Rate of mixing for equilibrium states in negative curvature and trees

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Abstract In this survey based on the book by the authors [BPP], we recall the Patterson-Sullivan construction of equilibrium states for the geodesic flow on negatively curved orbifolds or tree quotients, and discuss their mixing properties, emphazising the rate of mixing for (not necessarily compact) tree quotients via coding by countable (not necessarily finite) topological shifts. We give a new construction of numerous nonuniform tree lattices such that the (discrete time) geodesic flow on the tree quotient is exponentially mixing with respect to the maximal entropy measure: we construct examples whose tree quotients have an arbitrary space of ends or an arbitrary (at most exponential) growth type.[1]

1 A Patterson-Sullivan construction of equilibrium states

We refer to [PPS, Chap. 3, 6, 7] and [BPP, Chap. 2, 3, 4] for details and complements on this section.

Let X be (see [BPP] for a more general framework)

• either a complete, simply connected Riemannian manifold $\widetilde{M}$ with dimension m at least 2 and pinched sectional curvature at most -1,

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• or (the geometric realisation of) a simplicial tree $\mathbb{X}$ whose vertex degrees are uniformly bounded and at least 3. In this case, we respectively denote by $E\mathbb{X}$ and $V\mathbb{X}$ the sets of vertices and edges of $\mathbb{X}$. For every edge e, we denote by $o(e), t(e), \overline{e}$ its original vertex, terminal vertex and opposite edge.

Let us fix an indifferent basepoint x_* in $\widetilde{M}$ or in $V\mathbb{X}$.

Recall (see for instance [BH]) that a geodesic ray or line in X is an isometric map from $[0, +\infty[$ or $\mathbb{R}$ respectively into X, that two geodesic rays are *asymptotic* if they stay at bounded distance one from the other, and that the *boundary at infinity* of X is the space $\partial_\infty X$ of asymptotic classes of geodesic rays in X endowed with the quotient topology of the compact-open topology. When $X = \widetilde{M}$, up to a translation factor, two asymptotic geodesic rays converge exponentially fast one to the other, and $\partial_\infty \widetilde{M}$ is homeomorphic to the sphere $\mathbb{S}_{m-1}$ of dimension $m-1$. When X is a tree, up to a translation factor, two asymptotic geodesic rays coincide after a certain time, and $\partial_\infty \widetilde{M}$ is homeomorphic to a Cantor set.

For every x in X, the Gromov-Bourdon *visual distance* d_x on $\partial_\infty X$ seen from x (inducing the topology of $\partial_\infty X$) is defined by

$$d_x(\xi, \eta) = \lim_{t \to +\infty} e^{\frac{1}{2}(d(\xi_t, \eta_t) - d(x, \xi_t) - d(x, \eta_t))} ,$$

where $\xi, \eta \in \partial_\infty X$ and $t \mapsto \xi_t, \eta_t$ are any geodesic rays converging to ξ, η respectively. The visual distances seen from two points of X are Lipschitz equivalent.

Let Γ be a discrete group of isometries of X which is *nonelementary*, that is, does not preserve a subset of cardinality at most 2 in $X \cup \partial_\infty X$. When $X = \widetilde{M}$, this is equivalent to Γ not being virtually nilpotent. When X is a tree, we furthermore assume that X has no nonempty proper invariant subtree (this is not an important restriction, as one may always replace X by its unique minimal nonempty invariant subtree), and that Γ does not map an edge to its opposite one.

The *limit set* $\Lambda\Gamma$ of Γ is the smallest nonempty closed invariant subset of $\partial_\infty X$, which is the complement of the orbit Γx_* in its closure $\overline{\Gamma x_*}$, in the compactification $X \cup \partial_\infty X$ of X by its boundary at infinity.

Examples. (1) Let $\widetilde{M}$ be a symmetric space with negative curvature, e.g. the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$, and let Γ be an arithmetic lattice in $\mathrm{Isom}(\widetilde{M})$, e.g. $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acting by homographies on the upper halfplane model of $\mathbb{H}^2_\mathbb{R}$ with constant curvature -1 (see for instance [Kat], and [Mar] for a huge amount of examples).
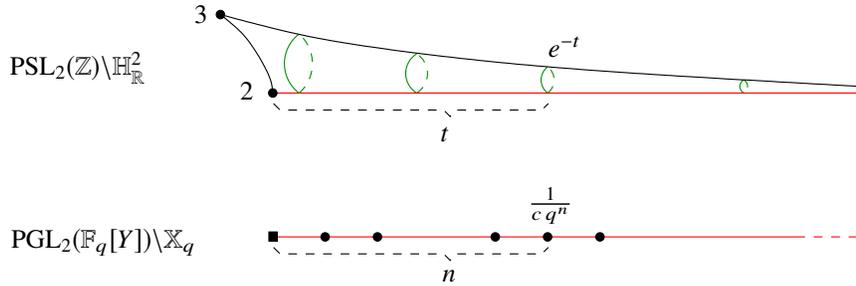
(2) For every prime power q, let $\mathbb{X}$ be the regular tree of degree $q + 1$, and let $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[Y])$, acting on $\mathbb{X}$ seen as the Bruhat-Tits tree $\mathbb{X}_q$ of PGL_2 over the local field $\mathbb{F}_q((Y^{-1}))$ (see for example [Ser], and [BaL] for a huge amount of examples).

Note that the pictures of the quotients $\Gamma \backslash X$ are very similar in the above two special examples, in particular

• the lengths of the closed horocycle quotients in $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2_{\mathbb{R}}$ go exponentially to 0 (they are equal to e^{-t} where t is the distance of the horocycle quotient to the orbifold point of order 2),

• the orders of the vertex stabilisers along a geodesic ray in $\mathbb{X}_q$ lifting the quotient ray $\mathrm{PGL}_2(\mathbb{F}_q[Y]) \backslash \mathbb{X}_q$ increase exponentially (they are equal to $c\,q^n$ where c is a constant and n is the distance of the vertex to the origin of the ray), see for instance [BPP, §15.2].

Remark. Note that we allow torsion in Γ, as this is in particular important in the tree case; we allow $\Gamma \backslash X$ to be noncompact; and we allow Γ not to be a lattice. These allowances give in the tree case the possibility to have almost any (metrisable, compact, totally disconnect) space of ends and almost any type of asymptotic growth of the quotient $\Gamma \backslash X$ (linear, polynomial, exponential, etc), see loc. cit.

Recall that Γ is a lattice in X if either the Riemannian volume $\mathrm{Vol}(\Gamma \backslash \widetilde{M})$ of the quotient orbifold $\Gamma \backslash \widetilde{M}$ is finite, or if the *graph of groups volume*

$$\mathrm{Vol}(\Gamma \backslash\!\backslash \mathbb{X}) = \sum_{[x] \in \Gamma \backslash V\mathbb{X}} \frac{1}{\mathrm{Card}(\Gamma_x)}$$

(where Γ_x is the stabiliser of x in Γ) of the quotient graph of groups $\Gamma \backslash\!\backslash \mathbb{X}$ is finite. Note the analogy, in the two special examples above, between the computation of (most of) the volume of $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}^2_{\mathbb{R}}$ as a converging integral of the lengths of the closed horocycle quotients and of the volume of $\mathrm{PGL}_2(\mathbb{F}_q[Y]) \backslash\!\backslash \mathbb{X}_q$ (which does converge by a geometric mean argument).

The phase space. Let $\mathscr{G}X$ be the space of geodesic lines $\ell : \mathbb{R} \to X$ in X, such that, when X is a tree, $\ell(0)$ is a vertex, endowed with the $\mathrm{Isom}(X)$-invariant distance (inducing its topology) defined by

$$d(\ell, \ell') = \int_{-\infty}^{+\infty} d(\ell(t), \ell'(t))\, e^{-2|t|}\, dt\,,$$

and with the $\mathrm{Isom}(X)$-equivariant *geodesic flow*, which is the one-parameter group of homeomorphisms

$$\mathfrak{g}^t : \ell \mapsto \{s \mapsto \ell(s + t)\}$$

for all $\ell \in \mathscr{G}X$, with continuous time parameter $t \in \mathbb{R}$ if $X = \widetilde{M}$ and discrete time parameter $t \in \mathbb{Z}$ if X is a tree. We again call *geodesic flow* and denote by $(\mathfrak{g}^t)_t$ the quotient flow on the *phase space* $\Gamma \backslash \mathscr{G}X$.

Note that the map from the unit tangent bundle $T^1\widetilde{M}$ endowed with Sasaki's metric to $\mathscr{G}\widetilde{M}$, which associates to a unit tangent vector v the unique geodesic line whose

tangent vector at time $t = 0$ is v, is an $\mathrm{Isom}(\widetilde{M})$-equivariant bi-Hölder-continuous[2] homeomorphism, by which we identify the two spaces from now on.

Potentials on the phase space. We now introduce the supplementary data (with physical origin) that we will consider on our phase space. Assume first that $X = \widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a *potential*, that is, a Γ-invariant, bounded[3] Hölder-continuous real map on $T^1\widetilde{M}$. Two potentials $\widetilde{F}, \widetilde{F}^* : T^1\widetilde{M} \to \mathbb{R}$ are *cohomologous* (see for instance [Livš]) if there exists a Hölder-continuous, bounded, differentiable along flow lines, Γ-invariant function $\widetilde{G} : T^1\widetilde{M} \to \mathbb{R}$, such that, for every $v \in T^1\widetilde{M}$,

$$\widetilde{F}^*(v) - \widetilde{F}(v) = \frac{d}{dt}_{|t=0} \widetilde{G}(\mathrm{g}^t v) \ .$$

For every $x, y \in \widetilde{M}$, let us define (with the obvious convention of being 0 if $x = y$) the integral of $\widetilde{F}$ between x and y, called the *amplitude* of $\widetilde{F}$ between x and y, to be

$$\int_x^y \widetilde{F} = \int_0^{d(x,y)} \widetilde{F}(\mathrm{g}^t v) \ dt$$
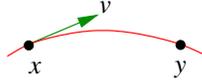
and v is the tangent vector to the geodesic segment from x to y.

Now assume that X is a tree. Let $\widetilde{c} : E\mathbb{X} \to \mathbb{R}$ be a (logarithmic) *system of conductances* (see for instance [Zem]), that is, a Γ-invariant, bounded real map on $E\mathbb{X}$. Two systems of conductances $\widetilde{c}, \widetilde{c}^* : E\mathbb{X} \to \mathbb{R}$ are *cohomologous* if there exists a Γ-invariant function $\widetilde{f} : V\mathbb{X} \to \mathbb{R}$, such that for every $e \in E\mathbb{X}$

$$\widetilde{c}^*(e) - \widetilde{c}(e) = f(t(e)) - f(o(e)) \ .$$

For every $\ell \in \mathscr{G}X$, we denote by $e_0^+(\ell) = \ell([0,1]) \in E\mathbb{X}$ the first edge followed by ℓ, and we define $\widetilde{F} : \mathscr{G}X \to \mathbb{R}$ as the map $\ell \mapsto \widetilde{c}(e_0^+(\ell))$. For every $x, y \in V\mathbb{X}$, we now define the *amplitude* of $\widetilde{F}$ between x and y, to be

$$\int_x^y \widetilde{F} = \sum_{i=1}^k \widetilde{c}(e_i) \ dt$$
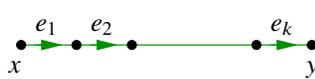
if $(e_1, e_2, \ldots, e_k)$ is the geodesic edge path in $\mathbb{X}$ between x and y.

In both cases, we will denote by $F : \Gamma\backslash\mathscr{G}X \to \mathbb{R}$ the function on the phase space induced by $\widetilde{F}$ by taking the quotient modulo Γ, that we call the *potential* on $\Gamma\backslash\mathscr{G}X$. Note that we make no assumption of reversibility on F.

Cohomological invariants. Let us now introduce three cohomological invariants of the potentials on the phase space.

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[2] In order to deal with noncompactness issues, a map f between two metric spaces is *Hölder-continuous* if there exist $c, c' > 0$ and $\alpha \in \ ]0, 1]$ such that for every x, y in the source space, if $d(x, y) \le c$, then $d(f(x), f(y)) \le c' d(x, y)^\alpha$.

[3] see [BPP, §3.2] for a weakening of this assumption

The *pressure* of F is the physical complexity associated with the potential F defined by

$$P_F = \sup_{\mu \ (\mathrm{g}^t)_t\text{-invariant proba on } \Gamma\backslash\mathscr{G}X} \left(h_\mu + \int_{\Gamma\backslash\mathscr{G}X} F \, d\mu \right)$$

where h_μ is the metric entropy[4] of μ for the time 1 map g^1 of the geodesic flow.

The *critical exponent* of F is the weighted (by the exponential amplitudes) orbital growth rate of the group Γ, defined by

$$\delta_F = \lim_{n\to+\infty} \frac{1}{n} \ln \left(\sum_{\gamma\in\Gamma, \ n-1<d(x_*,\gamma x_*)\le n} \exp\left(\int_{x_*}^{\gamma x_*} \widetilde{F} \right) \right).$$

Note that the critical exponent δ_0 of the zero potential is the usual critical exponent of the group Γ (see for instance [Pau]). We have $\delta_F \in\ ]-\infty,+\infty[$ since

$$\delta_0 + \inf \widetilde{F} \le \delta_F \le \delta_0 + \sup \widetilde{F} \ .$$

Note that $\delta_{F\circ\iota} = \delta_F$ where $\iota : \mathscr{G}X \to \mathscr{G}X$ is the involutive *time reversal map* defined by $\ell \mapsto \{t \mapsto \ell(-t)\}$.

The *period* for the potential F of a periodic orbit $\mathscr{O}$ of the geodesic flow $(\mathrm{g}^t)_t$ on $\Gamma\backslash\mathscr{G}X$ is $\int_{\mathscr{O}} F = \int_{\ell(0)}^{\ell(t_{\mathscr{O}})} \widetilde{F}$ where $\ell \in \mathscr{G}X$ maps to $\mathscr{O}$ and

$$t_{\mathscr{O}} = \inf\{t > 0 \ : \ \Gamma\mathrm{g}^t\ell = \Gamma\ell\}$$

is the *length* of the periodic orbit $\mathscr{O}$. The *Gurevich pressure* of F is the growth rate of the exponentials of periods for F of the periodic orbits, defined by

$$\mathscr{P}_F^{\mathrm{Gur}} = \lim_{n\to+\infty} \frac{1}{n} \ln \sum_{\mathscr{O} \ : \ t_{\mathscr{O}}\le n, \ \mathscr{O}\cap W\neq\emptyset} \exp\left(\int_{\mathscr{O}} F \right),$$

where the sum is taken over the periodic orbits $\mathscr{O}$ of $(\mathrm{g}^t)_t$ on $\Gamma\backslash\mathscr{G}X$ with length at most n and meeting W, where W is any relatively compact open subset of $\Gamma\backslash\mathscr{G}X$ meeting the nonwandering set of the geodesic flow (recall that we made no assumption of compactness on the phase space).

Note that the above three limits exist, and are independent of the choices of x_* and W, and depend only on the cohomology class of the potential F.

The following result proved in [PPS, Theo. 4.1 and 6.1] extends the case of the zero potential due to Otal and Peigné [OP].

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[4] The metric entropy h_μ is the upper bound, for all measurable countable partitions ξ of $\Gamma\backslash\mathscr{G}X$, of

$$\lim_{k\to+\infty} \frac{1}{k} H_\mu(\xi \vee \cdots \vee g^{-k}\xi)$$

where $H_\mu(\xi) = -\sum_{E\in\xi} \mu(E) \ln \mu(E)$ is Shannon's entropy of the countable partition ξ, see for instance [KH], and the join $\xi \vee \xi'$ of two partitions ξ and ξ' is the partition by the nonempty intersections of an element of ξ and an element of ξ'.

Theorem 1 (Paulin-Pollicott-Schapira) *If $X = \widetilde{M}$ has pinched sectional curvatures with uniformly bounded derivatives, [5] then*

$$\boxed{P_F = \delta_F = \mathscr{P}_F^{\mathrm{Gur}}.}$$

Note that the dynamics of the geodesic flow $(\mathfrak{g}^t)_t$ on the phase space $\Gamma\backslash\mathscr{G}X$ is very chaotic. In particular, there are lots of $(\mathfrak{g}^t)_t$-invariant measures on $\Gamma\backslash\mathscr{G}X$. We give two basic examples, and we will then contruct, using potentials, a huge family of such measures.

Examples. (1) If $X = \widetilde{M}$, then the *Liouville measure* m_{Liou} on $T^1M = \Gamma\backslash(T^1\widetilde{M})$ is the measure on T^1M which disintegrates, with respect to the canonical footpoint projection $T^1M \to M$, over the Riemannian measure vol_M of the Riemannian orbifold $M = \Gamma\backslash\widetilde{M}$, with conditional measures on the fibers the spherical measures $\mathrm{vol}_{T_x^1M}$ on the (orbifold) unit tangent spheres at the points x in M:

$$\boxed{dm_{\mathrm{Liou}}(v) = \int_{x\in M} d\,\mathrm{vol}_{T_x^1M}(v)\ \ d\,\mathrm{vol}_M(x).}$$

(2) For every periodic orbit $\mathscr{O}$ of the geodesic flow $(\mathfrak{g}^t)_t$ on $\Gamma\backslash\mathscr{G}X$, we denote by $\mathscr{L}_{\mathscr{O}}$ the Lebesgue measure[6] (when $X = \widetilde{M}$) or counting measure (when X is a tree) of $\mathscr{O}$. This is a $(\mathfrak{g}^t)_t$-invariant measure on $\Gamma\backslash\mathscr{G}X$ with support $\mathscr{O}$.

The main class of invariant measures we will study is the following one, and the terminology has been mostly introduced by Sinai, Ruelle, Bowen, see for instance [Rue]. A $(\mathfrak{g}^t)_t$-invariant probability measure μ on the phase space $\Gamma\backslash\mathscr{G}X$ is an *equilibrium state* for the potential F if it realizes the upper bound defining the pressure of F, that is, if

$$h_\mu + \int_{\Gamma\backslash\mathscr{G}X} F\ d\mu = P_F\ .$$

The remainder of this section is devoted to the problems of **existence, uniqueness and explicit construction** of equilibrium states.

Gibbs cocycles. As for instance defined by Hamenstädt, the (normalised) *Gibbs cocycle* of the potential F is the function $C : \partial_\infty X \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ when $X = \widetilde{M}$ or the function $C : \partial_\infty X \times V\mathbb{X} \times V\mathbb{X} \to \mathbb{R}$ when X is a tree, defined by the following limit of difference of amplitudes for the renormalised potential

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[5] This assumption on the derivatives was forgotten in the statements of [OP, PPS], but is used in the proofs.

[6] If the length of $\mathscr{O}$ is T and if $v \in T^1\widetilde{M}$ maps into $\mathscr{O}$ by the canonical projection $T^1\widetilde{M} \to T^1M$, the Lebesgue measure $\mathscr{L}_{\mathscr{O}}$ of $\mathscr{O}$ is the pushforward by $t \mapsto \Gamma\mathfrak{g}^t v$ of the Lebesgue measure on $[0, T]$.

$$C_\xi(x, y) = \lim_{t \to +\infty} \int_y^{\xi_t} (\widetilde{F} - \delta_F) - \int_x^{\xi_t} (\widetilde{F} - \delta_F),$$
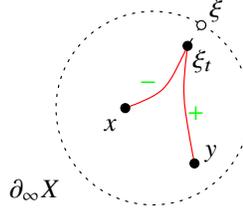
where $t \mapsto \xi_t$ is any geodesic ray converging to ξ. The limit does exist. The Gibbs cocycle is Γ-invariant (for the diagonal action) and locally Hölder-continuous. It does satisfy the cocycle property $C_\xi(x, z) = C_\xi(x, y) + C_\xi(y, z)$ for all x, y, z. Furthermore, there exist constants $c_1, c_2 > 0$ (depending only on the bounds of $\widetilde{F}$ and on the pinching of the sectional curvature, when $X = \widetilde{M}$) such that if $d(x, y) \leq 1$, then $C_\xi(x, y) \leq c_1 d(x, y)^{c_2}$. See [BPP, §3.4].

Patterson densities. A (normalised) *Patterson density* of the potential F is a Γ-equiv-ariant family $(\mu_x)_{x \in X}$ of pairwise absolutely continuous (positive, Borel) measures on $\partial_\infty X$, whose support is $\Lambda\Gamma$, such that

$$\gamma_* \mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_\xi(x, y)} \tag{1}$$

for every $\gamma \in \Gamma$, for all $x, y \in X$, and for (almost) every $\xi \in \partial_\infty X$.

Patterson densities do exist and they satisfy the following Mohsen's shadow lemma (see for instance [BPP, §4.1]):
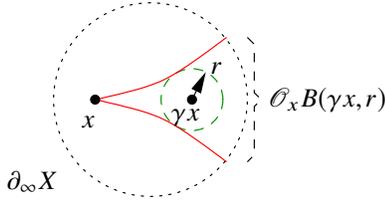
Define the *shadow* $\mathcal{O}_x E$ seen from x of a subset E of X as the set of points at infin-ity of the geodesic rays from x through E. Then for every $x \in X$, if $r > 0$ is large enough, there exists $\kappa > 0$ such that for every $\gamma \in \Gamma$, we have

$$\boxed{\frac{1}{\kappa} \, \exp\left(\int_x^{\gamma x} (\widetilde{F} - \delta_F) \right) \leq \mu_x\left(\mathcal{O}_x B(\gamma x, r)\right) \leq \kappa \, \exp\left(\int_x^{\gamma x} (\widetilde{F} - \delta_F) \right).} \tag{2}$$

Gibbs measures. The *Hopf parametrisation* of X at x_* is the map from $\mathcal{G}X$ to $(\partial_\infty X \times \partial_\infty X - \text{Diag}) \times R$, where $R = \mathbb{R}$ if $X = \widetilde{M}$ and $R = \mathbb{Z}$ if X is a tree, defined by

$$\ell \mapsto (\ell_-, \ell_+, t)$$
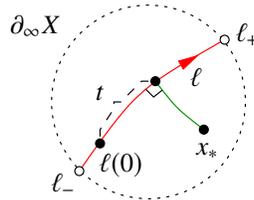
where ℓ_-, ℓ_+ are the original and terminal points at infinity of the geodesic line ℓ, and t is the algebraic distance along ℓ between the footpoint $\ell(0)$ and the closest point to x_* on the geodesic line. It is a Hölder-continuous homeomorphism (for the previously defined distances). Up to translations on the third factor, it does not depend on the basepoint x_* and is Γ-invariant, see for instance [BPP, §2.3 and §3.1]. The geodesic flow acts by translations on the third factor.

Let $(\mu_x)_{x \in X}$ and $(\mu_x^\iota)_{x \in X}$ be Patterson densities for the potentials F and $F \circ \iota$ respectively, where $\iota : \Gamma\ell \mapsto \Gamma\{t \mapsto \ell(-t)\}$ is the time reversal on the phase space $\Gamma\backslash\mathscr{G}X$. We denote by C^ι the Gibbs cocycle of the potential $F \circ \iota$. We denote by dt the Lebesgue or counting measure on R. The measure on $\mathscr{G}X$ defined using the Hopf parametrisation at x_* by

$$d\widetilde{m}_F(\ell) = \frac{d\mu_{x_*}^\iota(\ell_-) \, d\mu_{x_*}(\ell_+) \, dt}{\exp\left(C_{\ell_-}^\iota(x_*, \ell(0)) + C_{\ell_+}(x_*, \ell(0))\right)}$$

is a σ-finite nonzero measure on $\mathscr{G}X$. By Equation (1) and by the invariance of the measure dt under translations, it is independent of the choice of basepoint x_*, hence is Γ-invariant and $(\mathfrak{g}^t)_t$-invariant. Therefore it induces a σ-finite nonzero $(\mathfrak{g}^t)_t$-invariant measure on $\Gamma\backslash\mathscr{G}X$, called the *Gibbs measure* on the phase space and denoted by m_F.

Examples. (1) When $F = 0$, then the Gibbs measure is called the Bowen-Margulis measure (see for instance [Rob]).

(2) When $X = \widetilde{M}$ and $\widetilde{F}$ is the *unstable Jacobian*, that is, for every $v \in T^1\widetilde{M}$,

$$\widetilde{F}^{\mathrm{su}}(v) = -\frac{d}{dt}_{|t=0} \ln\left(\begin{array}{c}\text{Jacobian of restriction of } \mathfrak{g}^t \text{ to} \\ \text{strong unstable leaf } W^{su}(v)\end{array}\right),$$
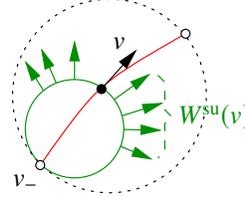
we have the following result (see [PPS, §7], in particular for weaker assumptions). When M has variable sectional curvature, the Liouville measure and the Bowen-Margulis measure might be quite different. The following result in particular says that the huge family of Gibbs measures interpolates between the Liouville measure and the Bowen-Margulis measure. This sometimes provides common proofs of properties satisfied by both the Liouville measure and the Bowen-Margulis measure.

Theorem 2 (Paulin-Pollicott-Schapira) *If $X = \widetilde{M}$ has pinched sectional curvatures with uniformly bounded derivatives, then $\widetilde{F}^{\mathrm{su}}$ is Hölder-continuous and bounded. If $\widetilde{M}$ has a cocompact lattice and if $(\mathfrak{g}^t)_t$ is completely conservative [7] for the Liouville measure, then*

$$m_{F_{\mathrm{su}}} = m_{\mathrm{Liou}}.$$

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[7] That is, every wandering set has measure zero.

The following result, due to Bowen and Ruelle when M is compact and to Otal-Peigné [OP] when $F = 0$, completely solves the problems of existence, uniqueness and explicit construction of equilibrium states, see [PPS, §6].

Theorem 3 (Paulin-Pollicott-Schapira) *Assume that $X = \widetilde{M}$ has pinched sectional curvatures with uniformly bounded derivatives.*[8] *If the Gibbs measure m_F is finite, then $\overline{m_F} = \frac{m_F}{\|m_F\|}$ is the unique equilibrium state. Otherwise, there is no equilibrium state.*

We refer to Section 3.2 for an analogous statement when X is a tree, whose proof uses completely different techniques.

2 Basic ergodic properties of Gibbs measures

We refer to [PPS, Chap. 3, 5, 8] and [BPP, Chap. 4] for details and complements on this section.

2.1 The Gibbs property

In this section, we justify the terminology of Gibbs measures used above.

For every $\ell \in \Gamma \backslash \mathscr{G}X$, say $\ell = \Gamma \widetilde{\ell}$, for every $r > 0$ and for all $t, t' \geq 0$, the (Bowen or) *dynamical ball* $B(\ell; t, t', r)$ in the phase space $\Gamma \backslash \mathscr{G}X$ centered at ℓ with parameters t, t', r is the image in $\Gamma \backslash \mathscr{G}X$ of the set of geodesic lines in $\mathscr{G}X$ following the lift $\widetilde{\ell}$ at distance less than r in the time interval $[-t', t]$, that is, the image in $\Gamma \backslash \mathscr{G}X$ of

$$B(\widetilde{\ell}; t, t', r) = \left\{ \ell' \in \mathscr{G}X \; : \; \sup_{s \in [-t', t]} d_X(\widetilde{\ell}(s), \ell'(s)) < r \right\}.$$

The following definition of the Gibbs property is well adapted to the possible non-compactness of the phase space $\Gamma \backslash \mathscr{G}X$. A $(\mathfrak{g}^t)_t$-invariant measure m' on $\Gamma \backslash \mathscr{G}X$ satisfies the *Gibbs property* for the potential F with *Gibbs constant* $c(F) \in \mathbb{R}$ if for every compact subset K of $\Gamma \backslash \mathscr{G}X$, there exists $r > 0$ and $c_{K,r} \geq 1$ such that for all $t, t' \geq 0$ large enough, for every ℓ in $\Gamma \backslash \mathscr{G}X$ with $\mathfrak{g}^{-t'} \ell, \mathfrak{g}^t \ell \in K$, we have

$$\frac{1}{C_{K,r}} \leq \frac{m'\big(B(\ell; t, t', r)\big)}{e^{\int_{-t'}^{t} (F(\mathfrak{g}^t \ell) - c(F)) \, dt}} \leq C_{K,r}.$$

The following result is due to [PPS, §3.8] when $X = \widetilde{M}$ and [BPP, §4.2] in general.

Proposition 4 *The Gibbs measure m_F satisfies the Gibbs property for F with Gibbs constant $c(F)$ equal to the critical exponent δ_F.*

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[8] This assumption on the derivatives was forgotten in the statements of [OP, PPS].

Let us give a sketch of its proof, which explains the decorrelation of the influence of the two points at infinity of the geodesic lines, using the fact that the Gibbs measure is absolutely continuous with respect to a product measure in the Hopf parametrisation. The key geometric lemma is the following one.

Lemma 5 *For every $r > 0$, there exists $t_r > 0$ such that for all $t, t' \geq t_r$ and $\ell \in \mathscr{G}X$, we have, using the Hopf parametrisation at the footpoint $\ell(0)$,*

$$\mathscr{O}_{\ell(0)} B(\ell(-t'), r) \times \mathscr{O}_{\ell(0)} B(\ell(t), r) \times\ ]-1, 1[\ \subset\ B(\ell; t, t', 2r + 2)$$

$$B(\ell; t, t', r)\ \subset\ \mathscr{O}_{\ell(0)} B(\ell(-t'), 2r) \times \mathscr{O}_{\ell(0)} B(\ell(t), 2r) \times\ ]-r, r[\ .$$

Let us give a proof-by-picture of the first claim, the second one being similar. See the following picture. If a geodesic line ℓ' has its points at infinity ℓ'_- and ℓ'_+ in the shadows seen from $\ell(0)$ of $B(\ell(-t'), r)$ and $B(\ell(-t'), r)$ respectively, then by the properties of triangles in negatively curved spaces, if t and t' are large, then the image of ℓ' is close to the union of the images of the geodesic rays from $\ell(0)$ to ℓ_- and ℓ_+. The control on the time parameter in Hopf parametrisation then says that ℓ' is staying at bounded distance from ℓ in the time interval $[-t', t]$.
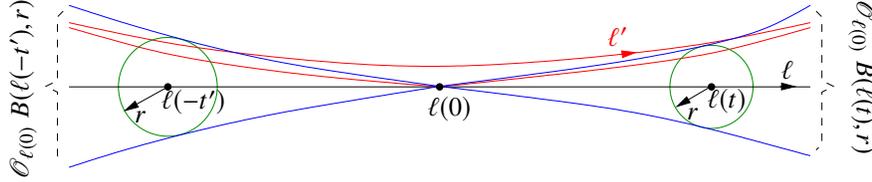
We now conclude the proof of Proposition 4 by using the boundedness of the Gibbs cocycles C and C^ι on a given compact subset K in order to control the denominator in the formula giving $\widetilde{m}_F$, and by using Mohsen's shadow lemma (see Equation (2)) which estimates the Patterson measures of shadows of balls.

2.2 Ergodicity

In this section, we study the ergodicity property of the Gibbs measures under the geodesic flow in the phase space.

The *Poincaré series* of the potential F is

$$Q_F(s) = \sum_{\gamma \in \Gamma} \exp\left(\int_{x_*}^{\gamma x_*} (\widetilde{F} - s)\right).$$

It depends on the basepoint x_*, but its convergence or divergence does not. It converges if $s > \delta_F$ and diverges for $s < \delta_F$, by the definition of the critical exponent δ_F.

The following result has a long history, and we refer for instance to [PPS, §5] and [BPP, §4.2] for proofs, and proofs of its following two corollaries.

Theorem 6 (Hopf-Tsuji-Sullivan-Roblin) *The following assertions are equivalent.*

(1) *The Poincaré series of F diverges at the critical exponent of $F : Q_F(\delta_F) = +\infty$.*

(2) *The group action $(\partial_\infty X \times \partial_\infty X - \mathrm{Diag}, \mu^\iota_{x_*} \otimes \mu_{x_*}, \Gamma)$ is ergodic and completely conservative.*

(3) *The geodesic flow on the phase space endowed with the Gibbs measure $(\Gamma \backslash \mathscr{G} X, m_F, (\mathfrak{g}^t)_t))$ is ergodic and completely conservative.*

Corollary 7 *If $Q_F(\delta_F) = +\infty$, then there exists a Patterson density for F, unique up to a positive scalar. It is atomless, and the diagonal in $\partial_\infty X \times \partial_\infty X$ has measure 0 for the product measure $\mu^\iota_{x_*} \otimes \mu_{x_*}$.*

Let us give a sketch of the very classical proof of the first claim of this corollary.

Existence. Using the properties of negatively curved spaces, one can prove, denoting by $\mathscr{D}_x$ the Dirac mass at a point x, that one can take

$$\mu_x = \lim_{s_i \to \delta_F^+} \frac{1}{Q_F(s_i)} \sum_{\gamma \in \Gamma} \exp\Big(\int_x^{\gamma x_*} (\widetilde{F} - s_i) \Big) \ \mathscr{D}_{\gamma x_*},$$

where the atomic measure before taking the limit is, when $x = x_*$, a probability measure, hence has, for some sequence $(s_i)_{i \in \mathbb{N}}$ in $]\delta_F, +\infty[$ converging to δ_F, a weakstar converging subsequence in the compact space of probability measures on the compact space $X \cup \partial_\infty X$.

Uniqueness. Let $(\mu'_x)_x$ be another Patterson density. Up to positive scalars, we may assume that μ_{x_*} and μ'_{x_*} are probability measures. Then $(\omega_x = \frac{1}{2}(\mu_x + \mu'_x))_x$ is a Patterson density, μ_{x_*} is absolutely continuous with respect to ω_{x_*}, and by ergodicity, the Radon-Nikodym derivative $\frac{d\mu_{x_*}}{d\omega_{x_*}}$ is almost everywhere constant, hence the probability measures μ_{x_*} and ω_{x_*} are equal, hence $\mu_{x_*} = \mu'_{x_*}$.

Corollary 8 *If m_F is finite, then $Q_F(\delta_F) = +\infty$ (hence $(\mathfrak{g}^t)_t$) is ergodic) and the normalised Gibbs measure $\overline{m_F} = \frac{m_F}{\|m_F\|}$ is a cohomological invariant of the potential F.*

2.3 Mixing

In this section, we study the mixing property of the Gibbs measures under the geodesic flow in the phase space. Recall that the *length spectrum* for the action of Γ on X is the subgroup of $\mathbb{R}$ (hence of $\mathbb{Z}$ when X is a tree) generated by the set of lengths of the closed geodesic in $\Gamma \backslash X$ (or, in dynamical terms, of the set of lengths of periodic orbits of the geodesic flow on the phase space). See for instance [PPS, §8.1] when $X = \widetilde{M}$ and [BPP, §4.4] when X is a tree for a proof of the following result, which crucially uses the fact that the Gibbs measure is absolutely continuous with respect to a product measure in the Hopf parametrisation.

Theorem 9 (Babillot) *If the Gibbs measure m_F is finite, then the following assertions are equivalent.*
(1) *The Gibbs measure m_F is mixing under the geodesic flow $(\mathfrak{g}^t)_t$.*
(2) *The geodesic flow $(\mathfrak{g}^t)_t$ is topologically mixing on its nonwandering set in the phase space.*
(3) *The length spectrum of Γ is dense in $\mathbb{R}$ if $X = \widetilde{M}$ or equal to $\mathbb{Z}$ if X is a tree.*

We summarise in the following result the known properties of the rate of mixing of the geodesic flow in the manifold case when $X = \widetilde{M}$ (see [BPP, §9.1]), refering to Section 3 for the tree case, whose proof turns out to be quite different.

Let $\alpha \in \,]0,1]$ and let $\mathscr{C}_{\mathrm{b}}^{\alpha}(Z)$ be the Banach space[9] of bounded α-Hölder-continuous functions on a metric space Z. When $X = \widetilde{M}$, we will say that the (continuous time) geodesic flow on the phase space $T^1 M = \Gamma\backslash T^1\widetilde{M}$ is *exponentially mixing for the α-Hölder regularity* or that it has *exponential decay of α-Hölder correlations* for the potential F if there exist two constants $c', \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}_{\mathrm{b}}^{\alpha}(T^1 M)$ and $t \in \mathbb{R}$, we have

$$\left| \int_{T^1 M} \phi \circ \mathfrak{g}^{-t}\, \psi\, d\overline{m_F} - \int_{T^1 M} \phi\, d\overline{m_F} \int_{T^1 M} \psi\, d\overline{m_F} \right| \le c'\, e^{-\kappa |t|}\, \|\phi\|_\alpha\, \|\psi\|_\alpha \,.$$

Theorem 10 *Assume that $X = \widetilde{M}$ and that $M = \Gamma\backslash\widetilde{M}$ is compact. Then the geodesic flow on the phase space $T^1 M$ has exponential decay of Hölder correlations if*
- *M is two-dimensional, by [Dol],*
- *M is $1/9$-pinched and $F = 0$, by [GLP, Coro. 2.7],*
- *the potential F is the unstable Jacobian F^{su}, so that, up to a positive scalar, m_F is the Liouville measure m_{Liou}, by [Live], see also [Tsu], [NZ, Coro. 5] who give more precise estimates,*
- *M is locally symmetric by [Sto], see also [MO, LiP] for some noncompact cases.*

Note that this gives only a very partial picture of the rate of mixing of the geodesic flow in negative curvature, and it would be interesting to have a complete result. Stronger results exist for the Sobolev regularity when $\widetilde{M}$ is a symmetric space, $F = 0$ and Γ is an arithmetic lattice (the Gibbs measure then coincides, up to a multiplicative constant, with the Liouville measure): see for instance [KM, Theorem 2.4.5], using spectral gap properties given by [Clo, Theorem 3.1]. But this still does not give a complete answer.

---

[9] Recall that its norm (taking into account the possible noncompactness of Z) is given by

$$\|f\|_\alpha = \|f\|_\infty + \sup_{\substack{x,\,y \in Z \\ 0 < d(x,\,y) \le 1}} \frac{|f(x) - f(y)|}{d(x,\,y)^\alpha} \,.$$

3 Coding and rate of mixing for geodesic flows on trees

We refer to [BPP, Chap. 5 and 9.2] for details and complements on this section.

From now on, we assume that X is (the geometric realisation of) a simplicial tree $\mathbb{X}$, and we write $\mathscr{G}\mathbb{X}$ instead of $\mathscr{G}X$. We consider the discrete group Γ, the system of conductances $\widetilde{c}$ and the associated potential F on the phase space $\Gamma\backslash\mathscr{G}\mathbb{X}$ as introduced in Section 1.

The study of the rate of mixing of the (discrete time) geodesic flow on the phase space uses coding theory. But since, as explained, we make no assumption of compactness on the phase space, and no hypothesis of being without torsion on the group Γ in the huge class of examples described in Section 1, the coding theory requires more sophisticated tools than subshifts of finite type.

3.1 Coding

Let $\mathscr{A}$ be a countable discrete set, called an *alphabet*, and let $A = (A_{i,j})_{i,j\in\mathscr{A}}$ be an element in $\{0,1\}^{\mathscr{A}\times\mathscr{A}}$, called a *transition matrix*. The (two-sided, countable state) *topological shift*[10] with alphabet $\mathscr{A}$ and transition matrix A is the topological dynamical system (Σ, σ), where Σ, called the *shift space*, is the closed subset of the topological product space $\mathscr{A}^{\mathbb{Z}}$ of A-*admissible* two-sided infinite sequences, defined by

$$\Sigma = \left\{ x = (x_n)_{n\in\mathbb{Z}} \in \mathscr{A}^{\mathbb{Z}} \ : \ \forall\, n \in \mathbb{Z}, \quad A_{x_n, x_{n+1}} = 1 \right\},$$

and $\sigma : \Sigma \to \Sigma$ is the (two-sided) *shift* defined by

$$\forall\, x \in \Sigma, \ \forall\, n \in \mathbb{Z}, \quad (\sigma(x))_n = x_{n+1}\ .$$

We endow Σ with the distance

$$d(x, x') = \exp\left(-\sup\left\{n \in \mathbb{N} \ : \ \forall\, i \in \{-n, \ldots, n\}, \ x_i = x'_i\right\}\right)\ .$$

Let us denote by $\mathbb{Y}$ the (countable) quotient graph[11] $\Gamma\backslash\mathbb{X}$. For every vertex or edge $x \in V\mathbb{Y} \cup E\mathbb{Y}$, we fix a lift $\widetilde{x}$ in $V\mathbb{X} \cup E\mathbb{X}$, and we define $G_x = \Gamma_{\widetilde{x}}$ to be the stabiliser of $\widetilde{x}$ in Γ.

---

[10] We prefer not to use the frequent terminology of *topological Markov shift* as it could be misleading, many probability measures invariant under general topological shifts do not satisfy the Markov chain property that the probability to pass from one state to another depends only on the previous state, not of all past states.

[11] The fact that the canonical projection is a morphism of graphs is the reason why we assumed Γ to be acting without mapping an edge to its inverse.

For every $e \in E\mathbb{Y}$, we assume that $\widetilde{\overline{e}} = \overline{\widetilde{e}}$. But there is no reason in general for the equality $\widetilde{t(e)} = t(\widetilde{e})$ to hold. We fix $g_e \in \Gamma$ mapping $\widetilde{t(e)}$ to $t(\widetilde{e})$ (which does exist), and we denote by $\rho_e : G_e = \Gamma_{\widetilde{e}} \to \Gamma_{\widetilde{t(e)}} = G_{t(e)}$ the conjugation $g \mapsto g_e^{-1} g \, g_e$ by g_e on G_e (noticing that the stabiliser $\Gamma_{\widetilde{e}}$ is contained in the stabiliser $\Gamma_{t(\widetilde{e})}$).

$$g_e \in \Gamma$$
$$\widetilde{t(e)} \bullet \qquad \xrightarrow[\widetilde{e}]{} \bullet \, t(\widetilde{e})$$
$$p : \mathbb{X} \to \mathbb{Y} = \Gamma \backslash \mathbb{X}$$
$$\xrightarrow{e} \bullet \, t(e)$$
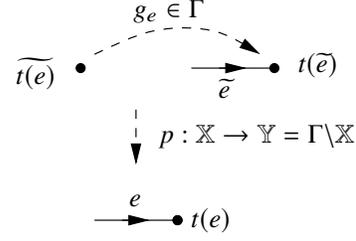
Let us try to code a geodesic line in the phase space $\Gamma \backslash \mathscr{G}\mathbb{X}$. The natural starting point is to write it as $\Gamma \ell$ for some $\ell \in \mathscr{G}X$, that is, to choose one of its lifts. We then have to construct a coding which is independent of the choice of this lift. For every $i \in \mathbb{Z}$, let us denote by $f_i = \ell([i, i+1])$ the i-th edge followed by ℓ, and by e_i (also denoted by $e_{i+1}^-(\ell)$ for later use) its image by the canonical $p : \mathbb{X} \to \mathbb{Y} = \Gamma \backslash \mathbb{X}$, which seems fit to be a natural part of the coding of ℓ. Since we will need to translate through our coding the fact that ℓ is geodesic, hence has no backtracking, the edge e_{i+1} (also denoted by $e_{i+1}^+(\ell)$ for later use) following e_i seems to have a role to play.

$$\ell([i, i+1])$$
fix γ_i \qquad fix γ_{i+1} \qquad $p : \mathbb{X} \to \mathbb{Y}$ \qquad $h_{i+1}(\ell) \in G_{t(e_i)}$

$\widetilde{e_i}$ \qquad $\widetilde{e_{i+1}}$ \qquad g_{e_i} \quad $g_{\overline{e_{i+1}}}$ \qquad $\widetilde{t(e_i)}$

in $\mathbb{X}$ \qquad\qquad\qquad in $\mathbb{Y} = \Gamma \backslash \mathbb{X}$
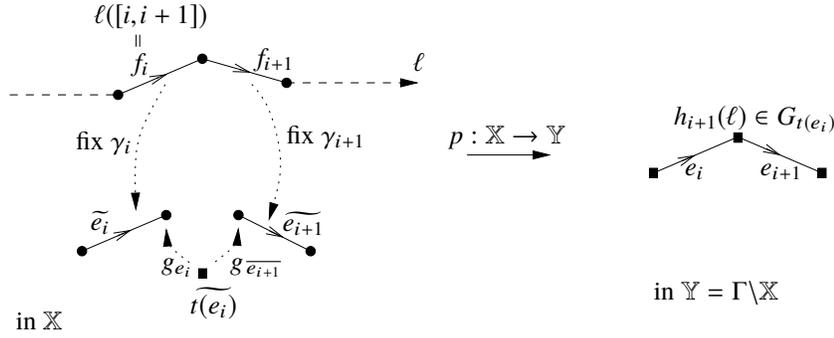
Since the terminal point of f_i is the original point of f_{i+1}, the terminal point of e_i is naturally also the original point of e_{i+1}. But there is no reason for the terminal point of the choosen lift $\widetilde{e_i}$ to also be the original point of the choosen lift $\widetilde{e_{i+1}}$. Since f_i and $\widetilde{e_i}$ both map by p to e_i, we may fix $\gamma_i \in \Gamma$ such that $\gamma_i f_i = \widetilde{e_i}$, for every $i \in \mathbb{Z}$.

Now, note that the vertex stabilizers in Γ of vertices of $\mathbb{X}$ are in general nontrivial (and we explained in Section 1 that it is important to allow them to become very large in order to have numerous dynamically interesting noncompact quotients of simplicial trees). The construction (see the above diagram) provides a natural element $g_{e_i}^{-1} \gamma_i \gamma_{i+1}^{-1} g_{\overline{e_{i+1}}}$ which stabilises the lifted vertex $\widetilde{t(e_i)}$, hence belongs to $G_{t(e_i)}$. Since we made choices for the elements γ_i, the element $g_{e_i}^{-1} \gamma_i \gamma_{i+1}^{-1} g_{\overline{e_{i+1}}}$ gives a well-defined double class $h_{i+1}(\ell)$ in $\rho_{e_i}(G_{e_i}) \backslash G_{t(e_i)} / \rho_{\overline{e_{i+1}}}(G_{e_{i+1}})$, which also seems fit to be another natural piece of the coding of ℓ.

It turns out that this construction is indeed working. We take as alphabet the (countable) set

$$\mathscr{A} = \left\{ (e^-, h, e^+) \; : \; \begin{array}{l} e^\pm \in E\mathbb{Y} \text{ with } t(e^-) = o(e^+) \\ h \in \rho_{e^-}(G_{e^-}) \backslash G_{o(e^+)} / \rho_{\overline{e^+}}(G_{e^+}) \text{ with } h \neq [1] \text{ if } \overline{e^+} = e^- \end{array} \right\} .$$

This last assumption of conditional nontriviality of the double class codes the fact that ℓ being a geodesic line, the edge f_{i+1} is not the opposite edge of f_i, though e_{i+1} might be the opposite edge of e_i. And since in the tree $\mathbb{X}$, being locally geodesic implies being geodesic, it is very reasonable that we have captured through our coding all the geodesic properties of the geodesic lines and translated them into symbolic terms. We take as transition matrix A over the alphabet $\mathscr{A}$ the matrix with entries

$$A_{(e^-, h, e^+), (e'^-, h', e'^+)} = \begin{cases} 1 \text{ if } e^+ = e'^- \\ 0 \text{ otherwise,} \end{cases}$$

which just says that we are glueing together the coding of pairs of consecutive edges of the geodesic line. Note that since the tree is locally finite, the transition matrix A has finitely many nonzero entries on each row and column, hence the associated shift space Σ is locally compact.

We then refer to [BPP, §5.2] for a proof of the following result, though almost everything is in the above picture! We denote by $F_{\text{symb}} : \Sigma \to \mathbb{R}$ the locally constant map which associates to $\big((e_i^-, h_i, e_i^+)\big)_{i \in \mathbb{Z}}$ the image $\widetilde{c}(\widetilde{e_0^+})$ by the system of conductances of the lift of its first edge.

Theorem 11 *The map*

$$\Theta : \begin{cases} \Gamma \backslash \mathscr{G}\mathbb{X} \longrightarrow \Sigma \\ \Gamma\ell \quad \mapsto \quad \big((e_i^-(\ell), h_i(\ell), e_i^+(\ell))\big)_{i \in \mathbb{Z}} \end{cases}$$

is a bilipschitz homeomorphism, conjugating the time 1 map of the (discrete time) geodesic flow $(\mathfrak{g}^t)_{t \in \mathbb{Z}}$ to the shift σ. Furthermore,

(1) *(Σ, σ) is topologically transitive,* [12]

(2) *if the Gibbs measure m_F is finite and if the length spectrum of Γ is equal to $\mathbb{Z}$, then the probability measure $\mathbb{P} = \Theta_* \overline{m_F}$ is mixing for the shift σ on Σ,*

(3) *the measure $\mathbb{P}$ satisfies the Gibbs property on (Σ, σ) with Gibbs constant δ_F for the potential F_{symb}.* [13]

(4) *if $(Z_n : x \mapsto x_n)_{n \in \mathbb{Z}}$ is the canonical random process in symbolic dynamics, then the pair $((Z_n)_{n \in \mathbb{Z}}, \mathbb{P})$ is not always a Markov chain.*

---

[12] This comes from the assumption that there is no nontrivial proper Γ-invariant subtree in $\mathbb{X}$, since then $\partial_\infty X = \Lambda\Gamma$, implying that the nonwandering set of the geodesic flow $(\mathfrak{g}^t)_{t \in \mathbb{Z}}$ is the full phase space $\Gamma \backslash \mathscr{G}\mathbb{X}$.

[13] That is, with a formulation adapted to the possibility that the alphabet $\mathscr{A}$ may be infinite, for every finite subset E of the alphabet $\mathscr{A}$, there exists $C_E \geq 1$ such that for all $p \leq q$ in $\mathbb{Z}$ and for every $x = (x_n)_{n \in \mathbb{Z}} \in \Sigma$ such that $x_p, x_q \in E$, we have

$$\frac{1}{C_E} \leq \frac{\mathbb{P}([x_p, x_{p+1}, \ldots, x_{q-1}, x_q])}{e^{-\delta_F(q-p+1) + \sum_{n=p}^{q} F_{\text{symb}}(\sigma^n x)}} \leq C_E .$$

where $[x_p, x_{p+1}, \ldots, x_{q-1}, x_q]$ is the cylinder $\{(y_n)_{n \in \mathbb{Z}} \in \Sigma \; : \; \text{if } p \leq n \leq q \text{ then } y_n = x_n\}$.

This last claim has lead to an erratum in the paper [Kwo]. The pair $((Z_n)_{n \in \mathbb{Z}}, \mathbb{P})$ is not a Markov chain for instance in Example (2) at the beginning of Section 1, when $\mathbb{X} = \mathbb{X}_q$ and $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[Y])$.[14]

3.2 Variational principle for simplicial trees

The first corollary of the coding results in the previous section is the following existence and uniqueness result of equilibrium states for the geodesic flow on the phase space $\Gamma \backslash \mathscr{G}\mathbb{X}$ for the potential F.

Corollary 12 *If m_F is finite, then $\overline{m_F} = \frac{m_F}{\|m_F\|}$ is the unique equilibrium state for F under the geodesic flow $(\mathfrak{g}^t)_{t \in \mathbb{Z}}$ on $\Gamma \backslash \mathscr{G}\mathbb{X}$, and furthermore*

$$\boxed{P_F = \delta_F.}$$

We only give a sketch of a proof, refering to [BPP, §5.4] for a complete one. We use the coding given in Theorem 11 with its properties (in particular the fact that it satisfies the Gibbs property for a symbolic potential related to the potential F).

Let (Σ, σ) be a topological shift, with countable alphabet $\mathscr{A}$. A σ-invariant probability measure m on Σ is a *weak [15] Gibbs measure* for a map $\phi : \Sigma \to \mathbb{R}$ with Gibbs constant $c(m) \in \mathbb{R}$ if for every $a \in \mathscr{A}$, there exists a constant $c_a \geq 1$ such that for all $n \in \mathbb{N} - \{0\}$ and x in the cylinder $[a] = \{y = (y_n)_{n \in \mathbb{Z}} \in \Sigma \ : \ y_0 = a\}$ such that $\sigma^n(x) = x$, we have

$$\frac{1}{c_a} \leq \frac{m([x_0, x_1, \ldots, x_{n-1}])}{e^{\sum_{i=0}^{n-1} (\phi(\sigma^i x) - c(m))}} \leq c_a \ .$$

The following result of Buzzi is proved in [BPP, Appendix], with a much weaker regularity assumption on ϕ, and it concludes the proof of Corollary 12.

Theorem 13 (Buzzi) *Let (Σ, σ) be a topological shift and $\phi : \Sigma \to \mathbb{R}$ a bounded Hölder-continuous function. If m is a weak Gibbs measure for ϕ with Gibbs constant $c(m)$, then $P_\phi = c(m)$ and m is the unique equilibrium state for the potential ϕ.*

---

[14] As noticed by J.-P. Serre [Ser], the image of almost every geodesic line of $\mathbb{X}$ in the quotient ray $\Gamma \backslash X$ is a broken line which makes infinitely many back-and-forths from the origin of the quotient ray.
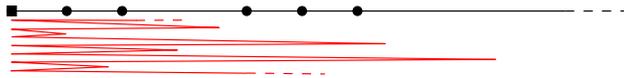
There is absolutely no way to predict the probability of behaviour of the geodesic line image at a given time in terms of its recent past probabilities (except that when it starts to go down, it has to go down all the way to the origin).

[15] The terminology comes from the fact that the assumptions bear only on the periodic points of σ.

3.3 Rate of mixing for simplicial trees

Let us first recall the definition of an exponential mixing rate for discrete time dynamical systems.

Let (Z, m, T) be a dynamical system with (Z, m) a metric probability space and let $T : Z \to Z$ be a (not necessarily invertible) measure preserving map. For all $n \in \mathbb{N}$ and $\phi, \psi \in \mathbb{L}^2(m)$, the (well-defined) n-th *correlation coefficient* of ϕ, ψ is

$$\operatorname{cov}_{m, n}(\phi, \psi) = \int_Z (\phi \circ T^n) \, \psi \, dm - \int_Z \phi \, dm \int_Z \psi \, dm \;.$$

Let $\alpha \in \;]0, 1]$. As for the case of flows in Section 2.3, we will say that the dynamical system (Z, m, T) is *exponentially mixing for the α-Hölder regularity* or that it has *exponential decay of α-Hölder correlations* if there exist $c', \kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}_{\mathrm{b}}^\alpha(Z)$ and $n \in \mathbb{N}$, we have

$$|\operatorname{cov}_{m, n}(\phi, \psi)| \leq c' \, e^{-\kappa n} \, \|\phi\|_\alpha \, \|\psi\|_\alpha \;.$$

Note that this property is invariant under measure preserving conjugations of dynamical systems by bilipschitz homeomorphisms. In our case, T will be either the time 1 map of the geodesic flow $(\mathfrak{g}^t)_{t \in \mathbb{Z}}$ on the phase space $Z = \Gamma \backslash \mathscr{G}\mathbb{X}$ or the two-sided shift σ on a two-sided topological shift space Σ or (see below) the one-sided shift σ_+ on a one-sided topological shift space Σ_+.

The following result is one of the new results contained in the book [BPP]. For every finite subset E in $\Gamma \backslash V\mathbb{X}$, let $\tau_E : \Gamma \backslash \mathscr{G}\mathbb{X} \to \mathbb{N} \cup \{+\infty\}$ be the first positive passage time of geodesic lines in E, that is, the map

$$\ell \mapsto \inf\{n \in \mathbb{N} - \{0\} \; : \; \mathfrak{g}^n \ell(0) \in E\} \;.$$

The following result says that if the tree quotient contains a finite subset in which the geodesic lines with large return times have an exponentially decreasing mass, then the (discrete time) geodesic flow on the phase space has exponential decay of correlations. This condition turns out to be quite easy to check on practical examples, see for instance [BPP, §9.2].

Theorem 14 *If m_F is finite and mixing for $(\mathfrak{g}^t)_{t \in \mathbb{Z}}$, if there exist a finite subset E in $\Gamma \backslash V\mathbb{X}$ and $c'', \kappa' > 0$ such that*

$$\forall \, n \in \mathbb{N}, \quad m_F(\{\ell \in \Gamma \backslash \mathscr{G}\mathbb{X} \; : \; \ell(0) \in E, \tau_E(\ell) \geq n\}) \leq c'' e^{-\kappa' n} \;,$$

then for every $\alpha \in \;]0, 1]$, the (discrete time) dynamical system $(\Gamma \backslash \mathscr{G}\mathbb{X}, m_F, (\mathfrak{g}^t)_{t \in \mathbb{Z}})$ is exponentially mixing for the α-Hölder regularity.

The hypothesis of Theorem 14 is for instance satisfied for Example (2) at the beginning of Section 1 with $\mathbb{X} = \mathbb{X}_q$ and $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[Y])$, taking E consisting of the origin of the modular ray $\Gamma \backslash \mathbb{X}_q$, and using the exponential decay of the stabilisers orders along a lift of the modular ray in $\mathbb{X}_p$. In this case, the quotient graph $\Gamma \backslash \mathbb{X}$ has

linear growth. We gave in [BPP, page 193] examples where the quotient graph $\Gamma\backslash\mathbb{X}$ has exponential growth.

Here is an example where the quotient graph has quadratic growth, for every even $q \geq 2$. The tree $\mathbb{X}$ is the regular tree of degrees $q + 2$. The vertex group of the top-left vertex x_* of the quotient graph is $\mathbb{Z}/(\frac{q}{2} + 1)\mathbb{Z}$. A set E as in Theorem 14 consists of the three vertices at distance at most 1 from x_*. The vertex group of a vertex at distance at distance at least 1 from x_*, on the $(m + 1)$-th horizontal and $(n + 1)$-th vertical is $(\mathbb{Z}/q^n\mathbb{Z}) \times (\mathbb{Z}/(q + 1)^m\mathbb{Z})$. The number at the beginning of each edge represents the index of the edge group inside the vertex group of its origin.
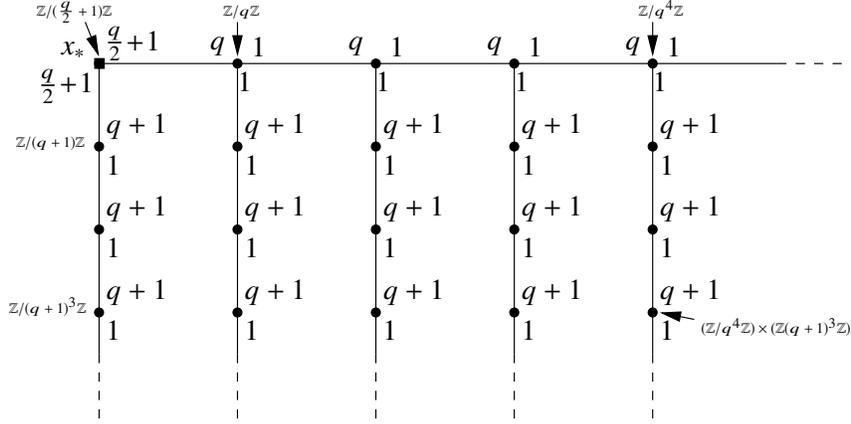
Recall that two *growth functions* f and f', that is, two increasing maps from $\mathbb{N}$ to $\mathbb{N} - \{0\}$, are *equivalent* if there exist two integers $c \geq 1$ and $c' \geq 0$ such that for every $n \in \mathbb{N}$ large enough, we have $f(\lfloor \frac{1}{c} n - c' \rfloor) \leq f'(n) \leq f(c\,n + c')$. The *type of growth* of an infinite, connected, locally finite graph Y is the equivalence class of the map $n \mapsto \text{Card } B_{VY}(v_0, n)$, which does not depend on the choice of a base point $v_0 \in VY$, nor on the quasi-isometry type of Y.

It is well known (see for instance [Cho, Hug] or [GNS, §6.2]) that every totally disconnected compact metric space is homeomorphic to the boundary at infinity of a simplicial tree with uniformly bounded degrees, and that any increasing positive integer sequence $(a_n)_{n\in\mathbb{N}}$ with at most exponential speed (that is, there exists $k \in \mathbb{N}$ such that $a_{n+1} \leq ka_n$ for every $n \in \mathbb{N}$) is, up to the above equivalence, the sequence of orders of the balls of an infinite rooted simplicial tree with uniformly bounded degrees. Hence the following result (not contained in [BPP]) says that we can realize any space of ends, or any at most exponential type of growth, in the quotient graph of an action of a group on a tree satisfying the hypothesis of Theorem 14.

Proposition 15 *For every rooted tree $(\mathscr{T}, *)$ with uniformly bounded degrees such that $\mathscr{T} \neq \{*\}$, there exists a simplicial tree $\mathbb{X}$ and a discrete group Γ of automorphisms of $\mathbb{X}$ as in the beginning of Section 1 such that Γ is a lattice, $\Gamma\backslash\mathbb{X}$ is the union of $\mathscr{T}$ with a loop at $*$, and the geodesic flow $(\mathfrak{g}^t)_{t\in\mathbb{Z}}$ is exponentially mixing for the α-Hölder regularity on $\Gamma\backslash\mathscr{G}\mathbb{X}$ for the zero potential.*

Proof. We refer for instance to [Ser, §I.5] for background on graphs of groups.

Let us fix $q \in \mathbb{N}$ large enough compared with the maximum degree d of $\mathscr{T}$. We define a graph of groups $(\mathscr{T} \cup \{e_*, \overline{e_*}\}, G_\bullet)$ with underlying graph the union of $\mathscr{T}$ with a loop glued at the root $*$ as follows. Let $G_{e_*} = \{1\}$. For every vertex v of $\mathscr{T}$ at distance n of the root $*$, we define $G_v = \mathbb{Z}/q^n\mathbb{Z}$. For every edge $e \neq e_*, \overline{e_*}$ whose closest vertex to the root $*$ is at distance n from $*$, we define $G_e = \mathbb{Z}/q^n\mathbb{Z}$. For every edge $e \neq e_*, \overline{e_*}$ pointing away from the root, we define the monomorphism $G_e \rightarrow G_{o(e)}$ to be the identity, and the monomorphism $G_e \rightarrow G_{t(e)}$ to be the multiplication by q map, so that the index of G_e in $G_{o(e)}$ is 1 and the index of G_e in $G_{t(e)}$ is q.

Let Γ and $\mathbb{X}$ be respectively the fundamental group (using the root as the basepoint) and the Bass-Serre tree of the graph of groups $(\mathscr{T}, G_\bullet)$. Then the degrees of the vertices of $\mathbb{X}$ are at least 3 and at most $q + d - 1$, and for every n, we have

$$\sum_{x \in V\mathscr{T} \, : \, d(x,*)=n} \frac{1}{|G_x|} \leq d^n/q^n \ . \tag{3}$$

Since q is large compared to d, this implies that the volume of $(\mathscr{T}, G_\bullet)$ is finite, hence Γ is a lattice.

Since the potential is the zero potential, the Gibbs measure m_0 is the Bowen-Margulis measure. Note that m_0 is finite since Γ is a lattice, by [BPP, Prop. 4.16]. Since we glued a loop at the root, there exists an element in Γ whose translation length is equal to 1, hence the length spectrum of Γ is equal to $\mathbb{Z}$. By Theorem 9, this implies that m_0 is mixing for the geodesic flow $(\mathfrak{g}^t)_t$. If $E = \{*\}$ is the singleton in $V\mathscr{T}$ consisting of the root, since q is large compared to d, Equation (3) then shows that the hypothesis of Theorem 14 is satisfied, and this concludes the proof of Proposition 15. $\qquad\square$

We conclude this survey with a sketch of proof of Theorem 14, sending to [BPP, §9.2] for a complete proof. We thank Omri Sarig for a key idea in the proof of this theorem.

Step 1. The first step consists in passing from the geometric dynamical system to a two-sided symbolic dynamical system, using Section 3.1.

Let $\mathscr{A}, A, \Sigma, \sigma, \Theta, \mathbb{P}$ be as given in Theorem 11 for the coding of the (discrete time) geodesic flow on the phase space $\Gamma \backslash \mathscr{G}\mathbb{X}$. Let $\pi_+ : \Sigma \rightarrow \mathscr{A}^{\mathbb{N}}$ be the natural projection defined by $(x_n)_{n \in \mathbb{Z}} \mapsto (x_n)_{n \in \mathbb{N}}$. Let

$$\mathscr{E} = \{(e^-, h, e^+) \in \mathscr{A} \ : \ t(e^-) = o(e^+) \in E\}$$

which is a finite subset of the alphabet, and let $\tau_{\mathscr{E}} : \Sigma \rightarrow \mathbb{N}$ be the first positive passage time in $\mathscr{E}$ of the two-sided shift orbits, that is, the map

$$x = (x_n)_{n \in \mathbb{Z}} \mapsto \inf\{n \in \mathbb{N} - \{0\} \ : \ x_n \in \mathscr{E}\} \ .$$

The rate of mixing statement for two-sided symbolic dynamical system, that we will prove in Step 2, is the following one.

Theorem 16 *Let $(\mathscr{A}, A, \Sigma, \sigma)$ be a locally compact transitive two-sided topological shift, and let $\mathbb{P}$ be a mixing σ-invariant probability measure with full support on Σ. Assume that*

(1) for every $n \in \mathbb{N}$ and for every A-admissible finite sequence $w = (w_0, \dots, w_n)$ in $\mathscr{A}$, the (measure theoretic) Jacobian of the map

$$f_w : \{(x_k)_{k \in \mathbb{N}} \in \pi_+(\Sigma) \ : \ x_0 = w_n\}$$
$$\to \{(y_k)_{k \in \mathbb{N}} \in \pi_+(\Sigma) \ : \ y_0 = w_0, \dots, y_n = w_n\}$$

defined by $(x_0, x_1, x_2, \dots) \mapsto (w_0, \dots, w_n, x_1, x_2, \dots)$, with respect to the restrictions of the pushforward measure $(\pi_+)_* \mathbb{P}$, is constant;
(2) there exist a finite subset $\mathscr{E}$ of $\mathscr{A}$ and $c'', \kappa' > 0$ such that for every $n \in \mathbb{N}$, we have

$$\mathbb{P}\big(\{x \in \Sigma \ : \ x_0 \in \mathscr{E} \text{ and } \tau_{\mathscr{E}}(x) \geq n\}\big) \leq c'' \, e^{-\kappa' n} \ .$$

Then $(\Sigma, \sigma, \mathbb{P})$ has exponential decay of α-Hölder correlations.

Theorem 14 follows from Theorem 16 by using the coding given in Theorem 11. The verification of Assertion (2) is immediate as it corresponds to the assumption of Theorem 14. The one of Assertion (1) is a bit technical, using a strengthened version of Mohsen's shadow lemma for trees.

Step 2. The second step consists in passing from the two-sided symbolic dynamical system to a one-sided symbolic dynamical system.

Let (Σ_+, σ_+) be the one-sided topological shift with the same alphabet $\mathscr{A}$ and same transition matrix A as the two-sided one in the statement of Theorem 16, with $\Sigma_+ = \pi_+(\Sigma)$ where π_+ is the natural projection, and let $\mathbb{P}_+ = (\pi_+)_* \mathbb{P}$. Let $\tau_{\mathscr{E},+} : \Sigma_+ \to \mathbb{N}$ be the first positive passage time in $\mathscr{E}$ of the one-sided shift orbits, that is, the map $(x_n)_{n \in \mathbb{N}} \mapsto \inf\{n \in \mathbb{N} - \{0\} \ : \ x_n \in \mathscr{E}\}$. Recall that the *cylinders* in Σ_+ are the subsets defined for $k \in \mathbb{N}$ and $w_0, \dots w_k \in \mathscr{A}$ by

$$[w_0, \dots, w_k] = \{x = (x_n)_{n \in \mathbb{N}} \in \Sigma_+ \ : \ x_0 = w_0, \dots, x_k = w_k\} \ .$$

The rate of mixing statement for one-sided symbolic dynamical system, that we will prove in Step 3, is the following one.

Theorem 17 *Let $(\mathscr{A}, A, \Sigma_+, \sigma_+)$ be a locally compact transitive one-sided topological shift, and let $\mathbb{P}_+$ be a mixing σ-invariant probability measure with full support on Σ_+. Assume that*

(1) for every $n \in \mathbb{N}$ and for every A-admissible finite sequence $w = (w_0, \dots, w_n)$ in $\mathscr{A}$, the Jacobian of the map between cylinders

$$f_w : [w_n] \to [w_0, \dots, w_n]$$

defined by $(x_0, x_1, x_2, \dots) \mapsto (w_0, \dots, w_n, x_1, x_2, \dots)$, with respect to the restrictions of $\mathbb{P}_+$, is constant;
(2) there exist a finite subset $\mathscr{E}$ of $\mathscr{A}$ and $c'', \kappa' > 0$ such that for every $n \in \mathbb{N}$, we have

$$\mathbb{P}_+\big(\{x \in \Sigma_+ \ : \ x_0 \in \mathscr{E} \text{ and } \tau_{\mathscr{E},+}(x) \geq n\}\big) \leq c'' \, e^{-\kappa' n} \ .$$

Then $(\Sigma_+, \sigma_+, \mathbb{P}_+)$ has exponential decay of α-Hölder correlations.

Theorem 16 follows from Theorem 17 by a classical argument due to Sinai and Bowen (and explained to the authors by Buzzi), saying that if the one-sided symbolic dynamical system $(\Sigma_+, \sigma_+, (\pi_+)_*\mathbb{P})$ is exponentially mixing, then so is the two-sided symbolic dynamical system $(\Sigma, \sigma, \mathbb{P})$.

Step 3. The third and final step that we sketch is a proof of Theorem 17, using as main tool a Young's tower argument.

We implicitly throw away from Σ_+ the measure zero subset of points $x \in \Sigma_+$ whose orbit under the shift σ_+ does not pass infinitely many times in the open nonempty finite union of fundamental cylinders

$$\Delta_0 = \bigcup_{a \in \mathscr{E}} [a] \ .$$

We denote by $\Phi : \Sigma_+ \to \Delta_0$ the first positive time passage map, which is defined by $x \mapsto \sigma_+^{\tau_{\mathscr{E},+}(x)}(x)$. We denote by W the set of excursions outside $\mathscr{E}$, that is, the set of A-admissible finite sequences $(w_0, \ldots, w_n)$ in $\mathscr{A}$ such that $w_0, w_n \in \mathscr{E}$ and $w_i \notin \mathscr{E}$ for $1 \leq i \leq n-1$.

We have the following properties.

(1) The set $\{[a] \ : \ a \in \mathscr{E}\}$ is a finite measurable partition of Δ_0. For every $a \in \mathscr{E}$, the set $\{[w] \ : \ w \in W, w_0 = a\}$ is a countable measurable partition of $[a]$.

(2) For every $w \in W$, the first positive passage time $\tau_{\mathscr{E},+}$ is positive on every excursion cylinder $[w]$, and if w_n is the last letter of the finite sequence w, then the restriction $\Phi|_{[w]} \colon [w] \to [w_n]$ is a bijection with constant Jacobian with respect to $\mathbb{P}_+$ (actually much less is needed in order to apply Young's arguments).

(3) The first positive time passage map Φ satisfies strong dilations properties on the excursion cylinders. More precisely, for every excursion $w = (w_0, \ldots, w_n) \in W$, for every $k \leq n-1$, for all $x, y \in [w]$, we have $d(\Phi(x), \Phi(y)) \geq e \, d(x, y)$ and $d(\sigma_+^k x, \sigma_+^k y)) < d(\Phi(x), \Phi(y))$.

Let us fix $\alpha \in \ ]0,1]$. Then an adaptation of [You, Theo. 3] implies that there exists $\kappa > 0$ such that for all $\phi, \psi \in \mathscr{C}_{\mathrm{b}}^\alpha(\Sigma_+)$, there exists $c_{\phi,\psi} > 0$ such that for every $n \in \mathbb{N}$, we have

$$|\operatorname{cov}_{\mathbb{P}_+, n}(\phi, \psi)| \leq c_{\phi,\psi} \, e^{-\kappa n} \ .$$

An argument using the Principle of Uniform Boundedness due to Chazotte then allows us to take $c_{\phi,\psi} = c' \, \|\phi\|_\alpha \, \|\psi\|_\alpha$ for some constant $c' > 0$.

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