theorem 15.4, we will use Lemma 3.9 when $X = |\mathbb{X}_v|_1$ is the geometric realisation of the simplicial tree $\mathbb{X}_v$, $\alpha = \alpha^-_{e,\gamma}$ is[20] the common perpendicular between $\mathbb{D}^-$ and $\gamma\mathbb{D}^+$ for $\gamma \in \Gamma$ (when it exists), and $\rho = \rho_\gamma$ is any extension of α to a geodesic ray, or rather to a generalised geodesic line isometric exactly on $[0, +\infty[$. Under the assumptions of Theorem 15.4, we have by Lemma 3.9

$$\frac{({q_v}^2-1)(q_v+1)}{2\ {q_v}^3}\ \frac{\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\ {q_v}^{-n} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}} \Delta_{\rho_\gamma} \ \overset{*}{\rightharpoonup}\ \widetilde{\sigma}^+_{\mathbb{D}^-}\ , \tag{15.13}$$

with, if Γ is geometrically finite, an error term when evaluated on $\widetilde{\psi} \in \mathscr{C}^\beta_c(\check{\mathscr{G}}\mathbb{X}_v)$ of the form $\mathrm{O}(e^{-\kappa\, n}\ \|\widetilde{\psi}\|_\beta)$ for some $\kappa > 0$ small enough (depending in particular on $\beta \in\ ]0,1]$).

From now on in this book, for every subtree $\mathbb{D}$ of $\mathbb{X}_v$ with geometric realisation $D = |\mathbb{D}|_1$, we endow $\partial_\infty\mathbb{X}_v - \partial_\infty D$ with the distance-like map d_D defined in Equation (3.8). We use this map d_D in order to define both the β-Hölder-continuity of maps with values in $(\partial_\infty\mathbb{X}_v - \partial_\infty D, d_D)$ and the β-Hölder-norm of a function defined on $(\partial_\infty\mathbb{X}_v - \partial_\infty D, d_D)$.

Remark 15.6. With the notation of Theorem 15.4, we will use Proposition 3.10 when $\mathbb{X} = \mathbb{X}_v$ and $\mathbb{D} = \mathbb{D}^-$. Under the assumptions of Theorem 15.4, with ρ_γ any extension to a geodesic ray of the common perpendicular $\alpha^-_{e,\gamma}$ between $\mathbb{D}^-$ and $\gamma\mathbb{D}^+$ for $\gamma \in \Gamma$, since pushing forward measures on $\partial^1_+\mathbb{D}^-$ by the homeomorphism $\partial^+ : \rho \mapsto \rho_+$ introduced in Proposition 3.10 is continuous, we have by Equation (15.13)

$$\frac{({q_v}^2-1)(q_v+1)}{2\ {q_v}^3}\ \frac{\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\ {q_v}^{-n} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}} \Delta_{(\rho_\gamma)_+} \ \overset{*}{\rightharpoonup}\ (\partial^+)_*\widetilde{\sigma}^+_{\mathbb{D}^-}\ . \tag{15.14}$$

If Γ is geometrically finite, for all $\beta \in\ ]0,1]$ and $\psi \in \mathscr{C}^\beta_c(\partial_\infty\mathbb{X}_v - \partial_\infty\mathbb{D}^-)$, using the error term in Equation (15.13) with regularity $\frac{\beta}{2}$ when evaluated on $\widetilde{\psi} = \psi\circ\partial^+$, which belongs to $\mathscr{C}^{\frac{\beta}{2}}_c(\check{\mathscr{G}}\mathbb{X}_v)$ by the first claim of Proposition 3.10, we have by the last claim of Proposition 3.10 an error term in Equation (15.14) evaluated on ψ of the form $\mathrm{O}(e^{-\kappa\, n}\ \|\psi\|_\beta)$ for some $\kappa > 0$ small enough.

[20]the generalised geodesic line isometric exactly on $[0, d(\mathbb{D}^-, \gamma\mathbb{D}^+)]$ parametrising

A stronger assumption than the Hölder regularity is the locally constant regularity, that has been defined at the end of Section 3.1, and is applicable here since the involved spaces are totally disconnected. Several error terms estimates in the literature use this stronger regularity (see for instance [AtGP, KemaPS]). The following result of decay of correlations under locally constant regularity follows from Corollary 9.6 by Remark 3.11 and since[21] any lattice of $\mathrm{PGL}_2(K_v)$ is geometrically finite.[22]

Proposition 15.7. *Assume that Γ is a lattice of $\mathrm{PGL}_2(K_v)$, and let $\beta \in \,]0,1]$.*

(1) If $L_\Gamma = \mathbb{Z}$, there exist $C, \kappa > 0$ such that for every $\epsilon \in \,]0,1]$, for all ϵ-locally constant maps $\phi, \psi : \Gamma\backslash\mathscr{G}\mathbb{X}_v \to \mathbb{R}$ and $n \in \mathbb{Z}$, we have

$$\Big|\int_{\Gamma\backslash\mathscr{G}\mathbb{X}_v} \phi\circ \mathtt{g}^{-n}\;\psi\; d\,m_{\rm BM} - \frac{1}{\|m_{\rm BM}\|}\int_{\Gamma\backslash\mathscr{G}\mathbb{X}_v} \phi\; d\,m_{\rm BM}\int_{\Gamma\backslash\mathscr{G}\mathbb{X}_v} \psi\; d\,m_{\rm BM}\Big| \\ \leqslant C\; e^{-\kappa|n|}\;\|\phi\|_{\epsilon\,{\rm lc},\,\beta}\;\|\psi\|_{\epsilon\,{\rm lc},\,\beta}\;.$$

(2) If $L_\Gamma = 2\mathbb{Z}$, then there exist $C, \kappa > 0$ such that for every $\epsilon \in \,]0,1]$, for all ϵ-locally constant maps $\phi, \psi : \Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}_v \to \mathbb{R}$ and $n \in \mathbb{Z}$, we have

$$\Big|\int_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}} \phi\circ \mathtt{g}^{-2n}\;\psi\; d\,m_{\rm BM} - \frac{1}{m_{\rm BM}(\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X})}\int_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}} \phi\; d\,m_{\rm BM}\int_{\Gamma\backslash\mathscr{G}_{\rm even}\mathbb{X}} \psi\; d\,m_{\rm BM}\Big| \\ \leqslant\; C\; e^{-\kappa|n|}\;\|\phi\|_{\epsilon\,{\rm lc},\,\beta}\;\|\psi\|_{\epsilon\,{\rm lc},\,\beta}\;. \quad \Box$$

We will not use the following result in this book, but its Assertion (2) is used in the announcement [BrPP] which only considers the locally constant regularity.

Theorem 15.8. *Assume that Γ is a lattice of $\mathrm{PGL}_2(K_v)$, and let $\beta \in \,]0,1]$.*

(1) If $L_\Gamma = \mathbb{Z}$, there exists $\kappa > 0$ such that for every $\epsilon \in \,]0,1]$ and every ϵ-locally constant map $\widetilde{\psi} : \breve{\mathscr{G}}\mathbb{X}_v \to \mathbb{R}$, we have, as $n \to +\infty$,

$$\frac{(q_v-1)}{(q_v+1)\;q_v}\;\frac{\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;q_v^{\;-n}\sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}}\widetilde{\psi}(\alpha^-_{e,\,\gamma}) \\ = \int_{\breve{\mathscr{G}}\mathbb{X}_v}\widetilde{\psi}\;d\widetilde{\sigma}^+_{\mathbb{D}^-} + \mathrm{O}\big(e^{-\kappa\,n}\;\|\widetilde{\psi}\,\|_{\epsilon\,{\rm lc},\,\beta}\big)\;.$$

(2) If $L_\Gamma = 2\mathbb{Z}$, there exists $\kappa > 0$ such that for every $\epsilon \in \,]0,1]$ and every ϵ-locally constant map $\widetilde{\psi} : \breve{\mathscr{G}}\mathbb{X}_v \to \mathbb{R}$, we have, as $n \to +\infty$,

$$\frac{(q_v^{\;2}-1)(q_v+1)}{2\;q_v^{\;3}}\;\frac{\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;q_v^{\;-n}\sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}}\widetilde{\psi}(\alpha^-_{e,\,\gamma}) \\ = \int_{\breve{\mathscr{G}}\mathbb{X}_v}\widetilde{\psi}\;d\widetilde{\sigma}^+_{\mathbb{D}^-} + \mathrm{O}\big(e^{-\kappa\,n}\;\|\widetilde{\psi}\,\|_{\epsilon\,{\rm lc},\,\beta}\big)\;.$$

Proof. This result follows by replacing in the proof of Theorem 12.16 (or rather Remark (ii) following its proof) the use of the exponential decay of β-Hölder correlations given by Corollary 9.6 by the use of Proposition 15.7. $\Box$

[21]See the end of Section 2.6.

[22]Note that being a lattice of $\mathrm{PGL}_2(K_v)$ is much stronger than been a lattice in $\mathrm{Aut}(\mathbb{X}_v)$.

We conclude this Section by giving an algebraic second proof of Theorem 15.8, in the following situation. Let K_v be the completion of a function field K over $\mathbb{F}_q$ with respect to a valuation v of K and let Γ be a nonuniform[23] lattice of $G_v = \mathrm{PGL}_2(K_v)$. We only obtain a version using an exponent $\beta \geqslant \ln q_v$ for the locally constant norm. For simplicity, we consider only Assertion (1) of Theorem 15.8, though Assertion (2) xould be treated similarly.

The group G_v acts (on the left) on the complex vector space of maps ψ from $\Gamma\backslash G_v$ to $\mathbb{R}$, by right translation on the source: For every $g \in G_v$, we have $g\psi : x \mapsto \psi(xg)$. A function $\psi : \Gamma\backslash G_v \to \mathbb{R}$ is *algebraically locally constant* if there exists a compact-open subgroup U of $\mathrm{PGL}_2(\mathscr{O}_v)$ which leaves ψ invariant:

$$\forall\, g \in U, \quad g\psi = \psi \;,$$

or equivalently, if ψ is constant on each orbit of U under the right action of G_v on $\Gamma\backslash G_v$. Note that ψ is then continuous, since the orbits of U are compact-open subsets. We define

$$d_\psi = \dim\big(\,\mathrm{Vect}_{\mathbb{R}}(\mathrm{PGL}_2(\mathscr{O}_v)\psi)\big)$$

as the dimension of the complex vector space generated by the images of ψ under the elements of $\mathrm{PGL}_2(\mathscr{O}_v)$, which is finite, and even satisfies

$$d_\psi \leqslant [\mathrm{PGL}_2(\mathscr{O}_v) : U] \;.$$

We define the alc-*norm* of every bounded algebraically locally constant map $\psi : \Gamma\backslash G_v \to \mathbb{R}$ by

$$\|\psi\|_{\mathrm{alc}} = \sqrt{d_\psi}\,\|\psi\|_\infty \;.$$

Though the alc-norm does not satisfy the triangle inequality, we have $\|\lambda\,\psi\|_{\mathrm{alc}} = |\lambda|\,\|\psi\|_{\mathrm{alc}}$ for every $\lambda \in \mathbb{R}$. We denote by $\mathrm{alc}(\Gamma\backslash G_v)$ the vector space of bounded algebraically locally constant maps ψ from $\Gamma\backslash G_v$ to $\mathbb{R}$.

For every $n \in \mathbb{N}$, let U_n be the compact-open subgroup of $\mathrm{PGL}_2(\mathscr{O}_v)$ which is the kernel of the morphism $\mathrm{PGL}_2(\mathscr{O}_v) \to \mathrm{PGL}_2(\mathscr{O}_v/{\pi_v}^n\mathscr{O}_v)$ of reduction modulo ${\pi_v}^n$. Note that any compact-open subgroup U of $\mathrm{PGL}_2(\mathscr{O}_v)$ contains U_n for some $n \in \mathbb{N}$. Hence $\psi : \Gamma\backslash G_v \to \mathbb{R}$ is algebraically locally constant if and only if there exists $n \in \mathbb{N}$ such that ψ is constant on each right orbit of U_n. For every $n \in \mathbb{N}$, since the order of $\mathrm{PGL}_2(\mathscr{O}_v/{\pi_v}^n\mathscr{O}_v)$ is at most the order of $(\mathscr{O}_v/{\pi_v}^n\mathscr{O}_v)^4$, which is ${q_v}^{4n}$, if $\psi : \Gamma\backslash G_v \to \mathbb{R}$ is constant on each right orbit of U_n, then

$$\|\psi\|_{\mathrm{alc}} \leqslant {q_v}^{2n}\,\|\psi\|_\infty \;. \tag{15.15}$$

Recall[24] that we have a natural homeomorphism $\Xi : \Gamma g \underline{A}(\mathscr{O}_v) \mapsto \Gamma g\,\ell^*$ between the double coset space $\Gamma\backslash G_v/\underline{A}(\mathscr{O}_v)$ and $\Gamma\backslash\mathscr{G}\mathbb{X}_v$. We denote by $p_{\mathscr{G}} : \Gamma\backslash G_v \to \Gamma\backslash\mathscr{G}\mathbb{X}_v$ the composition map of the canonical projection $\big(\Gamma\backslash G_v\big) \to \big(\Gamma\backslash G_v/\underline{A}(\mathscr{O}_v)\big)$ and of Ξ. By Equation (15.4), for all $x \in \Gamma\backslash G_v$ and $n \in \mathbb{N}$, we have

$$p_{\mathscr{G}}\,(x\;{a_v}^n) = \mathsf{g}^n\,p_{\mathscr{G}}(x) \;. \tag{15.16}$$

Lemma 15.9. *For every $\epsilon \in \,]0,1]$, for every ϵ-locally constant function $\psi : \Gamma\backslash\mathscr{G}\mathbb{X}_v \to \mathbb{R}$, if $n = \lceil -\frac{1}{2}\ln\epsilon\rceil$, then the map $\psi\circ p_{\mathscr{G}} : \Gamma\backslash G_v \to \mathbb{R}$ is U_n-invariant and*

$$\|\psi\circ p_{\mathscr{G}}\|_{\mathrm{alc}} \leqslant {q_v}^2\,\|\psi\|_{\epsilon\,\mathrm{lc},\,\ln q_v} \;. \tag{15.17}$$

[23] This assumption is introduced in order to apply the following Theorem 15.10. Note that the existence of a nonuniform lattice in $G_v = \mathrm{PGL}_2(K_v)$ forces the characteristic to be positive, see for instance [Lub1].

[24] See Section 15.1.

Proof. Let ϵ, ψ, n be as in the statement. Let us first prove that if $\ell, \ell' \in \mathscr{G}\mathbb{X}_v$ satisfy $\ell_{[-n,+n]} = \ell'_{[-n,+n]}$, then $d(\ell, \ell') \leqslant \epsilon$.

If $\ell_{[-n,+n]} = \ell'_{[-n,+n]}$, then $d(\ell(t), \ell'(t)) = 0$ for $t \in [-n, n]$ and by the triangle inequality $d(\ell(t), \ell'(t)) \leqslant 2(|t| - n)$ if $|t| \geqslant n$, hence

$$\begin{aligned} d(\ell, \ell') &\leqslant 2 \int_n^{+\infty} 2\,(t-n)\, e^{-2\,t}\, dt = 2\, e^{-2\,n} \int_0^{+\infty} u\, e^{-u} \,\frac{du}{2} = e^{-2\,n} \\ &\leqslant e^{-2(-\frac{1}{2}\ln \epsilon)} = \epsilon\ , \end{aligned}$$

as wanted.

In order to prove that $\psi \circ p_{\mathscr{G}} : \Gamma\backslash G_v \to \mathbb{R}$ is U_n-invariant, let $x, x' \in \Gamma\backslash G_v$ be such that $x' \in x\, U_n$. Since U_n acts by the identity map on the ball of radius n in the Bruhat-Tits tree $\mathbb{X}_v$, the geodesic lines $p_{\mathscr{G}}(x)$ and $p_{\mathscr{G}}(x')$ in $\Gamma\backslash\mathscr{G}\mathbb{X}_v$ coincide (at least) on $[-n, n]$. Hence, as we saw in the beginning of the proof, we have $d(p_{\mathscr{G}}(x), p_{\mathscr{G}}(x')) \leqslant \epsilon$. Therefore $\psi(p_{\mathscr{G}}(x)) = \psi(p_{\mathscr{G}}(x'))$ since ψ is ϵ-locally constant.

Now, using Equation (15.15), we have

$$\begin{aligned} \|\psi \circ p_{\mathscr{G}}\|_{\rm alc} &\leqslant {q_v}^{2n}\, \|\psi \circ p_{\mathscr{G}}\|_\infty \\ &\leqslant {q_v}^{2(1-\frac{1}{2}\ln\epsilon)}\, \|\psi\|_\infty = {q_v}^2\, \epsilon^{-\ln q_v}\, \|\psi\|_\infty = {q_v}^2\, \|\psi\|_{\epsilon\, lc,\, \ln q_v}\ . \quad \Box \end{aligned}$$

Now, we will use an algebraic result of exponential decay of correlations, Theorem 15.10 (see for instance [AtGP]). We first recall some definitions and notation, useful for its statement.

Recall that the left action of the locally compact unimodular group G_v on the locally compact space $\mathscr{G}\mathbb{X}_v$ is continuous and transitive, and that its stabilisers are compact hence unimodular. Since Γ is a lattice, the (Borel, positive, regular) Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ on $\mathscr{G}\mathbb{X}_v$ is G_v-invariant (see Proposition 4.16 (2)). Hence by [Wei1] (see also [GoP, Lem. 5]), there exists a unique Haar measure on G_v, which disintegrates by the evaluation map $\widetilde{p_{\mathscr{G}}} : G_v \to \mathscr{G}\mathbb{X}_v$ defined by $g \mapsto g\ell_*$, with conditional measure on the fiber over $\ell = g\ell_* \in \mathscr{G}\mathbb{X}_v$ the probability Haar measure on the stabiliser $g\underline{A}(\mathscr{O}_v)g^{-1}$ of ℓ under G_v. Hence, taking the quotient under Γ and normalising in order to have probability measures, if μ_v is the right G_v-invariant probability measure on $\Gamma\backslash G_v$, we have

$$(p_{\mathscr{G}})_*\mu_v = \frac{m_{\rm BM}}{\|m_{\rm BM}\|}\ . \tag{15.18}$$

For every $g \in G_v$, we denote by $R_g : \Gamma\backslash G_v \to \Gamma\backslash G_v$ the right action of g, and for all bounded continuous functions $\widetilde{\psi}, \widetilde{\psi}'$ on $\Gamma\backslash G_v$, we define

$${\rm cov}_{\mu_v,\, g}(\widetilde{\psi}, \widetilde{\psi}') = \int_{\Gamma\backslash G_v} \widetilde{\psi} \circ R_g\ \widetilde{\psi}'\, d\mu_v\ -\ \int_{\Gamma\backslash G_v} \widetilde{\psi}\, d\mu_v \int_{\Gamma\backslash G_v} \widetilde{\psi}'\, d\mu_v\ .$$

Note that by Equations (15.18) and (15.16), for all bounded continuous functions $\psi, \psi' : \Gamma\backslash\mathscr{G}\mathbb{X}_v \to \mathbb{R}$ and $n \in \mathbb{Z}$, we have[25]

$${\rm cov}_{\frac{m_{\rm BM}}{\|m_{\rm BM}\|},\, n}(\psi, \psi') = {\rm cov}_{\mu_v,\, {a_v}^n}(\psi \circ p_{\mathscr{G}}, \psi' \circ p_{\mathscr{G}})\ . \tag{15.19}$$

[25] See Section 9.2 for a definition of ${\rm cov}_{\mu,\, n}$.

Recall that the adjoint representation of $G_v = \mathrm{PGL}_2(K_v)$ is the continuous morphism $G_v \to \mathrm{GL}(\mathscr{M}_2(K_v))$ defined by $[h] \mapsto \{x \mapsto hxh^{-1}\}$, which is independent of the choice of the representative $h \in \mathrm{GL}_2(K_v)$ of $[h] \in \mathrm{PGL}_2(K_v)$. For every $g \in G_v$, we denote by $|g|_v$ the operator norm of the adjoint representation of g. For instance, recalling that $a_v = \begin{bmatrix} 1 & 0 \\ 0 & \pi_v^{-1} \end{bmatrix}$, we have, for all $n \in \mathbb{Z}$,

$$|{a_v}^n|_v = {q_v}^{|n|} \; . \tag{15.20}$$

We refer for instance to [AtGP] for the following result of exponential decay of correlations.

Theorem 15.10. *Let Γ be a nonuniform lattice of G_v. There exist $C', \kappa' > 0$ such that, for all bounded locally constant functions $\widetilde{\psi}, \widetilde{\psi}' : \Gamma\backslash G_v \to \mathbb{R}$ and $g \in G_v$,*

$$\big|\operatorname{cov}_{\mu_v,\, g}(\widetilde{\psi}, \widetilde{\psi}')\big| \leqslant C' \,\|\widetilde{\psi}\|_{\rm alc}\, \|\widetilde{\psi}'\|_{\rm alc}\, |g|_v{}^{-\kappa'} \; . \quad \Box \tag{15.21}$$

Proposition 15.7 (1) and therefore Theorem 15.8 (1) with $\beta \geqslant \ln q_v$ follows from this result applied to $\widetilde{\psi} = \psi \circ p_{\mathscr{G}}$, $\widetilde{\psi}' = \psi' \circ p_{\mathscr{G}}$ and $g = {a_v}^n$ by using Equations (15.19), (15.17) and (15.20) and by taking $C = C' q_v^4$ and $\kappa = \kappa' \ln q_v$. $\Box$

Remark. There is a similar relationship between locally constant functions on K_v in an algebraic sense and the ones in the metric sense.

The additive group $(K_v, +)$ acts on the complex vector space of functions from K_v to $\mathbb{R}$, by translations on the source: for all $y \in K_v$ and $\psi : K_v \to \mathbb{R}$, the function $y \cdot \psi$ is equal to $x \mapsto \psi(x + y)$. A function $\psi : K_v \to \mathbb{R}$ is *algebraically locally constant* if there exists $k \in \mathbb{N}$ such that ψ is invariant under the action of the compact-open subgroup $(\pi_v)^k \mathscr{O}_v$ of K_v, that is, if for all $x \in K_v$ and $y \in (\pi_v)^k \mathscr{O}_v$, we have $\psi(x + y) = \psi(x)$. Note that a locally constant function from K_v to $\mathbb{R}$ is continuous.

For any locally constant function $\psi : K_v \to \mathbb{R}$, the complex vector space $\mathrm{Vect}_{\mathbb{R}}(\mathscr{O}_v \cdot \psi)$ generated by the images of ψ under the elements of $\mathscr{O}_v$ is finite dimensional. Its dimension d_ψ satisfies, with k as above,

$$d_\psi \leqslant [\mathscr{O}_v : (\pi_v)^k \mathscr{O}_v] = {q_v}^k \; .$$

We define the alc-*norm* of every bounded algebraically locally constant function $\psi : K_v \to \mathbb{R}$ by

$$\|\psi\|_{\rm alc} = \sqrt{d_\psi}\; \|\psi\|_\infty \; .$$

Though the alc-norm does not satisfy the triangle inequality, we have $\|\lambda\psi\|_{\rm alc} = |\lambda|\, \|\psi\|_{\rm alc}$ for every $\lambda \in \mathbb{R}$, and the set of bounded algebraically locally constant maps from K_v to $\mathbb{R}$ is a real vector space.

Actually, a function $\psi : K_v \to \mathbb{R}$ is algebraically locally constant if and only if it is locally constant. More precisely, for every $\epsilon \in \,]0, 1]$, since the closed balls of radius q_v^{-k} in K_v are the orbits by translations under $(\pi_v)^k \mathscr{O}_v$, every ϵ-locally constant function $\psi : K_v \to \mathbb{R}$ is constant under the additive action of $(\pi_v)^k \mathscr{O}_v$ for $k = \lceil \frac{-\ln \epsilon}{\ln q_v} \rceil$, hence

$$\|\psi\|_{\rm alc} \leqslant \|\psi\|_{\epsilon\,{\rm lc},\, \frac{1}{2}} \; .$$

15.5 Geometrically finite lattices with infinite Bowen-Margulis measure

This Section is a digression from the theme of arithmetic applications, in which we use the Nagao lattice defined in Section 15.2 in order to construct a geometrically finite discrete group of automorphisms of a simplicial tree which has infinite Bowen-Margulis measure. This example was promised after Proposition 4.16, where we saw that such examples do not exist for uniform trees.

We will equivariantly change the lengths of the edges of a simplicial tree $\mathbb{X}$ endowed with a geometrically finite (nonuniform) lattice Γ in order to turn $\mathbb{X}$ into a metric tree in which the group Γ remains a geometrically finite lattice, but now has a geometrically finite subgroup with infinite Bowen-Margulis measure. This example is an adaptation of the negatively curved manifold example of [DaOP, §4]. The simplicial example is obtained as a modification of the metric tree example.

Theorem 15.11. *There exists a geometrically finite discrete group of automorphisms of a metric tree with constant degrees, whose Bowen-Margulis measure is infinite.*

There exists a geometrically finite discrete group of automorphisms of a simplicial tree with uniformly bounded degrees whose Bowen-Margulis measure is infinite.

Proof. Let $v = v_\infty$ be the valuation at infinity of $K = \mathbb{F}_q(Y)$, $K_v = \mathbb{F}_q((Y^{-1}))$, $\mathscr{O}_v = \mathbb{F}_q[[Y^{-1}]]$ and $R_v = \mathbb{F}_q[Y]$. Let $\mathbb{X}_v$ be the Bruhat-Tits tree of (PGL_2, K_v) with base point $*_v = [\mathscr{O}_v \times \mathscr{O}_v]$. Let $\Gamma_v = \mathrm{PGL}_2(R_v)$, which is a lattice of $\mathbb{X}_v$, with quotient graph of groups the modular ray $\Gamma_v \backslash\backslash \mathbb{X}_v$ described in Section 15.2. We denote by $(y_i)_{i=-1,0,\dots}$ the ordered vertices along $\Gamma_v \backslash\backslash \mathbb{X}_v$ with vertex stabilisers $(\Gamma_i)_{i=-1,0,\dots}$, and by $(e_i)_{i\in\mathbb{N}}$ the ordered edges along $\Gamma_v \backslash\backslash \mathbb{X}_v$ (pointing away from the origin of the modular ray).

The subgroup

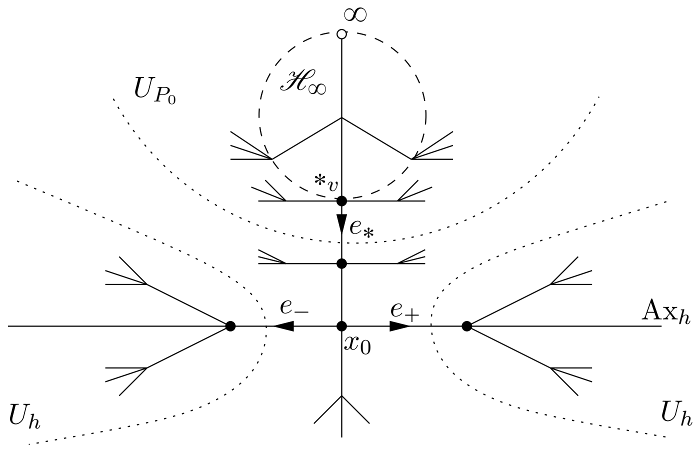
$$P = \bigcup_{i\geqslant 0} \Gamma_i = \left\{ \begin{bmatrix} a & Q \\ 0 & d \end{bmatrix} : Q \in \mathbb{F}_q[Y], a, d \in \mathbb{F}_q^\times \right\}$$

is the stabiliser in Γ of $\infty \in \partial_\infty \mathbb{X}_v$. Let P_0 be the finite index subgroup of P consisting of the elements $\begin{bmatrix} 1 & Q \\ 0 & 1 \end{bmatrix}$ with $Q(0) = 0$. Observing that $d(\gamma *_v, *_v) = 2(i+1)$ for any $\gamma \in \Gamma_i - \Gamma_{i-1}$ and that the cardinality of $(\Gamma_i - \Gamma_{i-1}) \cap P_0$ is $(q-1)q^{i+1}$, it is easy to see that the Poincaré series

$$\mathscr{Q}_{P_0,\,0,\,*_v,\,*_v}(s) = \sum_{\gamma \in P_0} e^{-s\, d(*_v, \gamma *_v)}$$

of the discrete (though elementary) subgroup P_0 of $\mathrm{Isom}(\mathbb{X}_v)$ is (up to a multiplicative constant) equal to $\sum_{i=0}^{\infty} q^i e^{-2si}$, which gives $\delta_{P_0} = \frac{\ln q}{2}$ for the critical exponent of P_0 on $\mathbb{X}_v$.

Let h be an element of Γ_v which is loxodromic on $\mathbb{X}_v$ and whose fixed points belong to the open subset $Y^{-1}\mathscr{O}_v$ of $K_v = \partial_\infty \mathbb{X}_v - \{\infty\}$. Hence the horoball $\mathscr{H}_\infty$ centred at $\infty \in \partial_\infty \mathbb{X}_v$, whose horosphere contains $*_v$, is disjoint from the translation axis Ax_h of h. Note that the stabiliser of $\mathscr{H}_\infty$ in Γ_v is P and that P_0 acts freely on the edges exiting $\mathscr{H}_\infty$. Let $x_0 \in V\mathbb{X}_v$ be the closest point on Ax_h to $\mathscr{H}_\infty$, let e_* be the edge with origin $*_v$ pointing towards x_0, and let e_-, e_+ be the two edges with origin x_0 on Ax_h.



Let U_h be the set of points x in $V\mathbb{X}_v - \{x_0\}$ such that the geodesic segment from x_0 to x starts either by the edge e_- or by e_+. Let U_{P_0} be the set of points y in $V\mathbb{X}_v - \{t(e_*)\}$ such that the geodesic segment from $t(e_*)$ to y starts by the edge $\overline{e_*}$. We have

(1) $U_h \cap U_{P_0} = \varnothing$ and $x_0 \notin U_h \cup U_{P_0}$,

(2) $h^k(V\mathbb{X}_v - U_h) \subset U_h$ for every $k \in \mathbb{Z} - \{0\}$ and $w(V\mathbb{X}_v - U_{P_0}) \subset U_{P_0}$ for every $w \in P_0 - \{\mathrm{id}\}$,

(3) $d(x, y) = d(x, x_0) + d(x_0, y)$ for all $x \in U_h$ and $y \in U_{P_0}$.

Let Γ' be the subgroup of Γ_v generated by P_0 and h. By a ping-pong argument, Γ' is a free product of P_0 and of the infinite cyclic group generated by h, and Γ' is geometrically finite (see for instance [Mask, Theorem C.2 (xi)] for Kleinian groups). Hence every element γ in $\Gamma' - \{e\}$ may be written uniquely as a word $w_0 h^{n_0} w_1 h^{n_1} \dots w_k h^{n_k}$ with $k \in \mathbb{N}$, $w_i \in P_0, n_i \in \mathbb{Z}$ with $w_i \neq e$ if $i \neq 0$ and $n_i \neq 0$ if $i \neq k$. Using the above properties, we have by induction

$$d(x_0, w_0 h^{n_0} w_1 h^{n_1} \dots w_k h^{n_k} x_0) = \sum_{0 \leqslant i \leqslant k} d(x_0, h^{n_i} x_0) + \sum_{0 \leqslant i \leqslant k} d(x_0, w_i x_0)\ . \tag{15.22}$$

Let $\lambda : E\mathbb{X}_v \to \mathbb{R}_+$ be the Γ'-invariant length map on the set of edges of $\mathbb{X}_v$ such that for every $i \in \mathbb{N}$, the length of $e \in E\mathbb{X}_v$ is 1 if e is not contained in $\bigcup_{\gamma \in \Gamma'} \gamma \mathscr{H}_\infty$, and otherwise, if e maps to e_i or to $\overline{e_i}$ under the canonical map $\mathbb{X}_v \to \Gamma_v \backslash \mathbb{X}_v$, then $\lambda(e) = 1 + \ln \frac{i+1}{i}$ if $i \geqslant 1$ and $\lambda(e) = 1$ if $i = 0$. Note that the distance in the metric graph $|\mathbb{X}_v|_\lambda$ from $*_v$ to the vertex on the geodesic ray from $*_v$ to ∞ originally at distance i from $*_v$ is now $i + \ln i$. The distances along the translation axis of h have not changed. Equation (15.22) remains valid with the new distance. Let us denote by d' this (new) distance on $|\mathbb{X}_v|_\lambda$, and by $\mathscr{Q}'_{\Gamma'}(s) = \mathscr{Q}'_{\Gamma', 0, x_0, x_0}(s)$ and $\mathscr{Q}'_{P_0}(s)$ the Poincaré series for the actions of Γ' and P_0 on $(|\mathbb{X}_v|_\lambda, d')$ (taking (x_0, x_0) as pair of basepoints and the zero system of conductances).

Let us now prove that the discrete subgroup Γ' of automorphisms of the metric tree $(\mathbb{X}_v, \lambda)$ (with degrees all equal to $q+1$) satisfies the first claim of Theorem 15.11.

By Γ'-invariance of λ, the group Γ' remains a discrete subgroup of $\mathrm{Aut}(\mathbb{X}_v, \lambda)$. The elements of $\Gamma'\infty$, or equivalently, the points at infinity of the horoballs in the Γ'-equivariant family of horoballs $(\gamma \mathscr{H}_\infty)_{\gamma \in \Gamma'/\Gamma'_\infty}$ with pairwise disjoint interiors in $\mathbb{X}_v$, remain bounded parabolic fixed points of Γ', and the other limit points remain conical limit points of Γ'. Hence Γ' remains a geometrically finite discrete subgroup of $\mathrm{Aut}(\mathbb{X}_v, \lambda)$.

The Poincaré series $\mathscr{Q}'_{P_0}$ is (up to a multiplicative and additive constant) $\sum_{i=1}^{\infty} q^i\, e^{-2si}\, i^{-2s}$, which has the same critical exponent $\delta_{P_0} = \frac{\ln q}{2}$ as previously, but it is easy to see that the discrete group P_0 is now of convergence type if $q \geqslant 3$.

Let $\lambda(h)$ be the (old and new) translation length of h. Using Equation (15.22) and partitionning the above words by the number of nonzero powers of h they contain, we have, for every $s \geqslant 0$,

$$\begin{aligned}\mathscr{Q}'_{\Gamma'}(s) &= \sum_{\gamma\in\Gamma'} e^{-s\, d'(x_0,\gamma x_0)}\\ &\leqslant \Big(1+\sum_{k=1}^{+\infty}\Big(\sum_{i\in\mathbb{Z}-\{0\}} e^{-s\,|i|\,\lambda(h)} \sum_{w\in P_0} e^{-s\, d'(x_0,wx_0)}\Big)^k\Big) \sum_{w\in P_0} e^{-s\, d'(x_0,wx_0)}\,.\end{aligned}$$

At $s=\delta_{P_0}$, the above sums over $i\in\mathbb{Z}-\{0\}$ and over $w\in P_0$ converge, and by replacing h by a high enough power if necessary, we may assume that $\sum_{i\in\mathbb{Z}-\{0\}} e^{-s\,|i|\lambda(h)} \sum_{w\in P_0} e^{-s\, d'(x_0,wx_0)} < 1$ if $s=\delta_{P_0}$, which makes the Poincaré series of Γ' converge at $s=\delta_{P_0}$. Since P_0 is a subgroup of Γ' with critical exponent δ_P, we have that $\mathscr{Q}'_{\Gamma'}(s) \geqslant \mathscr{Q}'_{P_0}(s) = +\infty$ if $s<\delta_{P_0}$. We conclude that the critical exponent of Γ' is equal to δ_{P_0} and that Γ' is of convergence type. By Corollary 4.7 (1), the Bowen-Margulis measure of Γ' is infinite.

In order to prove the second claim of Theorem 15.11, we first define a new length map $\lambda : E\mathbb{X}_v \to \mathbb{R}_+$ which coincides with the previous one on every edge e of $\mathbb{X}_v$, unless e maps to e_i or to $\overline{e_i}$ for any $i\in\mathbb{N}$ under the canonical map $\mathbb{X}_v \to \Gamma\backslash\mathbb{X}_v$, in which case we set

$$\lambda(e) = 1 + \lfloor \ln(i+1)\rfloor - \lfloor \ln i\rfloor$$

if $i\geqslant 1$ and $\lambda(e)=1$ if $i=0$ (where $\lfloor\cdot\rfloor$ is the largest previous integer map). This map λ now has values in $\{1,2\}$, and we subdivide each edge of length 2 into two edges of length 1. The tree $\mathbb{Y}$ thus obtained has uniformly bounded degrees (although it is no longer a uniform tree), and the group Γ' defines a geometrically finite discrete subgroup of $\mathrm{Aut}(\mathbb{Y})$ with infinite Bowen-Margulis measure. To see that P_0 is again of convergence type, observe that each distance in the simplicial case appearing in the Poincaré series differs by at most 1 from the corresponding distance in the metric tree case. The remainder of the argument is the same as in the metric tree case. □

Chapter 16

Equidistribution and counting of rational points in completed function fields

Let K be a (global) function field over $\mathbb{F}_q$ of genus g, let v be a (normalised discrete) valuation of K, let K_v be the associated completion of K and let R_v be the affine function ring associated with v.[1] In this Chapter, we prove analogues of the classical results on the counting and equidistribution towards the Lebesgue measure on $\mathbb{R}$ of the Farey fractions $\frac{p}{q}$ with $(p,q) \in \mathbb{Z}\times(\mathbb{Z}-\{0\})$ relatively prime.[2] In particular, we prove various equidistribution results of locally finite families of elements of K towards the Haar measure on K_v, using the geometrical work on equidistribution of common perpendiculars done in Section 11.4 and recalled in Section 15.4.

16.1 Counting and equidistribution of non-Archimedean Farey fractions

The first result of this Section is an analog in function fields of the equidistribution of Farey fractions to the Lebesgue measure in $\mathbb{R}$, see the Introduction, and for example [PaP14b, p. 978] for the precise statement and a geometric proof. For every $(x_0, y_0) \in R_v \times R_v - \{(0,0)\}$, let

$$m_{v,\,x_0,\,y_0} = \mathrm{Card}\{a \in (R_v)^\times \;:\; \exists\, b \in x_0 R_v \cap y_0 R_v,\ (a-1)x_0 y_0 - b x_0 \in y_0^2 R_v\}\,.$$

For future use, note that by Equation (14.3)

$$m_{v,\,1,\,0} = q - 1\,. \tag{16.1}$$

For every $(a,b) \in R_v \times R_v$ and every subgroup H of $\mathrm{GL}_2(R_v)$, let $H_{(a,b)}$ be the stabiliser of (a,b) for the linear action of H on $R_v \times R_v$.

Theorem 16.1. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, and let $(x_0, y_0) \in R_v \times R_v - \{(0,0)\}$. Let*

$$c = \frac{({q_v}^2 - 1)\,(q_v + 1)\,\zeta_K(-1)\,m_{v,\,x_0,\,y_0}\,(N\langle x_0, y_0\rangle)^2\,[\mathrm{GL}_2(R_v) : G]}{(q-1)\,q^{\,g-1}\,q_v^3\,[\mathrm{GL}_2(R_v)_{(x_0,y_0)} : G_{(x_0,y_0)}]}\,.$$

[1] See Section 14.2 for definitions and background.

[2] See for instance [Nev], as well as [PaP14b] for an approach using methods similar to the ones in this text.

Then, as $s \to +\infty$,

$$c\, s^{-2} \sum_{(x,\,y) \in G(x_0,\, y_0),\ |y|_v \leqslant s} \Delta_{\frac{x}{y}} \ \overset{*}{\rightharpoonup}\ \mathrm{Haar}_{K_v}\ .$$

For every $\beta \in \,]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every β-Hölder-continuous function $\psi : K_v \to \mathbb{R}$ with compact support,[3] as for instance if $\psi : K_v \to \mathbb{R}$ is locally constant with compact support (see Remark 3.11), there is an error term in the equidistribution claim of Theorem 16.1 evaluated on ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\beta)$.

It is remarkable that due to the general nature of our geometrical tools, we are able to work with any finite index subgroup G of $\mathrm{GL}_2(R_v)$, and not only with its congruence subgroups. In this generality, the usual techniques (for instance involving analysis of Eisenstein series) are not likely to apply. Also note that the Hölder regularity for the error term is a much weaker assumption than the locally constant one that is usually obtained by analytic number theory methods.

Theorem 1.13 in the Introduction follows from this result, by taking $K = \mathbb{F}_q(Y)$ (so that $g = 0$), $v = v_\infty$, $q_v = q$, $(x_0, y_0) = (1, 0)$ and $s = q^t$, and by using Equations (14.6) and (16.1) in order to simplify the constant.

Before proving Theorem 16.1, let us give a counting result which follows from this equidistribution result.

The additive group R_v acts on $R_v \times R_v$ by the horizontal shears (transvections):

$$\forall\, z \in R_v,\ \forall\, (x, y) \in R_v \times R_v, \quad z \cdot (x, y) = (x + zy, y)\ ,$$

and this action preserves the absolute value $|y|_v$ of the vertical coordinate y. Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, and let $x_0, y_0 \in R_v$. Let $R_{v,G}$ be the finite index additive subgroup of R_v consisting of the elements $x \in R_v$ such that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G$, acting by horizontal shears on $G(x_0,\, y_0)$. Note that $R_{v,G} = R_v$ if $G = \mathrm{GL}_2(R_v)$. We may then define a *counting function* $\Psi_{G,\,x_0,y_0}$ of the elements of K in an orbit by homographies under G, as

$$\Psi_{G,\,x_0,\,y_0}(s) = \mathrm{Card}\ \ R_{v,\,G}\backslash\big\{(x,\ y) \in G(x_0,\ y_0),\ \ |y|_v \leqslant s\big\}\ .$$

Corollary 16.2. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, and let $(x_0, y_0) \in R_v \times R_v - \{(0,0)\}$. Then there exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\Psi_{G,\,x_0,y_0}(s) = \frac{(q-1)\, q^{2g-2}\, q_v^3\, [\mathrm{GL}_2(R_v)_{(x_0,y_0)} : G_{(x_0,y_0)}]\, [R_v : R_{v,\,G}]}{({q_v}^2-1)\, (q_v+1)\, \zeta_K(-1)\, m_{v,\,x_0,\,y_0}\, (N\langle x_0, y_0\rangle)^2\, [\mathrm{GL}_2(R_v) : G]}\, s^2 + \mathrm{O}(s^{2-\kappa})\ .$$

Proof. This follows from Theorem 16.1, by considering the locally constant characteristic function of a closed and open fundamental domain of K_v modulo the action by translations of $R_{v,\,G}$, and by using Lemma 14.4 with $I = R_v$. □

Let us fix some notation for this Section. For every subgroup H of $\mathrm{GL}_2(R_v)$, we denote by $\overline{H}$ its image in $\Gamma_v = \mathrm{PGL}_2(R_v)$. Let $\mathbb{X}_v$ be the Bruhat-Tits tree[4] of (PGL_2, K_v), which is regular of degree $q_v + 1$. Let

$$r = \frac{x_0}{y_0} \in K \cup \{\infty\}\,.$$

[3] where K_v is endowed with the distance $(x, y) \mapsto |x - y|_v$

[4] See Section 15.1.

If $y_0 = 0$, let $g_r = \mathrm{id} \in \mathrm{GL}_2(K)$, and if $y_0 \neq 0$, let

$$g_r = \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(K) \;.$$

Proof of Theorem 16.1. We apply Theorem 15.4 with $\Gamma = \overline{G}$, $\mathbb{D}^- = \mathscr{H}_\infty$ and $\mathbb{D}^+ = g_r\mathscr{H}_\infty$. Recall that $\mathscr{H}_\infty$ is the horoball in $\mathbb{X}_v$ centred at ∞ whose boundary contains $*_v$ (see Section 15.1).

Note that Γ has finite index in Γ_v and, in particular, it is a lattice of $\mathbb{X}_v$. By [Ser3, II.1.2, Coro.], for all $x \in V\mathbb{X}_v$ and $\gamma \in \mathrm{GL}_2(R_v)$, the distance $d(x, \gamma x)$ is even since $v(\det \gamma) = 0$. Hence by the equivalence in Equation (4.17), the length spectrum L_{Γ_v} of Γ_v is $2\mathbb{Z}$. The length spectrum of Γ is also $2\mathbb{Z}$, since it is contained in L_{Γ_v}.

Note that $\mathbb{D}^+$ is a horoball in $\mathbb{X}_v$ centred at $r = \frac{x_0}{y_0} \in \partial_\infty\mathbb{X}_v$, by Lemma 15.1. The stabiliser $\Gamma_{\mathbb{D}^-}$ of $\mathbb{D}^-$ (respectively $\Gamma_{\mathbb{D}^+}$ of $\mathbb{D}^+$) coincides with the stabiliser Γ_∞ of $\infty \in \partial_\infty\mathbb{X}_v$ (respectively the stabiliser Γ_r of r) in Γ. Note that the families $\mathscr{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite, since Γ_v, and hence its finite index subgroup Γ, is geometrically finite,[5] and since ∞ and $r \in K$ are bounded parabolic limit points of Γ_v, hence of its finite index subgroup Γ.

For every $\gamma \in \Gamma/\Gamma_r$ such that $\mathbb{D}^-$ and $\gamma\mathbb{D}^+$ are disjoint, let $\alpha^-_{e,\gamma}$ be the generalised geodesic line, isometric exactly on $[0, d(\mathbb{D}^-, \gamma\mathbb{D}^+)]$, whose image is the common perpendicular between $\mathbb{D}^-$ and $\gamma\mathbb{D}^+$, and let ρ_γ be the geodesic ray starting (at time $t = 0$) from $\alpha^-_{e,\gamma}(0)$ and ending at the point at infinity $\gamma \cdot r$ of $\gamma\mathbb{D}^+$. Note that ρ_γ and $\alpha^-_{e,\gamma}$ coincide on $[0, d(\mathbb{D}^-, \gamma\mathbb{D}^+)]$.

Since the Patterson densities of lattices of $\mathbb{X}_v$ have total mass $\frac{q_v+1}{q_v}$ by Proposition 15.2 (2), they are normalised as in Theorem 15.4. Then by Equation (15.14), we have

$$\lim_{n\to+\infty} \frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3}\;\frac{\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;{q_v}^{-n} \sum_{\substack{\gamma\in\Gamma/\Gamma_r\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}} \Delta_{(\rho_\gamma)_+} \;=\; (\partial^+)_*\widetilde{\sigma}^+_{\mathbb{D}^-} \;. \tag{16.2}$$

Furthermore, for every $\beta \in \,]0, \frac{1}{\ln q_v}]$, by the comment following Equation (15.14), we have an error term of the form $\mathrm{O}(e^{-\kappa\, n}\, \|\psi\|_{\beta \ln q_v})$ for some $\kappa > 0$ in the above formula when evaluated on $\psi \in \mathscr{C}_c^{\beta \ln q_v}(\partial_\infty\mathbb{X}_v - \{\infty\})$, where $\partial_\infty\mathbb{X}_v - \{\infty\}$ is endowed with Hamenstädt's distance[6] $d_{\mathscr{H}_\infty}$. Hence we have an error term $\mathrm{O}(e^{-\kappa\, n}\, \|\psi\|_\beta)$ for some $\kappa > 0$ in the above formula when evaluated on $\psi \in \mathscr{C}_c^\beta(K_v)$, where $K_v = \partial_\infty\mathbb{X}_v - \{\infty\}$ is endowed with the distance $(x, y) \mapsto |x - y|_v$, see Equation (15.3).

By Proposition 15.2 (3), we have

$$(\partial^+)_*\widetilde{\sigma}^+_{\mathbb{D}^-} = \mathrm{Haar}_{K_v} \;.$$

Hence Equation (16.2) gives, with the appropriate error term,

$$\lim_{n\to+\infty} \frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3}\;\frac{\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;{q_v}^{-n} \sum_{\substack{\gamma\in\Gamma/\Gamma_r\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}} \Delta_{\gamma\cdot r} \;=\; \mathrm{Haar}_{K_v} \;. \tag{16.3}$$

[5] See Section 15.2.

[6] See Equation (2.12), and note that with $D = \mathscr{H}_\infty$, Hamenstädt's distance $d_{\mathscr{H}_\infty}$ coincides with the distance-like function d_D introduced in Equation (3.8) and used in Remark 15.6, since D is a horoball.

Let $g \in \mathrm{GL}_2(K)$ be such that $g\infty \neq \infty$. This condition is equivalent to asking that the $(2,1)$-entry $c = c(g)$ of g is nonzero. By Lemma 15.1, the signed distance between the horospheres $\mathscr{H}_\infty$ and $g\mathscr{H}_\infty$ is

$$d(\mathscr{H}_\infty, g\mathscr{H}_\infty) = -2\, v(c) = 2\, \frac{\ln |c|_v}{\ln q_v}\ . \tag{16.4}$$

If $y_0 \neq 0$, then $(x, y) = g(x_0, y_0)$ if and only if $(\frac{x}{y_0}, \frac{y}{y_0}) = gg_r(1, 0)$, and the $(2,1)$-entry of gg_r is $\frac{y}{y_0}$. If $y_0 = 0$ (which implies that $g_r = \mathrm{id}$ and $x_0 \neq 0$), then $(x, y) = g(x_0, y_0)$ if and only if $(\frac{x}{x_0}, \frac{y}{x_0}) = g(1, 0)$, and the $(2,1)$-entry of $g = gg_r$ is $\frac{y}{x_0}$. Let

$$z_0 = \begin{cases} y_0 & \text{if } y_0 \neq 0 \\ x_0 & \text{otherwise.} \end{cases}$$

By Equation (16.4), the signed distance between $\mathbb{D}^- = \mathscr{H}_\infty$ and $g\mathbb{D}^+ = g\, g_r \mathscr{H}_\infty$ is

$$d(\mathbb{D}^-, g\mathbb{D}^+) = \frac{2}{\ln q_v}\ \ln \Big|\frac{y}{z_0}\Big|_v\ .$$

By discreteness, there are only finitely many double classes $[g] \in G_{(1,0)}\backslash G / G_{(x_0,y_0)}$ such that $\mathbb{D}^- = \mathscr{H}_\infty$ and $g\mathbb{D}^+ = g\, g_r \mathscr{H}_\infty$ are not disjoint. Let $Z(G)$ be the centre of G, which is finite. Since $Z(G)$ acts trivially on $\mathbb{P}_1(K_v)$, the map $G/G_{(x_0,y_0)} \to \Gamma/\Gamma_r$ induced by the canonical map $\mathrm{GL}_2(R_v) \to \mathrm{PGL}_2(R_v)$ is $|Z(G)|$-to-1. Using the change of variable

$$s = |z_0|_v\ {q_v}^{\frac{n}{2}}\ ,$$

so that ${q_v}^{-n} = {|z_0|_v}^2\ s^{-2}$, Equation (16.3) gives, with the appropriate error term,

$$\begin{aligned}\lim_{s\to+\infty} \frac{({q_v}^2-1)\ (q_v+1)\ {|z_0|_v}^2}{2\ {q_v}^3\ |Z(G)|}\ \frac{\mathrm{Vol}(\Gamma\backslash\!\backslash \mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\ s^{-2} \sum_{(x,\,y)\in G(x_0,\,y_0),\ \ |y|_v\leqslant s} \Delta_{\frac{x}{y}} \\ = \mathrm{Haar}_{K_v}\ .\end{aligned} \tag{16.5}$$

The order of the centre $Z(GL_2(R_v)) = (R_v)^\times\,\mathrm{id}$ is $q-1$ by Equation (14.3). The map $\mathrm{GL}_2(R_v)/G \to \Gamma_v/\Gamma$ induced by the canonical map $\mathrm{GL}_2(R_v) \to \mathrm{PGL}_2(R_v)$ is hence $\frac{q-1}{|Z(G)|}$-to-1. By Equation (15.7), we hence have

$$\begin{aligned}\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v) &= [\Gamma_v : \Gamma]\,\mathrm{Vol}(\Gamma_v\backslash\!\backslash\mathbb{X}_v) = 2\ \zeta_K(-1)\,[\Gamma_v:\Gamma] \\ &= \frac{2}{q-1}\ \zeta_K(-1)\ |Z(G)|\ [\mathrm{GL}_2(R_v) : G]\ .\end{aligned} \tag{16.6}$$

Theorem 16.1 follows from Equations (16.5) and (16.6) and from Lemma 16.3 below. □

Lemma 16.3. *We have*

$$\|\sigma^-_{\mathscr{D}^+}\| = \frac{q^{\,g-1}\ {|z_0|_v}^2\ [\mathrm{GL}_2(R_v)_{(x_0,y_0)} : G_{(x_0,y_0)}]}{m_{v,\,x_0,\,y_0}\ (N\langle x_0, y_0\rangle)^2}\ .$$

Proof. Let γ_r be the image of g_r in $\mathrm{PGL}_2(K)$. Let us define $\Gamma' = {\gamma_r}^{-1}\Gamma\gamma_r$, which is a finite index subgroup in $\Gamma'_v = {\gamma_r}^{-1}\Gamma_v\gamma_r$ and a lattice of $\mathbb{X}_v$. Since γ_r maps ∞ to r, the point ∞ is a bounded parabolic limit point of Γ', and we have $(\Gamma')_\infty = {\gamma_r}^{-1}\Gamma_r\gamma_r$. Since the canonical map $\mathrm{GL}_2(R_v) \to \mathrm{PGL}_2(R_v)$ is injective on the stabiliser $\mathrm{GL}_2(R_v)_{(x_0,y_0)}$, we have

$$[(\Gamma'_v)_\infty : (\Gamma')_\infty] = [(\Gamma_v)_r : \Gamma_r] = [\mathrm{GL}_2(R_v)_{(x_0,y_0)} : G_{(x_0,y_0)}] \,.$$

Since the Patterson density of a lattice does not depend on the lattice (see Proposition 4.16 (1)), the skinning measures $\widetilde{\sigma}^{\pm}_{\mathscr{H}}$ of a given horoball $\mathscr{H}$ do not depend on the lattice. Thus

$$\gamma_*\, \widetilde{\sigma}^{\pm}_{\mathscr{H}} = \widetilde{\sigma}^{\pm}_{\gamma\mathscr{H}}$$

for every $\gamma \in \mathrm{Aut}(\mathbb{X}_v)$. Let $\mathscr{D}^+_1 = (\gamma'\mathscr{H}_\infty)_{\gamma'\in\Gamma'/\Gamma'_\infty}$, which is a locally finite Γ'-equivariant family of horoballs. We hence have, using Proposition 15.2 (4) for the third equality,

$$\begin{aligned}\|\sigma^-_{\mathscr{D}^+}\| &= \|\sigma^-_{\gamma_r\mathscr{D}^+_1}\| = \|\sigma^-_{\mathscr{D}^+_1}\| = \mathrm{Haar}_{K_v}((\Gamma')_\infty\backslash K_v)\\ &= [(\Gamma'_v)_\infty : (\Gamma')_\infty]\ \mathrm{Haar}_{K_v}((\Gamma'_v)_\infty\backslash K_v)\\ &= [\mathrm{GL}_2(R_v)_{(x_0,y_0)} : G_{(x_0,y_0)}]\ \mathrm{Haar}_{K_v}((\Gamma'_v)_\infty\backslash K_v)\,. \end{aligned} \tag{16.7}$$

Every element in the stabiliser of ∞ in $\mathrm{PGL}_2(K_v)$ can be uniquely written in the form $\alpha = \begin{bmatrix} a & b\\ 0 & 1\end{bmatrix}$ with $(a,b) \in (K_v)^\times \times K_v$. Note that

$$\begin{pmatrix} r & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & -r\end{pmatrix} = \begin{pmatrix} br+1 & ar-br^2-r\\ b & a-br\end{pmatrix}\,.$$

When $x_0 = 0$ or $y_0 = 0$, we have $\alpha \in \Gamma'_v$ if and only if

$$b \in R_v \quad\text{and}\quad a \in (R_v)^\times\,.$$

When $x_0, y_0 \neq 0$, we have $\alpha \in \Gamma'_v$ if and only if $\gamma_r\alpha{\gamma_r}^{-1} \in \Gamma_v$, hence if and only if

$$b \in R_v \cap \frac{1}{r}R_v, \quad a \in (R_v)^\times, \quad ar - br^2 - r \in R_v\,.$$

Let U'_∞ be the kernel of the group morphism from $(\Gamma'_v)_\infty$ to $(K_v)^\times$ sending $\begin{bmatrix} a & b\\ 0 & 1\end{bmatrix}$ to a, and let m_v be its index in $(\Gamma'_v)_\infty$. If $x_0 = 0$ or $y_0 = 0$, then m_v is equal to $|(R_v)^\times|$, so that, by Equation (14.3),

$$m_v = |(R_v)^\times| = |(\mathbb{F}_q)^\times| = q-1\,.$$

If $x_0, y_0 \neq 0$, we have

$$m_v = \mathrm{Card}\,\{a \in (R_v)^\times \;:\; \exists\, b \in R_v \cap \frac{1}{r}R_v,\;\; ar - br^2 - r \in R_v\}\,.$$

Note that the notation m_v coincides with the constant $m_{v,\,x_0,\,y_0}$ defined before the statement of Theorem 16.1 in both cases.

If $I_{(x_0,y_0)}$ is the nonzero fractional ideal

$$I_{(x_0,y_0)} = \begin{cases} R_v & \text{if } x_0 = 0 \text{ or } y_0 = 0\,,\\ R_v \cap \frac{1}{r}R_v \cap \frac{1}{r^2}R_v & \text{otherwise,}\end{cases}$$

then

$$U'_\infty = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \;:\; b \in I_{(x_0,y_0)} \right\}.$$

Therefore by Lemma 14.4,

$$\operatorname{Haar}_{K_v}((\Gamma'_v)_\infty \backslash K_v) = \frac{\operatorname{Haar}_{K_v}(I_{(x_0,y_0)} \backslash K_v)}{[(\Gamma'_v)_\infty : U'_\infty]} = \frac{q^{\,g-1}\, N(I_{(x_0,y_0)})}{m_v} \,. \tag{16.8}$$

Let $(x_0) = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}(x_0)}$ and $(y_0) = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}(y_0)}$ be the prime decompositions of the principal ideals (x_0) and (y_0). By the formulas of the prime decompositions of intersections, sums and products of ideals in Dedekind rings (see for instance [Nar, §1.1] and Equation (14.4)), we have

$$(x_0^2) \cap (x_0 y_0) \cap (y_0^2) = (x_0^2) \cap (y_0^2) = \prod_{\mathfrak{p}} \mathfrak{p}^{2\max\{\nu_{\mathfrak{p}}(x_0),\, \nu_{\mathfrak{p}}(y_0)\}}$$

and

$$\langle x_0, y_0 \rangle = \prod_{\mathfrak{p}} \mathfrak{p}^{\min\{\nu_{\mathfrak{p}}(x_0),\, \nu_{\mathfrak{p}}(y_0)\}} \,.$$

By the definition of the ideal $I_{(x_0,y_0)}$, by the multiplicativity of the norm, and by Equation (14.5), we hence have if $x_0 \neq 0$ and $y_0 \neq 0$

$$\frac{N(I_{(x_0,y_0)})\; (N\langle x_0, y_0\rangle)^2}{|y_0|_v^{\;2}} = N\Big(\big((x_0^2) \cap (x_0 y_0) \cap (y_0^2)\big) \langle x_0, y_0\rangle^2 (x_0)^{-2} (y_0)^{-2}\Big) = 1 \,. \tag{16.9}$$

If $x_0 = 0$ or $y_0 = 0$, then

$$N(I_{(x_0,y_0)}) = N(R_v) = 1 \,. \tag{16.10}$$

Lemma 16.3 follows from Equations (16.7), (16.8) and (16.9) if $x_0 \neq 0$ and $y_0 \neq 0$ or (16.10) if $x_0 = 0$ or $y_0 = 0$. □

Let us state one particular case of Theorem 16.1 in an arithmetic setting, using a congruence sugbroup.

Theorem 16.4. *Let I be a nonzero ideal of R_v. Then as $t \to +\infty$, we have*

$$\frac{(q_v^{\,2} - 1)\, (q_v + 1)\, \zeta_K(-1)\, N(I) \prod_{\mathfrak{p}|I} (1 + \frac{1}{N(\mathfrak{p})})}{q^{\,g-1}\, q_v^{\,3}}\, (q_v)^{-2t} \sum_{\substack{(x,y) \in R_v \times I \\ \langle x, y\rangle = R_v,\; v(y) \geqslant -t}} \Delta_{\frac{x}{y}} \;\overset{*}{\rightharpoonup}\; \operatorname{Haar}_{K_v} \,,$$

where the product ranges over the prime factors $\mathfrak{p}$ of the ideal I. Furthermore, if

$$\Psi(t) = \operatorname{Card}\; R_v \backslash \big\{ (x,y) \in R_v \times I \;:\; \langle x, y \rangle = R_v,\; v(y) \geqslant -t \big\} \,,$$

then there exists $\kappa > 0$ such that, as $t \to +\infty$,

$$\Psi(t) = \frac{q^{\,2g-2}\, q_v^{\,3}}{(q_v^{\,2} - 1)\, (q_v + 1)\, \zeta_K(-1)\, N(I) \prod_{\mathfrak{p}|I}(1 + \frac{1}{N(\mathfrak{p})})}\, q_v^{\;2t} + \operatorname{O}(q_v^{\;(2-\kappa)t}) \,.$$

For every $\beta \in \,]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every $\psi \in \mathscr{C}_c^\beta(K_v)$ there is an error term in the above equidistribution claim evaluated on ψ, of the form $\operatorname{O}(q_v^{-\kappa\, t}\, \|\psi\|_\beta)$.

Proof. The counting claim is deduced from the equidistribution claim in the same way that Corollary 16.2 is deduced from Theorem 16.1, noting that the action of R_v by horizontal shears preserves $R_v \times I$.

In order to prove the equidistribution claim, we apply Theorem 16.1 with $(x_0, y_0) = (1, 0)$ and with G the Hecke congruence subgroup

$$G_I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2(R_v) \;:\; c \in I \right\}, \tag{16.11}$$

which is the preimage of the upper triangular subgroup of $\operatorname{GL}_2(R_v/I)$ by reduction modulo I. In this case, the constant m_{v,x_0,y_0} appearing in the statement of Theorem 16.1 is equal to $q-1$ by Equation (16.1). The group G_I has finite index in $\operatorname{GL}_2(R_v)$. The following result is well-known to arithmetic readers (see for instance [Shi, page 24] when R_v is replaced by $\mathbb{Z}$), we only give a sketch of proof (indicated to us by J.-B. Bost) for the sake of the geometer readers.

Lemma 16.5. *We have*

$$[\operatorname{GL}_2(R_v) : G_I] = N(I) \prod_{\mathfrak{p}|I} (1 + \frac{1}{N(\mathfrak{p})}) \,,$$

where the product ranges over the prime factors $\mathfrak{p}$ *of the ideal* I.

Proof. Recall that we denote by $|E|$ the cardinality of a finite set E. For every commutative ring A with finite group of invertible elements $A^\times$, we have a disjoint union

$$\operatorname{GL}_2(A) = \bigcup_{a \in A^\times} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \operatorname{SL}_2(A) \,.$$

Hence $[\operatorname{GL}_2(A) : \operatorname{SL}_2(A)] = |A^\times|$. Since $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ belongs to G_I for all $a \in (R_v)^\times$, we have

$$G_I = \bigcup_{a \in (R_v)^\times} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} G_I \cap \operatorname{SL}_2(R_v) \,,$$

so that $[\operatorname{GL}_2(R_v) : G_I] = [\operatorname{SL}_2(R_v) : G_I \cap \operatorname{SL}_2(R_v)]$.

The group morphism of reduction modulo I from $\operatorname{SL}_2(R_v)$ to $\operatorname{SL}_2(R_v/I)$ is onto, by an argument of further reduction to the various prime power factors of I and of lifting elementary matrices. The order of the upper triangular subgroup of $\operatorname{SL}_2(R_v/I)$ is $|(R_v/I)^\times|\,|R/I|$, where $(R_v/I)^\times$ is the group of invertible elements of the ring R_v/I (that we will see again below). Hence

$$\begin{aligned}[\operatorname{GL}_2(R_v) : G_I] = [\operatorname{SL}_2(R_v) : G_I \cap \operatorname{SL}_2(R_v)] &= \frac{|\operatorname{SL}_2(R_v/I)|}{|(R_v/I)^\times|\,|R/I|} \\ &= \frac{|\operatorname{GL}_2(R_v/I)|}{|(R_v/I)^\times|^2 |R/I|} \,. \end{aligned} \tag{16.12}$$

By the multiplicativity of the norm and by the Chinese remainder theorem,[7] one reduces the result to the case when $I = \mathfrak{p}^n$ is the n-th power of a fixed prime ideal $\mathfrak{p}$ with norm $N(\mathfrak{p}) = N$, where $n \in \mathbb{N}$. Note that since $R_v/\mathfrak{p}$ is a field, we have

$$|\operatorname{GL}_1(R_v/\mathfrak{p})| = |(R_v/\mathfrak{p})^\times| = |R_v/\mathfrak{p}| - 1 = N - 1$$

and

$$|\operatorname{GL}_2(R_v/\mathfrak{p})| = (|R_v/\mathfrak{p}|^2 - 1)(|R_v/\mathfrak{p}|^2 - |R_v/\mathfrak{p}|) = N^2(N-1)^2\big(1 + \frac{1}{N}\big)\ .$$

For $k = 1$ or $k = 2$, the kernel of the morphism of reduction modulo $\mathfrak{p}$ from $\operatorname{GL}_k(R_v/I) = \operatorname{GL}_k(R_v/\mathfrak{p}^n)$ to $\operatorname{GL}_k(R_v/\mathfrak{p})$ has order $N^{k^2(n-1)}$. Hence

$$|\operatorname{GL}_2(R_v/I)| = N^{4(n-1)}N^2(N-1)^2\big(1 + \frac{1}{N}\big)\ ,$$

and

$$|(R_v/I)^\times| = N^{n-1}(N-1)\ .$$

Therefore, by Equation (16.12), we have

$$[\operatorname{GL}_2(R_v) : G_I] = \frac{N^{4(n-1)}N^2(N-1)^2\big(1+\frac{1}{N}\big)}{N^{2(n-1)}(N-1)^2N^n} = N^n\big(1 + \frac{1}{N}\big)\ .$$

This proves the result. □

We can now conclude the proof of Theorem 16.4. Note that $\operatorname{GL}_2(R_v)_{(1,0)} = (G_I)_{(1,0)}$. The result then follows from Theorem 16.1 and its Corollary 16.2, using the change of variables $s = (q_v)^t$, since

$$G_I(1,0) = \{(x,y) \in R_v \times I\ :\ \langle x, y\rangle = R_v\}\ . \quad \square$$

The following result is a particular case of Theorem 16.4.

Corollary 16.6. *Let P_0 be a nonzero element of the polynomial ring $R = \mathbb{F}_q[Y]$ over $\mathbb{F}_q$, and let $P_0 = a_0\prod_{i=1}^k (P_i)^{n_i}$ be the prime decomposition of P_0. Then as $t \to +\infty$,*

$$\frac{(q+1)\prod_{i=1}^k q^{n_i \deg P_i}(1+q^{-\deg P_i})}{(q-1)\,q^2}\; q^{-2t} \sum_{\substack{(P,Q)\in R\times(P_0R)\\ PR+QR=R,\ \deg Q\leqslant t}} \Delta_{\frac{P}{Q}} \;\overset{*}{\rightharpoonup}\; \operatorname{Haar}_{\mathbb{F}_q((Y^{-1}))}\ .$$

For every $\beta \in\,]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every $\psi \in \mathscr{C}_c^\beta\big(\mathbb{F}_q((Y^{-1}))\big)$ there is an error term in the above equidistribution claim evaluated on ψ, of the form $\operatorname{O}(q^{\,-\kappa\, t}\,\|\psi\|_\beta)$.

Proof. In this statement, we use the standard convention that $k = 0$ if P_0 is constant, $a_0 \in (\mathbb{F}_q)^\times$ and $P_i \in R$ is monic.

We apply the first claim of Theorem 16.4 with $K = \mathbb{F}_q(T)$ and $v = v_\infty$ so that $g = 0$, $q_v = q$ and $R_v = R$, and with $I = P_0R$, so that $N(I) = \prod_{i=1}^k q^{n_i \deg P_i}$. The result follows from Equation (14.6). □

[7] saying that the rings R_v/I and $\prod_{\mathfrak{p}} R_v/\mathfrak{p}^{v_\mathfrak{p}(I)}$ are isomorphic, see for instance [Nar, page 11]

16.2 Mertens's formula in function fields

In this Section, we recover the function field analogue of Mertens's classical formula on the average order of the Euler function. We begin with a more general counting and equidistribution result.

Let $\mathfrak{m}$ be a (nonzero) fractional ideal of R_v, with norm $N(\mathfrak{m})$. Note that the action of the additive group R_v on $K_v \times K_v$ by the horizontal shears $z \cdot (x, y) = (x + zy, y)$ preserves $\mathfrak{m} \times \mathfrak{m}$. We consider the counting function $\psi_{\mathfrak{m}} : [0, +\infty[\to \mathbb{N}$ defined by

$$\psi_{\mathfrak{m}}(s) = \operatorname{Card}\big(R_v \backslash \big\{(x, y) \in \mathfrak{m} \times \mathfrak{m} \;:\; 0 < N(\mathfrak{m})^{-1} N(y) \leqslant s,\ \langle x, y\rangle = \mathfrak{m}\big\}\big)\,.$$

Note that $\psi_{\mathfrak{m}}$ depends only on the ideal class of $\mathfrak{m}$ and thus we can assume in the computations that $\mathfrak{m}$ is integral, that is, contained in R_v.

Corollary 16.7. *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\psi_{\mathfrak{m}}(s) = \frac{(q-1)\, q^{2g-2}\, {q_v}^3}{({q_v}^2 - 1)\,(q_v + 1)\,\zeta_K(-1)\, m_{v,\, x_0,\, y_0}}\; s^2 + \mathrm{O}(s^{2-\kappa})\,,$$

where $\mathfrak{m} = \langle x_0, y_0\rangle$. Furthermore, as $s \to +\infty$,

$$\frac{({q_v}^2 - 1)\,(q_v + 1)\,\zeta_K(-1)\, m_{v,\, x_0,\, y_0}}{(q-1)\, q^{g-1}\, {q_v}^3}\; s^{-2} \sum_{\substack{(x,y) \in \mathfrak{m} \times \mathfrak{m} \\ N(\mathfrak{m})^{-1} N(y) \leqslant s,\ \langle x, y\rangle = \mathfrak{m}}} \Delta_{\frac{x}{y}} \overset{*}{\rightharpoonup} \operatorname{Haar}_{K_v}\,.$$

For every $\beta \in \,]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every $\psi \in \mathscr{C}_c^{\beta}(K_v))$ there is an error term in the above equidistribution claim evaluated on ψ, of the form $\mathrm{O}(s^{-\kappa}\,\|\psi\|_\beta)$.

Theorem 1.14 in the Introduction follows from this result, by taking $K = \mathbb{F}_q(Y)$ (so that $g = 0$) and $v = v_\infty$ (so that $q_v = q$). In order to simplify the constant, we use Equation (14.6) and the fact that the ideal class number of K, that equals the number of orbits of $\mathrm{PGL}_2(\mathbb{F}_q[Y])$ on $\mathbb{P}^1(\mathbb{F}_q(Y))$, is 1. Thus, if $\mathfrak{m} = \langle x_0, y_0\rangle$ then the constant $m_{v,\, x_0,\, y_0}$ is equal to $m_{v,\,1,\,0}$, which is $q - 1$ by Equation (16.1).

Proof. Every nonzero ideal I in R_v is of the form $I = xR_v + yR_v$ for some $(x, y) \in R_v \times R_v - \{(0, 0)\}$, see for instance [Nar, Coro. 5, page 11]. For all (x, y) and (z, w) in $R_v \times R_v$, we have $x\,R_v + y\,R_v = z\,R_v + w\,R_v$ if and only if $(z, w) \in \mathrm{GL}_2(R_v)(x, y)$. The ideal class group of K corresponds bijectively to the set $\mathrm{PGL}_2(R_v)\backslash\mathbb{P}^1(K)$ of cusps of the quotient graph of groups $\mathrm{PGL}_2(R_v)\backslash\backslash\mathbb{X}_v$,[8] by the map induced by $I = x\,R_v + y\,R_v \mapsto [x : y] \in \mathbb{P}^1(K)$.

Given a fixed ideal $\mathfrak{m}$ in R_v, we apply Theorem 16.1 with $G = \mathrm{GL}_2(R_v)$ and $(x_0, y_0) \in R_v \times R_v - \{(0, 0)\}$ a fixed pair such that $x_0\,R_v + y_0\,R_v = \mathfrak{m}$, so that $G(x_0, y_0) = \mathfrak{m} \times \mathfrak{m}$. Using therein the change of variable $s \mapsto N(\mathfrak{m})s$ and Equation (14.5), the result follows from Theorem 16.1 and its Corollary 16.2. □

As already encountered in the proof of Lemma 16.5, the *Euler function* φ_{R_v} of R_v is defined on the set of (nonzero, integral) ideals I of R_v by setting[9]

$$\varphi_{R_v}(I) = \operatorname{Card}((R_v/I)^\times)\,,$$

[8] where $\mathbb{X}_v$ is the Bruhat-Tits tree of (PGL_2, K_v)

[9] See for example [Ros, §1].

and we denote $\varphi_{R_v}(y) = \varphi_{R_v}(y\,R_v)$ for every $y \in R_v$. Thus, by the definition of the action of R_v on $R_v \times R_v$ by horizontal shears, we have

$$\begin{aligned}\psi_{R_v}(s) &= \sum_{y\in R_v,\ 0<N(y)\leqslant s} \operatorname{Card}\{x \in R_v/yR_v \ :\ \langle x, y\rangle = R_v\}\\ &= \sum_{y\in R_v,\ 0<N(y)\leqslant s} \varphi_{R_v}(y)\,. \qquad (16.13)\end{aligned}$$

As a particular application of Corollary 16.7, we get a well-known asymptotic result on the number of relatively prime polynomials in $\mathbb{F}_q[Y]$. The Euler function of the ring of polynomials $R = \mathbb{F}_q[Y]$ is then the map $\phi_q : R - \{0\} \to \mathbb{N}$ defined by

$$\phi_q(Q) = \big|\big(R/QR\big)^\times\big| = \operatorname{Card}\big\{P \in R : \langle P, Q\rangle = R,\ \deg P < \deg Q\big\}\,.$$

Note that $\phi_q(\lambda Q) = \phi_q(Q)$ for every $\lambda \in (\mathbb{F}_q)^\times$.

Corollary 16.8 (Mertens's formula for polynomials)**.** *We have*

$$\lim_{n\to+\infty} \frac{1}{q^{2n}} \sum_{Q\in\mathbb{F}_q[X],\,\deg Q\leqslant n} \phi_q(Q) = \frac{q\,(q-1)}{q+1}\,.$$

Proof. We apply the first claim of Corollary 16.7, in the special case when $K = \mathbb{F}_q(T)$ and $v = v_\infty$ so that $g = 0$, $q_v = q$ and $R_v = R$, and with $\mathfrak{m} = R_v$, so that $m_{v,\,x_0,\,y_0} = q - 1$, in order to obtain the asymptotic value of $\psi_{R_v}(s)$ with the change of variable $s = q^n$. The result follows from Equations (16.13) and (14.6). □

The above result is an analog of Mertens's formula when K is replaced by $\mathbb{Q}$ and R_v by $\mathbb{Z}$, see [HaW, Theo. 330]. See also [Grot, Satz 2], [Cos, §4.3], as well as [PaP14b] and [PaP17a, §5] for further developments.

A much more precise result than Corollary 16.8 can be obtained by purely number theoretical means as follows. The average value of ϕ_q is computed in [Ros, Prop. 2.7]: For $n \geqslant 1$,

$$\sum_{\deg f=n,\ f \text{ monic}} \phi_q(f) = q^{2n}\big(1 - \frac{1}{q}\big)\,.$$

This gives $\sum_{\deg f=n} \phi_q(f) = q^{2n}\frac{(q-1)^2}{q}$, so that

$$\sum_{0<\deg f\leqslant n} \phi_q(f) = \sum_{k=1}^{n} q^{2k}\frac{(q-1)^2}{q} = q(q-1)^2\frac{q^{2n}-1}{q^2-1} = \frac{q(q-1)(q^{2n}-1)}{(q+1)}\,,$$

from which Corollary 16.8 easily follows.

Chapter 17

Equidistribution and counting of quadratic irrational points in non-Archimedean local fields

Let K_v be a non-Archimedean local field, with valuation v, valuation ring $\mathscr{O}_v$, choice of uniformiser π_v, and residual field k_v of order q_v; let $\mathbb{X}_v$ be the Bruhat-Tits tree of (PGL_2, K_v).[1] In this Chapter, we give counting and equidistribution results in $K_v = \partial_\infty \mathbb{X}_v - \{\infty\}$ of an orbit under a lattice of $\mathrm{PGL}_2(K_v)$ of a fixed point of a loxodromic element of this lattice. We use these results to deduce equidistribution and counting results of quadratic irrational elements in non-Archimedean local fields.

When $\mathbb{X}_v$ is replaced by a real hyperbolic space, or by a more general simply connected complete Riemannian manifold with negative sectional curvature, there are numerous quantitative results on the density of such an orbit, see the works of Patterson, Sullivan, Hill, Velani, Stratmann, Hersonsky-Paulin, Parkkonen-Paulin. See for instance [PaP16] for references. The arithmetic applications when $\mathbb{X}_v$ is replaced by the upper halfspace model of the real hyperbolic space of dimension 2, 3 or 5 are counting and equidistribution results of quadratic irrational elements in $\mathbb{R}$, $\mathbb{C}$ and the Hamiltonian quaternions. See for instance [PaP12, Coro. 3.10] and [PaP14b].

17.1 Counting and equidistribution of loxodromic fixed points

An element $\gamma \in \mathrm{PGL}_2(K_v)$ is said to be *loxodromic* if it is loxodromic[2] on the (geometric realisation of the) simplicial tree $\mathbb{X}_v$. Its translation length is

$$\lambda(\gamma) = \min_{x \in V\mathbb{X}_v} d(x, \gamma x) > 0\,,$$

and the subset

$$\mathrm{Ax}_\gamma = \{x \in V\mathbb{X}_v \;:\; d(x,\gamma x) = \lambda(\gamma)\}$$

is the image of a (discrete) geodesic line in $\mathbb{X}_v$, which we call the *(discrete) translation axis* of γ. The points at infinity of Ax_γ are denoted by γ^- and γ^+, chosen so that γ translates away

[1] See Sections 14.1 and 15.1.
[2] See Section 2.1.

from γ^- and towards γ^+ on Ax_γ. Note that for every $\gamma' \in \mathrm{PGL}_2(K_v)$, we have

$$\gamma' \,\mathrm{Ax}_\gamma = \mathrm{Ax}_{\gamma'\gamma\,(\gamma')^{-1}} \quad \text{and} \quad \gamma'\,\gamma^{\pm} = (\gamma'\gamma\,(\gamma')^{-1})^{\pm}\ .$$

If Γ is a discrete subgroup of $\mathrm{PGL}_2(K_v)$ and if α is one of the two fixed points of a loxodromic element of Γ, we denote the other fixed point of this element by α^σ. Since Γ is discrete, the translation axes of two loxodromic elements of Γ coincide if they have a common point at infinity. Hence α^σ is uniquely defined. For every $\gamma \in \Gamma$, we hence have

$$(\gamma\cdot\alpha)^\sigma = \gamma\cdot(\alpha^\sigma)\ . \tag{17.1}$$

We define the *complexity* $h(\alpha)$ of the loxodromic fixed point α by

$$h(\alpha) = \frac{1}{|\alpha-\alpha^\sigma|_v} \tag{17.2}$$

if $\alpha, \alpha^\sigma \neq \infty$, and by $h(\alpha) = 0$ if α or α^σ is equal to ∞. We define $\iota_\alpha \in \{1,2\}$ by $\iota_\alpha = 2$ if there exists an element $\gamma\in\Gamma$ such that $\gamma\cdot\alpha = \alpha^\sigma$, and $\iota_\alpha = 1$ otherwise.[3]

Following [Ser3, II.1.2], we denote by $\mathrm{PGL}_2(K_v)^+$ the kernel of the group morphism $\mathrm{PGL}_2(K_v) \to \mathbb{Z}/2\mathbb{Z}$ defined by $\gamma = [g] \mapsto v(\det g) \mod 2$. The definition does not depend on the choice of a representative $g \in \mathrm{GL}_2(K_v)$ of an element $\gamma\in\mathrm{PGL}_2(K_v)$, since

$$v\Big(\det\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\Big) = 2\,v(\lambda)$$

is even for every $\lambda \in (K_v)^\times$. Note that when K_v is the completion of a function field over $\mathbb{F}_q$ endowed with a valuation v, with associated affine function ring R_v, the group $\Gamma_v = \mathrm{PGL}_2(R_v)$ is contained in $\mathrm{PGL}_2(K_v)^+$: For every $g\in\mathrm{GL}_2(R_v)$, since $\det g \in (R_v)^\times = (\mathbb{F}_q)^\times$,[4] we have $v(\det g) = 0$.

The following result proves the equidistribution in K_v of the loxodromic fixed points with complexity at most s in a given orbit by homographies under a lattice in $\mathrm{PGL}_2(K_v)$ as $s\to+\infty$, and its associated counting result. If $\xi\in\partial_\infty\mathbb{X}_v = \mathbb{P}_1(K_v)$ and Γ is a subgroup of $\mathrm{PGL}_2(K_v)$, we denote by Γ_ξ the stabiliser in Γ of ξ.

Theorem 17.1. *Let Γ be a lattice in $\mathrm{PGL}_2(K_v)^+$, and let $\gamma_0\in\Gamma$ be a loxodromic element of Γ. Then as $s\to+\infty$,*

$$\frac{(q_v+1)^2\ \mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{2\,q_v^2\ \mathrm{Vol}(\Gamma_{\gamma_0^-}\backslash\!\backslash\,\mathrm{Ax}_{\gamma_0})}\, s^{-1} \sum_{\alpha\in\Gamma\cdot\gamma_0^-,\ h(\alpha)\leqslant s} \Delta_\alpha \ \overset{*}{\rightharpoonup}\ \mathrm{Haar}_{K_v}$$

and there exists $\kappa = \kappa_\Gamma > 0$ such that

$$\mathrm{Card}\{\alpha\in(\Gamma\cdot\gamma_0^-)\cap\mathscr{O}_v\ :\ h(\alpha)\leqslant s\} = \frac{2\,q_v^2\ \mathrm{Vol}(\Gamma_{\gamma_0^-}\backslash\!\backslash\,\mathrm{Ax}_{\gamma_0})}{(q_v+1)^2\ \mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}\, s + \mathrm{O}(s^{1-\kappa})\ .$$

[3]Recall that the groups $\mathrm{GL}_2(K_v)$ and $\mathrm{PGL}_2(K_v)$ act on $\mathbb{P}^1(K_v) = K_v\cup\{\infty\}$ by homographies, and that these actions are denoted by $\cdot$, see Section 15.1.

[4]See Equation (14.3).

For every $\beta \in \,]0, \frac{1}{\ln q_v}]$, there is an error term of the form $\operatorname{O}(s^{-\kappa}\|\psi\|_\beta)$ for some $\kappa > 0$ in the equidistribution claim evaluated on any β-Hölder-continuous function $\psi : \mathscr{O}_v \to \mathbb{C}$.

Proof. The second result follows from the first one by integrating on the characteristic function of the compact-open subset $\mathscr{O}_v$, whose Haar measure is 1.

In order to prove the equidistribution result, we apply Theorem 15.4 with $\mathbb{D}^- = \{*_v\}$ and $\mathbb{D}^+ = \operatorname{Ax}_{\gamma_0}$. The families $\mathscr{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite, since Γ is discrete and the stabiliser $\Gamma_{\mathbb{D}^+}$ of $\mathbb{D}^+$ acts cocompactly on $\mathbb{D}^+$. Furthermore, $\|\sigma^-_{\mathscr{D}^+}\|$ is finite and nonzero by Equation (8.12). Since Γ is contained in $\operatorname{PGL}_2(K_v)^+$, the length spectrum L_Γ of Γ is contained in $2\mathbb{Z}$ by [Ser3, II.1.2, Coro.]. Hence, it is equal to $2\mathbb{Z}$ by the equivalence given by Equation (4.17).

For every $\gamma \in \Gamma$ such that $d(\mathbb{D}^-, \gamma\mathbb{D}^+) > 0$,[5] let $\alpha^-_{e,\gamma}$ be the generalised geodesic line, isometric exactly on $[0, d(\mathbb{D}^-, \gamma\mathbb{D}^+)]$, whose image is the common perpendicular between $\mathbb{D}^-$ and $\gamma\mathbb{D}^+$, and let ρ_γ be the geodesic ray starting at time 0 from the origin of $\alpha^-_{e,\gamma}$ (which is $*_v$) with point at infinity $\gamma\cdot\gamma_0^-$. Since $\mathbb{X}_v$ is a tree and $\gamma\cdot\gamma_0^-$ is one of the two endpoints of $\gamma\mathbb{D}^+$, the geodesic segment $\alpha^-_{e,\gamma}|_{[0,d(\mathbb{D}^-,\gamma\mathbb{D}^+)]}$ is an initial subsegment of ρ_γ.[6] Therefore, by Equation (15.14), for the weak-star convergence of measures on $\partial^1_+\mathbb{D}^-$, we have

$$\lim_{n\to+\infty} \frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3}\,\frac{\operatorname{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\,{q_v}^{-n} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}} \Delta_{\gamma\cdot\gamma_0^-} \;=\; (\partial^+)_*\widetilde{\sigma}^+_{\mathbb{D}^-}\,. \tag{17.3}$$

Furthermore, for every $\beta \in \,]0, \frac{1}{\ln q_v}]$, by the comment following Equation (15.14) and since Γ, being a lattice in $\operatorname{PGL}_2(K_v)$, is geometrically finite,[7] we have an error term of the form $\operatorname{O}(e^{-\kappa\, n}\,\|\psi\|_{\beta\ln q_v})$ for some $\kappa>0$ in the above formula when evaluated on $\psi \in \mathscr{C}_c^{\beta\ln q_v}(\partial_\infty\mathbb{X}_v)$, where $\partial_\infty\mathbb{X}_v$ is endowed with the visual distance d_{*_v}.[8] Note that on $\mathscr{O}_v$, the visual distance d_{*_v} and the distance $(x,y)\mapsto |x-y|_v$ are related by

$$|x-y|_v = d_{\mathscr{H}_\infty}(x,y)^{\ln q_v} = d_{*_v}(x,y)^{\ln q_v}\ ,$$

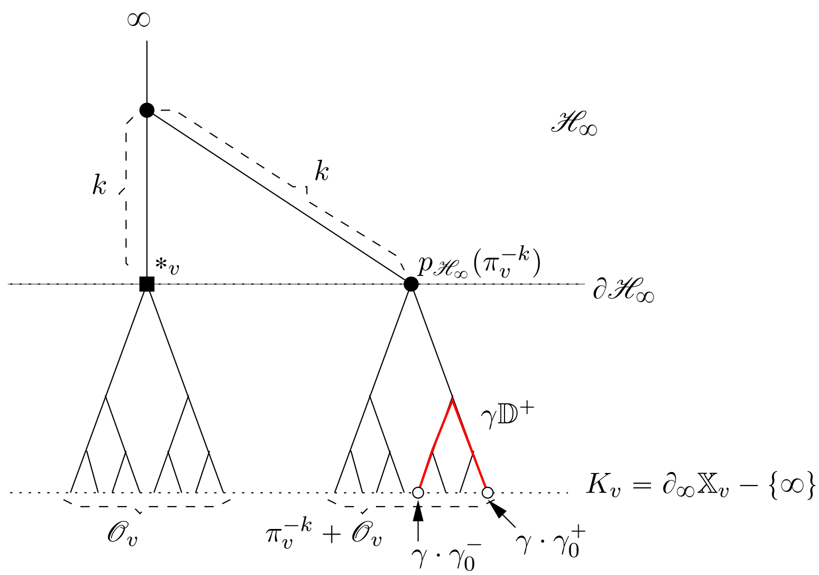
using Equation (15.2) for the first equality. Hence we have an error term $\operatorname{O}(e^{-\kappa\, n}\,\|\psi\|_\beta)$ for some $\kappa>0$ in the above formula when evaluated on $\psi\in\mathscr{C}_c^\beta(\mathscr{O}_v)$, where $\mathscr{O}_v$ is endowed with the distance $(x,y)\mapsto|x-y|_v$.

[5] that is, such that $*_v \notin \gamma\mathbb{D}^+$

[6] It connects $*_v$ to its closest point $P_{\gamma\mathbb{D}^+}(*_v)$ on $\gamma\mathbb{D}^+$, with $P_\cdot(\cdot)$ defined in Section 2.4.

[7] See the end of Section 2.6.

[8] Note that when $D = \{x\}$ is a singleton, the distance-like map d_D used in Remark 15.6 coincides with the visual distance d_x, as said after Equation (3.8).



Let us fix for the moment $k \in \mathbb{N}$. For every $\xi \in \pi_v^{-k} + \mathscr{O}_v$, we have $|\xi|_v = q_v^{-v(\xi)} = q_v^k$ if $k \geqslant 1$ and $|\xi|_v \leqslant 1$ if $k = 0$. By restricting the measures to the compact-open subset $\pi_v^{-k} + \mathscr{O}_v$ and by Proposition 15.2 (1), we have, with the appropriate error term when $k = 0$,

$$\lim_{n\to+\infty} \frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3}\;\frac{\operatorname{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;{q_v}^{-n} \sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ \gamma\cdot\gamma_0^-\in\pi_v^{-k}+\mathscr{O}_v\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}} \Delta_{\gamma\cdot\gamma_0^-}$$
$$= q_v^{-2\,k}\;\operatorname{Haar}_{K_v}\big|_{(\pi_v)^{-k}+\mathscr{O}_v}\;. \tag{17.4}$$

If $\beta \in \Gamma$ is loxodromic and satisfies $\beta^- \in \pi_v^{-k} + \mathscr{O}_v$ and $\beta^+ \notin \pi_v^{-k} + \mathscr{O}_v$, then the translation axis of β passes at distance at most $2k$ from $*_v$, since it passes through $P_{\mathscr{H}_\infty}(\pi_v^{-k})$ which is the closest point on $\mathscr{H}_\infty$ to π_v^{-k}. If $\beta \in \Gamma$ is loxodromic and satisfies $\beta^-, \beta^+ \in \pi_v^{-k} + \mathscr{O}_v$, then

$$d(*_v, \operatorname{Ax}_\beta) = 2k + d(\mathscr{H}_\infty, \operatorname{Ax}_\beta)\;.$$

Furthermore, we have, by Equations (15.2) and (2.12)

$$|\beta^- - \beta^+|_v = d_{\mathscr{H}_\infty}(\beta^-, \beta^+)^{\ln q_v} = q_v^{-d(\mathscr{H}_\infty,\,\operatorname{Ax}_\beta)}\;.$$

Therefore by the definition of the complexity in Equation (17.2), we have for these elements

$$h(\beta^-) = \frac{1}{|\beta^- - \beta^+|_v} = q_v^{d(\mathscr{H}_\infty,\,\operatorname{Ax}_\beta)} = q_v^{d(*_v,\,\operatorname{Ax}_\beta)-2k}\;. \tag{17.5}$$

Since the family $\mathscr{D}^+ = (\gamma\mathbb{D}^+)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}}$ is locally finite, there are only finitely many elements $\gamma \in \Gamma/\Gamma_{\mathbb{D}^+}$ such that $\gamma\mathbb{D}^+ = \operatorname{Ax}_{\gamma\gamma_0\gamma^{-1}}$ is at distance at most $2k$ from $*_v$. Hence for all but finitely many $\gamma \in \Gamma/\Gamma_{\mathbb{D}^+}$ such that $\gamma\cdot\gamma_0^- = (\gamma\gamma_0\gamma^{-1})^- \in \pi_v^{-k} + \mathscr{O}_v$, we have $\gamma\cdot\gamma_0^+ = (\gamma\gamma_0\gamma^{-1})^+ \in \pi_v^{-k} + \mathscr{O}_v$ and, using Equation (17.5) with $\beta = \gamma\gamma_0\gamma^{-1}$,

$$h(\gamma\cdot\gamma_0^-) = q_v^{d(\mathbb{D}^-,\gamma\mathbb{D}^+)-2k}\;.$$

Therefore, using the change of variable $s = {q_v}^{n-2k}$, Equation (17.4) becomes

$$\lim_{s\to+\infty} \frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3}\;\frac{\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;s^{-1}\sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ \gamma\cdot\gamma_0^-\in\pi_v^{-k}+\mathscr{O}_v\\ 0<h(\gamma\cdot\gamma_0^-)\leqslant s}} \Delta_{\gamma\cdot\gamma_0^-} \;=\; \mathrm{Haar}_{K_v}{}_{|(\pi_v)^{-k}+\mathscr{O}_v}\;. \tag{17.6}$$

Note that the stabiliser $\Gamma_{\gamma_0^-}$ of γ_0^- in Γ has index $\iota_{\gamma_0^-}$ in $\Gamma_{\mathbb{D}^+}$ by the definition of $\iota_{\gamma_0^-}$ and that $\Gamma/\Gamma_{\gamma_0^-}$ identifies with $\Gamma\cdot\gamma_0^-$ by the map $\gamma\Gamma_{\gamma_0^-}\mapsto\gamma\cdot\gamma_0^-$. Since $\big((\pi_v)^{-k}+\mathscr{O}_v\big)_{k\in\mathbb{N}}$ is a countable family of pairwise disjoint compact-open subsets covering K_v, and since the support of any continuous function with compact support is contained in finitely many elements of this family, we have

$$\lim_{s\to+\infty} \frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3\;\iota_{\gamma_0^-}}\;\frac{\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;s^{-1}\sum_{\substack{\alpha\in\Gamma\cdot\gamma_0^-\\ 0<h(\alpha)\leqslant s}} \Delta_{\alpha} \;=\; \mathrm{Haar}_{K_v}\;, \tag{17.7}$$

with the appropriate error term.

Recall that by Equation (8.12), if the Patterson measures are normalised to be probability measures, then

$$\|\sigma^-_{\mathscr{D}^+}\| = \frac{q_v-1}{q_v+1}\;\mathrm{Vol}(\Gamma_{\mathbb{D}^+}\backslash\!\backslash\mathbb{D}^+)\;.$$

Hence if instead the Patterson densities are normalised to have total mass $\frac{q_v+1}{q_v}$ as in Proposition 15.2 (2), then

$$\|\sigma^-_{\mathscr{D}^+}\| = \frac{q_v-1}{q_v}\;\mathrm{Vol}(\Gamma_{\mathbb{D}^+}\backslash\!\backslash\mathbb{D}^+)\;.$$

Note that, since $\iota_{\gamma_0^-}=[\Gamma_{\mathrm{Ax}_{\gamma_0}}:\Gamma_{\gamma_0^-}]$,

$$\mathrm{Vol}(\Gamma_{\gamma_0^-}\backslash\!\backslash\,\mathrm{Ax}_{\gamma_0}) = \iota_{\gamma_0^-}\;\mathrm{Vol}(\Gamma_{\mathrm{Ax}_{\gamma_0}}\backslash\!\backslash\,\mathrm{Ax}_{\gamma_0})\;.$$

Equation (17.7) thus gives the equidistribution result in Theorem 17.1. □

In the following two Sections, we use Theorem 17.1 to deduce counting and equidistribution results of elements of non-Archimedean local fields that are quadratic irrational over appropriate subfields, when an appropriate algebraic complexity tends to infinity.

17.2 Counting and equidistribution of quadratic irrationals in positive characteristic

Let K be a (global) function field over $\mathbb{F}_q$, let v be a (normalised discrete) valuation of K, let K_v be the associated completion of K and let R_v be the affine function ring associated with v.[9]

An element $\beta\in K_v$ is *quadratic irrational* over K if $\beta\notin K$ and β is a root of a quadratic polynomial $a\beta^2+b\beta+c$ for some $a,b,c\in K$ with $a\neq0$. The *Galois conjugate* β^σ of β is the other root of the same polynomial. Let

$$\mathtt{tr}(\beta)=\beta+\beta^\sigma \quad\text{and}\quad \mathtt{n}(\beta)=\beta\beta^\sigma$$

[9]See Section 14.2.

be the relative *trace* and relative *norm* of β. It is easy to check that $\beta^\sigma \neq \beta$, as the following lemma shows.

Recall that for every field F, a polynomial $P \in F[Y]$ is *separable* if its roots in any algebraic closure of F are simple, and *inseparable* otherwise. It is well known (see for instance [Lan, §V.6]) that, with p the characteristic of F, any irreducible quadratic polynomial P over F is separable when $p \neq 2$, and is inseparable when $p = 2$ if and only if $P = a(Y^2 - b)$ with $a \in F^\times$ and $b \in F$ which is not a square in F.

Lemma 17.2. *An irreducible quadratic polynomial P over K which splits over K_v is separable.*

Proof. The result is immediate if q is odd. Otherwise, assume for a contradiction that P is inseparable, so that $P = a(X^2 - b)$ with $a \in F^\times$ and $b \in K$ not a square in K. Since P splits over K_v, the element b is a square in K_v. Since K_v is isomorphic to the field $\mathbb{F}_{q_v}((\pi_v))$ of formal Laurent series over $\mathbb{F}_{q_v}$ with variable the uniformiser π_v, which may be assumed to belong to K, there exist $m \in \mathbb{Z}$ and a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{F}_{q_v}$ such that, by the properties of the Frobenius automorphism $x \mapsto x^2$,

$$b = \Big(\sum_{n\in\mathbb{N}} a_n \pi_v^{m+k}\Big)^2 = \pi_v^{2m}\Big(\sum_{n\in\mathbb{N}} a_n^2 \pi_v^{2k}\Big)\;.$$

Since $b \in K$, this implies that $a_n^2 = 0$ for n large enough, hence that $a_n = 0$ for n large enough, so that b is a square in K, a contradiction. □

The next proposition gives a characterisation of quadratic irrationals over K.

Proposition 17.3. *Let $\beta \in K_v$. The following assertions are equivalent:*

(1) *β is quadratic irrational over K,*

(2) *β is a fixed point of a loxodromic element of* $\mathrm{PGL}_2(R_v)$.

Proof. The fact that (2) implies (1) is immediate since $\mathrm{PGL}_2(R_v)$ acts by homographies. The converse is classical once we know that $\beta \neq \beta^\sigma$, see for instance [PaP11b, Lem. 6.2] in the Archimedean case and [BerN] above its Section 5 when $K = \mathbb{F}_q(Y)$ and $v = v_\infty$. □

If $\beta \in K_v$ is quadratic irrational over K, its Galois conjugate β^σ is the other fixed point of any loxodromic element of $\mathrm{PGL}_2(R_v)$ fixing β, hence the notations β^σ in this Section and in Section 17.1 coincide.

The actions by homographies of the groups $\mathrm{GL}_2(R_v)$ and $\mathrm{PGL}_2(R_v)$ on $K_v \cup \{\infty\}$ preserve the set of quadratic irrationals over K. Contrary to the case of rational points, both groups act with infinitely many orbits.

The *complexity* of a quadratic irrational $\alpha \in K_v$ over K is

$$h(\alpha) = \frac{1}{|\alpha - \alpha^\sigma|_v}\;,$$

see for instance [HeP4, §6] for motivations and results when $K = \mathbb{F}_q(Y)$ and $v = v_\infty$. Note that this complexity is invariant under the action of the stabiliser $\mathrm{GL}_2(R_v)_\infty$ of ∞ in $\mathrm{GL}_2(R_v)$, which is its upper triangular subgroup. In particular, it is invariant under the action of R_v by translations.[10] In [PaP12], where K and $|\cdot|_v$ are replaced by $\mathbb{Q}$ and its Archimedean absolute

[10]This is a particular case of Proposition 17.4 (2) below.

value, there was, for convenience, an extra factor 2 in the numerator of the complexity, which is not needed here. We refer for instance to [PaP11b, Rem. 3, p. 136] for the connection of this complexity to the standard height, and to [PaP12, §4.2, 4.4] and [PaP11b, §6.1] for studies using this complexity.

The complexity $h(\cdot)$ satisfies the following elementary properties, giving in particular its behaviour under the action of $\mathrm{PGL}_2(R_v)$ by homographies on the quadratic irrationals in K_v over K. We also give the well-known computation of the Jacobian of the Haar measure for the change of variables given by homographies, and prove the invariance of a measure which will be useful in Section 18.1.

For all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_v)$ and $z \in K_v$ such that $g \cdot z \neq \infty$, let

$$jg(z) = \frac{|\det g|_v}{|c\, z + d|_v^2}\,.$$

Proposition 17.4. *Let $\alpha \in K_v$ be a quadratic irrational over K.*

(1) *We have* $h(\alpha) = \dfrac{1}{\sqrt{|\,\mathtt{tr}(\alpha)^2 - 4\,\mathtt{n}(\alpha)|_v}}$.

(2) *For every* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$ *with* $|\det g|_v = 1$, *we have*

$$h(g \cdot \alpha) = |\,\mathtt{n}(d + c\,\alpha)|_v\; h(\alpha)\ .$$

(3) *If* $Q_\alpha : R_v \times R_v \to [0, +\infty[$ *is the map* $(x, y) \mapsto |\,\mathtt{n}(x - y\,\alpha)|_v$, *then for every* $g \in \mathrm{GL}_2(R_v)$, *we have*

$$Q_{g\cdot\alpha} = \frac{h(\alpha)}{h(g\cdot\alpha)}\; Q_\alpha \circ g^{-1}\ .$$

In particular, if $g \in \mathrm{GL}_2(R_v)$ *fixes* α, *then*

$$Q_\alpha \circ g = Q_\alpha\ .$$

(4) *For all* $x, y, z \in K_v$ *and* $g \in \mathrm{GL}_2(K_v)$ *such that* $g \cdot x,\ g \cdot y,\ g \cdot z \neq \infty$, *we have*

$$|g \cdot x - g \cdot y|_v^2 \;=\; |x - y|_v^2\;\; jg(x)\;\; jg(y)$$

and

$$jg(z) = \frac{d(g^{-1})_*\,\mathrm{Haar}_{K_v}}{d\,\mathrm{Haar}_{K_v}}(z)\ .$$

(5) *The measure*

$$d\mu(z) = \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z - \alpha|_v\;|z - \alpha^\sigma|_v}$$

on $K_v - \{\alpha, \alpha^\sigma\}$ *is invariant under the stabiliser of* α *in* $\mathrm{PGL}_2(R_v)$

Proof. (1) This follows from the formula $(\alpha - \alpha^\sigma)^2 = (\alpha + \alpha^\sigma)^2 - 4\,\alpha\,\alpha^\sigma$.

(2) Since g has rational coefficients (that is, coefficients in K), we have

$$\begin{aligned} g\cdot\alpha-(g\cdot\alpha)^\sigma = g\cdot\alpha - g\cdot\alpha^\sigma &= \frac{a\alpha+b}{c\alpha+d}-\frac{a\alpha^\sigma+b}{c\alpha^\sigma+d}\\ &= \frac{(ad-bc)(\alpha-\alpha^\sigma)}{(c\alpha+d)(c\alpha^\sigma+d)} = \frac{(\det g)(\alpha-\alpha^\sigma)}{\mathtt{n}(d+c\alpha)}\;. \end{aligned}$$

Taking absolute values and inverses, this gives Assertion (2).

(3) Let $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(R_v)$. Note that $g^{-1}\cdot\alpha=\frac{d\alpha-b}{a-c\alpha}$. For all $x,y\in R_v$, we hence have

$$\begin{aligned} \mathtt{n}\big((ax+by)-(cx+dy)\alpha\big) &= \mathtt{n}\left(x(a-c\alpha)-y(d\alpha-b)\right)\\ &= \mathtt{n}\left(x(a-c\alpha)-y(a-c\alpha)\,g^{-1}\cdot\alpha\right)\\ &= \mathtt{n}(a-c\alpha)\;\mathtt{n}(x-y\,g^{-1}\cdot\alpha)\;. \end{aligned}$$

Taking absolute values and using Assertion (2), we have

$$Q_\alpha\circ g=\frac{h(g^{-1}\cdot\alpha)}{h(\alpha)}\,Q_{g^{-1}\cdot\alpha}\;.$$

Assertion (3) follows by replacing g by its inverse.

(4) Let $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(K_v)$. As seen in the proof of Assertion (2), we have

$$g\cdot x-g\cdot y=\frac{(\det g)(x-y)}{(c\,x+d)(c\,y+d)}\;.$$

Taking absolute values and squares, this gives the first claim of Assertion (4).

Recall that a homography $z\mapsto \frac{a\,z+b}{c\,z+d}$ is holomorphic[11] on $K_v-\{-\frac{d}{c}\}$, with derivative $z\mapsto\frac{ad-bc}{(cz+d)^2}$. Hence infinitesimally close to z, the homography acts (up to translations which leave the Haar measure invariant) by a homothety of ratio $\frac{ad-bc}{(cz+d)^2}$. By Equation (14.7), this proves that

$$d\,\mathrm{Haar}_{K_v}(g\cdot z)=\frac{|\det g|_v}{|c\,z+d|_v^2}\,d\,\mathrm{Haar}_{K_v}(z)\;,$$

as wanted.

(5) Let $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(R_v)$ fixing α. Note that an element of $\mathrm{GL}_2(R_v)$ which fixes α also fixes α^σ. By Assertion (4), we have

$$\begin{aligned} d\mu(g\cdot z) &= \frac{d\,\mathrm{Haar}_{K_v}(g\cdot z)}{|g\cdot z-\alpha|_v\;|g\cdot z-\alpha^\sigma|_v} = \frac{d\,\mathrm{Haar}_{K_v}(g\cdot z)}{|g\cdot z-g\cdot\alpha|_v\;|g\cdot z-g\cdot\alpha^\sigma|_v}\\ &= \frac{jg(z)\;d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha|_v\,\sqrt{jg(z)\;jg(\alpha)}\;|z-\alpha^\sigma|_v\,\sqrt{jg(z)\;jg(\alpha^\sigma)}}\\ &= \frac{1}{\sqrt{jg(\alpha)\;jg(\alpha^\sigma)}}\;d\mu(z)\;. \end{aligned}$$

[11] We refer for instance to [Ser4] for background on holomorphic functions on non-Archimedean local fields.

By Assertion (4) applied with $x = \alpha$ and $y = \alpha^\sigma$, we have

$$\sqrt{jg(\alpha)\, jg(\alpha^\sigma)} = \frac{|g\cdot\alpha - g\cdot\alpha^\sigma|_v}{|\alpha-\alpha^\sigma|_v} = 1\ .$$

The result follows. □

Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$. We say that a quadratic irrational $\beta \in K_v$ over K is *G-reciprocal* (simply *reciprocal* if $G = \mathrm{GL}_2(R_v)$) if some element of G maps β to β^σ. We define the *G-reciprocity index* $\iota_G(\beta)$ as 2 if β is G-reciprocal and 1 otherwise. Similarly, we say that a loxodromic element γ of G is *G-reciprocal* (simply *reciprocal* if $G = \mathrm{GL}_2(R_v)$) if there exists an element in G that switches the two fixed points of γ.

Proposition 17.5. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, and let γ be a loxodromic element of G. The following assertions are equivalent:*

(1) *γ is conjugate in G to $\gamma'\gamma^{-1}$ for some $\gamma' \in G$ pointwise fixing Ax_γ,*

(2) *the loxodromic element γ is G-reciprocal,*

(3) *the quadratic irrational γ^- is G-reciprocal.*

When $G = \mathrm{GL}_2(R_v)$, Assertions (1), (2) and (3) are also equivalent to

(4) *the image of $\gamma''\gamma$ in $\mathrm{PGL}_2(R_v)$, for some $\gamma'' \in G$ pointwise fixing Ax_γ, is conjugate to the image in $\mathrm{PGL}_2(R_v)$ of ${}^t\gamma$.*

Proof. Most of the proofs are similar to the ones when R_v, K and $|\cdot|_v$ are replaced by $\mathbb{Z}$, $\mathbb{Q}$ and its Archimedean absolute value, see for instance [PaP12]. We only give hints for the sake of completeness. Let $\alpha = \gamma^-$.

If α is G-reciprocal, then let $\beta \in G$ be such that $\beta\cdot\alpha = \alpha^\sigma$. Since $R_v \subset K$, we have $\beta\cdot\alpha^\sigma = \alpha$. Hence $\beta\gamma\beta^{-1}$ is a loxodromic element of G fixing α and α^σ, having the same translation length as γ, but translating in the opposite direction on Ax_γ. Hence $\gamma' = \beta\gamma\beta^{-1}\gamma$ fixes pointwise Ax_γ. Therefore (3) implies (1).

If $\beta \in G$ conjugates γ to $\gamma'\gamma^{-1}$ for some $\gamma' \in G$ pointwise fixing Ax_γ, then β preserves the set $\{\alpha, \alpha^\sigma\}$. Hence, it preserves the translation axis of γ but it switches α and α^σ since γ and $\gamma'\gamma^{-1}$ translate in opposite directions on Ax_γ. Therefore (1) implies (2).

The fact that (2) implies (3) is immediate, since $\alpha^\sigma = \gamma^+$.

The equivalence between (1) and (4) when $G = \mathrm{GL}_2(R_v)$ follows from the fact that the stabiliser of Ax_γ normalises the pointwise stabiliser of Ax_γ, and from the formula

$${}^t\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}^{-1}$$

which is valid over any field. □

The following result says that any orbit of a given quadratic irrational in K_v over K, by homographies under a given finite index subgroup of the modular group $\mathrm{PGL}_2(R_v)$, equidistributes to the Haar measure on K_v. Again, note that we are not assuming the finite index subgroup to be a congruence subgroup.

Theorem 17.6. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$. Let $\alpha_0 \in K_v$ be a quadratic irrational over K. Then, as $s \to +\infty$,*

$$\frac{(q_v+1)^2\,\zeta_K(-1)\,m_0\,[\mathrm{GL}_2(R_v):G]}{2\,q_v^2\,(q-1)\,|v(\mathrm{tr}\,g_0)|}\;s^{-1}\sum_{\alpha\in G\cdot\alpha_0\,:\,h(\alpha)\leqslant s}\Delta_\alpha\;\overset{*}{\rightharpoonup}\;\mathrm{Haar}_{K_v}\,,$$

where $g_0 \in G$ fixes α_0 with $v(\mathrm{tr}\,g_0) \neq 0$, and where m_0 is the index of $g_0^{\mathbb{Z}}$ in the stabiliser of α_0 in G. Furthermore, there exists $\kappa > 0$ such that, as $s \to +\infty$,

$$\mathrm{Card}\{\alpha\in(G\cdot\alpha_0)\cap\mathscr{O}_v\;:\;h(\alpha)\leqslant s\}=\frac{2\,q_v^2\,(q-1)\,|v(\mathrm{tr}\,g_0)|}{(q_v+1)^2\,\zeta_K(-1)\,m_0\,[\mathrm{GL}_2(R_v):G]}\;s+\mathrm{O}(s^{1-\kappa})\,.$$

For every $\beta \in \,]0, \frac{1}{\ln q_v}]$, there exists $\kappa > 0$ such that for every $\psi \in \mathscr{C}_c^\beta(K_v)$ there is an error term in the above equidistribution claim evaluated on ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\beta)$.

Proof. We apply Theorem 17.1 with Γ the image of G in $\Gamma_v = \mathrm{PGL}_2(R_v)$ and with γ_0 the image in Γ_v of the element g_0 introduced in the statement. Note that Γ, which is contained in Γ_v, is indeed contained in $\mathrm{PGL}_2(K_v)^+$.

By Equation (15.6), for every $g \in \mathrm{GL}_2(R_v)$, the translation length of g in $\mathbb{X}_v$ is $2\,|v(\mathrm{tr}\,g)|$, and g is loxodromic if and only if $v(\mathrm{tr}\,g) \neq 0$. This implies that g_0 exists, since G has finite index in $\mathrm{GL}_2(R_v)$, and such an element exists in $\mathrm{GL}_2(R_v)$ by Proposition 17.3. Up to replacing g_0 by its inverse, which changes neither $|v(\mathrm{tr}\,g_0)|$ nor m_0, we assume that $\gamma_0^- = \alpha_0$. Furthermore

$$\lambda(\gamma_0) = 2\,|v(\mathrm{tr}\,g_0)|\,.$$

Since the centre of $\mathrm{GL}_2(K_v)$ acts trivially by homographies, we have

$$G\cdot\alpha_0 = \Gamma\cdot\alpha_0\,.$$

For every $\alpha \in G\cdot\alpha_0$, the complexities $h(\alpha)$, when α is considered as a quadratic irrational or when α is considered as a loxodromic fixed point, coincide.

Since the centre $Z(G)$ of G acts trivially by homographies, by the definition of m_0 in the statement, we have

$$[\Gamma_{\gamma_0^-}:\gamma_0^{\mathbb{Z}}]=\frac{[G_{\alpha_0}:g_0^{\mathbb{Z}}]}{|Z(G)|}=\frac{m_0}{|Z(G)|}\,.$$

Therefore,

$$\begin{aligned}\mathrm{Vol}(\Gamma_{\gamma_0^-}\backslash\!\backslash \mathrm{Ax}_{\gamma_0}) &= \frac{1}{[\Gamma_{\gamma_0^-}:\gamma_0^{\mathbb{Z}}]}\;\mathrm{Vol}(\gamma_0^{\mathbb{Z}}\backslash\!\backslash \mathrm{Ax}_{\gamma_0}) = \frac{\lambda(\gamma_0)}{[\Gamma_{\gamma_0^-}:\gamma_0^{\mathbb{Z}}]}\\ &= \frac{2\,|v(\mathrm{tr}\,g_0)|\,|Z(G)|}{m_0}\,.\end{aligned} \tag{17.8}$$

Theorem 17.6 now follows from Theorem 17.1 using Equations (16.6) and (17.8). □

Example 17.7. (1) Theorem 1.15 in the Introduction follows from this result, by taking $K = \mathbb{F}_q(Y)$ and $v = v_\infty$, and by using Equation (14.6) in order to simplify the constant.

(2) Let G_I be the Hecke congruence subgroup associated with a nonzero ideal I of R_v, see Equation (16.11). By Lemma 16.5, we have, as $s \to +\infty$,

$$\frac{(q_v+1)^2\,\zeta_K(-1)\,m_0\,N(I)\prod_{\mathfrak{p}|I}(1+\frac{1}{N(\mathfrak{p})})}{2\,q_v^2\,(q-1)\,|v(\mathrm{tr}\,g_0)|}\;s^{-1}\sum_{\alpha\in G_I\cdot\alpha_0\,:\,h(\alpha)\leqslant s}\Delta_\alpha\;\overset{*}{\rightharpoonup}\;\mathrm{Haar}_{K_v}\,.$$

We conclude this Section by a characterisation of quadratic irrationals and reciprocal quadratic irrationals in the field of formal Laurent series $\mathbb{F}_q((Y^{-1}))$ in terms of continued fractions. When $\mathbb{F}_q[Y]$, $\mathbb{F}_q(Y)$ and v_∞ are replaced by $\mathbb{Z}$, $\mathbb{Q}$ and its Archimedean absolute value, we refer for instance to [Sarn] and [PaP12, Prop. 4.3] for characterisations of reciprocal quadratic irrationals.

Recall that Artin's *continued fraction expansion* of $f \in \mathbb{F}_q((Y^{-1})) - \mathbb{F}_q(Y)$ is the sequence $(a_i = a_i(f))_{i\in\mathbb{N}}$ in $\mathbb{F}_q[Y]$ with $\deg a_i > 0$ if $i > 0$ such that

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}\ .$$

See for instance the surveys [Las, Sch2], as well as [Pau3] for a geometric interpretation. We say that the continued fraction expansion of f is *eventually periodic* if there exist $n \in \mathbb{N}$ and $N \in \mathbb{N} - \{0\}$ such that $a_{n+i} = a_{n+N+i}$ for every $i \in \mathbb{N}$, and we write

$$f = [a_0, \dots, a_{n-1}, \overline{a_n, \dots, a_{n+N-1}}]\ .$$

Such a sequence $a_n, \dots, a_{n+N-1}$ is called a *period* of f, and if of minimal length, it is well defined up to cyclic permutation.

Two elements $\beta, \beta' \in \mathbb{F}_q((Y^{-1}))$ are in the same $\mathrm{PGL}_2(\mathbb{F}_q[Y])$-orbit if and only if their continued fraction expansions have equal tails up to an invertible element of $\mathbb{F}_q[Y]$ by [Sch2, Theo. 1] or [BerN, Theo. 1] (and even before that by [deM, Sect. IV.3]). More precisely, $\beta, \beta' \in K_v$ are in the same $\mathrm{PGL}_2(\mathbb{F}_q[Y])$-orbit if and only if there exist $m, n \in \mathbb{N}$ and $x \in \mathbb{F}_q^\times$ such that for every $k \in \mathbb{N}$, we have $a_{n+k}(\beta') = x^{(-1)^k}\, a_{m+k}(\beta)$.

Proposition 17.8. *Assume that $K = \mathbb{F}_q(Y)$ and $v = v_\infty$.*

(1) *An element $\alpha \in K_v - K$ is quadratic irrational over K if and only if its continued fraction expansion of β is eventually periodic, and if and only if it is a fixed point of a loxodromic element of $\mathrm{PGL}_2(\mathbb{F}_q[Y])$.*

(2) *A quadratic irrational $\alpha \in K_v$ is reciprocal if and only if the period $a_0, \dots, a_{N-1}$ of the continued fraction expansion of α is palindromic up to cyclic permutation and invertible elements, in the sense that there exist $x \in \mathbb{F}_q^\times$ and $p \in \mathbb{N}$ such that for $k = 0, \dots, N-1$, we have $a_{k+p} = x^{(-1)^k} a_{N-k-1}$ (with indices modulo N).*

Proof. (1) The equivalence of being quadratic irrational and having an eventually periodic continued fraction expansion is well-known, see for instance the survey [Las, Theo. 3.1]. The second part of the claim follows from Proposition 17.3.

(2) The proof is similar to the Archimedean case in [Per, §23].[12] For every quadratic irrational $f \in \mathbb{F}_q((Y^{-1}))$, up to the action of $\mathrm{GL}_2(\mathbb{F}_q[Y])$, we may assume that $f, (f^\sigma)^{-1} \in Y^{-1}\mathbb{F}_q[[Y^{-1}]]$

[12] See also [BerN, Coro. 1] by relating, using twice the period, what the authors call the $-$ continued fraction expansion to the standard expansion.

and $f = [0,\ \overline{a_1, a_2, \dots, a_n}]$. Then we may define by induction quadratic irrationals $f_2, \dots, f_n \in \mathbb{F}_q((Y^{-1}))$ over $\mathbb{F}_q(Y)$ such that

$$\frac{1}{f} = a_1 + f_2,\ \ \frac{1}{f_2} = a_2 + f_3,\ \dots,\ \frac{1}{f_{n+1}} = a_{n+1} + f_n,\ \ \frac{1}{f_n} = a_n + f\ .$$

Passing to the Galois conjugates, we have

$$\frac{1}{f^\sigma} = a_1 + f_2^\sigma,\ \ \frac{1}{f_2^\sigma} = a_2 + f_3^\sigma,\ \dots,\ \frac{1}{f_n^\sigma} = a_n + f^\sigma\ .$$

Taking these equations in the reverse order, we have

$$\frac{1}{-\frac{1}{f^\sigma}} = a_n - \frac{1}{f_n^\sigma},\ \ \frac{1}{-\frac{1}{f_n^\sigma}} = a_{n-1} - \frac{1}{f_{n-1}^\sigma},\ \dots,\ \frac{1}{-\frac{1}{f_2^\sigma}} = a_1 - \frac{1}{f^\sigma}\ ,$$

so that, since $-\frac{1}{f^\sigma} \in Y^{-1}\mathbb{F}_q[[Y^{-1}]]$, we have

$$-\frac{1}{f^\sigma} = [0,\ \overline{a_n, \dots, a_2, a_1}]\ .$$

Therefore $f^\sigma = [\overline{-a_n, \dots, -a_2, -a_1}]$. Thus, if f and f^σ are in the same orbit, the periods are palindromic up to cyclic permutation and invertible elements by [Sch2, Theo. 1], [BerN, Theo. 1]. □

17.3 Counting and equidistribution of quadratic irrationals in $\mathbb{Q}_p$

There are interesting arithmetic (uniform) lattices of $\mathrm{PGL}_2(\mathbb{Q}_p)$ constructed using quaternion algebras. In this Section, we study equidistribution properties of loxodromic fixed points elements of these lattices. See for instance [LedP] for an equidistribution result of the eigenvalues of the loxodromic elements. We use [Vig] as our standard reference on quaternion algebras.

Let F be a field and let $a, b \in F^\times$. Let $D = \left(\frac{a,b}{F}\right)$ be the quaternion algebra over F with basis $1, i, j, k$ as a F-vector space such that $i^2 = a$, $j^2 = b$ and $ij = ji = -k$. If $x = x_0 + x_1 i + x_2 j + x_3 k \in D$, then its *conjugate* is

$$\overline{x} = x_0 - x_1 i - x_2 j - x_3 k\,,$$

its (reduced) *norm* is

$$\mathtt{N}(x) = x\,\overline{x}\ = \overline{x}\,x = x_0^2 - a\,x_1^2 - b\,x_2^2 + ab\,x_3^2$$

and its (reduced) *trace* is

$$\mathtt{Tr}(x) = x + \overline{x} = 2\,x_0\,.$$

Let us fix two negative rational integers a, b and let $D = \left(\frac{a,b}{\mathbb{Q}}\right)$. For every field extension E of $\mathbb{Q}$, we denote by D_E the quaternion algebra $D \otimes_{\mathbb{Q}} E$ over E, and we say that D *splits over* E if the E-algebra $D \otimes_{\mathbb{Q}} E$ is isomorphic to $M_2(E)$. The assumption that a, b are negative implies that D does not split over $\mathbb{R}$. Furthermore, when $p \in \mathbb{N}$ is an odd prime, D splits over $\mathbb{Q}_p$ if and only if the equation $a\,x^2 + b\,y^2 = 1$ has a solution in $\mathbb{Q}_p$, see [Vig, page 32].

The *reduced discriminant* of D is

$$\mathrm{Disc}_D = \prod_{q \in \mathrm{Ram}(D)} q \,.$$

where $\mathrm{Ram}(D)$ is the finite set of primes p such that D does not split over $\mathbb{Q}_p$.

For instance, the quaternion algebra $D = \big(\frac{-1,-1}{\mathbb{Q}}\big)$ splits over $\mathbb{Q}_p$ if and only if $p \neq 2$, hence it has reduced discriminant 2.

Assume from now on that $p \in \mathbb{N}$ is a positive rational prime such that D splits over $\mathbb{Q}_p$ and, for simplicity, that $\mathbb{Q}_p$ contains square roots $\sqrt{a}$ and $\sqrt{b}$ of a and b. For example, if $a = b = -1$, this is satisfied if $p \equiv 1 \mod 4$. We then have an isomorphism of $\mathbb{Q}_p$-algebras $\theta = \theta_{a,\,b} : D_{\mathbb{Q}_p} \to M_2(\mathbb{Q}_p)$ defined by

$$\theta(x_0 + x_1 i + x_2 j + x_3 k) = \begin{pmatrix} x_0 + x_1\sqrt{a} & \sqrt{b}\,(x_2 + \sqrt{a}\,x_3) \\ \sqrt{b}\,(x_2 - \sqrt{a}\,x_3) & x_0 - x_1\sqrt{a} \end{pmatrix}, \tag{17.9}$$

so that

$$\det(\theta(x)) = \mathtt{N}(x) \quad \text{and} \quad \mathrm{tr}(\theta(x)) = \mathtt{Tr}(x)\,.$$

If the assumption on the existence of the square roots in $\mathbb{Q}_p$ is not satisfied, we can replace $\mathbb{Q}_p$ by an appropriate finite extension, and prove equidistribution results in this extension.

Let $\mathscr{O}$ be a $\mathbb{Z}\big[\frac{1}{p}\big]$-*order* in $D_{\mathbb{Q}_p}$, that is, a finitely generated $\mathbb{Z}\big[\frac{1}{p}\big]$-submodule of $D_{\mathbb{Q}_p}$ generating $D_{\mathbb{Q}_p}$ as a $\mathbb{Q}_p$-vector space, which is a subring of $D_{\mathbb{Q}_p}$. Let $\mathscr{O}^1$ be the group of elements of norm 1 in $\mathscr{O}$. Then the image $\Gamma^1_{\mathscr{O}}$ of $\theta(\mathscr{O}^1)$ in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is a cocompact lattice, see for instance [Vig, Sect. IV.1]. In fact, this lattice is contained in $\mathrm{PSL}_2(\mathbb{Q}_p)$, hence in $\mathrm{PGL}_2(\mathbb{Q}_p)^+$. In this Section 17.3, we denote by $\mathbb{X}_p$ the Bruhat-Tits tree of $(\mathrm{PSL}_2, \mathbb{Q}_p)$, which is $(p+1)$-regular.

The next result computes the covolume of this lattice.[13]

Proposition 17.9. *Let D be a quaternion algebra over $\mathbb{Q}$ which splits over $\mathbb{Q}_p$ and does not split over $\mathbb{R}$, and let $\mathscr{O}$ be a $\mathbb{Z}\big[\frac{1}{p}\big]$-order in $D_{\mathbb{Q}_p}$. If $\mathscr{O}_{\max}$ is a maximal $\mathbb{Z}\big[\frac{1}{p}\big]$-order in $D_{\mathbb{Q}_p}$ containing $\mathscr{O}$, then*

$$\mathrm{Vol}(\Gamma^1_{\mathscr{O}} \backslash\backslash \mathbb{X}_p) = [\mathscr{O}^1_{\max} : \mathscr{O}^1]\; \frac{p}{12} \prod_{q \,|\, \mathrm{Disc}_D} (q-1)\,.$$

Proof. We refer to [Vig, page 53] for the (common) definition of the *discriminant* $\mathrm{Disc}(\mathbb{Q}_p)$ of the local field $\mathbb{Q}_p$ and $\mathrm{Disc}(D_{\mathbb{Q}_p})$ of the quaternion algebra $D_{\mathbb{Q}_p}$ over the local field $\mathbb{Q}_p$. We will only use the facts that $\mathrm{Disc}(\mathbb{Q}_p) = 1$ as it easily follows from the definition, and that

$$\mathrm{Disc}(D_{\mathbb{Q}_p}) = \mathrm{Disc}(\mathbb{Q}_p)^4 \big(N(p\mathbb{Z}_p)\big)^2 = p^2 \tag{17.10}$$

which follows by [Vig, Lem. 4.7, page 53] and [Vig, Cor. 1.7, page 35] for the first equality and $N(p\mathbb{Z}_p) = \mathrm{Card}(\mathbb{Z}_p/p\mathbb{Z}_p) = \mathrm{Card}(\mathbb{Z}/p\mathbb{Z}) = p$ for the second one.

We refer to [Vig, Sect. II.4] for the definition of the *Tamagawa measure* μ_{T} on $X^\times$ when $X = D_{\mathbb{Q}_p}$ or $X = \mathbb{Q}_p$. It is a Haar measure of the multiplicative locally compact group $X^\times$, and understanding its explicit normalisation is the main point of this proposition. By [Vig,

[13] The index q ranges over the primes dividing Disc_D, that is, over the elements of $\mathrm{Ram}(D)$.

Lem. 4.6, page 52],[14] with dx the Haar measure on the additive group X,[15] with $\|x\|$ the *module* of the left multiplication by $x \in X^\times$ on the additive group X,[16] we have

$$d\mu_{\rm T}(x) = \frac{1}{\sqrt{{\rm Disc}(X)}\ \|x\|}\, dx\ .$$

By [Vig, proof of Lem. 4.3, page 50], identifying $D_{\mathbb{Q}_p}$ to $M_2(\mathbb{Q}_p)$ by θ, the measure of ${\rm GL}_2(\mathbb{Z}_p)$ for the measure $\frac{1}{(1-p^{-1})\ \|x\|}\, dx$ is $1-p^{-2}$. Hence, by scaling and by Equation (17.10), we have

$$\mu_{\rm T}({\rm GL}_2(\mathbb{Z}_p)) = \frac{(1-p^{-2})(1-p^{-1})}{\sqrt{{\rm Disc}(D_{\mathbb{Q}_p})}} = \frac{(p^2-1)(p-1)}{p^4}\,.$$

By [Vig, Lem. 4.3, page 49], the mass of $\mathbb{Z}_p^\times$ for the measure $\frac{1}{(1-p^{-1})\ \|x\|}\, dx$ on $\mathbb{Q}_p^\times$ is 1, hence by scaling

$$\mu_{\rm T}(\mathbb{Z}_p^\times) = \frac{1-p^{-1}}{\sqrt{{\rm Disc}(\mathbb{Q}_p)}} = \frac{p-1}{p}\,.$$

By [Vig, pages 53–54], since we have an exact sequence

$$1 \longrightarrow {\rm SL}_2(\mathbb{Q}_p) \longrightarrow {\rm GL}_2(\mathbb{Q}_p) \xrightarrow{\ \det\ } \mathbb{Q}_p^\times \longrightarrow 1\ ,$$

the Tamagawa measure of ${\rm GL}_2(\mathbb{Q}_p)$ disintegrates by the determinant over the Tamagawa measure of $\mathbb{Q}_p^\times$ with conditional measures the translates of a measure on ${\rm SL}_2(\mathbb{Q}_p)$, called the *Tamagawa measure* of ${\rm SL}_2(\mathbb{Q}_p)$ and again denoted by $\mu_{\rm T}$. Thus,

$$\mu_{\rm T}({\rm SL}_2(\mathbb{Z}_p)) = \frac{\mu_{\rm T}({\rm GL}_2(\mathbb{Z}_p))}{\mu_{\rm T}(\mathbb{Z}_p^\times)} = \frac{p^2-1}{p^3}$$

By Example 3 on page 108 of [Vig], since the $\mathbb{Z}[\frac{1}{p}]$-order $\mathscr{O}_{\max}$ is maximal, we have, with $G = \theta(\mathscr{O}^1_{\max})$,

$$\mu_{\rm T}(G\backslash\, {\rm SL}_2(\mathbb{Q}_p)) = \frac{1}{24}\,(1-p^{-2}) \prod_{q\,|\,{\rm Disc}_D} (q-1)\ .$$

Since ${\rm GL}_2(\mathbb{Q}_p)$ acts transitively on $V\mathbb{X}_p$ with stabiliser of the base point $* = [\mathbb{Z}_p \times \mathbb{Z}_p]$ the maximal compact subgroup ${\rm GL}_2(\mathbb{Z}_p)$,[17] and by the centred equation mid-page 116 of [Ser3], we have

$$\begin{aligned}{\rm Vol}(G\backslash\!\backslash\mathbb{X}_p) &= \sum_{[x]\in G\backslash V\mathbb{X}_p} \frac{1}{|G_x|} = \frac{\mu_{\rm T}(G\backslash\, {\rm GL}_2(\mathbb{Q}_p))}{\mu_{\rm T}({\rm GL}_2(\mathbb{Z}_p))} = \frac{\mu_{\rm T}(G\backslash\, {\rm SL}_2(\mathbb{Q}_p))}{\mu_{\rm T}({\rm SL}_2(\mathbb{Z}_p))}\\ &= \frac{p}{24} \prod_{q|\ {\rm Disc}_D} (q-1)\,.\end{aligned}$$

The natural homomorphism $G = \theta(\mathscr{O}^1_{\max}) \to \Gamma^1_{\mathscr{O}_{\max}}$ is 2-to-1, so that

$${\rm Vol}(\Gamma^1_{\mathscr{O}_{\max}}\backslash\!\backslash\mathbb{X}_p) = 2\ {\rm Vol}(G\backslash\!\backslash\mathbb{X}_p)\,.$$

[14] See more precisely the top of page 55 in op. cit.
[15] with a normalisation that does not need to be made precise
[16] so that $(M_x)_* dx = \|x\|\, dx$ where $M_x : y \mapsto xy$ is the left multiplication by x on X
[17] See Section 15.1.

Since $[\Gamma^1_{\mathscr{O}_{\max}} : \Gamma^1_{\mathscr{O}}] = [\mathscr{O}^1_{\max} : \mathscr{O}^1]$, Proposition 17.9 follows. □

Note that the fixed points z for the action on $\mathbb{P}^1(\mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$ by homographies of the elements in the image of $\theta(D)$ are quadratic over $\mathbb{Q}(\sqrt{a}, \sqrt{b})$. More precisely, $\frac{z}{\sqrt{b}}$ is quadratic over $\mathbb{Q}(\sqrt{a})$. An immediate application of Theorem 17.1, using Proposition 17.9, gives the following result of equidistribution of quadratic elements in $\mathbb{Q}_p$ over $\mathbb{Q}(\sqrt{a}, \sqrt{b})$.

Theorem 17.10. *Let Γ be a finite index subgroup of $\Gamma^1_{\mathscr{O}}$, and let $\gamma_0 \in \Gamma$ be a loxodromic element of Γ. Then as $s \to +\infty$,*

$$\frac{(p+1)^2\ \prod_{q|\,\mathrm{Disc}_D}(q-1)\ [\mathscr{O}^1_{\max} : \mathscr{O}^1]\ [\Gamma^1_{\mathscr{O}} : \Gamma]}{24\, p\ \mathrm{Vol}(\Gamma_{\gamma_0^-} \backslash\!\backslash \mathrm{Ax}_{\gamma_0})}\ s^{-1} \sum_{\alpha \in \Gamma\cdot\gamma_0^-,\ h(\alpha)\leqslant s} \Delta_\alpha \overset{*}{\rightharpoonup} \mathrm{Haar}_{\mathbb{Q}_p}\ ,$$

where $\mathscr{O}_{\max}$ is a maximal $\mathbb{Z}[\frac{1}{p}]$-order in $D_{\mathbb{Q}_p}$ containing $\mathscr{O}$, and there exists $\kappa > 0$ such that as $s \to +\infty$

$$\mathrm{Card}\{\alpha \in (\Gamma\cdot\gamma_0^-) \cap \mathbb{Z}_p\ :\ h(\alpha) \leqslant s\} = \frac{24\, p\ \mathrm{Vol}(\Gamma_{\gamma_0^-} \backslash\!\backslash \mathrm{Ax}_{\gamma_0})}{(p+1)^2\ \prod_{q|\,\mathrm{Disc}_D}(q-1)\ [\mathscr{O}^1_{\max} : \mathscr{O}^1]\ [\Gamma^1_{\mathscr{O}} : \Gamma]}\ s + \mathrm{O}(s^{1-\kappa})\ . \quad \square$$

Assume furthermore that the positive rational prime $p \in \mathbb{N}$ is such that $p \equiv 1 \mod 4$ and that the integer $\frac{p^2-1}{4}$ is not of the form $4^a(8b+7)$ for $a, b \in \mathbb{N}$ (for instance $p = 5$). By Legendre's three squares theorem (see for instance [Gros]), there exist $x'_1, x'_2, x'_3 \in \mathbb{Z}$ such that $\frac{p^2-1}{4} = {x'_1}^2 + {x'_2}^2 + {x'_3}^2$. Hence there are $x_1, x_2, x_3 \in 2\mathbb{Z}$ such that $p^2 - 1 = {x_1}^2 + {x_2}^2 + {x_3}^2$.

A standard consequence of Hensel's theorem says that when p is odd, a number $n \in \mathbb{Z}$ has a square root in $\mathbb{Z}_p$ if n is relatively prime to p and has a square root modulo p, see for instance [Kna, page 351]. Thus, $1 - p^2$ has a square root in $\mathbb{Z}_p$, that we denote by $\sqrt{1-p^2}$. As noticed above, since $p \equiv 1 \mod 4$, the element -1 has a square root in $\mathbb{Q}_p$, that we denote by ε. The element

$$\alpha_0 = \frac{\varepsilon\, x_1 + \sqrt{1-p^2}}{x_3 + \varepsilon\, x_2}$$

is a quadratic irrational in $\mathbb{Q}_p$ over $\mathbb{Q}(\varepsilon)$.

The following result is a counting and equidistribution result of quadratic irrationals over $\mathbb{Q}(\varepsilon)$ in $\mathbb{Q}_p$. We denote by α^σ the Galois conjugate of a quadratic irrational α in $\mathbb{Q}_p$ over $\mathbb{Q}(\varepsilon)$, and by

$$h(\alpha) = \frac{1}{|\alpha - \alpha^\sigma|_p}$$

the *complexity* of α.

Theorem 17.11. *Let $D = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ be Hamilton's quaternion algebra over $\mathbb{Q}$. Let $p \in \mathbb{N}$ be a positive rational prime with $p \equiv 1 \mod 4$ such that $p^2 - 1 = {x_1}^2 + {x_2}^2 + {x_3}^2$ for some $x_1, x_2, x_3 \in 2\mathbb{Z}$, and let $\mathscr{O}$ be the $\mathbb{Z}[\frac{1}{p}]$-order*[18]

$$\mathscr{O} = \big\{y \in \mathbb{Z}[\tfrac{1}{p}] + \mathbb{Z}[\tfrac{1}{p}]\ i + \mathbb{Z}[\tfrac{1}{p}]\ j + \mathbb{Z}[\tfrac{1}{p}]\ k\ :\ y \equiv 1 \mod 2\big\}$$

[18] This order plays an important role in the construction of Ramanujan graphs by Lubotzky, Phillips and Sarnak [LuPS1, LuPS2] (see also [Lub2, §7.4]), and in the explicit construction of free subgroups of SO(3) in order to construct Hausdorff-Banach-Tarsky paradoxical decompositions of the 2-sphere, see for instance [Lub2, page 11].

in $D_{\mathbb{Q}_p}$. Let Γ be a finite index subgroup of $\Gamma^1_{\mathscr{O}}$. Then as $s \to +\infty$,

$$\frac{(p+1)^2\,[\Gamma^1_{\mathscr{O}}:\Gamma]}{2\,p^2\,k_\Gamma}\;s^{-1}\sum_{\alpha\in\Gamma\cdot\alpha_0,\;h(\alpha)\leqslant s}\Delta_\alpha\;\overset{*}{\rightharpoonup}\;\mathrm{Haar}_{\mathbb{Q}_p}\;,$$

where k_Γ is the smallest positive integer such that $\begin{bmatrix}1+\varepsilon\,x_1 & -x_3+\varepsilon\,x_2\\ x_3+\varepsilon\,x_2 & 1-\varepsilon\,x_1\end{bmatrix}^{k_\Gamma}\in\Gamma$. Furthermore, there exists $\kappa>0$ such that as $s\to+\infty$

$$\mathrm{Card}\{\alpha\in(\Gamma\cdot\alpha_0)\cap\mathbb{Z}_p\;:\;h(\alpha)\leqslant s\}=\frac{2\,p^2\,k_\Gamma}{(p+1)^2\,[\Gamma^1_{\mathscr{O}}:\Gamma]}\;s+\mathrm{O}(s^{1-\kappa})\;.$$

Proof. The group $\mathscr{O}^\times$ of invertible elements of $\mathscr{O}$ is

$$\mathscr{O}^\times=\big\{x\in\mathscr{O}\;:\;\mathtt{N}(x)\in p^{\mathbb{Z}}\big\}\;.$$

The centre of $\mathscr{O}^\times$ is $Z(\mathscr{O}^\times)=\{\pm p^n\;:\;n\in\mathbb{Z}\}$ and the centre of $\mathscr{O}^1$ is $Z(\mathscr{O}^1)=\{\pm 1\}$. We identify $\mathscr{O}^1/Z(\mathscr{O}^1)$ with its image in $\mathscr{O}^\times/Z(\mathscr{O}^\times)$. The quotient group $\mathscr{O}^\times/Z(\mathscr{O}^\times)$ is a free group on $s=\frac{p+1}{2}$ generators $\gamma_1,\gamma_2,\dots,\gamma_s$, which are the images modulo $Z(\mathscr{O}^\times)$ of some elements of $\mathscr{O}$ of norm p, see for instance [Lub2, Coro. 2.1.11].[19]

Since $\mathtt{N}(p)=p^2$, any reduced word of even length in $S=\{\gamma_1^\pm,\gamma_2^\pm,\dots,\gamma_s^\pm\}$ belongs to $\mathscr{O}^1/Z(\mathscr{O}^1)$. Two distinct elements in S differ by a reduced word of length 2, and γ_1 does not belong to $\mathscr{O}^1/Z(\mathscr{O}^1)$. Hence $\{1,\gamma_1\}$ is a system of left coset representatives of $\mathscr{O}^1/Z(\mathscr{O}^1)$ in $\mathscr{O}^\times/Z(\mathscr{O}^\times)$, and the index of $\mathscr{O}^1/Z(\mathscr{O}^1)$ in $\mathscr{O}^\times/Z(\mathscr{O}^\times)$ is

$$[\mathscr{O}^\times/Z(\mathscr{O}^\times):\mathscr{O}^1/Z(\mathscr{O}^1)]=2\;. \tag{17.11}$$

Let

$$g_0=\begin{pmatrix}\frac{1+\varepsilon\,x_1}{p} & \frac{-x_3+\varepsilon\,x_2}{p}\\ \frac{x_3+\varepsilon\,x_2}{p} & \frac{1-\varepsilon\,x_1}{p}\end{pmatrix}\;.$$

By the definition of the isomorphism θ in Equation (17.9) (with $\sqrt{a}=\sqrt{b}=\varepsilon$) and of the integers x_1,x_2,x_3, the element g_0 belongs to $\theta(\mathscr{O})$ since x_1,x_2,x_3 are even (and p is odd), and $\det g_0=1$. Hence $g_0\in\theta(\mathscr{O}^1)$. Its fixed points for its action by homography on $\mathbb{P}^1(\mathbb{Q}_p)$ are, by an easy computation,

$$\frac{\varepsilon\,x_1\pm\sqrt{1-p^2}}{x_3+\varepsilon\,x_2}\;.$$

In particular, α_0 is one of these two fixed points. Note that $\operatorname{tr} g_0=\frac{2}{p}$, hence $|v_p(\operatorname{tr} g_0)|=1$, and the image $[g_0]$ of g_0 in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is a primitive loxodromic element of $\Gamma^1_{\mathscr{O}}$.

Let us define

$$\gamma_0=[g_0]^{u\,k_\Gamma}$$

where $u\in\{\pm1\}$ is chosen so that $\gamma_0^-=\alpha_0$ and where k_Γ is defined in the statement of Theorem 17.11. Since Γ has finite index in $\Gamma^1_{\mathscr{O}}$, some power of $[g_0]$ does belong to Γ, hence k_Γ exists (and note that $k_\Gamma=1$ if $\Gamma=\Gamma^1_{\mathscr{O}}$). By the minimality of k_Γ, the element γ_0 is a primitive loxodromic element of Γ. We will apply Theorem 17.1 to this γ_0.

[19] The group $\mathscr{O}^\times/Z(\mathscr{O}^\times)$ is denoted by $\Lambda(2)$ in [Lub2, page 11].

The algebra isomorphism θ induces a group isomorphism from $\mathscr{O}^\times/Z(\mathscr{O}^\times)$ onto its image in $\mathrm{PGL}_2(\mathbb{Q}_p)$, that we denote by $\Gamma^\times_{\mathscr{O}}$.[20] By [Lub2, Lem. 7.4.1], the group $\Gamma^\times_{\mathscr{O}}$ acts simply transitively on the vertices of the Bruhat-Tits tree $\mathbb{X}_p$.

In particular, $\Gamma^1_{\mathscr{O}}$ acts freely on $\mathbb{X}_p$, and by Equation (17.11), we have

$$\begin{aligned}\mathrm{Vol}(\Gamma^1_{\mathscr{O}}\backslash\backslash\mathbb{X}_p) &= [\Gamma^\times_{\mathscr{O}} : \Gamma^1_{\mathscr{O}}]\ \mathrm{Vol}(\Gamma^\times_{\mathscr{O}}\backslash\backslash\mathbb{X}_p)\\ &= [\mathscr{O}^\times/Z(\mathscr{O}^\times) : \mathscr{O}^1/Z(\mathscr{O}^1)]\ \mathrm{Card}\big(\Gamma^\times_{\mathscr{O}}\backslash V\mathbb{X}_p\big) = 2\ .\end{aligned} \tag{17.12}$$

Again since $\Gamma^1_{\mathscr{O}}$ (hence Γ) acts freely on $\mathbb{X}_p$, since γ_0 is primitive loxodromic in Γ, and by Equation (15.6), we have

$$\begin{aligned}\mathrm{Vol}(\Gamma_{\gamma_0^-}\backslash\backslash\,\mathrm{Ax}_{\gamma_0}) &= \mathrm{Card}\big(\Gamma_{\gamma_0^-}\backslash V\mathrm{Ax}_{\gamma_0}\big) = \lambda(\gamma_0)\\ &= k_\Gamma\ \lambda([g_0]) = 2\ k_\Gamma\ |v_p(\mathrm{tr}\, g_0)| = 2\ k_\Gamma\ .\end{aligned} \tag{17.13}$$

Using Equations (17.12) and (17.13), the result now follows from Theorem 17.1. □

[20]This group is denoted by $\Gamma(2)$ in [Lub2, page 95].

Chapter 18

Equidistribution and counting of crossratios

We use the same notation as in Chapter 17: K_v is a non-Archimedean local field, with valuation v, valuation ring $\mathscr{O}_v$, choice of uniformiser π_v, residual field k_v of order q_v, and $\mathbb{X}_v$ is the Bruhat-Tits tree of (PGL_2, K_v). Let Γ be a lattice in $\mathrm{PGL}_2(K_v)$.

In this Chapter, we give counting and equidistribution results in $K_v = \partial_\infty \mathbb{X}_v - \{\infty\}$ of orbit points under Γ, using a complexity defined using crossratios, which is different from the one in Chapter 17. We refer to [PaP14b] for the development when K_v is $\mathbb{R}$ or $\mathbb{C}$ with its standard absolute value.

Recall that the *crossratio* of four pairwise distinct points a, b, c, d in $\mathbb{P}_1(K_v) = K_v \cup \{\infty\}$ is

$$[a, b, c, d] = \frac{(c-a)\,(d-b)}{(c-b)\,(d-a)} \in (K_v)^\times \ ,$$

with the standard conventions when one of the points is ∞. Adopting Ahlfors's terminology in the complex case, the *absolute crossratio* of four pairwise distinct points $a, b, c, d \in \mathbb{P}_1(K_v) = K_v \cup \{\infty\}$ is

$$|a, b, c, d|_v = |[a, b, c, d]|_v = \frac{|c-a|_v\,|d-b|_v}{|c-b|_v\,|d-a|_v} \,,$$

with conventions analogous to the previous ones when one of the points is ∞. As in the classical case, the crossratio and the absolute crossratio are invariant under the diagonal projective action of $\mathrm{GL}_2(K_v)$ on the set of quadruples of pairwise distinct points in $\mathbb{P}_1(K_v)$.[1]

18.1 Counting and equidistribution of crossratios of loxodromic fixed points

Let $\alpha, \beta \in K_v$ be loxodromic fixed points of Γ. Recall that $\alpha^\sigma, \beta^\sigma$ is the other fixed point of a loxodromic element of Γ fixing α, β, respectively. The *relative height* of β with respect to α

[1]The logarithm in base q_v of this absolute crossratio is up to the order equal to the (logarithmic) crossratio $[\![\cdot, \cdot, \cdot, \cdot]\!]$ introduced in Section 2.6: More precisely, if $\xi_1, \xi_2, \xi_3, \xi_4$ are pairwise distinct points in the boundary of $\mathbb{X}_v$, then

$$[\![\xi_1, \xi_2, \xi_3, \xi_4]\!] = \log_{q_v} |\xi_1, \xi_4, \xi_3, \xi_2|_v \ .$$

We will not use this relationship in this book.

is[2]

$$h_\alpha(\beta) = \frac{1}{|\alpha-\alpha^\sigma|_v\,|\beta-\beta^\sigma|_v}\ \max\big\{|\beta-\alpha|_v\,|\beta^\sigma-\alpha^\sigma|_v,\ |\beta-\alpha^\sigma|_v\,|\beta^\sigma-\alpha|_v\big\}\ .$$

When $\beta \notin \{\alpha, \alpha^\sigma\}$, we have

$$h_\alpha(\beta) = \max\{|\alpha,\beta,\beta^\sigma,\alpha^\sigma|_v,\ |\alpha,\beta^\sigma,\beta,\alpha^\sigma|_v\} = \frac{1}{\min\{|\alpha,\beta,\alpha^\sigma,\beta^\sigma|_v,\ |\alpha,\beta^\sigma,\alpha^\sigma,\beta|_v\}}\ . \quad (18.1)$$

We will use the relative height as a complexity when β varies in a given orbit of Γ (and α is fixed).

The following properties of relative heights are easy to check using the definitions, the invariance properties of the crossratio and Equation (17.1).

Lemma 18.1. *Let $\alpha, \beta \in K_v$ be loxodromic fixed points of Γ. Then*

(1) $h_{\alpha^\rho}(\beta^\tau) = h_\alpha(\beta)$ for all $\rho, \tau \in \{\mathrm{id}, \sigma\}$.

(2) If $\beta \in \{\alpha, \alpha^\sigma\}$, then $h_\alpha(\beta) = 1$.

(3) $h_{\gamma\cdot\alpha}(\gamma\cdot\beta) = h_\alpha(\beta)$ for every $\gamma \in \Gamma$.

(4) $h_\alpha(\gamma\cdot\beta) = h_\alpha(\beta)$ for every $\gamma \in \mathrm{Stab}_\Gamma(\{\alpha, \alpha^\sigma\})$. □

The following result relates the relative height of two loxodromic fixed points with the distance between the two translation axes.

Proposition 18.2. *Let $\alpha, \beta \in K_v$ be loxodromic fixed points of Γ such that $\beta \notin \{\alpha, \alpha^\sigma\}$. Then*

$$h_\alpha(\beta) = q_v^{\,d(]\alpha,\alpha^\sigma[\,,\,]\beta,\beta^\sigma[)}\ .$$

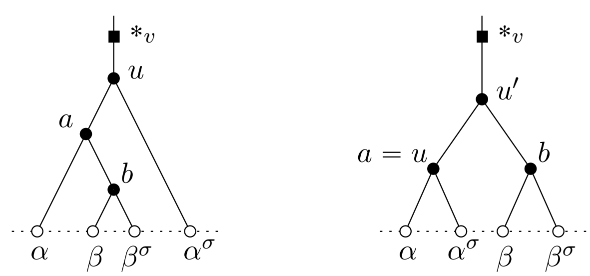
In particular, we have $h_\alpha(\beta) > 1$ if and only if the geodesic lines $]\alpha, \alpha^\sigma[$ and $]\beta, \beta^\sigma[$ in $\mathbb{X}_v$ are disjoint, and $h_\alpha(\beta) = 1$ otherwise (using Lemma 18.1 (2) when $\beta \in \{\alpha, \alpha^\sigma\}$).

Proof. Up to replacing $\alpha, \beta, \alpha^\sigma, \beta^\sigma$ by their images under a large enough power γ of a loxodromic element in Γ with attracting fixed point in $\mathscr{O}_v$, we may assume that these four points belong to $\mathscr{O}_v$. Note that γ exists since $\Lambda\Gamma = \partial_\infty\mathbb{X}_v$, and it preserves the relative height by Lemma 18.1 (3) as well as the distances between translation axes.

Let $A =]\alpha, \alpha^\sigma[$ and $B =]\beta, \beta^\sigma[$. Let u be the closest point to $*_v$ on A, so that

$$v(\alpha - \alpha^\sigma) = d(u, *_v)\ .$$

We will consider five configurations.



[2]The factor $|\alpha - \alpha^\sigma|_v$ in the denominator, that did not appear in [PaP14b] in the analogous definition for the case when K_v is $\mathbb{R}$ or $\mathbb{C}$, is there in order to simplify the statements below.

Case 1. First assume that A and B are disjoint. Let $[a, b]$ be the common perpendicular from A to B, with $a \in A$, so that, by the geometric interpretation of elements in $\mathscr{O}_v$ given in Section 15.1, we have

$$d(A, B) = d(a, b)\ .$$

First assume that $u \neq a$. Up to exchanging α, α^σ (which does not change $d(A, B)$ or $h_\alpha(\beta)$ by Lemma 18.1 (1)), we may assume that $a \in [u, \alpha[$. Then (see the picture on the left above),

$$v(\beta - \beta^\sigma) = d(b, *_v), \quad v(\alpha - \beta) = v(\alpha - \beta^\sigma) = d(a, *_v)$$

and

$$v(\alpha^\sigma - \beta) = v(\alpha^\sigma - \beta^\sigma) = d(u, *_v)\ .$$

Therefore

$$\begin{aligned}|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v &= |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v = q_v^{\,v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)}\\ &= q_v^{d(a,*_v)-d(b,*_v)} = q_v^{-d(a,b)} = q_v^{-d(A,B)}\ ,\end{aligned}$$

which proves the result by Equation (18.1).

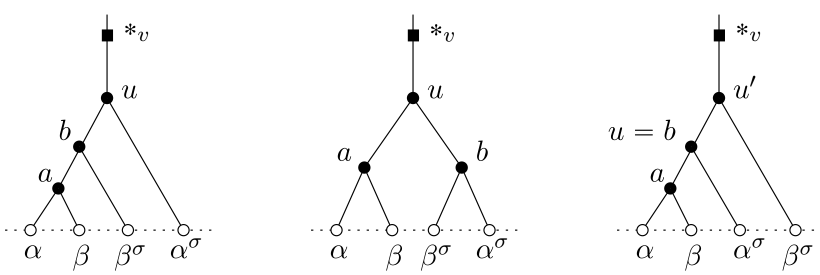
Assume on the contrary that $u = a$. Let $u' \in V\mathbb{X}_v$ be such that $[a, *_v] \cap [a, b] = [a, u']$. Note that $u' \in [*_v, b]$ since $\beta, \beta^\sigma \in \mathscr{O}_v$. Then (see the picture on the right above),

$$v(\beta - \beta^\sigma) = d(b, *_v), \quad v(\alpha - \beta) = v(\alpha - \beta^\sigma) = v(\alpha^\sigma - \beta) = v(\alpha^\sigma - \beta^\sigma) = d(u', *_v)\ .$$

Therefore

$$\begin{aligned}|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v &= |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v = q_v^{\,v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)}\\ &= q_v^{2d(u',*_v)-d(a,*_v)-d(b,*_v)} = q_v^{-d(a,b)} = q_v^{-d(A,B)}\ ,\end{aligned}$$

which proves the result by Equation (18.1).



Case 2. Now assume that A and B are not disjoint, so that

$$d(A, B) = 0\ .$$

Since $\beta \notin \{\alpha, \alpha^\sigma\}$, the intersection $A \cap B$ is a compact segment $[a, b]$ (possibly with $a = b$) in $\mathbb{X}_v$. Up to exchanging a and b, α and α^σ, as well as β and β^σ (which does not change $d(A, B)$ or $h_\alpha(\beta)$ by Lemma 18.1 (1)), we may assume that $\alpha, a, b, \alpha^\sigma$ and $\beta, a, b, \beta^\sigma$ are in this order on A and B respectively, and that $a \in [u, \alpha[$.

Assume first that $b \in]u, \alpha[$. Then (see the picture on the left above),

$$v(\alpha - \beta) = d(a, *_v), \quad v(\alpha - \beta^\sigma) = v(\beta - \beta^\sigma) = d(b, *_v)$$

and

$$v(\beta - \alpha^\sigma) = v(\beta^\sigma - \alpha^\sigma) = d(u, *_v) \; .$$

Therefore

$$|\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v = q_v^{\,v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} = q_v^{d(a,\,*_v)-d(b,\,*_v)} = q_v^{d(a,b)} \geqslant 1 \; ,$$

and

$$|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v = q_v^{\,v(\alpha-\beta^\sigma)+v(\alpha^\sigma-\beta)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} = q_v^0 = 1 = q_v^{-d(A,B)} \; ,$$

which proves the result by Equation (18.1).

Assume that $b \in \,]u, \alpha^\sigma[$. Then (see the picture in the middle above),

$$v(\alpha - \beta) = d(a, *_v), \quad v(\alpha^\sigma - \beta^\sigma) = d(b, *_v)$$

and

$$v(\alpha - \beta^\sigma) = v(\beta - \alpha^\sigma) = v(\beta - \beta^\sigma) = d(u, *_v) \; .$$

Therefore

$$\begin{aligned}|\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v &= q_v^{\,v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} \\ &= q_v^{d(a,\,*_v)+d(b,\,*_v)-2d(u,*_v)} = q_v^{d(a,b)} \geqslant 1 \; ,\end{aligned}$$

and

$$|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v = q_v^{\,v(\alpha-\beta^\sigma)+v(\alpha^\sigma-\beta)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} = q_v^0 = 1 = q_v^{-d(A,B)} \; ,$$

which proves the result by Equation (18.1).

Assume at last that $b = u$. Let $u' \in V\mathbb{X}_v$ be such that $[b, *_v] \cap [b, \beta^\sigma[\, = [b, u']$. Then (see the picture on the right above),

$$v(\alpha - \beta) = d(a, *_v), \quad v(\alpha^\sigma - \beta) = d(u, *_v)$$

and

$$v(\alpha - \beta^\sigma) = v(\beta - \beta^\sigma) = v(\alpha^\sigma - \beta^\sigma) = d(u', *_v) \; .$$

Therefore

$$\begin{aligned}|\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v &= q_v^{\,v(\alpha-\beta)+v(\alpha^\sigma-\beta^\sigma)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} \\ &= q_v^{d(a,\,*_v)-d(u,*_v)} = q_v^{d(a,b)} \geqslant 1 \; ,\end{aligned}$$

and

$$|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v = q_v^{\,v(\alpha-\beta^\sigma)+v(\alpha^\sigma-\beta)-v(\alpha-\alpha^\sigma)-v(\beta-\beta^\sigma)} = q_v^0 = 1 = q_v^{-d(A,B)} \; ,$$

which proves the result by Equation (18.1). □

The next result says that the relative height is an appropriate complexity on a given orbit under Γ of a loxodromic fixed point, and that the counting function we will study is well defined. We denote by Γ_ξ the stabiliser in Γ of a point $\xi \in \partial_\infty \mathbb{X}_v = \mathbb{P}_1(K_v)$.

Lemma 18.3. *Let $\alpha, \beta \in K_v$ be loxodromic fixed points of Γ. Then for every $s > 1$, the set*

$$E_s = \{\beta' \in \Gamma_\alpha \backslash \Gamma \cdot \beta \;:\; 1 < h_\alpha(\beta') \leqslant s\}$$

is finite.

Proof. The set E_s is well defined by Lemma 18.1 (4). Recall that a loxodromic fixed point is one of the two points at infinity of a unique translation axis. By local finiteness, there are, up to the action of the stabiliser of a fixed translation axis A, only finitely many images under Γ of another translation axis B at distance at most $\frac{\ln s}{\ln q_v}$ from A. Since the stabiliser of A contains the stabiliser of either of its points at infinity with index at most 2, the result then follows from Proposition 18.2. $\square$

We now state our main counting and equidistribution result of orbits of loxodromic fixed points, when the complexity is the relative height with respect to a fixed loxodromic fixed point.

Theorem 18.4. *Let Γ be a lattice in $\mathrm{PGL}_2(K_v)^+$. Let $\alpha_0, \beta_0 \in K_v$ be loxodromic fixed points of Γ. Then for the weak-star convergence of measures on $K_v - \{\alpha_0, \alpha_0^\sigma\}$, as $s \to +\infty$,*

$$\frac{(q_v+1)^2\ \mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)}{2\,q_v^2\ |\alpha_0-\alpha_0^\sigma|_v\ \mathrm{Vol}(\Gamma_{\beta_0}\backslash\backslash\,]\beta_0,\beta_0^\sigma[)}\ s^{-1}\sum_{\beta\in\Gamma\cdot\beta_0\;:\;h_{\alpha_0}(\beta)\leqslant s}\Delta_\beta \ \overset{*}{\rightharpoonup}\ \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha_0|_v\,|z-\alpha_0^\sigma|_v}\ .$$

Furthermore, there exists $\kappa > 0$ such that, as $s \to +\infty$,

$$\begin{aligned}&\mathrm{Card}\ \Gamma_{\alpha_0}\backslash\{\beta\in\Gamma\cdot\beta_0\ :\ h_{\alpha_0}(\beta)\leqslant s\}\\ &\quad=\frac{2\,q_v\,(q_v-1)\ \mathrm{Vol}(\Gamma_{\alpha_0}\backslash\backslash\,]\alpha_0,\alpha_0^\sigma[)\ \mathrm{Vol}(\Gamma_{\beta_0}\backslash\backslash\,]\beta_0,\beta_0^\sigma[)}{(q_v+1)^2\ \mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)}\ s+\mathrm{O}(s^{1-\kappa})\ .\end{aligned}$$

For every $\beta' \in\,]0,1]$, there exists $\kappa > 0$ such that for every $\psi \in \mathscr{C}_c^{\beta'}(K_v - \{\alpha_0, \alpha_0^\sigma\})$, where $K_v - \{\alpha_0, \alpha_0^\sigma\}$ is endowed with the distance-like map $d_{]\alpha_0,\alpha_0^\sigma[}$,[3] there is an error term in the equidistribution claim of Theorem 18.4 when evaluated on ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_{\beta'})$. This result applies for instance if $\psi : K_v - \{\alpha_0, \alpha_0^\sigma\} \to \mathbb{R}$ is locally constant with compact support, see Remark 3.11.

Proof. The proof of the equidistribution claim is similar to the one of Theorem 17.1. We now apply Theorem 15.4 with $\mathbb{D}^- := \,]\alpha_0, \alpha_0^\sigma[$ and $\mathbb{D}^+ := \,]\beta_0, \beta_0^\sigma[$. Since Γ is contained in $\mathrm{PGL}_2(K_v)^+$, the length spectrum L_Γ of Γ is equal to $2\mathbb{Z}$. The families $\mathscr{D}^\pm = (\gamma\mathbb{D}^\pm)_{\gamma\in\Gamma/\Gamma_{\mathbb{D}^\pm}}$ are locally finite, and $\|\sigma^-_{\mathscr{D}^+}\|$ is finite and nonzero.

Arguing as in the proof of Theorem 17.1,[4] we have

$$\lim_{n\to+\infty}\frac{({q_v}^2-1)(q_v+1)}{2\,{q_v}^3}\ \frac{\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\ {q_v}^{-n}\sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}}\Delta_{\gamma\cdot\beta_0}\ =\ (\partial^+)_*\widetilde{\sigma}^+_{\mathbb{D}^-} \tag{18.2}$$

for the weak-star convergence of measures on $\partial_\infty\mathbb{X}_v - \partial_\infty\mathbb{D}_-$. Futhermore, for every $\beta' \in\,]0,1]$, there exists $\kappa > 0$ such that for every β-Hölder-continuous function $\psi \in \mathscr{C}_c^{\beta'}(\partial_\infty\mathbb{X}_v - \partial_\infty\mathbb{D}_-)$, where $\partial_\infty\mathbb{X}_v - \partial_\infty\mathbb{D}_-$ is endowed with the distance-like map $d_{\mathbb{D}^-}$, there is an error term in the equidistribution statement of Equation (18.2) when evaluated on ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_{\beta'})$.

By Proposition 18.2, we have

$$h_{\alpha_0}(\gamma\cdot\beta_0) = {q_v}^{d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)}\ .$$

[3] See Equation (3.8).
[4] See Equation (17.3).

By Proposition 15.2 (5), we have

$$(\partial^+)_*\widetilde{\sigma}^+_{\mathbb{D}^-}(z) = \frac{|\alpha_0-\alpha_0^\sigma|_v}{|z-\alpha_0|_v\,|z-\alpha_0^\sigma|_v}\; d\,\mathrm{Haar}_{K_v}(z)$$

for z in the full measure subset $K_v-\{\alpha_0,\alpha_0^\sigma\}$ of $\partial_\infty\mathbb{X}_v$. Hence, using the change of variable $s={q_v}^n$, we have, with the appropriate error term,

$$\lim_{s\to+\infty}\frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3}\;\frac{\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{|\alpha_0-\alpha_0^\sigma|_v\;\|\sigma^-_{\mathscr{D}^+}\|}\;s^{-1}\sum_{\substack{\gamma\in\Gamma/\Gamma_{\mathbb{D}^+}\\ 1<h_{\alpha_0}(\gamma\cdot\beta_0)\leqslant s}}\Delta_{\gamma\cdot\beta_0}\;=\;\frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha_0|_v\,|z-\alpha_0^\sigma|_v}\;.$$

We again denote by ι_{α_0} the index

$$\iota_{\alpha_0}=[\Gamma_{\{\alpha_0,\,\alpha_0^\sigma\}}:\Gamma_{\alpha_0}]\;,$$

and similarly for β_0. Since the stabiliser Γ_{β_0} of β_0 in Γ has index ι_{β_0} in $\Gamma_{\mathbb{D}^+}$ and Γ/Γ_{β_0} identifies with $\Gamma\cdot\beta_0$ by the map $\gamma\mapsto\gamma\cdot\beta_0$, we have, with the appropriate error term,

$$\lim_{s\to+\infty}\frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3}\;\frac{\mathrm{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{|\alpha_0-\alpha_0^\sigma|_v\;\|\sigma^-_{\mathscr{D}^+}\|\;\iota_{\beta_0}}\;s^{-1}\sum_{\substack{\beta\in\Gamma\cdot\beta_0\\ 1<h_{\alpha_0}(\beta)\leqslant s}}\Delta_{\beta}\;=\;\frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha_0|_v\,|z-\alpha_0^\sigma|_v}\;.$$

As in the end of the proof of Theorem 17.1, we have

$$\|\sigma^-_{\mathscr{D}^+}\|=\frac{q_v-1}{q_v\;\iota_{\beta_0}}\;\mathrm{Vol}(\Gamma_{\beta_0}\backslash\!\backslash\,]\beta_0,\beta_0^\sigma[)\;.$$

This proves the equidistribution claim, and its error term.

In order to obtain the counting claim, we note that since $\widetilde{\sigma}^+_{\mathbb{D}^-}$ is invariant under the stabiliser in Γ of $\mathbb{D}^-$, hence under Γ_{α_0}, the measures on both sides of the equidistribution claim in Theorem 18.4 are invariant under Γ_{α_0}, see Proposition 17.4 (5) for the invariance of the right hand side. By Proposition 15.2 (5) and (6), and by the definition of ι_{α_0}, we have

$$\begin{aligned}\int_{\Gamma_{\alpha_0}\backslash(K_v-\{\alpha_0,\alpha_0^\sigma\})}\frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha_0|_v\,|z-\alpha_0^\sigma|_v}&=\frac{\iota_{\alpha_0}}{|\alpha_0-\alpha_0^\sigma|_v}\int_{\Gamma_{\mathbb{D}^-}\backslash\partial^1_+\mathbb{D}^-}d\widetilde{\sigma}^+_{\mathbb{D}^-}\\&=\frac{(q_v-1)\;\iota_{\alpha_0}\;\mathrm{Vol}(\Gamma_{\mathbb{D}^-}\backslash\!\backslash\mathbb{D}^-)}{q_v\;|\alpha_0-\alpha_0^\sigma|_v}\\&=\frac{(q_v-1)\;\mathrm{Vol}(\Gamma_{\alpha_0}\backslash\!\backslash\,]\alpha_0,\alpha_0^\sigma[)}{q_v\;|\alpha_0-\alpha_0^\sigma|_v}\;.\end{aligned}\qquad(18.3)$$

The counting claim follows by evaluating the equidistribution claim on the characteristic function ψ of a compact-open fundamental domain for the action of Γ_{α_0} on $K_v-\{\alpha_0,\alpha_0^\sigma\}$. This characteristic function is locally constant, hence β'-Hölder-continuous for the distance-like function $d_{\mathbb{D}^-}$, as seen end of Section 3.1. □

18.2 Counting and equidistribution of crossratios of quadratic irrationals

In this Section, we give two arithmetic applications of Theorem 18.4.

Let us first consider an application in positive characteristic. Let K be a (global) function field over $\mathbb{F}_q$, let v be a (normalised discrete) valuation of K, let K_v be the associated completion of K and let R_v be the affine function ring associated with v.[5] Given two quadratic irrationals $\alpha, \beta \in K_v$ over K, with Galois conjugates $\alpha^\sigma, \beta^\sigma$ respectively, such that $\beta \notin \{\alpha, \alpha^\sigma\}$, we define the *relative height* of β with respect to α by

$$h_\alpha(\beta) = \frac{1}{\min\{|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_v,\ |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_v\}}\,. \tag{18.4}$$

The following result says that the orbit of any quadratic irrational in K_v over K, by homographies under a given finite index subgroup of the modular group $\mathrm{PGL}_2(R_v)$, equidistributes, when its complexity is given by the relative height with respect to another fixed quadratic irrational α_0. The limit measure is absolutely continuous with respect to the Haar measure on K_v and it is invariant under the stabiliser of α_0 in $\mathrm{PGL}_2(R_v)$ by Proposition 17.4 (5).

Theorem 18.5. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$. Let $\alpha_0, \beta_0 \in K_v$ be quadratic irrationals over K. Let g_0, h_0 be elements in G fixing α_0, β_0 with $v(\mathrm{tr}\, g_0), v(\mathrm{tr}\, h_0) \neq 0$, and let m_0, n_0 be the index of $g_0^{\mathbb{Z}}, h_0^{\mathbb{Z}}$ in the stabiliser of α_0, β_0 in G respectively. Then, as $s \to +\infty$,*

$$\frac{(q_v+1)^2\,\zeta_K(-1)\;n_0\;[\mathrm{GL}_2(R_v):G]}{2\;q_v^2\;(q-1)\;|\alpha_0-\alpha_0^\sigma|_v\;|v(\mathrm{tr}\,h_0)|}\;s^{-1}\sum_{\beta\in G\cdot\beta_0\;:\;h_{\alpha_0}(\beta)\leqslant s}\Delta_\beta \;\overset{*}{\rightharpoonup}\; \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha_0|_v\,|z-\alpha_0^\sigma|_v}\,,$$

and there exists $\kappa > 0$ such that, as $s \to +\infty$,

$$\begin{aligned}&\mathrm{Card}\;\Gamma_{\alpha_0}\backslash\{\beta \in G\cdot\beta_0\;:\;h_{\alpha_0}(\beta)\leqslant s\}\\ &\quad= \frac{4\;q_v\;(q_v-1)\;(q-1)\;|v(\mathrm{tr}\,g_0)|\;|v(\mathrm{tr}\,h_0)|\;|Z(G)|}{(q_v+1)^2\;\zeta_K(-1)\;m_0\;n_0\;[\mathrm{GL}_2(R_v):G]}\;s + \mathrm{O}(s^{1-\kappa})\,.\end{aligned}$$

Proof. This follows, as in the proof of Theorem 17.6, from Theorem 18.4 using Equations (16.6) and (17.8), as well as Equation (18.3) for the counting claim. □

Example 18.6. (1) Theorem 1.16 in the Introduction follows from this result, by taking $K = \mathbb{F}_q(Y)$ and $v = v_\infty$ (so that $q_v = q$), and by using Equation (14.6) in order to simplify the constant.

(2) If G_I is the Hecke congruence subgroup associated with a nonzero ideal I of R_v (see Equation (16.11)), using Lemma 16.5, we have, as $s \to +\infty$,

$$\frac{(q_v+1)^2\,\zeta_K(-1)\;n_0\;N(I)\prod_{\mathfrak{p}|I}(1+\frac{1}{N(\mathfrak{p})})}{2\;q_v^2\;(q-1)\;|\alpha_0-\alpha_0^\sigma|_v\;|v(\mathrm{tr}\,h_0)|}\;s^{-1}\sum_{\beta\in G_I\cdot\beta_0\;:\;h_{\alpha_0}(\beta)\leqslant s}\Delta_\beta \;\overset{*}{\rightharpoonup}\; \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha_0|_v\,|z-\alpha_0^\sigma|_v}\,.$$

The second arithmetic application of Theorem 18.4 is in $\mathbb{Q}_p$. We use the notation of Section 17.3.

[5]See Section 14.2.

Let $p \in \mathbb{N}$ be a positive rational prime with $p \equiv 1 \mod 4$ such that $\frac{p^2-1}{4}$ is not of the form $4^a(8b+7)$ for $a, b \in \mathbb{N}$ (for instance $p = 5$). Let ε be a square root of -1 in $\mathbb{Q}_p$. Let $x_1, x_2, x_3 \in 2\mathbb{Z}$ be such that $p^2 - 1 = {x_1}^2 + {x_2}^2 + {x_3}^2$. We again consider

$$\alpha_0 = \frac{\varepsilon\, x_1 + \sqrt{1-p^2}}{x_3 + \varepsilon\, x_2}\ ,$$

which is a quadratic irrational in $\mathbb{Q}_p$ over $\mathbb{Q}(\varepsilon)$. We denote by α^σ the Galois conjugate of a quadratic irrational α in $\mathbb{Q}_p$ over $\mathbb{Q}(\varepsilon)$, and by

$$h_\alpha(\beta) = \frac{1}{\min\{|\alpha, \beta, \alpha^\sigma, \beta^\sigma|_p,\ |\alpha, \beta^\sigma, \alpha^\sigma, \beta|_p\}} \tag{18.5}$$

the *relative height* of a quadratic irrational β in $\mathbb{Q}_p$ over $\mathbb{Q}(\varepsilon)$ with respect to α, assuming that $\beta \notin \{\alpha, \alpha^\sigma\}$. We again consider Hamilton's quaternion algebra $D = (\frac{-1,-1}{\mathbb{Q}})$ over $\mathbb{Q}$ and its $\mathbb{Z}\big[\frac{1}{p}\big]$-order

$$\mathscr{O} = \big\{y \in \mathbb{Z}\big[\tfrac{1}{p}\big] + \mathbb{Z}\big[\tfrac{1}{p}\big]\, i + \mathbb{Z}\big[\tfrac{1}{p}\big]\, j + \mathbb{Z}\big[\tfrac{1}{p}\big]\, k\ :\ y \equiv 1 \mod 2\big\}\ .$$

The following result says that the orbit of α_0 in $\mathbb{Q}_p$ by homographies under a given finite index subgroup of the arithmetic group $\Gamma^1_{\mathscr{O}}$ (defined in Section 17.3) equidistributes, when its complexity is given by the relative height with respect to α_0, to a measure absolutely continuous with respect to the Haar measure on $\mathbb{Q}_p$.

Theorem 18.7. *With the above notation, let Γ be a finite index subgroup of $\Gamma^1_{\mathscr{O}}$. Then, as $s \to +\infty$,*

$$\frac{(p+1)^2\, [\Gamma^1_{\mathscr{O}} : \Gamma]}{2\, p^2\, k_\Gamma\, |\alpha_0 - \alpha_0^\sigma|_p}\, s^{-1} \sum_{\alpha \in \Gamma \cdot \alpha_0\ :\ h_{\alpha_0}(\alpha) \leqslant s} \Delta_\alpha \ \overset{*}{\rightharpoonup}\ \frac{d\,\mathrm{Haar}_{\mathbb{Q}_p}(z)}{|z - \alpha_0|_p\, |z - \alpha_0^\sigma|_p}\ ,$$

where k_Γ is the smallest positive integer such that $\begin{bmatrix} 1 + \varepsilon\, x_1 & -x_3 + \varepsilon\, x_2 \\ x_3 + \varepsilon\, x_2 & 1 - \varepsilon\, x_1 \end{bmatrix}^{k_\Gamma} \in \Gamma$. Furthermore, there exists $\kappa > 0$ such that, as $s \to +\infty$,

$$\mathrm{Card}\ \ \Gamma_{\alpha_0}\backslash\{\alpha \in \Gamma \cdot \alpha_0\ :\ h_{\alpha_0}(\alpha) \leqslant s\} = \frac{4\, p\, (p-1)\, (k_\Gamma)^2}{(p+1)^2\, [\Gamma^1_{\mathscr{O}} : \Gamma]}\, s + \mathrm{O}(s^{1-\kappa})\ .$$

Proof. This follows, as in the proof of Theorem 17.11, from Theorem 18.4 using Equations (17.12) and (17.13), as well as Equation (18.3) and again Equation (17.13) for the counting claim. □

Chapter 19

Equidistribution and counting of integral representations by quadratic norm forms

In the final Chapter of this book, we give another arithmetic equidistribution and counting result of rational elements in non-Archimedean local fields of positive characteristic, again using our geometric equidistribution and counting results of common perpendiculars in trees summarised in Section 15.4. We use here a complexity defined using the norm forms associated with fixed quadratic irrationals. In particular, the complexity in this Chapter is different from that used in the Mertens type of results in Section 16.1. We refer for instance to [PaP14b, §5.3] for motivations and results in the Archimedean case, and also to [GoP] for higher dimensional norm forms.

Let K be a (global) function field over $\mathbb{F}_q$ of genus g, let v be a (normalised discrete) valuation of K, let K_v be the associated completion of K and let R_v be the affine function ring associated with v.[1] Let $\alpha \in K_v$ be a quadratic irrational over K. The *norm form* $\mathtt{n}_\alpha$ associated with α is the quadratic form $K \times K \to K$ defined by

$$(x, y) \mapsto \mathtt{n}(x - y\alpha) = (x - y\alpha)(x - y\alpha^\sigma) = x^2 - xy\,\mathtt{tr}(\alpha) + y^2\,\mathtt{n}(\alpha)\;.$$

See Proposition 17.4 (3) for elementary transformation properties under elements of $\mathrm{GL}_2(R_v)$ of this norm form.

A pair $(x, y) \in R_v \times R_v$ is an *integral representation* of an element $z \in K$ by the quadratic norm form $\mathtt{n}_\alpha$ if $\mathtt{n}_\alpha(x, y) = z$. The following result describes the projective equidistribution as $s \to +\infty$ of the integral representations by $\mathtt{n}_\alpha$ of elements with absolute value at most s. For every $(x_0, y_0) \in R_v \times R_v$, let $H_{(x_0,y_0)}$ be the stabiliser of (x_0, y_0) for the linear action of any subgroup H of $\mathrm{GL}_2(R_v)$ on $R_v \times R_v$. We use the notation $N\langle x_0, y_0\rangle$ for the norm of the ideal $\langle x_0, y_0\rangle$ generated by x_0, y_0 (see Section 14.2) and the notation $m_{v,\,x_0,\,y_0}$ introduced above Theorem 16.1.

Theorem 19.1. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, let $\alpha \in K_v$ be a quadratic irrational over K, and let $(x_0, y_0) \in R_v \times R_v - \{(0, 0)\}$. Let*

$$c' = \frac{(q_v - 1)\;(q_v + 1)^2\;\zeta_K(-1)\;m_{v,\,x_0,\,y_0}\;(N\langle x_0,\,y_0\rangle)^2\;[\mathrm{GL}_2(R_v) : G]}{q_v^3\;(q - 1)\;q^{g-1}\;[\mathrm{GL}_2(R_v)_{(x_0,\,y_0)} : G_{(x_0,\,y_0)}]}\;.$$

[1] See Section 14.2.

Then for the weak-star convergence of measures on $K_v - \{\alpha, \alpha^\sigma\}$, we have

$$\lim_{s\to+\infty} c'\, s^{-1} \sum_{(x,y)\in G(x_0,y_0),\ |\mathtt{n}(x-y\alpha)|_v\leqslant s} \Delta_{\frac{x}{y}} \;=\; \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha|_v\,|z-\alpha^\sigma|_v}\ .$$

For every $\beta \in\,]0,1]$, there exists $\kappa > 0$ such that for every $\psi \in \mathscr{C}_c^\beta(K_v - \{\alpha,\alpha^\sigma\})$, where $K_v - \{\alpha,\alpha^\sigma\}$ is endowed with the distance-like map $d_{]\alpha,\alpha^\sigma[}$,[2] there is an error term in the equidistribution claim of Theorem 19.1 when evaluated on ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\beta)$. This holds for instance if $\psi : K_v - \{\alpha,\alpha^\sigma\} \to \mathbb{R}$ is locally constant with compact support (see Remark 3.11).

Examples 19.2. (1) Let $(x_0,y_0) = (1,0)$, $K = \mathbb{F}_q(Y)$ and $v = v_\infty$ (so that $g = 0$ and $q_v = q$). Theorem 1.17 in the Introduction follows from Theorem 19.1, using Equations (14.6) and (16.1) to simplify the constant c'.

(2) Let $(x_0,y_0) = (1,0)$ and let $G = G_I$ be the Hecke congruence subgroup of $\mathrm{GL}_2(R_v)$ defined in Equation (16.11). The index in $[\mathrm{GL}_2(R_v) : G_I]$ is given by Lemma 16.5 and G_I satisfies $(G_I)_{(1,0)} = \mathrm{GL}_2(R_v)_{(1,0)}$. For every nonzero ideal I of R_v, for the weak-star convergence of measures on $K_v - \{\alpha,\alpha^\sigma\}$, we have

$$\lim_{s\to+\infty} c_I\, s^{-1} \sum_{(x,y)\in R_v\times I,\ \langle x,y\rangle = R_v,\ |\mathtt{n}(x-y\alpha)|_v\leqslant s} \Delta_{\frac{x}{y}} \;=\; \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha|_v\,|z-\alpha^\sigma|_v}\ ,$$

where

$$c_I = \frac{(q_v-1)\,(q_v+1)^2\,\zeta_K(-1)\,N(I)\prod_{\mathfrak{p}|I}(1+\frac{1}{N(\mathfrak{p})})}{q_v^3\,q^{g-1}}\ .$$

(3) This third example is only interesting when the ideal class number is larger than 1. Given any fractional ideal $\mathfrak{m}$ of R_v, taking $(x_0,y_0) \in R_v \times R_v$ such that the fractional ideals $\langle x_0,y_0\rangle$ and $\mathfrak{m}$ have the same ideal class and $G = \mathrm{GL}_2(R_v)$, using the change of variables $s \mapsto sN(\mathfrak{m})^2$ in the statement of Theorem 19.1, for the weak-star convergence of measures on $K_v - \{\alpha,\alpha^\sigma\}$, with the same error term as for Theorem 19.1, we have

$$\lim_{s\to+\infty} c_{\mathfrak{m}}\, s^{-1} \sum_{(x,y)\in \mathfrak{m}\times\mathfrak{m},\ \langle x,y\rangle = \mathfrak{m},\ N(\mathfrak{m})^{-2}|\mathtt{n}(x-y\alpha)|_v\leqslant s} \Delta_{\frac{x}{y}} \;=\; \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha|_v\,|z-\alpha^\sigma|_v}\ ,$$

where

$$c_{\mathfrak{m}} = \frac{(q_v-1)\,(q_v+1)^2\,\zeta_K(-1)\,m_{v,x_0,y_0}}{q_v^3\,(q-1)\,q^{g-1}}\ .$$

Before proving Theorem 19.1, let us give a counting result which follows from it. Any subgroup of G acts on the left on any orbit of G. Furthermore, the stabiliser G_α of α in G preserves the map $(x,y) \mapsto |\mathtt{n}(x-y\alpha)|_v$, by Proposition 17.4 (3). We may then define a *counting function* $\Psi'(s) = \Psi'_{G,\alpha,x_0,y_0}(s)$ of elements of $R_v \times R_v$ in a linear orbit under a finite index subgroup G of $\mathrm{GL}_2(R_v)$ on which the absolute value of the norm form associated with α is at most s, as

$$\Psi'(s) = \mathrm{Card}\ \ G_\alpha\backslash\big\{(x,\,y)\in G(x_0,\,y_0),\ \ |\mathtt{n}(x-y\alpha)|_v\leqslant s\big\}\ .$$

[2] See Equation (3.8).

Corollary 19.3. *Let G be a finite index subgroup of $\mathrm{GL}_2(R_v)$, let $\alpha \in K_v$ be a quadratic irrational over K, and let $(x_0, y_0) \in R_v \times R_v - \{(0,0)\}$. Let $g_0 \in G_\alpha$ with $v(\mathrm{tr}\, g_0) \neq 0$ and let m_0 be the index of $g_0^{\mathbb{Z}}$ in G_α. Let*

$$c'' = \frac{2\, q_v^2\, (q-1)\, q^{g-1}\, |Z(G)|\, |v(\mathrm{tr}\, g_0)|\, [\mathrm{GL}_2(R_v)_{(x_0,\, y_0)} : G_{(x_0,\, y_0)}]}{(q_v+1)^2\, \zeta_K(-1)\, |\alpha - \alpha^\sigma|_v\, m_0\, m_{v,\, x_0,\, y_0}\, (N\langle x_0,\, y_0\rangle)^2\, [\mathrm{GL}_2(R_v) : G]} \;.$$

Then there exists $\kappa > 0$ such that, as $s \to +\infty$,

$$\Psi'(s) = c''\, s + \mathrm{O}(s^{1-\kappa}) \;.$$

Proof. Using Equation (18.3) (with Γ the image of G in $\mathrm{PGL}_2(R_v)$) and Equation (17.8), we have

$$\int_{G_\alpha \backslash (K_v - \{\alpha, \alpha^\sigma\})} \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha|_v\, |z-\alpha^\sigma|_v} = \frac{2\, (q_v - 1)\, |Z(G)|\, |v(\mathrm{tr}\, g_0)|}{q_v\, |\alpha - \alpha^\sigma|_v\, m_0} \;.$$

The corollary then follows by applying the equidistribution claim in Theorem 19.1 to the characteristic function of a compact-open fundamental domain of $K_v - \{\alpha, \alpha^\sigma\}$ modulo the action by homographies of G_α. □

Example 19.4. Let $(x_0, y_0) = (1, 0)$, $K = \mathbb{F}_q(Y)$, $v = v_\infty$ (so that $g = 0$ and $q_v = q$) and $G = \mathrm{GL}_2(\mathbb{F}_q[Y])$. Using Equations (14.6) and (16.1), Proposition 17.4 (1), the change of variable $s = q^t$ and the fact that $|Z(G)| = q - 1$ in order to simplify the constant c'' of Corollary 19.3, and recalling the expression of the absolute value at ∞ in terms of the degree from Section 14.2, we get the following counting result: For every integral quadratic irrational $\alpha \in \mathbb{F}_q((Y^{-1}))$ over $\mathbb{F}_q(Y)$, there exists $\kappa > 0$ such that, as $t \to +\infty$,

$$\begin{aligned}&\mathrm{Card}\;\; {}_{\mathrm{GL}_2(\mathbb{F}_q[Y])_\alpha}\Big\backslash \left\{ (x,\, y) \in \mathbb{F}_q[Y] \times \mathbb{F}_q[Y] \;:\; \begin{array}{l} \langle x,\, y\rangle = \mathbb{F}_q[Y], \\ \deg(x^2 - xy\,\mathtt{tr}(\alpha) + y^2\,\mathtt{n}(\alpha)) \leqslant t \end{array} \right\} \\ &\quad = \frac{2\, q\, (q-1)^3}{m_0\, (q+1)}\, \deg(\mathrm{tr}\, g_0)\, q^{-\frac{1}{2}\deg(\mathtt{tr}(\alpha)^2 - 4\,\mathtt{n}(\alpha))}\; q^t + \mathrm{O}(q^{t-\kappa}) \;,\end{aligned}$$

where $g_0 \in \mathrm{GL}_2(\mathbb{F}_q[Y])$ fixes α with $\deg(\mathrm{tr}\, g_0) \neq 0$ and m_0 is the index of $g_0^{\mathbb{Z}}$ in the stabiliser $\mathrm{GL}_2(\mathbb{F}_q[Y])_\alpha$ of α in $\mathrm{GL}_2(\mathbb{F}_q[Y])$.

Proof of Theorem 19.1. The proof is similar to that of Theorem 16.1. Let $r = \frac{x_0}{y_0} \in K \cup \{\infty\}$. If $y_0 = 0$, let $g_r = \mathrm{id} \in \mathrm{GL}_2(K)$, and if $y_0 \neq 0$, let

$$g_r = \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(K) \;.$$

We apply Theorem 15.4 with $\Gamma = \overline{G}$ the image of G in $\mathrm{PGL}_2(R_v)$, $\mathbb{D}^- = \,]\alpha, \alpha^\sigma[$ the (image of any) geodesic line in $\mathbb{X}_v$ with points at infinity α and α^σ, and $\mathbb{D}^+ = \gamma_r \mathscr{H}_\infty$, where γ_r is the image of g_r in $\mathrm{PGL}_2(R_v)$.

We have $L_{\Gamma_v} = 2\mathbb{Z}$ and the family $\mathscr{D}^+ = (\gamma \mathbb{D}^+)_{\gamma \in \Gamma / \Gamma_{\mathbb{D}^+}}$ is locally finite, as seen in the beginning of the proof of Theorem 16.1. The family $\mathscr{D}^- = (\gamma \mathbb{D}^-)_{\gamma \in \Gamma / \Gamma_{\mathbb{D}^-}}$ is locally finite as seen in the beginning of the proof of Theorem 17.1.

By Proposition 15.2 (5), we have (on the full measure subset $K_v - \{\alpha, \alpha^\sigma\}$ of $\partial_\infty \mathbb{X}_v$)

$$(\partial^+)_* \widetilde{\sigma}^+_{\mathbb{D}^-} = \frac{|\alpha - \alpha^\sigma|_v}{|z-\alpha|_v\, |z-\alpha^\sigma|_v}\, d\,\mathrm{Haar}_{K_v}(z) \,.$$

As in order to obtain Equation (16.3), since the point at infinity of $\gamma\mathbb{D}^+$ is $\gamma\cdot r$, we have, with an error term for every $\beta\in\,]0,1]$ of the form $\operatorname{O}(s^{-\kappa}\|\psi\|_\beta)$ for some $\kappa>0$ when evaluated on $\psi\in\mathscr{C}_c^\beta(\partial_\infty\mathbb{X}_v-\partial_\infty\mathbb{D}^-)$,

$$\lim_{n\to+\infty}\ \frac{({q_v}^2-1)(q_v+1)}{2\,q_v^3}\ \frac{\operatorname{Vol}(\Gamma\backslash\!\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\ {q_v}^{-n}\sum_{\substack{\gamma\in\Gamma/\Gamma_r\\ 0<d(\mathbb{D}^-,\,\gamma\mathbb{D}^+)\leqslant n}}\Delta_{\gamma\cdot r}$$
$$=\ \frac{|\alpha-\alpha^\sigma|_v}{|z-\alpha|_v\,|z-\alpha^\sigma|_v}\,d\operatorname{Haar}_{K_v}(z)\ . \qquad (19.1)$$

We use the following result in order to switch from counting over elements $\gamma\in\Gamma/\Gamma_r$ for which $0<d(\mathbb{D}^-,\gamma\mathbb{D}^+)\leqslant t$ to counting over integral representations with bounded value of the norm form. See [PaP11a, page 1054] for the analogous result for the real hyperbolic 3-space and indefinite binary Hermitian forms.

Lemma 19.5. *Let $g\in\operatorname{GL}_2(R_v)$ and let γ be the image of g in $\operatorname{PGL}_2(K)$. Let $z_0=y_0$ if $y_0\neq0$ and $z_0=x_0$ otherwise. Let $(x,y)=g(x_0,y_0)$. If $d(\mathbb{D}^-,\gamma\mathbb{D}^+)>0$, then*

$$d(\mathbb{D}^-,\gamma\mathbb{D}^+)=\frac{1}{\ln q_v}\ \ln\Big(|\,\mathtt{n}(x-y\alpha)|_v\ \frac{h(\alpha)}{|z_0|_v^2}\Big)\ .$$

Proof. We start by showing that

$$g\,g_r(1,0)=\Big(\frac{x}{z_0},\frac{y}{z_0}\Big)\ .$$

Indeed, if $y_0\neq0$, we have

$$g\,g_r(1,0)=g(r,1)=\frac{1}{y_0}\ g(x_0,y_0)$$

and otherwise

$$g\,g_r(1,0)=g(1,0)=\frac{1}{x_0}\ g(x_0,0)=\frac{1}{x_0}\ g(x_0,y_0)\,.$$

In particular,

$$(g\,g_r)^{-1}=\begin{pmatrix}* & *\\ -\frac{y}{z_0} & \frac{x}{z_0}\end{pmatrix}\ .$$

Note that $g\,g_r\in\operatorname{GL}_2(K)$ and $|\det(g\,g_r)|_v=|\det g|_v\ |\det g_r|_v=1$ since $g\in\operatorname{GL}_2(R_v)$. By Proposition 17.4 (2), we hence have

$$h((g\,g_r)^{-1}\cdot\alpha)=\Big|\,\mathtt{n}\Big(\frac{x}{z_0}-\frac{y}{z_0}\ \alpha\Big)\Big|_v\ h(\alpha)=|\,\mathtt{n}(x-y\ \alpha)|_v\ \frac{h(\alpha)}{|z_0|_v^2}\ . \qquad (19.2)$$

With $\beta_{\,.}(\cdot,\cdot)$ the Busemann function defined in Equation (2.5), we use the signed distance $d(L,H)=\min_{x\in L}\beta_\xi(x,x_H)$ between a geodesic line L and a horoball H centred at $\xi\neq L^\pm$, where x_H is any point of the boundary of H. Now, by Equations (15.2) and (2.12), we have

$$\begin{aligned}d(\mathbb{D}^-,\gamma\mathbb{D}^+)&=d(]\alpha,\alpha^\sigma[\,,\gamma\gamma_r\mathscr{H}_\infty)=d\big(](g\,g_r)^{-1}\cdot\alpha,(g\,g_r)^{-1}\cdot\alpha^\sigma[\,,\mathscr{H}_\infty\big)\\&=v\big((g\,g_r)^{-1}\cdot\alpha-(g\,g_r)^{-1}\cdot\alpha^\sigma\big)\\&=\frac{-\ln\big|(g\,g_r)^{-1}\cdot\alpha-(g\,g_r)^{-1}\cdot\alpha^\sigma\big|_v}{\ln q_v}=\frac{\ln h((g\,g_r)^{-1}\cdot\alpha)}{\ln q_v}\ . \qquad (19.3)\end{aligned}$$

Combining Equations (19.2) and (19.3) gives the result. $\square$

By discreteness, there are only finitely many double classes $[g] \in G_\alpha \backslash G/G_{(x_0,y_0)}$ such that $\mathbb{D}^- = \,]\alpha, \alpha^\sigma[$ and $g\mathbb{D}^+ = g\, g_r \mathscr{H}_\infty$ are not disjoint. Let $Z(G)$ be the centre of G, which is finite. Since $Z(G)$ acts trivially on $\mathbb{P}_1(K_v)$, the map $G/G_{(x_0,y_0)} \to \Gamma/\Gamma_r$ induced by the canonical map $\mathrm{GL}_2(R_v) \to \mathrm{PGL}_2(R_v)$ is $|Z(G)|$-to-1. Using the change of variable

$$s = \frac{{|z_0|_v}^2}{h(\alpha)}\, {q_v}^n \;,$$

and Lemma 19.5 since $\gamma \cdot r = \frac{x}{y}$ with the notation of this lemma, Equation (19.1) gives

$$\lim_{s\to+\infty} \frac{({q_v}^2-1)\,(q_v+1)\,{|z_0|_v}^2}{2\,{q_v}^3\,|Z(G)|}\;\frac{\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)}{\|\sigma^-_{\mathscr{D}^+}\|}\;s^{-1}\sum_{(x,\,y)\in G(x_0,\,y_0),\;\;|\,\mathtt{n}(x-y\alpha)|_v\leqslant s}\Delta_{\frac{x}{y}} = \frac{d\,\mathrm{Haar}_{K_v}(z)}{|z-\alpha|_v\,|z-\alpha^\sigma|_v}\;,$$

with the appropriate error term. Replacing $\mathrm{Vol}(\Gamma\backslash\backslash\mathbb{X}_v)$ and $\|\sigma^-_{\mathscr{D}^+}\|$ by their values respectively given by Equation (16.6) and Lemma 16.3, the claim of Theorem 19.1 follows. $\square$

Appendix A

A weak Gibbs measure is the unique equilibrium, by J. Buzzi

In this Appendix, for a transitive topological Markov shift endowed with a Hölder-continuous potential, we prove that a weak Gibbs measure is the unique equilibrium measure.

A.1 Introduction

Let $\sigma : \Sigma \to \Sigma$ be a topological Markov shift (possibly one- or two-sided), see for instance Section 5.1. More precisely, we consider the one-sided and two-sided vertex-shifts defined by a countable oriented graph G with set of vertices V_G and set of arrows $A_G \subset V_G \times V_G$. We assume that Σ is transitive, that is, that G is connected (as an oriented graph).

We denote by $\mathbb{P}(\Sigma)$ the set of σ-invariant probability measures on Σ and by $\mathbb{P}_{\rm erg}(\Sigma)$ the subset of ergodic ones. Recall that, for all $n \in \mathbb{N}$, the *n-cylinders* are the following subsets of Σ, where x varies in Σ:

$$C_n(x) = [x_0, \ldots, x_{n-1}] = \{y \in \Sigma \ : \ \forall\, k \in \{0, \ldots, n-1\},\ y_k = x_k\}\ ,$$

so that the 1-cylinders are $[v] = \{y \in \Sigma \ : \ y_0 = v\}$ for all $v \in V_G$. The points of Σ admitting $n \in \mathbb{N}$ as period under the shift σ form the set

$$\mathrm{Fix}_n(\Sigma) = \{x \in \Sigma \ : \ \sigma^n x = x\}\ .$$

We fix a *potential* on Σ, that is, a continuous function $\phi : \Sigma \to \mathbb{R}$. We do not assume that ϕ is bounded. We define $\phi^- = \max\{-\phi, 0\}$ and, for all $n \in \mathbb{N} - \{0\}$,

$$\mathrm{var}_n(\phi) = \sup_{x,y\in\Sigma\,:\,\forall\, k\in\{0,\ldots,n-1\},\ x_k=y_k} |\phi(y) - \phi(x)|$$

if (Σ, σ) is one-sided and otherwise

$$\mathrm{var}_n(\phi) = \sup_{x,y\in\Sigma\,:\,\forall\, k\in\{-n+1,\ldots,n-1\},\ x_k=y_k} |\phi(y) - \phi(x)|\ .$$

We say that ϕ has *summable variations* if $\sum_{n\geqslant 1} \mathrm{var}_n(\phi) < \infty$. This is in particular the case if ϕ is Hölder-continuous. Let $S_n\phi = \sum_{i=0}^{n-1} \phi \circ \sigma^i$ for all $n \in \mathbb{N}$.

Definition A.1. *A* weak Gibbs measure *for the potential ϕ is a σ-invariant Borel probability measure m on Σ such that there exists a number $c(m) \in \mathbb{R}$ such that for every $v \in V_G$, there exists $C \geqslant 1$ with*

$$\forall\, n \geqslant 1,\ \forall x \in \mathrm{Fix}_n(\Sigma) \cap [v], \quad C^{-1} \leqslant \frac{m(C_n(x))}{\exp\big(S_n\phi(x) - c(m)n\big)} \leqslant C\,. \tag{A.1}$$

Note that $c(m)$ is then unique; it is called the *Gibbs constant* of m. Let us stress that we do not assume the so-called *Big Image Property* [Sar1] and hence using the above weakened Gibbs property (that is, allowing C to depend on v) is necessary.

Note that if Σ is locally compact, that is, if every vertex of G has finite degree (the number of arrows arriving or leaving from the given vertex), then the above condition is equivalent to the fact that for any nonempty compact subset K in Σ, there exists $C \geqslant 1$ with

$$\forall\, n \geqslant 1,\ \forall\, x \in \mathrm{Fix}_n(\Sigma) \cap K, \quad C^{-1} \leqslant \frac{m(C_n(x))}{\exp\big(S_n\phi(x) - c(m)n\big)} \leqslant C\,.$$

The *pressure* $P(\phi, \nu)$ of an element $\nu \in \mathbb{P}(\Sigma)$ such that $\int \phi^-\, d\nu < +\infty$ is

$$P(\phi, \nu) = h_\nu(\sigma) + \int \phi\, d\nu\,.$$

An *equilibrium measure* μ_{eq} for (Σ, ϕ) is an element $\mu_{eq} \in \mathbb{P}(\Sigma)$ such that $\int \phi^-\, d\mu_{eq} < +\infty$ and

$$P(\phi, \mu_{eq}) = \sup\{P(\phi, \nu) : \nu \in \mathbb{P}(\Sigma) \text{ and } \int \phi^-\, d\nu < +\infty\}\,.$$

The *Gurevič pressure* is

$$P_G(\phi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{Fix}_n(\Sigma)\cap[v]} e^{S_n\phi(x)}$$

for any vertex $v \in V_G$. Note that the Gurevič pressure does not depend on v. Let us recall a few results on the above notions.

Theorem A.2 (Iommi-Jordan [IJ, Theorem 2.2])**.** *If ϕ has summable variations, the following variational principle holds:*

$$P_G(\phi) = \sup\{P(\phi, \nu) : \nu \in \mathbb{P}(\Sigma) \textit{ and } \int \phi^-\, d\nu < +\infty\}\,. \quad \square$$

Theorem A.3 (Buzzi-Sarig [BuS, Theorem 1.1])**.** *Assume that ϕ has summable variations.*

If $P_G(\phi) < \infty$, then there exists at most one equilibrium measure.

If there exists an equilibrium measure μ, then $d\mu = h\, d\nu$ where $h : \Sigma \to \mathbb{R}$ is a continuous, positive function and ν is a positive measure with full support on Σ such that

- *$L_\phi h = e^{P_G(\phi)} h$, and $L^*_\phi \nu = e^{P_G(\phi)} \nu$ where L_ϕ is the transfer operator defined by $L_\phi u\,(x) = \sum_{y \in \sigma^{-1}x} e^{\phi(y)}\, u(y)$.*
- *ν is finite on each cylinder.* $\square$

We note that [BuS] assumed $\sup \phi < \infty$, but this was only used to justify the variational principle and so this condition can be removed by using Theorem A.2.

We now state the main result of this appendix.

Theorem A.4. *Let (Σ, σ) be a one-sided transitive topological Markov shift and let $\phi : \Sigma \to \mathbb{R}$ be a potential with summable variations. Let m be a σ-invariant probability measure on Σ such that $\int \phi^- dm < +\infty$.*

Then m is a weak Gibbs measure if and only if it is an equilibrium measure. In this case, the Gibbs constant $c(m)$ is equal to the Gurevič pressure and the equilibrium measure is unique.

By a classical argument that follows, this result extends to two-sided topological Markov shifts (up to a slight strengthening of the regularity assumption on ϕ, still satisfied if ϕ is Hölder-continuous).

Corollary A.5. *Let (Σ, σ) be a two-sided transitive topological Markov shift and let $\phi : \Sigma \to \mathbb{R}$ be a potential with $\sum_{n \geqslant 1} n \ \mathrm{var}_n(\phi) < \infty$. Let m be a σ-invariant probability measure on Σ such that $\int \phi^- dm < +\infty$.*

Then m is a weak Gibbs measure if and only if it is an equilibrium measure. In this case, the Gibbs constant $c(m)$ is equal to the Gurevič pressure and the equilibrium measure is unique.

Remark. The case of the full shift $\mathbb{N}^{\mathbb{Z}}$ has been treated in [PeSZ, Sec. 3]. More generally, assuming the Big Image Property, the above result follows from [Sar1] and [BuS] along the lines of [PeSZ].

Proof of Corollary A.5. Let (Σ, σ), ϕ, and m be as in the statement of this Corollary. Let $\pi : \Sigma \to \Sigma_+$ with $(x_n)_{n \in \mathbb{Z}} \mapsto (x_n)_{n \in \mathbb{N}}$ be the obvious factor map onto the one-sided topological Markov shift (Σ_+, σ_+) defined by the same graph G as for (Σ, σ), called the *natural extension*.

First, we replace ϕ by a potential $\overline{\phi}$ depending only on future coordinates. The proof of [Bowe2, Lemma 1.6] applies to our non-compact setting without changes. To be more precise, for each vertex $a \in V_G$, choose $z^a \in \Sigma$ with $z^a_0 = a$. Define $r : \Sigma \to \Sigma$ by $r(x) = y$ with $y_n = x_n$ for $n \geqslant 0$ and $y_n = z^{x_0}_n$ for $n \leqslant 0$. For every $x \in \Sigma$, let

$$u(x) = \sum_{k \geqslant 0} (\phi \circ \sigma^k - \phi \circ \sigma^k \circ r)(x) \ .$$

This defines a bounded real function on Σ since $|\phi \circ \sigma^k - \phi \circ \sigma^k \circ r| \leqslant \mathrm{var}_{k+1}(\phi)$ and ϕ has summable variations. Moreover, u itself has summable variations since, given $x, y \in \Sigma$ with $x_k = y_k$ for $|k| < n$, we have

$$\begin{aligned} |u(x) - u(y)| \leqslant & \sum_{0 \leqslant k < \lfloor n/2 \rfloor} \Big(|\phi(\sigma^k x) - \phi(\sigma^k y)| + |\phi(\sigma^k(rx)) - \phi(\sigma^k(ry))| \Big) \\ & + \sum_{k \geqslant \lfloor n/2 \rfloor} \Big(|\phi \circ \sigma^k(x) - \phi \circ \sigma^k(rx)| + |\phi \circ \sigma^k(y) - \phi \circ \sigma^k(ry)| \Big) \\ \leqslant & \ 4 \sum_{k \geqslant \lfloor n/2 \rfloor} \mathrm{var}_{k+1}(\phi) \ , \end{aligned}$$

so that

$$\sum_{n \geqslant 1} \mathrm{var}_n(u) \leqslant 8 \sum_{k \geqslant 1} k \, \mathrm{var}_k(\phi) < \infty \ .$$

Now define $\overline{\phi} : \Sigma \to \mathbb{R}$ by

$$\overline{\phi} = \phi + u \circ \sigma - u \ .$$

The function $\overline{\phi}$ is continuous with summable variations. Following [Bowe2], let us prove that $\overline{\phi} = \overline{\phi} \circ r$. We have

$$\begin{aligned}\overline{\phi} &= \phi + \sum_{k \geqslant 0} (\phi \circ \sigma^{k+1} - \phi \circ \sigma^k \circ r \circ \sigma) - \sum_{k \geqslant 0} (\phi \circ \sigma^k - \phi \circ \sigma^k \circ r) \\ &= \phi - \phi - \sum_{k \geqslant 0} \Big(\phi \circ \sigma^k \circ r \circ \sigma - \phi \circ \sigma^k \circ r \Big) \\ &= \sum_{k \geqslant 0} \Big(\phi \circ \sigma^k \circ r - \phi \circ \sigma^k \circ r \circ \sigma \Big) \ .\end{aligned}$$

Now, $r^2 = r$ and $r \circ \sigma \circ r = r \circ \sigma$. Hence $\overline{\phi} \circ r = \overline{\phi}$ as claimed. Thus, $\overline{\phi}$ induces on the one-sided shift a function $\widetilde{\phi} : \Sigma_+ \to \mathbb{R}$ defined by

$$\tilde{\phi} : (x_n)_{n \in \mathbb{N}} \mapsto \overline{\phi}(\dots z_{-2}^{x_0} z_{-1}^{x_0} x_0 x_1 \dots) \ ,$$

satisfying $\overline{\phi} = \widetilde{\phi} \circ \pi$.

To conclude, observe that $S_n \overline{\phi}(x) - S_n \phi(x) = S_n(u \circ \sigma - u)(x) = 0$ if $x \in \mathrm{Fix}_n(\sigma)$, and that cylinders defined by the same finite words have the same measure for an invariant probability measure m on the two-sided shift (Σ, σ) and for its image $\pi_* m$ on the one-sided shift (Σ_+, σ_+). Therefore m is a weak Gibbs measure for ϕ if and only if $\pi_* m$ is a weak Gibbs measure for $\widetilde{\phi}$, and their Gibbs constant are then equal.

By construction $\pi_* m(\widetilde{\phi}) = m(\overline{\phi}) = m(\phi)$ since m is invariant. As it is well-known, the natural extension π preserves the entropy. Thus, the measure m is an equilibrium measure with respect to ϕ if and only if $\pi_* m$ is an equilibrium measure with respect to $\widetilde{\phi}$.

The reduction to one-sided topological Markov shifts is thus complete. □

A.2 Proof of the main result Theorem A.4

The uniqueness of the equilibrium state is given by Theorem A.3. We need to prove that weak Gibbs measures and equilibrium measures coincide under the integrability assumption on ϕ^- and that the number $c(m)$ is equal to the Gurevič pressure.

Step 1. If m is an equilibrium measure, then it is a weak Gibbs measure.

This is a routine consequence of Theorem A.3. Our definition of an equilibrium measure m enforces $\int \phi^- \, dm < +\infty$ (hences excludes the concomitance of $h_m(\sigma) = +\infty$ and $\int \phi \, dm = -\infty$).

Recall from Theorem A.3 that $dm = h \, d\nu$ with h and ν as mentionned. For $v \in V_G$ and $x \in \mathrm{Fix}_n(\Sigma) \cap [v]$, we have

$$\begin{aligned}m(C_n(x)) &= \int h \, \mathbb{1}_{C_n(x)} \, d\nu = e^{-n \, P_G(\phi)} \int h \, \mathbb{1}_{C_n(x)} \, d((L_\phi^*)^n \nu) \\ &= e^{-n \, P_G(\phi)} \int L_\phi^n (h \, \mathbb{1}_{C_n(x)}) \, d\nu \ .\end{aligned}$$

By definition,

$$L_\phi^n (h \, \mathbb{1}_{C_n(x)})(z) = \exp(S_n \phi(x_0 \dots x_{n-1} z)) \, h(x_0 \dots x_{n-1} z)$$

for all $z \in \sigma^n(C_n(x)) = \sigma([v])$ (and $L_\phi^n(h\,\mathbb{1}_{C_n(x)})(z) = 0$ otherwise). Hence[1]

$$m(C_n(x)) = e^{-n\,P_G(\phi)} \exp\Big(S_n\phi(x) \pm \sum_{k=1}^{n} \mathrm{var}_k(\phi)\Big) \int_{\sigma([v])} h\,d\nu\ .$$

As $0 < \int_{\sigma([v])} h\,d\nu < +\infty$ and $\sum_{k=1}^{+\infty} \mathrm{var}_k(\phi) < +\infty$, the measure m is a weak Gibbs measure for ϕ with Gibbs constant $c(m) = P_G(\phi)$.

We now turn to the converse implication. Let m be a weak Gibbs measure for ϕ such that $\int \phi^-\,dm < +\infty$.

The weak Gibbs condition only controls the cylinders that start and end with the same symbol. Passing to an induced system (that is, considering a first return map on a 1-cylinder) will remove this restriction. More precisely, let $a \in V_G$ be a vertex of G and let μ be an invariant probability measure on (Σ, σ) with $\mu([a]) > 0$. The *induced system* on the 1-cylinder $[a] = \{x \in \Sigma \ :\ x_0 = a\}$ is the map $\overline{\sigma} : [a] \to [a]$ (almost everywhere) defined as follows:

- let $\tau(x) = \inf\{n \geqslant 1\ :\ \sigma^n x \in [a]\}$ be the *first-return time* in $[a]$, that we also denote by $\tau_{[a]}(x)$ when we want to emphasize $[a]$;
- let $\overline{\sigma}(x) = \sigma^{\tau(x)}(x)$ if $\tau(x) < \infty$;
- let $\overline{\mu}(B) = \mu(B \cap [a])/\mu([a])$ for every Borel subset B of Σ be the restriction of μ to $[a]$ normalized to be a probability measure.

We also define $\tau^0(x) = 0$ and by induction $\tau^{n+1}(x) = \tau(x) + \tau^n(\overline{\sigma}x)$ for every $n \in \mathbb{N}$. Note that $\overline{\sigma}$ can only be iterated on the subset

$$\{x \in [a]\ :\ \forall\, n \geqslant 1,\ \tau^n(x) < \infty\}\ .$$

By Poincaré's recurrence theorem, this is a full measure subset of $[a]$, hence the distinction will be irrelevant for our purposes.

The *induced partition* is

$$\beta = \{[a, \xi_1, \dots, \xi_{n-1}, a] \neq \varnothing : n \geqslant 1, \xi_i \neq a\}\ .$$

We note that $\overline{\sigma} : [a] \to [a]$ is topologically Bernoulli with respect to the partition β (that is, $\overline{\sigma} : b \to [a]$ is a homeomorphism for each $b \in \beta$). For every integer $N \geqslant 1$, we define the N-th *iterated partition* β^N of β by

$$\beta^N = \{b_0 \cap \overline{\sigma}^{-1} b_1 \cap \dots \cap \overline{\sigma}^{-N+1} b_{N-1} \neq \varnothing\ :\ b_0, \dots, b_{N-1} \in \beta\}$$

and we write $\beta^N(x)$ for the element of the partition β^N that contains x.

Step 2. The topological Markov shift may be assumed to be topologically mixing.

This follows from the spectral decomposition for topological Markov shifts, see for instance [BuS, Lem. 2.2].

[1] We use, for all $u, v, d \geqslant 0$ and $c > 0$, the notation $u = c^{\pm d} v$ if $\frac{1}{c^d} v \leqslant u \leqslant c^d v$.

Step 3. The Gibbs property implies full support and ergodicity.

Let A be a σ-invariant ($\sigma^{-1}(A) = A$) measurable subset of Σ with $m(A) > 0$ and let us prove that $m(A) = 1$.

Observe that the Gibbs property, together with the transitivity of Σ, implies that any cylinder has positive measure for m, hence that m has full support. Let $a \in V_G$ be such that $m(A \cap [a]) > 0$.

As $m([a]) > 0$, we may consider the induced system on $[a]$. Let $N \geqslant 1$. When f is a homeomorphism between topological spaces, let f^* denote the pushforwards of measures by f^{-1}. First note that, for almost every $x \in [a]$ and every $N \in \mathbb{N} - \{0\}$, since $\overline{\sigma}^N$ is an homeomorphism from $\beta^N(x)$ onto $[a]$, we have

$$\frac{m(A \cap [a])}{m([a])} = \frac{m(\overline{\sigma}^N(A \cap \beta^N(x)))}{m(\overline{\sigma}^N(\beta^N(x)))} = \frac{\int_{\beta^N(x)\cap A} \frac{d(\overline{\sigma}^N)^* m}{dm}\, dm}{\int_{\beta^N(x)} \frac{d(\overline{\sigma}^N)^* m}{dm}\, dm} \ .$$

Now, observe that for m-almost every $y \in \beta^N(x)$:

$$\frac{d(\overline{\sigma}^N)^* m}{dm}(y) = \lim_{n\to\infty} \frac{m(\sigma^{\tau^N(y)}[y_0, \ldots, y_n])}{m([y_0, \ldots, y_n])} = C^{\pm 2} e^{-S_{\tau^N(y)}\phi(y) + \tau^N(y)\, c(m)} \ .$$

Hence, since τ^N is constant on $\beta^N(x)$,

$$\frac{m(A \cap [a])}{m([a])} = C^{\pm 4} \ \ e^{\pm \sum_{k\geqslant 1} \operatorname{var}_k(\phi)} \ \ \frac{m(A \cap \beta^N(x))}{m(\beta^N(x))} \ .$$

By Doob's increasing martingale convergence theorem (see for instance [Pet]), for m-almost every $x \in [a] - A$, the ratio on the right hand side converges to 0 as $N \to \infty$. Thus $[a]$ is contained in A modulo m. Therefore $A = \bigcup_{a\in W}[a]$ modulo m for some subset W of V_G.

Since Σ is topologically mixing, for any vertex b, the intersection $[a] \cap \sigma^{-i}[b] \cap \sigma^{-j}[a]$ is not empty for some integers $0 < i < j$. Pick some point x in that set. By invariance, $m([b]) \geqslant m(\sigma^i(C_j(x))) \geqslant m(C_j(x))$. But this last number is positive by the weak Gibbs property. Thus $[b]$ is contained in A modulo m. Hence $m(A) = 1$, proving the ergodicity of m.

Step 4. The Gurevič pressure $P_G(\phi)$ is equal to $c(m)$, hence is finite. Furthermore $h_m(\sigma) < \infty$ and $\phi \in \mathbb{L}^1(m)$.

Fix $v \in V_G$ and let $K = [v]$. Note that $m(K) > 0$. The ergodicity of m gives a Cesaro convergence: as $n \to \infty$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} m(K \cap \sigma^{-k}K) \longrightarrow m(K)^2 > 0 \ .$$

The Gibbs property implies that, for all $n \geqslant 1$,

$$\begin{aligned} m(K \cap \sigma^{-n}K) &= C^{\pm 1} \sum_{x\in \mathrm{Fix}_n(\Sigma)\cap K} e^{S_n\phi(x) - c(m)n} \\ &= C^{\pm 1}\Big(\sum_{x\in \mathrm{Fix}_n(\Sigma)\cap K} e^{S_n\phi(x)}\Big)\, e^{-c(m)n} \ . \end{aligned} \tag{A.2}$$

If we write Z_n for the term between the parenthesis, we have by the definition of the Gurevič pressure:

$$P_G(\phi) = \limsup_{n\to\infty} \frac{1}{n} \log Z_n \ .$$

As the value of the left hand side of Equation (A.2) is less than one, we see that $c(m) \geqslant P_G(\phi)$. If this was a strict inequality, then the left hand side of Equation (A.2) would converge to zero, contradicting its Cesaro convergence to $m(K)^2 > 0$. Therefore $P_G(\phi) = c(m)$.

Since $c(m)$ is finite, so is $P_G(\phi)$. Hence Theorem A.2 implies that, for any $\nu \in \mathbb{P}(\Sigma)$ with $\int \phi^- \, d\nu < +\infty$, we have $h_\nu(\sigma) < \infty$ and ϕ is ν-integrable. In particular, this holds for $\nu = m$, which finishes the proof of Step 4.

Step 5. If the mean entropy $H_{\overline{\mu}}(\beta) = -\sum_{b\in\beta} \overline{\mu}(b) \log \overline{\mu}(b)$ is finite, then

$$h_{\overline{\mu}}(\overline{\sigma}) = -\int \log \frac{d\overline{\mu}}{d\overline{\sigma}^*\overline{\mu}} \, d\overline{\mu} \ ,$$

where $\overline{\sigma}^*\overline{\mu}$ is the measure on Σ defined by $B \mapsto \sum_{b\in\beta} \overline{\mu}(\overline{\sigma}(B \cap b))$, with respect to which μ is absolutely continuous: $\mu \ll \overline{\sigma}^*\overline{\mu}$.

This is a classical formula, sometimes called the *Rokhlin formula* (see for instance [BuS]), which follows from the computation of the entropy in terms of the information function when the mean entropy is finite

$$h_{\overline{\mu}}(\overline{\sigma}) = -\int \sum_{b\in\beta} \mathbb{1}_b(x) \lim_{n\to\infty} \log \mathbb{E}_{\overline{\mu}}(\mathbb{1}_b \mid \overline{\sigma}^{-1}\beta \vee \cdots \vee \overline{\sigma}^{-n}\beta)(x) \, d\overline{\mu}(x)$$

and from the identity, for $x \in b$,

$$\mathbb{E}_{\overline{\mu}}(\mathbb{1}_b \mid \overline{\sigma}^{-1}\beta \vee \cdots \vee \overline{\sigma}^{-n}\beta)(x) = \frac{\overline{\mu}(\beta^{n+1}(x))}{\overline{\mu}(\beta^n(\overline{\sigma}x))} = \frac{\overline{\mu}(\beta^{n+1}(x))}{\overline{\sigma}^*\overline{\mu}(\beta^{n+1}(x))} \ .$$

The absolute continuity follows from a direct computation and ensures that the integral above is well-defined.

Step 6. For all $a \in V_G$, $N \geqslant 1$ and $\mu \in \mathbb{P}_{\text{erg}}(\Sigma)$ with $\int \phi^- \, d\mu < +\infty$, we have

$$h_\mu(\sigma) = -\mu([a]) \int_{[a]} \frac{1}{N} \log \frac{d\,\overline{\mu}}{d((\overline{\sigma}^N)^*\,\overline{\mu})} \, d\,\overline{\mu} \tag{A.3}$$

We use arguments from the proof of [BuS, Theorem 1.1]: the key is to see that the induced partition β of $[a]$ has finite mean entropy for the induced measure $\bar{\mu}$ using a Bernoulli approximation.

Let us consider the Bernoulli measure $\overline{\mu}_B$ for $([a], \overline{\sigma})$ defined by

$$\overline{\mu}_B\Big(\bigcap_{i=0}^{n-1} \overline{\sigma}^{-i} B_i\Big) = \prod_{i=0}^{n-1} \overline{\mu}(B_i)$$

for all $B_i \in \beta$. We construct from it an invariant measure μ_B on (Σ, σ): For every Borel subset A, let

$$\mu_B(A) = \mu([a]) \int_{[a]} \sum_{i=0}^{\tau_{[a]}-1} \mathbb{1}_A \circ \sigma^i \, d\,\overline{\mu}_B \ .$$

Note that μ_B is ergodic, since $\overline{\mu}_B$ is ergodic and $\bigcup_{i\geqslant 1} \overline{\sigma}^{-i}([a])$ has full measure. Define $\overline{\phi} = \sum_{i=0}^{\tau_{[a]}-1} \phi \circ \sigma^i$. Let $C' = \sup_{k\geqslant 1} \mathrm{var}_k(\phi)$, which is finite since ϕ has summable variations. Since every $b \in \beta$ is a cylinder of length $\tau_{[a]}(x)+1$ for every $x \in b$, the conditional expectation

$$\mathbb{E}_{\overline{\mu}}(\overline{\phi}\,|\,\beta) = \sum_{b\in\beta} \mathbb{1}_b \, \frac{1}{\overline{\mu}(b)} \int_b \overline{\phi}\, d\overline{\mu}$$

satisfies $\|\overline{\phi} - \mathbb{E}_{\overline{\mu}}(\overline{\phi}\,|\,\beta)\|_\infty \leqslant C'$. Hence

$$\begin{aligned}\int \phi\, d\mu_B = \mu([a]) \int_{[a]} \overline{\phi}\, d\,\overline{\mu}_B &\geqslant \mu([a]) \int_{[a]} \left(\mathbb{E}_{\overline{\mu}}(\overline{\phi}\,|\,\beta) - C'\right) d\,\overline{\mu}_B \\ &\geqslant \mu([a]) \int_{[a]} \overline{\phi}\, d\,\overline{\mu} - C' = \int \phi\, d\mu - C' > -\infty\ .\end{aligned}$$

Therefore, the last paragraph of the proof of Step 4 applies to $\nu = \mu_B$ and $h_{\mu_B}(\sigma) < +\infty$. Since μ_B is ergodic, Abramov's formula yields

$$h_{\mu_B}(\sigma) = \mu([a])\, h_{\overline{\mu}_B}(\overline{\sigma})\ .$$

Since $\overline{\mu}_B$ is Bernoulli, the right hand side of this equality is equal to

$$\mu([a])\, H_{\overline{\mu}_B}(\beta) = \mu([a]) H_{\overline{\mu}}(\beta)\ ,$$

so that $H_{\overline{\mu}}(\beta)$ is proven to be finite. Thus, Step 5 applies:

$$h_{\overline{\mu}}(\overline{\sigma}) = -\int_{[a]} \log \frac{d\overline{\mu}}{d(\overline{\sigma}^*\overline{\mu})}\, d\overline{\mu}\ .$$

This formula extends to $h_{\overline{\mu}}(\overline{\sigma}^N)$ for all integers $N \geqslant 1$. Using Abramov's formula this time for μ and $\overline{\mu}$ (since μ is ergodic), we have

$$h_\mu(\sigma) = \mu([a])\, h_{\overline{\mu}}(\overline{\sigma}) = \frac{\mu([a])}{N}\, h_{\overline{\mu}}(\overline{\sigma}^N) = -\mu([a]) \int_{[a]} \frac{1}{N} \log \frac{d\overline{\mu}}{d((\overline{\sigma}^N)^*\overline{\mu})}\, d\overline{\mu}\ ,$$

as claimed.

Step 7. The entropy of m is equal to $c(m) - \int \phi\, dm$.

In order to prove this, we apply Step 6 with $\mu = m$ (which is possible, since m has been proven to be ergodic in Step 3). As in the proof of Step 3, the Radon-Nikodym derivative is almost everywhere

$$\frac{d\overline{m}}{d((\overline{\sigma}^N)^*\overline{m})}(x) = \lim_{n\to\infty} \frac{\overline{m}(\beta^n(x))}{\overline{m}(\overline{\sigma}^N(\beta^n(x)))} = C^{\pm 2} \exp\left(S_{\tau^N(x)}\phi(x) - c(m)\tau^N(x)\right)\ .$$

Therefore, using Step 6 and the fact that $m_{|[a]} = m([a])\,\overline{m}$, we have

$$h_m(\sigma) = \lim_{N\to\infty} \frac{1}{N}\Big(\int_{[a]} \left(c(m)\,\tau^N(x) - S_{\tau^N(x)}\phi(x)\right) dm \pm 2\log C\Big)\ . \tag{A.4}$$

Note that $\tau^N(x)$ can be seen as a Birkhoff sum for the induced system on $[a]$ and the function τ and that, by Kac's theorem (see for instance [Pet, Sect. 2.4]),

$$\int_{[a]} \tau \, d\overline{m} = m([a])^{-1} .$$

Therefore, Birkhoff's ergodic theorem yields, with convergence in $\mathbb{L}^1(\overline{m})$,

$$\lim_{N\to\infty} \frac{c(m)\,\tau^N(x)}{N} = \frac{c(m)}{m([a])} .$$

In order to analyze the second term in Equation (A.4), let $\hat{\phi}(x) = \sum_{k=0}^{\tau(x)-1} \phi(\sigma^k x)$ and observe that, by a variation of the proof of Kac's theorem, $\hat{\phi} \in \mathbb{L}^1(\overline{m})$ with $\overline{m}(\hat{\phi}) = m([a])^{-1} m(\phi)$. Indeed, passing to the natural extension, one can assume the system to be invertible and use the partition modulo m given by

$$\bigcup_{n\geqslant 1,\, 0\leqslant k<n} \sigma^k(\{x \in [a] \ : \ \tau(x) = n\}) .$$

Since $S_{\tau^N(x)}\phi(x)$ coincides with the Birkhoff sum $\bar{S}_N\hat{\phi}(x)$ for the induced system, Birkhoff's ergodic theorem yields, with convergence in $\mathbb{L}^1(\overline{m})$,

$$\lim_{N\to\infty} \frac{1}{N} S_{\tau^N(x)}\phi(x) = \lim_{N\to\infty} \frac{1}{N} \bar{S}_N\hat{\phi}(x) = m([a])^{-1} m(\phi) .$$

The claim follows.

Step 8. Conclusion: any weak Gibbs measure is an equilibrium measure and $c(m) = P_G(\phi)$.

Steps 4 and 7 prove that $h_m(\sigma) + \int \phi \, dm$ is well-defined and equal to $c(m)$, which by Step 4 is equal to $P_G(\phi)$, which is equal to $\sup\{P(\phi,\nu) : \nu \in \mathbb{P}(\Sigma) \text{ and } \int \phi^- \, d\nu < +\infty\}$ by Theorem A.2, so that m is an equilibrium measure. This completes the proof of Theorem A.4. □

List of Symbols

∞	standard point at infinity $[1:0]$ of a projective line	272
$\overset{*}{\rightharpoonup}$	weak-star convergence of measures on locally compact spaces	22
$\mathbb{1}_A$	characteristic function of a subset A	22
$\sim\, = \,\sim_{\mathscr{D}}$	equivalence relation on index set of an equivariant family $\mathscr{D}$	148
$\lVert f\rVert_\alpha$	α-Hölder norm of $f \in \mathscr{C}_{\rm c}^{\alpha}(Z)$	58
$\lVert\psi\rVert_\ell$	Sobolev $W^{\ell,2}$-norm of $\psi \in \mathscr{C}_{\rm c}^{\ell}(N)$	164
$\lvert\cdot\rvert_v$	(normalised) absolute value associated with a valuation v	268
$\lvert w\rvert$	length of the cylinder $[w]$ associated with an admissible sequence w	168
$*_v$	base point $*_v = [\mathscr{O}_v \times \mathscr{O}_v]$ of the Bruhat-Tits tree $\mathbb{X}_v$	271
$\underline{A}(\mathscr{O}_v)$	maximal compact-open diagonal subgroup of $\mathrm{PGL}_2(K_v)$	273
$\mathrm{Aut}(\mathbb{X},\lambda)$	automorphism group (edge-preserving, without inversion) of a metric tree $(\mathbb{X},\lambda)$	41
$\mathrm{Aut}\,\mathbb{X}$	automorphism group (without inversion) of a simplicial tree $\mathbb{X}$	41
$B(x,r)$	closed ball of center x and radius r in a metric space	22
$B^{\pm}(w,\eta')$	Hamenstädt's ball of radius $\eta' > 0$ with center any geodesic line extension of $w \in \mathscr{G}_{\pm}X$	35
$B(\ell;T,T',r)$	dynamical ball in the space of geodesic lines $\mathscr{G}X$	81
$\mathbf{C}$	geometrically connected smooth projective curve over $\mathbb{F}_q$	267
${}^cA$	complementary set of a subset A	22
$\widetilde{c}_F$	system of conductances on $\mathbb{X}$ associated with a potential $\widetilde{F}$	74
c_F	system of conductances on $\Gamma\backslash\mathbb{X}$ associated with a potential F	74
$c(g)$	period for a system of conductances c of a closed orbit g for the geodesic flow	259
$\mathscr{C}_{\rm c}(Z)$	space of real-valued continuous maps with compact support on Z	22
$\mathscr{C}_{\rm b}^{\alpha}(Z)$	space of bounded α-Hölder-continuous real-valued functions on Z	58
$\mathscr{C}_{\rm b}^{k,\,\alpha}(Z)$	space of real-valued functions on Z with bounded α-Hölder-continuous derivatives of order at most k along the flow	174
$\mathscr{C}_{\rm c}^{\alpha}(Z)$	space of α-Hölder-continuous real-valued functions with compact support on Z	58

$HB_-(w)$	unstable horoball of $w \in \mathscr{G}_- X$	35
$H_+(w)$	stable horosphere of $w \in \mathscr{G}_+ X$	35
$H_-(w)$	unstable horosphere of $w \in \mathscr{G}_- X$	35
$\mathbb{H}^n_{\mathbb{C}}$	complex hyperbolic space of dimension n	86
ι	antipodal map $w \mapsto \{t \mapsto w(-t)\}$ on $\breve{\mathscr{G}}X$, as well as on T^1X	30
ι	antipodal map $\Gamma w \mapsto \{t \mapsto \Gamma w(-t)\}$ on $\Gamma\backslash\breve{\mathscr{G}}X$, as well as on $\Gamma\backslash T^1X$	31
ι_α	reciprocity index of a loxodromic fixed point α	302
$\iota_G(\beta)$	G-reciprocity index of a quadratic irrational β	309
$\operatorname{Isom}(X)$	isometry group of X	26
$\mathscr{I}_v$	set of classes of fractional ideals of R_v	275
K	global function field over $\mathbb{F}_q$	267
K_v	completion of function field K for the valuation v	267
k_v	residual field of the valuation v on K	267
$\lambda(\gamma)$	translation length of an isometry γ of X	26
$\lambda(g)$	length of a periodic orbit g	259
$\Lambda\Gamma$	limit set of a discrete group of isometries Γ of X	26
$\Lambda_c\Gamma$	conical limit set of a discrete group of isometries Γ of X	26
$\mathbb{L}^2(\mathbb{Y}, G_*)$	Hilbert space of square integrable maps on $V\mathbb{Y}$ for the measure $\operatorname{vol}_{(\mathbb{Y},G_*)}$	44
$\mathscr{L}_g$	Lebesgue measure along a periodic orbit g	259
lk	links of vertices in simplicial trees	272
L_Γ	length spectrum of action of Γ on X	95
ln	natural logarithm (with $\ln(e) = 1$)	22
$\ell_\pm$	positive/negative endpoint of geodesic line ℓ	30
ℓ^*	standard basepoint in space of geodesic lines $\mathscr{G}\mathbb{X}_v$	273
$m_{\mathscr{D}}(x)$	multiplicity of a vertex x with respect to an equivariant family $\mathscr{D}$	157
$\widetilde{m}_F$	Gibbs measure on the space of geodesic lines $\mathscr{G}X$	80
m_F	Gibbs measure on the quotient space of geodesic lines $\Gamma\backslash\mathscr{G}X$	80
$\widetilde{m}_F$	Gibbs measure on the space of discrete geodesic lines $\mathscr{G}\mathbb{X}$	91
m_F	Gibbs measure on the quotient space of discrete geodesic lines $\Gamma\backslash\mathscr{G}\mathbb{X}$	91
$\overline{m_F}$	renormalised Gibbs measure $m_F/\|m_F\|$ on $\Gamma\backslash\mathscr{G}X$	163
$\overline{m_c}$	renormalised Gibbs measure $m_c/\|m_c\|$ on $\Gamma\backslash\mathscr{G}\mathbb{X}$	165
$(\mu^\pm_x)_{x\in X}$	(normalised) Patterson density for the pair $(\Gamma, \widetilde{F}^\pm)$	77
$\mu_{W^\pm(w)}$	skinning measures on the strong stable or strong unstable leaf $W^\pm(w)$	145
$(\mu^{\rm Haus}_x)_{x\in X}$	Hausdorff measures of the visual distances d_x on $\Lambda\Gamma$	94
$N(I)$	(absolute) norm of a nonzero ideal I in a Dedekind ring	269
$\mathscr{N}_\epsilon A$	closed ϵ-neighbourhood of a subset A of a metric space	22

$\sigma^{+}_{\mathscr{D}}$	outer skinning measure on $\Gamma\backslash\check{\mathscr{G}}X$ of a family of closed convex subsets $\mathscr{D}$	149
$\widetilde{\sigma}^{\pm}_{\mathscr{D}}$	outer/inner skinning measure on $\check{\mathscr{G}}X$ of a family of closed convex subsets $\mathscr{D}$	149
$\widetilde{\sigma}^{\pm}_{\Omega}$	outer/inner skinning measure of a family $\Omega = (\Omega_i)_{i\in I}$ of subsets of $(\partial^1_{\pm} D_i)_{i\in I}$	149
t	terminal vertex map $E\mathbb{X} \to V\mathbb{X}$ in a graph $\mathbb{X}$	40
$T\pi$	first edge map of a discrete time geodesic line	42
T^1X	space of germs at $t = 0$ of geodesic lines in X	31
$\mathtt{tr}(\beta)$	relative trace of quadratic irrational β	305
$\mathtt{Tr}(x)$	reduced trace of a quaternion x	312
$\mathrm{Tvol}_{(\mathbb{Y},\,G_*)}$	volume form on the set of edges of a graph of finite groups $(\mathbb{Y},\, G_*)$	44
$\mathrm{Tvol}_{(\mathbb{Y},\,G_*,\lambda)}$	volume form of the set of edges of a metric graph of finite groups $(\mathbb{Y},\, G_*, \lambda)$	45
$\mathrm{TVol}(\mathbb{Y},\, G_*)$	total volume of the set of edges of a graph of finite groups $(\mathbb{Y},\, G_*)$	44
$\mathrm{TVol}(\mathbb{Y},\, G_*, \lambda)$	total volume of the set of edges of a metric graph of finite groups $(\mathbb{Y},\, G_*, \lambda)$	45
$\mathscr{U}^{\pm}_{D}$	domain of the fibration $f^{\pm}_{D}$	37
$V^{\pm}_{w,\,\eta,\,\eta'}$	dynamical neighbourhoods of a point $w \in \mathscr{G}_{\pm}X$	37
$\mathscr{V}^{\pm}_{\eta,\,\eta'}(\Omega^{\mp})$	dynamical neighbourhood of a subset $\Omega^{\mp}$ of $\mathscr{G}_{\pm}X$	38
v_{∞}	valuation at infinity of $\mathbb{F}_q(Y)$	266
v_{ℓ}	germ at $t = 0$ of a geodesic line ℓ	31
$\mathrm{vol}_{(\mathbb{Y},\,G_*)}$	volume form on the set of vertices of a graph of finite groups $(\mathbb{Y},\, G_*)$	44
$\mathrm{Vol}(\mathbb{Y},\, G_*)$	volume of a graph of finite groups $(\mathbb{Y},\, G_*)$	44
$V\mathbb{X}$	set of vertices of a graph $\mathbb{X}$	40
$V_{\rm even}\mathbb{X}$	set of vertices of a pointed graph $(\mathbb{X}, x_0)$ at even distances from x_0	95
$V_{\rm odd}\mathbb{X}$	set of vertices of a pointed graph $(\mathbb{X}, x_0)$ at odd distances from x_0	95
$w_{\pm}$	positive/negative endpoint of generalised geodesic line w	30
$W^{+}(w)$	strong stable leaf of $w \in \mathscr{G}_{+}X$	32
$W^{0+}(w)$	stable leaf of $w \in \mathscr{G}_{+}X$	35
$W^{-}(w)$	strong unstable leaf of $w \in \mathscr{G}_{-}X$	32
$W^{0-}(w)$	unstable leaf of $w \in \mathscr{G}_{-}X$	35
$\lvert\mathbb{X}\rvert_{\lambda}$	geometric realisation of a metric tree $(\mathbb{X}, \lambda)$	41
$\mathbb{X}_v$	Bruhat-Tits tree of (PGL_2, K_v)	271
$\zeta_K(s)$	Dedekind's zeta function of a function field K	269

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