# Equilibrium states in negative curvature

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February 25, 2014.

**Abstract**

With their origin in thermodynamics and symbolic dynamics, Gibbs measures are crucial tools to study the ergodic theory of the geodesic flow on negatively curved manifolds. We develop a framework (through Patterson-Sullivan densities) allowing us to get rid of compactness assumptions on the manifold, and prove many existence, uniqueness and finiteness results of Gibbs measures. We give many applications, to the Variational Principle, the counting and equidistribution of orbit points and periods, the unique ergodicity of the strong unstable foliation and the classification of Gibbs densities on some Riemannian covers. [1]

## 1 Introduction

With their thermodynamic origin, Gibbs measures (or states) are very useful in symbolic dynamics over finite alphabets (see for instance [Rue3, Zin, Kel]). Sinai, Bowen and Ruelle introduced them in hyperbolic dynamics (which, via coding theory, has strong links to symbolic dynamics), in particular in order to study the weighted distribution of periodic orbits and the equilibrium states for given potentials (see for instance [Sin, Bow4, BoR, PaPo2]). For instance, this allows the dynamical analysis of the geodesic flow of negatively curved Riemannian manifolds, provided its non-wandering set is compact. As a first step to venture beyond the compact case, Gibbs measures in symbolic dynamics over countable alphabets have been developed by Sarig [Sar1, Sar3]. But since no coding theory which does not lose geometric information is known for general non-compact manifolds, this methodology is not well adapted to the non-compact case. Note that a coding-free approach of transfer operators, in particular via improved spectral methods, was put in place by Liverani, Baladi, Gouëzel and others (see for instance, restricting the references to the case of flows, [Live, BuL, Tsu1, Tsu2, BaLi, GLP], as well as [AvG] for an example in a non locally homogeneous and non-compact situation). In this text, we construct and study geometrically Gibbs measures for the geodesic flow of negatively curved Riemannian manifolds, without compactness assumption.

By considering the action on a universal Riemannian cover of its covering group, this study is strongly related to the (weighted) distribution of orbits of discrete groups of isometries of negatively curved simply connected Riemannian manifolds. After work of Huber and Selberg (in particular through his trace formula) in constant curvature, Margulis (see for instance [Marg]) made a breakthrough, albeit in the unweighted lattice case, to

[1]**Keywords:** geodesic flow, negative curvature, Gibbs state, periods, orbit counting, Patterson density, pressure, variational principle, strong unstable foliation. **AMS codes:** 37D35, 53D25, 37D40, 37A25, 37C35, 53C12

give the precise asymptotic growth of the orbits. Patterson [Patt] and Sullivan [Sul1], in the surface and constant curvature case respectively, have introduced a nice approach using measures at infinity, giving in particular a nice construction of the measure of maximal entropy in the cocompact case. The Patterson-Sullivan theory extends to variable curvature and non lattice case, and an optimal reference (in the unweighted case) is due to Roblin [Rob1].

After preliminary work of Hamenstädt [Ham2] on Gibbs cocycles, of Ledrappier [Led2], Coudène [Cou2] and Schapira [Sch3] using a slightly different approach, and of Mohsen [Moh] (their contributions will be explained as the story unfolds), the aim of this book is to develop an optimal theory of Gibbs measures for the dynamical study of geodesic flows in general negatively curved Riemannian manifolds, and of weighted distribution of orbits of general discrete groups of isometries of negatively curved simply connected Riemannian manifolds.

Let us give a glimpse of our results, starting by describing the players. Let $M$ be a complete connected Riemannian manifold with pinched negative curvature. Let $F : T^1M \to \mathbb{R}$ be a Hölder-continuous map, called a *potential*, which is going to help us define the various weights. In order to simplify the exposition of this introduction, we assume here that $F$ is invariant by the antipodal map $v \mapsto -v$ (see the main body of the text for complete statements). Let $p : \widetilde{M} \to M$ be a universal Riemannian covering map, with covering group $\Gamma$ and sphere at infinity $\partial_\infty \widetilde{M}$, and let $\widetilde{F} = F \circ p$. We make no compactness assumption on $M$: we only assume $\Gamma$ to be non-elementary (that is, non virtually nilpotent). (In the body of the text, we will also allow $M$ to be a good orbifold, hence $\Gamma$ to have torsion.) Let $\phi = (\phi_t)_{t\in\mathbb{R}}$ be the geodesic flow on $T^1M$ and $\widetilde{\phi} = (\widetilde{\phi}_t)_{t\in\mathbb{R}}$ the one on $T^1\widetilde{M}$. For all $x, y \in \widetilde{M}$, let us define $\int_x^y \widetilde{F} = \int_0^{d(x,\,y)} \widetilde{F}(\widetilde{\phi}_t v)\, dt$ where $v$ is the unit tangent vector at $x$ to a geodesic from $x$ through $y$. For every periodic orbit $g$ of $\phi$, let $\mathscr{L}_g$ be the Lebesgue measure along $g$ and $\int_g F = \mathscr{L}_g(F)$ the *period* of $g$ for the potential $F$.

We will study three numerical invariants of the weighted dynamics.

• Let $x, y \in \widetilde{M}$, and $c > 0$ large enough. The *critical exponent* of $(\Gamma, F)$ is an exponential growth rate of the orbit points of $\Gamma$ weighted by the potential $F$:

$$\delta_{\Gamma,\,F} = \lim_{t\to+\infty} \frac{1}{t} \log \sum_{\gamma\in\Gamma,\; t-c\le d(x,\,\gamma y)\le t} e^{\int_x^{\gamma y} \widetilde{F}}\ .$$

• For every $t \geq 0$, let $\mathscr{P}\mathrm{er}(t)$ be the set of periodic orbits of $\phi$ with length at most $t$, and $\mathscr{P}\mathrm{er}'(t)$ its subset of primitive ones. Let $W$ be a relatively compact open subset of $T^1M$ meeting the non-wandering set of $\phi$, and $c > 0$ large enough. The *Gurevich pressure* of $(M, F)$ is an exponential growth rate of the closed geodesics of $M$ weighted by the potential $F$:

$$P_{Gur}(M, F) = \lim_{t\to+\infty} \frac{1}{t} \log \sum_{g\in\mathscr{P}\mathrm{er}(t)-\mathscr{P}\mathrm{er}(t-c),\; g\cap W\neq\emptyset} e^{\int_g F}\ .$$

The restriction for the periodic orbits under consideration to meet a given compact set is necessary: for instance if $M$ is an infinite cyclic cover of a compact manifold, and $F$ is invariant under the cyclic covering group, then $T^1M$ has periodic orbits with the same length and same period going out of every compact subset, and the left hand side would otherwise be $+\infty$ for $t, c$ large enough.

• Let $\mathscr{M}$ be the set of $\phi$-invariant probability measures on $T^1M$ and let $h_m(\phi)$ be the (metric, that is measure-theoretic) entropy of the geodesic flow $\phi$ with respect to $m \in \mathscr{M}$. The *topological pressure* of $(\phi, F)$ is the upper bound of the sum of the entropy and the averaged potential of all states:

$$P_{top}(\phi, F) = \sup_{m \in \mathscr{M}} \Big( h_m(\phi) + \int_{T^1M} F\,dm \Big) .$$

When $F = 0$, the quantity $\delta_{\Gamma,\,F}$ is the usual critical exponent $\delta_\Gamma$ of $\Gamma$, and $P_{top}(\phi, F)$ is the topological entropy $h_{top}(\phi)$ of $\phi$. We prove in Subsection 4.2 and 4.3 that the limits defining $\delta_{\Gamma,\,F}$ and $P_{Gur}(M, F)$ exist (a result of Roblin [Rob2] when $F = 0$) and are independent of $x, y, W$ and $c > 0$ large enough (depending on $x, y, W$), and that, when $\delta_{\Gamma,\,F} > 0$, the sums in their definition may be taken respectively over $\{\gamma \in \Gamma,\ d(x, \gamma y) \leq t\}$ and $\{g \in \mathscr{P}\mathrm{er}(t),\ g \cap W \neq \emptyset\}$.

We have not assumed $\Gamma$ to be finitely generated, so we prove in Subsection 4.4 that $\delta_{\Gamma,\,F}$ is the upper bound of the critical exponents with potential $F$ of the free finitely generated semi-groups in $\Gamma$ (of Schottky type), a result due to Mercat [Mer] if $F = 0$. For Markov shifts on countable alphabets, the Gurevich pressure has been introduced by Gurevich [Gur1, Gur2] when the potential is equal to 0, and by Sarig [Sar1, Sar3, Sar2] in general. The Gurevich pressure had not been studied much in a non symbolic non-compact context before this work.

Our first result, generalising in particular results of Manning [Mann] ($\delta_\Gamma = h_{top}(\phi)$), Ruelle [Rue2] and Parry [Par3] when $M$ is compact, is the following one (see Subsection 4.3 and Chapter 6).

**Theorem 1.1** *We have*

$$P_{Gur}(M, F) = \delta_{\Gamma,\,F} = P_{top}(\phi, F) .$$

Let us now define the Gibbs measures (see Chapter 3). We use Mohsen's extension (whose compactness assumptions are not necessary) of the Patterson-Sullivan construction, see [Moh], which is more convenient than the one of [Led2, Cou2, Sch3]. The (normalised) *Gibbs cocycle* of $(M, F)$ is the map $C$ from $\partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M}$ to $\mathbb{R}$ defined by

$$(\xi, x, y) \mapsto C_\xi(x, y) = \lim_{t \to +\infty} \int_y^{\xi_t} (\widetilde{F} - \delta_{\Gamma,\,F}) - \int_x^{\xi_t} (\widetilde{F} - \delta_{\Gamma,\,F}) ,$$

where $t \mapsto \xi_t$ is any geodesic ray converging to a given $\xi \in \partial_\infty \widetilde{M}$. When $F = 0$, we have $C_\xi(x, y) = \delta_\Gamma \beta_\xi(x, y)$ where $\beta$ is the Busemann cocycle. A (normalised) *Patterson density* of $(\Gamma, F)$ is a family of pairwise absolutely continuous finite positive nonzero Borel measures $(\mu_x)_{x \in \widetilde{M}}$ on the sphere at infinity $\partial_\infty \widetilde{M}$ such that, for all $\gamma \in \Gamma$, $x, y \in \widetilde{M}$ and for $\mu_y$-almost every $\xi \in \partial_\infty \widetilde{M}$,

$$\gamma_* \mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_\xi(x,\,y)} .$$

Mohsen has extended Patterson's construction of such a family, as well as Sullivan's shadow lemma.

Fix $x_0 \in \widetilde{M}$. For every $v \in T^1\widetilde{M}$, let $v_-, \pi(v), v_+$ be respectively the point at $-\infty$, the origin, the point at $+\infty$ of the geodesic line defined by $v$, and let $t$ be the distance between $\pi(v)$ and the closest point to $x_0$ on this line. Define a measure $\widetilde{m}_F$ on $T^1\widetilde{M}$ by

$$d\widetilde{m}_F(v) = \frac{d\mu_{x_0}(v_-)}{e^{C_{v_-}(\pi(v),\, x_0)}}\ \frac{d\mu_{x_0}(v_+)}{e^{C_{v_+}(\pi(v),\, x_0)}}\ dt\ .$$

This measure on $T^1\widetilde{M}$ is invariant under $\Gamma$ and $\widetilde{\phi}$, hence induces a measure on $T^1M$ invariant under $\phi$, called the *Gibbs measure* (or *Gibbs state*) of the potential $F$. The Gibbs measures are Radon measures (that is, they are locally finite Borel positive measures), but they are not always finite.

In symbolic dynamics, the Gibbs measures are shift-invariant measures, which give to any cylinder a weight defined by the Birkhoff sum of the potential. Here, the analogs of the cylinders are the neighbourhoods of a (pointed and oriented) geodesic line, defined by small neighbourhoods of its two points at infinity, that we also weight according to the Birkhoff integral of the potential in order to define our Gibbs measures. In Subsection 3.8, we prove that our Gibbs measures indeed satisfy the Gibbs property on the dynamical balls.

We have a huge choice of potential functions. For instance, we can define

$$F^{\rm su} = -\frac{d}{dt}_{|t=0} \log \operatorname{Jac}\big(\phi_{t|W^{\rm su}(v)}\big)(v)\ ,$$

the negative of the pointwise exponential growth rate of the Jacobian of the geodesic flow restricted to the strong unstable manifold (see Chapter 7). When $M$ has dimension $n$ and constant curvature $-1$, then $F^{\rm su}$ is constant, equal to $-(n-1)$. Some references use the opposite sign convention on the potential to define the topological pressure as $\sup_{m\in\mathscr{M}}\big(h_m(\phi) - \int_{T^1M} F\,dm\big)$, hence the unstable Jacobian needs also to be defined with the opposite sign as the one above.

When $M$ is compact, we recover, up to a scalar multiple, for $F = 0$ (the Patterson-Sullivan construction of) the Bowen-Margulis measure, and for $F = F^{\rm su}$ the Liouville measure. Using Gibbs measures is a way of interpolating between these measures, and gives a wide range of applications.

We will prove (see Chapter 7) new results about when the Liouville measure is a Gibbs measure. Note that there are many examples of non-compact (with infinite volume) Riemannian covers of compact Riemannian manifolds with constant sectional curvature $-1$ and with ergodic (hence conservative) geodesic flow for the Liouville measure, see for instance [Ree].

**Theorem 1.2** *If $\widetilde{M}$ is a Riemannian cover of a compact manifold, and if the geodesic flow $\phi$ of $M$ is conservative with respect to the Liouville measure, then the Liouville measure is, up to a scalar multiple, the Gibbs measure for the potential $F^{\rm su}$ and*

$$P_{Gur}(M, F^{\rm su}) = \delta_{\Gamma,\, F^{\rm su}} = P_{top}(\phi, F^{\rm su}) = 0\ .$$

Let us now state (a simplified version of) the main results of our book. An *equilibrium state* for the potential $F$ is a $\phi$-invariant probability measure $m$ on $T^1M$ such that $h_m(\phi) + \int_{T^1M} F\,dm$ is equal to the topological pressure $P_{top}(\phi, F)$ (hence realising the upper bound defining it). The first result (see Chapter 6) says that the equilibrium states are the Gibbs states (normalised to probability measures).

**Theorem 1.3 (Variational Principle)** *If $m_F$ is finite, then $\frac{m_F}{\|m_F\|}$ is the unique equilibrium state. Otherwise, there exists no equilibrium state.*

When $\Gamma$ is convex-cocompact, this result is due to Bowen and Ruelle [BoR]. Their approach, using symbolic dynamics, called the *thermodynamic formalism*, is described in many references, and has produced many extensions for various situations (see for instance [Rue3, Zin, Kel]), under some compactness assumption (except for the work of Sarig already mentioned). For general $\Gamma$ but when $F = 0$, the result is due to Otal and Peigné [OtP], and we will follow their proof.

Up to translating $F$ by a constant, which does not change the Patterson density nor the Gibbs measure, we assume in the following four results 1.4 to 1.7, that $\delta_{\Gamma,\,F} > 0$ (see the main body of this book for the appropriate statements when $\delta_{\Gamma,\,F} \leq 0$). The next result, of equidistribution of (ordered) pairs of Dirac masses on an orbit weighted by the potential towards the product of Patterson measures, is due to Roblin [Rob1] when $F = 0$, and we will follow his proof (as well as for the next three results). We denote by $\mathscr{D}_z$ the unit Dirac mass at a point $z \in \widetilde{M}$.

**Theorem 1.4 (Two point orbital equidistribution)** *If $m_F$ is finite and mixing, then we have, as $t$ goes to $+\infty$,*

$$\delta_{\Gamma,\,F}\;\|m_F\|\;e^{-\delta_{\Gamma,\,F}\;t}\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}}\;\mathscr{D}_{\gamma^{-1}x}\otimes\mathscr{D}_{\gamma y}\;\overset{*}{\rightharpoonup}\;\mu_y\otimes\mu_x\;.$$

**Corollary 1.5 (Orbital counting)** *If $m_F$ is finite and mixing, then, as $t$ goes to $+\infty$,*

$$\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}}\;\sim\;\frac{\|\mu_y\|\;\|\mu_x\|}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{\delta_{\Gamma,\,F}\;t}\;.$$

When $M$ is compact and $F = 0$, this corollary is due to Margulis. When $F = 0$, it is due to Roblin in this generality. Theorem 1.4 also gives counting asymptotics of orbit points $\gamma x_0$ staying, as well as their reciprocal $\gamma^{-1}x_0$, in given cones, see Subsection 9.2 (improving, under stronger hypotheses, the logarithmic growth result of Subsection 4.2).

The next two results are due to Bowen [Bow2] when $M$ is compact (or even $\Gamma$ convex-cocompact), see also [PaPo1, Par2]. But our proof is very different even under this compactness assumption, in particular we avoid any coding theory, hence any symbolic dynamics, and any zeta function or transfer operator. They are due to Roblin [Rob1] when $F = 0$ in this generality. The assumptions in the second statement cannot be removed: for instance if $M$ is an infinite cyclic cover of a closed manifold (which is not geometrically finite), and $F$ is invariant under the cyclic covering group, then the Gibbs measure is infinite, $T^1M$ has infinitely many periodic orbits with the same length and same period, and the left hand side is $+\infty$ for $t$ large enough. Due to the rarity of non geometrically finite examples with finite and mixing Bowen-Margulis measure, we do not know if the assumption that $\Gamma$ is geometrically finite can be removed.

**Theorem 1.6 (Equidistribution of periodic orbits)** *If $m_F$ is finite and mixing, then, as $t$ goes to $+\infty$,*

$$\delta_{\Gamma,\,F}\;e^{-\delta_{\Gamma,\,F}\;t}\sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\int_g F}\;\mathscr{L}_g\;\overset{*}{\rightharpoonup}\;\frac{m_F}{\|m_F\|}\;.$$

**Corollary 1.7 (Counting of periodic orbits)** *If $\Gamma$ is geometrically finite and if $m_F$ is finite and mixing, then, as $t$ goes to $+\infty$,*

$$\sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\int_g F} \;\sim\; \frac{e^{\delta_{\Gamma,\,F}\; t}}{\delta_{\Gamma,\,F}\; t}\;.$$

Let us now describe additional results. Consider the Poincaré series of $(\Gamma, F)$

$$Q_{\Gamma,\,F,\,x,\,y}(s) = \sum_{\gamma\in\Gamma} \; e^{\int_x^{\gamma y}(\widetilde{F}-s)}\;,$$

which, independently of $x, y \in \widetilde{M}$, converges if $s > \delta_{\Gamma,\,F}$ and diverges if $s < \delta_{\Gamma,\,F}$. In Chapter 5, extending works of Hopf, Tsuji, Sullivan, Roblin when $F = 0$ and following the proofs of [Rob1], we give criteria for the ergodicity and non ergodicity of the geodesic flow on $T^1M$ endowed with a Gibbs measure.

**Theorem 1.8** *The following conditions are equivalent*

*(i) The Poincaré series of $(\Gamma, F)$ diverges at $s = \delta_{\Gamma,\,F}$.*

*(ii) The conical limit set of $\Gamma$ has positive measure with respect to $\mu_x$.*

*(iii) The dynamical system $(\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}, \Gamma, \mu_x \otimes \mu_x)$ is ergodic and conservative.*

*(iv) The dynamical system $(T^1M, \phi, m_F)$ is ergodic and conservative.*

We discuss several applications in Subsection 5.3. In particular, if the Poincaré series diverges at $s = \delta_{\Gamma,\,F}$, then the normalised Patterson density has no atom and is unique up to a scalar multiple, and the Gibbs measure on $T^1M$ is unique up to a scalar multiple and ergodic when finite.

In view of (iii), it would be interesting to study if $(\partial_\infty\widetilde{M}, \mu_x)$ is a (weak) $\Gamma$-boundary in the sense of Burger-Mozes [BuM] or Bader-Furman [BaF, BaFS].

In Subsection 8.2, we give a criterion for the finiteness of $m_F$ when $M$ is geometrically finite, extending the one in [DaOP] when $F = 0$, as in Coudène [Cou2]. The results 1.4 to 1.7 require $m_F$ to be finite and mixing. But given the finiteness assumption, the mixing one is mild. Indeed, by Babillot's result [Bab2, Theo. 1], since Gibbs measures are quasi-product measures, if $m_F$ is finite, then $m_F$ is mixing if and only if the geodesic flow $\phi$ is topologically mixing on $T^1M$, a purely topological condition which is seemingly easier to check and is conjecturally always satisfied.

It is indeed another interesting feature of Gibbs measures, that they have strong ergodic properties. As soon as they are finite, they are ergodic and even mixing as just said (when $\phi$ is topologically mixing on $T^1M$). If in addition $P_{top}(\phi, F) > \sup F$ (for example if $\sup F - \inf F < h_{top}(\phi)$), then the entropy of the Gibbs measure $m_F$ is positive. These properties are natural, since the geodesic flow in negative curvature is a hyperbolic flow, and therefore should have strong stochastic properties. However, generically (in Baire's sense), Coudène-Schapira [CS] have announced that the invariant probability measures for $\phi$ are ergodic, but not mixing and of entropy zero. Therefore, contrarily to the generic case, the Gibbs measures provide an explicit family of good measures reflecting the strong stochastic properties of the geodesic flow.

Consider the cocycle $c_F$ defined on the (ordered) pairs of vectors $v, w \in T^1M$ in the same leaf of the strong unstable foliation $\mathscr{W}^{\rm su}$ of the geodesic flow $\phi$ in $T^1M$ by

$$c_F : (w, v) \mapsto \lim_{t\to+\infty} \int_0^t (F(\phi_{-s}w) - F(\phi_{-s}v))\, ds\ .$$

A nonzero family $(\nu_T)_T$ of locally finite (positive Borel) measures on the transversals $T$ to $\mathscr{W}^{\rm su}$, stable by restrictions, is *$c_F$-quasi-invariant* if for every holonomy map $f : T \to T'$ along the leaves of $\mathscr{W}^{\rm su}$ between two transversals $T, T'$ to $\mathscr{W}^{\rm su}$, we have

$$\frac{d\, f_*\nu_T}{d\,\nu_{T'}}(f(x)) = e^{c_F(f(x),\,x)}\ .$$

We will prove in Chapter 10, as in [Rob1, Chap. 6] when $F = 0$ and in [Sch1, Chap. 8.2.2] (unpublished), the following unique ergodicity result.

**Theorem 1.9** *If $m_F$ is finite and mixing, then there exists, up to a scalar multiple, one and only one $c_F$-quasi-invariant family of transverse measures $(\mu_T)_T$ for $\mathscr{W}^{\rm su}$ such that $\mu_T$ gives full measure to the set of points of $T$ which are negatively recurrent under the geodesic flow $\phi$ in $T^1M$.*

We give an explicit construction of $(\mu_T)_T$ in terms of the above Patterson densities, and we deduce from Theorem 1.9, when $M$ is geometrically finite, the classification of the ergodic $c_F$-quasi-invariant families of transverse measures. This generalises works when $F = 0$ of Furstenberg, Dani, Dani-Smillie, Ratner for $M$ a surface, and of Roblin for general $M$.

The main tool, which follows from Babillot's work, is the following equidistribution result in $T^1M$ of the strong unstable leaves towards the Gibbs measure $m_F$. For every $v \in T^1M$, let $W^{\rm su}(v)$ be its strong unstable leaf for $\phi$ and let $(\mu_{W^{\rm su}(v)})_{v\in T^1M}$ be the family of conditional measures on the strong unstable leaves of the Gibbs measure $m_F$ associated to the potential $F$. Note that conditional measures are in general defined almost everywhere, and up to normalisation. Here, the nice structure of the Gibbs measure allows to define them everywhere and not up to normalisation (though they depend on the harmless normalisation of the Patterson measures). Then for every relatively compact Borel subset $K$ of $W^{\rm su}(v)$ (containing a positively recurrent vector in its interior), for every $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$, we have

$$\lim_{t\to+\infty} \frac{1}{\int_K e^{c_F(w,v)}\, d\mu_{W^{\rm su}(v)}(w)} \int_K \psi\circ\phi_t(w)\, e^{c_F(w,v)}\, d\mu_{W^{\rm su}(v)}(w) = \frac{1}{\|m_F\|}\int_{T^1M} \psi\, dm_F\ .$$

We also prove and use the sub-exponential growth of the mass of strong unstable balls for the conditional measures on the strong unstable leaves (we may no longer have polynomial growth as when $F = 0$, though we have no example in mind besides divergent leaves).

In the final chapter 11, we extend the work of Babillot-Ledrappier [BaL], Hamenstädt [Ham4] and Roblin [Rob3] when $F = 0$ to study the Gibbs states on Galois Riemannian covers of $M$.

Let $\chi : \Gamma \to \mathbb{R}$ be a (real) character of $\Gamma$. For all $x, y \in \widetilde{M}$, let us define the *(twisted) Poincaré series of* $(\Gamma, F, \chi)$ by

$$Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s) = \sum_{\gamma\in\Gamma}\ e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-s)}\ .$$

The *(twisted) critical exponent of* $(\Gamma, F, \chi)$ is the element $\delta_{\Gamma,\,F,\,\chi}$ in $[-\infty, +\infty]$ defined by

$$\delta_{\Gamma,\,F,\,\chi} = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma,\; n-1<d(x,\,\gamma y)\leq n} e^{\chi(\gamma)+\int_x^{\gamma y} \widetilde{F}} \;.$$

When $F = 0$ and $M$ is compact, since $H^1(M;\mathbb{R}) \simeq \operatorname{Hom}(\Gamma, \mathbb{R})$, we recover the Poincaré series associated with a cohomology class of $M$ introduced in [Bab1], and the critical exponent of $(\Gamma, F, \chi)$ is then called the *cohomological pressure*.

A *(twisted) Patterson density* of dimension $\sigma \in \mathbb{R}$ for $(\Gamma, F, \chi)$ is a family of finite nonzero (positive Borel) measures $(\mu_x)_{x\in\widetilde{M}}$ on $\partial_\infty\widetilde{M}$ such that, for every $\gamma \in \Gamma$, for all $x, y \in \widetilde{M}$, for every $\xi \in \partial_\infty\widetilde{M}$, we have

$$\gamma_*\mu_x = \; e^{-\chi(\gamma)}\; \mu_{\gamma x}\;,$$

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{\lim_{t\to+\infty} \int_x^{\xi_t}(\widetilde{F}-\sigma) - \int_y^{\xi_t}(\widetilde{F}-\sigma)}\;,$$

where $t \mapsto \xi_t$ is any geodesic ray converging to $\xi$.

Using the above-mentioned extension of the Hopf-Tsuji-Sullivan-Roblin theorem (Theorem 1.8), as well as (an immediate extension to the case with potential of) the Fatou-Roblin radial convergence theorem for the ratio of the total masses of two Patterson densities (see [Rob3, Théo. 1.2.2]), we prove in Chapter 11, amongst other results, that the topological pressure is unchanged by taking an amenable cover of $M$, and we give a classification result of the Patterson densities on nilpotent covers of $M$, for instance when $M$ is compact.

**Theorem 1.10** *Let $\Gamma'$ be a normal subgroup of $\Gamma$, and $F' : \Gamma'\backslash T^1\widetilde{M} \to \mathbb{R}$ be the map induced by $\widetilde{F}$.*

*(1) If the quotient group $\Gamma/\Gamma'$ is amenable, then $\delta_{\Gamma,\,F} = \delta_{\Gamma',\,F'}$.*

*(2) Let $\chi : \Gamma \to \mathbb{R}$ be a character of $\Gamma$. If $\delta_{\Gamma,\,F,\,\chi} < +\infty$ and $Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s)$ diverges at $s = \delta_{\Gamma,\,F,\,\chi}$ (for instance if $\Gamma$ is convex-cocompact), then there exists a unique (up to a scalar multiple) twisted Patterson density $\mu_{\Gamma,\,F,\,\chi} = (\mu_{\Gamma,\,F,\,\chi,\,x})_{x\in\widetilde{M}}$ of dimension $\delta_{\Gamma,\,F,\,\chi}$ for $(\Gamma, F, \chi)$.*

*(3) If $\Gamma$ is convex-cocompact and $\Gamma/\Gamma'$ is nilpotent, then the set of ergodic Patterson densities for $(\Gamma', F')$ is the set of multiples of $\mu_{\Gamma,\,F,\,\chi}$ for the characters $\chi$ of $\Gamma$ vanishing on $\Gamma'$.*

We conclude this introduction by saying a few words concerning again the motivations, the framework, and possible developments.

As already alluded to, the main motivation comes from statistical physics, including one-dimensional lattice gasses, interactions, ensembles and transfer matrix, from the point of view of equilibrium states for chaotic dynamics under general potentials, pressure computations, weighted distribution of periods, etc. The restriction to compact state spaces was starting to be cumbersome. From a purely dynamical system point of view, using potentials is a way to study the weighting of the dynamics: a subset of the state space where the potential is very negative tends to have a small measure with respect to the equilibrium state, by the maximisation process involved in the Variational Principle. More precisely, for all $v, w \in T^1M$, if $\phi^{\mathbb{R}}w$ stays in a subset of $T^1M$ where $F$ is small, and $\phi^{\mathbb{R}}v$

stays in a subset of $T^1M$ where $F$ is large, the ratio $m_F(B(v,T,r))/m_F(B(w,T,r))$ of the masses for the equilibrium state of the potential $F$ of the dynamical balls centred at $v, w$ should tend exponentially to 0. Furthermore, the family of Gibbs measures, when the potential is varying, is a large family having excellent dynamical properties, including a quasi-product structure with respect to the stable/unstable manifolds, which provides a wide playground with many possible applications, including multi-fractal analysis.

The framework of pinched negatively curved orbifolds, instead of quotients $\Gamma\backslash X$ of CAT$(-1)$ metric spaces $X$ as in [Rob1], comes precisely from the necessity of controlling the potential. Though the quotient space $\Gamma\backslash\mathscr{G}X$ of the space $\mathscr{G}X$ of geodesic lines in $X$ is the natural replacement for the state space which is unit tangent bundle, it is problematic to define the potential $F$ on this replacement $\Gamma\backslash\mathscr{G}X$, if one wants to be able to integrate $F$ along pieces of the orbits, as two geodesic lines might share a nontrivial segment. The lower bound on the curvature is also useful to control the Hölder structure on the unit tangent bundle (see Lemma 2.3) and the divergence on the potential on long pieces of orbits which are initially close (see Lemma 2.5 (ii)). We refer to [BrPP] where the authors introduce, in the case of metric trees (which lie at the other end of the range of CAT$(-1)$ spaces), the new tools necessary to deal with the potential in these singular state spaces.

Here is, amongst many possible others, a list of research themes connected to the above presentation, some of which the authors plan to work on (see also [PaP4] for applications to counting geodesic arcs with weights, and [PauP] for Khintchine type results for the above equilibrium states).

1. It would be interesting to know when the Patterson densities associated to potentials (defined above and in Subsection 3.6) are harmonic measures for random walks on an orbit of $\Gamma$, see [CM] when $M$ is compact. It would also be interesting to extend to the case with nonzero potential the study of the spectral theory of Patterson densities, as done by Patterson and Sullivan in constant curvature (starting by studying the meromorphic extension of the Poincaré series associated to the potential, possibly using techniques of [GLP] or the recent microlocal approach of [DZ]).

2. It would be interesting to study on which conditions on two potentials their associated Patterson densities belong to the same measure class (that is, are absolutely continuous with respect to each other). Random walk techniques often allow to construct singular measures (see for instance [KaiL, GaMT]), hence this problem could be linked to the previous one.

3. It would be interesting to study when our Gibbs measures (defined above and in Subsection 3.7) are harmonic measures associated to elliptic second-order differential operators on $M$ (see for instance [Ham3] when $M$ is compact).

4. Recall (see for instance [PoY, Chap. 16], [AaN]) that a measure preserving flow $(\psi_t)_{t\in\mathbb{R}}$ of a measured space $(X,\mu)$ is 2-*recurrent* if for every measurable subset $B$ in $X$ with $\mu(B)>0$, there exists $n\in\mathbb{N}-\{0\}$ such that
$$\mu(\psi_{-n}B\cap B\cap\psi_n B)>0\ .$$
(We do not know whether we may replace $n\in\mathbb{N}-\{0\}$ by $t\in[1,+\infty[$ and get an equivalent definition.) For every $d\in\mathbb{N}-\{0\}$, the notion of $d$-recurrence (replacing

the above centred formula by $\mu(B \cap \psi_{-n}B \cap \cdots \cap \psi_{-dn}B) > 0$) has been introduced by Furstenberg in his proof of Szemeredi's theorem, and the 1-recurrence property is the conservativity property (see Subsection 5.2). Furstenberg proved that if $\mu$ is a probability measure, then $(\psi_t)_{t\in\mathbb{R}}$ is $d$-recurrent for every $d \in \mathbb{N} - \{0\}$. But when $\mu$ is infinite, there are examples of transformations which are 1-recurrent but not 2-recurrent (see for instance [AaN]). It would be interesting to know when the geodesic flow on $T^1M$ is multiply recurrent with respect to the Gibbs measure associated to a given potential. More precisely, let $N$ be a compact connected negatively curved Riemannian manifold; let $N' = G'\backslash\widetilde{N}$ where $G'$ is a (non necessarily normal) subgroup of the covering group $G$ of a universal Riemannian cover $\widetilde{N} \to N$; let $S$ be a finite generating set of $G$; and let $X(G, S)$ be the Cayley graph of $G$ with respect to $S$. It would be interesting to know when exactly the geodesic flow of $N'$ is ergodic or 2-recurrent, with respect to the Liouville measure, or to any given Gibbs measure. We expect the ergodicity to hold exactly when the appropriate (possibly simple for the Liouville measure) random walk on the Schreier graph $G'\backslash X(G, S)$ of $G$ relative to $G'$ is recurrent. We expect the 2-recurrence to be more restrictive, as Aaronson and Nakada (private communication) claim that the geodesic flow on the $\mathbb{Z}^d$-cover of a compact connected hyperbolic manifold is 2-recurrent for the Liouville measure if and only if $d = 0$ or $d = 1$, and Rees proved in [Ree] that it is ergodic (and hence conservative by [Aar, Prop. 1.2.1]) for the Liouville measure if and only if $d = 0, 1, 2$.

5. It would be interesting to extend our work on equilibrium states and Gibbs measures from the geodesic flow to the magnetic flows, seen as a deformation family of the geodesic flow, as studied for instance by Paternain and Grognet [Gro].

6. It would be interesting to know under which conditions on $\Gamma$ we have $\delta_{\Gamma,\,F^{\rm su}} \leq 0$ or, as when $M$ is compact, $\delta_{\Gamma,\,F^{\rm su}} = 0$ (see Section 7 for partial results).

7. As suggested by the referee, it would be interesting to generalise Lalley's result [Lal], using our techniques, to prove that a weighted closed geodesic chosen at random amongst the ones with lengths at most $T$ is close with high probability to the equilibrium state. This can be made precise by Large deviation results (see Y. Kifer [Kif], as well as [Pol] for some extension to Gibbs measures, both in the compact case), as well as by Central Limit theorems and multidimensional ones, that would be interesting to establish in our framework. The key point, if one wants to follow Kifer's steps, would be to study the upper-continuity of the pressure of the measures (in the non-compact case, which could be delicate, considering the countable state shift counterexamples).

8. It would be interesting to study the continuous dependence (using for instance Wasserstein-type distances, as in [Vil]) or higher regularity dependence (using the differential calculus of Ambrosio-Gigli-Savaré [AGS] or others) on the potential $F$ of its associated Gibbs measure $m_F$. In another direction, it would be interesting to know whether the integral against a fixed Hölder-continuous reference function of a Gibbs measure varies analytically with the potential, possibly using ideas of [BuL] and [GLP]. One could also deform the Riemannian metric, as was done in [KatKPW, Con], both in the compact case.

9. It would be interesting to study the exponential decay of correlations for equilibrium

states in our general framework (see the works of Dolgopyat, Liverani, Giulietti-Liverani-Pollicott [GLP], and the first serious result in the (geometric) non-compact case of [MO]).

10. It would be interesting to study the ergodic theory of the Gibbs measures (starting with their existence, uniqueness and explicit construction!) for the Teichmüller flow on the quotient of the space $\mathscr{Q}^1(\Sigma)$ of unit-norm holomorphic quadratic differentials on a closed connected orientable surface $\Sigma$ of genus at least two by a non-elementary subgroup $\Gamma$ of the mapping class group of $\Sigma$, for a potential $F$ on $\mathscr{Q}^1(\Sigma)$ invariant under $\Gamma$. Patterson-Sullivan measures (without potential) on Thurston's boundary of Teichmüller spaces have been constructed for instance by Hamenstädt [Ham5] and Gekhtman [Gek]. Note that, relating this last question to the first two ones, V. Gadre [Gad] has proved that the harmonic measures on Thurston's boundary of Teichmüller spaces (whose existence is due to Kaimanovich-Masur), coming from random walks with finite support on the mapping class groups, are singular with respect to the Lebesgue measure, see also [GaMT].

*Acknowledgments.* The origin of this work is a fruitful stay of the second author as invited professor at the Ecole Normale Supérieure in 2009 during the first author's term there. The first author thanks the University of Warwick for visits during which part of the work took place. We thank J. Parkkonen for his corrections on a preliminary version of this text, J. Buzzi for his comments, and M. Brin, S. Crovisier, Y. Coudène, F. Ledrappier, J. Aaronson for their help with Chapter 7. The first author thanks the Mittag-Leffler Institute where Chapter 10 was written. The third author benefited from the ANR grant JCJC10-0108 Geode. Most of all, we would like to thank the anonymous referee for her/his tremendous work on the first version of this book. By her/his extremely thorough reading (her/his report was 26 pages long), she/he has corrected and improved many results, allowing to remove many superfluous hypotheses, and providing complete proofs in several instances. In particular,

- the statements 3.3 (viii), 4.3, 4.4, 4.6, 5.13, 9.5, 9.12, 11.15,
- the remark (2) following Definition 3.15, and what it allows later on,
- the deduction of 9.4 from 9.1; 9.10 from 9.6, 9.7, 9.8 and 9.9; 9.14 from 9.11; 9.17 from 9.16,
- the removal of the boundedness assumption on the potential to obtain the equality between its pressure and its critical exponent (see Equation (87)), and
- a major improvement of Theorem 7.2 (where the present conservativity assumption was originally replaced by Furstenberg's 2-recurrence property),
- a complete reorganisation of Section 11,

should be attributed to her/him.

# Contents

## 2 Background on negatively curved manifolds

In this text, the triple $(\widetilde{M}, \Gamma, \widetilde{F})$ will denote the following data.

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature $-b^2 \leq K \leq -1$, where $b \geq 1$. Let $\Gamma$ be a non-elementary (see the definition of non-elementary below) discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map, called a *potential* (see the definition of the Hölder-continuity we will use in Subsection 2.1).

We refer for instance to [BaGS, BrH] for general background on negatively curved manifolds. We only recall in this chapter the notation and results about them that we will use in this text.

**General notation.** Here is some general notation that will be used in this text.

Let $A$ be a subset of a set $E$. We denote by $\mathbb{1}_A : E \to \{0, 1\}$ the characteristic (or indicator) function of $A$: $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$ otherwise. We denote by ${}^cA = E - A$ the complementary subset of $A$ in $E$.

We denote by $\log$ the natural logarithm (with $\log(e) = 1$).

For every real number $x$, we denote by $\lfloor x \rfloor$ the integral part of $x$, that is the largest integer not greater that $x$.

Given two maps $f, g : [0, +\infty[ \to ]0, +\infty[$, we will write $f \asymp g$ if there exists $c > 0$ such that for every $t \geq 0$, we have $\frac{1}{c}\, f(t) \leq g(t) \leq c\, f(t)$.

For every smooth map $f : N \to N'$ between smooth manifolds, we denote by $Tf : TN \to TN'$ its tangent map.

We denote by $\|\mu\|$ the total mass of a finite positive measure $\mu$.

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are measurable spaces, $f : X \to Y$ a measurable map, and $\mu$ is a measure on $X$, we denote by $f_*\mu$ the image measure of $\mu$ by $f$, with $f_*\mu(B) = \mu(f^{-1}(B))$ for every $B \in \mathscr{B}$.

For every integer $n \geq 2$, we denote by $\mathbb{H}^n_{\mathbb{R}}$ (any model of) the real hyperbolic space of dimension $n$ (its sectional curvature is constant with value $-1$).

Given a topological space $X$, we denote by $\mathscr{C}_c(X; \mathbb{R})$ the vector space of continuous maps $f : X \to \mathbb{R}$ with compact support.

### 2.1 Uniform local Hölder-continuity

Recall that two distances $d$ and $d'$ on a set $E$ are (uniformly locally) *Hölder-equivalent* if there exist $c, \epsilon > 0$ and $\alpha \in ]0, 1]$ such that for all $x, y \in E$ with $d(x, y) \leq \epsilon$, we have

$$\frac{1}{c}\, d(x, y)^{\frac{1}{\alpha}} \leq d'(x, y) \leq c\, d(x, y)^{\alpha} .$$

This relation is an equivalence relation on the set of distances on $E$, and a *Hölder structure* on $E$ is the choice of such an equivalence class. Note that two Hölder-equivalent distances on a set $E$ induce the same topology on $E$. A metric space will be endowed with the Hölder structure of its distance. A map $f : E \to E'$ between sets endowed with Hölder structures is *Hölder-continuous* if for any distances $d$ and $d'$ in the Hölder structures of $E$ and $E'$, there exist $c, \epsilon > 0$ and $\alpha \in ]0, 1]$ such that for all $x, y \in E$ with $d(x, y) \leq \epsilon$, we have

$$d'(f(x), f(y)) \leq c\, d(x, y)^{\alpha} .$$

We will call $c, \epsilon$ and $\alpha$ the *Hölder constants* of $f$, even though they are not unique and depend on the choices of the distances $d$ and $d'$ in their equivalence class.

A map $f : E \to E'$ between sets endowed with Hölder structures is *locally Hölder-continuous* if for every $x \in E$, there exists a neighbourhood $U$ of $x$ such that the restriction of $f$ to $U$ is Hölder-continuous.

A Hölder structure on a set $E$ is *natural* under a group of bijections $G$ of $E$ if any element of $G$ is Hölder-continuous.

**Remarks.** Note that this notion of Hölder-continuity is a global one, and that some other texts only require the above centred inequality to hold, for some given $c, \alpha$ and for every $x$, only if $y$ is close enough to $x$ (that is, in some ball of centre $x$ whose radius might be smaller than 1 and might depend on $x$). When the topology on $E$ defined by $d$ is compact, these two definitions coincide, but we are trying to avoid any compactness assumption in this text. For instance in symbolic dynamics, the potentials may be only required to have this second property (or even less, see for instance [Kel]): since two sequences in the same strong stable leaf are eventually equal, the Gibbs cocycle is then well-defined. But the fact that the Gibbs cocycle in our case is well defined will use this global Hölder-continuity (see Subsection 3.3).

Let $f : X \to Y$ be a map between metric spaces. If $X$ is geodesic, then the definition of the Hölder-continuity of $f$ is equivalent to its following strengthened version: $f$ is *Hölder-continuous* if and only if there exist $c > 0$ and $\alpha \in \,]0, 1]$ such that for all $x, y \in E$ with $d(x, y) \leq 1$, we have

$$d(f(x), f(y)) \leq c\, d(x, y)^\alpha \ .$$

We will use this definition throughout this text, whenever the source space is geodesic.

Another consequence of the global Hölder-continuous property of $f$ is that $f$ then has at most linear growth: if $X$ is a geodesic space, the above condition implies that for all $x, y$ in $X$ with $d(x, y) \geq 1$, we have $d(f(x), f(y)) \leq 3c\, d(x, y)$. This will turn out to be useful in particular in Subsection 10.3.

### 2.2 Boundary at infinity, isometries and the Busemann cocycle

We denote by $d$ or $d_{\widetilde{M}}$ the Riemannian distance on $\widetilde{M}$. We denote by $\partial_\infty \widetilde{M}$ the boundary at infinity of $\widetilde{M}$. We endow $\widetilde{M} \cup \partial_\infty \widetilde{M}$ with the cone topology, homeomorphic to the closed unit ball of $\mathbb{R}^n$, where $n$ is the dimension of $\widetilde{M}$. For every $x \in \widetilde{M}$, recall that the Gromov-Bourdon *visual distance* $d_x$ on $\partial_\infty \widetilde{M}$ seen from $x$ (see [Bou]) is defined by

$$d_x(\xi, \eta) = \lim_{t \to +\infty} e^{\frac{1}{2}(d(\xi_t, \eta_t) - d(x, \xi_t) - d(x, \eta_t))} \ , \tag{1}$$

where $t \mapsto \xi_t, \eta_t$ are any geodesic rays ending at $\xi, \eta$ respectively. By the triangle inequality, for all $x, y \in \widetilde{M}$ and $\xi, \eta \in \partial_\infty \widetilde{M}$, we have

$$e^{-d(x,\, y)} \leq \frac{d_x(\xi, \eta)}{d_y(\xi, \eta)} \leq e^{d(x,\, y)} \ . \tag{2}$$

We endow $\partial_\infty \widetilde{M}$ with the Hölder structure which is the equivalence class of any visual distance. It is natural under the isometry group of $\widetilde{M}$ and its induced topology on $\partial_\infty \widetilde{M}$ coincides with the cone topology.

For the following definitions, let $r > 0$, $x \in \widetilde{M} \cup \partial_\infty \widetilde{M}$, $A \subset \widetilde{M}$ and $B \subset \partial_\infty \widetilde{M}$. We denote by $\mathscr{O}_x A$ the *shadow of $A$ seen from $x$*, that is, the subset of $\partial_\infty \widetilde{M}$ consisting of the endpoints of the geodesic rays (if $x \in \widetilde{M}$) or lines (if $x \in \partial_\infty \widetilde{M}$) starting from $x$ and meeting $A$. Note that $\mathscr{O}$ stands for "ombre". We denote by $\mathscr{C}_x B$ the *cone on $B$ with vertex $x$* in $\widetilde{M}$, that is, the union of the geodesic rays or lines starting from $x$ and ending at $B$. We denote by $\mathscr{N}_r A$ the open $r$-neighbourhood of $A$, that is,

$$\mathscr{N}_r A = \{x \in \widetilde{M} \ : \ d(x, A) < r\} \ .$$

We denote by $\mathscr{N}_r^{\,-} A$ the *open $r$-interior* of $A$, that is,

$$\mathscr{N}_r^{\,-} A = \{x \in \widetilde{M} \ : \ d(x, {}^c\!A) > r\} \ ,$$

which is contained in $A$. These sets satisfy the following equivariance properties: for every isometry $\gamma$ of $\widetilde{M}$, we have

$$\gamma \mathscr{O}_x A = \mathscr{O}_{\gamma x}(\gamma A), \quad \gamma \mathscr{C}_x B = \mathscr{C}_{\gamma x}(\gamma B), \quad \gamma \mathscr{N}_r A = \mathscr{N}_r(\gamma A), \quad \gamma \mathscr{N}_r^{\,-} A = \mathscr{N}_r^{\,-}(\gamma A) \ .$$

For every isometry $\gamma$ of $\widetilde{M}$, we denote by

$$\ell(\gamma) = \inf_{x \in \widetilde{M}} d(x, \gamma x)$$

the *translation length* of $\gamma$ on $\widetilde{M}$. Recall that $\gamma$ is *elliptic* if it has a fixed point in $\widetilde{M}$, *parabolic* if non elliptic and $\ell(\gamma) = 0$, and *loxodromic* otherwise. In this last case, we denote by

$$\mathrm{Axe}_\gamma = \{x \in \widetilde{M} \ : \ d(x, \gamma x) = \ell(\gamma)\}$$

the *translation axis* of $\gamma$, which is isometric to $\mathbb{R}$.

The isometry group $\mathrm{Isom}(\widetilde{M})$ of $\widetilde{M}$ is endowed with the compact-open topology. Let $\Gamma'$ be a discrete group of isometries of $\widetilde{M}$. Its *limit set*, which is the set of accumulation points in $\partial_\infty \widetilde{M}$ of any orbit of $\Gamma'$ in $\widetilde{M}$, will be denoted by $\Lambda\Gamma'$, and the convex hull in $\widetilde{M}$ of this limit set by $\mathscr{C}\Lambda\Gamma'$. Recall that $\Gamma'$ is *non-elementary* if $\mathrm{Card}(\Lambda\Gamma') \geq 3$. Since $\widetilde{M}$ has pinched negative curvature, the following assertions are equivalent (see for instance [Bowd]):

- $\Gamma'$ is non-elementary,
- $\Gamma'$ contains a free subgroup of rank 2,
- $\Gamma'$ is not virtually solvable,
- $\Gamma'$ is not virtually nilpotent.

Also recall that $\Gamma'$ is *convex-cocompact* if $\Gamma'$ is non-elementary and $\Gamma' \backslash \mathscr{C}\Lambda\Gamma'$ is compact.

A loxodromic element $\gamma \in \Gamma'$ is *primitive* in $\Gamma'$ if there are no $\alpha \in \Gamma'$ and $k \geq 2$ such that $\gamma = \alpha^k$. Every loxodromic element of $\Gamma'$ is a power of a primitive element of $\Gamma'$.

If $\Gamma'$ is non-elementary, then $\Lambda\Gamma'$ is the closure of the set of fixed points of loxodromic elements of $\Gamma'$, and it is the smallest nonempty, closed, $\Gamma'$-invariant subset of $\partial_\infty \widetilde{M}$. In particular, if $\Gamma''$ is an infinite normal subgroup of $\Gamma'$, then

$$\Lambda\Gamma'' = \Lambda\Gamma' \ .$$

We will use the following dynamical properties of the action of $\Gamma'$ on its limit set $\Lambda\Gamma'$: if $\Gamma'$ is non-elementary, then

- the action of $\Gamma'$ on $\Lambda\Gamma'$ is minimal (that is, every orbit is dense);
- the diagonal action of $\Gamma'$ on $\Lambda\Gamma' \times \Lambda\Gamma'$ is transitive (that is, there exists a dense orbit);
- the set of pairs of fixed points of the loxodromic elements of $\Gamma'$ is dense in $\Lambda\Gamma' \times \Lambda\Gamma'$.

The *conical* (or *radial*) *limit set* $\Lambda_c\Gamma'$ of $\Gamma'$ is the set of points $\xi \in \partial_\infty \widetilde{M}$ such that there exists a sequence of orbit points of $x$ under $\Gamma'$ converging to $\xi$ while staying at bounded distance from a geodesic ray ending at $\xi$. We have $\Lambda_c\Gamma' = \Lambda\Gamma'$ if $\Gamma'$ is convex-cocompact. If $\Gamma'$ is non-elementary, the conical limit set $\Lambda_c\Gamma'$ of $\Gamma'$ is dense in its limit set $\Lambda\Gamma'$, and the point at infinity of a geodesic ray $\rho$ in $\widetilde{M}$ belongs to $\Lambda_c\Gamma'$ if and only if the image of $\rho$ in $M$ comes back in some compact subset at times tending to $+\infty$.

A point $\xi$ in $\partial_\infty \widetilde{M}$ is a *Myrberg point* of $\Gamma'$ (see for instance [Tuk]) if for all $\eta \neq \eta'$ in $\Lambda\Gamma$ and $x \in \widetilde{M}$, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\Gamma'$ such that $\lim_{n\to\infty} \gamma_n x = \eta$ and $\lim_{n\to\infty} \gamma_n \xi = \eta'$. We will denote by $\Lambda_{\rm Myr}\Gamma'$ the set of Myrberg points of $\Gamma'$. It is clearly a subset of $\Lambda_c\Gamma'$, and even a proper subset of $\Lambda_c\Gamma'$, since the fixed points of the loxodromic elements of $\Gamma'$ are conical limit points but not Myrberg points.

In this text, we are not assuming $\Gamma$ to be torsion free.

Note that the action of $\Gamma$ on $\Lambda\Gamma$ is not necessarily faithful (viewing the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$ as the intersection with the horizontal plane of the ball model of the real hyperbolic space $\mathbb{H}^3_\mathbb{R}$, consider the subgroup generated by a non-elementary discrete isometry group of $\mathbb{H}^2_\mathbb{R}$ and the hyperbolic reflexion of $\mathbb{H}^3_\mathbb{R}$ with fixed point set $\mathbb{H}^2_\mathbb{R}$). But the pointwise stabiliser $\operatorname{Fix}_\Gamma(\Lambda\Gamma)$ of $\Lambda\Gamma$ in $\Gamma$ is a finite normal subgroup of $\Gamma$.

**Lemma 2.1** *The set of fixed points in $\Lambda\Gamma$ of any element $\gamma$ of $\Gamma - \operatorname{Fix}_\Gamma(\Lambda\Gamma)$ is a closed subset with empty interior in $\Lambda\Gamma$.*

**Proof.** This is well known, but difficult to locate in a reference. We may assume $\gamma$ to be elliptic. If $\gamma$ fixes a fixed point of a loxodromic element $\alpha$, then $\gamma$ pointwise fixes its translation axis $\mathrm{Axe}_\alpha$: otherwise, $(\alpha^{-n}\gamma\alpha^n)_{n\in\mathbb{N}}$ is a sequence of pairwise distinct elements of $\Gamma$ mapping any given point in $\mathrm{Axe}_\alpha$ at bounded distance from itself, which contradicts the discreteness of $\Gamma$. Since the set of pairs of endpoints of the translation axes of loxodromic elements of $\Gamma$ is dense in $\Lambda\Gamma \times \Lambda\Gamma$, if $\gamma$ pointwise fixes an open subset of $\Lambda\Gamma$, then $\gamma$ is the identity on $\Lambda\Gamma$. $\Box$

In particular, by Baire's theorem, the union of the sets of fixed points in $\Lambda\Gamma$ of the elements of $\Gamma - \operatorname{Fix}_\Gamma(\Lambda\Gamma)$ is a $\Gamma$-invariant Borel subset, whose complement is a dense $G_\delta$-set. We refer to Lemma 2.2 for the consequences of this lemma for the fixed points of the elements of $\Gamma$ inside $T^1\widetilde{M}$.

The *Busemann cocycle* is the map $\beta : \partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$, defined by

$$(\xi, x, y) \mapsto \beta_\xi(x,y) = \lim_{t\to+\infty} d(x,\xi_t) - d(y,\xi_t) \;,$$

where $t \mapsto \xi_t$ is any geodesic ray ending at $\xi$. For every $\xi \in \partial_\infty \widetilde{M}$, the *horospheres centred at* $\xi$ are the level sets of the map $y \mapsto \beta_\xi(y, x_0)$ from $\widetilde{M}$ to $\mathbb{R}$, and the (closed) *horoballs centred at* $\xi$ are its sub-level sets, for any $x_0 \in \widetilde{M}$.

### 2.3 The geometry of the unit tangent bundle

We denote by $\pi : T^1\widetilde{M} \rightarrow \widetilde{M}$ the unit tangent bundle of $\widetilde{M}$, and we endow $T^1\widetilde{M}$ with Sasaki's Riemannian metric (see below). We denote by $\iota : T^1\widetilde{M} \rightarrow T^1\widetilde{M}$ the flip map $v \mapsto -v$.

Consider the Riemannian orbifold $M = \Gamma\backslash\widetilde{M}$ and let us denote by $T^1M$ and $TT^1M$ the quotient Riemannian orbifolds $\Gamma\backslash T^1\widetilde{M}$ and $\Gamma\backslash TT^1\widetilde{M}$. We denote by $F : T^1M \rightarrow \mathbb{R}$ and again by $\iota : T^1M \rightarrow T^1M$ and $\pi : T^1M \rightarrow M$ the quotient maps of $\widetilde{F}$, $\iota$ and $\pi$.

For every $v \in T^1\widetilde{M}$, we denote by $v_-$ and $v_+$ the points at $-\infty$ and $+\infty$ of the geodesic line $\ell_v : \mathbb{R} \rightarrow \widetilde{M}$ with $\dot{\ell}_v(0) = v$; we have $(\iota v)_\pm = v_\mp$.

We denote by $\widetilde{\Omega}\Gamma$ (respectively $\widetilde{\Omega}_c\Gamma$) the closed (respectively Borel) $\Gamma$-invariant set of elements $v \in T^1\widetilde{M}$ such that $v_-$ and $v_+$ both belong to $\Lambda\Gamma$ (respectively $\Lambda_c\Gamma$), and by $\Omega\Gamma$ and $\Omega_c\Gamma$ their images by the canonical projection $T^1\widetilde{M} \rightarrow T^1M$.

**Lemma 2.2** *The union $A$ of the sets of fixed points in $\widetilde{\Omega}\Gamma$ of the elements of $\Gamma - \mathrm{Fix}_\Gamma(\Lambda\Gamma)$ is $\Gamma$-invariant, closed, and properly contained in $\widetilde{\Omega}\Gamma$.*

**Proof.** The invariance of $A$ is clear and its closeness follows by discreteness. The set of fixed points in $\widetilde{\Omega}\Gamma$ of a given element of $\Gamma - \mathrm{Fix}_\Gamma(\Lambda\Gamma)$ is closed, with empty interior in $\widetilde{\Omega}\Gamma$ (otherwise, its set of points at $+\infty$ would have nonempty interior in $\Lambda\Gamma$, which would contradict Lemma 2.1). The set $A$ is hence properly contained in $\widetilde{\Omega}\Gamma$, since its complement is a dense $G_\delta$-subset of $\widetilde{\Omega}\Gamma$, by Baire's theorem. □

For every $x \in \widetilde{M}$, let $\iota_x : \partial_\infty\widetilde{M} \rightarrow \partial_\infty\widetilde{M}$ be the *antipodal map with respect to* $x$, that is the involutive map $\xi \mapsto v_-$ where $v$ is the unique element of $T^1\widetilde{M}$ such that $\pi(v) = x$ and $v_+ = \xi$. Note that for every isometry $\gamma$ of $\widetilde{M}$ and for every $x$ in $\widetilde{M}$, we have

$$\iota_{\gamma x} = \gamma \circ \iota_x \circ \gamma^{-1} .$$

Let us now recall a parametrisation of $T^1\widetilde{M}$ in terms of the boundary at infinity of $\widetilde{M}$. Let $\partial^2_\infty\widetilde{M} = (\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}) - \Delta$, where $\Delta$ is the diagonal in $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$. For every base point $x_0$ in $\widetilde{M}$, the space $T^1\widetilde{M}$ may be identified with $\partial^2_\infty\widetilde{M} \times \mathbb{R}$, by the map which maps a unit tangent vector $v$ to the triple $(v_-, v_+, t)$ where $t$ is the algebraic distance on $\ell_v(\mathbb{R})$ (oriented from $v_-$ to $v_+$) between $\ell_v(0)$ and the closest point of $\ell_v(\mathbb{R})$ to $x_0$. This parametrisation (called the *Hopf parametrisation* defined by $x_0$) differs from the one defined by another base point $x'_0$ only by an additive term on the third factor (independent of the time $t$).

We end this subsection by giving some information on the Sasaki metric on $T\widetilde{M}$ (see for instance [Bal, Chap. IV]). Recall that it has the following defining properties: The vector bundle $TT\widetilde{M} \rightarrow T\widetilde{M}$ has a Whitney sum decomposition $TT\widetilde{M} = V \oplus H$ such that, for every $v \in T\widetilde{M}$, the direct sum decomposition of the fibres

$$T_vT\widetilde{M} = V_v \oplus H_v$$

is orthogonal for Sasaki's scalar product, and if $\pi : T\widetilde{M} \rightarrow \widetilde{M}$ also denotes the full tangent bundle of $\widetilde{M}$,

- $T\pi_{|H_v} : H_v \rightarrow T_{\pi(v)}\widetilde{M}$ is an isometric linear isomorphism,
- $V_v = \mathrm{Ker}\, T_v\pi = T_v(T_{\pi(v)}\widetilde{M}) = T_{\pi(v)}\widetilde{M}$ (equality as Euclidean spaces),

• if $\nabla$ is the Levi-Civita connection of $\widetilde{M}$, if $p_V : TT\widetilde{M} \to V$ is the bundle map which is the linear projection from $T_vT\widetilde{M}$ on $V_v$ parallel to $H_v$ for every $v \in T\widetilde{M}$, then for every vector field $X : \widetilde{M} \to T\widetilde{M}$ on $\widetilde{M}$, with $TX : T\widetilde{M} \to TT\widetilde{M}$ its tangent map, we have

$$\nabla_v X = p_V \circ TX(v)\ .$$

The map $\pi : T^1\widetilde{M} \to \widetilde{M}$ is a Riemannian submersion with totally geodesic fibres, and if $\gamma$ is an isometry of $\widetilde{M}$, then its tangent map, still denoted by $\gamma$, is an isometry of $T^1\widetilde{M}$. The flip map $\iota$ is also an isometry of the Sasaki metric. The Riemannian distance $d_S$ induced by the Sasaki metric satisfies (see [Sol] for generalisations)

$$d_S(v,w) = \inf_{\alpha} \sqrt{\ell(\alpha)^2 + \|\ (v - \|_\alpha w)\ \|^2}\ , \tag{3}$$

where the lower bound is taken on the smooth paths $\alpha : [0,1] \to \widetilde{M}$ of length $\ell(\alpha)$ from $\pi(w)$ to $\pi(v)$ and $\|_\alpha$ is the parallel transport along $\alpha$. Note that for all $v, w \in T^1\widetilde{M}$

$$d_S(v,w) \geq d(\pi(v), \pi(w))\ . \tag{4}$$

We will endow $T^1\widetilde{M}$ with its quite usual distance $d = d_{T^1\widetilde{M}}$, defined as follows: for all $v, v' \in T^1\widetilde{M}$,

$$d(v,v') = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} d(\pi(\phi_t v), \pi(\phi_t v'))\, e^{-t^2}\ dt\ . \tag{5}$$

Note that for all $v \in T^1\widetilde{M}$ and $t \in \mathbb{R}$, we have

$$d(v, \phi_t v) = |t|\ ,$$

and for all $v, v' \in T^1\widetilde{M}$ and $\gamma \in \mathrm{Isom}(\widetilde{M})$, we have

$$d(\gamma\, v, \gamma\, v') = d(v,v') \quad \text{and} \quad d(\iota\, v, \iota\, v') = d(v,v')\ . \tag{6}$$

We will sometimes consider another distance $d' = d'_{T^1\widetilde{M}}$ on $T^1\widetilde{M}$, defined by (using the convexity of the distance in $\widetilde{M}$ to obtain the second equality)

$$\begin{aligned} \forall\ v, w \in T^1\widetilde{M}, \quad d'_{T^1\widetilde{M}}(v,w) &= \max_{s \in [-1,0]} d_{\widetilde{M}}\big(\pi(\phi_s v), \pi(\phi_s w)\big) \\ &= \max\big\{ d_{\widetilde{M}}\big(\pi(v), \pi(w)\big), d_{\widetilde{M}}\big(\pi(\phi_{-1} v), \pi(\phi_{-1} w)\big)\big\}\ . \end{aligned} \tag{7}$$

The distance $d'$ is (Lipschitz-)equivalent to the Riemannian distance $d_S$ induced by the Sasaki metric on $T^1\widetilde{M}$, since $\widetilde{M}$ has pinched negative sectional curvature (see for instance [Bal, page 70]). Note that the flip map $\iota$, which is an isometry of the Sasaki metric, is hence Lipschitz for this metric, with Lipschitz constant depending only on the bounds on the sectional curvature.

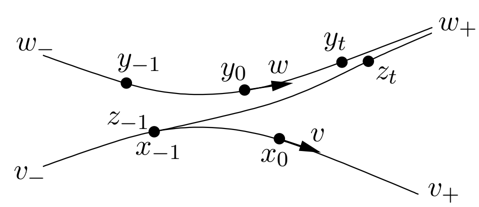
The following result should be well known, but we provide a proof in the absence of a precise reference.

**Lemma 2.3** *The distance $d$ on $T^1\widetilde{M}$ is Hölder-equivalent to the Riemannian distance $d_S$ induced by Sasaki's Riemannian metric.*

In particular, when we will consider Hölder-continuous functions $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$, we may use any one of the three distances $d_S$, $d = d_{T^1\widetilde{M}}$ and $d' = d'_{T^1\widetilde{M}}$, remembering that the Hölder constants depend on this distance.

**Proof.** It is sufficient to prove that the distances $d$ and $d'$ on $T^1\widetilde{M}$ are Hölder-equivalent.

Let $v, w \in T^1\widetilde{M}$. Define $x_t = \pi(\phi_t v)$ and $y_t = \pi(\phi_t w)$ for every $t \in \mathbb{R}$, as well as $\epsilon_0 = d(x_0, y_0)$ and $\epsilon_{-1} = d(x_{-1}, y_{-1})$. Let $t \mapsto z_t$ be the geodesic ray parametrised by $[-1, +\infty[$ from $x_{-1}$ to the point at infinity $w_+$.




By the definition of $d'$, for every $t \in [-1, 0]$, we have

$$d(x_t, y_t) \le d'(v, w) = \max\{\epsilon_0, \epsilon_{-1}\} \ .$$

By convexity, for every $t \ge -1$, we have $d(z_t, y_t) \le \epsilon_{-1}$. Since $\widetilde{M}$ has a finite lower bound on its curvature, there exists a constant $c_1 > 0$ such that for every $t \ge 0$, we have $d(x_t, z_t) \le d(x_0, z_0)\, e^{c_1 t}$. Hence, by the triangle inequality, for every $t \ge 0$, we have

$$d(x_t, y_t) \le \epsilon_{-1} + \epsilon_0\, e^{c_1 t} \ .$$

Similarly, considering the geodesic ray from $x_0$ to $w_-$, for every $t \le -1$, we have $d(x_t, y_t) \le \epsilon_0 + \epsilon_{-1}\, e^{c_1(1-t)}$. Therefore $\sqrt{\pi}\, d(v, w)$ is bounded from above by

$$\int_{-\infty}^{-1} (\epsilon_0 + \epsilon_{-1}\, e^{c_1(1-t)})\, e^{-t^2}\, dt + \int_{-1}^{0} \max\{\epsilon_0, \epsilon_{-1}\}\, e^{-t^2}\, dt + \int_0^{+\infty} (\epsilon_{-1} + \epsilon_0\, e^{c_1 t})\, e^{-t^2}\, dt \ ,$$

and there exists a constant $c_2 > 0$ such that $d(v, w) \le c_2\, (\epsilon_{-1} + \epsilon_0) \le 2\, c_2\, d'(v, w)$.

Conversely, assume that $d'(v, w) \le 1$. Then $\epsilon_0, \epsilon_1 \le 1$ and

$$\begin{aligned}
\sqrt{\pi}\, d(v, w) &\ge \int_0^{\epsilon_0/4} d(x_t, y_t)\, e^{-t^2}\, dt + \int_{-1}^{-1+\epsilon_{-1}/4} d(x_t, y_t)\, e^{-t^2}\, dt \\
&\ge \int_0^{\epsilon_0/4} (\epsilon_0 - 2t)\, e^{-t^2}\, dt + \int_{-1}^{-1+\epsilon_{-1}/4} (\epsilon_{-1} - 2(t+1))\, e^{-t^2}\, dt \\
&\ge \frac{\epsilon_0}{4} \times \frac{\epsilon_0}{2} \times \frac{1}{e} \ + \ \frac{\epsilon_{-1}}{4} \times \frac{\epsilon_{-1}}{2} \times \frac{1}{e} \ge \frac{1}{16\, e}(\epsilon_0 + \epsilon_{-1})^2 \ge \frac{1}{16\, e}(d'(v, w))^2 \ .
\end{aligned}$$

This proves the result. $\Box$

## 2.4 Geodesic flow, (un)stable foliations and the Hamenstädt distances

We denote by $(\phi_t : T^1\widetilde{M} \to T^1\widetilde{M})_{t \in \mathbb{R}}$ the geodesic flow of $\widetilde{M}$, which satisfies $\iota \circ \phi_t = \phi_{-t} \circ \iota$. In the Hopf parametrisation, the geodesic flow is the action of $\mathbb{R}$ by translation on the third factor. We denote again by $\phi_t : T^1M \to T^1M$ the quotient map of $\phi_t$.

Given $x \ne y$ in $\widetilde{M}$, the unit tangent vector at $x$ *pointing towards* $y$ is the unique element $v \in T^1_x\widetilde{M}$ such that there exists $t > 0$ with $\pi(\phi_t v) = y$.

By [Ebe, Coro. 3.8], the closed subset $\Omega\Gamma$ defined in the previous subsection is the (topological) *non-wandering set* of the geodesic flow in $T^1M$, that is, the set of points $v \in T^1M$ such that, for every neighbourhood $U$ of $v$, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$

converging to $+\infty$ such that, for every $n \in \mathbb{N}$, the intersection $U \cap \phi_{t_n} U$ is nonempty. The support of any Borel probability measure on $T^1 M$ which is invariant under the geodesic flow is contained in $\Omega\Gamma$. Note that $\Omega_c\Gamma$ is dense in $\Omega\Gamma$, and is the *two-sided recurrent set* of the geodesic flow in $T^1 M$, that is, the set of unit tangent vectors whose geodesic line comes back in some compact subset of $M$ at times tending to $+\infty$ and to $-\infty$. In this book, we will say that an element $v \in T^1 M$ is positively (respectively negatively) *recurrent* (under the geodesic flow) if there exists a compact subset $K$ of $T^1 M$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ converging to $+\infty$ such that $\phi_{t_n} v$ (respectively $\phi_{-t_n} v$) belongs to $K$.

We now recall the definitions and properties of the dynamical foliations associated to the geodesic flow on $T^1\widetilde{M}$. For every $v \in T^1\widetilde{M}$, we define the *strong stable leaf* of $v$, *strong unstable leaf* of $v$, *(weak or central) stable leaf* of $v$ and *(weak or central) unstable leaf* of $v$ respectively by

$$W^{\rm ss}(v) = \{w \in T^1\widetilde{M} \;:\; \lim_{t\to+\infty} d(\phi_t v, \phi_t w) = 0\}\;,$$

$$W^{\rm su}(v) = \{w \in T^1\widetilde{M} \;:\; \lim_{t\to-\infty} d(\phi_t v, \phi_t w) = 0\}\;,$$

$$W^{\rm s}(v) = \{w \in T^1\widetilde{M} \;:\; \exists s \in \mathbb{R},\; \lim_{t\to+\infty} d(\phi_{t+s} v, \phi_t w) = 0\}\;,$$

$$W^{\rm u}(v) = \{w \in T^1\widetilde{M} \;:\; \exists s \in \mathbb{R},\; \lim_{t\to-\infty} d(\phi_{t+s} v, \phi_t w) = 0\}\;.$$

Another set of notation is sometimes used, replacing $W^{\rm ss}, W^{\rm su}, W^{\rm s}$ and $W^{\rm u}$ by respectively $W^{\rm s}, W^{\rm s}, W^{\rm cs}$ and $W^{\rm cu}$, which is easier to relate to direct sums of the appropriate tangent spaces, but we will keep the above one, which is quite traditional.

These four families define continuous foliations of $T^1\widetilde{M}$ with smooth leaves, denoted by $\widetilde{\mathscr{W}}^{\rm ss}$, $\widetilde{\mathscr{W}}^{\rm su}$, $\widetilde{\mathscr{W}}^{\rm s}$, $\widetilde{\mathscr{W}}^{\rm u}$ and called the *strong stable, strong unstable, stable, unstable foliations* of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1\widetilde{M}$.

The smooth submanifolds $\pi(W^{\rm ss}(v))$ and $\pi(W^{\rm su}(v))$ of $\widetilde{M}$ are the horospheres passing through $\pi(v)$ with centres respectively $v_+$ and $v_-$, and conversely $W^{\rm ss}(v)$ and $W^{\rm su}(v)$ are the subsets of $T^1\widetilde{M}$ consisting respectively of the inner and outer unit normal vectors to these two horospheres. Note that $W^{\rm s}(v)$ and $W^{\rm u}(v)$ are invariant under the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$. Furthermore, for every $v \in T^1\widetilde{M}$, the map $w \mapsto w_+$ from $W^{\rm su}(v)$ to $\partial_\infty\widetilde{M} - \{v_-\}$ is a homeomorphism, and the map $w \mapsto w_-$ from $W^{\rm ss}(v)$ to $\partial_\infty\widetilde{M} - \{v_+\}$ is also a homeomorphism.

For every $v \in T^1 M$, denote again by $W^{\rm ss}(v)$ (respectively $W^{\rm su}(v)$, $W^{\rm s}(v)$, $W^{\rm u}(v)$) the image by the canonical projection $T^1\widetilde{M} \to T^1 M$ of $W^{\rm ss}(\widetilde{v})$ (respectively $W^{\rm su}(\widetilde{v})$, $W^{\rm s}(\widetilde{v})$, $W^{\rm u}(\widetilde{v})$), where $\widetilde{v}$ is any lift of $v$ to $T^1\widetilde{M}$. Note that $W^{\rm ss}(\iota v) = \iota(W^{\rm su}(v))$, $W^{\rm su}(\iota v) = \iota(W^{\rm ss}(v))$, $W^{\rm s}(\iota v) = \iota(W^{\rm u}(v))$, $W^{\rm u}(\iota v) = \iota(W^{\rm s}(v))$ for every $v \in T^1\widetilde{M}$ or $v \in T^1 M$. The sets $W^{\rm ss}(v)$ (respectively $W^{\rm su}(v)$, $W^{\rm s}(v)$, $W^{\rm u}(v)$) define a partition of $T^1 M$, denoted by $\mathscr{W}^{\rm ss}$ (respectively $\mathscr{W}^{\rm su}$, $\mathscr{W}^{\rm s}$, $\mathscr{W}^{\rm u}$), which is a foliation outside the image in $T^1 M$ of the set of fixed points of elements of $\Gamma - \{\rm id\}$ in $T^1\widetilde{M}$.

Let us also recall a natural family of distances on the strong stable leaves of $T^1\widetilde{M}$ (see for instance [HeP1, Appendix], slightly modifying the definition of [Ham1], as well as [HeP2, §2.2] for a generalisation when a horosphere is replaced by the boundary of any nonempty closed convex subset). For every $w \in T^1\widetilde{M}$, let $d_{W^{\rm su}(w)}$ be the *Hamenstädt distance* on the strong unstable leaf of $w$, defined as follows: for all $v, v' \in W^{\rm su}(w)$,

$$d_{W^{\rm su}(w)}(v, v') = \lim_{t\to+\infty} e^{\frac{1}{2} d(\pi(\phi_t v),\, \pi(\phi_t v')) - t}\;.$$

This limit exists, and the Hamenstädt distance is a distance inducing the original topology on $W^{\rm su}(w)$. For all $v, v' \in W^{\rm su}(w)$ and for every isometry $\gamma$ of $\widetilde{M}$, we have

$$d_{W^{\rm su}(\gamma w)}(\gamma v, \gamma v') = d_{W^{\rm su}(w)}(v, v')\ .$$

For all $w \in T^1\widetilde{M}$, $s \in \mathbb{R}$ and $v, v' \in W^{\rm su}(w)$, we have

$$d_{W^{\rm su}(\phi_s w)}(\phi_s v, \phi_s v') = e^s\ d_{W^{\rm su}(w)}(v, v')\ . \tag{8}$$

The following lemma compares the Hamenstädt distance $d_{W^{\rm su}(w)}$ with the distance $d$ on $\widetilde{M}$ and the distance $d$ on $T^1\widetilde{M}$.

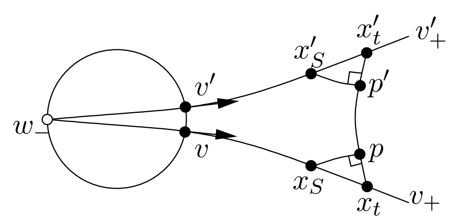
**Lemma 2.4** *There exists $c > 0$ such that, for all $w \in T^1\widetilde{M}$ and $v, v' \in W^{\rm su}(w)$, we have*

$$\max\{\ \frac{1}{c}\ d(v, v'),\ d(\pi(v), \pi(v'))\ \} \le d_{W^{\rm su}(w)}(v, v') \le e^{\frac{1}{2}d(\pi(v),\, \pi(v'))}\ . \tag{9}$$

**Proof.** We give proofs only for the sake of completeness. The inequality on the right hand side follows from the triangle inequality.

The inequality $d(v, v') \le c\ d_{W^{\rm su}(w)}(v, v')$ for some universal constant $c > 0$ is due to [PaP2, Lem. 3], as follows.

We may assume that $v \ne v'$. Let $x_t = \pi(\phi_t v)$ and $x'_t = \pi(\phi_t v')$. By the convexity properties of the distance in $\widetilde{M}$, the map from $\mathbb{R}$ to $\mathbb{R}$ defined by $t \mapsto d(x_t, x'_t)$ is increasing, with image $]0, +\infty[$. Let $S \in \mathbb{R}$ be such that $d(x_S, x'_S) = 1$. For every $t \ge S$, let $p$ and $p'$ be the closest point projections of $x_S$ and $x'_S$ respectively on the geodesic segment $[x_t, x'_t]$.




In the hyperbolic upper half-plane $\mathbb{H}^2_{\mathbb{R}}$, the points $i$, $-1+2i$ and $1+2i$ are pairwise at distance $2\log\frac{1+\sqrt{5}}{2} < 1$. Hence by convexity, if $x, x'$ are the points of $\mathbb{H}^2_{\mathbb{R}}$ with horizontal coordinates $1, -1$ respectively, with same vertical coordinate and at hyperbolic distance 1, then the distance from $x, x'$ to the geodesic line with endpoints $1, -1$ is at most 1. By comparison, we therefore have $d(p, x_S),\ d(p', x'_S) \le 1$. Hence, by convexity and the triangle inequality,

$$\begin{aligned} d(x_t, x'_t) &\ge d(x_t, p) + d(p', x'_t) \\ &\ge d(x_t, x_S) - 1 + d(x'_t, x'_S) - 1 = 2(t - S - 1)\ . \end{aligned}$$

Thus by the definition of the Hamenstädt distance $d_{W^{\rm su}(w)}$, we have

$$d_{W^{\rm su}(w)}(v, v') \ge e^{-S-1}\ . \tag{10}$$

By the triangle inequality, if $t \ge S$, then

$$d(x_t, x'_t) \le d(x_t, x_S) + d(x_S, x'_S) + d(x'_S, x'_t) = 2(t - S) + 1\ .$$

Since $\widetilde{M}$ is CAT$(-1)$, if $t \le S$, we have by comparison

$$d(x_t, x'_t) \le e^{t-S}\ d(x_S, x'_S) = e^{t-S}\ .$$

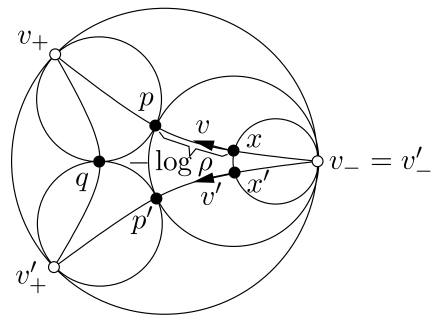
Therefore, by the definition of the distance $d$ on $T^1\widetilde{M}$ (see Equation (5)), we have

$$d(v,v') \le \int_{-\infty}^{S} e^{t-S}\, e^{-t^2}\, dt + \int_{S}^{+\infty} (2(t-S)+1)\, e^{-t^2}\, dt = \mathrm{O}(e^{-S})\ .$$

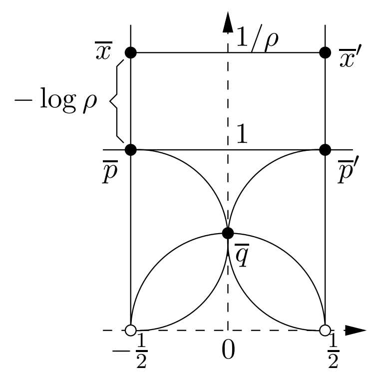
The result hence follows from Equation (10).

The last inequality $d(\pi(v),\pi(v')) \le d_{W^{\rm su}(w)}(v,v')$ is due to [PaP4, §2], as follows.

Let $x = \pi(v)$, $x' = \pi(v')$ and $\rho = d_{W^{\rm su}(w)}(v,v')$. Consider the ideal triangle $\Delta$ with vertices $v_+, v'_+$ and $v_- = v'_-$. Let $p \in \,]v_-, v_+[\,$, $p' \in \,]v'_-, v'_+[$ and $q \in \,]v_+, v'_+[$ be the pairwise tangency points of horospheres centred at the vertices of $\Delta$: $\beta_{v_-}(p,p') = 0$, $\beta_{v_+}(p,q) = 0$ and $\beta_{v'_+}(p',q) = 0$. By definition of the Hamenstädt distance, the algebraic distance from $x$ to $p$ on the geodesic line $]v_-, v_+[$ (oriented from $v_-$ to $v_+$) is $-\log\rho$.




Consider the ideal triangle $\overline{\Delta}$ in the hyperbolic upper half-plane $\mathbb{H}^2_{\mathbb{R}}$, with vertices $-\frac{1}{2}$, $\frac{1}{2}$ and $\infty$. Let $\overline{p} = (-\frac{1}{2}, 1)$, $\overline{p}' = (\frac{1}{2}, 1)$ and $\overline{q} = (0, \frac{1}{2})$ be the pairwise tangency points of horospheres centred at the vertices of $\overline{\Delta}$. Let $\overline{x}$ and $\overline{x}'$ be the point at algebraic (hyperbolic) distance $-\log\rho$ from $\overline{p}$ and $\overline{p}'$, respectively, on the upwards oriented vertical line through them. By comparison, we have $d(x,x') \le d(\overline{x}, \overline{x}') \le 1/e^{-\log\rho} = \rho$. $\square$




We define analogously the *Hamenstädt distance* $d_{W^{\rm ss}(w)}$ on the strong stable leaf of $w \in T^1\widetilde{M}$ by

$$d_{W^{\rm ss}(w)}(v,v') = \lim_{t\to+\infty} e^{\frac{1}{2}d(\pi(\phi_{-t}v),\ \pi(\phi_{-t}v'))-t}\ .$$

It satisfies analogous properties, in particular that for all $w \in T^1\widetilde{M}$, $s \in \mathbb{R}$ and $v, v' \in W^{\rm ss}(w)$, we have

$$d_{W^{\rm ss}(\phi_s w)}(\phi_s v, \phi_s v') = e^{-s}\, d_{W^{\rm ss}(w)}(v,v')\ . \tag{11}$$

The Hamenstädt distances on the strong stable and strong unstable leaves are related as follows: for all $w \in T^1\widetilde{M}$ and $v, v' \in W^{\rm su}(w)$, we have

$$d_{W^{\rm ss}(\iota\, w)}(\iota\, v, \iota\, v') = d_{W^{\rm su}(w)}(v,v')\ . \tag{12}$$

We denote by $B^{\rm su}(v,r)$ the open ball of centre $v$ and radius $r$ for the Hamenstädt distance $d_{W^{\rm su}(v)}$ on $W^{\rm su}(v)$, and similarly for $B^{\rm ss}(v,r)$. They are in general different from the open balls of centre $v$ and radius $r$ for the induced Riemannian distances, that we denote by respectively $\mathscr{B}^{\rm su}(v,r)$ and $\mathscr{B}^{\rm ss}(v,r)$. The reader is advised not to mix the two notations: in general variable curvature, the Riemannian balls do not have the exact scaling property that the Hamenstädt balls satisfy (see Equation (13) and (14)), hence using the second ones is preferable in most of the cases. We will only use the Riemannian balls in Subsection 6.2, for the purpose of staying close to some reference. The distances $d_{W^{\rm su}(v)}$ and balls $B^{\rm su}(v,r)$ were denoted by $d_{H_+(v)}$ and $B^+(v,r)$ in [Rob1] and [Sch3]. For all $v\in T^1\widetilde{M}$, $r>0$, $\gamma\in\Gamma$ and $t\in\mathbb{R}$, by Equation (8), we have

$$\gamma B^{\rm su}(v,r)=B^{\rm su}(\gamma v,r)\;,$$

and

$$\phi_t\big(B^{\rm su}(v,r)\big)=B^{\rm su}(\phi_t v,e^t r)\;. \tag{13}$$

Similarly, for all $\gamma\in\Gamma$ and $t\in\mathbb{R}$, the open balls $B^{\rm ss}(v,r)$ of centre $v\in T^1\widetilde{M}$ and radius $r>0$ for the Hamenstädt distance $d_{W^{\rm ss}(v)}$ on the strong stable leaves $W^{\rm ss}(v)$ satisfy $\gamma B^{\rm ss}(v,r)=B^{\rm ss}(\gamma v,r)$ and

$$\phi_t\big(B^{\rm ss}(v,r)\big)=B^{\rm ss}(\phi_t v,e^{-t} r)\;. \tag{14}$$

By Equation (12), we have

$$\iota\, B^{\rm su}(v,r)=B^{\rm ss}(-v,r)\;\text{ and }\;\iota\, B^{\rm ss}(v,r)=B^{\rm su}(-v,r)\;. \tag{15}$$

### 2.5 Some exercises in hyperbolic geometry

We end this chapter by the following (well known, see for instance [PaP1]) exercises in hyperbolic geometry.

**Lemma 2.5** *(i) For all $x,y,z$ in $\widetilde{M}$ such that $d(x,z)=d(y,z)$, for every $t\in[0,d(x,z)]$, if $x_t$ (respectively $y_t$) is the point on $[x,z]$ (respectively $[y,z]$) at distance $t$ from $x$ (respectively $y$), then*

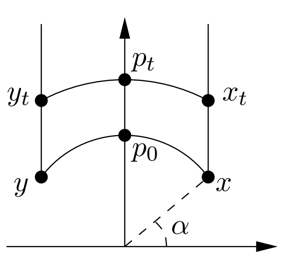
$$d(x_t,y_t)\le e^{-t}\sinh d(x,y)\;.$$

*(ii) For every $\alpha>0$, there exists $\beta>0$ (depending only on $\alpha$ and the bounds on the sectional curvature of $\widetilde{M}$) such that for all $x,y,z$ in $\widetilde{M}$ such that $d(x,z)=d(y,z)\ge 2$ and $d(x,y)\le\alpha$, for every $t\in[0,d(x,z)]$, with $x_t$ and $y_t$ defined as above,*

$$d(x_t,y_t)\le\beta\, d(x_1,y_1)\, e^{-t}\;.$$

**Proof.** (i) We may assume that $x\neq y$, otherwise $x_t=y_t$ and the result is trivially true. Since the sectional curvature of $\widetilde{M}$ is at most $-1$, and by comparison, we may assume that $\widetilde{M}$ is the hyperbolic upper half-space $\mathbb{H}^2_{\mathbb{R}}$. Let $m$ be the orthogonal projection of $z$ on the geodesic line through $x$ and $y$, which is the midpoint of $[x,y]$ by symmetry.

Replacing $z$ by the point at infinity of the geodesic ray starting from $m$ and passing through $z$ (any one perpendicular to $[x,y]$ if $m=z$) increases $d(x_t,y_t)$ and does not change $d(x,y)$. Hence we may assume that $z$ is the point at infinity in $\mathbb{H}^2_{\mathbb{R}}$, and that $x$ and $y$ lie on the (Euclidean) circle of centre 0 and radius 1, have same (Euclidean) height, with the real part of $x$ positive, the real part of $y$ negative.




Let $p_t$ be the intersection point of $[x_t,y_t]$ with the vertical axis, which is by symmetry the midpoint of $[x_t,y_t]$. If $\alpha$ is the (Euclidean) angle at 0 between the horizontal axis and the (Euclidean) line from 0 passing through $x$, then an easy computation in hyperbolic geometry (see also [Bea], page 145) gives $\sinh d(x,p_0)=\cos\alpha/\sin\alpha$.

Similarly, $\sinh d(x_t, p_t) = \cos\alpha/(e^t \sin\alpha)$. So that

$$d(x_t, y_t) = 2\,d(x_t, p_t) \leq 2 \sinh d(x_t, p_t) = 2\,e^{-t} \sinh d(x, p_0) = 2\,e^{-t} \sinh \frac{d(x,y)}{2}\ ,$$

which proves the first result, as $2 \sinh \frac{t}{2} \leq \sinh t$ if $t \geq 0$.

(ii) We start by the observation that for all $a > 0$ and $s \in [0, a]$, we have $s \leq \sinh s \leq s\,\frac{\sinh a}{a}$.

By (i), we hence only have to prove that there exists a constant $c > 0$ such that $d(x_1, y_1) \geq c\,d(x, y)$. By comparison, we may assume that $\widetilde{M}$ is the real hyperbolic plane with constant curvature $-b^2$. By the hyperbolic sine formula in this plane, we have

$$\frac{\sinh\big(b\ \frac{d(x_1,y_1)}{2}\big)}{\sinh\big(b\ d(x_1, z)\big)} = \frac{\sinh\big(b\ \frac{d(x,y)}{2}\big)}{\sinh\big(b\ d(x, z)\big)}\ .$$

Since the map $s \mapsto \frac{\sinh s}{\sinh(s+b)}$ is non decreasing as $b > 0$, since $d(x, z) = d(x_1, z) + 1 \geq 2$, and since $d(x_1, y_1) \leq d(x, y) \leq \alpha$ by convexity, we hence have, by the preliminary observation, that

$$d(x_1, y_1) \geq \frac{\sinh b}{\sinh(2\,b)}\ \frac{\frac{b\,\alpha}{2}}{\sinh \frac{b\,\alpha}{2}}\ d(x, y)\ ,$$

as required. □

**Corollary 2.6** *(i) For every $T \in [0, +\infty[\,$, for all $v$ and $w$ in $T^1\widetilde{M}$ such that $\pi(\phi_T v) = \pi(\phi_T w)$, for every $t \in [0, T]$,*

$$d'_{T^1\widetilde{M}}(\phi_t v, \phi_t w) \leq e^{-t} \sinh\big(2 + d(\pi(v), \pi(w))\big)\ .$$

*(ii) For every $\alpha' > 0$, there exists $\beta' > 0$ (depending only on $\alpha'$ and the bounds on the sectional curvature of $\widetilde{M}$) such that for every $T \in [1, +\infty[\,$, for all $v$ and $w$ in $T^1\widetilde{M}$ such that $\pi(\phi_T v) = \pi(\phi_T w)$ and $d(\pi(v), \pi(w)) \leq \alpha'$, for every $t \in [0, T]$,*

$$d'_{T^1\widetilde{M}}(\phi_t v, \phi_t w) \leq \beta'\, d(\pi(v), \pi(w))\, e^{-t}\ .$$

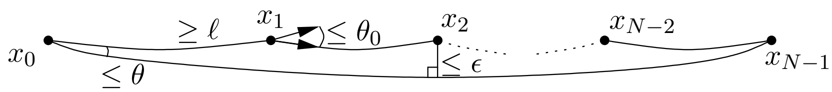
**Proof.** (i) By the definition of the distance $d'_{T^1\widetilde{M}}$ (see Equation (7)), this follows from Lemma 2.5 (i), applied to $x = \pi(\phi_{-1} v)$, $y = \pi(\phi_{-1} w)$ and $z = \pi(\phi_T v)$, since $d(x, y) \leq d(\pi(v), \pi(w)) + 2$.

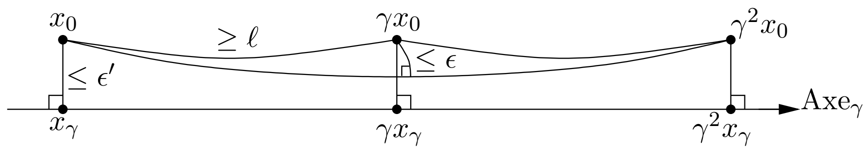
(ii) This follows similarly from Lemma 2.5 (ii), with $\alpha = \alpha' + 2$, since with $x, y, z$ as above and $x_1, y_1$ as in this lemma, we have $x_1 = \pi(v), y_1 = \pi(w)$ and $d(x, z) = d(y, z) = T + 1 \geq 2$. □

Also recall without proof the following well known results in complete simply connected Riemannian manifolds with sectional curvature at most $-1$ (variants of Anosov's closing lemma).

**Lemma 2.7** *For all $\ell, \epsilon, \theta > 0$, there exists $\theta_0 = \theta_0(\ell, \epsilon, \theta) > 0$ such that for every piecewise geodesic path $\omega = \bigcup_{0 \leq i < N} [x_i, x_{i+1}]$ in $\widetilde{M}$ with $N \in \mathbb{N} \cup \{+\infty\}$, if its exterior angle at $x_i$ is at most $\theta_0$ for $0 < i < N - 1$ and if the length of each segment $[x_i, x_{i+1}]$ is at least $\ell$, then*

- *$\omega$ is contained in the $\epsilon$-neighbourhood of the geodesic segment $\omega' = [x_0, x_N]$, if $N$ is finite, or of the geodesic ray $\omega' = [x_0, \xi]$ where $\xi \in \partial_\infty \widetilde{M}$ is the point at infinity to which the sequence $(x_i)_{i \in \mathbb{N}}$ converges, if $N$ is infinite;*
- *the angles between $\omega$ and $\omega'$ at the finite endpoints of $\omega'$ are at most $\theta$.*




**Lemma 2.8** *For all $\ell, \epsilon' > 0$, there exists $\epsilon = \epsilon(\ell, \epsilon') \in \,]0,1]$ with $\lim_{\epsilon' \to 0} \epsilon = 0$ such that for every isometry $\gamma$ of any proper geodesic CAT$(-1)$-space $X$, for every $x_0$ in $X$, if $d(x_0, \gamma x_0) \geq \ell$ and $d(\gamma x_0, [x_0, \gamma^2 x_0]) \leq \epsilon$, then $\gamma$ is loxodromic and $d(x_0, \mathrm{Axe}_\gamma) \leq \epsilon'$ where $\mathrm{Axe}_\gamma$ is the translation axis of $\gamma$ in $X$.*

## 2.6 Pushing measures by branched covers

In this text, we will often construct measures on $T^1M$ starting from $\Gamma$-invariant measures on $T^1\widetilde{M}$. Given a Galois covering $f : X \to Y$ of topological spaces, and a (positive Borel) measure $m$ on $X$ invariant under the covering group, it is well known that there exists a unique measure on $Y$ (which is not the push-forward measure $f_*m$) such that the map $f$ locally preserves the measure. In the case of ramified covers, the analogous construction is not so well known (see for instance [PaP4, §2.4]). The fact that we are not assuming $\Gamma$ to be torsion free requires it.

Let $\widetilde{X}$ be a locally compact metrisable space, endowed with a proper (but not necessarily free) action of a discrete group $G$. Let $p : \widetilde{X} \to X = G\backslash\widetilde{X}$ be the canonical projection. Let $\widetilde{\mu}$ be a locally finite $G$-invariant measure on $\widetilde{X}$.

Note that the map $N$ from $\widetilde{X}$ to $\mathbb{N} - \{0\}$ sending a point $x \in X$ to the order of its stabiliser in $G$ is upper semi-continuous. In particular, for every $n \geq 1$, the $G$-invariant subset $\widetilde{X}_n = N^{-1}(\{n\})$ is locally closed, hence locally compact metrisable. With $X_n = p(\widetilde{X}_n)$, the restriction $p_{|\widetilde{X}_n} : \widetilde{X}_n \to X_n$ is a local homeomorphism. Since $\widetilde{\mu}$ is $G$-invariant, there exists a unique measure $\mu_n$ on $X_n$ such that the map $p_{|\widetilde{X}_n}$ locally preserves the measure. Now, considering a measure on $X_n$ as a measure on $X$ with support in $X_n$, let us define

$$\mu = \sum_{n \geq 1} \frac{1}{n}\, \mu_n\ ,$$

which is a locally finite measure on $X$, called the measure *induced by* $\widetilde{\mu}$ on $X$. Note that the factor $1/n$, besides being natural, is necessary to get the continuity of $\widetilde{\mu} \mapsto \mu$, see below.

Note that if $\widetilde{\mu}$ gives measure 0 to the set $N^{-1}([2, +\infty[)$ of fixed points of nontrivial elements of $G$, then $\mu = \mu_1$, and the above construction is not really needed.

If $\widetilde{X}'$ is another locally compact metrisable space, endowed with a proper action of a discrete group $G'$, if $\widetilde{\mu}'$ is a locally finite $G'$-invariant measure on $\widetilde{X}'$, with induced measure $\mu'$ on $G'\backslash X'$, then the measure on the product of the quotient spaces $G\backslash X \;\times\; G'\backslash X'$, induced by the product measure $\widetilde{\mu} \otimes \widetilde{\mu}'$, is the product $\mu \otimes \mu'$ of the induced measures.

The following result is easy to check.

**Lemma 2.9** *The map $\widetilde{\mu} \mapsto \mu$ from the space of locally finite measures on $\widetilde{X}$ to the space of locally finite measures on $X$, both endowed with their weak-star topologies, is continuous.*

We conclude this subsection with an example which indicates why the factor $1/n$ before $\mu_n$ in the formula defining $\mu$ is necessary. Let $G$ be the finite group of rotations of order $n$ in the real plane $\widetilde{X} = \mathbb{C}$, so that the quotient map $p : \mathbb{C} \to X = G\backslash\mathbb{C}$ is a branched cover of order $n$. Denote by $\mathscr{D}_x$ the unit Dirac mass at a point $x$. For all $\rho \geq 0$, let $\widetilde{\mu}_\rho = \sum_{k=0}^{n-1} \mathscr{D}_{\rho\, e^{ik\theta}}$, which is a locally finite $G$-invariant measure on $\mathbb{C}$, which depends continuously on $\rho$. If $\rho > 0$, then, by the regular cover situation, the measure induced by $\widetilde{\mu}_\rho$ is the unit Dirac mass at the image of $\rho$ in $X$, which converges, as $\rho$ tends to 0, to the unit Dirac mass at the image $[0]$ of 0 in $X$. Since $\widetilde{\mu}_0$ is $n$ times the unit Dirac mass at 0, in order to have the required continuity property, we do have to define the measure induced by $\widetilde{\mu}_0$ as $1/n$ times the push-forward of $\widetilde{\mu}_0$ by $0 \mapsto [0]$.

# 3 A Patterson-Sullivan theory for Gibbs states

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$; $\Gamma$ is a non-elementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map. We use the notation $M = \Gamma\backslash\widetilde{M}$, $T^1M = \Gamma\backslash T^1\widetilde{M}$ and $F : T^1M \to \mathbb{R}$ (the map induced by $\widetilde{F}$) introduced in Chapter 2.

We recall in this chapter the construction of the Gibbs measure on the unit tangent bundle of a negatively curved manifold associated with a given potential, due to O. Mohsen [Moh]. More precisely, the subsections 3.4, 3.5, 3.7, 3.8, 3.9 are new (except for the strong influence of [Ham2]), the others are extracted (up to an adaptation to the non-cocompact case) from [Moh]. We refer to [Ham2, Led2, Cou2, Sch3] for different approaches. The Gibbs cocycles used in these four references are the same up to signs as the one we will define in Subsection 3.3. Note that the last three references use $\sum_{\gamma\in\Gamma} e^{s\int_x^{\gamma y}\widetilde{F}}$ for their Poincaré series, whereas we use $\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}-s)}$. This additive instead of multiplicative approach greatly simplifies the techniques.

## 3.1 Potential functions and their periods

For all $x, y$ in $\widetilde{M}$, let us define

$$\int_x^y \widetilde{F} = \int_0^{d(x,y)} \widetilde{F}(\phi_t(v))\,dt\,,$$

where $v$ is a unit tangent vector such that $\pi(v) = x$ and $\pi(\phi_{d(x,y)}v) = y$ (unique if $y \neq x$).

**Remark.** A general extension of the techniques of this work when $\widetilde{M}$ is any complete locally compact CAT$(-1)$ metric space as in [Rob1] (even assuming that its geodesic segments are extendible to geodesic lines) seems difficult at this point. The correct analog of the unit tangent bundle for such an extension is probably not the usual one of the space of all geodesic lines. Indeed, there could be many geodesic lines containing a given segment $[x, y]$. Hence defining $\int_x^y \widetilde{F}$ would require a choice of one of these geodesic lines, or restrictions on $\widetilde{F}$ or some averaging process, hence appropriate additional information in order not to lose the invariance under the group $\Gamma$. A better analog would be the space of germs of geodesic segments, but the geodesic flow then does not extend. See the case of trees in [BrPP].

Note that

$$\forall\, \gamma\in\Gamma,\quad \int_{\gamma x}^{\gamma y}\widetilde{F}=\int_x^y\widetilde{F}\qquad\text{and}\qquad \int_y^x\widetilde{F}=\int_x^y\widetilde{F}\circ\iota\;. \tag{16}$$

For every loxodromic element $\gamma$ in $\Gamma$, the *period of* $\gamma$ for the potential $F$ is

$$\mathrm{Per}_F(\gamma)=\int_x^{\gamma x}\widetilde{F}$$

for any point $x$ on the translation axis of $\gamma$ in $\widetilde{M}$. Note that, for all $n\in\mathbb{N}-\{0\}$ and $\alpha\in\Gamma$, we have

$$\mathrm{Per}_F(\alpha\gamma\alpha^{-1})=\mathrm{Per}_F(\gamma),\quad \mathrm{Per}_F(\gamma^n)=n\;\mathrm{Per}_F(\gamma)\quad\text{and}\quad \mathrm{Per}_F(\gamma^{-1})=\mathrm{Per}_{F\circ\iota}(\gamma)\;. \tag{17}$$

Let $\widetilde{F}^*:T^1\widetilde{M}\rightarrow\mathbb{R}$ be another Hölder-continuous $\Gamma$-invariant map. Say that $\widetilde{F}^*$ is *cohomologous* to $\widetilde{F}$ (see for instance [Livs]) if there exists a Hölder-continuous $\Gamma$-invariant map $\widetilde{G}:T^1\widetilde{M}\rightarrow\mathbb{R}$, differentiable along every flow line, such that

$$\widetilde{F}^*(v)-\widetilde{F}(v)=\frac{d}{dt}_{|t=0}\widetilde{G}(\phi_t v)\;.$$

We will say that $\widetilde{F}^*$ is *cohomologous to* $\widetilde{F}$ *via* $\widetilde{G}$ when we want to emphasise $\widetilde{G}$. Two cohomologous potentials have the same periods: if $\widetilde{F}^*$ is cohomologous to $\widetilde{F}$, then, denoting by $F^*$ the map induced on $T^1M$ by $\widetilde{F}^*$, for every loxodromic element $\gamma\in\Gamma$, we have

$$\mathrm{Per}_F(\gamma)=\mathrm{Per}_{F^*}(\gamma)\;.$$

**Remark 3.1** *Let $\widetilde{F},\widetilde{F}^*:T^1\widetilde{M}\rightarrow\mathbb{R}$ be Hölder-continuous $\Gamma$-invariant maps such that* $\mathrm{Per}_F(\gamma)=\mathrm{Per}_{F^*}(\gamma)$ *for every loxodromic element $\gamma\in\Gamma$. Then $\widetilde{F}^*$ is cohomologous to $\widetilde{F}$ in restriction to $\widetilde{\Omega}\Gamma$.*

**Proof.** Since the diagonal action of $\Gamma$ on $\Lambda\Gamma\times\Lambda\Gamma$ is topologically transitive, the action of the geodesic flow on the (topological) non-wandering set $\Omega\Gamma$ is topologically transitive. The validity of Anosov's closing lemma (see Lemma 2.7) does not require any compactness assumption on $M$. The proof of Livšic's theorem in [KatH, Theo. 19.2.4] then extends. □

Note that if $\widetilde{F}^*$ and $\widetilde{F}$ are cohomologous via the map $\widetilde{G}$, then $\widetilde{F}^*\circ\iota$ and $\widetilde{F}\circ\iota$ are cohomologous via the map $-\widetilde{G}\circ\iota$ (beware the sign). We will say that $\widetilde{F}$ (or $F$) is *reversible* if $\widetilde{F}$ is cohomologous to $\widetilde{F}\circ\iota$.

We end this subsection by the following technical lemma on potential functions (see also [Cou2, Lem. 3] with the multiplicative approach).

**Lemma 3.2** *For every $r_0>0$, there exist two constants $c_1>0$ and $c_2\in\,]0,1]$ (depending, besides the $r_0$ dependence of $c_1$, only on the Hölder constants of $\widetilde{F}$ and the bounds on the sectional curvature of $\widetilde{M}$) such that for all $x,y,z$ in $\widetilde{M}$, we have*

$$\Big|\int_x^z\widetilde{F}\;-\int_y^z\widetilde{F}\,\Big|\le c_1\,e^{d(x,\,y)}+d(x,y)\max_{\pi^{-1}(B(x,\,d(x,\,y)))}|\widetilde{F}|\;,$$

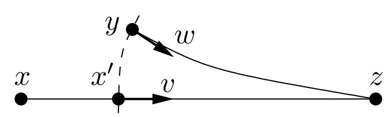
*and if furthermore* $d(x,y)\le r_0$*, then*

$$\Big|\int_x^z \widetilde F\;-\int_y^z \widetilde F\,\Big|\le c_1\,d(x,y)^{c_2}+d(x,y)\max_{\pi^{-1}(B(x,\,d(x,y)))}|\widetilde F|\;.$$

Note that in the first c entered equation (as well as the one in Lemma 3.4), the constant $c_1$ actually does not depend on any $r_0$, hence could be given some other name. But the formulation of the statement helps to keep at a minimum the number of constants, and facilitates future simultaneous references to both c entered equation.

**Proof.** By symmetry, we may assume that $d(x,z)\ge d(y,z)$. The result is true if $y=z$, hence we assume that $y\neq z$.

Let $x'$ be the point on $[x,z]$ at distance $d(y,z)$ from $z$. Let $v$ (respectively $w$) be the unit tangent vector at $x'$ (respectively $y$) pointing towards $z$.




The closest point $p$ of $y$ on $[x,z]$ lies in $[x',z]$ by convexity. Hence $d(x,x')\le d(x,p)\le d(x,y)$, since closest point maps do not increase distances. Since the distance function from a given point to a point varying on a geodesic line is convex, we have $d(x',y)\le d(x,y)$.

Let $c>0$ and $\alpha\in\,]0,1]$ be the Hölder constants of $\widetilde F$ for the distance $d'=d'_{T^1\widetilde M}$ on $T^1\widetilde M$ defined in Equation (7), that is $|\widetilde F(u)-\widetilde F(u')|\le c\,d'(u,u')^\alpha$ for all $u,u'$ in $T^1\widetilde M$ with $d'(u,u')\le 1$. By Corollary 2.6 (i), there exists a universal constant $c'>0$ (for instance $c'=\frac{e^2}{2}$) such that for every $t\in[0,d(y,z)]$, we have $d'(\phi_t v,\phi_t w)\le c' e^{d(x',\,y)-t}$.

Let $t\in[0,d(y,z)]$. If $d'(\phi_t v,\phi_t w)\le 1$, then

$$|\widetilde F(\phi_t v)-\widetilde F(\phi_t w)|\le c\,d'(\phi_t v,\phi_t w)^\alpha\le c\,\big(c'\,e^{d(x',\,y)-t}\big)^\alpha\;.$$

If $d'(\phi_t v,\phi_t w)\ge 1$, then by the remark at the end of Subsection 2.1, we have

$$|\widetilde F(\phi_t v)-\widetilde F(\phi_t w)|\le 3\,c\,d'(\phi_t v,\phi_t w)\le 3\,c\,c'\,e^{d(x',\,y)-t}\;.$$

Hence,

$$\begin{aligned}\Big|\int_x^z \widetilde F-\int_y^z \widetilde F\,\Big|&=\Big|\int_{-d(x,\,x')}^{d(y,\,z)}\widetilde F(\phi_t v)\,dt-\int_0^{d(y,\,z)}\widetilde F(\phi_t w)\,dt\,\Big|\\ &\le\int_{-d(x,\,x')}^{0}|\widetilde F(\phi_t v)|\,dt+\int_0^{d(y,\,z)}|\widetilde F(\phi_t v)-\widetilde F(\phi_t w)|\,dt\\ &\le d(x,x')\max_{\pi^{-1}(B(x,\,d(x,x')))}|\widetilde F|+\int_0^{+\infty}c\,\big(c'\,e^{d(x',\,y)-t}\big)^\alpha+3\,c\,c'\,e^{d(x',\,y)-t}\,dt\\ &\le d(x,y)\max_{\pi^{-1}(B(x,\,d(x,y)))}|\widetilde F|+\big(3\,c\,c'+\frac{c(c')^\alpha}{\alpha}\big)\,e^{d(x,\,y)}\;.\end{aligned}$$

The first result follows. The second may be proved similarly, using Corollary 2.6 (ii). □

**Remarks.** (1) When $x,y,z\in\mathscr{C}\Lambda\Gamma$, we may replace, in each assertion of the above lemma, $\max_{\pi^{-1}(B(x,\,d(x,y)))}|\widetilde F|$ by $\max_{\pi^{-1}(B(x,\,d(x,y))\cap\mathscr{C}\Lambda\Gamma)}|\widetilde F|$.

(2) Using the equality $\int_x^{x'}\widetilde F-\int_y^{y'}\widetilde F=\big(\int_x^{x'}\widetilde F-\int_y^{x'}\widetilde F\big)+\big(\int_{x'}^{y}\widetilde F\circ\iota-\int_{y'}^{y}\widetilde F\circ\iota\big)$ and the fact that $\iota$ is Lipschitz with constants depending only on the bounds on the sectional curvature of $\widetilde M$, we have a similar control on $\big|\int_x^{x'}\widetilde F-\int_y^{y'}\widetilde F\,\big|$ for all $x,x',y,y'$ in $\widetilde M$, in terms of $\max\{d(x,y),d(x',y')\}$.

## 3.2 The Poincaré series and the critical exponent of $(\Gamma, F)$

Let us fix $x, y \in \widetilde{M}$. The *Poincaré series* of $(\Gamma, F)$ is the map $Q_{\Gamma,\,F} = Q_{\Gamma,\,F,\,x,\,y} : \mathbb{R} \to [0, +\infty]$ defined by

$$Q_{\Gamma,\,F,\,x,\,y}(s) = \sum_{\gamma \in \Gamma} e^{\int_x^{\gamma y} (\widetilde{F} - s)} .$$

The *critical exponent* of $(\Gamma, F)$ is the element $\delta_{\Gamma,\,F}$ in $[-\infty, +\infty]$ defined by

$$\delta_{\Gamma,\,F} = \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{\gamma \in \Gamma,\; n-1 < d(x, \gamma y) \leq n} e^{\int_x^{\gamma y} \widetilde{F}} .$$

When $F = 0$, the Poincaré series $Q_\Gamma = Q_{\Gamma,\,x,\,y}$ of $(\Gamma, 0)$ is the usual *Poincaré series* of $\Gamma$, and the critical exponent $\delta_{\Gamma,\,0}$ is the *critical exponent* $\delta_\Gamma$ of $\Gamma$ (which belongs to $]0, +\infty[$ since $\Gamma$ is non-elementary and $\widetilde{M}$ has pinched negative curvature, see for instance [Rob1]).

We will prove in the following Lemma 3.3 (v) that $\delta_{\Gamma,\,F} > -\infty$. If $\delta_{\Gamma,\,F} < +\infty$, we say that $(\Gamma, F)$ is *of divergence type* if the series $Q_{\Gamma,\,F}(\delta_{\Gamma,\,F})$ diverges, and *of convergence type* otherwise.

If $\delta_{\Gamma,\,F} < +\infty$, the *normalised potential* is the Hölder-continuous map $\widetilde{F} - \delta_{\Gamma,\,F}$ on $T^1\widetilde{M}$ (or its induced map $F - \delta_{\Gamma,\,F}$ on $T^1 M$). Some references use the normalised potential with the opposite sign.

Since $\widetilde{F}$ is Hölder-continuous (see Remark (2) at the end of the previous Subsection 3.1), the critical exponent $\delta_{\Gamma,\,F}$ of $(\Gamma, F)$ does not depend on $x, y$, and satisfies the following elementary properties.

**Lemma 3.3** *(i) The Poincaré series of $(\Gamma, F)$ converges if $s > \delta_{\Gamma,\,F}$ and diverges if $s < \delta_{\Gamma,\,F}$. The Poincaré series of $(\Gamma, F)$ diverges at $\delta_{\Gamma,\,F}$ if and only if the Poincaré series of $(\Gamma, F + \kappa)$ diverges at $\delta_{\Gamma,\,F+\kappa}$, for every $\kappa \in \mathbb{R}$, and in particular*

$$\forall\, \kappa \in \mathbb{R}, \quad \delta_{\Gamma,\,F+\kappa} = \delta_{\Gamma,\,F} + \kappa . \tag{18}$$

*(ii) We have*

$$\forall\, s \in \mathbb{R},\; Q_{\Gamma,\,F\circ\iota,\,x,\,y}(s) = Q_{\Gamma,\,F,\,y,\,x}(s) \quad \text{and} \quad \delta_{\Gamma,\,F\circ\iota} = \delta_{\Gamma,\,F} . \tag{19}$$

*(iii) If $\Gamma'$ is a non-elementary subgroup of $\Gamma$, denoting by $F' : \Gamma' \backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$, we have*

$$\delta_{\Gamma',\,F'} \leq \delta_{\Gamma,\,F} .$$

*(iv) We have the upper and lower bounds*

$$\delta_\Gamma + \inf_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} \;\leq\; \delta_{\Gamma,\,F} \;\leq\; \delta_\Gamma + \sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} .$$

*(v) We have $\delta_{\Gamma,\,F} > -\infty$.*
*(vi) The map $F \mapsto \delta_{\Gamma,\,F}$ is convex, sub-additive and 1-Lipschitz for the uniform norm on the vector space of real continuous maps on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, that is, if $\widetilde{F^*} : T^1\widetilde{M} \to \mathbb{R}$ is another Hölder-continuous $\Gamma$-invariant map, inducing $F^* : \Gamma\backslash T^1\widetilde{M} \to \mathbb{R}$, if $\delta_{\Gamma,\,F}, \delta_{\Gamma,\,F^*} < +\infty$, then*

$$|\; \delta_{\Gamma,\,F^*} - \delta_{\Gamma,\,F} \;| \leq \sup_{v \in \pi^{-1}(\mathscr{C}\Lambda\Gamma)} |\; \widetilde{F^*}(v) - \widetilde{F}(v) \;| ,$$

$$\delta_{\Gamma,\,F+F^*} \leq \delta_{\Gamma,\,F} + \delta_{\Gamma,\,F^*}\ ,$$

*and, for every $t \in [0,1]$,*

$$\delta_{\Gamma,\,tF+(1-t)F^*} \leq t\,\delta_{\Gamma,\,F} + (1-t)\,\delta_{\Gamma,\,F^*}\ .$$

*(vii) For every $c > 0$, we have*

$$\delta_{\Gamma,\,F} = \limsup_{n\to+\infty} \frac{1}{n}\log \sum_{\gamma\in\Gamma,\ n-c<d(x,\gamma y)\leq n} e^{\int_x^{\gamma y}\widetilde{F}}\ , \tag{20}$$

*and if $\delta_{\Gamma,\,F} \geq 0$, then*

$$\delta_{\Gamma,\,F} = \limsup_{n\to+\infty} \frac{1}{n}\log \sum_{\gamma\in\Gamma,\ d(x,\gamma y)\leq n} e^{\int_x^{\gamma y}\widetilde{F}}\ . \tag{21}$$

*(viii) If $\Gamma''$ is a discrete cocompact group of isometries of $\widetilde{M}$ such that $F$ is $\Gamma''$-invariant, denoting by $F'' : \Gamma''\backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$, we have*

$$\delta_{\Gamma,\,F} \leq \delta_{\Gamma'',\,F''}\ .$$

*(ix) If $\Gamma$ is the infinite cyclic group*[2] *generated by a loxodromic isometry $\gamma$ of $\widetilde{M}$, then $(\Gamma, F)$ is of divergence type and*

$$\delta_{\Gamma,\,F} = \max\Big\{\,\frac{\mathrm{Per}_F(\gamma)}{\ell(\gamma)},\ \frac{\mathrm{Per}_{F\circ\iota}(\gamma)}{\ell(\gamma)}\,\Big\}\,.$$

*Furthermore,* $\sum_{\gamma\in\Gamma,\ d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}} \asymp \begin{cases} e^{\delta_{\Gamma,\,F}\,t} & \text{if } \delta_{\Gamma,\,F} > 0 \\ t & \text{if } \delta_{\Gamma,\,F} = 0 \\ 1 & \text{if } \delta_{\Gamma,\,F} < 0 \end{cases}$ .

We will prove in Subsection 4.2 that the upper limits in Equation (21) if $c > 0$ is large enough, and in Equation (21) if $\delta_{\Gamma,\,F} > 0$ are in fact limits.

An interesting problem is to study whether or not the critical exponent $\delta_{\Gamma,\,F}$ of $(\Gamma, F)$ is equal to the upper bound of the critical exponents $\delta_{\Gamma_0,\,F_0}$ where $\Gamma_0$ ranges over the convex-cocompact subgroups of $\Gamma$ and $F_0 : \Gamma_0\backslash T^1\widetilde{M} \to \mathbb{R}$ is the map induced by $\widetilde{F}$. Replacing subgroups by subsemigroups, we answer this positively in Subsection 4.4.

**Proof.** The verifications of (i), (ii) and (iii) are elementary. For instance, Equation (19) follows from both parts of Equation (16) and the change of variable $\gamma \mapsto \gamma^{-1}$ in the Poincaré series.

To prove Assertion (iv), note that if $x$ is a point in the convex hull of the limit set $\mathscr{C}\Lambda\Gamma$, then, for every $\gamma \in \Gamma$, the geodesic segment between $x$ and $\gamma x$ is contained in $\mathscr{C}\Lambda\Gamma$. Hence

$$d(x,\gamma x)\Big(\inf_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} - s\Big) \leq \int_x^{\gamma x} (\widetilde{F} - s) \leq d(x,\gamma x)\Big(\sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} \widetilde{F} - s\Big)\,.$$

[2] Note that $\Gamma$ is elementary, but we can still define the Poincaré series of $(\Gamma, F)$ and its critical exponent.

This proves Assertion (iv) for instance by taking the exponential, summing over $\gamma\in\Gamma$ and using the first assertion of (i).

To prove Assertion (v), let $\Gamma'$ be a non-elementary convex-cocompact subgroup of $\Gamma$ (for instance a Schottky subgroup of $\Gamma$). Denote by $F':\Gamma'\backslash T^1\widetilde{M}\rightarrow\mathbb{R}$ the map induced by $\widetilde{F}$. Since $|\widetilde{F}|$ is $\Gamma$-invariant and bounded on compact subsets of $\widetilde{M}$, by Assertion (iv), we have $\delta_{\Gamma',\,F'}>-\infty$. Assertion (v) then follows from Assertion (iii).

To prove Assertion (vi), let $c=\sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)}|\widetilde{F}-\widetilde{F^*}|$ and $x\in\mathscr{C}\Lambda\Gamma$. We have

$$\int_x^{\gamma x}(\widetilde{F}-(s+c))\leq\int_x^{\gamma x}(\widetilde{F^*}-s)\leq\int_x^{\gamma x}(\widetilde{F}-(s-c))\;,$$

and the first claim follows. To prove the two remaining claims of (vi), for all $s>\delta_{\Gamma,\,F}$ and $s^*>\delta_{\Gamma,\,F^*}$, we have

$$Q_{\Gamma,\,F,\,x,\,y}(s)Q_{\Gamma,\,F^*,\,x,\,y}(s^*)\geq Q_{\Gamma,\,F+F^*,\,x,\,y}(s+s^*)$$

and, by the convexity of the exponential,

$$Q_{\Gamma,\,tF+(1-t)F^*,\,x,\,y}(ts+(1-t)s^*)\leq tQ_{\Gamma,\,F,\,x,\,y}(s)+(1-t)Q_{\Gamma,\,F^*,\,x,\,y}(s^*)\;.$$

Hence $s+s^*\geq\delta_{\Gamma,\,F+F^*}$ and $\delta_{\Gamma,\,tF+(1-t)F^*}\leq ts+(1-t)s^*$ by the first assertion of (i). The result follows by letting $s$ and $s^*$ converge respectively to $\delta_{\Gamma,\,F}$ and $\delta_{\Gamma,\,F^*}$.

Assertion (vii) is also elementary. For its first claim, one cuts the interval $]t-c,t]$ at the points $t-k$ for $k=0,\cdots,\lfloor c\rfloor$ if $c\geq 1$, or the interval $]t-1,t]$ at the points $t-ck$ for $k=0,\cdots,\lfloor\frac{1}{c}\rfloor$ if $c\leq 1$, and one uses the inequalities $\max_{1\leq i\leq N}a_i\leq\sum_{1\leq i\leq N}a_i\leq N\max_{1\leq i\leq N}a_i$.

For the second claim, the right hand side of Equation (21) is clearly not smaller than $\delta_{\Gamma,\,F}$. To prove the other inequality, for every $\epsilon>0$, let $n_0\in\mathbb{N}$ be such that $\sum_{n-1<d(x,\,\gamma y)\leq n}e^{\int_x^y F}\leq e^{(\delta_{\Gamma,\,F}+\epsilon)n}$ for all $n\geq n_0$. Then for all $n\geq n_0$, we have

$$\sum_{d(x,\,\gamma y)\leq n}e^{\int_x^y F}\leq\sum_{k=0}^{n}e^{(\delta_{\Gamma,\,F}+\epsilon)k}+\mathrm{O}(1)\leq c'\;e^{(\delta_{\Gamma,\,F}+\epsilon)n}$$

for some $c'>0$, since $\delta_{\Gamma,\,F}+\epsilon>0$. The result follows.

Let us prove Assertion (viii). Since $\Gamma''\backslash\widetilde{M}$ is compact, there exists $c\geq 0$ such that for every $\gamma\in\Gamma$, there exists $\gamma''\in\Gamma''$ be such that $d(\gamma x,\gamma''x)\leq c$. Hence for every $s>\delta_{\Gamma'',\,F''}$, we have, with $r_0=c$ and $c_1$ as in Lemma 3.2 and by $\Gamma$-equivariance of $\widetilde{F}$,

$$\begin{aligned}Q_{\Gamma,\,F,\,x,\,x}(s)&\leq\sum_{\gamma\in\Gamma}e^{\int_x^{\gamma''x}(\widetilde{F}-s)+c_1\,e^{d(\gamma x,\,\gamma''x)}+d(\gamma x,\,\gamma''x)\max_{\pi^{-1}(B(\gamma x,\,d(\gamma x,\,\gamma''y)))}|\widetilde{F}|}\\&\leq e^{c_1\,e^c+c\,\max_{\pi^{-1}(B(x,\,c))}|\widetilde{F}|}\;Q_{\Gamma'',\,F'',\,x,\,x}(s)<+\infty\;.\end{aligned}$$

Hence $\delta_{\Gamma,\,F}\leq s$, and the result follows by letting $s$ tend to $\delta_{\Gamma'',\,F''}$.

Finally, in order to prove Assertion (ix), if $x$ belongs to the translation axis of $\gamma$, we have

$$\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}(\widetilde{F}-s)} = \sum_{n\in\mathbb{N}} e^{\int_x^{\gamma^n x}(\widetilde{F}-s)} + \sum_{n\in\mathbb{N}-\{0\}} e^{\int_x^{\gamma^{-n} x}(\widetilde{F}-s)}$$
$$= \sum_{n\in\mathbb{N}} e^{n(\operatorname{Per}_F(\gamma)-s\,\ell(\gamma))} + \sum_{n\in\mathbb{N}-\{0\}} e^{n(\operatorname{Per}_{F\circ\iota}(\gamma)-s\,\ell(\gamma))}\;.$$

Hence $Q_{\Gamma,\,F,\,x,\,x}(s)$ converges if and only if $\operatorname{Per}_F(\gamma)-s\,\ell(\gamma)<0$ and $\operatorname{Per}_{F\circ\iota}(\gamma)-s\,\ell(\gamma)<0$, which proves the first two claims of Assertion (xi). We have, again if $x$ belongs to the translation axis of $\gamma$,

$$\sum_{\gamma\in\Gamma,\;d(x,\,\gamma x)\le t} e^{\int_x^{\gamma x}\widetilde{F}} = \sum_{0\le k\le t/\ell(\gamma)} e^{k\;\operatorname{Per}_F(\gamma)} + \sum_{0<k\le t/\ell(\gamma)} e^{k\;\operatorname{Per}_{F\circ\iota}(\gamma)}\;.$$

The last claim follows, by a standard geometric series argument. □

Here are a few immediate consequences. The convergence or divergence of the Poincaré series $Q_{\Gamma,\,F}(s)$ is independent of the points $x$ and $y$. The Poincaré series $Q_{\Gamma,\,F\circ\iota}(s)$ converges if and only if $Q_{\Gamma,\,F}(s)$ does.

The critical exponent $\delta_{\Gamma,\,F}$ of $(\Gamma,F)$ is positive if $\inf_{T^1\widetilde{M}}\widetilde{F}>-\delta_\Gamma$. For instance, $\delta_{\Gamma,\,F}>0$ if $F$ is a small perturbation of zero (more precisely if $\|F\|_\infty=\sup_{T^1\widetilde{M}}|\widetilde{F}|<\delta_\Gamma$), or if $F\ge 0$, since $\Gamma$ is non-elementary and therefore $\delta_\Gamma>0$. Recall that $\delta_\Gamma<+\infty$ since $\widetilde{M}$ has pinched curvature. Hence if $F$ is bounded from above, then $\delta_{\Gamma,\,F}<+\infty$.

When $\delta_{\Gamma,\,F}<+\infty$, the additive convention for the Poincaré series and the fact that its interesting behaviour occurs at $s=\delta_{\Gamma,\,F}$ suggest it is meaningful to consider the normalised potential $F-\delta_{\Gamma,\,F}$. By Equation (18), the critical exponent of the normalised potential is 0. Moreover, adding a large enough constant to $F$ does not change the normalised potential and allows the critical exponent of $(\Gamma,F)$ to be positive (this reduction will be used repeatedly in many results of the chapters 9 and 10).

The relationship between $\delta_{\Gamma,\,F}$ and $\delta_{\Gamma,\,sF}$ for $s>0$ seems unclear in general.

**Remark.** Let $\widetilde{F}^*:T^1\widetilde{M}\to\mathbb{R}$ be another Hölder-continuous $\Gamma$-invariant map, which is cohomologous to $\widetilde{F}$ via $\widetilde{G}$. Let $F^*$ be the map induced on $T^1M=\Gamma\backslash T^1\widetilde{M}$ by $\widetilde{F}^*$. Note that for all $x,y\in\widetilde{M}$ and $\gamma\in\Gamma$, by $\Gamma$-invariance of $\widetilde{G}$,

$$\Big|\int_x^{\gamma y}\widetilde{F}-\int_x^{\gamma y}\widetilde{F}^*\Big| = \big|\widetilde{G}(\phi_{d(x,\gamma y)}v)-\widetilde{G}(v)\big| \le 2\sup_{w\,\in\,T^1_x\widetilde{M}\,\cup\,T^1_y\widetilde{M}}|\widetilde{G}(w)|\;,$$

where $v$ is a unit tangent vector such that $\pi(v)=x$ and $\pi(\phi_{d(x,\gamma y)}v)=\gamma y$. Hence $F$ and $F^*$ have the same critical exponent:

$$\delta_{\Gamma,\,F^*}=\delta_{\Gamma,\,F}\;.$$

Furthermore, $(\Gamma,F^*)$ is of divergence type if and only if $(\Gamma,F)$ is of divergence type.

### 3.3 The Gibbs cocycle of $(\Gamma,F)$

By the Hölder-continuity of $F$ and the properties of asymptotic geodesic rays in $\widetilde{M}$, there exists a well-defined map $C_F:\partial_\infty\widetilde{M}\times\widetilde{M}\times\widetilde{M}\to\mathbb{R}$, called the *Gibbs cocycle* for the potential $F$, defined by

$$(\xi, x, y) \mapsto C_{F,\xi}(x,y) = \lim_{t\to+\infty} \int_y^{\xi_t} \widetilde{F} - \int_x^{\xi_t} \widetilde{F}$$

for $t \mapsto \xi_t$ any geodesic ray ending at $\xi$. Do note the apparent order reversal of $x$ and $y$ in this formula, which allows us to have the required sign in front of the Busemann cocycle in Equation (24). We will compare our cocycle with the one introduced by Hamenstädt in [Ham2] at the end of Subsection 3.5.

If $x$ belongs to the geodesic ray from $y$ to $\xi$, then

$$C_{F,\xi}(x,y) = \int_y^x \widetilde{F} \; . \tag{22}$$

In particular, for every $w \in T^1\widetilde{M}$, for all $x$ and $y$ on the image of the geodesic line defined by $w$, with $w_-, x, y, w_+$ in this order, we have

$$C_{F\circ\iota,\, w_-}(x,y) = \int_y^x \widetilde{F}\circ\iota = \int_x^y \widetilde{F} = C_{F,\, w_+}(y,x) = -C_{F,\, w_+}(x,y) \; . \tag{23}$$

Note that when $F = -1$, the Gibbs cocycle equals the Busemann cocycle (defined at the end of Subsection 2.2)

$$\beta_\xi(x,y) = \lim_{t\to+\infty} d(x,\xi_t) - d(y,\xi_t)$$

with $t \mapsto \xi_t$ as above. Hence, for every $s \in \mathbb{R}$,

$$C_{F-s,\,\xi}(x,y) = C_{F,\,\xi}(x,y) + s\;\beta_\xi(x,y) \; . \tag{24}$$

The Gibbs cocycle satisfies the following cocycle property: for all $\xi$ in $\partial_\infty\widetilde{M}$ and $x, y, z$ in $\widetilde{M}$,

$$C_{F,\xi}(x,z) = C_{F,\xi}(x,y) + C_{F,\xi}(y,z) \;\text{ and }\; C_{F,\xi}(x,y) = -\,C_{F,\xi}(y,x) \; , \tag{25}$$

and the following invariance property: for all $\gamma$ in $\Gamma$, $\xi$ in $\partial_\infty\widetilde{M}$ and $x, y$ in $\widetilde{M}$,

$$C_{F,\,\gamma\xi}(\gamma x, \gamma y) = C_{F,\xi}(x,y) \; . \tag{26}$$

**Remarks.** (1) If $\widetilde{F}^* : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map, which is cohomologous to $\widetilde{F}$, then the cocycle $C_{F^*} - C_F$ is a coboundary. Indeed, and more precisely, for all $x, y \in \widetilde{M}$, for every $\xi \in \partial_\infty\widetilde{M}$, if $\widetilde{F}^*$ is cohomologous to $\widetilde{F}$ via the map $\widetilde{G} : T^1\widetilde{M} \to \mathbb{R}$, if $v_{z\xi}$ is the tangent vector at the origin of the geodesic ray from a point $z \in \widetilde{M}$ to $\xi$, then

$$C_{F^*,\xi}(x,y) - C_{F,\xi}(x,y) = \widetilde{G}(v_{x\xi}) - \widetilde{G}(v_{y\xi}) \; . \tag{27}$$

In particular, $C_{F^*} - C_F$ is bounded on $\partial_\infty\widetilde{M} \times \widetilde{M} \times \widetilde{M}$ if $\widetilde{G}$ is bounded.

(2) Note that the Gibbs cocycle $C_F$ entirely determines the potential function $\widetilde{F}$, since for every $v \in T^1\widetilde{M}$, we have

$$\widetilde{F}(v) = \lim_{t\to 0^+} \frac{1}{t}\, C_{F,\,v_+}(\pi(v), \pi(\phi_t v)) \; .$$

(3) For every $x_0 \in \widetilde{M}$, the map $(\xi, \gamma) \mapsto C_{F,\,\xi}(x_0, \gamma x_0)$ from $\Lambda\Gamma \times \Gamma$ to $\mathbb{R}$ determines, up to a cohomologous potential, the restriction of $\widetilde{F}$ to $\widetilde{\Omega}\Gamma$.

Indeed, let $y_0$ be the closest point to $x_0$ on the translation axis of a loxodromic element $\gamma \in \Gamma$. Let $\gamma_+$ be the attractive fixed point of $\gamma$. By the equations (25), (26) and (22), we have

$$C_{F,\,\gamma^+}(x_0, \gamma x_0) = C_{F,\,\gamma^+}(x_0, y_0) + C_{F,\,\gamma^+}(y_0, \gamma y_0) + C_{F,\,\gamma^+}(\gamma y_0, \gamma x_0) = \int_{y_0}^{\gamma y_0} \widetilde{F}\;.$$

Hence

$$\mathrm{Per}_F(\gamma) = C_{F,\,\gamma^+}(x_0, \gamma x_0)\;.$$

The result now follows from Remark 3.1.

**Lemma 3.4** *For every $r_0 > 0$, there exist $c_1, c_2, c_3, c_4 > 0$ with $c_2, c_4 \leq 1$ (depending only on $r_0$, the Hölder constants of $\widetilde{F}$ and the bounds on the sectional curvature of $\widetilde{M}$) such that the following assertions hold.*

*(1) For all $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty \widetilde{M}$,*

$$|\, C_{F,\,\xi}(x, y)\,| \leq\; c_1\, e^{d(x,\,y)} \;+\; d(x, y) \max_{\pi^{-1}(B(x,\,d(x,\,y)))} |\widetilde{F}|\;,$$

*and if furthermore $d(x, y) \leq r_0$, then*

$$|\, C_{F,\,\xi}(x, y)\,| \leq c_1\, d(x, y)^{c_2} + d(x, y) \max_{\pi^{-1}(B(x,\,d(x,\,y)))} |\widetilde{F}|\;.$$

*(2) For every $r \in [0, r_0]$, for all $x, y'$ in $\widetilde{M}$, for every $\xi$ in the shadow $\mathscr{O}_x B(y', r)$ of the ball $B(y', r)$ seen from $x$, we have*

$$\Big|\; C_{F,\,\xi}(x, y') + \int_x^{y'} \widetilde{F}\;\Big| \leq c_3\; r^{c_4} + 2r \max_{\pi^{-1}(B(y',\,r))} |\widetilde{F}|\;.$$

**Proof.** (1) This follows from Lemma 3.2 by letting $z$ tend to $\xi$ along a geodesic ray.

(2) Let $t \mapsto \xi_t$ be the geodesic ray starting from $x$ and ending at $\xi$, and let $p$ be the closest point on it to $y'$, which satisfies $d(y', p) \leq r$. Since $\iota$ is Lipschitz for $d'_{T^1M}$ (see the paragraph before Lemma 2.3) with constant depending only on the bounds on the sectional curvature, the assertion (2) follows from the second assertion of Lemma 3.2, using

$$\int_x^{\xi_t} \widetilde{F} - \int_{y'}^{\xi_t} \widetilde{F} - \int_x^{y'} \widetilde{F} = \Big(\int_p^x \widetilde{F}\circ\iota - \int_{y'}^x \widetilde{F}\circ\iota\Big) + \Big(\int_p^{\xi_t} \widetilde{F} - \int_{y'}^{\xi_t} \widetilde{F}\Big)\,.\quad \Box$$

**Remark.** By Remark (1) at the end of Subsection 3.1, when $x, y \in \mathscr{C}\Lambda\Gamma$ and $\xi \in \Lambda\Gamma$, we may replace $\max_{\pi^{-1}(B(x,\,d(x,y)))} |\widetilde{F}|$ by $\max_{\pi^{-1}(B(x,\,d(x,y)) \cap \mathscr{C}\Lambda\Gamma)} |\widetilde{F}|$ in Assertion (1) of the above lemma.

**Proposition 3.5** *(1) The map $C_F : \partial_\infty \widetilde{M} \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ is continuous.*

*(2) Assume that $\widetilde{F}$ is bounded. Then $C_F$ is locally Hölder-continuous, and the maps $\xi \mapsto C_{F,\,\xi}(x, y)$ for all $x, y \in \widetilde{M}$ and $(x, y) \mapsto C_{F,\,\xi}(x, y)$ for every $\xi \in \partial_\infty \widetilde{M}$ are Hölder-continuous.*

In particular, the Busemann cocycle (obtained by taking $F = -1$) is locally Hölder-continuous.

If we assume only that $\widetilde{F}$ is bounded on $\mathscr{C}\Lambda\Gamma$, then the same proof gives that the restriction of $C_F$ to $\Lambda\Gamma \times \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ is locally Hölder-continuous.

**Proof.** Let $x, y, x', y' \in \widetilde{M}$ and $\xi, \xi' \in \partial_\infty \widetilde{M}$.

(1) For every $t \geq 0$, let $x_t$ be the point at distance $t$ on the geodesic ray $[x, \xi[$. Let $y_t$, $x'_t$ and $y'_t$ be the closest points to $x_t$ on respectively $[y, \xi[$, $[x', \xi'[$ and $[y', \xi'[$. By (a version with $z \in \partial_\infty \widetilde{M}$ of) Lemma 2.5 (i), there exists $c'_1 > 0$ (depending only on $d(x, y)$) such that $d(x_t, y_t) \leq c'_1\, e^{-t}$.

By the linear growth property of Hölder-continuous functions (see the end of Subsection 2.1), there exists $c'_2 > 0$ (depending only on $\widetilde{F}(x)$ and the Hölder constants of $\widetilde{F}$) such that $\max_{\pi^{-1}(B(x_t,2))} |\widetilde{F}| \leq c'_2(t+1)$ for every $t \geq 0$.

Let $\epsilon > 0$. Let $c_1, c_2$ be the positive constants so that Lemma 3.2 and Lemma 3.4 (1) hold true for $r_0 = 1$. Let us fix $t$ large enough (which depends only on $\epsilon$, $d(x, y)$, $\widetilde{F}(x)$ and the Hölder constants of $\widetilde{F}$), so that $2\, c'_1\, e^{-t} \leq 1$ and $c_1(2\, c'_1\, e^{-t})^{c_2} + 2\, c'_1\, e^{-t}\, c'_2(t + 1) \leq \epsilon$.

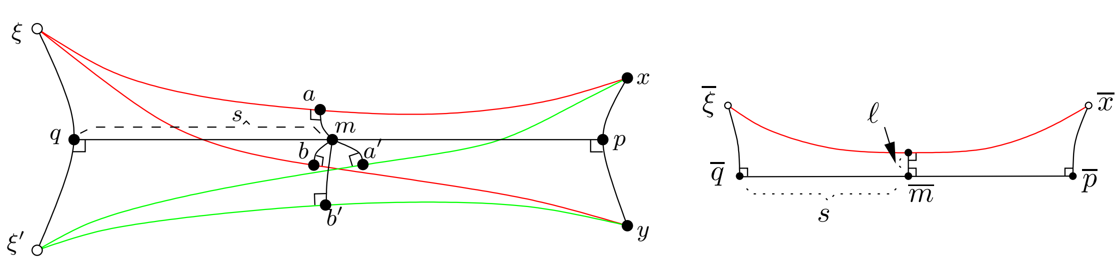
By the continuity property of geodesic rays and closest point maps, if $\xi'$, $x'$ and $y'$ are close enough to $\xi$, $x$ and $y$ respectively, we have $d(x, x'), d(y, y'), d(x_t, x'_t), d(y_t, y'_t) \leq c'_1\, e^{-t}$ and $d(x'_t, y'_t) \leq 2\, c'_1\, e^{-t}$. Hence, using the cocycle property (25) and Equation (22) for the following equality,

$$
\begin{aligned}
&\mid C_{F,\,\xi'}(x', y') - C_{F,\,\xi}(x, y) \mid \\
&= \Big| \int_{y'}^{y'_t} \widetilde{F} + C_{F,\,\xi'}(x'_t, y'_t) - \int_{x'}^{x'_t} \widetilde{F} - \int_{y}^{y_t} \widetilde{F} - C_{F,\,\xi}(x_t, y_t) + \int_{x}^{x_t} \widetilde{F} \Big| \\
&\leq \Big| \int_{y'}^{y'_t} \widetilde{F} - \int_{y}^{y_t} \widetilde{F} \Big| + \Big| \int_{x'}^{x'_t} \widetilde{F} - \int_{x}^{x_t} \widetilde{F} \Big| + \mid C_{F,\,\xi'}(x'_t, y'_t) \mid + \mid C_{F,\,\xi}(x_t, y_t) \mid .
\end{aligned}
$$

Then by the definition of $t$ and by Lemma 3.2 and Lemma 3.4 (1), we have $\mid C_{F,\,\xi'}(x', y') - C_{F,\,\xi}(x, y) \mid \leq 6\,\epsilon$, and Assertion (1) follows.

(2) To prove Assertion (2), let us first prove that there exist constants $c > 0$ and $\alpha \in\, ]0, 1[$ (depending only on $d(x, y)$, on the Hölder constants of $\widetilde{F}$, on the upper bound of $|\widetilde{F}|$ and on the bounds on the sectional curvature of $\widetilde{M}$) such that if $d_x(\xi, \xi') \leq e^{-d(x,y)-2}$, then

$$
|C_{F,\,\xi'}(x, y) - C_{F,\,\xi}(x, y)| \leq c\; d_x(\xi, \xi')^\alpha \; .
$$




Let $[p, q]$ be the shortest arc between $[x, y]$ and $]\xi, \xi'[$ , with $p \in [x, y]$. Let $m$ be the midpoint of $[p, q]$ and $s = d(p, m) = d(m, q) = \frac{1}{2} d(p, q)$. Let $a, b, a', b'$ be the closest points to $m$ on respectively $[x, \xi[$ , $[y, \xi[$ , $[x, \xi'[$ , $[y, \xi'[$ .

Since $q$ is the closest point to $p$ on $]\xi, \xi'[$, and by comparison, the distance from $q$ to $[p, \xi[$ and to $[p, \xi'[$ is at most $\log(1+\sqrt{2})$. Hence by the triangle inequality and the definition of the visual distance (see Equation (1)), we have

$$e^{-2s} \leq d_p(\xi, \xi') \leq e^{-2s+2\log(1+\sqrt{2})} \ .$$

By the properties of the visual distances (see Equation (2)), we have

$$e^{-d(x,\,y)} \leq e^{-d(x,\,p)} \leq \frac{d_x(\xi, \xi')}{d_p(\xi, \xi')} \leq e^{d(x,\,p)} \leq e^{d(x,\,y)} \ .$$

In particular,

$$s \geq -\frac{1}{2} \log d_p(\xi, \xi') \geq \frac{1}{2}(-\log d_x(\xi, \xi') - d(x, y)) \geq 1 \ .$$

By comparison, since the angle at $q$ between $p$ and $\xi$ is $\frac{\pi}{2}$, and since, if $p \neq x$, the angle at $p$ between $q$ and $x$ is at least $\frac{\pi}{2}$ (and exactly $\frac{\pi}{2}$ if $p \neq y$), the distance from $m$ to $a$ is at most the distance $\ell$ in the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ between the midpoint $\overline{m}$ of a segment $[\overline{p}, \overline{q}]$ of length $2s$ to a geodesic line between $\overline{\xi}, \overline{x} \in \partial_\infty \mathbb{H}^2_{\mathbb{R}}$ where the angle at $\overline{q}$ between $\overline{p}$ and $\overline{\xi}$ and the angle at $\overline{p}$ between $\overline{q}$ and $\overline{x}$ are exactly $\frac{\pi}{2}$. By a well known formula in hyperbolic geometry (see for instance [Bea, page 157]), we have $\sinh \ell = \frac{1}{\sinh s}$. Note that for all $s \geq 1$, we have $\sinh s \geq \frac{e^s}{4}$. Hence

$$d(a, m) \leq \ell \leq \sinh \ell = \frac{1}{\sinh s} \leq 4\, e^{-s} \leq 4\, d_p(\xi, \xi')^{\frac{1}{2}} \leq 4\, e^{\frac{d(x,\,y)}{2}}\, d_x(\xi, \xi')^{\frac{1}{2}} \ .$$

Since the computations are unchanged by permuting $x$ and $y$, as well as $\xi$ and $\xi'$, we have

$$\max\{d(a, m),\, d(b, m),\, d(a', m),\, d(b', m)\} \leq 4\, e^{\frac{d(x,\,y)}{2}}\, d_x(\xi, \xi')^{\frac{1}{2}} \ .$$

Hence by the triangle inequality,

$$\max\{d(a, b),\, d(a', b'),\, d(a, a'),\, d(b, b')\} \leq 8\, e^{\frac{d(x,\,y)}{2}}\, d_x(\xi, \xi')^{\frac{1}{2}} \leq 8\, e^{\frac{d(x,\,y)}{2}} \ . \tag{28}$$

Now, using respectively

- the cocycle property (25),
- Equation (22),
- the triangle inequality,
- Lemma 3.2 and Lemma 3.4 with $r_0 = 8\, e^{\frac{d(x,\,y)}{2}}$, for some constants $c_1 > 0$ and $c_2 \in\, ]0, 1[$ (depending only on $r_0$, the Hölder constants of $\widetilde{F}$, and the bounds on the sectional curvature of $\widetilde{M}$) and $\kappa = \sup_{\widetilde{M}} |\widetilde{F}|$,
- Equation (28),

we have

$$
\begin{aligned}
&|C_{F,\,\xi'}(x,y)-C_{F,\,\xi}(x,y)|\\
&=|-C_{F,\,\xi'}(a',x)+C_{F,\,\xi'}(a',b')+C_{F,\,\xi'}(b',y)+C_{F,\,\xi}(a,x)-C_{F,\,\xi}(a,b)-C_{F,\,\xi}(b,y)|\\
&=\Big|-\int_x^{a'}\widetilde{F}+C_{F,\,\xi'}(a',b')+\int_y^{b'}\widetilde{F}+\int_x^{a}\widetilde{F}-C_{F,\,\xi}(a,b)-\int_y^{b}\widetilde{F}\Big|\\
&\leq\Big|\int_x^{a}\widetilde{F}-\int_x^{a'}\widetilde{F}\;\Big|+|C_{F,\,\xi'}(a',b')|+|C_{F,\,\xi}(a,b)|+\Big|\int_y^{b}\widetilde{F}-\int_y^{b'}\widetilde{F}\;\Big|\\
&\leq c_1\,d(a,a')^{c_2}+\kappa\,d(a,a')+c_1\,d(a',b')^{c_2}+\kappa\,d(a',b')\\
&\qquad+c_1\,d(a,b)^{c_2}+\kappa\,d(a,b)+c_1\,d(b,b')^{c_2}+\kappa\,d(b,b')\\
&\leq 8^{1+c_2}\max\{c_1,\kappa\}\;e^{\frac{c_2\,d(x,y)}{2}}\;d_x(\xi,\xi')^{\frac{c_2}{2}}\;.
\end{aligned}
$$

This proves the claim made at the beginning of this proof, as well as the second claim in the statement of Proposition 3.5 (2).

Now, by Equation (25), we have

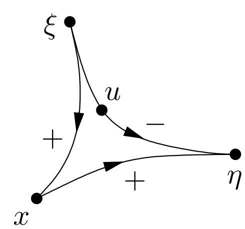
$$|C_{F,\,\xi'}(x',y')-C_{F,\,\xi}(x,y)|\leq|C_{F,\,\xi'}(x,y)-C_{F,\,\xi}(x,y)|+|C_{F,\,\xi'}(x,x')|+|C_{F,\,\xi'}(y,y')|\;.$$

Using the initial claim and again Lemma 3.4 (1), the result follows. □

### 3.4 The potential gap of $(\Gamma, F)$

For all $x\in\widetilde{M}$ and $(\xi,\eta)\in\partial^2_\infty\widetilde{M}$, let us define the $F$*-gap seen from* $x$ between $\xi$ and $\eta$ by

$$D_{F,\,x}(\xi,\eta)=\exp\frac{1}{2}\Big(\lim_{t\to+\infty}\int_x^{\eta_t}\widetilde{F}-\int_{\xi_t}^{\eta_t}\widetilde{F}+\int_{\xi_t}^{x}\widetilde{F}\Big)\;,$$




where $t\mapsto\xi_t$, $t\mapsto\eta_t$ are any geodesic rays ending at $\xi$ and $\eta$ respectively (see the picture on the right). By the Hölder-continuity of $F$ and the properties of asymptotic geodesic rays in $\widetilde{M}$, the limit does exist and is independent of the choices.

We denote by

$$D_F:\widetilde{M}\times\partial^2_\infty\widetilde{M}\to\mathbb{R}$$

the map defined by $(x,\xi,\eta)\mapsto D_{F,\,x}(\xi,\eta)$, which we will call the *gap map* of the potential $F$. We will compare our map $D_F$ with the analogous one introduced by Hamenstädt in [Ham2] at the end of Subsection 3.5. When $F$ is the constant function with value $-1$, then $D_{F,\,x}(\xi,\eta)=d_x(\xi,\eta)$, where $d_x$ is the visual distance on $\partial_\infty\widetilde{M}$ seen from $x$ (see Subsection 2.2). So that for every $s\in\mathbb{R}$, we have

$$D_{F-s,\,x}(\xi,\eta)=D_{F,\,x}(\xi,\eta)\;d_x(\xi,\eta)^s\;.$$

The gap map $D_F$ satisfies the following elementary properties.

**Lemma 3.6** *(1) For any point $u$ on the geodesic line between $\xi$ and $\eta$, we have*

$$D_{F,\,x}(\xi,\eta)=e^{-\frac{1}{2}(C_{F,\,\eta}(x,u)+C_{F\circ\iota,\,\xi}(x,u))}\;. \tag{29}$$

*For every* $\gamma\in\Gamma$,

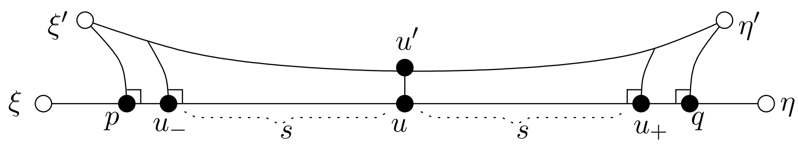
$$D_{F,\,\gamma x}(\gamma\xi,\gamma\eta)=D_{F,\,x}(\xi,\eta)\quad\text{and}\quad D_{F,\,x}(\eta,\xi)=D_{F\circ\iota,\,x}(\xi,\eta)\;. \tag{30}$$

*(2) If* $\widetilde{F}$ *is bounded, the gap map* $D_F:\widetilde{M}\times\partial^2_\infty\widetilde{M}\to\mathbb{R}$ *is locally Hölder-continuous.*

**Proof.** (1) This is immediate, using Equation (16).

(2) Let us first prove that there exists $c>0$ such that for all $(\xi,\eta),(\xi',\eta')\in\partial^2_\infty\widetilde{M}$ and for every point $u$ on the geodesic line $(\xi\eta)$ between $\xi$ and $\eta$, if $d_u(\xi,\xi'),d_u(\eta,\eta')\le e^{-2}$, then there exists $u'$ on the geodesic line $(\xi'\eta')$ such that

$$d(u,u')\le c\;\max\{d_u(\xi,\xi'),d_u(\eta,\eta')\}\;.$$




Define

$$s=\min\{-\log d_u(\xi,\xi'),-\log d_u(\eta,\eta')\}-\log(1+\sqrt{2})\ge 1\;.$$

Let $p$ be the closest point to $\xi'$ on the geodesic ray $[u,\xi[$, which by comparison is at distance at most $\log(1+\sqrt{2})$ from the geodesic line $(\xi\xi')$. By the triangle inequality and the definition of the visual distance (see Equation (1)), we have $d(u,p)\ge-\log d_u(\xi,\xi')-\log(1+\sqrt{2})$. Similarly, the closest point $q$ to $\eta'$ on $[u,\eta[$ satisfies $d(u,q)\ge-\log d_u(\eta,\eta')-\log(1+\sqrt{2})$. By convexity, the point $u_-$ at distance $s$ from $u$ on $[u,\xi[$ and the point $u_+$ at distance $s$ from $u$ on $[u,\eta[$ are the orthogonal projections of points on the extended geodesic line $[\xi',\eta']$. By comparison (as in the proof of Proposition 3.5 (2) using the formulae of [Bea, page 157]), the point $u$ is at distance at most $4\,e^{-s}$ from the geodesic line $(\xi'\eta')$. The initial claim follows.

Now, for all $x,x'\in\widetilde{M}$ and $(\xi,\eta),(\xi',\eta')\in\partial^2_\infty\widetilde{M}$, fix $u$ on the geodesic line $(\xi\eta)$, and let $u'$ be the point defined above, which exists if $\xi'$ and $\eta'$ are close enough to $\xi$ and $\eta$ respectively. By Equation (29), we have

$$|D_{F,\,x}(\xi,\eta)-D_{F,\,x'}(\xi',\eta')|=|e^{-\frac{1}{2}(C_{F,\,\eta}(x,u)+C_{F\circ\iota,\,\xi}(x,u))}-e^{-\frac{1}{2}(C_{F,\,\eta'}(x',u')+C_{F\circ\iota,\,\xi'}(x,u'))}|\;.$$

Assertion (2) of Lemma 3.6 then follows from Proposition 3.5 (2). □

**Remarks.** (1) Let $\widetilde{F}':T^1\widetilde{M}\to\mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map, which is cohomologous to $\widetilde{F}$ via the map $\widetilde{G}:T^1\widetilde{M}\to\mathbb{R}$. Then for every $x\in\widetilde{M}$, for all $\xi,\eta\in\partial_\infty\widetilde{M}$, it follows from the equations (29) and (27), and from the fact that the potentials $\widetilde{F}'\circ\iota$ and $\widetilde{F}\circ\iota$ are cohomologous via the map $-\widetilde{G}\circ\iota$, that

$$D_{F',\,x}(\xi,\eta)=D_{F,\,x}(\xi,\eta)\;e^{\frac{1}{2}(\widetilde{G}(v_{x\xi})-\widetilde{G}\circ\iota(v_{x\eta}))}\;. \tag{31}$$

In particular, if $F$ and $F\circ\iota$ are cohomologous, then by Equation (30), the ratio $\frac{D_{F,\,x}(\eta,\xi)}{D_{F,\,x}(\xi,\eta)}$ is uniformly bounded in $(\eta,\xi)$, and also in $(x,\eta,\xi)$ if the function $\widetilde{G}$ such that $\widetilde{F}$ and $\widetilde{F}\circ\iota$ are cohomologous via $\widetilde{G}$ is bounded.

(2) By Equation (29), by the cocycle property of $C_F$ and by Lemma 3.4 (1), for all $x, y$ in $\widetilde{M}$, there exists a constant $c_{x,\,y} > 0$ such that, for all $\xi, \eta \in \partial_\infty \widetilde{M}$, we have

$$\frac{1}{c_{x,\,y}} \leq \frac{D_{F,\,x}(\xi, \eta)}{D_{F,\,y}(\xi, \eta)} \leq c_{x,\,y}\ .$$

(3) By the thin triangles property of CAT$(-1)$ spaces, if $p$ is the closest point to $x$ on the geodesic line between $\xi$ and $\eta$ and $r = \log(1+\sqrt{2})$, we have $\eta, \xi \in \mathscr{O}_x B(p, r)$. Hence by Lemma 3.4 (2), if $\widetilde{F}$ is bounded, there exists a constant $c_5 \geq 1$ (depending only on $\|\widetilde{F}\|_\infty$ and on the Hölder constants of $\widetilde{F}$) such that, for all $x \in \widetilde{M}$ and $\xi, \eta \in \partial_\infty \widetilde{M}$, we have

$$\frac{1}{c_5} \leq \frac{D_{F,\,x}(\xi, \eta)}{e^{\frac{1}{2}(\int_x^p \widetilde{F} + \int_x^p \widetilde{F}\circ\iota)}} \leq c_5\ . \tag{32}$$

These inequalities are still satisfied, when $\widetilde{F}$ is only assumed to be bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, if $x \in \mathscr{C}\Lambda\Gamma$ and $\xi, \eta \in \Lambda\Gamma$.

(4) Let us prove in this remark that, under some assumptions on the potential, a minor modification of the gap map $D_F$ yields a distance on $\Lambda\Gamma$ (see also [Sch1, Section 2.6]).

**Lemma 3.7** *Let $x \in \mathscr{C}\Lambda\Gamma$. Assume that $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$. Assume that $\widetilde{F}$ is reversible or that there exists a constant $c > 0$ such that, for all $y, z \in \mathscr{C}\Lambda\Gamma$ such that $y \in [x, z]$, we have*

$$\int_x^z \widetilde{F} \leq c + \int_x^y \widetilde{F}\ . \tag{33}$$

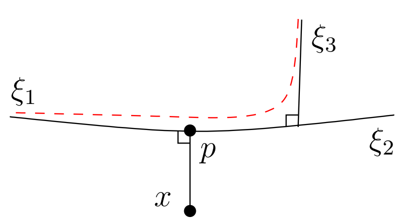
*Then for every $\epsilon > 0$ small enough, there exist a distance $d_{F,\,x,\,\epsilon}$ on $\Lambda\Gamma$ and $c_\epsilon > 0$ such that for all $\xi, \eta \in \Lambda\Gamma$, we have*

$$\frac{1}{c_\epsilon}\, D_{F,\,x}(\xi, \eta)^\epsilon \leq d_{F,\,x,\,\epsilon}(\xi, \eta) \leq c_\epsilon\, D_{F,\,x}(\xi, \eta)^\epsilon\ . \tag{34}$$

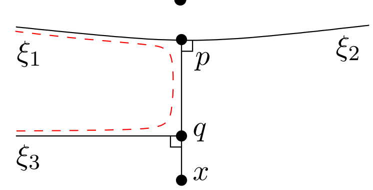
We will give, in the proof of Proposition 3.13, conditions on $(\Gamma, F)$ implying that the hypothesis (33) is satisfied when $F$ is a normalised potential. Note that the hypothesis (33) does hold when $\widetilde{F} \leq 0$ (for any $\Gamma$), as for instance when $\widetilde{F} = \widetilde{F}^{\rm su}$ (see Chapter 7).

**Proof.** By the techniques of approximation by trees in CAT$(-1)$-spaces (see for instance [GdlH, §2.2]), there exists a universal constant $c' > 0$ such that for all $x \in \widetilde{M}$ and $\xi_1, \xi_2, \xi_3 \in \partial_\infty \widetilde{M}$ such that $\xi_1 \neq \xi_2$, if $p$ is the closest point to $x$ on the geodesic line between $\xi_1$ and $\xi_2$, then at least one of the two following cases holds:

- the distance between $p$ and the closest point to $x$ on the geodesic line between $\xi_3$ and one of $\xi_1, \xi_2$ is at most $c'$,




- or the distance between the closest point $q$ to $\xi_3$ on the segment $[x, p]$ and the closest point to $x$ on the geodesic line between $\xi_3$ and $\xi_1$ is at most $c'$.



Note that $p$, as well as $q$ in the second case, is contained in $\mathscr{C}\Lambda\Gamma$ if $\xi_1, \xi_2, \xi_3 \in \Lambda\Gamma$.

Note that in the second case, if $\widetilde{F}$ and $\widetilde{F} \circ \iota$ are cohomologous via $\widetilde{G}$, with $v_{p,\,q}$ a unit tangent vector at $p$ to the geodesic segment $[p, q]$ (unique if $p \neq q$, with $v_{q,\,p} = -v_{p,\,q}$ if $p = q$), we have, for every $\gamma \in \Gamma$, and by the $\Gamma$-invariance of $\widetilde{G}$,

$$\Big|\int_x^{\gamma x} \widetilde{F} - \int_{\gamma x}^{x} \widetilde{F}\,\Big| = \Big|\int_x^{\gamma x} (\widetilde{F} - \widetilde{F} \circ \iota)\Big| = |\,\widetilde{G}(-v_{\gamma x,\, x}) - \widetilde{G}(v_{x,\,\gamma x})| \leq 2 \max_{T^1_x \widetilde{M}} |\,\widetilde{G}\,|\,. \tag{35}$$

Note that Formula (32) is also valid, up to enlarging $c_5$, if $p$ is replaced by any point $p'$ at distance less than $c'$ from $p$. Indeed, with $F' = F$ or $F' = F \circ \iota$, by Lemma 3.2, the quantity $\big|\int_x^{p'} \widetilde{F}' - \int_x^{p} \widetilde{F}'\big|$ is bounded by a constant depending only on $c'$ and $\kappa = \sup_{\pi^{-1}(\mathscr{N}_{c'}(\mathscr{C}\Lambda\Gamma))} |\,\widetilde{F}\,|$. Furthermore, $\kappa$ is finite since $|\,\widetilde{F}\,|$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$ and has at most linear growth by the remark at the end of Subsection 2.1.

Hence, by Equation (32), considering the two above cases (the first one being easier) and using in the second case either Formula (35) or Formula (33), we see that there exists a constant $c_6$ such that for all $\xi_1, \xi_2, \xi_3 \in \Lambda\Gamma$,

$$D_{F,\,x}(\xi_1, \xi_2) \leq c_6 \max\{D_{F,\,x}(\xi_1, \xi_3), D_{F,\,x}(\xi_2, \xi_3)\}\,.$$

The constant $c_6$ depends on $\sup_{v \in \pi^{-1}(\mathscr{C}\Lambda\Gamma)} |\widetilde{F}(v)|$, on the Hölder constants of $\widetilde{F}$ and either on $\max_{v\,\in T^1_x\widetilde{M}} |\,\widetilde{G}\,|$ if $\widetilde{F}$ and $\widetilde{F} \circ \iota$ are assumed to be cohomologous via $\widetilde{G}$ or on the constant $c$ appearing in Equation (33). Note that, also by Formula (32),

$$D_{F,\,x}(\xi_1, \xi_2) \leq {c_5}^2 D_{F,\,x}(\xi_2, \xi_1)\,.$$

These weak ultrametric triangle inequality and weak symmetry imply (see for instance [GdlH, §7.3]) that there exists $c_7 \in \,]0, 1]$ (depending only on $c_5, c_6$) such that for every $\epsilon \in \,]0, c_7]$, there exist a distance $d_{F,\,x,\,\epsilon}$ on $\Lambda\Gamma$ and a constant $c_\epsilon > 0$ (depending only on $\epsilon, c_5, c_6$) such that for all $\xi, \eta \in \Lambda\Gamma$, we have

$$\frac{1}{c_\epsilon}\, D_{F,\,x}(\xi, \eta)^\epsilon \leq d_{F,\,x,\,\epsilon}(\xi, \eta) \leq c_\epsilon\, D_{F,\,x}(\xi, \eta)^\epsilon\,.$$

This proves Lemma 3.7. □

By the above Remark (2), for all $x, y \in \mathscr{C}\Lambda\Gamma$, the distances $d_{F,\,x,\,\epsilon}$ and $d_{F,\,y,\,\epsilon}$ are equivalent. Note that if $F$ is constant, equal to $-1$, then we may take $c_7 = 1$ and $d_{F,\,x,\,\epsilon} = {d_x}^\epsilon$, where $d_x$ is the visual distance on $\partial_\infty \widetilde{M}$ (see for instance [Bou]).

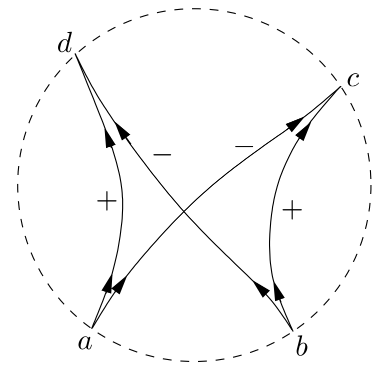
Note that if in the assumptions of Lemma 3.7, we replace $\mathscr{C}\Lambda\Gamma$ by $\widetilde{M}$, then we get a distance $d_{F,\,x,\,\epsilon}$ on the full boundary $\partial_\infty \widetilde{M}$ instead of just $\Lambda\Gamma$, that satisfies Equation (34).

### 3.5 The crossratio of $(\Gamma, F)$

Let $\partial^4_\infty \widetilde{M}$ be the space of pairwise distinct quadruples of points in $\partial_\infty \widetilde{M}$. The *crossratio* of the potential $F$ is the map from $\partial^4_\infty \widetilde{M}$ to $\mathbb{R}$ defined by

$$[a, b, c, d]_F = \exp\, \frac{1}{2} \lim_{t \to +\infty} \Big(\int_{a_t}^{d_t} \widetilde{F} - \int_{b_t}^{d_t} \widetilde{F} + \int_{b_t}^{c_t} \widetilde{F} - \int_{a_t}^{c_t} \widetilde{F}\Big)\,,$$

where $t \mapsto a_t, t \mapsto b_t, t \mapsto c_t, t \mapsto d_t$ are geodesic rays converging to $a, b, c, d$ respectively (see the figure on the right). The limit does exist and is independent of the choices of these geodesic rays, again by the Hölder-continuity of $F$ and the properties of asymptotic geodesic rays in $\widetilde{M}$. An easy cancellation argument shows that, for every $x \in \widetilde{M}$,




$$[a, b, c, d]_F = \frac{D_{F,x}(a, c) D_{F,x}(b, d)}{D_{F,x}(a, d) D_{F,x}(b, c)} . \qquad (36)$$

In particular, the right hand side of this equation does not depend on $x$.

We summarise the elementary properties of the crossratio of $F$ in the next statement.

**Lemma 3.8** *(i) The crossratio of $F$ is a positive map from $\partial^4_\infty \widetilde{M}$ to $\mathbb{R}$ such that, for every $(a, b, c, d) \in \partial^4_\infty \widetilde{M}$,*

*(1)* ($\Gamma$-invariance) *for every $\gamma \in \Gamma$, we have $[\gamma a, \gamma b, \gamma c, \gamma d]_F = [a, b, c, d]_F$;*

*(2) $[a, b, c, d]_F = [b, a, d, c]_F$;*

*(3) $[a, b, c, d]_F = {[a, b, d, c]_F}^{-1}$;*

*(4) $[a, b, c, d]_{F\circ\iota} = [c, d, a, b]_F$;*

*(5) $[a, b, c, d]_F = [a, b, c, \xi]_F \; [a, b, \xi, d]_F$ for every $\xi \in \partial_\infty \widetilde{M} - \{a, b, c, d\}$.*

*(ii) If the potential $F$ is bounded, then its crossratio is locally Hölder-continuous.*

*(iii) If $\widetilde{F}' : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map, cohomologous to $\widetilde{F}$, then their associated crossratios coincide:*

$$[a, b, c, d]_F = [a, b, c, d]_{F'} .$$

*(iv) For every loxodromic element $\gamma$ in $\Gamma$, with attractive and repulsive fixed points $\gamma_+$ and $\gamma_-$ respectively on $\partial_\infty \widetilde{M}$, we have*

$$\lim_{n \to +\infty} \frac{1}{n} \log [\gamma_-, \gamma^n \xi, \xi, \gamma_+]_F = \frac{1}{2}\big(\operatorname{Per}_F \gamma + \operatorname{Per}_F(\gamma^{-1})\big) ,$$

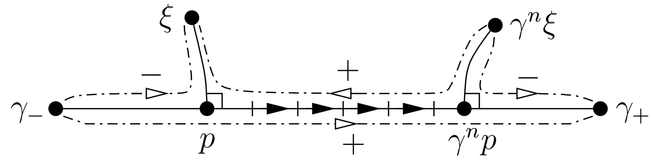
*uniformly for $\xi$ in a compact subset of $\partial_\infty \widetilde{M} - \{\gamma_-, \gamma_+\}$.*

Assertion (5) is a multiplicative cocycle property in the last two variables of the crossratio of $F$. Using (4) and (5), it follows that the crossratio of $F$ is also a multiplicative cocycle in the first two variables, that is

$$[a, b, c, d]_F = [a, \xi, c, d]_F \; [\xi, b, c, d]_F$$

for every $\xi \in \partial_\infty \widetilde{M} - \{a, b, c, d\}$.

**Proof.** The assertions (2), (3) and (4) follow easily from the definition of the crossratio of $F$. The equations (36) and (30) imply the assertions (1) and (5). Assertion (ii) follows from Equation (36) and Lemma 3.6 (2). The equations (36) and (31) imply Assertion (iii).




To prove the last assertion, let us define $p$ to be the closest point to $\xi$ on the translation axis $\mathrm{Axe}_\gamma$ of $\gamma$. Note that $\gamma^n p$ is then the closest point to $\gamma^n\xi$ on $\mathrm{Axe}_\gamma$. By hyperbolicity, the four geodesic lines along which one integrates $\widetilde{F}$ to define the crossratio are at bounded distance from the union of $\mathrm{Axe}_\gamma$ and of the geodesic rays $]\xi, p]$ and $]\gamma^n\xi, \gamma^n p]$. Furthermore, the segment between $p$ and $\gamma^n p$ stays very close to the geodesic line between $\xi$ and $\gamma^n\xi$ except for a segment of bounded length near each of its endpoints. By definition of the periods,

$$\int_p^{\gamma^n p} \widetilde{F} = n \,\mathrm{Per}_F\, \gamma \quad \text{and} \quad \int_{\gamma^n p}^{p} \widetilde{F} = \int_p^{\gamma^{-n} p} \widetilde{F} = n\, \mathrm{Per}_F(\gamma^{-1})$$

After dividing by $n$, the other contributions vanish as $n$ goes to $+\infty$, by Lemma 3.4 (1) and since $\widetilde{F}$ is bounded on the $\log(1+\sqrt{2})$-neighbourhood of $\mathrm{Axe}_\gamma$. The result follows. $\Box$

**Remarks.** (1) Let us compare our maps $C_{F,\xi}, D_{F,x}, [\cdot,\cdot,\cdot,\cdot]_F$ with those introduced in Hamenstädt's paper [Ham2] (which assumes that $\Gamma$ is torsion free, $M = \Gamma\backslash\widetilde{M}$ is compact and $F$ and $F\circ\iota$ are cohomologous, requirements we do not assume in this book). Let $\zeta : T^1\widetilde{M}\times\mathbb{R}\to\mathbb{R}$ be the map defined by

$$\zeta(v,t) = \int_0^t \widetilde{F}(\phi_s v)\, ds\ .$$

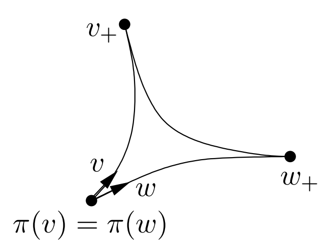
This map is locally Hölder-continuous, invariant under $\Gamma$ (acting trivially on $\mathbb{R}$) and satisfies $\zeta(v, s+t) = \zeta(v,s)+\zeta(\phi_s(v),t)$. Then Hamenstädt's map $k_\zeta : \widetilde{M}\times\widetilde{M}\times\partial_\infty\widetilde{M}\to\mathbb{R}$ defined in Lemma 1.1 in loc. cit. has the opposite sign to our Gibbs cocycle: for all $x, y\in\widetilde{M}$ and $\xi\in\partial_\infty\widetilde{M}$, we have

$$k_\zeta(x,y,\xi) = -C_{F,\xi}(x,y)\ . \tag{37}$$

Let

$$DT\widetilde{M} = \{(v,w)\in T^1\widetilde{M}\times T^1\widetilde{M}\ :\ \pi(v)=\pi(w)\ \text{ and }\ v\neq w\}\ .$$

Hamenstädt constructs (just above Lemma 1.2 in loc. cit.) a locally Hölder-continuous $\Gamma$-invariant map $\alpha_\zeta : DT\widetilde{M}\to\mathbb{R}$, and it may be proved using the equations (29) and (37) that when $(v,w)\in DT\widetilde{M}$ (see the picture on the right),




$$\alpha_\zeta(v,w) - \log D_{F,\pi(v)}(v_+, w_+)$$

is uniformly bounded (and even equal to 0 if $F\circ\iota = F$).

(2) Assume in this remark that $F$ and $F\circ\iota$ are cohomologous. It follows from the assertions (4), (iii) and (iv) of the above Lemma 3.8 and from the last equality in Equation (17) that the crossratio of $F$ then satisfies the extra symmetry

$$[a,b,c,d]_F = [c,d,a,b]_F \ \text{ for every }\ (a,b,c,d)\in\partial^4_\infty\widetilde{M}\ ,$$

and that

$$\lim_{n\to+\infty} \frac{1}{n} \log\,[\gamma_-, \gamma^n\xi, \xi, \gamma_+]_F = \operatorname{Per}_F \gamma\ .$$

In particular, the crossratio of $F$ then determines the periods of $F$. By Remark 3.1, the crossratio of the potential function $F$ determines the cohomology class of the restriction of $F$ to the (topological) non-wandering set $\Omega\Gamma$. When $\Gamma$ is torsion free and cocompact, this reproves one inclusion in [Ham2, Theo. A].

### 3.6 The Patterson densities of $(\Gamma, F)$

Let $\sigma \in \mathbb{R}$. A *Patterson density* of dimension $\sigma$ for $(\Gamma, F)$ is a family of finite nonzero (positive Borel) measures $(\mu_x)_{x\in\widetilde{M}}$ on $\partial_\infty\widetilde{M}$, such that, for every $\gamma\in\Gamma$, for all $x, y \in \widetilde{M}$, for every $\xi \in \partial_\infty\widetilde{M}$, we have

$$\gamma_*\mu_x = \mu_{\gamma x}\ , \tag{38}$$

$$d\mu_x(\xi) = e^{-C_{F-\sigma,\,\xi}(x,\,y)}\ d\mu_y(\xi)\ . \tag{39}$$

In particular, the measures $\mu_x$ for $x \in \widetilde{M}$ are in the same measure class. The Radon-Nikodym derivative $\frac{d\mu_x}{d\mu_y}$ is only defined almost everywhere, but by the continuity of the map $\xi \mapsto C_{F-\sigma,\,\xi}(x,\,y)$ (see Proposition 3.5 (1)), we may and we will take $\frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_{F-\sigma,\,\xi}(x,\,y)}$ for every $\xi \in \partial_\infty\widetilde{M}$.

Note that a Patterson density of dimension $\sigma$ for $(\Gamma, F)$ is also a Patterson density of dimension $\sigma + s$ for $(\Gamma, F + s)$, for every $s \in \mathbb{R}$. If $F = 0$, then we get the usual notion of a Patterson density for the group $\Gamma$, see for instance [Patt, Sul1, Nic, Bou, Rob1].

Proposition 3.5 (1) (or Lemma 3.4 (1)) implies that the map $x \mapsto \mu_x$ from $\widetilde{M}$ to the space of finite measures on $\partial_\infty\widetilde{M}$, endowed with the weak-star topology, is continuous, as for every $x \in \widetilde{M}$ and $\epsilon > 0$, if $y$ is close enough to $x$, then $\frac{d\mu_x}{d\mu_y} \in [e^{-\epsilon}, e^{\epsilon}]$ by Equation (39).

The support of the measure $\mu_x$ is independent of $x \in \widetilde{M}$ by Equation (39). It is a closed nonempty $\Gamma$-invariant subset of $\partial_\infty\widetilde{M}$ by Equation (38). Hence it contains the limit set $\Lambda\Gamma$ of $\Gamma$. We will be mostly interested in the case when the support of $\mu_x$ is equal to $\Lambda\Gamma$, as this is the case when $\sigma = \delta_{\Gamma,\,F} < +\infty$ and $(\Gamma, F)$ is of divergence type, by Corollary 5.12. We will prove in Corollary 5.14 that there exists a Patterson density of dimension $\sigma$ for $(\Gamma, F)$ whose support contains strictly $\Lambda\Gamma$ if and only if $\Lambda\Gamma \neq \partial_\infty\widetilde{M}$ and the Poincaré series $\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$ converges. The basic existence result is the following one.

**Proposition 3.9 (Patterson)** *If $\delta_{\Gamma,\,F} < \infty$, then there exists at least one Patterson density of dimension $\delta_{\Gamma,\,F}$ for $(\Gamma, F)$, with support exactly equal to $\Lambda\Gamma$.*

**Proof.** This is proved in [Moh] (see also [Ham2]), following Patterson's construction. Since we will need the construction, we recall it here.

Fix a point $y$ in $\widetilde{M}$. For every $z \in \widetilde{M}$, let $\mathscr{D}_z$ be the unit Dirac mass at $z$. Let $h : [0, +\infty[\ \to\ ]0, +\infty[$ be a nondecreasing map such that

- for every $\epsilon > 0$, there exists $r_\epsilon \geq 0$ such that $h(t + r) \leq e^{\epsilon t} h(r)$ for all $t \geq 0$ and $r \geq r_\epsilon$;
- if $\overline{Q}_{x,\,y}(s) = \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}-s)}\ h(d(x, \gamma y))$, then $\overline{Q}_{x,\,y}(s)$ diverges if and only if the inequality $s \leq \delta_{\Gamma,\,F}$ holds.

If $(\Gamma, F)$ is of divergence type, we may take $h = 1$ constant. Define the measure

$$\mu_{x,\,s} = \frac{1}{\overline{Q}_{y,\,y}(s)} \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y}(\widetilde{F}-s)}\ h(d(x,\gamma y))\ \mathscr{D}_{\gamma y}$$

on $\widetilde{M}$. It is proved in [Moh] that such a map $h$ exists, and that there exists a sequence $(s_k)_{k\in\mathbb{N}}$ in $]\delta_{\Gamma,\,F}, +\infty[$ converging to $\delta_{\Gamma,\,F}$ such that for every $x \in \widetilde{M}$, the sequence of measures $(\mu_{x,\,s_k})_{k\in\mathbb{N}}$ weak-star converges to a measure $\mu_x$ on the compact space $\widetilde{M}\cup\partial_\infty\widetilde{M}$, such that $(\mu_x)_{x\in\widetilde{M}}$ is a Patterson density of dimension $\delta_{\Gamma,\,F}$ for $(\Gamma, F)$. Since $\overline{Q}_{y,\,y}(\delta_{\Gamma,\,F}) = +\infty$ and since the support of $\mu_{x,\,s}$ in $\widetilde{M}\cup\partial_\infty\widetilde{M}$ is equal to $\Gamma y \cup \Lambda\Gamma$, the support of $\mu_x$ is contained in $\Lambda\Gamma$, hence equal to $\Lambda\Gamma$. □

We will often denote by $(\mu_{F,\,x})_{x\in\widetilde{M}}$ a density as in Proposition 3.9. We will prove in Subsection 5.3 that if $(\Gamma, F)$ is of divergence type, then $(\mu_{F,\,x})_{x\in\widetilde{M}}$ is unique up to a scalar multiple. Note that the definition of this density involves only the normalised potential $F - \delta_{\Gamma,\,F}$. Hence the assumption that $\delta_{\Gamma,\,F}$ is positive is harmless, by Equation (18), and we will often make it in Chapter 9 and Chapter 10.

The main basic tool concerning Patterson densities, which will be very useful, is the following lemma, proved in [Moh] (see also [Cou2, Lem. 4] with the multiplicative rather than additive convention) along the lines of Sullivan's shadow lemma (see [Rob1, p. 10]). We will prove a more general result in Proposition 11.1 in Subsection 11.1.

**Lemma 3.10 (Mohsen's shadow lemma)** *Let $\sigma \in \mathbb{R}$, let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma$ for $(\Gamma, F)$, and let $K$ be a compact subset of $\widetilde{M}$. If $R$ is large enough, there exists $C > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$,*

$$\frac{1}{C}\ e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)} \le \mu_x\big(\mathscr{O}_x B(\gamma y, R)\big) \le C\ e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)}\ .$$

**Proof.** Let $\sigma$, $(\mu_x)_{x\in\widetilde{M}}$ and $K$ be as in the statement. Let us prove that if $R'$ is large enough, there exists $C' > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$, we have

$$\frac{1}{C'} \le \mu_{\gamma y}(\mathscr{O}_x B(\gamma y, R')) \le C'\ . \tag{40}$$

Assuming this, let us prove Lemma 3.10. By Equation (39), we have

$$\mu_x(\mathscr{O}_x B(\gamma y, R')) = \int_{\xi\in\mathscr{O}_x B(\gamma y,\,R')} e^{-C_{F-\sigma,\,\xi}(x,\,\gamma y)}\ d\mu_{\gamma y}(\xi)\ .$$

By Lemma 3.4 (2) applied with $r = r_0 = R'$, since $\widetilde{F}$ is continuous and $\Gamma$-invariant, hence is uniformly bounded on $\Gamma\pi^{-1}(\mathscr{N}_{R'}K))$, which contains $\pi^{-1}(B(\gamma y, R'))$ for all $\gamma \in \Gamma$ and $y \in K$, there exists $C'' > 0$ such that for all $x, y \in K$ and $\xi \in \mathscr{O}_x B(\gamma y, R')$,

$$\Big|\, C_{F-\sigma,\,\xi}(x, \gamma y) + \int_x^{\gamma y}(\widetilde{F} - \sigma)\,\Big| \le C''\ .$$

Hence

$$\begin{aligned} e^{-C''}\ e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)}\ &\mu_{\gamma y}(\mathscr{O}_x B(\gamma y, R')) \\ &\le \mu_x(\mathscr{O}_x B(\gamma y, R')) \\ &\qquad \le e^{C''}\ e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)}\ \mu_{\gamma y}(\mathscr{O}_x B(\gamma y, R'))\ . \end{aligned}$$

Therefore the result follows with $R = R'$ and $C = C' e^{C''}$.

Let us now prove the upper bound in Equation (40). Fix $y_0 \in K$, and let

$$C''' = \sup_{x,\,y \in K,\ \xi \in \partial_\infty \widetilde{M}} |C_{F-\sigma,\,\xi}(x,y)| \;,$$

which is finite by Lemma 3.4 (1), since $K$ is compact. Then, using Equation (38) for the equality and Equation (39) for the last inequality, we have

$$\mu_{\gamma y}(\mathscr{O}_x B(\gamma y, R')) \leq \|\mu_{\gamma y}\| = \|\mu_y\| \leq e^{C'''}\ \|\mu_{y_0}\| \;,$$

and the upper bound holds if $C' \geq e^{C'''}\ \|\mu_{y_0}\|$.

Finally, to prove the lower bound in Equation (40), we assume for a contradiction that there exist sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ in $K$, $(\gamma_i)_{i\in\mathbb{N}}$ in $\Gamma$, and $(R_i)_{i\in\mathbb{N}}$ in $]0,+\infty[$ converging to $+\infty$, such that

$$\lim_{i\to+\infty}\ \mu_{\gamma_i y_i}(\mathscr{O}_{x_i} B(\gamma_i y_i, R_i))\ = 0\;. \tag{41}$$

Up to a subsequence, we have $\lim x_i = x \in K$, $\lim y_i = y \in K$ and $\lim \gamma_i^{-1} x_i = \xi \in \widetilde{M} \cup \partial_\infty \widetilde{M}$. By Equation (41), the shadow $\mathscr{O}_{x_i} B(\gamma_i y_i, R_i)$ is different from $\partial_\infty \widetilde{M}$, hence $d(x_i, \gamma_i y_i) \geq R_i$, which tends to $+\infty$ as $i \to +\infty$. Therefore $\xi \in \partial_\infty \widetilde{M}$.

Let $V$ be a relatively compact open subset of $\partial_\infty \widetilde{M} - \{\xi\}$. Then $V$ is contained in $\mathscr{O}_{\gamma_i^{-1} x_i} B(y_i, R_i)$ for $i$ large enough, since $\lim R_i = +\infty$ and $(y_i)_{i\in\mathbb{N}}$ stays in the compact subset $K$ (this is the original key remark on shadows by Sullivan). For every $i \in \mathbb{N}$, we have, using Equation (38) for the first equality and the equivariance property of shadows for the second one,

$$\begin{aligned}\mu_y(V) &= (\gamma_i^{-1})_* \mu_{\gamma_i y}(V) \leq (\gamma_i^{-1})_* \mu_{\gamma_i y}(\mathscr{O}_{\gamma_i^{-1} x_i} B(y_i, R_i)) = \mu_{\gamma_i y}(\mathscr{O}_{x_i} B(\gamma_i y_i, R_i))\\ &\leq e^{C'''} \mu_{\gamma_i y_i}(\mathscr{O}_{x_i} B(\gamma_i y_i, R_i))\;,\end{aligned}$$

this last inequality by the invariance property (26) of the Gibbs cocycle and by Equation (39). By Equation (41), letting $i \to +\infty$, we hence have $\mu_y(V) = 0$. Therefore the measure $\mu_y$ is supported in $\{\xi\}$. Since the support of a Patterson density of $(\Gamma, F)$ is $\Gamma$-invariant, this implies that $\xi$ is fixed by $\Gamma$. This is a contradiction since $\Gamma$ is non-elementary. $\Box$

A first corollary of Mohsen's shadow lemma shows the inexistence of a Patterson density of dimension less than the critical exponent of $(\Gamma, F)$. Its proof is similar to the one given by Sullivan [Sul1] in the case $F = 0$ and constant curvature (see also [Rob2, page 147]).

**Corollary 3.11** *Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma$ for $(\Gamma, F)$.*

*(1) For all $x, y \in \widetilde{M}$, there exists $c > 0$ such that for every $n \in \mathbb{N}$, we have*

$$\sum_{\gamma\in\Gamma\,:\, n-1 < d(x,\gamma y) \leq n} e^{\int_x^{\gamma y} (\widetilde{F} - \sigma)} \leq c\;.$$

*(2) We have $\sigma \geq \delta_{\Gamma,\,F}$.*

In particular, if $\delta_{\Gamma,\,F} = +\infty$, there exists no Patterson density for $(\Gamma, F)$.

We will describe in Proposition 5.11 the exact set of elements $\sigma \in \mathbb{R}$ for which there exists a Patterson density of dimension $\sigma$ for $(\Gamma, F)$ with support equal to $\Lambda\Gamma$.

**Proof.** Let $x, y \in \widetilde{M}$. Let $R$ and $C$ be as in Mohsen's shadow lemma 3.10 for $(\mu_x)_{x\in\widetilde{M}}$ and $K = \{x, y\}$. Let $\kappa = \operatorname{Card}\{\gamma \in \Gamma \;:\; d(y, \gamma y) \leq 1 + 3R\}$, which is finite by discreteness. For every $n \in \mathbb{N}$, let $\Gamma_n = \{\gamma \in \Gamma \;:\; n - 1 < d(x, \gamma y) \leq n\}$.

If $\gamma \in \Gamma_n$ and $\xi \in \mathscr{O}_x B(\gamma y, R)$, if $p_\gamma$ is the closest point to $\gamma y$ on the geodesic ray from $x$ to $\xi$, then $n - R - 1 \leq d(x, p_\gamma) \leq n$ by the triangle inequality. Hence, for all $\gamma, \gamma' \in \Gamma_n$, if $\xi \in \mathscr{O}_x B(\gamma y, R) \cap \mathscr{O}_x B(\gamma' y, R)$, then, again by the triangle inequality,

$$d(\gamma' y, \gamma y) \leq d(\gamma' y, p_{\gamma'}) + \big|d(x, p_{\gamma'}) - d(x, p_\gamma)\big| + d(p_\gamma, \gamma y) \leq 1 + 3R\;.$$

Thus, for every $n \in \mathbb{N}$, a point $\xi \in \partial_\infty \widetilde{M}$ belongs to at most $\kappa$ subsets $\mathscr{O}_x B(\gamma y, R)$ as $\gamma$ ranges over $\Gamma_n$. Therefore

$$\sum_{\gamma\in\Gamma_n} \mu_x\big(\mathscr{O}_x B(\gamma y, R)\big) \leq \kappa\; \mu_x\Big(\bigcup_{\gamma\in\Gamma_n} \mathscr{O}_x B(\gamma y, R)\Big)\;.$$

Now, by the lower bound in Lemma 3.10, and since $\mu_x$ is a finite measure,

$$\begin{aligned}\sum_{\gamma\in\Gamma_n} e^{\int_x^{\gamma y} \widetilde{F}} &\leq e^{\sigma\, n} \sum_{\gamma\in\Gamma_n} e^{\int_x^{\gamma y} (\widetilde{F}-\sigma)} \leq C\; e^{\sigma\, n} \sum_{\gamma\in\Gamma_n} \mu_x\big(\mathscr{O}_x B(\gamma y, R)\big)\\ &\leq C\kappa\; \|\mu_x\|\; e^{\sigma\, n}\;. \end{aligned}\tag{42}$$

The first assertion follows. By taking the logarithm, dividing by $n$ and taking the upper limit as $n$ goes to $+\infty$, the second assertion then follows, by the definition of the critical exponent of $(\Gamma, F)$. □

Let us give another consequence of Mohsen's shadow lemma 3.10, concerning the doubling properties of the Patterson densities. Recall (see for instance [Hei, page 3]) that a measured metric space $(X, d, \mu)$, that is a metric space $(X, d)$ endowed with a Borel positive measure $\mu$, is *doubling* if there exists $c \geq 1$ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2\,r)) \leq c\, \mu(B(x, r))$$

where $B(x, r)$ is the ball of centre $x$ and radius $r$ in $X$. Note that, up to changing $c$, the number 2 may be replaced by any constant larger than 1.

**Proposition 3.12** *Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma$ for $(\Gamma, F)$.*

*(1) For every compact subset $K$ of $\widetilde{M}$, for every $R > 0$ large enough, there exists $C = C(R) > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$, we have*

$$\mu_x(\mathscr{O}_x B(\gamma y, 5R)) \leq C\, \mu_x(\mathscr{O}_x B(\gamma y, R))\;.$$

*(2) For every $x \in \widetilde{M}$, if $\Gamma$ is convex-cocompact, then the measured metric space $(\Lambda\Gamma, d_x, \mu_x)$ is doubling.*

**Proof.** (1) This is immediate from Mohsen's shadow lemma 3.10

(2) Let $x \in \widetilde{M}$. Recall (see the beginning of Subsection 2.2) that $d_x$ is the visual distance on $\partial_\infty \widetilde{M}$ seen from $x$, and we again denote by $d_x$ its restriction to $\Lambda\Gamma$. Denote by $B_x(\xi, r)$ the ball of centre $\xi$ and radius $r$ for $d_x$. For every $\xi \in \partial_\infty \widetilde{M}$, let $\rho_\xi : [0, +\infty[ \to \widetilde{M}$ be the geodesic ray from $x$ to $\xi$.

By for instance [HeP2, Lem. 2.1], for every $s > 0$, there exists $a(s) \geq 1$ such that for every $t$ large enough, for every $\xi \in \partial_\infty \widetilde{M}$, we have

$$B(\xi, s\, e^{-t}) \subset \mathscr{O}_x B(\rho_\xi(t), s) \subset B(\xi, a(s) e^{-t}) \;.$$

If $\Gamma$ is convex-cocompact, there exists $\Delta > 0$ such that for all $\xi \in \Lambda\Gamma$ and $t \geq 0$, there exists $\gamma_t \in \Gamma$ with $d(\rho_\xi(t), \gamma_t x) \leq \Delta$. Let $m \in \mathbb{N} - \{0\}$ be such that $5^m \geq 2\, a(1 + 2\Delta)$. Let $C' = \prod_{i=0}^{m-1} C(5^i)$. For every $r > 0$ small enough, let $t = -\log r + \log a(1 + 2\Delta)$, which is large enough. Then

$$B_x(\xi, 2\, r) \subset B_x(\xi, 5^m e^{-t}) \subset \mathscr{O}_x B(\rho_\xi(t), 5^m) \subset \mathscr{O}_x B(\gamma_t x, 5^m(1 + \Delta)) \;,$$

and

$$\mathscr{O}_x B(\gamma_t x, 1 + \Delta) \subset \mathscr{O}_x B(\rho_\xi(t), 1 + 2\Delta) \subset B(\xi, a(1 + 2\Delta) e^{-t}) = B(\xi, r) \;.$$

Using $m$ times Assertion (1), we have

$$\mu_x\big(\mathscr{O}_x B(\gamma_t x, 5^m(1 + \Delta))\big) \leq C' \mu_x\big(\mathscr{O}_x B(\gamma_t x, 1 + \Delta)\big) \;.$$

Hence if $r > 0$ is small enough, for every $\xi \in \Lambda\Gamma$, we have

$$\mu_x(B(\xi, 2\, r)) \leq C' \mu_x(B(\xi, r)) \;.$$

By compactness and since the support of $\mu_x$ is $\Lambda\Gamma$, this proves the result. $\square$

Here is yet another consequence of Mohsen's shadow lemma 3.10, concerning the closeness of the gap map defined in Subsection 3.4 to an actual distance.

**Proposition 3.13** *Assume that $\delta = \delta_{\Gamma,\, F} < +\infty$. Let $x \in \mathscr{C}\Lambda\Gamma$. Assume that $\Gamma$ is convex-cocompact or that $\widetilde{F}$ is reversible and $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$. Then for every $\epsilon > 0$ small enough, there exist a distance $d_{F-\delta,\, x,\, \epsilon}$ on $\Lambda\Gamma$ and $c_\epsilon > 0$ such that for all $\xi, \eta \in \Lambda\Gamma$, we have*

$$\frac{1}{c_\epsilon}\, D_{F-\delta,\, x}(\xi, \eta)^\epsilon \leq d_{F-\delta,\, x,\, \epsilon}(\xi, \eta) \leq c_\epsilon\, D_{F-\delta,\, x}(\xi, \eta)^\epsilon \;.$$

*Furthermore, if $\sup_{T^1\widetilde{M}} \widetilde{F} < \delta$, then $d_{F-\delta,\, x,\, \epsilon}$ induces the original topology on $\Lambda\Gamma$.*

We do not know if the assumption $\sup_{T^1\widetilde{M}} \widetilde{F} < \delta$ is necessary for the last claim.

**Proof.** The case $\widetilde{F}$ is reversible and $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$ follows from Lemma 3.7. Assume that $\Gamma$ is convex-cocompact. Then $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, say by $\kappa \geq 0$. Let $\kappa'$ be the diameter of $\Gamma\backslash\mathscr{C}\Lambda\Gamma$. For all $x, y, z \in \mathscr{C}\Lambda\Gamma$ such that $y \in [x, z]$, let $\alpha, \beta \in \Gamma$ be such that $d(y, \alpha x), d(z, \beta x) \leq \kappa'$. Then by Lemma 3.2 with $r_0 = \kappa'$, there exists $c > 0$

(depending only on $\kappa$, $\kappa'$, the Hölder constants of $\widetilde{F}$ and the bounds on the sectional curvature) such that, with $\widetilde{F}' = \widetilde{F} - \delta$,

$$\begin{aligned}\int_x^z \widetilde{F}' &= \int_x^y \widetilde{F}' + \int_y^z \widetilde{F}' \\ &\leq \int_x^y \widetilde{F}' + \int_{\alpha x}^{\beta x} \widetilde{F}' + \Big| \int_y^z \widetilde{F}' - \int_{\alpha x}^z \widetilde{F}' \Big| + \Big| \int_{\alpha x}^z \widetilde{F}' - \int_{\alpha x}^{\beta x} \widetilde{F}' \Big| \\ &\leq \int_x^y \widetilde{F}' + \int_{\alpha x}^{\beta x} \widetilde{F}' + c \;\leq \int_x^y \widetilde{F}' + \sup_{\gamma\in\Gamma} \int_x^{\gamma x} \widetilde{F}' + c \,.\end{aligned}$$

The existence of a Patterson density of dimension $\sigma = \delta_{\Gamma,\,F} < +\infty$, its finiteness, and the lower bound in Mohsen's shadow lemma imply that for all $x, y \in \widetilde{M}$,

$$\sup_{\gamma\in\Gamma} \int_x^{\gamma y} (\widetilde{F} - \delta) < +\infty \,.$$

This proves that Equation (33) holds for the normalised potential $F - \delta$. Hence, the first claim of Proposition 3.13 follows by Lemma 3.7.

Let us now prove the final claim of Proposition 3.13. The function $\widetilde{F} - \delta$ is bounded from below and from above by a negative constant. Hence by Equation (32), there exist $c, c' > 0$ such that ${d_x}^c \leq D_{F-\delta,\,x} \leq {d_x}^{c'}$ on $\Lambda\Gamma \times \Lambda\Gamma$. The claim that the distance $d_{F-\delta,\,x,\,\epsilon}$ on $\Lambda\Gamma$ induces its topology therefore follows from the fact that the visual distance $d_x$ induces the topology on $\partial_\infty \widetilde{M}$. $\square$

**Remark.** Let $\widetilde{F}^* : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map, which is cohomologous to $\widetilde{F}$ via the map $\widetilde{G} : T^1\widetilde{M} \to \mathbb{R}$. Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma$ for $(\Gamma, F)$. Then the family of measures $(\mu^*_x)_{x\in\widetilde{M}}$ defined by setting, for all $x \in \widetilde{M}$ and $\xi \in \partial_\infty \widetilde{M}$,

$$d\mu^*_x(\xi) = e^{-\widetilde{G}(v_{x\xi})}\, d\mu_x(\xi) \,,$$

where $v_{x\xi}$ is the tangent vector at $x$ of the geodesic ray from $x$ to $\xi$, is also a Patterson density of dimension $\sigma$ for $(\Gamma, F^*)$. Indeed, the invariance property (38) for $(\mu^*_x)_{x\in\widetilde{M}}$ follows from the one for $(\mu_x)_{x\in\widetilde{M}}$ and the invariance of $\widetilde{G}$. And the absolutely continuous property (39) for $(\mu^*_x)_{x\in\widetilde{M}}$ follows from the one for $(\mu_x)_{x\in\widetilde{M}}$ and Equation (27).

## 3.7 The Gibbs states of $(\Gamma, F)$

Let $\sigma \in \mathbb{R}$, let $(\mu^\iota_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma$ for $(\Gamma, F\circ\iota)$, and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of the same dimension $\sigma$ for $(\Gamma, F)$. Using the Hopf parametrisation with respect to any base point $x_0$ of $\widetilde{M}$, we define the *Gibbs measure on $T^1\widetilde{M}$ associated with the pair of Patterson densities* $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ (of dimension $\sigma$) as the measure $\widetilde{m}$ on $T^1\widetilde{M}$ given by

$$d\,\widetilde{m}(v) = \frac{d\mu^\iota_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt}{D_{F-\sigma,\,x_0}(v_-, v_+)^2} \,.$$

Using Equation (29), the following expression of this measure, again in the Hopf parametrisation $v \mapsto (v_-, v_+, t)$ with respect to the base point $x_0$, will be useful

$$d\,\widetilde{m}(v) = e^{C_{F\circ\iota-\sigma,\,v_-}(x_0,\,\pi(v)) + C_{F-\sigma,\,v_+}(x_0,\,\pi(v))}\; d\mu^\iota_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt \,. \qquad (43)$$

The measure $\widetilde{m}$ on $T^1\widetilde{M}$ is independent of $x_0$ by the equations (43), (39) and (25), hence is invariant under the action of $\Gamma$ by the equations (38) and (30). It is invariant under the geodesic flow. Hence (see Subsection 2.6) it defines a measure $m$ on $T^1M = \Gamma\backslash T^1\widetilde{M}$ which is invariant under the quotient geodesic flow. We call $m$ the *Gibbs measure on* $T^1M$ *associated with the pair of Patterson densities* $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$. We will justify the terminology in Subsection 3.8.

The (Borel positive) measure

$$d\lambda(\xi,\eta) = \frac{d\mu^\iota_{x_0}(\xi)\, d\mu_{x_0}(\eta)}{D_{F-\sigma,\, x_0}(\xi,\eta)^2} \tag{44}$$

on $\partial^2_\infty\widetilde{M}$, which is locally finite and invariant under the diagonal action of $\Gamma$ on $\partial^2_\infty\widetilde{M}$, is a *geodesic current* for the action of $\Gamma$ on the Gromov-hyperbolic proper metric space $\widetilde{M}$ in the sense of Ruelle-Sullivan-Bonahon (see for instance [Bon] and references therein).

When $\delta_{\Gamma,\,F} < \infty$, we will denote by $\widetilde{m}_F$, and call the *Gibbs measure on* $T^1\widetilde{M}$ *with potential* $F$, the Gibbs measure on $T^1\widetilde{M}$ associated with any given pair of densities $\big((\mu_{F\circ\iota,\,x})_{x\in\widetilde{M}}, (\mu_{F,\,x})_{x\in\widetilde{M}}\big)$ (which have the same dimension $\delta_{\Gamma,\,F}$ by Equation (19)). Its induced measure on $T^1M$ will be denoted by $m_F$ and called the *Gibbs measure on* $T^1M$ *with potential* $F$. By the uniqueness statement in Subsection 5.3, if $(\Gamma, F)$ is of divergence type (we will prove in Corollary 5.15 that this is the case for instance if $m_F$ is finite), then $\widetilde{m}_F$ and $m_F$ are uniquely defined, up to a scalar multiple.

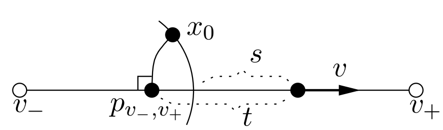
Since the supports of the Patterson densities $(\mu_{F\circ\iota,\,x})_{x\in\widetilde{M}}$ and $(\mu_{F,\,x})_{x\in\widetilde{M}}$ are equal to $\Lambda\Gamma$, the support of $\widetilde{m}_F$ is the set $\widetilde{\Omega}\Gamma$ of tangent vectors to the geodesic lines in $\widetilde{M}$ whose endpoints both lie in $\Lambda\Gamma$. Using the Hopf parametrisation, this support corresponds to $\big((\partial^2_\infty\widetilde{M})\cap(\Lambda\Gamma\times\Lambda\Gamma)\big)\times\mathbb{R}$. Hence the support of $m_F$ is the (topological) non-wandering set $\Omega\Gamma$ of the geodesic flow in $T^1M$.

Since only the normalised potential $F-\delta_{\Gamma,\,F}$ is involved in the definition of $\widetilde{m}_F$ and $m_F$, the assumption that $\delta_{\Gamma,\,F}$ is positive is harmless, by Equation (18). We will often make this assumption in Chapter 9 and Chapter 10.

**Remark (1)** Since $d\widetilde{m}(v) = \frac{d\mu^\iota_{x_0}(v_-)\, d\mu_{x_0}(v_+)\, dt}{D_{F\circ\iota-\sigma,\, x_0}(v_+,v_-)^2}$ by Equation (30), the measure $\iota_*\widetilde{m}$ is the Gibbs measure on $T^1\widetilde{M}$ associated with the switched pair of Patterson densities $\big((\mu_x)_{x\in\widetilde{M}}, (\mu^\iota_x)_{x\in\widetilde{M}}\big)$ for $(\Gamma,F)$ and $(\Gamma, F\circ\iota)$ respectively (and similarly on $T^1M$).

**Remark (2)** Another parametrisation of $T^1\widetilde{M}$ (used for instance in Subsection 3.9) also depending on the choice of a base point $x_0$ in $\widetilde{M}$, is the map from $T^1\widetilde{M}$ to $\partial^2_\infty\widetilde{M}\times\mathbb{R}$ sending $v$ to $(v_-, v_+, s)$ where $v_-$ and $v_+$ are as above and $s = \beta_{v_-}(\pi(v), x_0)$ (we may also use the different time parameter $s = \beta_{v_+}(x_0, \pi(v))$).

For every $(\eta,\xi)$ in $\partial^2_\infty\widetilde{M}$, let $p_{\eta,\,\xi}$ be the closest point to $x_0$ on the geodesic line between $\eta$ and $\xi$.




For every $v\in T^1\widetilde{M}$, with $(v_-, v_+, t)$ the original Hopf parametrisation, since

$$s - t = \beta_{v_-}(\pi(v), x_0) - \beta_{v_-}(\pi(v), p_{v_-,v_+}) = \beta_{v_-}(p_{v_-,v_+}, x_0)$$

depends only on $v_-$ and $v_+$, the measures $d\mu^\iota_{x_0}(v_-)\,d\mu_{x_0}(v_+)\,dt$ and $d\mu^\iota_{x_0}(v_-)\,d\mu_{x_0}(v_+)\,ds$ are equal. We hence may (and will in the proof of Proposition 3.18) use the second one to define the Gibbs measure associated with the above pair of densities.

**Remark (3)** It is sometimes useful (see for instance the proof of Theorem 10.4) to consider the following more general construction. Let $x_0, x'_0 \in \widetilde{M}$ and $\sigma^\iota, \sigma \in \mathbb{R}$ with possibly $\sigma^\iota \neq \sigma$. Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma^\iota$ for $(\Gamma, F\circ\iota)$, and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma$ for $(\Gamma, F)$. Using the Hopf parametrisation with respect to $x_0$ (the original one or the one defined in the previous Remark (2)), we define the *Gibbs measure* associated with the pair of densities $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ as the measure $\widetilde{m}$ on $T^1\widetilde{M}$ given by

$$d\,\widetilde{m}(v) = e^{C_{F\circ\iota-\sigma^\iota,\,v_-}(x'_0,\,\pi(v)) + C_{F-\sigma,\,v_+}(x_0,\,\pi(v))}\;d\mu^\iota_{x'_0}(v_-)\,d\mu_{x_0}(v_+)\,dt\;. \tag{45}$$

It is again independent of $x_0, x'_0$ by the equations (39) and (25), hence is invariant under the action of $\Gamma$ by the equations (38) and (26), and defines a measure $m$ on $T^1M$, by Subsection 2.6. But it is no longer invariant under the geodesic flow, as it now satisfies, for all $t\in\mathbb{R}$ and $v\in T^1\widetilde{M}$, using Equation (22) to get the second equality,

$$\begin{aligned} d(\phi_t)_*\widetilde{m}(v) &= e^{C_{F\circ\iota-\sigma^\iota,\,v_-}(\pi(v),\,\pi(\phi_{-t}v)) + C_{F-\sigma,\,v_+}(\pi(v),\,\pi(\phi_{-t}v))}\;d\,\widetilde{m}(v)\;.\\ &= e^{(\sigma^\iota-\sigma)t}\;d\,\widetilde{m}(v)\;. \end{aligned}$$

When $F=0$, these measures have been considered for instance by Burger [Bur] and Knieper [Kni] in particular cases, and by Roblin [Rob1, page 12] in general, and are sometimes called *Burger-Roblin measures*, see [OhS, Kim]. When $F$ is non-constant, see for instance [Sch1].

**Remark (4)** The Gibbs measure $m_F$ is not changed if we replace $F$ by a cohomologous potential. Indeed, let $\widetilde{F}^* : T^1\widetilde{M}\to\mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map cohomologous to $\widetilde{F}$ via the map $\widetilde{G} : T^1\widetilde{M}\to\mathbb{R}$. We have $\delta_{\Gamma,\,F^*} = \delta_{\Gamma,\,F}$ by the remark at the end of Subsection 3.2. Choose the Patterson densities of the remark at the end of Subsection 3.6 to construct $\widetilde{m}_{F^*}$ (such a choice is not important when $(\Gamma, F)$ is of divergence type, see Subsection 5.3). By Equation (43), by the remark at the end of Subsection 3.6 and by Equation (27), we then have

$$\widetilde{m}_{F^*} = \widetilde{m}_F\;.$$

### 3.8 The Gibbs property of Gibbs states

Let us indicate why our terminology of Gibbs measures (introduced in Subsection 3.7) is appropriate, by explicitly making an analogy with the symbolic dynamics case and by proving they satisfy the Gibbs property on the dynamical balls of the geodesic flow (see Remark (3) in the beginning of Chapter 10 for other explanations).

Gibbs measures were first introduced in statistical mechanics, and are naturally associated, via the thermodynamic formalism (see for instance [Rue3, Kel, Zin]), to symbolic dynamics. Let us recall the definitions of the pressure of a potential in a general dynamical system, and its equilibrium states, adapting them to the non-compact case, as in [Sar1] for the countable Markov shift case.

If $\phi' = (\phi'_t)_{t\in\mathbb{Z}}$ or $\phi' = (\phi'_t)_{t\in\mathbb{R}}$ is a discrete-time or continuous-time one-parameter group of homeomorphisms of a metric space $X'$ and if $F' : X'\to\mathbb{R}$ is a continuous map, let us define the *pressure* $P(F')$ *of the potential* $F'$ as the upper bound of $h_{m'}(\phi')+\int_{T^1M} F'\,dm'$, on all $\phi'$-invariant probability measures $m'$ on $X'$ such that the negative part $\max\{0,-F'\}$ of $F'$ is $m'$-integrable, where $h_{m'}(\phi')=h_{m'}(\phi'_1)$ is the (metric) entropy of the one-parameter group of homeomorphisms $\phi'$ with respect to $m'$. Define an *equilibrium state* of $F'$ as any such $m'$ realising this upper bound. When $\phi'=\phi$ is the geodesic flow on $X'=T^1M$, we will come back to these definitions in Chapter 6.

Let us now recall what is a Gibbs measure in symbolic dynamics. Let $A$ be a countable set, called an *alphabet* (see [Sar1, Sar3, Sar2] for the infinite case), endowed with the discrete distance $d(a,a')=1$ if $a\neq a'$, and $d(a,a')=0$ otherwise. Let $\Sigma$ be the product topological space $A^{\mathbb{Z}}$, which is compact if $A$ is finite. Let us endow it with the distance (inducing its topology)

$$d((x_i)_{i\in\mathbb{Z}},(x'_i)_{i\in\mathbb{Z}})=\sum_{i\in\mathbb{Z}} e^{-i^2} d(x_i,x'_i)\;.$$

Let $\sigma:\Sigma\rightarrow\Sigma$ be the (full) *shift map*, which is the homeomorphism of $\Sigma$ defined by $(x_i)_{i\in\mathbb{Z}}\mapsto(y_i)_{i\in\mathbb{Z}}$ where $y_i=x_{i+1}$ for every $i\in\mathbb{Z}$. Note that if $\pi':\Sigma\rightarrow A$ is the map $(x_i)_{i\in\mathbb{Z}}\mapsto x_0$, then for all $x,x'\in\Sigma$, we have

$$d(x,x')=\sum_{i\in\mathbb{Z}}\; d(\pi'(\sigma^i x),\pi'(\sigma^i x'))\;e^{-i^2}$$

(compare with Equation (5)). For all $n,n'\in\mathbb{N}$ and $a_{-n'},\dots,a_n$ in $A$, let

$$[a_{-n'},\dots,a_n]=\{(x_i)_{i\in\mathbb{Z}}\in\Sigma\;:\;x_i=a_i\;\text{ for }-n'\leq i\leq n\}\;,$$

which is called a *cylinder* in $\Sigma$. Recall that given $(x_i)_{i\in\mathbb{Z}}\in\Sigma$, the set of cylinders $\{[x_{-n'},\dots,x_n]\;:\;n,n'\in\mathbb{N}\}$ is a (natural) neighbourhood basis of $(x_i)_{i\in\mathbb{Z}}$ (even if we take $n'=n$). Finally, let $F:\Sigma\rightarrow\mathbb{R}$ be a Hölder-continuous map. For instance when $A=\{-1,+1\}$, the energy map $F:\Sigma\rightarrow\mathbb{R}$ defined by $(x_i)_{i\in\mathbb{Z}}\mapsto x_0(x_1+x_{-1})$ describes the one-dimensional Ising model (see for instance [Kel, page 96]).

The Gibbs measures are then shift-invariant measures, whose mass of a cylinder is a weight defined by integrating the normalised potential along the corresponding piece of orbit. More precisely (compare for instance with [Kel, page 100]), a *Gibbs measure of potential* $F$ is a $\sigma$-invariant measure $m'$ on $\Sigma$ such that there exists $c'(F)\in\mathbb{R}$ such that, for every finite subset $A'$ of $A$, there exists $c\geq 1$ such that for all $x=(x_i)_{i\in\mathbb{Z}}\in\Sigma$ and $n,n'\in\mathbb{N}$ with $x_{-n'},x_n\in A'$, we have

$$\frac{1}{c}\leq\frac{m'([x_{-n'},\dots,x_n])}{e^{\sum_{i=-n'}^{n}F(\sigma^i x)-(n+n'+1)c'(F)}}\leq c\;. \tag{46}$$

In this symbolic framework, when the alphabet $A$ is finite, with $\phi'=(\sigma^i)_{i\in\mathbb{Z}}$, $X'=\Sigma$ and $F'=F$, it is well-known (see for instance [Bow4, Rue3, Kel, Zin] as well as [HaR]) that a probability Gibbs measure of $F$ exists (with constant $c'(F)$ equal to the pressure of $F$), is unique, and is the unique equilibrium state of $F$ (see [Sar1, Sar3, Sar2] for what remains true in the countable alphabet case).

The analogy between the geodesic flow $\phi=(\phi_t)_{t\in\mathbb{R}}$ and the full shift $(\sigma^i)_{i\in\mathbb{Z}}$ proceeds as follows. For all $v\in T^1\widetilde{M}$ and $r>0$, $T,T'\geq 0$, let us define

$$B(v;T,T',r)=\{w\in T^1\widetilde{M}\;:\;\sup_{t\in[-T',T]} d(\pi(\phi_t v),\pi(\phi_t w))<r\}\;,$$

called a *dynamical (or Bowen) ball* around $v$. They satisfy the following invariance properties: for all $s \in [-T', T]$ and $\gamma \in \Gamma$,

$$\phi_s B(v; T, T', r) = B(\phi_s v; T - s, T' + s, r) \quad \text{and} \quad \gamma B(v; T, T', r) = B(\gamma v; T, T', r) \ .$$

Note the following inclusion properties of the dynamical balls: if $r \leq s$, $T \geq S$, $T' \geq S'$, then $B(v; T, T', r)$ is contained in $B(v; S, S', s)$.

**Lemma 3.14** *(1) For all $r' \geq r > 0$, there exists $T_{r,r'} \geq 0$ such that for all $v \in T^1\widetilde{M}$ and $T, T' \geq 0$, the dynamical ball $B(v; T + T_{r,r'}, T' + T_{r,r'}, r')$ is contained in $B(v; T, T', r)$.*

*(2) If $B(v; T, T', r)$ meets $B(u; S, S', r)$ and $T \geq S$, $T' \geq S'$, then $v$ belongs to the dynamical ball $B(u; S, S', 2r)$.*

**Proof.** (1) This follows by the properties of long geodesic segments with endpoints at bounded distance in a CAT$(-1)$ space (see for instance [BrH]).

(2) Let $z \in B(v; T, T', r) \cap B(u; S, S', r)$. By the triangle inequality and since $T \geq S$, $T' \geq S'$, we have

$$\max_{t \in [-S', S]} d(\pi(\phi_t v), \pi(\phi_t u)) \leq \max_{t \in [-T', T]} d(\pi(\phi_t v), \pi(\phi_t z)) + \max_{t \in [-S', S]} d(\pi(\phi_t z), \pi(\phi_t u)) < 2r \ .$$

Hence $v \in B(u; S, S', 2r)$. □

It is easy to see that for every fixed $r > 0$, the set $\{B(v; T, T', r) \ : \ T, T' \geq 0\}$ is a (natural) neighbourhood basis of $v \in T^1\widetilde{M}$ (even if we take $T' = T$), analogous to the set of cylindrical neighbourhoods of a sequence in $\Sigma$. We will see in Lemma 3.17 below that, as it was mentioned in the introduction, these (small) neighbourhoods of $v$ are defined by (small) neighbourhoods of the two points at infinity $v_-, v_+$ of the geodesic line defined by $v$. For every $v \in T^1M$, let us define $B(v; T, T', r')$ as the image by the canonical projection $T^1\widetilde{M} \to T^1M = \Gamma\backslash T^1\widetilde{M}$ of $B(\widetilde{v}; T, T', r')$, for any preimage $\widetilde{v}$ of $v$ in $T^1\widetilde{M}$.

Adapting the definition of [KatH] to the non-compact case, we want to consider the measures on $T^1M$ invariant under the geodesic flow, whose masses of appropriate dynamical balls are weights defined by integrating the normalised potential along the corresponding pieces of orbits, as follows.

**Definition 3.15** *A $\phi$-invariant measure $m'$ on $T^1M$* satisfies the Gibbs property *for the potential $F$ with constant $c(F) \in \mathbb{R}$ if for every compact subset $K$ of $T^1M$, there exist $r > 0$ and $c_{K,r} \geq 1$ such that for all large enough $T, T' \geq 0$, for every $v \in T^1M$ with $\phi_{-T'}v, \phi_T v \in K$, we have*

$$\frac{1}{c_{K,r}} \leq \frac{m'(B(v; T, T', r))}{e^{\int_{-T'}^{T}(F(\phi_t v) - c(F))\, dt}} \leq c_{K,r} \ .$$

**Remarks.** (1) The constant $c(F)$ is then uniquely defined.

(2) Since

$$\frac{m'(B(v; T, T', r))}{e^{\int_{-T'}^{T}(F(\phi_t v) - c(F))\, dt}} = \frac{m'(B(\phi_{-T'}v; T + T', 0, r))}{e^{\int_0^{T+T'}(F(\phi_t(\phi_{-T'}v)) - c(F))\, dt}}$$

by invariance of the measures, and by a compactness argument, and again by this invariance property, a $\phi$-invariant measure $m'$ on $T^1M$ satisfies the Gibbs property for the potential

$F$ with constant $c(F) \in \mathbb{R}$ if and only if for every compact subset $K$ of $T^1M$, there exist $r > 0$ and $c_{K,r} \geq 1$ such that one of the following three assertions holds

- for all large enough $T \geq 0$, for every $v \in T^1M$ with $v, \phi_T v \in K$, we have
$$\frac{1}{c_{K,r}} \leq \frac{m'(B(v;T,0,r))}{e^{\int_0^T (F(\phi_t v)-c(F))\,dt}} \leq c_{K,r} \;;$$
- for all $T \geq 0$ and $v \in T^1M$ with $v, \phi_T v \in K$, we have
$$\frac{1}{c_{K,r}} \leq \frac{m'(B(v;T,0,r))}{e^{\int_0^T (F(\phi_t v)-c(F))\,dt}} \leq c_{K,r} \;;$$
- for all $T, T' \geq 0$ and $v \in T^1M$ with $\phi_{-T'}v, \phi_T v \in K$, we have
$$\frac{1}{c_{K,r}} \leq \frac{m'(B(v;T,T',r))}{e^{\int_{-T'}^T (F(\phi_t v)-c(F))\,dt}} \leq c_{K,r} \;.$$

Furthermore, fixing $T_0 \geq 0$, a $\phi$-invariant measure $m'$ on $T^1M$ satisfies the Gibbs property for the potential $F$ with constant $c(F) \in \mathbb{R}$ if and only if for every compact subset $K$ of $T^1M$, there exist $r > 0$ and $c_{K,r,T_0} \geq 1$ such that for all large enough $T \geq 0$, for every $v \in T^1M$ with $\phi_{-T_0}v, \phi_T v \in K$, we have
$$\frac{1}{c_{K,r,T_0}} \leq \frac{m'(B(v;T,T_0,r))}{e^{\int_{-T_0}^T (F(\phi_t v)-c(F))\,dt}} \leq c_{K,r,T_0} \;;$$

(3) Using Lemma 3.14 and this equivalence of definitions, it is easy to check that, up to changing the constant $c_{K,r}$, the Gibbs property does not depend on the constant $r > 0$.

Indeed let $r' \geq r > 0$ and let $T_{r,r'}$ be as in Lemma 3.14. Since $|F|$ is bounded, say by $\kappa \geq 0$, on the compact subset $\bigcup_{t \in [-T_{r,r'}, T_{r,r'}]} \phi_t K$ for every compact subset $K$ of $T^1M$, we have, for all $T, T' \geq 0$ and for all $v \in T^1M$ with $\phi_{-T'}v, \phi_T v \in K$,
$$e^{-2\,T_{r,r'}(\kappa+|c(F)|)}\; \frac{m'(B(v;T+T_{r,r'},T'+T_{r,r'},r'))}{e^{\int_{-T'-T_{r,r'}}^{T+T_{r,r'}} (F(\phi_t v)-c(F))\,dt}} \leq \frac{m'(B(v;T,T',r))}{e^{\int_{-T'}^T (F(\phi_t v)-c(F))\,dt}} \leq \frac{m'(B(v;T,T',r'))}{e^{\int_{-T'}^T (F(\phi_t v)-c(F))\,dt}} \;.$$

Hence the Gibbs property for $r'$ implies the Gibbs property for $r$. The converse implication is proved by noticing that, for all $T, T' \geq 0$ large enough (and in particular at least $T_{r,r'}$) and $v \in T^1M$ with $\phi_{-T'}v, \phi_T v \in K$,
$$\frac{m'(B(v;T,T',r))}{e^{\int_{-T'}^T (F(\phi_t v)-c(F))\,dt}} \leq \frac{m'(B(v;T,T',r'))}{e^{\int_{-T'}^T (F(\phi_t v)-c(F))\,dt}} \leq e^{2\,T_{r,r'}(\kappa+|c(F)|)}\; \frac{m'(B(v;T-T_{r,r'},T'-T_{r,r'},r))}{e^{\int_{-T'+T_{r,r'}}^{T-T_{r,r'}} (F(\phi_t v)-c(F))\,dt}} \;.$$

(4) We will prove in Proposition 7.13 that, under some recurrence properties (as a very particular case when $M$ is compact), there exists at most one $\phi$-invariant measure on $T^1M$ which satisfies the Gibbs property for the potential $F$.

The following result shows that our terminology of Gibbs measures is indeed appropriate.

**Proposition 3.16** *Assume that $\delta_{\Gamma,\,F} < +\infty$. Let $m$ be the Gibbs measure on $T^1M$ associated with a pair of Patterson densities $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ of dimension $\delta_{\Gamma,\,F}$ for $(\Gamma, F\circ\iota)$ and $(\Gamma, F)$ respectively. Then $m$ satisfies the Gibbs property for the potential $F$, with constant $c(F) = \delta_{\Gamma,\,F}$.*

In Chapter 6, we will prove that the critical exponent $\delta_{\Gamma,\,F}$ is equal to the pressure $P(\Gamma, F)$, so that $c(F) = P(\Gamma, F)$ in accordance with the case of symbolic dynamics over a finite alphabet.

**Proof.** We start with the following lemma, which describes the dynamical balls in terms of points at infinity.

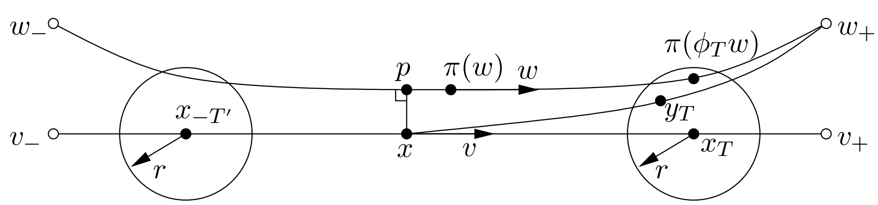
**Lemma 3.17** *For all $r > 0$ and $T, T' \geq 0$, for every $v \in T^1\widetilde{M}$, using the Hopf parametrisation with base point $x = \pi(v)$ and setting $x_T = \pi(\phi_T v)$ and $x_{-T'} = \pi(\phi_{-T'}v)$, we have*

$$B(v; T, T', r) \subset \mathscr{O}_x B(x_{-T'}, 2r) \times \mathscr{O}_x B(x_T, 2r) \times \,]-r, r[\ .$$

*Conversely, for every $r > 0$, there exists $T_r > 0$ such that for all $T, T' \geq T_r$ and $v \in T^1\widetilde{M}$, using the Hopf parametrisation with base point $x = \pi(v)$ and taking $x_{-T'}$ and $x_T$ as above, we have*

$$\mathscr{O}_x B(x_{-T'}, r) \times \mathscr{O}_x B(x_T, r) \times \,]-1, 1[\ \subset B(v; T, T', 2r + 2)\ .$$

**Proof.** To prove the first claim, for every $w \in B(v; T, T', r)$, let $p$ be the closest point to $x$ on the geodesic line defined by $w$ and let $y_T$ be the point at distance $T$ from $x$ on the geodesic ray $[x, w_+[$ .




Since the closest point maps do not increase the distances, we have

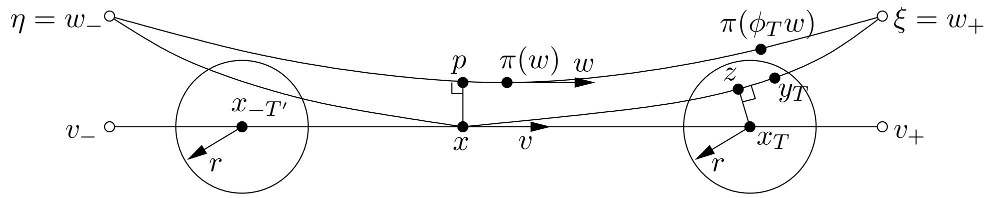
$$d(p, \pi(w)) \leq d(\pi(v), \pi(w)) < r\ .$$

By the triangle inequality and by convexity, we have

$$d(y_T, x_T) \leq d(y_T, \pi(\phi_T w)) + d(\pi(\phi_T w), \pi(\phi_T v)) \leq d(x, \pi(w)) + r < 2r\ ,$$

hence $w_+$ belongs to $\mathscr{O}_x B(x_T, 2r)$. With a similar argument for $w_-$, this proves the first claim.

To prove the second claim, let $\eta \in \mathscr{O}_x B(x_{-T'}, r)$ and let $\xi \in \mathscr{O}_x B(x_T, r)$, which is different from $\eta$ if $T$ and $T'$ are larger than a constant depending only on $r$. Let $p$ be the closest point to $x$ on the geodesic line $]\eta, \xi[$. Since the height of a vertex of a geodesic triangle in a CAT$(-1)$ space whose comparison angle is close to $\pi$ is small, if $T$ and $T'$ are larger than a constant depending only on $r$, we have $d(x, p) \leq 1$. Let $w \in T^1\widetilde{M}$ be such that $d(p, \pi(w)) < 1$ and $w_- = \eta$, $w_+ = \xi$.




Let us prove that $w \in B(v; T, T', 2r+2)$, which yields the second claim. By convexity, we only have to prove that $d(\pi(\phi_{-T'}w), x_{-T'}) < 2r+2$ and $d(\pi(\phi_T w), x_T) < 2r+2$. Since the argument for the first inequality is similar, we only prove the second inequality.

Let $z$ be the closest point to $x_T$ on the geodesic ray $[x, \xi[$, which satisfies $d(z, x_T) \leq r$. Note that $d(x,z) \geq d(x, x_T) - d(x_T, z) \geq T - r$ by the inverse triangle inequality. Let $y_T$ be the point at distance $T$ from $x$ on the geodesic ray $[x, w_+[$, which satisfies

$$d(\pi(\phi_T w), y_T) \leq d(\pi(w), x) \leq d(\pi(w), p) + d(p, x) < 2$$

by convexity. By the convexity of balls, we have $z \in [x, y_T]$. Then

$$\begin{aligned} d(\pi(\phi_T w), x_T) &\leq d(\pi(\phi_T w), y_T) + d(y_T, x_T) < 2 + d(y_T, z) + d(z, x_T) \\ &\leq 2 + \big(d(y_T, x) - d(x,z)\big) + r \leq 2 + r + T - (T - r) = 2r + 2 \,. \quad \Box \end{aligned}$$

Now, let $\widetilde{m}$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ of dimension $\delta = \delta_{\Gamma,F}$ for $(\Gamma, F\circ\iota)$ and $(\Gamma, F)$ respectively.

Let $\widetilde{K}$ be a compact subset of $T^1\widetilde{M}$. Let $r > 2$ and $C > 0$ be such that the conclusion of Mohsen's shadow lemma 3.10 holds true for $R = \frac{r-2}{2}$ and $R = 2r$, for the Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$, and for all $x, y$ in the compact subset $\pi(\widetilde{K})$ of $\widetilde{M}$. Let $T_* = T_{\frac{r-2}{2}}$ be given by Lemma 3.17. Let $x = \pi(v)$, which stays in the compact set $\mathscr{N}_{T_*}(\pi(\widetilde{K}))$.

Let us prove that there exists $c' > 0$ such that for all $T \geq 0$ large enough (in particular $T \geq T_*$) and for every $v \in T^1\widetilde{M}$ such that $\phi_{-T_*}v \in \widetilde{K}$ and $\phi_T v \in \Gamma\widetilde{K}$, we have

$$\frac{1}{c'} \leq \frac{\widetilde{m}(B(v; T, T_*, r))}{e^{\int_{-T_*}^{T} (\widetilde{F}(\phi_t v) - \delta)\, dt}} \leq c' \,. \tag{47}$$

Note that $\phi_{-T_*}B(v; T, T_*, r)$ is contained in a compact subset of $T^1\widetilde{M}$ (depending only on $\widetilde{K}$ and $r$). Hence the multiplicity of the restriction to $\phi_{-T_*}B(v; T, T_*, r)$ of the canonical projection $T^1\widetilde{M} \to T^1 M$ is bounded, by the discreteness of $\Gamma$. Since $\widetilde{m}$ is $\phi$-invariant, Equation (47) hence implies Proposition 3.16, using Remark (2) following Definition 3.15.

By Lemma 3.4 (1) with $r_0 = r$, there exists a constant $c_1 > 0$ such that for all $v \in T^1\widetilde{M}$, $T \geq 0$ and $w \in B(v; T, T_*, r)$, since $d(x, \pi(w)) \leq r$, we have

$$\begin{aligned} &\max\big\{\, |\, C_{F\circ\iota-\delta,\, w_-}(x, \pi(w))\,|,\ |\, C_{F-\delta,\, w_+}(x, \pi(w))\,|\,\big\} \\ &\leq\ c_1' =\ c_1\, e^r\ +\ r \max_{\pi^{-1}(\mathscr{N}_{r+T_*}(\pi(\widetilde{K})))} |\widetilde{F} - \delta| \,. \end{aligned}$$

Note that $c_1'$ is finite since $\widetilde{F}$ is bounded on compact subsets. We use the Hopf parametrisation with respect to the base point $x$. Let $x_T = \pi(\phi_T v)$ and $x_{-T_*} = \pi(\phi_{-T_*}v)$, which belong to $\Gamma\pi(\widetilde{K})$.

By Equation (43) and the second claim in Lemma 3.17, we hence have

$$\widetilde{m}(B(v;T,T_*,r)) \geq 2\,e^{-2\,c_1'}\;\mu_x^\iota(\mathscr{O}_x B(x_{-T_*},\frac{r-2}{2}))\;\mu_x(\mathscr{O}_x B(x_T,\frac{r-2}{2})).$$

Mohsen's shadow lemma 3.10 and Equation (16) imply that

$$\begin{aligned}\widetilde{m}(B(v;T,T_*,r)) &\geq 2\,e^{-2\,c_1'}\,C^{-2}\;e^{\int_x^{x_{-T_*}}(\widetilde{F}\circ\iota-\delta)}\;e^{\int_x^{x_T}(\widetilde{F}-\delta)}\\ &= 2\,e^{-2\,c_1'}\,C^{-2}\;e^{\int_{-T_*}^{T}(\widetilde{F}(\phi_t v)-\delta)\,dt}\;.\end{aligned}$$

This proves the lower bound in Equation (47).

The proof of the upper bound is similar. By Equation (43) and the first claim in Lemma 3.17, we have

$$\widetilde{m}(B(v;T,T_*,r)) \leq 2\,r\,e^{-2\,c_1'}\;\mu_x^\iota(\mathscr{O}_x B(x_{-T_*},2\,r))\;\mu_x(\mathscr{O}_x B(x_T,2\,r)).$$

Mohsen's shadow lemma 3.10 and Equation (16) imply that

$$\begin{aligned}\widetilde{m}(B(v;T,T_*,r)) &\leq 2\,r\,e^{2\,c_1'}\,C^{2}\;e^{\int_x^{x_{-T_*}}(\widetilde{F}\circ\iota-\delta)}\;e^{\int_x^{x_T}(\widetilde{F}-\delta)}\\ &= 2\,r\,e^{2\,c_1'}\,C^{2}\;e^{\int_{-T_*}^{T}(\widetilde{F}(\phi_t v)-\delta)\,dt}\;.\end{aligned}$$

This proves the upper bound in Equation (47), and Proposition 3.16 follows. □

**Remark.** A more usual definition of the dynamical balls is the following one. For every $T \geq 0$, consider the $\Gamma$-invariant distance $d'_{\phi,T}$ on $T^1\widetilde{M}$ defined by

$$\forall\; v,w \in T^1\widetilde{M},\quad d'_{\phi,\,T}(v,w) = \max_{t\in[0,T]} d'(\phi_t v,\phi_t w)\;,$$

where $d' = d'_{T^1\widetilde{M}}$ is the distance on $T^1\widetilde{M}$ defined in Equation (7). For all $\epsilon > 0$ and $T \geq 0$, let

$$B'(v;\epsilon,T) = \{w \in T^1\widetilde{M}\;:\;\max_{t\in[0,T]} d'(\phi_t v,\phi_t w)\;<\epsilon\}$$

be the open ball with centre $v \in T^1\widetilde{M}$ and radius $\epsilon$ for this distance $d'_{\phi,\,T}$. By the definition of $d'$, we have

$$B'(v;\epsilon,T) = B(v;T,-1,\epsilon)\;.$$

In particular, by the remarks (2) and (3) following Definition 3.15, a $\phi$-invariant measure $m'$ on $T^1M$ satisfies the Gibbs property for the potential $F$ with constant $c(F)$ if and only if for every compact subset $K$ of $T^1M$, there exists (or equivalently for every) $\epsilon > 0$, there exists $c_{K,\,\epsilon} \geq 1$ such that for every (or equivalently for every large enough) $T \geq 0$, for every $v \in T^1M$ with $v, \phi_T v \in K$, we have

$$\frac{1}{c_{K,\,\epsilon}} \leq \frac{m'(B'(v;\epsilon,T))}{e^{\int_0^T(F(\phi_t v)-c(F))\,dt}} \leq c_{K,\,\epsilon}\;.$$

Note that if $d' = d'_{T^1\widetilde{M}}$ is replaced by any Hölder-equivalent distance $d''$ (for instance the Riemannian distance $d_S$ induced by the Sasaki metric, or the distance $d = d_{T^1\widetilde{M}}$ defined in Equation (5)), and if $B''(v;\epsilon,T)$ are the corresponding dynamical balls, then there exist $\epsilon_0, c>0$ and $\alpha\in\,]0,1]$ such that, for all $\epsilon\in\,]0,\epsilon_0]$ and $T\geq 0$,

$$B'(v;\frac{1}{c}\,\epsilon^{\frac{1}{\alpha}},T)\subset B''(v;\epsilon,T)\subset B'(v;c\,\epsilon^{\alpha},T)\;.$$

Since $\widetilde{M}$ has pinched negative curvature and by the divergence properties of the geodesic lines, for all $r>0$, there exist $\epsilon_r, a_r, b_r, c_r, T_r>0$ such that for all $\epsilon\in\,]0,\epsilon_r]$ and $T\geq T_r$,

$$\begin{aligned}B(v;T-a_r\log\epsilon+c_r,-a_r\log\epsilon+c_r,r)&\\ \subset B'(v;\epsilon,T)\subset&\\ &B(v;T-b_r\log\epsilon-c_r,-b_r\log\epsilon-c_r,r)\;.\end{aligned}$$

When $\widetilde{M}$ has constant curvature $-1$, we may take $a_r=b_r=1$. But in general variable curvature, as $\epsilon$ tends to 0, the relationship between the dynamical balls $B'(v;\epsilon,T)$ and our ones $B'(v;T,T',r)$ becomes less precise.

### 3.9 Conditional measures of Gibbs states on strong (un)stable leaves

The aim of this subsection is to describe the conditional measures of the Gibbs measures constructed in Subsection 3.7 on the leaves of the strong unstable and strong stable foliations. We refer for instance to [DeM, Chap. 3, §70] for general information on the disintegration of measures.

We start by describing the family of measures on the strong unstable and strong stable leaves which will allow us to determine the conditional measures of the Gibbs measures.

Let $\sigma\in\mathbb{R}$. A Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\sigma$ for $(\Gamma,F)$ defines a family of measures $(\mu_{W^{\rm su}(v)})_{v\in T^1\widetilde{M}}$ on the strong unstable leaves, in the following way. Recall that for every $v\in T^1\widetilde{M}$, the map $w\mapsto w_+$ from $W^{\rm su}(v)$ to $\partial_\infty\widetilde{M}-\{v_-\}$ is a homeomorphism. This allows us to define a measure $\mu_{W^{\rm su}(v)}$ on $W^{\rm su}(v)$ by

$$d\mu_{W^{\rm su}(v)}(w)=e^{C_{F-\sigma,\,w_+}(x_0,\,\pi(w))}\,d\mu_{x_0}(w_+)\;,\tag{48}$$

where $x_0$ is any point of $\widetilde{M}$. This measure is nonzero, since $\Gamma$ being non-elementary, the support of $\mu_{x_0}$ is not reduced to $\{v_-\}$. By the absolute continuity property (39) of $(\mu_x)_{x\in\mathbb{R}}$ and by the cocycle property (25) of the Gibbs cocycle $C_{F-\sigma}$, the measure $\mu_{W^{\rm su}(v)}$ does not depend on the choice of $x_0$. By the invariance properties (38) and (26), the family of measures $(\mu_{W^{\rm su}(v)})_{v\in T^1\widetilde{M}}$ is $\Gamma$-equivariant: for all $v\in T^1\widetilde{M}$ and $\gamma\in\Gamma$,

$$\gamma_*\mu_{W^{\rm su}(v)}=\mu_{W^{\rm su}(\gamma v)}\;.\tag{49}$$

Note that $\mu_{W^{\rm su}(v)}=\mu_{W^{\rm su}(v')}$ if $v$ and $v'$ are in the same strong unstable leaf. If the support of $\mu_x$ is $\Lambda\Gamma$, the support of $\mu_{W^{\rm su}(v)}$ is

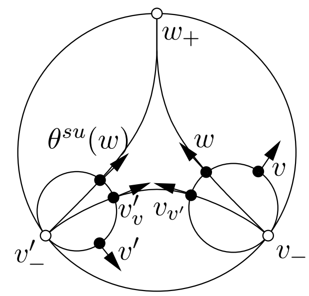
$$\operatorname{Supp}(\mu_{W^{\rm su}(v)})=\{w\in W^{\rm su}(v)\ :\ w_+\in\Lambda\Gamma\}\;.\tag{50}$$

We will also consider the measure $\mu_{W^{\rm su}(v)}$ on $W^{\rm su}(v)$ as a measure on $T^1\widetilde{M}$ with support contained in $W^{\rm su}(v)$. The map $v\mapsto\mu_{W^{\rm su}(v)}$ is continuous for the weak-star topology on the space of measures on $T^1\widetilde{M}$. The family of measures $(\mu_{W^{\rm su}(v)})_{v\in T^1\widetilde{M}}$ uniquely determines the Patterson density $(\mu_x)_{x\in\widetilde{M}}$, by Equation (48). The geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ preserves the measure class of these measures: for all $t\in\mathbb{R}$ and $w\in W^{\rm su}(v)$, we have

$$\frac{d\,(\phi_t)_*\mu_{W^{\rm su}(v)}}{d\,\mu_{W^{\rm su}(\phi_t v)}}(\phi_t w) = e^{C_{F-\sigma,\,w_+}(\pi(\phi_t w),\,\pi(w))} = e^{\int_0^t (\widetilde{F}(\phi_s w)-\sigma)\,ds}\;, \tag{51}$$

(which is equal to $e^{\int_{\pi(w)}^{\pi(\phi_t w)}(\widetilde{F}-\sigma)}$ if $t\geq 0$ and to $e^{-\int_{\pi(\phi_t w)}^{\pi(w)}(\widetilde{F}-\sigma)}$ otherwise, by Equation (22)).

For all $v,v'\in T^1\widetilde{M}$ such that $v_-\neq v'_-$, let $v_{v'}$ be the unique element of $W^{\rm su}(v)$ such that $(v_{v'})_+ = v'_-$. The map $\theta^{su}=\theta^{su}_{v',v}: W^{\rm su}(v)-\{v_{v'}\}\to W^{\rm su}(v')-\{v'_v\}$ sending $w$ to the unique element $\theta^{su}(w)$ such that $\theta^{su}(w)_+ = w_+$, is a homeomorphism. An easy computation using Equation (48) and Equation (25) shows that for every $w\in W^{\rm su}(v)-\{v_{v'}\}$, we have




$$\frac{d\,(\theta^{su})_*\mu_{W^{\rm su}(v)}}{d\,\mu_{W^{\rm su}(v')}}(\theta^{su}(w)) = e^{C_{F-\sigma,\,w_+}(\pi(\theta^{su}(w)),\,\pi(w))}\;. \tag{52}$$

Conversely, a family of nonzero measures $(\mu_{W^{\rm su}(v)})_{v\in T^1\widetilde{M}}$ on $T^1\widetilde{M}$, constant on the strong unstable leaves, with $\mu_{W^{\rm su}(v)}$ supported in $W^{\rm su}(v)$, satisfying the properties (49), (51) and (52) defines a unique Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\sigma$ for $(\Gamma,F)$ satisfying Equation (48).

By the equivariance property (49), the family $(\mu_{W^{\rm su}(v)})_{v\in T^1\widetilde{M}}$ induces on $T^1M$ a weak-star continuous family of measures, which we will denote by $(\mu_{W^{\rm su}(v)})_{v\in T^1M}$, constant on the strong unstable leaves, with $\mu_{W^{\rm su}(v)}$ supported in $W^{\rm su}(v)$ for every $v\in T^1M$.

A Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\sigma$ for $(\Gamma,F)$ also defines a family of measures $(\mu_{W^{\rm u}(v)})_{v\in T^1\widetilde{M}}$ on the unstable leaves, in the following way.

Let $x_0\in\widetilde{M}$ and $v\in T^1\widetilde{M}$. We have a homeomorphism from $W^{\rm su}(v)\times\mathbb{R}$ to $W^{\rm u}(v)$, defined by $(w,t)\mapsto w'=\phi_t w$, whose inverse is the map

$$w'\mapsto \big(w=\phi_{\beta_{v_-}(\pi(v),\,\pi(w'))}w',\, t=\beta_{v_-}(\pi(w'),\pi(v))\big)\;.$$

For future use, note that these homeomorphisms, as $v$ ranges over $T^1\widetilde{M}$, satisfy the following equivariance property: For every $\gamma\in\Gamma$, the following diagram commutes:

$$\begin{array}{ccc} W^{\rm su}(v)\times\mathbb{R} & \rightarrow & W^{\rm u}(v)\\ {}^{\gamma\times{\rm id}}\downarrow & & \downarrow^{\gamma}\\ W^{\rm su}(\gamma v)\times\mathbb{R} & \rightarrow & W^{\rm u}(\gamma v)\;. \end{array}$$

Using this homeomorphism from $W^{\rm su}(v)\times\mathbb{R}$ to $W^{\rm u}(v)$, as well as the homeomorphism from the unstable leaf $W^{\rm u}(v)$ to $(\partial_\infty\widetilde{M}-\{v_-\})\times\mathbb{R}$ defined by

$$w'\mapsto \big(w'_+,t=\beta_{v_-}(\pi(w'),\pi(v))\big)\;,$$

we may define a nonzero (positive Borel) measure $\mu_{W^{\rm u}(v)}$ on $W^{\rm u}(v)$, by

$$\begin{aligned} d\mu_{W^{\rm u}(v)}(w') &= e^{C_{F-\sigma,\,w_+}(\pi(w),\,\pi(\phi_t w))}\; d\mu_{W^{\rm su}(v)}(w)dt \\ &= e^{C_{F-\sigma,\,w'_+}(x_0,\,\pi(w'))}\; d\mu_{x_0}(w'_+)dt\;. \end{aligned} \tag{53}$$

In particular, the second formula is independent of the choice of $x_0$. Note that when $\widetilde{F}=0$, the first equality simplifies as

$$d\mu_{W^{\rm u}(v)}(w') = e^{-\sigma\, t}\; d\mu_{W^{\rm su}(v)}(w)dt\;.$$

By the invariance property (26) of the Gibbs cocycle and by Equation (49), the family of measures $(\mu_{W^{\rm u}(v)})_{v\in T^1\widetilde{M}}$ is equivariant under $\Gamma$: for all $v\in T^1\widetilde{M}$ and $\gamma\in\Gamma$,

$$\gamma_*\mu_{W^{\rm u}(v)} = \mu_{W^{\rm u}(\gamma v)}\;. \tag{54}$$

Note that $\mu_{W^{\rm u}(v)}=\mu_{W^{\rm u}(v')}$ if $v$ and $v'$ are in the same unstable leaf, by Equation (53) and since the parametrisation of $W^{\rm u}(v)$ by $(\partial_\infty\widetilde{M}-\{v_-\})\times\mathbb{R}$ is only changed, when passing from $v$ to $v'$, by adding a constant to the parameter $t$, as the Lebesgue measure $dt$ is invariant under translations. If the support of $\mu_x$ is $\Lambda\Gamma$, the support of $\mu_{W^{\rm u}(v)}$ is

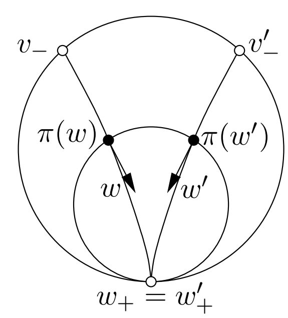
$$\operatorname{Supp}(\mu_{W^{\rm u}(v)}) = \{w\in W^{\rm u}(v)\;:\;w_+\in\Lambda\Gamma\}\;. \tag{55}$$

The map $v\mapsto\mu_{W^{\rm u}(v)}$ is continuous for the weak-star topology on the space of measures on $T^1\widetilde{M}$. The family of measures $(\mu_{W^{\rm u}(v)})_{v\in T^1\widetilde{M}}$ uniquely determines the family of measures $(\mu_{W^{\rm su}(v)})_{v\in T^1\widetilde{M}}$, hence the Patterson density $(\mu_x)_{x\in\widetilde{M}}$. The geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ preserves the measure class of these measures: for all $t\in\mathbb{R}$ and $w\in W^{\rm u}(v)$, we have

$$\frac{d\,(\phi_t)_*\mu_{W^{\rm u}(v)}}{d\,\mu_{W^{\rm u}(v)}}(\phi_t w) = e^{\int_0^t(\widetilde{F}(\phi_s w)-\sigma)\,ds}\;. \tag{56}$$

For all $v,v'\in T^1\widetilde{M}$, the map $\theta^u=\theta^u_{v',v}$ : $W^{\rm u}(v)-\{w\in W^{\rm u}(v)\;:\;w_+=v'_-\}\to W^{\rm u}(v')-\{w'\in W^{\rm u}(v')\;:\;w'_+=v_-\}$ sending $w$ to the unique element $w'=\theta^u(w)$ in $W^{\rm ss}(w)$ such that $w'_-=v'_-$ is a homeomorphism. An easy computation using Equation (52) shows that for every $w\in W^{\rm u}(v)$ with $w_+\neq v'_-$, we have

$$\frac{d\,(\theta^u)_*\mu_{W^{\rm u}(v)}}{d\,\mu_{W^{\rm u}(v')}}(\theta^u(w)) = e^{C_{F-\sigma,\,w_+}(\pi(\theta^u(w)),\,\pi(w))}\;. \tag{57}$$




Similarly, given $\sigma^\iota\in\mathbb{R}$ and a Patterson density $(\mu^\iota_x)_{x\in\widetilde{M}}$ of dimension $\sigma^\iota$ for $(\Gamma,F\circ\iota)$, we define weak-star continuous families $(\mu^\iota_{W^{\rm ss}(v)})_{v\in T^1\widetilde{M}}$ and $(\mu^\iota_{W^{\rm s}(v)})_{v\in T^1\widetilde{M}}$ of nonzero measures on $T^1\widetilde{M}$, constant on respectively the strong stable leaves and stable leaves, equivariant under $\Gamma$, as follows:

- using the homeomorphism from $W^{\rm ss}(v)$ to $\partial_\infty\widetilde{M}-\{v_+\}$ defined by $w\mapsto w_-$, we set

$$d\mu^\iota_{W^{\rm ss}(v)}(w) = e^{C_{F\circ\iota-\sigma^\iota,\,w_-}(x_0,\,\pi(w))}\; d\mu^\iota_{x_0}(w_-)\;, \tag{58}$$

which is independent of $x_0 \in \widetilde{M}$,

• using the homeomorphism from $W^{\rm ss}(v)\times\mathbb{R}$ to $W^{\rm s}(v)$, defined by $(w,t)\mapsto w'=\phi_t w$, and the homeomorphism from $W^{\rm s}(v)$ to $(\partial_\infty \widetilde{M}-\{v_+\})\times\mathbb{R}$ defined by $w'\mapsto \big(w'_-, t=\beta_{v_+}(\pi(v),\pi(w'))\big)$, we set

$$d\mu^{\iota}_{W^{\rm s}(v)}(w') = e^{C_{F\circ\iota-\sigma^\iota,\,w_-}(\pi(w),\,\pi(\phi_t w))}\; d\mu^{\iota}_{W^{\rm ss}(v)}(w)dt \tag{59}$$

$$= e^{C_{F\circ\iota-\sigma^\iota,\,w'_-}(x_0,\,\pi(w'))}\; d\mu^{\iota}_{x_0}(w'_-)dt\;. \tag{60}$$

As above, for all $v\in T^1\widetilde{M}$ and $\gamma\in\Gamma$, we have

$$\gamma_*\mu^{\iota}_{W^{\rm ss}(v)} = \mu^{\iota}_{\gamma W^{\rm ss}(v)} \quad\text{and}\quad \gamma_*\mu^{\iota}_{W^{\rm s}(v)} = \mu^{\iota}_{\gamma W^{\rm s}(v)}\;. \tag{61}$$

As above, if the support of $\mu^\iota_x$ is $\Lambda\Gamma$, then the support of $\mu^\iota_{W^{\rm ss}(v)}$ is $\{w\in W^{\rm ss}(v)\;:\;w_-\in\Lambda\Gamma\}$, and, for every $t\in\mathbb{R}$,

$$\forall\, w\in W^{\rm ss}(v),\quad \frac{d\,(\phi_t)_*\mu^{\iota}_{W^{\rm ss}(v)}}{d\,\mu^{\iota}_{W^{\rm ss}(\phi_t v)}}(\phi_t w) = e^{-\int_0^t (\widetilde{F}(\phi_s w)-\sigma^\iota)\,ds}\;,$$

$$\forall\, w\in W^{\rm s}(v),\quad \frac{d\,(\phi_t)_*\mu^{\iota}_{W^{\rm s}(v)}}{d\,\mu^{\iota}_{W^{\rm s}(v)}}(\phi_t w) = e^{-\int_0^t (\widetilde{F}(\phi_s w)-\sigma^\iota)\,ds}\;.$$

For all $v,v'\in T^1\widetilde{M}$, the map $\theta^s:\{w\in W^{\rm s}(v)\;:\;w_-\neq v'_+\}\to\{w'\in W^{\rm s}(v')\;:\;w'_-\neq v_+\}$ sending $w$ to the unique element $w'$ in $W^{\rm su}(w)\cap W^{\rm s}(v')$, is a homeomorphism. For every $w$ in the domain of $\theta^s$, we have

$$\frac{d\,(\theta^s)_*\mu^{\iota}_{W^{\rm s}(v)}}{d\,\mu^{\iota}_{W^{\rm s}(v')}}(\theta^s(w)) = e^{C_{F\circ\iota-\sigma^\iota,\,w_-}(\pi(\theta^s(w)),\,\pi(w))}\;. \tag{62}$$

Now, before determining the conditional measures of the Gibbs measures, we describe the local product structures of $T^1\widetilde{M}$ defined by the strong unstable and strong stable foliations, giving the local fibrations allowing to disintegrate the Gibbs measures.

Fix $w\in T^1\widetilde{M}$, and let

$$U_w = T^1\widetilde{M} - (W^{\rm s}(-w)\cup W^{\rm u}(-w)) = \{v\in T^1\widetilde{M}\;:\, v_+\neq w_-, v_-\neq w_+\}$$

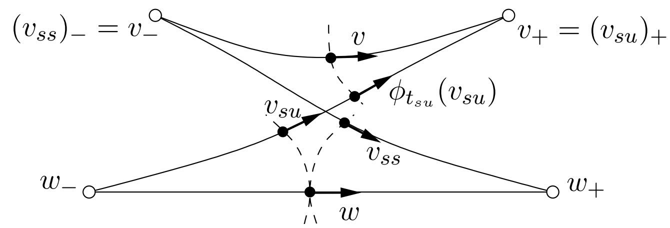
be the open neighbourhood of $w$ in $T^1\widetilde{M}$ consisting of the unit tangent vectors that belong neither to the stable nor to the unstable leaf of $-w$. Note that $U_w$ is dense in $T^1\widetilde{M}$. For every $v\in U_w$,

• let $v_{su}$ be the intersection point of $W^{\rm su}(w)$ and $W^{\rm s}(v)$, which is the unique point $v_{su}$ of $W^{\rm su}(w)$ such that $(v_{su})_+=v_+$,

• let $v_{ss}$ be the intersection point of $W^{\rm ss}(w)$ and $W^{\rm u}(v)$, which is the unique point $v_{ss}$ of $W^{\rm ss}(w)$ such that $(v_{ss})_-=v_-$,

• and let $t_{su}\in\mathbb{R}$ be the unique real number such that $\phi_{t_{su}}(v_{su})\in W^{\rm ss}(v)$.

Then the map $v\mapsto(v_{ss},v_{su},t_{su})$ from $U_w$ to $W^{\rm ss}(w)\times W^{\rm su}(w)\times\mathbb{R}$ is a homeomorphism. Note that, when $v_-$ and $v_+$ are fixed, so is $(v_{su})_+$, and the difference of the time parameters $t-t_{su}$ between the Hopf parametrisation and this one is constant.




The image by this homeomorphism of the Gibbs measures is absolutely continuous with respect to product measures. Indeed, let $\sigma \in \mathbb{R}$ and let $\widetilde{m}$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\sigma$ for $(\Gamma, F\circ\iota)$ and $(\Gamma, F)$ respectively. Fix $x \in \widetilde{M}$. By definition of the measures on the strong unstable and stable leaf, we have

$$d\mu_{W^{\rm su}(w)}(v_{su}) = e^{C_{F-\sigma,\,(v_{su})_+}(x,\,\pi(v_{su}))}\; d\mu_x((v_{su})_+)$$

and

$$d\mu^\iota_{W^{\rm ss}(w)}(v_{ss}) = e^{C_{F\circ\iota-\sigma,\,(v_{ss})_-}(x,\,\pi(v_{ss}))}\; d\mu^\iota_x((v_{ss})_-)\;.$$

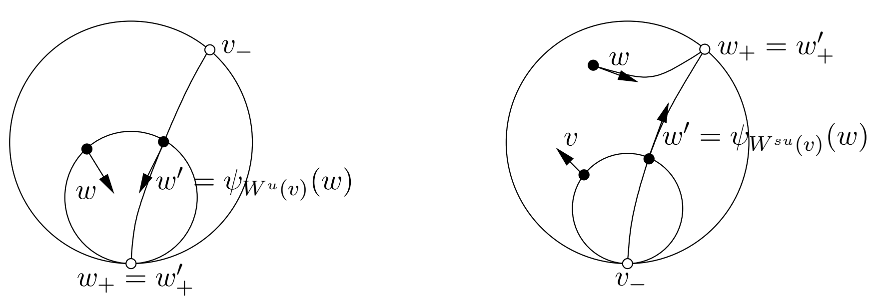
By Equation (43) and the cocycle property, for all $v \in U_w$, we hence have

$$d\widetilde{m}(v) = e^{C_{F\circ\iota-\sigma,\,v_-}(\pi(v_{ss}),\,\pi(v))+C_{F-\sigma,\,v_+}(\pi(v_{su}),\,\pi(v))}\; d\mu^\iota_{W^{\rm ss}(w)}(v_{ss})\, d\mu_{W^{\rm su}(w)}(v_{su})\, dt_{su}\;. \quad (63)$$

This quasi-product property, either in the form of Equation (43) or of Equation (63), is crucial in our book. In particular, if a measurable subset $A$ of $W^{\rm su}(w)$ has measure 0 for $\mu_{W^{\rm su}(w)}$, then the measurable set $\bigcup_{v\in A} W^{\rm s}(v)$ has measure 0 for $\widetilde{m}$.

A slightly different way of understanding this property is as follows. Let us fix $v \in T^1\widetilde{M}$. Let us define $U_{v_-} = \{w \in T^1\widetilde{M}\;:\; w_+ \neq v_-\}$, which is an open and dense subset of $T^1\widetilde{M}$, invariant under the geodesic flow. If $\mu_x(\{v_-\}) = 0$ (this does not depend on $x \in \widetilde{M}$), for instance if the measure $\mu_x$ has no atom, then $U_{v_-}$ has full measure with respect to $\widetilde{m}$. We have $U_{\gamma v_-} = \gamma U_{v_-}$ for every isometry $\gamma$ of $\widetilde{M}$, and in particular $U_{v_-}$ is invariant under the isometries of $\widetilde{M}$ fixing $v_-$.

The map $\psi_{W^{\rm u}(v)}$ from $U_{v_-}$ to $W^{\rm u}(v)$, sending $w \in U_{v_-}$ to the unique element $w'$ in $W^{\rm ss}(w)\cap W^{\rm u}(v)$, is a continuous fibration over the unstable leaf $W^{\rm u}(v)$, whose fibre over $w' \in W^{\rm u}(v)$ is precisely the strong stable leaf $W^{\rm ss}(w')$ of $w'$ (see the left hand picture below). This map depends only on the unstable leaf of $v$. For every isometry $\gamma$ of $\widetilde{M}$, we have $\psi_{W^{\rm u}(\gamma v)}\circ\gamma = \gamma\circ\psi_{W^{\rm u}(v)}$. For every $t \in \mathbb{R}$, we have $\psi_{W^{\rm u}(v)}\circ\phi_t = \phi_t\circ\psi_{W^{\rm u}(v)}$: the fibration $\psi_{W^{\rm u}(v)}$ commutes with the geodesic flow.



Similarly, for every $v \in T^1\widetilde{M}$, the map $\psi_{W^{\rm su}(v)}$ from $U_{v_-}$ to $W^{\rm su}(v)$, sending $w \in U_{v_-}$ to the unique element $w'$ in $W^{\rm s}(w) \cap W^{\rm su}(v)$, is a continuous fibration over the strong unstable leaf $W^{\rm su}(v)$, whose fibre over $w' \in W^{\rm su}(v)$ is precisely the stable leaf $W^{\rm s}(w')$ of $w'$ (see the right hand picture above). For every isometry $\gamma$ of $\widetilde{M}$, we have $\psi_{W^{\rm su}(\gamma v)} \circ \gamma = \gamma \circ \psi_{W^{\rm su}(v)}$. For every $t \in \mathbb{R}$, we have $\psi_{W^{\rm su}(v)} \circ \phi_t = \psi_{W^{\rm su}(v)}$: the fibration $\psi_{W^{\rm su}(v)}$ is invariant under the geodesic flow.

The aforementioned disintegration result, with respect to the strong unstable foliation, of the Gibbs measures, in the general setting of Remark (3) of Subsection 3.7, is the following one.

**Proposition 3.18** *Let $\widetilde{m}$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ of dimensions $\sigma^\iota$ and $\sigma$ for $(\Gamma, F\circ\iota)$ and $(\Gamma, F)$, respectively. Let $v \in T^1\widetilde{M}$.*

*(1) The restriction to $U_{v_-}$ of the measure $\widetilde{m}$ disintegrates by the fibration $\psi_{W^{\rm u}(v)}$ over the measure $\mu_{W^{\rm u}(v)}$, with conditional measure on the fibre $W^{\rm ss}(w')$ of $w'$ the measure $e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\; d\mu^\iota_{W^{\rm ss}(w')}(w)$: for every $w \in T^1\widetilde{M}$ such that $w_+ \neq v_-$, we have*

$$d\widetilde{m}(w) = \int_{w'\in W^{\rm u}(v)} e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\; d\mu^\iota_{W^{\rm ss}(w')}(w)\; d\mu_{W^{\rm u}(v)}(w')\;.$$

*(2) The restriction to $U_{v_-}$ of the measure $\widetilde{m}$ disintegrates by the fibration $\psi_{W^{\rm su}(v)}$ over the measure $\mu_{W^{\rm su}(v)}$, with conditional measure on the fibre $W^{\rm s}(w')$ of $w'$ the measure $e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\; d\mu^\iota_{W^{\rm s}(w')}(w)$: for every $w \in T^1\widetilde{M}$ such that $w_+ \neq v_-$, we have*

$$d\widetilde{m}(w) = \int_{w'\in W^{\rm su}(v)} e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\; d\mu^\iota_{W^{\rm s}(w')}(w)\; d\mu_{W^{\rm su}(v)}(w')\;.$$

We leave to the reader the analogous statements obtained by exchanging the stable and unstable foliations. When $F = 0$ and $(\mu^\iota_x)_{x\in\widetilde{M}} = (\mu_x)_{x\in\widetilde{M}}$, we recover the well-known fact, due to Margulis when $M$ is compact, that $\mu_{W^{\rm su}(v)}$ and $\mu_{W^{\rm ss}(v)}$ are the conditional measures of $\widetilde{m}$ along the leaves of the foliations $\widetilde{\mathscr{W}}^{\rm su}$ and $\widetilde{\mathscr{W}}^{\rm ss}$, respectively.

**Proof.** Fix $v \in T^1\widetilde{M}$. For every continuous map $\varphi \in \mathscr{C}_c(U_{v_-};\mathbb{R})$ with compact support in the domain $U_{v_-}$ of $\psi_{W^{\rm u}(v)}$, let

$$I_\varphi = \int_{w\in T^1\widetilde{M}} \varphi(w)\; d\widetilde{m}(w) = \int_{w\in U_{v_-}} \varphi(w)\; d\widetilde{m}(w)\;.$$

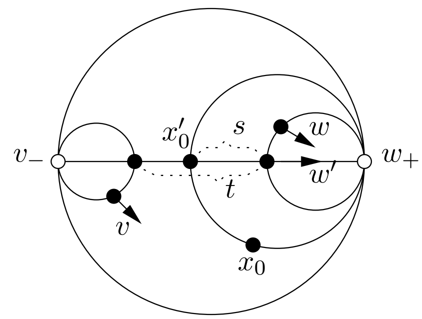
Using the Hopf parametrisation $w \mapsto \big(w_-, w_+, s = \beta_{w_+}(x_0, \pi(w))\big)$ as in Remark (2) of Subsection 3.7, and Equation (45) with $x'_0 = x_0$ fixed in $\widetilde{M}$, we have

$$I_\varphi = \int_{w_+\in\,\partial_\infty\widetilde{M}-\{v_-\}} \int_{w_-\in\,\partial_\infty\widetilde{M}-\{w_+\}} \int_{s\in\mathbb{R}} \varphi(w)\; e^{C_{F\circ\iota-\sigma^\iota,\,w_-}(x_0,\,\pi(w))+C_{F-\sigma,\,w_+}(x_0,\,\pi(w))}$$
$$d\mu^\iota_{x_0}(w_-)\, d\mu_{x_0}(w_+)\, ds\;.$$

For every $w \in U_{v_-}$, let $w' = \psi_{W^{\rm u}(v)}(w)$ be the unique point in $W^{\rm ss}(w) \cap W^{\rm u}(v)$. By the cocycle property of the Gibbs cocycle, and since $\pi(w)$ and $\pi(w')$ are in the same horosphere centred at $w_+ = w'_+$, we have

$$C_{F-\sigma,\,w_+}(x_0,\pi(w)) - C_{F-\sigma,\,w'_+}(x_0,\pi(w')) = C_{F-\sigma,\,w'_+}(\pi(w'),\pi(w))$$
$$= C_{F,\,w'_+}(\pi(w'),\pi(w)) \ .$$

When $w_+ = w'_+$ is fixed, the time parameter $s = \beta_{w_+}(x_0,\pi(w)) = \beta_{w_+}(x_0,\pi(w'))$ differs from the time parameter $t = \beta_{v_-}(\pi(w'),\pi(v))$ by a constant (equal to $\beta_{v_-}(\pi(v),x'_0)$ where $x'_0$ is the intersection point of the geodesic line $]v_-,w_+[$ and the horosphere centred at $w_+$ through $x_0$, by an easy computation), hence $ds = dt$.




By Equation (58) and Equation (53), we therefore have

$$I_\varphi = \int_{w'\in W^{\rm u}(v)}\int_{w\in W^{\rm ss}(w')} \varphi(w)\; e^{C_{F,\,w'_+}(\pi(w'),\,\pi(w))}\; d\mu^{\iota}_{W^{\rm ss}(w')}(w)\; d\mu_{W^{\rm u}(v)}(w') \ .$$

This is the first claim of Proposition 3.18. The second one is proved similarly. $\Box$

# 4 Critical exponent and Gurevich pressure

Let $(\widetilde M,\Gamma,\widetilde F)$ be as in the beginning of Chapter 2: $\widetilde M$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$; $\Gamma$ is a non-elementary discrete group of isometries of $\widetilde M$; and $\widetilde F : T^1\widetilde M \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map. The aim of this chapter is to prove results on the critical exponent $\delta_{\Gamma,\,F}$ which are valid under these general hypotheses (in particular, $\Gamma$ is only assumed to be non-elementary).

## 4.1 Counting orbit points and periodic geodesics

One way to study the repartition of an orbit of $\Gamma$ in $\widetilde M$ is to count the number of its elements in relatively compact subsets of $\widetilde M$. We will count them with weights given by the potential $\widetilde F$. We refer to Subsection 2.2 for the definition of the cone $\mathscr{C}_z B$ with vertex $z\in\widetilde M$ on $B\subset\partial_\infty\widetilde M$.

For all $s\geq 0$ and $c>0$, for all $x,y\in\widetilde M$ and for all open subsets $U$ and $V$ of $\partial_\infty\widetilde M$, let

$$G_{\Gamma,\,F,\,x,\,y,\,U,\,V}(t) = \sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\leq t\,,\;\gamma y\in\mathscr{C}_x U\,,\;\gamma^{-1}x\in\mathscr{C}_y V} e^{\int_x^{\gamma y}\widetilde F} \ ,$$

and

$$G_{\Gamma,\,F,\,x,\,y,\,U,\,V,\,c}(t) = \sum_{\gamma\in\Gamma\,:\,t-c<d(x,\gamma y)\leq t\,,\;\gamma y\in\mathscr{C}_x U\,,\;\gamma^{-1}x\in\mathscr{C}_y V} e^{\int_x^{\gamma y}\widetilde F} \ ,$$

The map $s\mapsto G_{\Gamma,\,F,\,x,\,y,\,U,\,V}(s)$ will be called the *bisectorial orbital counting function* of $(\Gamma,F,U,V)$, and $s\mapsto G_{\Gamma,\,F,\,x,\,y,\,U,\,V,\,c}(s)$ the *annular bisectorial orbital counting function* of $(\Gamma,F,U,V)$. When $V=\partial_\infty\widetilde M$, we denote them by $s\mapsto G_{\Gamma,\,F,\,x,\,y,\,U}(s)$ and

$s \mapsto G_{\Gamma,\,F,\,x,\,y,\,U,\,c}(s)$ and call them the *sectorial orbital counting function* and *annular sectorial orbital counting function* of $(\Gamma, F, U)$. When $U = V = \partial_\infty \widetilde{M}$, we denote them by $s \mapsto G_{\Gamma,\,F,\,x,\,y}(s)$ and $s \mapsto G_{\Gamma,\,F,\,x,\,y,\,c}(s)$, and call them the *orbital counting function* and *annular orbital counting function* of $(\Gamma, F)$. When $F = 0$, we recover the usual orbital counting functions of $\Gamma$, see for instance [Marg, Rob1] and [Bab3] as well as its references.

Other interesting counting functions are the ones counting periodic orbits of the geodesic flow on $T^1M$, again with weights given by the potential.

For every periodic (not necessarily primitive) orbit $g$ of length $\ell(g)$ of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M$, let $\mathscr{L}_g$ be the *Lebesgue measure along* $g$, that is, the measure on $T^1M$ with support $g$, such that, for every continuous map $f : T^1M \to \mathbb{R}$ and for any $v \in g$,

$$\mathscr{L}_g(f) = \int_0^{\ell(g)} f(\phi_t v)\; dt \;.$$

The *period of* $g$ *for the potential* $F$ is (for any $v \in g$)

$$\int_g F = \mathscr{L}_g(F) = \int_0^{\ell(g)} F(\phi_t v)\; dt \;.$$

One needs to be careful with the terminology and not confuse the length $\ell(g) = \mathscr{L}_g(1)$ of $g$ and its period $\int_g F = \mathscr{L}_g(F)$ (which depends on $F$).

For every $s \geq 0$, we denote by $\mathscr{P}\mathrm{er}(s) = \mathscr{P}\mathrm{er}_{T^1M}(s)$ the set of periodic orbits of the geodesic flow in $T^1M$ with length at most $s$; we allow non-primitive ones, that is, periodic orbits winding $n$ times around a primitive one, in which case its length and period are $n$ times the length and period of the primitive one, for any $n \in \mathbb{N}$. We denote by $\mathscr{P}\mathrm{er}'(s) = \mathscr{P}\mathrm{er}'_{T^1M}(s)$ the subset of primitive ones.

For every relatively compact open subset $W$ of $T^1M$, let us define the *period counting function* of $(\Gamma, F, W)$ by

$$s \mapsto Z_{\Gamma,\,F,\,W}(s) = \sum_{g \in \mathscr{P}\mathrm{er}(s),\; g\cap W \neq \emptyset} e^{\int_g F} \;,$$

and, for every $c > 0$, the *annular period counting function* of $(\Gamma, F, W)$ by

$$s \mapsto Z_{\Gamma,\,F,\,W,\,c}(s) = \sum_{g \in \mathscr{P}\mathrm{er}(s) - \mathscr{P}\mathrm{er}(s-c),\; g\cap W \neq \emptyset} e^{\int_g F} \;.$$

Since $M$ is not assumed to be compact, it is important to assume that $W$ is relatively compact in $T^1M$, since for instance when $M$ is an infinite Galois Riemannian cover of a compact negatively curved Riemannian manifold, and if $F$ is invariant under the covering group, then $T^1M$ contains infinitely many periodic orbits of the same period and of the same length at most $s$, for every $s$ large enough. Conversely, if $\Gamma$ is geometrically finite (see the definition of geometrically finite discrete groups of isometries in Subsection 8.2), there exists a relatively compact open subset of $T^1M$ meeting every periodic orbit. Note that $Z_{\Gamma,\,F,\,W}(s)$ is nonzero for $s$ large enough if and only if $W$ meets the (topological) non-wandering set $\Omega\Gamma$ of the geodesic flow in $T^1M$. The maps $c \mapsto G_{\Gamma,\,F,\,x,\,y,\,U,\,V,\,c}(t)$, $c \mapsto G_{\Gamma,\,F,\,x,\,y,\,U,\,c}(t)$, $c \mapsto G_{\Gamma,\,F,\,x,\,y,\,c}(t)$ and $c \mapsto Z_{\Gamma,\,F,\,W,\,c}(s)$ are nondecreasing.

We define the *Gurevich pressure* of $(\Gamma, F)$ by

$$P_{Gur}(\Gamma,\, F) = \limsup_{s\to+\infty}\ \frac{1}{s}\log Z_{\Gamma,\,F,\,W,\,c}(s)\ , \tag{64}$$

where $W$ is any relatively compact open subset of $T^1M$ meeting $\Omega\Gamma$ and $c$ any positive real number. The fact that the Gurevich pressure does not depend on $c>0$ is proved as the first claim of Assertion (vii) of Lemma 3.3. We will prove in Subsection 4.3 that the Gurevich pressure does not depend on $W$ too, and that the above upper limit is a limit if $c$ is large enough (see also Remark 9.13 for the problem of counting primitive versus non-primitive periodic orbits). Note that $\int_g (F+\kappa) = \int_g F + \kappa\ell(g)$ for every periodic orbit $g$ of the geodesic flow on $T^1M$, and that $P_{Gur}(\Gamma, F+\kappa) = P_{Gur}(\Gamma, F) + \kappa$ for every $\kappa\in\mathbb{R}$.

The Gurevich pressure is an (exponential) asymptotic growth rate of the periodic orbits of the geodesic flow, weighted by the potential. For Markov shifts on countable alphabets, it has been introduced by Gurevich [Gur1, Gur2] when the potential vanishes, and by Sarig [Sar1, Sar3, Sar2] in general. These last two works have motivated our study. To our knowledge, the Gurevich pressure had not been studied in a non symbolic non-compact context.

We will give in Chapter 9 precise asymptotic results as $t$ goes to $+\infty$ of these counting functions, under stronger hypotheses on $(\widetilde{M}, \Gamma, \widetilde{F})$. In the next subsection, we will only give weaker (logarithmic) asymptotic results, valid in general.

For instance, the following (very weak) result is an easy consequence of Mohsen's shadow lemma 3.10.

**Corollary 4.1** *For every $c>0$, we have $G_{\Gamma,\,F,\,x,\,y,\,U,\,V,\,c}(t) = O(e^{\delta_{\Gamma,\,F}\,t})$ as $t$ goes to $+\infty$. If $\delta_{\Gamma,\,F}>0$, we have $G_{\Gamma,\,F,\,x,\,y,\,U,\,V,\,c}(t) \le G_{\Gamma,\,F,\,x,\,y,\,U,\,V}(t) = O(e^{\delta_{\Gamma,\,F}\,t})$ as $t$ goes to $+\infty$.*

**Proof.** Since $G_{\Gamma,\,F,\,x,\,y,\,U,\,V} \le G_{\Gamma,\,F,\,x,\,y}$ and $G_{\Gamma,\,F,\,x,\,y,\,U,\,V,\,c} \le G_{\Gamma,\,F,\,x,\,y,\,c}$, we may assume that $U=V=\partial_\infty\widetilde{M}$. We may also assume that $\delta_{\Gamma,\,F}<+\infty$. Let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma=\delta_{\Gamma,\,F}$ (see Proposition 3.9) for $(\Gamma, F)$. Then the result follows from Corollary 3.11 (1), using a geometric series argument when $\delta_{\Gamma,\,F}>0$. □

## 4.2 Logarithmic growth of the orbital counting functions

In this subsection, we give logarithmic counting results which, though less precise than the results in Subsection 9.2, are valid in a much greater generality (they only require $\Gamma$ to be non-elementary).

The first result shows that the upper limit defining the critical exponent $\delta_{\Gamma,\,F}$ of $(\Gamma, F)$ is in fact a limit. Its proof follows closely the proof of the main result of [Rob2] (corresponding to the case $F=0$).

**Theorem 4.2** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$, and $x,y\in\widetilde{M}$. Let $\Gamma$ be a non-elementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F}: T^1\widetilde{M}\to\mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map. If $\delta_{\Gamma,\,F}>0$, then*

$$\delta_{\Gamma,\,F} = \lim_{n\to+\infty}\ \frac{1}{n}\log\sum_{\gamma\in\Gamma,\ d(x,\gamma y)\le n} e^{\int_x^{\gamma y}\widetilde{F}}\ .$$

With the previous notation, this can be written as

$$\lim_{s\to+\infty}\ \frac{1}{s}\log\ G_{\Gamma,\,F,\,x,\,y}(s)=\delta_{\Gamma,\,F}\ ,$$

that is, the orbital counting function grows logarithmically as $s\mapsto e^{\delta_{\Gamma,\,F}\,s}$. We will prove in Corollary 4.5 that the sectorial and bisectorial counting functions grow similarly (under obvious conditions).

**Proof.** Let $\delta=\delta_{\Gamma,\,F}>0$. For every $x'\in\widetilde{M}$, with $a_n=\sum_{\gamma\in\Gamma,\ d(x',\,\gamma y)\le n}\ e^{\int_{x'}^{\gamma y}\widetilde{F}}$, we have seen in Equation (21) that $\delta=\limsup_{n\to+\infty}\ \frac{1}{n}\log a_n$.

For a contradiction, assume that $\liminf_{n\to+\infty}\ \frac{1}{n}\log a_n<\delta$. Note that this lower limit does not depend on $x'$. Hence there exist a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and $\sigma\in\,]-\infty,\delta[$ such that for every $x'\in\widetilde{M}$, for every $k\in\mathbb{N}$ large enough,

$$a_{n_k}\le e^{\sigma n_k}\ . \tag{65}$$

Let us construct a Patterson density of dimension $\sigma$ for $(\Gamma,F)$, which contradicts Corollary 3.11 (2) since $\sigma<\delta$.

Let $\mathscr{D}_z$ be the (unit) Dirac mass at a point $z\in\widetilde{M}$. For all $t\in[0,+\infty[$ and $x'\in\widetilde{M}$, let

$$\nu_{x',\,t}=\frac{\sum_{\gamma\in\Gamma,\ d(x',\,\gamma y)\le t}\ e^{\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)}\ \mathscr{D}_{\gamma y}}{\sum_{\gamma\in\Gamma,\ d(y,\,\gamma y)\le t}\ e^{\int_{y}^{\gamma y}(\widetilde{F}-\sigma)}}\ .$$

By compactness, for every $r\in\mathbb{N}$, there exists a sequence $\big(k_i(r)\big)_{i\in\mathbb{N}}$ of positive integers such that the probability measures $\nu_{y,\,n_{k_i(r)}-r}$ weak-star converge to a probability measure $\mu_{y,\,r}$. Since the Poincaré series $\sum_{\gamma\in\Gamma}\ e^{\int_{y}^{\gamma y}(\widetilde{F}-\sigma)}$ diverges (as $\sigma<\delta$), the support of the measure $\mu_{y,\,r}$ is $\Lambda\Gamma$. For every $x'\in\widetilde{M}$, let us define

$$d\mu_{x',\,r}(\xi)=e^{C_{F-\sigma,\,\xi}(x',y)}\ d\mu_{y,\,r}(\xi)\ ,$$

and let us prove that $\nu_{x',\,n_{k_i(r)}-r}$ weak-star converges to $\mu_{x',\,r}$ as $i\to+\infty$ if $r\ge d(x',y)$.

Note that if $r\ge d(x',y)$, for every $k\in\mathbb{N}$ large enough, by the triangle inequality and Equation (65), we have

$$\begin{aligned}&\Big\|\sum_{\gamma\in\Gamma,\ d(x',\,\gamma y)\le n_k-r} e^{\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)}\ \mathscr{D}_{\gamma y}-\sum_{\gamma\in\Gamma,\ d(y,\,\gamma y)\le n_k-r} e^{\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)}\ \mathscr{D}_{\gamma y}\Big\|\\ &\le\sum_{\gamma\in\Gamma,\ n_k-r-d(x',\,y)\le d(x',\,\gamma y)\le n_k-r+d(x',\,y)} e^{\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)}\le\ e^{(2r-n_k)\sigma}\ a_{n_k}\le e^{2r\sigma}\ .\end{aligned}$$

Since the denominator of $\nu_{x',\,n_{k_i(r)}-r}$ tends to $+\infty$ as $i\to+\infty$ and since

$$\Big|\int_{x'}^{\gamma y}(\widetilde{F}-\sigma)-\int_{y}^{\gamma y}(\widetilde{F}-\sigma)-C_{F-\sigma,\xi}(x',y)\Big|$$

is arbitrarily small if $\gamma y$ is close enough to $\xi\in\partial_\infty\widetilde{M}$, we have the convergence we looked for.

Now, for all $\gamma \in \Gamma$, $t \geq 0$ and $x' \in \widetilde{M}$, we clearly have $\gamma_* \nu_{x',t} = \nu_{\gamma x',t}$. Hence if $r \geq \max\{d(x', y), d(\gamma x', y)\}$, then $\gamma_* \mu_{x',r} = \mu_{\gamma x',r}$ by taking limits. By compactness, there exists a sequence $(r_j)_{j\in\mathbb{N}}$ in $[0, +\infty[$ converging to $+\infty$ such that the probability measures $\mu_{y,r_j}$ weak-star converge to a probability measure $\mu_y$. If $(\mu_{x'})_{x'\in\widetilde{M}}$ is the family of measures on $\partial_\infty \widetilde{M}$ defined by

$$d\mu_{x'}(\xi) = e^{C_{F-\sigma,\,\xi}(x',y)}\, d\mu_y(\xi) \;,$$

the sequence $(\mu_{x',r_j})_{j\in\mathbb{N}}$ weak-star converges to $\mu_{x'}$, and hence $(\mu_{x'})_{x'\in\widetilde{M}}$ is a Patterson density of dimension $\sigma$ for $(\Gamma, F)$, a contradiction. □

**Remark.** Here is a short proof of this result, using the sub-additivity (or rather super-multiplicativity) ideas of [DaPS]. Let us fix $x, y \in \widetilde{M}$. This paper proves that

- there exist a finite subset $P$ of $\Gamma$ and $c \geq 0$ such that for all $\alpha, \beta \in \Gamma$, there exists $\gamma \in P$ such that the Hausdorff distance between the geodesic segment $[x, \alpha\gamma\beta y]$ and $[x, \alpha y] \cup [\alpha y, \alpha\gamma x] \cup [\alpha\gamma x, \alpha\gamma\beta y]$ is at most $c$;
- let $(b_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that there exist $C > 0$ and $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, we have

$$b_n b_m \leq C \sum_{i=-N}^{N} b_{n+m+i} \;,$$

then, with $a_n = \sum_{k=0}^{n-1} b_n$, the limit of $a_n^{\frac{1}{n}}$ as $n \to +\infty$ exists (and hence is equal to its upper limit).

The first point implies that there exists $c' \geq 0$ (depending on $x, y$) such that for all $\alpha, \beta \in \Gamma$, there exists $\gamma \in P$ satisfying

$$\big|\, d(x, \alpha\gamma\beta y) - d(x, \alpha y) - d(x, \beta y) \,\big| \leq c'$$

and, since $\widetilde{F}$ is Hölder-continuous and bounded on compact subsets of $\widetilde{M}$,

$$\Big| \int_x^{\alpha\gamma\beta y} \widetilde{F} - \int_x^{\alpha y} \widetilde{F} - \int_x^{\beta y} \widetilde{F} \Big| \leq c' \;.$$

Hence the sequence $\big(b_n = \sum_{\gamma\in\Gamma,\; n-1<d(x,\gamma y)\leq n} \; e^{\int_x^{\gamma y} \widetilde{F}}\big)_{n\in\mathbb{N}}$ satisfies the assumptions of the second point, and Theorem 4.2 follows.

As mentioned by the referee (who provided its proof), we have the following version of Theorem 4.2 when the critical exponent is possibly nonpositive.

**Theorem 4.3** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$, and $x, y \in \widetilde{M}$. Let $\Gamma$ be a non-elementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map. For every $c > 0$ large enough, we have*

$$\delta_{\Gamma,\,F} = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma,\; n-c<d(x,\gamma y)\leq n} e^{\int_x^{\gamma y} \widetilde{F}} \;.$$

Note that $c$ depends on $x, y$. With the previous notation, this can be written as

$$\lim_{s \to +\infty} \frac{1}{s} \log \, G_{\Gamma,\, F,\, x,\, y,\, c}(s) = \delta_{\Gamma,\, F} \;,$$

that is, the annular orbital counting function grows logarithmically as $s \mapsto e^{\delta_{\Gamma,\, F}\, s}$. We will prove in Corollary 4.6 that the annular sectorial and bisectorial counting functions grow similarly (under obvious conditions).

**Proof.** Since $\Gamma$ is non elementary, let $\gamma_0$ be a loxodromic element of $\Gamma$, with translation axis $\mathrm{Axe}_{\gamma_0}$ and translation length $\ell(\gamma_0)$. Let $c = \ell(\gamma_0) + 2\, d(x, y) + 2\, d(y, \mathrm{Axe}_{\gamma_0})$. For every $n \in \mathbb{N}$ with $n \geq d(x, y)$, let us first prove that there exists $\gamma \in \Gamma$ such that $n - c \leq d(x, \gamma y) \leq n$. The set $\{k' \in \mathbb{N} \;:\; d(x, \gamma_0^{k'} y) \leq n\}$ being non empty and finite, let $k$ be its maximum and let us prove that $\gamma = \gamma_0^k$ satisfies the required inequalities. By the triangle inequality, for every $k' \in \mathbb{N}$, we have

$$-d(x, y) + k'\, \ell(\gamma_0) \leq d(x, \gamma_0^{k'} y) \leq d(x, y) + k'\, \ell(\gamma_0) + 2\, d(y, \mathrm{Axe}_{\gamma_0})\,.$$

Since $d(x, \gamma_0^{k+1} y) > n$ by the maximality of $k$, we have

$$\begin{aligned} d(x, \gamma_0^k y) &\geq -d(x, y) + k\, \ell(\gamma_0) = d(x, y) + (k+1)\ell(\gamma_0) + 2\, d(y, \mathrm{Axe}_{\gamma_0}) - c \\ &\geq d(x, \gamma_0^{k+1} y) - c > n - c\,, \end{aligned}$$

as required.

Now, consider the sequence $\big(b_n = \sum_{\gamma \in \Gamma,\; n - c < d(x, \gamma y) \leq n} \; e^{\int_x^{\gamma y} \widetilde{F}}\big)_{n \in \mathbb{N}}$, and note that $b_n > 0$ for all $n \geq d(x, y)$. As in the above short proof of Theorem 4.2, the result follows from the following version of its second point: Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers such that there exist $C > 0$ and $N < N'$ in $\mathbb{N}$ such that $b_n > 0$ for every $n \geq N'$ and for all $n, m \in \mathbb{N}$, we have

$$b_n b_m \leq C \sum_{i=-N}^{N} b_{n+m+i} \;,$$

then for every $N'' \geq N + N' - 1$, with $a_n = \sum_{k=n-N''}^{n} b_n$, the limit of $\frac{1}{n} \log \, a_n$ as $n \to +\infty$ exists.

By the standard comparison of a sum of a bounded number of nonnegative real numbers and their maximum, this follows from the following lemma, by taking $u_n = \frac{1}{C(2N+1)}\, b_n$.

**Lemma 4.4** *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative real numbers such that there exist $N < N'$ in $\mathbb{N}$ such that $u_n > 0$ for every $n \geq N'$ and for all $n, m \in \mathbb{N}$, we have*

$$u_n u_m \leq \max_{-N \leq i \leq N} u_{n+m+i} \;.$$

*Then, with $N'' \geq N + N'$ and $v_n = \max_{n - N'' < i \leq n} u_i$, the limit of $\frac{1}{n} \log \, v_n$ as $n \to +\infty$ exists.*

**Proof.** Let $\delta = \limsup_{n \to +\infty} \frac{1}{n} \log \, v_n$. Let us prove that $\liminf_{n \to +\infty} \frac{1}{n} \log \, v_n \geq \delta$, which implies the result. We may assume that $\delta > -\infty$. Let $\sigma < \delta$ and $\epsilon \in \,]0, 1]$ with $2\epsilon \neq \sigma$.

Let $n' > N'' + N$ be large enough so that $\max\{\frac{N|\sigma|}{n''+N},\ \frac{N''|\sigma|}{n''}\} \leq \epsilon$ for every $n'' \in \mathbb{N}$ with $n' - N'' \leq n'' \leq n'$, and $\frac{1}{n'}\log v_{n'} \geq \sigma$. There exists $n'' \in \mathbb{N}$ with $n' - N'' \leq n'' \leq n'$ and $v_{n'} = u_{n''}$, so that $n'' > N$ and

$$\frac{1}{n''}\log u_{n''} = \frac{n'}{n''}\,\frac{1}{n'}\log v_{n'} \geq \frac{n'}{n''}\,\sigma \geq \sigma - \frac{N''|\sigma|}{n''} \geq \sigma - \epsilon\,.$$

There exists by induction an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ starting with $n_0 = n''$, such that $|n_{k+1} - (n_k + n_0)| \leq N$ and $u_{n_k}\, u_{n_0} \leq u_{n_{k+1}}$ for every $k \in \mathbb{N}$. We have by induction $n_k \leq (k+1)(n_0 + N)$ and

$$u_{n_k} \geq u_{n_0}^{k+1} = u_{n''}^{k+1} \geq e^{(\sigma-\epsilon)\,(k+1)\,n''} \geq e^{(\sigma-\epsilon-\frac{N\,|\sigma|}{n''+N})\,n_k} \geq e^{(\sigma-2\,\epsilon)n_k}\,.$$

Now, for every

$$n \geq \max\big\{\frac{|\sigma-2\,\epsilon|(2N+N'+n'')}{\epsilon},\ \frac{(n''+N+1)|\log u_{N'}|}{\epsilon}\big\}\,,$$

let us prove that $\frac{1}{n}\log v_n \geq \sigma - 4\epsilon$, which implies the result. Let $k \in \mathbb{N}$ be maximal so that $n_k \leq n$. Again by induction, let $(m_i)_{i\in\mathbb{N}}$ be an increasing sequence such that $|m_{i+1} - (m_i + N')| \leq N$ and $u_{n_k}\, u_{N'}^{i+1} \leq u_{n_k+m_i}$ for every $i \in \mathbb{N}$. There exists $\ell \in \mathbb{N}$ such that $n - (N + N') < n_k + m_\ell \leq n$, and $\ell \leq m_\ell \leq n_{k+1} - n_k \leq n'' + N$ by the maximality of $k$. Then

$$n \geq n_k \geq n - (N+N') - m_\ell \geq n - (2N + N' + n'') \geq n\,\big(1 - \frac{\epsilon}{|\sigma - 2\,\epsilon|}\big)$$

and, since $N'' \geq N + N'$,

$$v_n \geq u_{n_k+m_\ell} \geq u_{n_k}\, u_{N'}^{\ell+1} \geq e^{(\sigma-2\,\epsilon)n_k}\, e^{-(\ell+1)|\log u_{N'}|} \geq e^{(\sigma-4\epsilon)\,n}\,,$$

as required. □

**Corollary 4.5** *With the assumptions of Theorem 4.2,*

*(1) if $U$ is an open subset of $\partial_\infty\widetilde{M}$ meeting $\Lambda\Gamma$, then $\lim_{t\to+\infty}\frac{1}{t}\log G_{\Gamma,\,F,\,x,\,y,\,U}(t) = \delta_{\Gamma,\,F}$;*

*(2) if $U$ and $V$ are any two open subsets of $\partial_\infty\widetilde{M}$ meeting the limit set $\Lambda\Gamma$, then $\lim_{t\to+\infty}\frac{1}{t}\log G_{\Gamma,\,F,\,x,\,y,\,U,\,V}(t) = \delta_{\Gamma,\,F}$.*

**Proof.** We adapt the proof of [Rob2, Coro. 1] which corresponds to the case $F = 0$. The validity of these statements for all open sets $U$ and $V$ is independent of $x$ and $y$, since for all $z, z' \in \widetilde{M}$ and all open subsets $W, W'$ of $\partial_\infty\widetilde{M}$ such that $\overline{W} \subset W'$, there exists a compact subset $K$ of $\widetilde{M}$ such that $\mathscr{C}_z W \subset K \cup \mathscr{C}_{z'} W'$, and since the intersection of $K$ with any orbit of $\Gamma$ is a finite. Recall that the convex hull $\mathscr{C}\Lambda\Gamma$ of $\Lambda\Gamma$ is $\Gamma$-invariant. We may assume that $y \in \mathscr{C}\Lambda\Gamma$, hence that the orbit of $y$ is contained in $\mathscr{C}\Lambda\Gamma$.

(1) For every $t \in [0, +\infty[$ , for every open subset $U'$ of $\widetilde{M} \cup \partial_\infty\widetilde{M}$ and for every $z \in \widetilde{M}$, let

$$a_{t,\,U',\,z} = \sum_{\gamma\in\Gamma\,:\,d(z,\gamma y)\leq t\,,\ \gamma y\in U'} e^{\int_z^{\gamma y}\widetilde{F}}\,.$$

Note that for every $\gamma \in \Gamma$, by Equation (16) and a change of variable in this sum, we have

$$a_{t,\,U',\,z} = a_{t,\,\gamma U',\,\gamma z}\ . \tag{66}$$

When $U'$ is the open set $\mathscr{C}_x U - \{x\}$, the difference $G_{\Gamma,\,F,\,x,\,y,\,U}(t) - a_{t,\,U',\,x}$ is equal to 1 if $x$ belongs to the $\Gamma$-orbit of $y$, and 0 otherwise. Since $G_{\Gamma,\,F,\,x,\,y,\,U}(t) \leq G_{\Gamma,\,F,\,x,\,y}(t)$ and by Theorem 4.2, we have

$$\limsup_{t\to+\infty} \frac{1}{t} \log G_{\Gamma,\,F,\,x,\,y,\,U}(t) \leq \delta_{\Gamma,\,F}\ .$$

Hence to prove the first assertion of Corollary 4.5, we only have to prove that

$$\liminf_{t\to+\infty} \frac{1}{t} \log a_{t,\,U',\,x} \geq \delta_{\Gamma,\,F}$$

for every open subset $U'$ of $\widetilde{M} \cup \partial_\infty \widetilde{M}$ meeting $\Lambda\Gamma$.

Let $U'$ be such an open subset. Since $\Gamma$ is non-elementary, every orbit in $\Lambda\Gamma$ is dense. Hence by compactness of $\Lambda\Gamma$, there exist $\gamma_1, \dots, \gamma_k$ in $\Gamma$ such that $\Lambda\Gamma \subset \bigcup_{i=1}^k \gamma_i U'$. In particular $\mathscr{C}\Lambda\Gamma - \bigcup_{i=1}^k \gamma_i U'$ is compact. Hence there exists $c_1' \geq 0$ such that for every $t \in [0, +\infty[$, we have

$$G_{\Gamma,\,F,\,x,\,y}(t) \leq c_1' + \sum_{i=1}^k a_{t,\,\gamma_i U',\,x}\ . \tag{67}$$

Let $r = \max_{1\leq i\leq k}\ d(x, \gamma_i x)$ and $t \in [r, +\infty[$. Note that for all $i \in \{1, \dots, k\}$ and $\gamma \in \Gamma$, by the triangle inequality, if $d(\gamma_i^{-1} x, \gamma y) \leq t - r$, then $d(x, \gamma y) \leq t$. Furthermore, by Lemma 3.2, there exists $c_2' > 0$ such that for all $i \in \{1, \dots, k\}$ and $\gamma \in \Gamma$, we have

$$\Big| \int_x^{\gamma y} \widetilde{F} - \int_{\gamma_i^{-1} x}^{\gamma y} \widetilde{F} \Big| \leq c_2'\ .$$

Hence, for all $i \in \{1, \dots, k\}$ and $t \in [r, +\infty[$, we have, using Equation (66),

$$a_{t-r,\,\gamma_i U',\,x} = a_{t-r,\,U',\,\gamma_i^{-1} x} \leq e^{c_2'}\, a_{t,\,U',\,x}\ .$$

Therefore, by Equation (67), we have

$$a_{t,\,U',\,x} \geq \frac{1}{k\, e^{c_2'}} \big( G_{\Gamma,\,F,\,x,\,y}(t - r) - c_1' \big)\ . \tag{68}$$

By taking the logarithm, dividing by $t$ and taking the lower limit as $t \to +\infty$, the result then follows from Theorem 4.2.

(2) The proof of the second assertion reduces to the first one, using a similar approach as the reduction from the first one to Theorem 4.2. For every $t \in [0, +\infty[$, for all open subsets $U', V'$ of $\widetilde{M} \cup \partial_\infty \widetilde{M}$ and for all $z, w \in \widetilde{M}$, we now introduce

$$b_{t,\,U',\,V',\,z,\,w} = \sum_{\gamma\in\Gamma\,:\, d(z,\gamma w)\leq t\,,\ \gamma w \in U',\ \gamma^{-1} z \in V'} e^{\int_z^{\gamma w} \widetilde{F}}\ ,$$

which satisfies $b_{t,\,U',\,V',\,z,\,w} = b_{t,\,U',\,\alpha V',\,z,\,\alpha w}$ for every $\alpha \in \Gamma$. As above, we only have to prove that

$$\liminf_{t\to+\infty} \frac{1}{t} \log b_{t,\,U',\,V',\,x,\,y} \geq \delta_{\Gamma,\,F}$$

for all open subsets $U', V'$ of $\widetilde{M} \cup \partial_\infty \widetilde{M}$ meeting $\Lambda\Gamma$.

Let $U', V'$ be two such open subsets. As above, there exist $\alpha_1, \dots, \alpha_\ell$ in $\Gamma$ such that $\Lambda\Gamma \subset \bigcup_{i=1}^{\ell} \alpha_i V'$, and hence there exists $c'_3 \geq 0$ such that for every $t \in [0, +\infty[$ , we have

$$a_{t,\,U',\,x} \leq c'_3 + \sum_{i=1}^{\ell} b_{t,\,U',\,\alpha_i V',\,x,\,y} \;.$$

Now let $r = \max_{1 \leq i \leq \ell} d(y, \alpha_i y)$. For all $i \in \{1, \dots, \ell\}$, $t \in [r, +\infty[$ and $\gamma \in \Gamma$, by the triangle inequality, if $d(x, \gamma\alpha_i^{-1} y) \leq t - r$, then $d(x, \gamma y) \leq t$. Furthermore, by Equation (16) and by Lemma 3.2, there exists $c'_4 > 0$ such that for all $i \in \{1, \dots, \ell\}$ and $\gamma \in \Gamma$, we have

$$\Big| \int_x^{\gamma y} \widetilde{F} - \int_x^{\gamma\alpha_i^{-1} y} \widetilde{F} \Big| = \Big| \int_y^{\gamma^{-1} x} \widetilde{F} \circ \iota - \int_{\alpha_i^{-1} y}^{\gamma^{-1} x} \widetilde{F} \circ \iota \Big| \leq c'_4 \;.$$

Hence, for all $i \in \{1, \dots, k\}$ and $t \in [r, +\infty[$ , we have

$$b_{t-r,\,U',\,\alpha_i V',\,x,\,y} = b_{t-r,\,U',\,V',\,x,\,\alpha_i^{-1} y} \leq e^{c'_4}\, b_{t,\,U',\,V',\,x,\,y} \;.$$

Therefore

$$b_{t,\,U',\,V',\,x,\,y} \geq \frac{1}{\ell\, e^{c'_4}} \big( a_{t-r,\,U',\,x} - c'_3 \big) \;,$$

and we conclude, using Assertion (1), as in the end of the proof of this assertion. □

We have the following version of Corollary 4.5 when the critical exponent is possibly nonpositive.

**Corollary 4.6** *With the assumptions of Theorem 4.3, if $c > 0$ is large enough,*

*(1) if $U$ is an open subset of $\partial_\infty \widetilde{M}$ meeting $\Lambda\Gamma$, then $\lim_{t \to +\infty} \frac{1}{t} \log G_{\Gamma,\,F,\,x,\,y,\,U,\,c}(t) = \delta_{\Gamma,\,F}$;*

*(2) if $U$ and $V$ are any two open subsets of $\partial_\infty \widetilde{M}$ meeting the limit set $\Lambda\Gamma$, then $\lim_{t \to +\infty} \frac{1}{t} \log G_{\Gamma,\,F,\,x,\,y,\,U,\,V,\,c}(t) = \delta_{\Gamma,\,F}$.*

**Proof.** (1) The proof is similar to that of Corollary 4.5 (1). We define, for $t, U', z$ as previously and $c > 0$ large enough,

$$a_{t,\,U',\,z,\,c} = \sum_{\gamma \in \Gamma \,:\, t-c < d(z, \gamma y) \leq t\,,\ \gamma y \in U'} e^{\int_z^{\gamma y} \widetilde{F}} \;.$$

We have $a_{t,\,U',\,z,\,c} = a_{t,\,\gamma U',\,\gamma z,\,c}$ for every $\gamma \in \Gamma$, and we also only have to prove that

$$\liminf_{t \to +\infty} \frac{1}{t} \log a_{t,\,U',\,x,\,c} \geq \delta_{\Gamma,\,F} \;,$$

since the converse inequality is immediate, as $G_{\Gamma,\,F,\,x,\,y,\,U,\,c} \leq G_{\Gamma,\,F,\,x,\,y,\,c}$. With $\gamma_1, \dots, \gamma_k, r$ and $c'_2$ as previously, since $t - r - (c - 2r) < d(\gamma_i^{-1} x, \gamma y) \leq t - r$ implies that $t - c < d(x, \gamma y) \leq t$, we have $a_{t-r,\,\gamma_i U',\,z,\,c-2r} \leq e^{c'_2}\, a_{t,\,U',\,z,\,c}$. Hence

$$a_{t,\,U',\,x,\,c} \geq \frac{1}{k\, e^{c'_2}} \big( G_{\Gamma,\,F,\,x,\,y,\,c-2r}(t - r) - c'_1 \big) \;,$$

and the result follows from Theorem 4.3 as previously.

(2) The proof is similar to that of Corollary 4.5 (2) by considering, for $t$, $U'$, $V'$, $z$, $r$, $\ell$, $c'_3$, $c'_4$ as in its proof and $c>0$ large enough,

$$b_{t,\,U',\,V',\,z,\,w,\,c} = \sum_{\gamma\in\Gamma\,:\,t-c<d(z,\,\gamma w)\leq t\,,\;\gamma w\in U',\;\gamma^{-1}z\in V'} e^{\int_z^{\gamma w}\widetilde{F}}\,, \tag{69}$$

and by proving as previously that

$$b_{t,\,U',\,V',\,x,\,y,\,c} \geq \frac{1}{\ell\; e^{c'_4}}\big(a_{t-r,\,U',\,x,\,c-2r} - c'_3\big)$$

so that

$$\liminf_{t\to+\infty}\frac{1}{t}\log b_{t,\,U',\,V',\,x,\,y,\,c} \geq \delta_{\Gamma,\,F}\,. \quad \Box \tag{70}$$

## 4.3 Equality between critical exponent and Gurevich pressure

The aim of this subsection is stated in its title: we now prove that the logarithmic growth rate of the periodic geodesics (weighted by the potential) is equal to the logarithmic growth rate of the orbit points (weighted by the potential).

**Theorem 4.7** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$, and $x,y\in\widetilde{M}$. Let $\Gamma$ be a non-elementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F}:T^1\widetilde{M}\to\mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map. Let $W$ be a relatively compact open subset of $T^1M$ meeting the (topological) non-wandering set $\Omega\Gamma$. Then*

$$P_{Gur}(\Gamma,F) = \delta_{\Gamma,\,F}\,.$$

*If $c>0$ is large enough, then*

$$P_{Gur}(\Gamma,F) = \lim_{s\to+\infty}\frac{1}{s}\log Z_{\Gamma,\,F,\,W,\,c}(s)\,,$$

*and if $P_{Gur}(\Gamma,F)>0$, then*

$$P_{Gur}(\Gamma,F) = \lim_{s\to+\infty}\frac{1}{s}\log Z_{\Gamma,\,F,\,W}(s)\,.$$

This proves in particular that the Gurevich pressure does not depend on $W$ and that the upper limit defining it (see Equation (64)) is a limit if $c>0$ is large enough.

**Proof.** Let $T\widetilde{p}:T^1\widetilde{M}\to T^1M=\Gamma\backslash T^1\widetilde{M}$ be the canonical projection.

Let us first prove that

$$\limsup_{s\to+\infty}\frac{1}{s}\log Z_{\Gamma,\,F,\,W,\,c}(s) \leq \delta_{\Gamma,\,F}$$

if $c>0$ is large enough, and that

$$\limsup_{s\to+\infty}\frac{1}{s}\log Z_{\Gamma,\,F,\,W}(s) \leq \delta_{\Gamma,\,F}$$

if $\delta_{\Gamma,\,F} > 0$. Since $W$ is relatively compact, there exists a compact subset $K$ of $\widetilde{M}$ such that $T\widetilde{p}(\pi^{-1}(K))$ contains $W$. Let $r$ be the diameter of $K$, and fix $x \in K$.

For all $s \geq 0$ and $g$ in $\mathscr{P}\mathrm{er}(s)$ such that $g \cap W \neq \emptyset$, let $\gamma_g$ be one of the loxodromic elements of $\Gamma$ whose translation axis $\mathrm{Axe}_{\gamma_g}$ meets $K$, whose translation length is the length $\ell(g)$ of $g$ and such that for every $y \in \mathrm{Axe}_{\gamma_g}$, the image by $T\widetilde{p}$ of the unit tangent vector at $y$ pointing towards $\gamma_g y$ belongs to $g$. Note that the number of these elements $\gamma_g$ is at least equal to the cardinality of the pointwise stabiliser of $\mathrm{Axe}_{\gamma_g}$, that is to the multiplicity of $g$.

Let $x_g$ be the closest point to $x$ on $\mathrm{Axe}_{\gamma_g}$, which satisfies $d(x, x_g) \leq r$ since $x \in K$ and $\mathrm{Axe}_{\gamma_g}$ meets $K$. We have by the triangle inequality

$$\ell(g) \leq d(x, \gamma_g x) \leq d(x, x_g) + d(x_g, \gamma_g x_g) + d(\gamma_g x, \gamma_g x_g) \leq \ell(g) + 2r \leq s + 2r\ .$$

Furthermore, by (two applications of) Lemma 3.2 with $r_0 = r$, there exists a constant $c'_5 \geq 0$ (depending only on $r$, the Hölder constants of $\widetilde{F}$, the bounds on the sectional curvature and $\max_{\pi^{-1}(B(x,r))} |\widetilde{F}|$) such that

$$\Big| \int_x^{\gamma_g x} \widetilde{F}\ - \int_g \widetilde{F}\ \Big| = \Big| \int_x^{\gamma_g x} \widetilde{F}\ - \int_{x_g}^{\gamma_g x_g} \widetilde{F}\ \Big| \leq c'_5\ .$$

Hence $Z_{\Gamma,\,F,\,W}(s) \leq e^{c'_5}\ G_{\Gamma,\,F,\,x,\,x}(s + 2r)$ and $Z_{\Gamma,\,F,\,W,\,c}(s) \leq e^{c'_5}\ G_{\Gamma,\,F,\,x,\,x,\,c+2r}(s + 2r)$, which proves our first claim, by Theorems 4.2 and 4.3.

Since $Z_{\Gamma,\,F,\,W,\,c} \leq Z_{\Gamma,\,F,\,W}$ for every $c > 0$, in order to prove Theorem 4.7, we now only have to prove that, if $c > 0$ is large enough,

$$\liminf_{s \to +\infty}\ \frac{1}{s} \log Z_{\Gamma,\,F,\,W,\,c}(s) \geq \delta_{\Gamma,\,F}\ .$$

Let $v \in T^1\widetilde{M}$ such that $T\widetilde{p}(v) \in W \cap \Omega\Gamma$, and now let $x = \pi(v)$. Note that the points $v_-$ and $v_+$ both belong to $\Lambda\Gamma$.

By standard arguments (see for instance Lemma 2.8 and [GdlH, page 150-151]), there exist $U'$ and $V'$ small enough neighbourhoods in $\widetilde{M} \cup \partial_\infty \widetilde{M}$ of $v_+$ and $v_-$ respectively, such that for every $\gamma \in \Gamma$ such that $\gamma x \in U'$ and $\gamma^{-1} x \in V'$, then $\gamma$ is a loxodromic element and $v$ is close to its translation axis $\mathrm{Axe}_\gamma$, in the sense that the point $x$ is at distance at most 1 from some point $x_\gamma$ in $\mathrm{Axe}_\gamma$, and that if $v_\gamma$ is the unit tangent vector at $x_\gamma$ pointing towards $\gamma x_\gamma$, then $p(v_\gamma) \in W$ (recall that $W$ is open). Note that $U'$ and $V'$ meet $\Lambda\Gamma$.

Let $s \geq 0$ and let $\gamma \in \Gamma$ be such that $s - (c-2) < d(x, \gamma x) \leq s$, $\gamma x \in U'$ and $\gamma^{-1} x \in V'$. Note that the orbit $g_\gamma$ under the geodesic flow of $T\widetilde{p}(v_\gamma)$ is periodic, and that its length satisfies

$$s - c < d(x, \gamma x) - 2 \leq \ell(g_\gamma) \leq d(x, \gamma x) \leq s\ ,$$

and as above, there exists $c'_6 \geq 0$ such that

$$\Big| \int_{g_\gamma} \widetilde{F}\ - \int_x^{\gamma x} \widetilde{F}\ \Big| = \Big| \int_{x_\gamma}^{\gamma x_\gamma} \widetilde{F}\ - \int_x^{\gamma x} \widetilde{F}\ \Big| \leq c'_6\ .$$

Hence $Z_{\Gamma,\,F,\,W,\,c}(s) \geq e^{-c'_6}\ b_{s,\,U',\,V',\,x,\,x,\,c-2}$ with the notation of Equation (69), which proves our second claim, by Equation (70). □

The next result is an obvious corollary of Theorem 4.7, also following from Remark 3.1 and the remark at the end of Subsection 3.2.

**Corollary 4.8** *Under the assumption of the previous theorem, if $\widetilde{F}^*: T^1\widetilde{M} \to \mathbb{R}$ is another Hölder-continuous $\Gamma$-invariant map with induced map $F^*: T^1M \to \mathbb{R}$, such that the periods of every $\gamma \in \Gamma$ for $\widetilde{F}$ and $\widetilde{F}^*$ are the same, then the critical exponents of $(\Gamma, F)$ and $(\Gamma, F^*)$ are the same:*

$$\delta_{\Gamma,\, F} = \delta_{\Gamma,\, F^*}\ .$$

### 4.4 Critical exponent of Schottky semigroups

The main aim of this subsection is to prove (see Theorem 4.11) that the critical exponent $\delta_{\Gamma,\, F}$ is the upper bound of the critical exponents $\delta_{G,\, F}$ (defined as for groups) of finitely generated subsemigroups $G$ of $\Gamma$, which have strong geometric properties (of Schottky type). The constructions of this subsection will also be useful for the proof of the Variational Principle in Subsection 6.2.

Though the similar problem for free subgroups instead of free subsemigroups is open in our case, it is in general a much harder one, when dealing with growth problems. For instance, there exist finitely generated soluble groups with exponential growth, and though the most frequent way to prove they have this type of growth is by proving that they contain free subsemigroups on (at least) two generators, they simply do not contain any free subgroup on two generators. A famous result due to Doyle is that there is a gap away from 2 for the critical exponents of the classical Schottky subgroups (which are free) of the non-elementary Kleinian groups (that is, of non-elementary discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$, seen as the orientation preserving group of isometries of the real hyperbolic space $\mathbb{H}^3_{\mathbb{R}}$ of dimension 3). Hence free subgroups have rigidity properties that free semigroups do not have (see for instance [Mer]).

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature at most $-1$. Let $G$ be a discrete semigroup of isometries of $\widetilde{M}$. We refer to [Mer] for general information about discrete semigroups of isometries of CAT$(-1)$-spaces.

Let $\widetilde{F}: T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous $G$-invariant map. We define the *limit set* $\Lambda G$ of $G$ (as the set of accumulation points in $\partial_\infty \widetilde{M}$ of any orbit $Gx_0$), the *Poincaré series*

$$Q_{G,\, F,\, x,\, y}(s) = \sum_{\gamma \in G} e^{\int_x^{\gamma y} (\widetilde{F} - s)}$$

of $(G, F)$, and the *critical exponent*

$$\delta_{G,\, F} = \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{\gamma \in G,\; n-1 < d(x,\, \gamma y) \leq n} e^{\int_x^{\gamma y} \widetilde{F}}$$

of $(G, F)$, exactly as for the case of groups.

Recall that a map $f: X \to Y$ between metric spaces is *quasi-isometric* if there exist $\lambda \geq 1$ and $c \geq 0$ such that for all $x, y \in X$, we have

$$-c + \frac{1}{\lambda}\, d(x, y) \leq d(f(x), f(y)) \leq \lambda\, d(x, y) + c\ .$$

If $G$ is generated (as a semigroup) by a finite set $S$, and is endowed with the (semigroup) word metric defined by $S$ (see below when $G$ is free on $S$), we say that $G$ is *convex-cocompact* if the map from $G$ to $\widetilde{M}$ defined by $g \mapsto gx_0$ is quasi-isometric (for any $x_0 \in \widetilde{M}$).

We first give a construction of convex-cocompact free semigroups of isometries of $\widetilde{M}$ (of Schottky type). We have not tried to get optimal constants.

**Proposition 4.9** *For every $\epsilon \in \, ]0,1]$, there exists $\theta > 0$ such that for every set $S$ of isometries of $\widetilde{M}$, for all $N \geq 5$ and $C \geq 6$, for all $x_0 \in \widetilde{M}$ and $v_0 \in T^1_{x_0}\widetilde{M}$ such that the following conditions are satisfied:*

*(1) $N \leq d(x_0, \alpha x_0) < N+1$ for every $\alpha \in S$,*

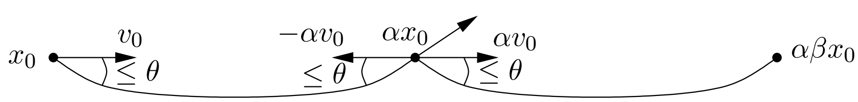
*(2) $d(\alpha x_0, \beta x_0) \geq C$ for all distinct $\alpha$ and $\beta$ in $S$,*

*(3) for every $\alpha \in S$, the angle at $x_0$ between $v_0$ and the unit tangent vector at $x_0$ pointing towards $\alpha x_0$, and the angle at $\alpha x_0$ between $-\alpha v_0$ and the unit tangent vector at $\alpha x_0$ pointing towards $x_0$, are at most $\theta$ (see the picture below),*

*then the semigroup $G$ generated by $S$ is free on $S$ and convex-cocompact, and for all $\gamma, \gamma'$ in $G$, the piecewise geodesic segment $[x_0, \gamma x_0] \cup [\gamma x_0, \gamma\gamma' x_0]$ is contained in the $\epsilon$-neighbourhood of $[x_0, \gamma\gamma' x_0]$. Furthermore*

$$k(N - 2\epsilon) \leq d(x_0, \gamma x_0) \leq k(N+1)$$

*for every word $\gamma$ in the elements of $S$ of length $k$. Lastly, for all words $\gamma$ and $\gamma'$ in the elements of $S$, none of them being an initial subword of the other, there exist $x \in [x_0, \gamma x_0]$ and $x' \in [x_0, \gamma' x_0]$ such that $d(x_0, x) = d(x_0, x')$ and $d(x, x') \geq 1 - 8\epsilon$.*




**Proof.** Fix $\epsilon \in \, ]0,1]$, and let $\theta = \min\{\frac{\pi}{4}, \frac{1}{2}\theta_0(5, \epsilon, \frac{\pi}{4})\}$, where $\theta_0(\cdot,\cdot,\cdot)$ has been defined in Lemma 2.7. Let $S, N, C, x_0, v_0$ be as in the statement. Note that the assumptions (1) and (2) imply that $S$ is finite. Before proving that $G$ is free and convex-cocompact, we start with a few preliminary remarks.

For every word $m = \alpha_1 \dots \alpha_k$ in elements of $S$, we denote by $\ell(m) = k \in \mathbb{N}$ its length (the only word with zero length being the empty word) and, for $0 \leq j \leq k$, by $m_j = \alpha_1 \dots \alpha_j$ its initial subword of length $j$ (in particular $m_0$ is the empty word). If $m = \alpha_1 \dots \alpha_k$ and $m' = \beta_1 \dots \beta_{k'}$ are two words in elements of $S$, let

$$j(m, m') = \max\{i \in \mathbb{N} \;:\; \forall\, i' \leq i, \;\; \alpha_{i'} = \beta_{i'}\}$$

be the length of the maximal common initial subword of $m$ and $m'$. It is 0 if and only if $m$ or $m'$ is the empty word or $\alpha_1 \neq \beta_1$. The distance between the words $m$ and $m'$ is

$$d(m, m') = \ell(m) + \ell(m') - 2j(m, m')\,.$$

Consider the piecewise geodesic path

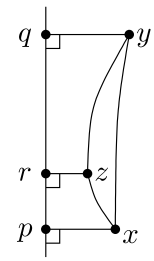
$$\omega_m = [m_0 x_0, m_1 x_0] \cup [m_1 x_0, m_2 x_0] \cup \dots \cup [m_{k-1} x_0, m_k x_0]$$

(with $\omega_m = \{x_0\}$ by convention if $m$ is the empty word). By Assumption (3), the exterior angles of $\omega_m$ at the points $m_i x_0$ for $1 \leq i \leq k-1$ are at most $2\theta$ (see the picture above). By

Assumption (1), the length of each segment of $\omega_m$ is at least $N \geq 5$. Hence, by Lemma 2.7 and the definition of $\theta$, the piecewise geodesic path $\omega_m$ is contained in the $\epsilon$-neighbourhood of $[x_0, mx_0]$. In particular, this proves the ante-penultimate assertion of Proposition 4.9, by convexity.

Since $N \geq 5$, $\epsilon \leq 1$ and $2\theta \leq \frac{\pi}{2}$, the closest points to $m_i x_0$ on $[x_0, mx_0]$ for $0 \leq i \leq k$ are in this order on this segment.

Indeed, if $x, y, z$ are points in $\widetilde{M}$ such that $N \leq d(x,y) \leq N+1$, $N \leq d(y,z) \leq N+1$, $\measuredangle_y(x,z) \geq \frac{\pi}{2}$, having closest points respectively $p, q, r$ at distance at most $\epsilon$ on a geodesic segment, with the absurd hypothesis that $r \in [p,q]$, then by the triangle inequality and since closest point maps do not increase distances, we have




$$\begin{aligned} d(x,z) &\leq d(p,r) + 2\epsilon = d(p,q) - d(r,q) + 2\epsilon \leq d(x,y) - d(z,y) + 4\epsilon \\ &\leq N + 1 - N + 4\epsilon = 1 + 4\epsilon \leq 5 \leq N\ . \end{aligned}$$

By an angle comparison, the angle $\measuredangle_y(x,z)$ is less than $\frac{\pi}{2}$, a contradiction. [Another argument is that since $\measuredangle_y(x,z) \geq \frac{\pi}{2}$, we have by a distance comparison that $d(x,z) \geq d(x,y) + d(y,z) - 2\log(1+\sqrt{2}) \geq 2N - 2\log(1+\sqrt{2}) > N$, a contradiction.]

We now give our last preliminary remark, which yields the penultimate assertion of Proposition 4.9. By the triangle inequality and Assumption (1), we have

$$d(mx_0, x_0) \leq \sum_{i=0}^{k-1} d(m_i x_0, m_{i+1} x_0) \leq k(N+1) = (N+1)\,\ell(m)\ ,$$

on one hand, and on the other hand, with $p_i$ the closest point to $m_i x_0$ on $[x_0, mx_0]$ for $0 \leq i \leq k$, using the above ordering remark,

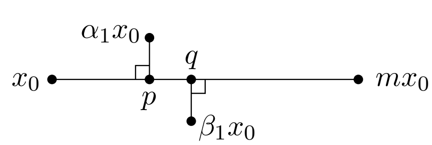
$$\begin{aligned} d(mx_0, x_0) = \sum_{i=0}^{k-1} d(p_i, p_{i+1}) &\geq \sum_{i=0}^{k-1} \big(d(m_i x_0, m_{i+1} x_0) - 2\epsilon\big) \\ &\geq k(N - 2\epsilon) = (N - 2\epsilon)\,\ell(m)\ . \end{aligned}$$

Note that $N - 2\epsilon > 0$ by the assumptions on $N$ and $\epsilon$. In particular, if $m$ is not the empty word, then $mx_0 \neq x_0$.

After these preliminary remarks, let us prove that $G$ is free on $S$. Let $m = \alpha_1 \dots \alpha_k$ and $m' = \beta_1 \dots \beta_{k'}$ be two distinct words in elements of $S$, and let us prove that $mx_0 \neq m'x_0$, which yields the result.

Assume for a contradiction that $mx_0 = m'x_0$. Let $j = j(m, m')$. Up to multiplying $m$ and $m'$ by the inverse of their maximal common initial subword, we may assume that $j = 0$. We have $\min\{k, k'\} > 0$, otherwise, up to exchanging $m$ and $m'$, we have $k \neq 0$ and $k' = 0$, so that $mx_0 = x_0$ and $m$ is not the empty word, a contradiction to the last preliminary remark. Hence $\alpha_1$ and $\beta_1$ exist and are distinct.

Let $p$ and $q$ be the closest points to $\alpha_1 x_0$ and $\beta_1 x_0$ on $[x_0, mx_0]$. Up to permuting $m$ and $m'$, we may assume that $x_0, p, q, mx_0$ are in this order on $[x_0, mx_0]$.



By the triangle inequality, since closest point maps do not increase distances, by Assumption (1) and by the properties of $\epsilon$ and $C$, we have

$$\begin{aligned} d(\alpha_1 x_0, \beta_1 x_0) &\leq d(p,q) + 2\epsilon = d(q,x_0) - d(p,x_0) + 2\epsilon \\ &\leq d(\beta_1 x_0, x_0) - d(\alpha_1 x_0, x_0) + 3\epsilon \leq N + 1 - N + 3\epsilon = 3\epsilon + 1 < C \,. \end{aligned}$$

This contradicts Assumption (2). Hence $G$ is indeed free on $S$.

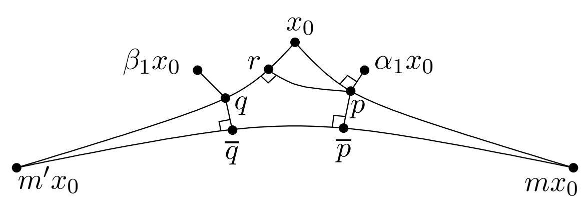
Let us now prove that the map from $G$ to $\widetilde{M}$ defined by $g \mapsto g x_0$ is quasi-isometric, which says that $G$ is convex-cocompact. Let $m = \alpha_1 \dots \alpha_k$ and $m' = \beta_1 \dots \beta_{k'}$ be two words in elements of $S$. Let us prove that

$$(N - 2\epsilon) d(m, m') - 2(\log(1+\sqrt{2}) + N - \epsilon) \leq d(m x_0, m' x_0) \leq (N+1)\, d(m, m') \,, \quad (71)$$

which implies the result.

Let $j = j(m, m')$. Up to multiplying $m$ and $m'$ by the inverse of their maximal common initial subword, we may assume that $j = 0$. By the last preliminary remark, we may assume that neither $m$ nor $m'$ is the empty word (we would not have to do this in the group case, but $G$ is here only a semigroup). We may also assume that $m \neq m'$. Hence $\alpha_1$ and $\beta_1$ exist and are distinct.

Let $p$ be the closest point to $\alpha_1 x_0$ on $[x_0, m x_0]$. Let $r$ and $q$ be the closest points to $p$ and $\beta_1 x_0$ respectively on $[x_0, m' x_0]$. Let $\overline{p}$ and $\overline{q}$ be the closest points to $p$ and $q$ respectively on $[m x_0, m' x_0]$.




Consider the geodesic triangle in the CAT$(-1)$-space $\widetilde{M}$ with vertices $m x_0, x_0, m' x_0$. Each of its sides is contained in the $\log(1+\sqrt{2})$-neighbourhood of the union of the two other sides.

Assume for a contradiction that $d(p, r) \leq \log(1+\sqrt{2})$. As when we proved that $G$ is free, we have

$$d(\alpha_1 x_0, \beta_1 x_0) \leq d(r,q) + 2\epsilon + \log(1+\sqrt{2}) = |d(q,x_0) - d(r,x_0)| + 2\epsilon + \log(1+\sqrt{2}) \,.$$

But by Assumption (1), we have $N - \epsilon \leq d(q, x_0) \leq N + 1$,

$$d(r, x_0) \leq d(p, x_0) \leq d(\alpha_1 x_0, x_0) \leq N + 1$$

and

$$\begin{aligned} d(r, x_0) &\geq d(p, x_0) - \log(1+\sqrt{2}) \geq d(\alpha_1 x_0, x_0) - \epsilon - \log(1+\sqrt{2}) \\ &\geq N - \epsilon - \log(1+\sqrt{2}) \,. \end{aligned}$$

In particular, $|d(q, x_0) - d(r, x_0)| \leq 1 + \epsilon + \log(1+\sqrt{2})$ and, since $\epsilon \leq 1$,

$$d(\alpha_1 x_0, \beta_1 x_0) \leq 1 + 3\epsilon + 2\log(1+\sqrt{2}) < 6 \,.$$

Since $C \geq 6$, this contradicts Assumption (2).

By the above property of the geodesic triangles, we hence have $d(p, \overline{p}) \leq \log(1+\sqrt{2})$. Similarly, $d(q, \overline{q}) \leq \log(1+\sqrt{2})$.

Let us prove the upper bound in Equation (71). Indeed, by the triangle inequality and the last preliminary remark, since $j(m, m') = 0$, we have

$$d(mx_0, m'x_0) \le d(mx_0, x_0) + d(x_0, m'x_0) \le (N+1)k + (N+1)k' = (N+1)d(m, m') \ .$$

Let us prove the lower bound in Equation (71). By convexity, the points $mx_0, \overline{p}, \overline{q}, m'x_0$ are in this order on $[mx_0, m'x_0]$. Hence by the triangle inequality, since $\alpha_1$ is the first letter of $m$ and $\beta_1$ is the first letter of $m'$, by the last preliminary remark, and since $j(m, m') = 0$, we have

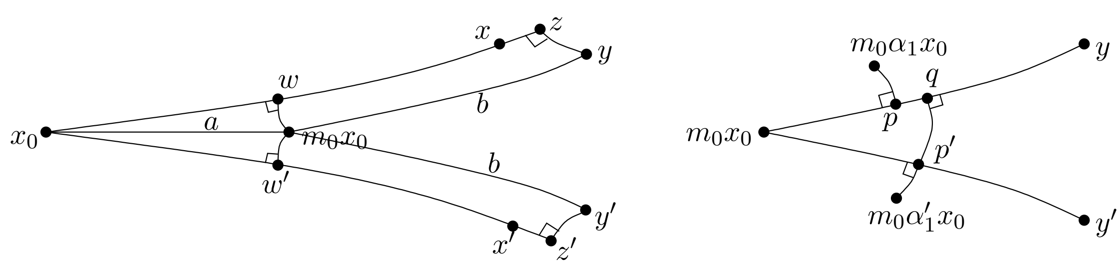
$$\begin{aligned} d(mx_0, m'x_0) &\ge d(mx_0, \overline{p}) + d(\overline{q}, m'x_0) \ge d(mx_0, p) + d(q, m'x_0) - 2\log(1+\sqrt{2}) \\ &\ge d(mx_0, \alpha_1 x_0) + d(\beta_1 x_0, m'x_0) - 2\log(1+\sqrt{2}) - 2\epsilon \\ &\ge (N-2\epsilon)(k-1) + (N-2\epsilon)(k'-1) - 2\log(1+\sqrt{2}) - 2\epsilon \\ &= (N-2\epsilon)d(m, m') - 2(\log(1+\sqrt{2}) + N - \epsilon) \ , \end{aligned}$$

as required.

Let us finally prove the last assertion of Proposition 4.9. Let $m$ and $m'$ be two words in the elements of $S$, none of them being an initial subword of the other. They are nontrivial (since the empty word is an initial subword of any word), and we may write $m = m_0\alpha_1 m_1$ and $m' = m_0\alpha'_1 m'_1$ where $m_0, m_1, m'_1$ are (possibly empty) words in the elements of $S$ with $\ell(m_0) = j(m, m')$ and $\alpha_1, \alpha'_1$ are distinct elements of $S$. Let

$$a = d(x_0, m_0x_0) \quad \text{and} \quad b = \min\{d(m_0x_0, mx_0),\ d(m_0x_0, m'x_0)\} \ .$$

Let $y$ and $y'$ be the points of $[m_0x_0, mx_0]$ and $[m_0x_0, m'x_0]$ respectively at distance $b$ from $m_0x_0$. Let $z$ and $z'$ be the closest points to $y$ and $y'$ on $[x_0, mx_0]$ and $[x_0, m'x_0]$ respectively. Let $w$ and $w'$ be the closest points to $m_0x_0$ on $[x_0, mx_0]$ and $[x_0, m'x_0]$ respectively. Let $x$ and $x'$ be the points on $[x_0, mx_0]$ and $[x_0, m'x_0]$ respectively at the same distance $a+b-3\epsilon$ from $x_0$. Let us prove that $x$ and $x'$ are well defined and satisfy the requirements of the last assertion of Proposition 4.9.




We have

$$\max\{d(m_0x_0, w), d(m_0x_0, w'), d(y, z), d(y', z')\} \le \epsilon$$

since $[x_0, m_0x_0] \cup [m_0x_0, mx_0]$ is contained in the $\epsilon$-neighbourhood of $[x_0, mx_0]$ and similarly upon replacing $m$ by $m'$.

By the last preliminary remark, we have

$$b = \min\{d(x_0, \alpha_1 m_1 x_0),\ d(x_0, \alpha'_1 m'_1 x_0)\} \ge N - 2\epsilon \ge 3\epsilon \ .$$

Hence $a+b-3\epsilon \geq 0$, and the points $x_0, w, z$ and $x_0, w', z'$ are in this order on $[x_0, mx_0]$ and $[x_0, m'x_0]$ respectively. By the triangle inequality, and since closest point maps do not increase the distances, we have

$$\begin{aligned}
d(x_0,x) = a+b-3\epsilon &= d(x_0,m_0x_0)+d(m_0x_0,y)-3\epsilon \\
&\leq d(x_0,w)+d(w,z)+2d(m_0x_0,w)+d(y,z)-3\epsilon \\
&\leq d(x_0,z) = d(x_0,w)+d(w,z) \leq a+b\ .
\end{aligned}$$

In particular, $a+b-3\epsilon \leq d(x_0,z) \leq d(x_0,mx_0)$ and $x$ is well defined. Similarly, $x'$ is well defined.

By the triangle inequality,

$$\begin{aligned}
d(x,y) &\leq d(x,z)+d(z,y) = (d(x_0,z)-d(x_0,x))+d(z,x) \\
&\leq a+b-(a+b-3\epsilon)+\epsilon = 4\epsilon\ .
\end{aligned}$$

Similarly, $d(x',y') \leq 4\epsilon$, hence again by the triangle inequality

$$d(x,x') \geq d(y,y')-d(x,y)-d(x',y') \geq d(y,y')-8\epsilon\ .$$

Let us prove that $d(y,y') \geq 1$, which implies the last assertion of Proposition 4.9.

Assume for a contradiction that $d(y,y') < 1$. Let $p$ and $p'$ be the closest points to $m_0\alpha_1x_0$ and $m_0\alpha'_1x_0$ on $[m_0x_0, mx_0]$ and $[m_0x_0, m'x_0]$ respectively. We have

$$\max\{d(m_0\alpha_1x_0,p),\ d(m_0\alpha'_1x_0,p')\} \leq \epsilon\ .$$

Up to permuting $m$ and $m'$, since $y = mx_0$ or $y' = m'x_0$ by the definition of $y$ and $y'$, we may assume that $p' \in [m_0x_0, y']$. Let $q$ be the closest point to $p'$ on $[m_0x_0, mx_0]$. By convexity, we have $d(p',q) \leq d(y',y)$. By the triangle inequality, since closest point maps do not increase distances, and by the hypothesis (1), we have

$$\begin{aligned}
d(p,q) = |d(m_0x_0,p)-d(m_0x_0,q)| &\leq |d(m_0x_0,p)-d(m_0x_0,p')|+d(p',q) \\
&\leq |d(m_0x_0,m_0\alpha_1x_0)-d(m_0x_0,m_0\alpha'_1x_0)|+\epsilon+d(p',q) \\
&< ((N+1)-N)+\epsilon+d(y',y) = 1+\epsilon+d(y',y)\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
d(\alpha_1x_0,\alpha'_1x_0) = d(m_0\alpha_1x_0,m_0\alpha'_1x_0) &\leq d(m_0\alpha_1x_0,p)+d(p,q)+d(q,p')+d(p',m_0\alpha'_1x_0) \\
&\leq 1+3\epsilon+2d(y',y) < 6 \leq C\ .
\end{aligned}$$

This contradicts the hypothesis (2). This ends the proof of Proposition 4.9. □

We now construct (in Proposition 4.10) finite subsets $S$ of $\Gamma$ satisfying the assumptions (1)-(3) of Proposition 4.9. These sets $S$ satisfy another property (4), which will allow us to prove in Theorem 4.11 that the critical exponent $\delta_{G,\,F}$, where $G$ is the subsemigroup generated by $S$, is close to the critical exponent $\delta_{\Gamma,\,F}$.

We will follow arguments contained in the proof of [OtP, Theo. 1], though the necessity of controlling the integral of the potential $\widetilde{F}$ along pieces of orbits of the geodesic flow, and the new application Theorem 4.11, require a much more precise construction. The origin of the techniques goes back to the paper [BiJ], as extended independently by U. Hamenstädt (unpublished) and [Pau1] to prove that the Hausdorff dimension of the radial limit set $\Lambda_c\Gamma$ of $\Gamma$ is the (standard) critical exponent of $\Gamma$.

**Proposition 4.10** *For all $\delta' \in \,]-\infty, \delta_{\Gamma,F}[$ and $\theta, C > 0$, there exist $x_0 \in \widetilde{M}$ and $v_0 \in T^1_{x_0}\widetilde{M}$ such that for every $L > 0$, there exist $N > L$ and a finite subset $S$ of $\Gamma$ such that*

*(1) $N \leq d(x_0, \alpha x_0) < N + 1$ for every $\alpha \in S$;*

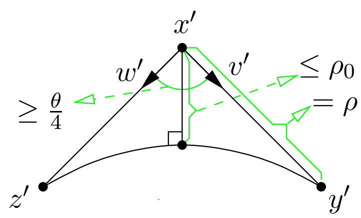
*(2) $d(\alpha x_0, \beta x_0) \geq C$ for all distinct $\alpha$ and $\beta$ in $S$;*

*(3) for every $\alpha \in S$, the angle at $x_0$ between $v_0$ and the unit tangent vector at $x_0$ pointing towards $\alpha x_0$, and the angle at $\alpha x_0$ between $-\alpha v_0$ and the unit tangent vector at $\alpha x_0$ pointing towards $x_0$, are at most $\theta$;*

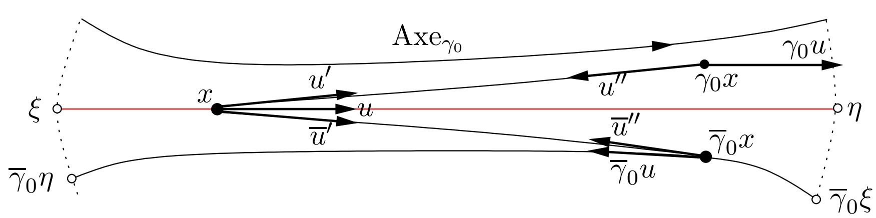
*(4) $\displaystyle\sum_{\alpha \in S} e^{\int_{x_0}^{\alpha x_0} \widetilde{F}} \geq e^{\delta' N}$.*

**Proof.** Let us fix a real constant $\delta' < \delta = \delta_{\Gamma,F}$ and positive ones $\theta, C$. First assume that $\delta' > 0$ (see the end of the proof to remove this assumption). We may assume that $\theta < \frac{\pi}{2}$. Let $\epsilon, \theta' > 0$ be small enough parameters, and let $L'$ be a large enough parameter.

Since $\widetilde{M}$ is CAT$(-1)$, there exist $\rho_0 = \rho_0(\theta) > 0$ and $\rho = \rho(\theta, \theta') \geq \rho_0 + 1$ such that for every $x' \in \widetilde{M}$, for all $v', w' \in T^1_{x'}\widetilde{M}$ whose angle $\measuredangle_{x'}(v', w')$ at $x'$ is a least $\frac{\theta}{4}$, if $y' = \pi(\phi_\rho v')$ and $z' = \pi(\phi_\rho w')$, then the angles at $y'$ and $z'$ of the geodesic triangle with vertices $x', y', z'$ are at most $\theta'$, and the distance from $x'$ to the opposite side $[y', z']$ is at most $\rho_0$.




Since $\Gamma$ is non-elementary, there exist two distinct points $\xi$ and $\eta$ in $\Lambda\Gamma$ such that the orbit of $(\xi, \eta)$ in $\Lambda\Gamma \times \Lambda\Gamma$ under the diagonal action of $\Gamma$ is dense in $\Lambda\Gamma \times \Lambda\Gamma$. Let $u$ be a unit tangent vector to the geodesic line from $\xi$ to $\eta$, and let $x = \pi(u)$.




Again since $\Gamma$ is non-elementary, the pairs of endpoints of translation axes of loxodromic elements of $\Gamma$ are dense in $\Lambda\Gamma \times \Lambda\Gamma$. In particular there exists $\gamma_0 \in \Gamma$ whose translation axis Axe$_{\gamma_0}$ has endpoints close to $\xi$ and $\eta$. Up to replacing $\gamma_0$ by a positive or negative power, we may assume that $\gamma_0$ translates from the endpoint close to $\xi$ towards the endpoint close to $\eta$, and that the geodesic segment $[x, \gamma_0 x]$ has length at least $L'$ and is contained in the $\epsilon$-neighbourhood of Axe$_{\gamma_0}$. Denote by $u'$ the unit tangent vector at $x$ pointing towards $\gamma_0 x$ and by $u''$ the unit tangent vector at $\gamma_0 x$ pointing towards $x$. We may also assume that the angles $\measuredangle_x(u, u')$ and $\measuredangle_{\gamma_0 x}(\gamma_0 u, -u'')$ are at most $\frac{\theta}{8}$. In particular, by the spherical triangle inequality, we have

$$\measuredangle_{\gamma_0 x}(u'', -\gamma_0 u') \leq \measuredangle_{\gamma_0 x}(u'', -\gamma_0 u) + \measuredangle_{\gamma_0 x}(-\gamma_0 u, -\gamma_0 u') \leq 2\,\frac{\theta}{8} \leq \frac{\theta}{2}\,. \tag{72}$$

By density and up to taking powers, there exists $\overline{\gamma}_0 \in \Gamma$ such that $\overline{\gamma}_0 \xi$ is close to $\eta$, $\overline{\gamma}_0 \eta$ is close to $\xi$, the geodesic segment $[x, \overline{\gamma}_0 x]$ has length at least $L'$, and, with $\overline{u}'$ the unit tangent vector at $x$ pointing towards $\overline{\gamma}_0 x$ and $\overline{u}''$ the unit tangent vector at $\overline{\gamma}_0 x$ pointing towards $x$, the angles $\angle_x(u, \overline{u}')$ and $\angle_{\overline{\gamma}_0 x}(\overline{\gamma}_0 u, \overline{u}'')$ are at most $\frac{\theta}{8}$.

Since $\widetilde{M}$ has pinched negative curvature and $\rho$ depends only on $\theta, \theta'$, there exists $\theta'' = \theta''(\epsilon, \theta, \theta') \in \,]0, \frac{\theta}{8}]$ such that for all $v', v''$ in $T^1_x\widetilde{M}$ with angle $\angle_x(v', v'')$ at most $\theta''$, for every $t \in [0, \rho]$ (or equivalently for $t = \rho$ by convexity), we have

$$d(\pi(\phi_t v'), \pi(\phi_t v'')) \leq \epsilon \;.$$

For every $n \in \mathbb{N}$, let

$$A_n = \{y \in \widetilde{M} \;:\; n \leq d(x, y) < n + 1\} \;.$$

We have

$$\limsup_{n \to +\infty} \sum_{\gamma \in \Gamma \,:\, \gamma x \in A_n} e^{\int_x^{\gamma x} (\widetilde{F} - \delta')} = +\infty \;.$$

Indeed, by the definition of the critical exponent $\delta = \delta_{\Gamma,\,F}$ of $(\Gamma, F)$ (at the beginning of Subsection 3.2), if $\delta'' \in \,]\delta', \delta[$, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\frac{1}{n_k} \log \sum_{\gamma\in\Gamma,\, \gamma x \in A_{n_k}} e^{\int_x^{\gamma y} \widetilde{F}} \geq \delta''$, and hence $\sum_{\gamma\in\Gamma \,:\, \gamma x\in A_{n_k}} e^{\int_x^{\gamma x} (\widetilde{F}-\delta')} \geq e^{(\delta''-\delta')n_k - \delta'}$, which tends to $+\infty$ as $k \to +\infty$.

Since the isometric action of $\Gamma$ on $\widetilde{M}$ is proper, the group $\Gamma$ is the union of finitely many subsets $E$ such that

$$d(\alpha x, \beta x) \geq C' = C + \max\{\, 2\,\rho_0,\; 2\, d(x, \gamma_0 x),\; 2\, d(x, \overline{\gamma}_0 x)\,\} \tag{73}$$

for all $\alpha \neq \beta$ in $E$. There exists such a subset $E_0$ satisfying

$$\limsup_{n \to +\infty} \sum_{\gamma \in E_0 \,:\, \gamma x \in A_n} e^{\int_x^{\gamma x} (\widetilde{F} - \delta')} = +\infty \;.$$

Consider the sequence of the finite subsets $Z_n$ of the unit tangent vectors at $x$ pointing towards the elements of $E_0 x \cap A_n$, for $n \in \mathbb{N}$. By looking at the accumulation points of these subsets and using the compactness of $T^1_x\widetilde{M}$, there exist $v, w$ in $T^1_x\widetilde{M}$, an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $[2\,\rho_0, +\infty[\, \cap \mathbb{N}$ and, for every $k \in \mathbb{N}$, a finite subset $S_k$ of $\Gamma$ such that

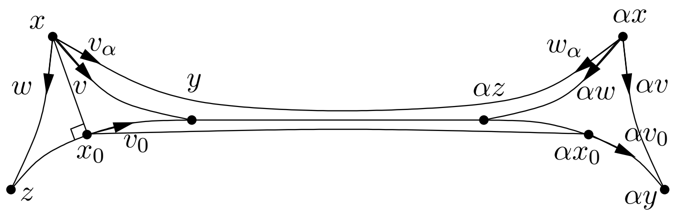
(i) for all $\alpha$ and $\beta$ in $S_k$, we have $n_k \leq d(x, \alpha x) \leq n_k + 1$ and $d(\alpha x, \beta x) \geq C'$ if $\alpha \neq \beta$;

(ii) for every $\alpha \in S_k$, the angle at $x$ between $v$ and the unit tangent vector $v_\alpha$ at $x$ pointing towards $\alpha x$, and the angle at $\alpha x$ between $\alpha w$ and the unit tangent vector $w_\alpha$ at $\alpha x$ pointing towards $x$, are at most $\theta''$;

(iii) $\displaystyle\sum_{\alpha \in S_k} e^{\int_x^{\alpha x} \widetilde{F}} \geq k\, e^{\delta' n_k}$.

The end of the proof of Proposition 4.10, which would be easy if $v = -w$, is subdivided in a few cases, the first of them being the easiest case when $v = -w$ almost holds, and the other ones dealing with the situation when the equality $v = -w$ is far from holding, and where we try to go back to the easiest case by rounding up finitely many geodesic segments in $M$ with the same origin and endpoint, and making them composable in an almost geodesic way.

**Case 1.** First assume that $\angle_x(v, w) \geq \frac{\theta}{4}$.




Let $y = \pi(\phi_\rho v)$ and $z = \pi(\phi_\rho w)$. Define $x_0$ to be the closest point to $x$ on $[y, z]$ and $v_0$ as the unit tangent vector at $x_0$ pointing towards $y$. In particular, by the definition of $\rho_0$, we have $d(x, x_0) \leq \rho_0$.

Let $k \in \mathbb{N}$ and $\alpha, \beta \in S_k$. By the triangle inequality and Assertion (i) above, we have

$$0 < n_k - 2\rho_0 \leq d(x_0, \alpha x_0) \leq n_k + 1 + 2\rho_0 \ , \tag{74}$$

and, if $\alpha \neq \beta$, by the definition of $C'$ in Equation (73),

$$d(\alpha x_0, \beta x_0) \geq C' - 2\rho_0 \geq C \ .$$

If $k$ is large enough, then $d(x, \alpha x)$ is large enough by Assertion (i) and, since $\measuredangle_x(v, v_\alpha) \leq \theta''$ and $\measuredangle_{\alpha x}(\alpha w, w_\alpha) \leq \theta''$ by Assertion (ii) above, the points $y$ and $\alpha z$ are, by the definition of $\theta''$, at distance at most $\epsilon$ to the geodesic segment $[x, \alpha x]$.

If two geodesic segments in a non-positively curved manifold have lengths at least 1 and have sufficiently close endpoints, then their tangent vectors (parametrising them proportionally to arc-length by $[0, 1]$) are uniformly close. In particular (or by using Lemma 3.2), since $\rho \geq 1$, if $\epsilon$ is small enough and if $k$ is large enough, there exists a constant $c_2 > 0$ (depending only on $\epsilon$, the Hölder constants of $\widetilde{F}$ and $\max_{\pi^{-1}(B(y,\,\epsilon)\,\cup\, B(z,\,\epsilon))} |\widetilde{F}|$) such that

$$\Big| \int_x^{\alpha x} \widetilde{F} - \int_x^{y} \widetilde{F} - \int_y^{\alpha z} \widetilde{F} - \int_{\alpha z}^{\alpha x} \widetilde{F} \,\Big| \leq c_2 \ . \tag{75}$$

By the triangle inequality and the definition of $\rho$, we have $d(x_0, y) \geq \rho - \rho_0 \geq 1$, and similarly $d(\alpha z, \alpha x_0) \geq 1$. If $k$ is large enough, we have, for every $\alpha \in S_k$,

$$d(y, \alpha z) \geq d(x, \alpha x) - 2\rho \geq n_k - 2\rho \geq 1 \ .$$

Recall that $d(y, [x, \alpha x]) \leq \epsilon$, $d(\alpha z, [x, \alpha x]) \leq \epsilon$, and that $\measuredangle_y(\phi_\rho v, \phi_{d(x_0, y)} v_0) \leq \theta'$ by the definition of $\rho$. Let $\epsilon' > 0$ be small enough, and assume that $\theta'$ and $\epsilon$ are small enough compared to $\theta$ and $\epsilon'$. Then, if $k$ is large enough, the piecewise geodesic path

$$\omega = [x_0, y] \cup [y, \alpha z] \cup [\alpha z, \alpha x_0]$$

has its exteriors angles at $y$ and $\alpha z$ at most $\theta_0(1, \epsilon', \theta)$, where $\theta_0(\cdot, \cdot, \cdot)$ has been defined in Lemma 2.7. Therefore by this lemma, for every $\alpha \in S_k$, the angles at $x_0$ between $v_0$ and the unit tangent vector at $x_0$ pointing towards $\alpha x_0$, and the angle at $\alpha x_0$ between $-\alpha v_0$ and the unit tangent vector at $\alpha x_0$ pointing towards $x_0$, are at most $\theta$; furthermore the piecewise geodesic path $\omega$ is at distance at most $\epsilon'$ from $[x_0, \alpha x_0]$.

As in the proof of Equation (75), there exists a constant $c_3 > 0$ (depending only on $\epsilon'$, the Hölder constants of $\widetilde{F}$ and $\max_{\pi^{-1}(B(y,\,\epsilon')\,\cup\, B(z,\,\epsilon'))} |\widetilde{F}|$) such that

$$\Big| \int_{x_0}^{\alpha x_0} \widetilde{F} - \int_{x_0}^{y} \widetilde{F} - \int_y^{\alpha z} \widetilde{F} - \int_{\alpha z}^{\alpha x_0} \widetilde{F} \,\Big| \leq c_3 \ . \tag{76}$$

With $c_4 = \rho \max_{\pi^{-1}(B(x,\rho))} |\widetilde{F}|$, since $d(x_0, y) \leq d(x,y) = \rho$ and similarly with $z$ instead of $y$, we have

$$\Big|\int_{x_0}^{y} \widetilde{F}\,\Big|,\ \Big|\int_{x}^{y} \widetilde{F}\,\Big|,\ \Big|\int_{\alpha z}^{\alpha x} \widetilde{F}\,\Big|,\ \Big|\int_{\alpha z}^{\alpha x_0} \widetilde{F}\,\Big| \leq c_4\ . \tag{77}$$

By the formulas (75), (76), (77), and by Assertion (iii) above, we have, if $k$ is large enough,

$$\sum_{\alpha\in S_k} e^{\int_{x_0}^{\alpha x_0}\widetilde{F}} \geq e^{-c_2-c_3-4c_4} \sum_{\alpha\in S_k} e^{\int_{x}^{\alpha x}\widetilde{F}} \geq k\ e^{-c_2-c_3-4c_4}\ e^{\delta' n_k}\ .$$

Hence, using Equation (74), there exists $N \in \mathbb{N} \cap\, ]n_k - 2\rho_0 - 1, n_k + 2\rho_0 + 2[$ such that

$$\sum_{\substack{\alpha\ \in\ S_k \\ N\ \leq\ d(x_0,\,\alpha x_0)\ <\ N+1}} e^{\int_{x_0}^{\alpha x_0}\widetilde{F}} \geq \frac{k\ e^{-c_2-c_3-4c_4}}{4\rho_0+3}\ e^{\delta' n_k} \geq \frac{k\ e^{-c_2-c_3-4c_4-\delta'(2\rho_0+2)}}{4\rho_0+3}\ e^{\delta' N}\ .$$

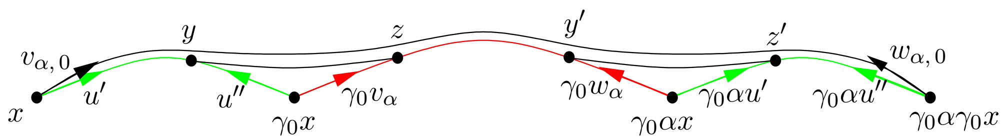
Given $L > 0$, if $k$ is large enough, then $\frac{k\ e^{-c_2-c_3-4c_4-\delta'(2\rho_0+2)}}{4\rho_0+3} \geq 1$ and $N \geq n_k - 2\rho_0 - 1 > L$, and defining

$$S = \{\alpha \in S_k\ :\ N \leq d(x_0, \alpha x_0) < N+1\}\ ,$$

the objects $x_0, v_0, N, S$ defined above satisfy the properties (1)-(4) required in Proposition 4.10.

**Case 2.** Assume that $\measuredangle_x(u, -v) \geq \frac{\theta}{2}$ and that $\measuredangle_x(u, w) \geq \frac{\theta}{2}$.

Consider the loxodromic element $\gamma_0$ of $\Gamma$ and the unit tangent vectors $u'$ and $u''$ defined above. We will define in due course the notation in the following figure.




Let $k \in \mathbb{N}$ and $\alpha \in S_k$. In order to get the following inequalities, we use:

- the spherical triangle inequality for the first one;
- the assumption of Case 2, the properties of $u'$ and $u''$, and Assertion (ii) above for the second one;
- and the assumption on $\theta''$ for the last one:

$$\begin{aligned}\measuredangle_{\gamma_0 x}(u'', \gamma_0 v_\alpha) &\geq \measuredangle_{\gamma_0 x}(-\gamma_0 u, \gamma_0 v) - \measuredangle_{\gamma_0 x}(-\gamma_0 u, u'') - \measuredangle_{\gamma_0 x}(\gamma_0 v_\alpha, \gamma_0 v) \\ &\geq \frac{\theta}{2} - \frac{\theta}{8} - \theta'' \geq \frac{\theta}{4}\ .\end{aligned}$$

Similarly, $\measuredangle_{\gamma_0\alpha x}(\gamma_0 w_\alpha, \gamma_0\alpha u') \geq \frac{\theta}{4}$.

Hence, by the definition of $\rho$, the angles at $y = \pi(\phi_\rho u'')$ and $z = \pi(\phi_\rho(\gamma_0 v_\alpha))$ of the geodesic triangle with vertices $\gamma_0 x, y, z$ are at most $\theta'$, and similarly the angles at $y' = \pi(\phi_\rho(\gamma_0 w_\alpha))$ and $z' = \pi(\phi_\rho(\gamma_0\alpha u'))$ of the geodesic triangle with vertices $\gamma_0\alpha x, y', z'$ are at most $\theta'$.

Assume that $L' \geq \rho + 1$, so that by the construction of $\gamma_0$

$$d(x,y) = d(x, \gamma_0 x) - \rho \geq L' - \rho \geq 1\ ,$$

and similarly $d(z', \gamma_0\alpha\gamma_0 x) \geq 1$. By the triangle inequality and the definition of $\rho$, we have $d(y,z) \geq 2(\rho - \rho_0) \geq 2$, and similarly $d(y',z') \geq 2$. Assume that $k$ is large enough, in particular so that we have

$$d(z,y') = d(\gamma_0 x, \gamma_0\alpha x) - d(\gamma_0 x, z) - d(y', \gamma_0\alpha x) \geq n_k - 2\rho \geq 1 \ .$$

Let $\epsilon' > 0$ be small enough, and assume that $\theta'$ is small enough, so that in particular $\theta' \leq \theta_0(1, \epsilon', \frac{\theta}{2})$. The piecewise geodesic path

$$\omega = [x,y] \cup [y,z] \cup [z,y'] \cup [y',z'] \cup [z', \gamma_0\alpha\gamma_0 x]$$

has exterior angles at $y, z, y', z'$ at most $\theta'$. Hence, by Lemma 2.7, the angles at $x$ between $u'$ and the unit tangent vector $v_{\alpha,0}$ at $x$ pointing towards $\gamma_0\alpha\gamma_0 x$, and the angle at $\gamma_0\alpha\gamma_0 x$ between $\gamma_0\alpha u''$ and the unit tangent vector $w_{\alpha,0}$ at $\gamma_0\alpha\gamma_0 x$ pointing towards $x$, are at most $\frac{\theta}{2}$:

$$\measuredangle_x(u', v_{\alpha,0}) \leq \frac{\theta}{2}, \quad \measuredangle_{\gamma_0\alpha\gamma_0 x}(\gamma_0\alpha u'', w_{\alpha,0}) \leq \frac{\theta}{2} \ . \tag{78}$$

Furthermore the piecewise geodesic path $\omega$ is at distance at most $\epsilon'$ from $[x, \gamma_0\alpha\gamma_0 x]$.

Hence, as in Case 1, there exists a constant $c_5 > 0$ (depending only on $\epsilon'$, the Hölder constants of $\widetilde{F}$ and $\max_{\pi^{-1}(B(y,\epsilon')\cup B(z,\epsilon')\cup B(y',\epsilon')\cup B(z',\epsilon'))} |\widetilde{F}|$) such that

$$\Big| \int_x^{\gamma_0\alpha\gamma_0 x} \widetilde{F} - \int_x^y \widetilde{F} - \int_y^z \widetilde{F} - \int_z^{y'} \widetilde{F} - \int_{y'}^{z'} \widetilde{F} - \int_{z'}^{\gamma_0\alpha\gamma_0 x} \widetilde{F} \Big| \leq c_5 \ . \tag{79}$$

By invariance and additivity, we have

$$\int_x^{\alpha x} \widetilde{F} = \int_{\gamma_0 x}^{\gamma_0\alpha x} \widetilde{F} = \int_{\gamma_0 x}^{z} \widetilde{F} + \int_z^{y'} \widetilde{F} + \int_{y'}^{\gamma_0\alpha x} \widetilde{F} \ . \tag{80}$$

Note that $d(x,y) \leq d(x, \gamma_0 x)$, $d(y,z) \leq 2\rho$ and $d(y',z') \leq 2\rho$. Therefore, with $c_6 = (d(x,\gamma_0 x) + 2\rho) \max_{\pi^{-1}(B(x,\, d(x,\gamma_0 x)+2\rho))} |\widetilde{F}|$, we have

$$\Big| \int_{\gamma_0 x}^{z} \widetilde{F} \Big|, \ \Big| \int_{y'}^{\gamma_0\alpha x} \widetilde{F} \Big|, \ \Big| \int_x^y \widetilde{F} \Big|, \ \Big| \int_y^z \widetilde{F} \Big|, \ \Big| \int_{y'}^{z'} \widetilde{F} \Big|, \ \Big| \int_{z'}^{\gamma_0\alpha\gamma_0 x} \widetilde{F} \Big| \leq c_6 \ .$$

Hence, by the formulas (79) and (80), we have

$$\int_x^{\gamma_0\alpha\gamma_0 x} \widetilde{F} \geq \Big( \int_x^{\alpha x} \widetilde{F} \Big) - c_5 - 6c_6 \ .$$

Assertion (iii) above implies that

$$\sum_{\alpha \in S_k} e^{\int_x^{\gamma_0\alpha\gamma_0 x} \widetilde{F}} \geq k \, e^{-c_5 - 6c_6} \, e^{\delta' n_k} \ .$$

Define

$$x_0 = x \quad \text{and} \quad v_0 = u' \ .$$

By Equation (78), for $k$ large enough and for every $\alpha \in S_k$, we have

$$\measuredangle_{x_0}(v_0, v_{\alpha,0}) \leq \frac{\theta}{2} \leq \theta \ .$$

By the spherical triangle inequality and since the isometry $\gamma_0\alpha$ preserves the angles for the first inequality, and by the formulas (72) and (78) for the second one, we have

$$\measuredangle_{\gamma_0\alpha\gamma_0 x_0}(-\gamma_0\alpha\gamma_0 u', w_{\alpha,0}) \leq \measuredangle_{\gamma_0 x}(-\gamma_0 u', u'') + \measuredangle_{\gamma_0\alpha\gamma_0 x}(\gamma_0\alpha u'', w_{\alpha,0}) \leq \frac{\theta}{2} + \frac{\theta}{2} = \theta \ .$$

For $k$ large enough and for all $\alpha, \beta \in S_k$, we have by the triangle inequality and Assertion (i) above

$$n_k - 2\, d(x, \gamma_0 x) \leq d(x_0, \gamma_0\alpha\gamma_0 x_0) \leq n_k + 1 + 2\, d(x, \gamma_0 x) \ ,$$

and, if $\alpha \neq \beta$, by the definition of $C'$ in Equation (73),

$$\begin{aligned} d(\gamma_0\alpha\gamma_0 x_0, \gamma_0\beta\gamma_0 x_0) &= d(\alpha\gamma_0 x, \beta\gamma_0 x) \geq d(\alpha x, \beta x) - d(\alpha x, \alpha\gamma_0 x) - d(\beta x, \beta\gamma_0 x) \\ &\geq C' - 2\, d(x, \gamma_0 x) \geq C \ . \end{aligned}$$

As in Case 1, there exists $N \in \mathbb{N} \cap \,]n_k - 2\, d(x, \gamma_0 x) - 1, n_k + 2\, d(x, \gamma_0 x) + 2[$ such that

$$\begin{aligned} \sum_{\substack{\alpha \,\in\, S_k \\ N \,\leq\, d(x_0,\, \gamma_0\alpha\gamma_0 x_0) \,<\, N+1}} e^{\int_{x_0}^{\gamma_0\alpha\gamma_0 x_0} \widetilde{F}} &\geq \frac{k\ e^{-c_5 - 6c_6}}{4\, d(x, \gamma_0 x) + 3}\, e^{\delta' n_k} \\ &\geq \frac{k\ e^{-c_5 - 6c_6 - \delta'(2\, d(x,\gamma_0 x)+2)}}{4\, d(x, \gamma_0 x) + 3}\, e^{\delta' N} \ . \end{aligned}$$

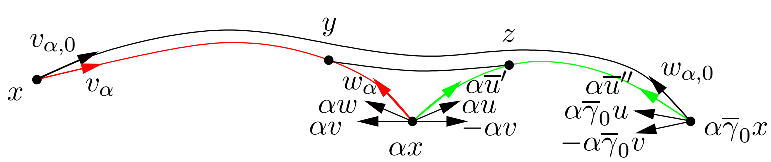
Given $L > 0$, if $k$ is large enough, then $\frac{k\ e^{-c_5-6c_6-\delta'(2\, d(x,\gamma_0 x)+2)}}{4\, d(x,\gamma_0 x)+3} \geq 1$ and $N \geq n_k - 2\, d(x, \gamma_0 x) - 1 > L$, and defining

$$S = \{\gamma_0\alpha\gamma_0 \ : \ \alpha \in S_k \ \text{ and } \ N \leq d(x_0, \gamma_0\alpha\gamma_0 x_0) < N + 1\} \ ,$$

the objects $x_0, v_0, N, S$ defined above satisfy the properties (1)-(4) required in Proposition 4.10.

**Case 3.** Assume that $\measuredangle_x(v, w) < \frac{\theta}{4}$ and that $\measuredangle_x(u, -v) < \frac{\theta}{2}$.

Consider the element $\overline{\gamma}_0$ of $\Gamma$ and the unit tangent vectors $\overline{u}'$ and $\overline{u}''$ defined above.




Let $k \in \mathbb{N}$ and $\alpha \in S_k$. By the spherical triangle inequality, by the assumptions of Case 3, by Assertion (ii) above, by the properties of $\overline{u}'$, and since $\theta'' \leq \frac{\theta}{8}$ and $\theta \leq \frac{\pi}{2}$, we have

$$\begin{aligned} \measuredangle_{\alpha x}(w_\alpha, \alpha\overline{u}') &\geq \measuredangle_{\alpha x}(\alpha v, -\alpha v) \\ &\qquad - \measuredangle_{\alpha x}(\alpha v, \alpha w) - \measuredangle_{\alpha x}(\alpha w, w_\alpha) - \measuredangle_{\alpha x}(-\alpha v, \alpha u) - \measuredangle_{\alpha x}(\alpha u, \alpha\overline{u}') \\ &\geq \pi - \frac{\theta}{4} - \theta'' - \frac{\theta}{2} - \frac{\theta}{8} \geq \pi - \theta \geq \frac{\pi}{2} \geq \frac{\theta}{4} \ . \end{aligned}$$

As in Case 2, if $L'$ and $k$ are large enough, for every $\epsilon'$ small enough, if $\theta'$ is small enough, so that in particular $L' \geq \rho + 1$, $n_k \geq \rho + 1$, $\theta' \leq \theta_0(1, \epsilon', \frac{\theta}{4})$, with $y = \pi(\phi_\rho w_\alpha)$ and $z = \pi(\phi_\rho(\alpha \overline{u}'))$, considering the piecewise geodesic path

$$w = [x, y] \cup [y, z] \cup [z, \alpha\overline{\gamma}_0 x] \;,$$

we have for every $\alpha \in S_k$:

- the angles at $y$ and $z$ of the geodesic triangle with vertices $\alpha x, y, z$ are at most $\theta'$, by the definition of $\rho$, hence the exterior angles of $\omega$ at $y, z$ are at most $\theta' \leq \theta_0(1, \epsilon', \frac{\theta}{4})$;
- we have

$$d(x, y) \geq n_k - \rho \geq 1 \;, \quad d(y, z) \geq 2(\rho - \rho_0) \geq 2 \;,$$
$$d(z, \alpha\overline{\gamma}_0 x) = d(\alpha x, \alpha\overline{\gamma}_0 x) - \rho \geq L' - \rho \geq 1 \;.$$

hence we may apply Lemma 2.7 to $\omega$ with $\ell = 1$;

- the angles at $x$ between $v_\alpha$ and the unit tangent vector $v_{\alpha,0}$ at $x$ pointing towards $\alpha\overline{\gamma}_0 x$, and the angle at $\alpha\overline{\gamma}_0 x$ between $\alpha\overline{u}''$ and the unit tangent vector $w_{\alpha,0}$ at $\alpha\overline{\gamma}_0 x$ pointing towards $x$, are at most $\frac{\theta}{4}$;
- the path $\omega$ is at distance at most $\epsilon'$ from $[x, \alpha\overline{\gamma}_0 x]$;
- there exists a constant $c_7 > 0$ (depending only on $\epsilon'$, the Hölder constants of $\widetilde{F}$ and $\max_{\pi^{-1}(B(y,\epsilon') \cup B(z,\epsilon'))} |\widetilde{F}|$) such that

$$\Big| \int_x^{\alpha\overline{\gamma}_0 x} \widetilde{F} - \int_x^y \widetilde{F} - \int_y^z \widetilde{F} - \int_z^{\alpha\overline{\gamma}_0 x} \widetilde{F} \Big| \leq c_7 \;;$$

- if $c_8 = (d(x, \overline{\gamma}_0 x) + 2\rho) \max_{\pi^{-1}(B(x, d(x,\overline{\gamma}_0 x) + 2\rho))} |\widetilde{F}|$, we have

$$\Big| \int_y^{\alpha x} \widetilde{F} \Big|, \Big| \int_y^z \widetilde{F} \Big|, \Big| \int_z^{\alpha\overline{\gamma}_0 x} \widetilde{F} \Big| \leq c_8 \;.$$

Since $\int_x^{\alpha x} \widetilde{F} = \int_x^y \widetilde{F} + \int_y^{\alpha x} \widetilde{F}$, and by the last two points, we have

$$\int_x^{\alpha\overline{\gamma}_0 x} \widetilde{F} \geq \Big( \int_x^{\alpha x} \widetilde{F} \Big) - c_7 - 3\, c_8 \;.$$

Assertion (iii) above implies that

$$\sum_{\alpha \in S_k} e^{\int_x^{\alpha\overline{\gamma}_0 x} \widetilde{F}} \geq k\, e^{-c_7 - 3\, c_8}\, e^{\delta' n_k} \;.$$

Define

$$x_0 = x \quad \text{and} \quad v_0 = v \;.$$

By Assertion (ii) above and the third point above, we have

$$\measuredangle_{x_0}(v_0, v_{\alpha,0}) \leq \measuredangle_x(v, v_\alpha) + \measuredangle_x(v_\alpha, v_{\alpha,0}) \leq \theta'' + \frac{\theta}{4} \leq \theta \;.$$

By the assumptions of Case 3, by the properties of $\overline{u}''$, and by the third point above, we have

$$\begin{aligned} \measuredangle_{\alpha\overline{\gamma}_0 x}(-\alpha\overline{\gamma}_0 v_0, w_{\alpha,0}) &\leq \measuredangle_{\alpha\overline{\gamma}_0 x}(-\alpha\overline{\gamma}_0 v_0, \alpha\overline{\gamma}_0 u) + \measuredangle_{\alpha\overline{\gamma}_0 x}(\alpha\overline{\gamma}_0 u, \alpha\overline{u}'') + \measuredangle_{\alpha\overline{\gamma}_0 x}(\alpha\overline{u}'', w_{\alpha,0}) \\ &\leq \frac{\theta}{2} + \frac{\theta}{8} + \frac{\theta}{4} \leq \theta \;. \end{aligned}$$

For all $\alpha, \beta \in S_k$, by the triangle inequality and Assertion (i) above, we have

$$n_k - d(x, \overline{\gamma}_0 x) \leq d(x_0, \alpha\overline{\gamma}_0 x_0) \leq n_k + 1 + d(x, \overline{\gamma}_0 x) \;,$$

and, if $\alpha \neq \beta$, by the definition of $C'$ in Equation (73),

$$d(\alpha\overline{\gamma}_0 x_0, \beta\overline{\gamma}_0 x_0) \geq d(\alpha x, \beta x) - 2\, d(x, \overline{\gamma}_0 x) \geq C' - 2\, d(x, \overline{\gamma}_0 x) \geq C \;.$$

As in Case 1, there exists $N \in \mathbb{N} \cap \,]n_k - d(x, \overline{\gamma}_0 x) - 1, n_k + d(x, \overline{\gamma}_0 x) + 2[$ such that

$$\sum_{\substack{\alpha \,\in\, S_k \\ N \,\leq\, d(x_0,\, \alpha\overline{\gamma}_0 x_0) \,<\, N+1}} e^{\int_{x_0}^{\alpha\overline{\gamma}_0 x_0} \widetilde{F}} \geq \frac{k\; e^{-c_7 - 3c_8 - \delta'(d(x,\overline{\gamma}_0 x)+2)}}{2\, d(x, \overline{\gamma}_0 x) + 3}\; e^{\delta' N} \;.$$

Given $L > 0$, if $k$ is large enough, then $\frac{k\; e^{-c_7 - 3c_8 - \delta'(d(x,\overline{\gamma}_0 x)+2)}}{2\, d(x,\overline{\gamma}_0 x)+3} \geq 1$ and $N \geq n_k - d(x, \overline{\gamma}_0 x) - 1 > L$, and defining

$$S = \{\alpha\overline{\gamma}_0 \;:\; \alpha \in S_k \;\text{ and }\; N \leq d(x_0, \alpha\overline{\gamma}_0 x_0) < N+1\} \;,$$

the objects $x_0, v_0, N, S$ defined above satisfy the properties (1)-(4) required in Proposition 4.10.

**Case 4.** The remaining case, when $\measuredangle_x(v, w) < \frac{\theta}{4}$ and $\measuredangle_x(u, w) < \frac{\theta}{2}$, follows similarly to Case 3: we take $x_0 = x, v_0 = w$ and $S$ the appropriate subset of $\{\overline{\gamma}_0\alpha \;:\; \alpha \in S_k\}$.

This ends the proof of Proposition 4.10 under the assumption that $\delta' > 0$. Up to adding the same big enough constant to $\widetilde{F}$ and to $\delta'$, the conclusion of each of the four steps above (as $k$ was allowed to be taken large enough) proves that Proposition 4.10 remains valid when $\delta' \leq 0$. $\square$

The main objective of this subsection is to show the following result.

**Theorem 4.11** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature at most $-1$. Let $\Gamma$ be a non-elementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map.*

*Then the critical exponent $\delta_{\Gamma,\,F}$ is equal to the upper bound of the critical exponents $\delta_{G,\,F}$, where $G$ ranges over the convex-cocompact free subsemigroups of $\Gamma$.*

When $F = 0$, this result (with $\widetilde{M}$ any proper geodesic CAT$(-1)$-space) is due to Mercat [Mer].

**Proof.** It is immediate that $\delta_{\Gamma,\,F} \geq \delta_{G,\,F}$ for any subsemigroup $G$ of $\Gamma$. Hence $\delta_{\Gamma,\,F}$ is at least the upper bound of the critical exponents $\delta_{G,\,F}$ of the convex-cocompact free subsemigroups $G$ of $\Gamma$. We now prove the converse inequality. Up to adding a constant to $F$, we may assume that $\delta_{\Gamma,\,F} > 0$.

Let $s \in \,]0, \delta_{\Gamma,\,F}[$ and $\delta' \in \,]s, \delta_{\Gamma,\,F}[$. Let $\epsilon \in \,]0, 1]$ be small enough. Let $\theta > 0$ (depending only on $\epsilon$) be given by Proposition 4.9. Let $x_0 \in \widetilde{M}$ and $v_0 \in T^1\widetilde{M}$ be given by Proposition 4.10 (depending on $\delta', \theta, C$).

Since two geodesic segments of lengths at least 1 which are close enough have their unit tangent vectors close (see Lemma 3.2), there exists $c > 0$ (depending only on $\epsilon$, $x_0$, the Hölder constants of $\widetilde{F}$ and on $\max_{\pi^{-1}(B(x_0,1))} |\widetilde{F} - s|$) such that for all $x, y, z \in \Gamma x_0$ and for every piecewise geodesic segment $[x, y] \cup [y, z]$, with segments of length at least 1, which is contained in the $\epsilon$-neighbourhood of $[x, z]$, we have

$$\Big| \int_x^z (\widetilde{F} - s) \;-\; \int_x^y (\widetilde{F} - s) \;-\; \int_y^z (\widetilde{F} - s) \,\Big| \leq c\,.$$

Let $L = \max\{5, \frac{c+s}{\delta'-s}\}$. Let $N > L$ and $S$ a finite subset of $\Gamma$ be given by Proposition 4.10 (depending on $\delta', \theta, L, C$). By Proposition 4.9, whose assumptions are satisfied by the first three assertions of Proposition 4.10, the semigroup $G$ generated by $S$ is free on $S$ and convex-cocompact, and for all $\gamma \in G$ and $\alpha \in S$, the piecewise geodesic segment $[x_0, \gamma x_0] \cup [\gamma x_0, \gamma\alpha x_0]$, whose segments have length at least 1, is contained in the $\epsilon$-neighbourhood of $[x_0, \gamma\alpha x_0]$. In particular, by the definition of $c$ and by invariance, we have

$$\Big| \int_{x_0}^{\gamma\alpha x_0} (\widetilde{F} - s) \;-\; \int_{x_0}^{\gamma x_0} (\widetilde{F} - s) \;-\; \int_{x_0}^{\alpha x_0} (\widetilde{F} - s) \,\Big| \leq c\,. \tag{81}$$

For every $k \in \mathbb{N}$, denote by $S_k$ the set of words of length $k$ in elements of $S$. Let

$$\Sigma_k = \sum_{\gamma \in S_k} e^{\int_{x_0}^{\gamma x_0} (\widetilde{F} - s)}\,,$$

so that, since $G$ is the free semigroup generated by $S$, we have

$$Q_{G,\,F,\,x_0,\,x_0}(s) = \sum_{k \in \mathbb{N}} \Sigma_k\,.$$

By Assertion (1) of Proposition 4.10, for every $\alpha \in S$, we have $-s d(x_0, \alpha x_0) \geq -s(N+1)$. By Equation (81), by Assertion (4) of Proposition 4.10, and since $N > L \geq \frac{c+s}{\delta'-s}$, we hence have

$$\begin{aligned}
\Sigma_{k+1} = \sum_{\gamma \in S_k,\; \alpha \in S} e^{\int_{x_0}^{\gamma\alpha x_0} (\widetilde{F} - s)} &\geq e^{-c} \sum_{\gamma \in S_k,\; \alpha \in S} e^{\int_{x_0}^{\gamma x_0} (\widetilde{F} - s) + \int_{x_0}^{\alpha x_0} (\widetilde{F} - s)} \\
&= e^{-c}\Sigma_k \sum_{\alpha \in S} e^{\int_{x_0}^{\alpha x_0} (\widetilde{F} - s)} \geq e^{-c-s}\Sigma_k\, e^{(\delta'-s)N} \geq \Sigma_k\,.
\end{aligned}$$

Therefore the Poincaré series $Q_{G,\,F,\,x_0,\,x_0}(s)$ diverges. Hence $\delta_{G,\,F} \geq s$. As $s < \delta_{\Gamma,\,F}$ was arbitrary, this proves Theorem 4.11. □

# 5 A Hopf-Tsuji-Sullivan-Roblin theorem for Gibbs states and applications

Let $\widetilde{M}, \Gamma, \widetilde{F}$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$; $\Gamma$ is a non-elementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map. The aim of this chapter is to give criteria for the ergodicity and non ergodicity of the geodesic flow on the space $T^1M = \Gamma\backslash T^1\widetilde{M}$ endowed with a Gibbs measure. We also discuss several applications, in particular to the uniqueness of a Gibbs measure on $T^1M$ as soon as there exists a finite such measure.

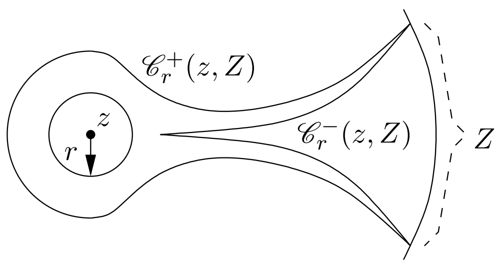
## 5.1 Some geometric notation

We introduce in this subsection the notation for several geometric objects that will be used in the proofs of the subsections 5.2 and 9.1.

Though in this Section 5, this notation is not needed for the statement of the main results 5.3 and 5.4, hence could logically be put at the beginning of their proofs, we prefer to make a separate initial subsection, precisely to facilitate the references to it in Section 9. The reader may skip to the beginning of Subsection 5.2 and come back here when needed.

For every $z \in \widetilde{M}$, every subset $Z$ of $\partial_\infty \widetilde{M}$ and every $r > 0$, let us define the *$r$-thickened* and *$r$-thinned* cones over $Z$ with vertex $z$ as

$$\mathscr{C}_r^+(z, Z) = \mathscr{N}_r\Big(\bigcup_{z' \in B(z,r)} \mathscr{C}_{z'} Z\Big) \quad\text{and}\quad \mathscr{C}_r^-(z, Z) = \mathscr{N}_r^-\Big(\bigcap_{z' \in B(z,r)} \mathscr{C}_{z'} Z\Big) .$$

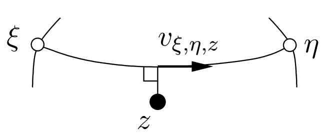



For all $r > 0$ and $z \in \widetilde{M}$, we will say that $v \in T^1\widetilde{M}$ is *$r$-close to $z$* if there exists $s \in \left]-\frac{r}{2}, \frac{r}{2}\right[$ such that $\pi(\phi_s v)$ is the closest point to $z$ on the geodesic line defined by $v$, and $d(z, \pi(\phi_s v)) < r$. For every $\gamma \in \Gamma$, the vector $\gamma v$ is $r$-close to $\gamma z$ if and only if $v$ is $r$-close to $z$.

For every subset $Z$ of $\partial_\infty \widetilde{M}$, let $K^+(z, r, Z)$ be the set of $v \in T^1\widetilde{M}$ such that $v_+ \in Z$ and $v$ is $r$-close to $z$, and $K^-(z, r, Z) = \iota K^+(z, r, Z)$. Note that $\gamma K^\pm(z, r, Z) = K^\pm(\gamma z, r, \gamma Z)$ for every $\gamma \in \Gamma$. Also denote by

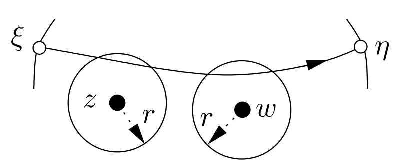
$$K(z, r) = K^+(z, r, \partial_\infty \widetilde{M}) = K^-(z, r, \partial_\infty \widetilde{M})$$

the relatively compact open set of vectors $v$ that are $r$-close to $z$, which satisfies $\gamma K(z, r) = K(\gamma z, r)$ for every isometry $\gamma$ of $\widetilde{M}$.




For distinct points $\xi$ and $\eta$ in $\partial_\infty \widetilde{M}$, let $v_{\xi,\,\eta,\,z}$ be the unit tangent vector based at the closest point to $z$ on the geodesic line between $\xi$ and $\eta$ pointing towards $\eta$. We have $\gamma v_{\xi,\,\eta,\,z} = v_{\gamma\xi,\,\gamma\eta,\,\gamma z}$ for every isometry $\gamma$ of $\widetilde{M}$.

For all $r > 0$ and $z, w \in \widetilde{M}$ such that $d(z, w) > 2r$, let us define $L_r(z, w)$ as the set of $(\xi, \eta)$ in $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$ such that the geodesic line starting from $\xi$ and ending at $\eta$ meets first $B(z, r)$ and then $B(w, r)$. We have $\gamma L_r(z, w) = L_r(\gamma z, \gamma w)$ for every isometry $\gamma$ of $\widetilde{M}$.




For all $r > 0$ and $z, w \in \widetilde{M}$, let us define

$$\mathscr{O}_r^+(z, w) = \bigcup_{z' \in B(z,r)} \mathscr{O}_{z'} B(w, r) \quad\text{and}\quad \mathscr{O}_r^-(z, w) = \bigcap_{z' \in B(z,r)} \mathscr{O}_{z'} B(w, r) .$$

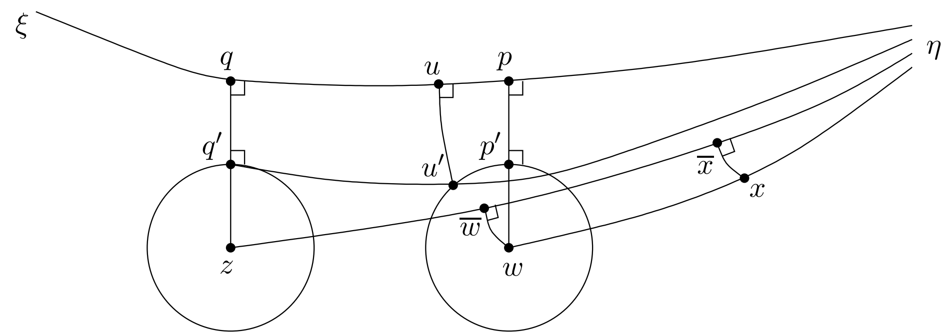
For all $r > 0$, $w \in \widetilde{M}$ and $z \in \partial_\infty \widetilde{M}$, let us define

$$\mathscr{O}_r^+(z,w) = \mathscr{O}_r^-(z,w) = \mathscr{O}_z B(w,r) \ .$$

Note that, for every fixed $w \in \widetilde{M}$, the closure of $\mathscr{O}_r^\pm(z,w)$ converges to the closure of $\mathscr{O}_r^\pm(z_0,w)$ in the space of closed subsets of $\widetilde{M} \cup \partial_\infty \widetilde{M}$, as $z$ tends to $z_0$ in $\widetilde{M} \cup \partial_\infty \widetilde{M}$, uniformly on $r$ in a bounded interval. We have $\gamma \mathscr{O}_r^\pm(z,w) = \mathscr{O}_r^\pm(\gamma z, \gamma w)$ for every isometry $\gamma$ of $\widetilde{M}$.

**Lemma 5.1** *For all $r > 0$ and $z, w \in \widetilde{M}$ such that $d(z,w) > 2r$, the set $L_r(z,w)$ contains $\mathscr{O}_r^-(w,z) \times \mathscr{O}_r^-(z,w)$.*

**Proof.** Let $(\xi,\eta) \in \mathscr{O}_r^-(w,z) \times \mathscr{O}_r^-(z,w)$ and let us prove that $(\xi,\eta) \in L_r(z,w)$.




For every $x$ on the geodesic ray $[w,\eta[$ , denote by $\overline{x}$ its closest point on the geodesic ray $[z,\eta[$ . Since $\eta \in \mathscr{O}_r^-(z,w) \subset \mathscr{O}_z B(w,r)$, the point $\overline{w}$ is at distance at most $r$ from $w$. Since closest point maps do not increase the distances, by convexity, by the triangle inequality and since $d(z,w) > 2r$, we have

$$d(z,x) \geq d(z,\overline{x}) \geq d(z,\overline{w}) \geq d(z,w) - d(w,\overline{w}) > 2r - r = r \ .$$

Hence $\eta \notin \mathscr{O}_w B(z,r) \supset \mathscr{O}_r^-(w,z)$. This implies that $\xi \neq \eta$, hence the geodesic line $]\xi,\eta[$ between $\xi$ and $\eta$ is well defined.

Let $p$ and $q$ be the closest points to $w$ and $z$ respectively on the geodesic line $]\xi,\eta[$. We need to prove that $\xi, q, p, \eta$ are in this order on $]\xi,\eta[$, and that $d(w,p), d(z,q) \leq r$.

Let $p'$ and $q'$ be the points on the geodesic segments $[w,p]$ and $[z,q]$ at distance $\min\{r, d(w,p)\}$ and $\min\{r, d(z,q)\}$ from $w$ and $z$, respectively, and let us prove that $p'$ and $q'$ are actually equal to $p$ and $q$, respectively (see the above picture, where for a contradiction we assume that neither equality holds). Note that $p'$ and $q'$ are the closest points to $p$ and $q$ on the balls $B(w,r)$ and $B(z,r)$ respectively.

Up to permuting $p$ and $q$ as well as $\xi$ and $\eta$, we may assume that $d(p,w) \geq d(q,z)$. Since $\eta \in \mathscr{O}_r^-(z,w)$, the geodesic ray $[q',\eta[$ meets $B(w,r)$ at at least one point $u'$. Let $u$ be the closest point to $u'$ on the geodesic line between $\xi$ and $\eta$. Since $d(z,w) > 2r$, we have $u' \neq q'$. Hence, by convexity,

$$d(q,q') \geq d(u,u') \geq d(p,p')$$

(the second inequality holds since $d(p',p)$ is by convexity the smallest distance between a point in the ball $B(w,r)$ and a point on the geodesic line $]\xi,\eta[$ ).

Assume for a contradiction that $q \neq q'$: the first inequality is then strict, and this contradicts the fact that $d(p,w) = r + d(p,p') \geq d(q,z) = r + d(q,q')$. Hence $q = q'$, therefore $p = p'$ and the points $z, w$ are at distance at most $r$ from the geodesic line $]\xi,\eta[$.

Since $d(q,z) \leq r$ and $\eta \in \mathscr{O}_r^-(z,w)$, the geodesic ray $[q,\eta[$ meets $B(w,r)$. Since $d(z,w) > 2r$, the point $q$ does not belong to the intersection of $B(w,r)$ and $[q,\eta[$. Hence by convexity, $p$ belongs to this intersection, and $\xi, q, p, \eta$ are in this order on the geodesic line $]\xi,\eta[$. Therefore $(\xi,\eta) \in L_r(z,w)$, as required. □

The following result that we state without proof is a simple extension of Mohsen's shadow lemma 3.10, with a similar proof (see [Rob1, p. 10]).

**Lemma 5.2** *Let $\sigma \in \mathbb{R}$, let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma$ for $(\Gamma, F)$, and let $x, y \in \widetilde{M}$. If $R$ is large enough, there exists $C > 0$ such that for every $\gamma \in \Gamma$,*

$$\frac{1}{C}\, e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)} \leq \mu_x\big(\mathscr{O}_x^- B(\gamma y, R)\big) \leq \mu_x\big(\mathscr{O}_x^+ B(\gamma y, R)\big) \leq C\, e^{\int_x^{\gamma y}(\widetilde{F}-\sigma)}\ .$$

## 5.2 The Hopf-Tsuji-Sullivan-Roblin theorem for Gibbs states

Fix $\sigma \in \mathbb{R}$. Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ be Patterson densities of the same dimension $\sigma$ for $(\Gamma, F\circ\iota)$ and $(\Gamma, F)$ respectively, and let $\widetilde{m}$ (respectively $m$) be their associated Gibbs measure on $T^1\widetilde{M}$ (respectively $T^1M$), see Subsection 3.7.

In this subsection, we give criteria for the ergodicity and non ergodicity of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M = \Gamma\backslash T^1\widetilde{M}$ endowed with Gibbs measures. These criteria generalise the classical Hopf-Tsuji-Sullivan-Roblin results for the Bowen-Margulis measures (that is, when $F = 0$). Their proofs, given at the end of this subsection, follow closely the general lines of those presented in [Rob1, Chap. 1E].

Let us recall some definitions on a dynamical system $(\Omega, G, \nu)$, where $\Omega$ is a Hausdorff locally compact $\sigma$-compact topological space, $G$ is a locally compact $\sigma$-compact topological group acting continuously on $\Omega$, and $\nu$ is a positive regular (hence $\sigma$-finite) Borel measure on $\Omega$, which is quasi-invariant under $G$.

Recall that $(\Omega, G, \nu)$ is *ergodic* if for any $G$-invariant measurable subset $A$ of $\Omega$, either $A$ or ${}^cA$ has zero measure for $\nu$.

Recall that $\mathbb{1}_A$ denotes the characteristic function of a subset $A$. A (measurably) *wandering set* of $(\Omega, G, \nu)$ is a Borel subset $E$ of $\Omega$ such that for $\nu$-almost every $\omega \in E$, the integral $\int_G \mathbb{1}_E(g\omega)\, dg$ is finite, where $dg$ is a fixed left Haar measure on $G$. The *dissipative part* of $(\Omega, G, \nu)$ (well defined up to a zero measure subset) is any $G$-invariant Borel subset which contains every wandering set up to a zero measure subset, and does not contain any non-wandering set of positive measure. Note that any countable union of wandering sets is contained in the dissipative part up to a zero measure subset. The *conservative part* of $(\Omega, G, \nu)$ is the complement of its dissipative part (and is also defined up to a zero measure subset). Recall that $(\Omega, G, \nu)$ is *conservative* (respectively *completely dissipative*) if its dissipative part (respectively conservative part) has zero measure.

The first result gives criteria for the non-ergodicity of the geodesic flow.

**Theorem 5.3** *The following assertions are equivalent.*

*(1) The Poincaré series of $(\Gamma, F)$ at the point $\sigma$ converges: $Q_{\Gamma,F}(\sigma) < +\infty$.*

*(2) There exists $x \in \widetilde{M}$ such that $\mu_x(\Lambda_c\Gamma) = 0$.*

*(3) There exists $x \in \widetilde{M}$ such that $\mu^\iota_x(\Lambda_c\Gamma) = 0$.*

*(4) There exists $x \in \widetilde{M}$ such that the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is non-ergodic and completely dissipative.*

*(5) The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is non-ergodic and completely dissipative.*

We may replace Assertion (2) by

*(2') For every $x \in \widetilde{M}$, we have $\mu_x(\Lambda_c\Gamma) = 0$.*

and Assertion (3) by

*(3') For every $x \in \widetilde{M}$, we have $\mu^\iota_x(\Lambda_c\Gamma) = 0$.*

since two elements in a Patterson density are pairwise absolutely continuous. By equivalence of measures, we may also replace "There exists $x \in \widetilde{M}$ such that" by "For every $x \in \widetilde{M}$," in Assertion (4).

**Remark.** This result indicates that the case when $Q_{\Gamma,\,F}(\sigma) < +\infty$ (and in particular the case when $(\Gamma, F)$ is not of divergence type and $\sigma = \delta_{\Gamma,\,F}$) is not interesting from the dynamical system viewpoint. It is also not the one that occurs the most frequently, as we will see later on. Furthermore, when $Q_{\Gamma,\,F}(\sigma) < +\infty$, uninteresting Patterson densities of dimension $\sigma$ for $(\Gamma, F)$ exist if $\Lambda\Gamma \neq \partial_\infty\widetilde{M}$, as follows. Fix $x_0$ in $\widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M} - \Lambda\Gamma$. Denote by $\mathscr{D}_\eta$ the (unit) Dirac mass at a point $\eta \in \partial_\infty\widetilde{M}$. Then for every $x \in X$, the series

$$\nu^\xi_x = \sum_{\gamma\in\Gamma} e^{-C_{F-\sigma,\,\gamma\xi}(x,\gamma x_0)}\ \mathscr{D}_{\gamma\xi}$$

converges. Indeed, its convergence is independent of $x$ and $x_0$ by the cocycle property of $C_{F-\sigma}$ (Equation (25)) and by Lemma 3.4 (1). Hence, for every fixed $r > 0$, we may assume that $x \in \mathscr{C}\Lambda\Gamma$ and that $x_0$ is close enough to $\xi$ so that $\xi \in \mathscr{O}_{\gamma^{-1}x}B(x_0, r)$ for every $\gamma \in \Gamma$. Then by the invariance property of $C_{F-\sigma}$ (Equation (26)) and by Lemma 3.4 (2), there exists $c > 0$ such that for every $\gamma \in \Gamma$, we have

$$-C_{F-\sigma,\,\gamma\xi}(x, \gamma x_0) = -C_{F-\sigma,\,\xi}(\gamma^{-1}x, x_0) \leq c + \int_{\gamma^{-1}x}^{x_0} (\widetilde{F} - \sigma) = c + \int_{x}^{\gamma x_0} (\widetilde{F} - \sigma)\,.$$

Hence the convergence (in norm) of the series $\nu^\xi_x$ follows from the convergence of the series $Q_{\Gamma,\,F,\,x,\,x_0}(\sigma)$. It is easy to check that $(\nu^\xi_x)_{x\in\widetilde{M}}$ is a Patterson density of dimension $\sigma$ for $(\Gamma, F)$. Taking linear combinations of them as $\xi$ varies and weak-star limits gives other examples. The measures $\nu^\xi_x$ are atomic and ergodic (but not properly ergodic, indeed of type $I_\infty$ in the Hopf-Krieger classification, see for instance [FM]: they give full measure to an infinite orbit of $\Gamma$).

The next result gives criteria for the ergodicity of the geodesic flow.

**Theorem 5.4** *The following assertions are equivalent.*

*(i) The Poincaré series of $(\Gamma, F)$ at the point $\sigma$ diverges: $Q_{\Gamma,\,F}(\sigma) = +\infty$.*

*(ii) There exists $x \in \widetilde{M}$ such that $\mu_x({}^c\Lambda_c\Gamma) = 0$.*

*(iii) There exists $x \in \widetilde{M}$ such that $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$.*

*(iv) There exists $x \in \widetilde{M}$ such that the dynamical system $(\partial^2_\infty \widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty \widetilde{M}})$ is ergodic and conservative.*

*(v) The dynamical system $(T^1 M, (\phi_t)_{t\in\mathbb{R}}, m)$ is ergodic and conservative.*

We may also replace Assertion (ii) by

*(ii') For every $x \in \widetilde{M}$, we have $\mu_x({}^c\Lambda_c\Gamma) = 0$.*

and Assertion (iii) by

*(iii') For every $x \in \widetilde{M}$, we have $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$.*

By equivalence of measures, we may also replace "There exists $x \in \widetilde{M}$ such that" by "For every $x \in \widetilde{M}$," in Assertion (iv).

The proof of these two theorems will be based on the following list of implications, as in the case $F = 0$ in [Rob1, Chap. 1E]. Let us fix $x \in \widetilde{M}$.

**Proposition 5.5** *(a) If $Q_{\Gamma,\,F}(\sigma) < +\infty$ then $\mu_x(\Lambda_c\Gamma) = 0$ and $\mu^\iota_x(\Lambda_c\Gamma) = 0$.*

*(b) If $\mu_x(\Lambda_c\Gamma) = 0$ and $\mu^\iota_x(\Lambda_c\Gamma) = 0$, then $(\partial^2_\infty \widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty \widetilde{M}})$ is completely dissipative.*

*(c) If $(\partial^2_\infty \widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty \widetilde{M}})$ is ergodic, then $\mu^\iota_x \otimes \mu_x$ is atomless, the diagonal of $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$ has zero measure for $\mu^\iota_x \otimes \mu_x$, and $(\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}, \Gamma, \mu^\iota_x \otimes \mu_x)$ is ergodic and conservative.*

*(d) For all $v \in T^1\widetilde{M}$, we have $v_+ \in \Lambda_c\Gamma$ if and only if there exists a relatively compact subset $K$ in $T^1\widetilde{M}$ such that $\int_0^\infty \mathbb{1}_{\Gamma K} \circ \phi_t(v)dt$ diverges. Therefore $(T^1 M, (\phi_t)_{t\in\mathbb{R}}, m)$ is conservative (respectively completely dissipative) if and only if $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$ and $\mu_x({}^c\Lambda_c\Gamma) = 0$ (respectively $\mu^\iota_x(\Lambda_c\Gamma) = 0$ and $\mu_x(\Lambda_c\Gamma) = 0$).*

*(e) The dynamical system $(T^1 M, (\phi_t)_{t\in\mathbb{R}}, m)$ is ergodic if and only if the dynamical system $(\partial^2_\infty \widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty \widetilde{M}})$ is ergodic.*

*(f) If $Q_{\Gamma,\,F}(\sigma) = +\infty$ then $\mu_x(\Lambda_c\Gamma) > 0$ and $\mu^\iota_x(\Lambda_c\Gamma) > 0$.*

*(g) If $\mu_x(\Lambda_c\Gamma) > 0$ then $\mu_x({}^c\Lambda_c\Gamma) = 0$.*

*(h) If $(T^1 M, (\phi_t)_{t\in\mathbb{R}}, m)$ is conservative then it is ergodic.*

**Proof.** (a) Mohsen's shadow lemma 3.10 applied to $(\mu_x)_{x\in\widetilde{M}}$ with $K = \{x\}$ shows the existence of $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, there exists $c_N > 0$ such that

$$\mu_x(\mathscr{O}_x(B(\gamma x, N))) \leq C_N \; e^{\int_x^{\gamma x} (\widetilde{F} - \sigma)} \,.$$

By the definition of $\Lambda_c\Gamma$, we have $\Lambda_c\Gamma = \bigcup_{N\in\mathbb{N},\; N\geq N_0} \Lambda(N)$ where

$$\Lambda(N) = \bigcap_{A \text{ finite subset of } \Gamma} \; \bigcup_{\gamma\in\Gamma - A} \mathscr{O}_x(B(\gamma x, N)) \,.$$

The fact that $\mu_x(\Lambda(N)) = 0$ for every $N \geq N_0$, which implies that $\mu_x(\Lambda_c\Gamma) = 0$, then follows by using the easy part of the Borel-Cantelli lemma.

Since $Q_{\Gamma,\,F}(\sigma) < +\infty$ if and only if $Q_{\Gamma,\,F\circ\iota}(\sigma) < +\infty$ (see Lemma 3.3 (ii)), the last claim follows similarly.

(b) By Fubini's theorem, the set $A = \{(\xi, \eta) \in \partial^2_\infty \widetilde{M} \;:\; \xi, \eta \notin \Lambda_c\Gamma\}$ has full measure with respect to $(\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty \widetilde{M}}$. For every $(\xi, \eta) \in A$, let $\Gamma_{\xi,\,\eta}$ be the finite nonempty set of $\alpha \in \Gamma$ such that the distance between $\alpha x$ and the geodesic line between $\xi$ and $\eta$ is minimal; it satisfies $\gamma\Gamma_{\xi,\,\eta} = \Gamma_{\gamma\xi,\,\gamma\eta}$ for every $\gamma \in \Gamma$. For any finite subset $F$ of $\Gamma$, we denote $S_F = \{(\xi, \eta) \in A \;:\; \Gamma_{\xi,\,\eta} = F\}$. We claim that the measurable subset $S_F$ is wandering, since if $\gamma S_F$ meets $S_F$ then $\gamma F = F$ and since the stabiliser (for the action by left translations of $\Gamma$ on itself) of a finite subset of $\Gamma$ is finite. As a countable union of wandering sets is contained in the dissipative part up to a zero measure subset, and since $A = \bigcup_F S_F$ has full measure, the result follows.

(c) Assume for a contradiction that $(\xi', \xi) \in \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$ is an atom for $\mu^\iota_x \otimes \mu_x$, that is, $\xi'$ is an atom for $\mu^\iota_x$ and $\xi$ is an atom for $\mu_x$. This implies that $\gamma\xi$ is an atom for $\mu_x$ for every $\gamma \in \Gamma$ by quasi-invariance. Let $\Gamma_{\xi'}$ be the stabiliser of $\xi'$ in $\Gamma$, and note that its limit set $\Lambda\Gamma_{\xi'}$ is finite (it has 0, 1 or 2 points). Since $\Gamma$ is non-elementary, there exists $\gamma_0 \in \Gamma$ such that $\gamma_0\xi \neq \xi'$. By ergodicity, the $\Gamma$-orbit $\mathscr{O}$ of $(\xi', \gamma_0\xi)$ in $\partial^2_\infty \widetilde{M}$ has full measure with respect to $(\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty \widetilde{M}}$. Since $\Gamma$ acts minimally in $\Lambda\Gamma$, there exists $\gamma_1 \in \Gamma$ such that $\gamma_1\xi$ does not lie in the countable closed set $\Gamma_{\xi'}(\gamma_0\xi) \cup \Lambda\Gamma_{\xi'} \cup \{\xi'\}$, whose complement in $\Lambda\Gamma$ is open and nonempty since $\Lambda\Gamma$ is uncountable. Note that $(\xi', \gamma_1\xi)$ does not belong to $\mathscr{O}$, since otherwise there exists $\gamma \in \Gamma$ such that $\xi' = \gamma\xi'$ and $\gamma_1\xi = \gamma\gamma_0\xi$, contradicting that $\gamma_1\xi$ does not belong to $\Gamma_{\xi'}(\gamma_0\xi)$. But $(\xi', \gamma_1\xi)$ is an atom of $(\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty \widetilde{M}}$, contradicting the fact that $\mathscr{O}$ has full measure with respect to this measure.

Since the product measure $\mu^\iota_x \otimes \mu_x$ is atomless, the diagonal of $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$ has measure zero for the product measure (by Fubini's theorem, since a finite measure has at most countably many atoms). Hence the dynamical system $(\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}, \Gamma, \mu^\iota_x \otimes \mu_x)$ is also ergodic. Since $\Gamma$ is countable and since $\mu^\iota_x \otimes \mu_x$ is atomless, this ergodic dynamical system is conservative (see for instance [Aar, page 51, Prop. 1.6.6]).

(d) The first claim that the conical limit points are the endpoints of the geodesic rays whose image in $M = \Gamma\backslash\widetilde{M}$ returns infinitely often in some relatively compact subset is immediate from the definition of $\Lambda_c\Gamma$.

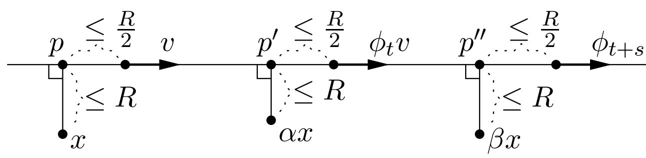
Note that the dynamical system $\mathscr{D} = (T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is conservative if and only if every relatively compact subset $K$ of $T^1M$ is contained in the conservative part of $\mathscr{D}$ up to $m$-measure 0. By the Halmos recurrence theorem (see for instance [Aar, page 15, Theo. 1.1.1] in the case of transformations), if $\mathscr{D}$ is conservative, $m$-almost every element of any relatively compact subset $K$ returns infinitely often (positively and negatively) in $K$. Hence for $\widetilde{m}$-almost every $v \in T^1\widetilde{M}$, we have $v_-, v_+ \in \Lambda_c\Gamma$. This holds if and only if $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$ and $\mu_x({}^c\Lambda_c\Gamma) = 0$, since $d\widetilde{m}(v)$ is a quasi-product measure of $d\mu^\iota_x(v_-)$, $d\mu_x(v_+)$ and $dt$. Conversely, if these two equalities hold, then by this quasi-product property, for $m$-almost every $v \in T^1M$, for every big enough relatively compact subset $K$ of $T^1M$, we have $\int_{\mathbb{R}} \mathbb{1}_K(\phi_t v)\, dt = +\infty$. Hence any (measurably) wandering set has $m$-measure 0, and $\mathscr{D}$ is conservative.

(e) This is immediate by the construction of $m$.

(f) Assume that $Q_{\Gamma,\,F}(\sigma) = +\infty$. Let $R > 0$ be large enough so that Lemma 5.2 (with $y = x$) holds. With $a \geq 0$ the maximum mass of an atom of $\mu^\iota_x$, also assume that $R$ is large enough so that for every $\gamma \in \Gamma$, we have $\mu^\iota_x(\mathscr{O}^-_R(\gamma x, x)) \geq c_7 = \frac{\|\mu^\iota_x\|-a}{2}$, with $\mathscr{O}^-_R(\gamma x, x)$ defined in Subsection 5.1. Let $K$ be the compact subset $K(x, R)$ of $T^1\widetilde{M}$ defined in Subsection 5.1. The proof of Assertion (f) relies on the following two lemmas.

**Lemma 5.6** *There exists $c_8 > 0$ such that for every sufficiently large $T > 0$*

$$\int_0^T \int_0^T \Big( \sum_{\alpha,\,\beta\in\Gamma} \widetilde{m}(K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K)\Big)\, ds\, dt \leq c_8 \Big( \sum_{\gamma\in\Gamma,\, d(x,\gamma x)\leq T} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)} \Big)^2 .$$




**Proof.** Let $s, t \geq 0$ and $\alpha, \beta \in \Gamma$ with $d(x, \alpha x), (\alpha x, \beta x), d(x, \beta x) > 2R$. By the definition of $K = K(x, R)$ in Subsection 5.1, by the triangle inequality (and since closest point projections on geodesic lines do not increase the distance), if $v \in K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K$, if $p, p', p''$ are the closest points to respectively $x, \alpha x, \beta x$ on the geodesic line defined by $v$, then $v_-, p, p', p'', v_+$ are in this order on this geodesic line (see the above picture), and

$$d(x, \alpha x) + d(\alpha x, \beta x) \leq d(x, \beta x) + 4R .$$

By the definition of $L_R(x, \gamma x)$ in Subsection 5.1, we also have $(v_-, v_+) \in L_R(x, \gamma x)$. Furthermore,

$$d(x, \alpha x) - 3R \leq t \leq d(x, \alpha x) + R \quad \text{and} \quad d(\alpha x, \beta x) - 3R \leq s \leq d(\alpha x, \beta x) + R .$$

Since $d(\alpha x, [x, \beta x]) \leq 2R$, we also observe by Lemma 3.2 that there exists a constant $c' \geq 0$ (depending only on $R$, on the Hölder constants of $\widetilde{F}$, on the bounds on the sectional curvature and on $\max_{\pi^{-1}(B(x,\,2R))} |\widetilde{F} - \sigma|$) such that

$$\Big| \int_x^{\beta x} (\widetilde{F} - \sigma) - \int_x^{\alpha x} (\widetilde{F} - \sigma) - \int_{\alpha x}^{\beta x} (\widetilde{F} - \sigma) \Big| \leq c' .$$

By the definitions of $m$ in Subsection 3.7, since the time parameter of any unit tangent vector $v \in K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K$ for Hopf's parametrisation with respect to the base point $x$ varies at most in an interval of length $R$, we have

$$\widetilde{m}(K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K) \leq R \int_{(\xi,\,\eta)\in L_R(x,\,\beta x)} \frac{d\mu^\iota_x(\xi)\; d\mu_x(\eta)}{D_{F-\sigma,\,x}(\xi, \eta)^2} .$$

Let $(\xi, \eta) \in L_R(x, \beta x)$ and $p$ be the closest point to $x$ on the geodesic line between $\xi$ and $\eta$. By Equation (29) saying that

$$D_{F-\sigma,\,x}(\xi, \eta) = e^{-\frac{1}{2}(C_{F-\sigma,\,\eta}(x,p) + C_{F\circ\iota-\sigma,\,\xi}(x,p))} ,$$

by Lemma 3.4 (1) implying that

$$|\,C_{F-\sigma,\,\eta}(x,p)\,| \le c_1 e^R + R \sup_{\pi^{-1}(B(x,\,R))} |\widetilde{F}-\sigma|$$

and similarly by replacing $F$ by $F\circ\iota$ and $\eta$ by $\xi$, there exists a constant $c>0$ such that

$$\frac{1}{c} \le D_{F-\sigma,\,x}(\xi,\eta) \le c\;. \tag{82}$$

Since $L_R(x,\beta x)\subset \partial_\infty\widetilde{M}\times \mathscr{O}^+_R(x,\beta x)$, and by Lemma 5.2, we hence have,

$$\widetilde{m}(K\cap\phi_{-t}\alpha K\cap\phi_{-s-t}\beta K) \le c^2\; C\, R\; \|\mu^\iota_x\|\; e^{\int_x^{\beta x}(\widetilde{F}-\sigma)}\;.$$

Therefore, since $m(K\cap\phi_{-t}\alpha K\cap\phi_{-s-t}\beta K)=0$ outside intervals of $s$ and $t$ of lengths $4R$, we have, up to increasing the constant $C$ in order to take into account the contribution of the elements $\alpha,\beta$ in $\Gamma$ with $d(x,\alpha x)$, $d(\alpha x,\beta x)$ or $d(x,\beta x)$ at most $2R$,

$$\begin{aligned}
&\int_0^T\int_0^T\Big(\sum_{\alpha,\,\beta\in\Gamma}\widetilde{m}(K\cap\phi_{-t}\alpha K\cap\phi_{-s-t}\beta K)\Big)\,ds\;dt\\
&\le (4R)^2 \sum_{\substack{\alpha\,\in\,\Gamma,\;d(x,\,\alpha x)\,\le\, T+3R\\ \beta\,\in\,\Gamma,\;d(\alpha x,\,\beta x)\,\le\, T+3R}} c^2\;C\,R\;\|\mu^\iota_x\|\;e^{c'}\;e^{\int_x^{\alpha x}(\widetilde{F}-\sigma)+\int_{\alpha x}^{\beta x}(\widetilde{F}-\sigma)}\\
&\le C_3\Big(\sum_{\gamma\in\Gamma,\;d(x,\gamma x)\le T+3R} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}\Big)^2\,,
\end{aligned}$$

for some constant $C_3>0$. By Corollary 3.11 (1), the sum

$$\sum_{\gamma\in\Gamma,\;T<d(x,\gamma x)\le T+3R} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$$

is uniformly bounded as $T$ tends to $+\infty$. Since the Poincaré series of $(\Gamma,F)$ diverges at the point $\sigma$, the result follows. □

**Lemma 5.7** *There exists $c_9>0$ such that for every sufficiently large $T>0$*

$$\int_0^T\Big(\sum_{\gamma\in\Gamma}\widetilde{m}(K\cap\phi_{-t}\gamma K)\Big)\,dt \ge c_9 \sum_{\gamma\in\Gamma,\,d(x,\gamma x)\le T} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}\;.$$

**Proof.** The proof is similar to the previous one. Let $\gamma\in\Gamma$ be such that $3R\le d(x,\gamma x)\le T-3R$. Then by the definition of $K=K(x,R)$, we have (see Subsection 5.1 for the definition of $v_{\xi,\,\eta,\,x}$)

$$\begin{aligned}
\int_0^T\widetilde{m}(K\cap\phi_{-t}\gamma K)\;dt &= \int_{(\xi,\,\eta)\in L_R(x,\,\gamma x)}\frac{d\mu^\iota_x(\xi)\;d\mu_x(\eta)}{D_{F-\sigma,\,x}(\xi,\eta)^2}\int_{-\frac{R}{2}}^{\frac{R}{2}}\int_0^T \mathbb{1}_{\gamma K}(\phi_{s+t}v_{\xi,\,\eta,\,x})\;dt\;ds\\
&= R^2\int_{(\xi,\eta)\in L_R(x,\gamma x)}\frac{d\mu^\iota_x(\xi)\;d\mu_x(\eta)}{D_{F-\sigma,\,x}(\xi,\eta)^2}\\
&\ge \frac{1}{c^2}\;R^2\;\mu^\iota_x(\mathscr{O}^-_R(\gamma x,x))\;\mu_x(\mathscr{O}^-_R(x,\gamma x))\;,
\end{aligned}$$

using the fact that $L_R(x,\gamma x)$ contains $\mathscr{O}_R^-(\gamma x,x)\times\mathscr{O}_R^-(x,\gamma x)$ (see Lemma 5.1) and the upper bound in Equation (82). By the assumption on $R$ at the beginning of the proof of Assertion (f), and by the lower bound in Lemma 5.2, we have

$$\int_0^T\sum_{\gamma\in\Gamma}\widetilde{m}(K\cap\phi_{-t}\gamma K)\;dt\geq\sum_{\gamma\in\Gamma,\,3R\leq d(x,\,\gamma x)\leq T-3R}\frac{1}{c^2}\;R^2\;c_7\;\frac{1}{C}\;e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}\;.$$

We conclude as in the end of the proof of the previous Lemma 5.6. □

Now, to prove Assertion (f) from these two lemmas, let $B$ be the image of $K$ in $T^1M=\Gamma\backslash T^1\widetilde{M}$, let $A_t=B\cap\phi_{-t}B$, and let $\nu$ be the restriction of the measure $m$ to the relatively compact set $B$. Then $\int_0^\infty\nu(A_t)\;dt=+\infty$ by Lemma 5.7 and the divergence of the Poincaré series of $(\Gamma,F)$ at the point $\sigma$. Furthermore, using simultaneously Lemma 5.6 and Lemma 5.7,

$$\int_0^T\int_0^T\nu(A_t\cap A_s)\;ds\;dt\leq 2\int_0^T\int_0^T\nu(A_t\cap A_{s+t})\;ds\;dt\leq 2\frac{c_8}{{c_9}^2}\Big(\int_0^T\nu(A_t)\;dt\Big)^2.$$

By the appropriate version of the Borel-Cantelli Lemma (see for instance [Rob1, page 20]), we deduce that the set $\{v\in B\;:\;\int_0^\infty\mathbb{1}_{A_t}(v)\;dt=+\infty\}$ has positive measure with respect to $\nu$. By Assertion (d) and the fact that $\widetilde{m}$ is a quasi-product measure, this implies that $\mu_x(\Lambda_c\Gamma)>0$. Since $Q_{\Gamma,\,F}(\sigma)$ diverges if and only if $Q_{\Gamma,\,F\circ\iota}(\sigma)$ diverges, we also have $\mu_x^\iota(\Lambda_c\Gamma)>0$.

(g) Observe first that, thanks to Proposition 3.12 (1), the measure $\mu_x$ has the following "doubling property of shadows": for every $R>0$ large enough, there exists $C=C(x,R)>0$ such that for every $\gamma\in\Gamma$, we have

$$\mu_x(\mathscr{O}_xB(\gamma x,5R))\leq C\,\mu_x(\mathscr{O}_xB(\gamma x,R))\;.$$

Therefore, we deduce as in [Rob1, page 22-23] the following lemma. For every $R>0$, let $\Lambda(R)$ be the set of $\xi\in\Lambda_c\Gamma$ that are limits of points in $\Gamma x$ at distance at most $R$ of the geodesic ray starting from $x$ and ending at $\xi$.

**Lemma 5.8** *If $R>0$ is large enough, for every $\mu_x$-integrable map $\varphi:\partial_\infty\widetilde{M}\to\mathbb{R}$, for $\mu_x$-almost every $\xi\in\Lambda(R)$, when $d(x,\gamma x)\to+\infty$ and $\xi\in\mathscr{O}_xB(\gamma x,R)$, we have*

$$\frac{1}{\mu_x(\mathscr{O}_xB(\gamma x,R))}\int_{\mathscr{O}_xB(\gamma x,R)}\varphi\,d\mu_x\to\varphi(\xi)\;.$$

Since $\Lambda_c\Gamma=\bigcup_{n\in\mathbb{N}}\Lambda(n)$ and by the assumption of assertion (g), choose $R>0$ large enough so that $\mu_x(\Lambda(R))>0$, and Lemma 3.10 with $K=\{x\}$ and Lemma 5.8 are satisfied. Let $B=\partial_\infty\widetilde{M}-\Lambda_c\Gamma$, let $\varphi=\mathbb{1}_B$ be the characteristic function of $B$ and let $\xi\in\Lambda(R)$. Then $\varphi(\xi)=0$ and there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\Gamma$ such that $d(x,\gamma_nx)\to+\infty$ and $\xi\in\mathscr{O}_xB(\gamma_nx,R)$. By Lemma 5.8 (taking $\xi$ for which it applies), we have

$$\lim_{n\to+\infty}\frac{\mu_x(B\cap\mathscr{O}_xB(\gamma_nx,R))}{\mu_x(\mathscr{O}_xB(\gamma_nx,R))}=0\;.$$

Using Equation (39) and Lemma 3.4 (2) to estimate the numerator, and Lemma 3.10 to estimate the denominator, there exists a constant $c>0$ (depending on $x,R,F$) such that

$$\frac{1}{c}\;\mu_{\gamma_nx}(B\cap\mathscr{O}_xB(\gamma_nx,R))\leq\frac{\mu_x(B\cap\mathscr{O}_xB(\gamma_nx,R))}{\mu_x(\mathscr{O}_xB(\gamma_nx,R))}\leq c\;\mu_{\gamma_nx}(B\cap\mathscr{O}_xB(\gamma_nx,R))\;.\quad(83)$$

By Equation (38) and the invariance of $B$ under $\Gamma$, we hence have

$$\lim_{n\to+\infty} \mu_x(B\cap \mathscr{O}_{\gamma_n^{-1}x}B(x,R)) = 0\,.$$

Choosing arbitrarily large values of $R$, we may construct an increasing sequence $(U_k)_{k\in\mathbb{N}}$ of nonempty open subsets of $\partial_\infty X$, with $\lim_{k\to+\infty}\mu_x(B\cap U_k)=0$, such that the diameter of the complement of $\partial_\infty X - U_k$ tends to 0. Up to extracting a subsequence, the closed subset $\partial_\infty X - U_k$ converges for the Hausdorff distance to a singleton $\{\zeta\}$. We have $\mu_x(B-\{\zeta\})=0$. Since $\Gamma$ is non-elementary, there exists $\gamma\in\Gamma$ such that $\gamma\zeta\neq\zeta$. By quasi-invariance of the measure $\mu_x$ and as $\Gamma$ preserves $B$, we hence have $\mu_x(B)=0$, as required.

(h) As in [Sul1, Rob1], the proof relies on well-known arguments of Hopf [Hop1]. The first part of it (before Lemma 5.9) could essentially be replaced by a reference to the nice paper [Cou3].

Fix a positive Lipschitz map $\rho: T^1M\to\mathbb{R}$ with $\|\rho\|_{\mathbb{L}^1(m)}=1$. A standard way of constructing such a $\rho$ is as follows. Let $d$ be the quotient distance on $T^1M$ of the Riemannian distance on $T^1\widetilde{M}$, whose closed balls are compact. Fix $v_0\in T^1M$ in the support of $m$. Let $f:[0,+\infty[\;\to\;]0,+\infty[$ be the (positive, continuous, non-increasing) map, affine on each $[n,n+1]$ (hence Lipschitz since the slopes are bounded), such that $f(n)=\frac{1}{2^n m(B(v_0,n+1))}$. Note that $v\mapsto f(d(v_0,v))$ is integrable, since if $A_n=\{v\in T^1M\;:\; n\le d(v_0,v)<n+1\}$, then

$$\int_{T^1M} f(d(v_0,v))\;dm(v)=\sum_{n=0}^{\infty}\int_{A_n} f(d(v_0,v))\;dm(v)\le\sum_{n=0}^{\infty}\frac{m(A_n)}{2^n\,m(B(v_0,n+1))}\le 2\,.$$

Take for $\rho$ the integrable map $v\mapsto f(d(v_0,v))$ renormalised to have $\mathbb{L}^1$-norm 1. Note that $\rho$ is Lipschitz, as the composition of two Lipschitz maps.

As $(T^1M,(\phi_t)_{t\in\mathbb{R}},m)$ is conservative (by the assumption of (h)), for $m$-almost every $v\in T^1M$, we have

$$\int_0^{+\infty}\rho\circ\phi_{-t}(v)dt=\int_0^{+\infty}\rho\circ\phi_t(v)dt=+\infty$$

(see for instance [Aar, Prop. 1.1.6, page 18] in the non invertible case). Let $\widetilde{\rho}\colon T^1\widetilde{M}\to\mathbb{R}$ be the lift of $\rho$ by the canonical projection $T^1\widetilde{M}\to\Gamma\backslash T^1\widetilde{M}=T^1M$.

For every Lipschitz map $h\colon T^1M\to\mathbb{R}$ with compact support, let $\widetilde{h}\colon T^1\widetilde{M}\to\mathbb{R}$ be its lift, and consider the following partially defined maps:

$$\widetilde{h}^*\colon v\mapsto\lim_{T\to+\infty}\frac{\int_0^T\widetilde{h}\circ\phi_t(v)\,dt}{\int_0^T\widetilde{\rho}\circ\phi_t(v)\,dt}\quad\text{and}\quad\widetilde{h}_*\colon v\mapsto\lim_{T\to+\infty}\frac{\int_0^T\widetilde{h}\circ\phi_{-t}(v)\,dt}{\int_0^T\widetilde{\rho}\circ\phi_{-t}(v)\,dt}\,.$$

We denote respectively by $\widetilde{D}^*$ and $\widetilde{D}_*$ the intersection of their domains of definition with the subsets (of full measure for $\widetilde{m}$) of $v\in T^1\widetilde{M}$ such that $\int_0^{+\infty}\widetilde{\rho}\circ\phi_t(v)\,dt=+\infty$ and $\int_0^{+\infty}\widetilde{\rho}\circ\phi_{-t}(v)\,dt=+\infty$. Note that $\widetilde{D}^*$, $\widetilde{D}_*$, $\widetilde{h}^*$ and $\widetilde{h}_*$ are invariant under $\Gamma$ and by the geodesic flow. Let $h^*$ and $h_*$ be the maps from the images of $\widetilde{D}^*$ and $\widetilde{D}_*$ in $T^1M$ to $\mathbb{R}$ induced by $\widetilde{h}^*$ and $\widetilde{h}_*$ respectively. By the Lipschitz continuity of $h$ and $\rho$, and by the fact that the distance $d(\phi_t v,\phi_t w)$ tends exponentially to 0 if $w\in W^{\rm ss}(v)$ as $t\to+\infty$ (respectively $w\in W^{\rm su}(v)$ as $t\to-\infty$), the domains $\widetilde{D}^*$ and $\widetilde{D}_*$ are saturated respectively by the strong stable and strong unstable leaves of the geodesic flow, and furthermore $\widetilde{h}^*(v)$ depends only on $v_+$ and $\widetilde{h}_*(v)$ on $v_-$.

Let $\mathscr{I}$ be the $\sigma$-algebra of $(\phi_t)_{t\in\mathbb{R}}$-invariant Borel subsets of $T^1M$, let $\mathbb{P}$ be the probability measure $\rho m$ and let $E(\varphi\,|\,\mathscr{I})$ be the conditional expectation of $\varphi\in\mathbb{L}^1(\mathbb{P})$ with respect to $\mathscr{I}$: the map $E(\varphi\,|\,\mathscr{I})$ is $\mathbb{P}$-integrable and almost everywhere invariant under $(\phi_t)_{t\in\mathbb{R}}$ and for every bounded measurable map $\psi:T^1M\to\mathbb{R}$ which is almost everywhere invariant under $(\phi_t)_{t\in\mathbb{R}}$, we have

$$\int_{T^1M}\varphi\,\psi\;d\mathbb{P}=\int_{T^1M}E(\varphi\,|\,\mathscr{I})\,\psi\;d\mathbb{P}\;. \tag{84}$$

As $(T^1M,(\phi_t)_{t\in\mathbb{R}},m)$ is conservative (by the assumption of (h)) and invertible, Hopf's ratio ergodic theorem (see for instance [Hop1], as well as [Zwe] for a short proof) implies that $\widetilde{D}^*$ and $\widetilde{D}_*$ have full measure with respect to $\widetilde{m}$, and that the maps $h^*$, $h_*$ and $E_{\rho m}(\frac{h}{\rho}\,|\,\mathscr{I})$ are defined and coincide $m$-almost everywhere. Let us denote by $\psi^*$ and $\psi_*$ two bounded measurable maps on $\partial_\infty\widetilde{M}$ such that $\widetilde{h}^*(v)=\psi^*(v_+)$ and $\widetilde{h}_*(v)=\psi^*(v_-)$ for $\widetilde{m}$-almost every $v\in T^1\widetilde{M}$.

**Lemma 5.9** *If $(T^1M,(\phi_t)_{t\in\mathbb{R}},m)$ is conservative, then the product measure $\mu^\iota_x\otimes\mu_x$ is atomless.*

**Proof.** Assume for a contradiction that $(\xi',\xi)$ is an atom of $\mu^\iota_x\otimes\mu_x$. Since $\Gamma$ is non-elementary, there exists $\gamma\in\Gamma$ such that $\gamma\xi'$ is different from $\xi$ and from the other fixed point of any loxodromic element of $\Gamma$ (if there is one) fixing $\xi$. Then by the quasi-invariance of $\mu^\iota_x$, the pair $(\gamma\xi',\xi)$ is another atom of $\mu^\iota_x\otimes\mu_x$. Let $\epsilon>0$ be small enough. Let $v\in T^1\widetilde{M}$ be such that $v_-=\gamma\xi'$ and $v_+=\xi$. Then the image of $\{\phi_t v\;:\;t\in[-\epsilon,\epsilon]\}$ in $T^1M$ is a measurable subset of positive measure with respect to $m$ which is a wandering set for the geodesic flow, since $\{\gamma\xi',\xi\}$ is not the set of fixed points of a loxodromic element. $\Box$

It follows from this lemma, as in the proof of Assertion (c), that the diagonal of $\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}$ has measure 0 for the product measure $\mu^\iota_x\otimes\mu_x$. Consider the probability measures $\overline{\mu}^\iota=\frac{\mu^\iota_x}{\|\mu^\iota_x\|}$ and $\overline{\mu}=\frac{\mu_x}{\|\mu_x\|}$. Since $\widetilde{m}$ is equivalent to the product measure $d\mu^\iota_x\,d\mu_x\,dt$ on $\partial^2_\infty\widetilde{M}\times\mathbb{R}$, we have, using Fubini's theorem, the Hopf parametrisation of Remark (2) of Subsection 3.7 (with $x_0=x$), and the fact that $h_*$ and $h^*$ coincide almost everywhere for the measure $d\overline{\mu}^\iota(v_-)\,d\overline{\mu}(v_+)\,dt$,

$$
\begin{aligned}
\int_{\partial_\infty\widetilde{M}} \psi^*(\eta)^2\,d\overline{\mu}(\eta) &= \int_0^1\int_{\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}} \psi^*(\eta)^2\,d\overline{\mu}^\iota(\xi)\,d\overline{\mu}(\eta)\,dt\\
&= \int_{\partial^2_\infty\widetilde{M}\times\mathbb{R}} \mathbb{1}_{[0,1]}(t)\ \psi^*(\eta)^2\,d\overline{\mu}^\iota(\xi)\,d\overline{\mu}(\eta)\,dt\\
&= \int_{T^1\widetilde{M}} \mathbb{1}_{[0,1]}(\beta_{v_-}(\pi(v),x))\ h^*(v)^2\,d\overline{\mu}^\iota(v_-)\,d\overline{\mu}(v_+)\,dt\\
&= \int_{T^1\widetilde{M}} \mathbb{1}_{[0,1]}(\beta_{v_-}(\pi(v),x))\ h_*(v)h^*(v)\,d\overline{\mu}^\iota(v_-)\,d\overline{\mu}(v_+)\,dt\\
&= \int_0^1\int_{\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}} \psi_*(\xi)\psi^*(\eta)\,d\overline{\mu}^\iota(\xi)\,d\overline{\mu}(\eta)\,dt\\
&= \int_{\partial_\infty\widetilde{M}} \psi_*(\xi)\,d\overline{\mu}^\iota(\xi)\int_{\partial_\infty\widetilde{M}}\psi^*(\eta)\,d\overline{\mu}(\eta)\\
&= \int_{\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}\times[0,1]} \psi_*(\xi)\,d\overline{\mu}^\iota(\xi)\,d\overline{\mu}(\eta)\,dt\int_{\partial_\infty\widetilde{M}}\psi^*(\eta)\,d\overline{\mu}(\eta)\\
&= \int_{\partial_\infty\widetilde{M}\times\partial_\infty\widetilde{M}\times[0,1]} \psi^*(\eta)\,d\overline{\mu}^\iota(\xi)\,d\overline{\mu}(\eta)\,dt\int_{\partial_\infty\widetilde{M}}\psi^*(\eta)\,d\overline{\mu}(\eta)\\
&= \Big(\int_{\partial_\infty\widetilde{M}}\psi^*(\eta)\,d\overline{\mu}(\eta)\Big)^2\,.
\end{aligned}
$$

By the equality case in the Cauchy-Schwarz inequality, we hence have that $\psi^*$ is constant $\mu_x$-almost everywhere. By Fubini's theorem and the fact that $\widetilde{m}$ is equivalent to the product measure $d\mu^\iota_x\,d\mu_x\,dt$ on $\partial^2_\infty\widetilde{M}\times\mathbb{R}$, the preimage of a $\mu_x$-measure zero subset of $\partial_\infty\widetilde{M}$ by $v\mapsto v_+$ is a $\widetilde{m}$-measure zero subset of $T^1\widetilde{M}$. We hence have that $\widetilde{h}^*$ is $\widetilde{m}$-almost everywhere constant.

Since the map $E(\frac{h}{\rho}\,|\,\mathscr{I})$ is $m$-almost everywhere constant, it is $\mathbb{P}$-almost everywhere equal to $\int_{T^1M}\frac{h}{\rho}\,d\mathbb{P}=\int_{T^1M}h\,dm$ by Equation (84). Let $\psi\in\mathbb{L}^1(\mathbb{P})$ be invariant under the geodesic flow. From the density in $\mathbb{L}^1(\mathbb{P})$ of the subspace of smooth maps with compact support (hence Lipschitz), we deduce that $\int_{T^1M}\psi^2\,d\mathbb{P}=(\int_{T^1M}\psi\,d\mathbb{P})^2$. By the equality case in the Cauchy-Schwarz inequality, this implies that $\psi$ is almost everywhere constant. Since $\mathbb{P}$ and $m$ are equivalent, this says that $(T^1M,(\phi_t)_{t\in\mathbb{R}},m)$ is ergodic. $\square$

**Proofs of Theorem 5.3 and Theorem 5.4.** We use the numbering of the assertions (1)-(5) of Theorem 5.3, (i)-(v) of Theorem 5.4 and (a)-(g) of Proposition 5.5.

The proof of Theorem 5.3 follows from the following implications:

- (1), (2) and (3) are equivalent, by (a) and (f).
- (5) implies (2), by (d).
- (2) and (3) imply (4), by (b) and (c).
- (2) and (3) and (4) imply (5) by (e) and (d).
- Finally, assume that Assertion (4) holds, let us prove that Assertion (1) holds. Since $(\partial^2_\infty\widetilde{M},\Gamma,(\mu^\iota_x\otimes\mu_x)_{|\partial^2_\infty\widetilde{M}})$ is non-ergodic, $(T^1M,(\phi_t)_{t\in\mathbb{R}},m)$ is non-ergodic by (e). By the contrapositive of (h), $(T^1M,(\phi_t)_{t\in\mathbb{R}},m)$ is not conservative. By the contrapositive of (d), we have $\mu_x({}^c\Lambda_c\Gamma)>0$ or $\mu^\iota_x({}^c\Lambda_c\Gamma)>0$. By the contrapositive of (g), which is also valid upon replacement of $\mu_x$ by $\mu^\iota_x$, we have $\mu_x(\Lambda_c\Gamma)=0$ or $\mu^\iota_x(\Lambda_c\Gamma)=0$. By the contrapositive of (f), this proves (1).

The proof of Theorem 5.4 follows from the following implications:

• (ii) and (iii) are equivalent, by (g) which is also valid upon replacement of $\mu_x$ by $\mu^\iota_x$, since (2) and (3) are.

• (i) implies (ii), by (f) and (g).

• (ii) and (iii) imply (v), by (d) and (h).

• (v) implies (iv), by (e) and (c).

• (iv) implies (i), by (c) and the contrapositives of (b) and (a). □

### 5.3 Uniqueness of Patterson densities and of Gibbs states

The first result in this subsection, generalising the case $F = 0$ due to Sullivan [Sul2], gives a useful criterion for being of divergence type.

**Corollary 5.10** *Assume that there exists a Patterson density $(\mu_x)_{x\in X}$ for $(\Gamma, F)$ of dimension $\sigma \in \mathbb{R}$ such that $\mu_x(\Lambda_c\Gamma) > 0$. Then $\sigma = \delta_{\Gamma,\,F}$ and $(\Gamma, F)$ is of divergence type.*

**Proof.** We have $\sigma \geq \delta_{\Gamma,\,F}$ by Corollary 3.11 (2). If $\sigma > \delta_{\Gamma,\,F}$, then the Poincaré series $Q_{\Gamma,\,F}$ converges at $\sigma$, and $\mu_x(\Lambda_c\Gamma) = 0$ by Theorem 5.3, a contradiction. Hence $\sigma = \delta_{\Gamma,\,F}$ and if $(\Gamma, F)$ is of convergence type, Theorem 5.3 gives the same contradiction. □

In particular, if $\Gamma$ is convex-cocompact, then Patterson's construction recalled in the proof of Proposition 3.9 shows that $(\Gamma, F)$ is of divergence type.

**Proposition 5.11** *Let $\sigma \in \mathbb{R}$.*

*(1) If $\Gamma$ is convex-cocompact, then there exists a Patterson density for $(\Gamma, F)$ of dimension $\sigma$ with support equal to $\Lambda\Gamma$ if and only if $\sigma = \delta_{\Gamma,\,F}$.*

*(2) If $\Gamma$ is not convex-cocompact, then there exists a Patterson density for $(\Gamma, F)$ of dimension $\sigma$ with support equal to $\Lambda\Gamma$ if and only if $\sigma \geq \delta_{\Gamma,\,F}$.*

**Proof.** By Corollary 3.11, we always have $\sigma \geq \delta_{\Gamma,\,F}$ if there exists a Patterson density for $(\Gamma, F)$ of dimension $\sigma$. Since $\Lambda\Gamma = \Lambda_c\Gamma$ when $\Gamma$ is convex-cocompact, the first claim follows from Patterson's construction (see Proposition 3.9) for the existence part when $\sigma = \delta_{\Gamma,\,F}$, and from the implication (1) implies (2) of Theorem 5.3 for the converse statement.

To prove the second claim, by Corollary 3.11 and Proposition 3.9, we only have to prove that given $\sigma > \delta_{\Gamma,\,F}$, there exists a Patterson density for $(\Gamma, F)$ of dimension $\sigma$ with support equal to $\Lambda\Gamma$. Since $\Gamma$ is not convex-cocompact, there exists a sequence $(y_i)_{i\in\mathbb{N}}$ in $\mathscr{C}\Lambda\Gamma$ such that $\lim_{i\to+\infty} d(y_i, \Gamma x_0) = +\infty$. For every $x \in \widetilde{M}$, consider the nonzero (since $\sigma > \delta_{\Gamma,\,F}$) measure

$$\mu_{x,\,i} = \frac{\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma y_i}(\widetilde{F}-\sigma)}\mathscr{D}_{\gamma y_i}}{\sum_{\gamma\in\Gamma} e^{\int_{x_0}^{\gamma y_i}(\widetilde{F}-\sigma)}}\,,$$

whose support is contained in $\mathscr{C}\Lambda\Gamma$, and whose total mass is bounded from above and below by a positive constant independent of $i$ by Lemma 3.2. The family $(\mu_{x,\,i})_{x\in\widetilde{M}}$ has an accumulation point (for the product topology of the weak-star topologies) $(\mu_x)_{x\in\widetilde{M}}$ of finite nonzero measures with support contained in (hence equal to) $\Lambda\Gamma$, which is easily proved to be a Patterson density for $(\Gamma, F)$ of dimension $\sigma$. □

The divergence type assumption has several nice consequences.

**Corollary 5.12** *Assume that $\delta_{\Gamma,\,F} < +\infty$ and that $(\Gamma, F)$ is of divergence type. Then there exists a Patterson density $(\mu_{F,\,x})_{x\in\widetilde M}$ of dimension $\delta_{\Gamma,\,F}$ for $(\Gamma, F)$, which is unique up to a scalar multiple. If $\mathscr{D}_z$ is the unit Dirac mass at a point $z \in \widetilde M$, then for all $x_0, x \in \widetilde M$, we have*

$$\mu_{F,\,x} = \|\mu_{F,\,x_0}\| \lim_{s\to\delta_{\Gamma,\,F}{}^+} \frac{1}{Q_{\Gamma,\,F,\,x_0,\,x_0}(s)} \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x_0}(\widetilde F - s)}\, \mathscr{D}_{\gamma x_0}\;, \tag{85}$$

*as well as*

$$\frac{\mu_{F,\,x}}{\|\mu_{F,\,x}\|} = \lim_{s\to\delta_{\Gamma,\,F}{}^+} \frac{1}{Q_{\Gamma,\,F,\,x,\,x}(s)} \sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}(\widetilde F - s)}\, \mathscr{D}_{\gamma x}\;. \tag{86}$$

*The support of the measures $\mu_{F,\,x}$ is $\Lambda\Gamma$ and furthermore $\mu_{F,\,x}(\Lambda\Gamma - \Lambda_c\Gamma) = 0$. We have $\mu_{F,\,x}(\Lambda\Gamma - \Lambda_{\rm Myr}\Gamma) = 0$.*

Equation (85) (from which Equation (86) follows) says that Patterson's construction of the Patterson densities (see the proof of Proposition 3.9) does not depend on choices of accumulation points (and follows from the uniqueness property). This is due to Patterson when $F = 0$. In the context of Gibbs measures (with the multiplicative convention, as explained in the beginning of Chapter 3), this has been first proved by Ledrappier [Led2].

**Proof.** The existence has been seen in Proposition 3.9. The uniqueness is classical (see [Sul1]): If $(\mu'_{F,\,x})_{x\in\widetilde M}$ is another Patterson density of dimension $\delta_{\Gamma,\,F}$ for $(\Gamma, F)$, then so is $(\mu_{F,\,x} + \mu'_{F,\,x})_{x\in\widetilde M}$. For any fixed $x \in \widetilde M$, let $\mu = \mu_{F,\,x}$, $\mu' = \mu'_{F,\,x}$ and $\nu = \mu + \mu'$. Note that $\mu$ is absolutely continuous with respect to $\nu$, and that the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ is $\Gamma$-invariant and measurable, hence is $\nu$-almost everywhere equal to a constant $\lambda > 0$ by ergodicity (see Theorem 5.4, which implies that any Patterson density of dimension $\delta_{\Gamma,\,F}$ is ergodic under $\Gamma$). Therefore $\mu = \lambda\nu$ and $\mu' = (\lambda^{-1} - 1)\mu$.

The fact that the support of $\mu_{F,\,x}$ is $\Lambda\Gamma$ follows from its construction (see Proposition 3.9), and the fact that $\mu_{F,\,x}$ gives full measure to the conical limit set follows from Theorem 5.4.

The proof that $\mu_x(\Lambda\Gamma - \Lambda_{\rm Myr}\Gamma) = 0$ is the same one as when $F = 0$ in [Rob1] (generalising [Tuk]): Let $\mathscr{B}$ be a countable basis of the topology of the support of the Gibbs measure $\widetilde m_F$ associated to $(\mu_{F\circ\iota,\,x})_{x\in\widetilde M}$ and $(\mu_{F,\,x})_{x\in\widetilde M}$, where $(\mu_{F\circ\iota,\,x})_{x\in\widetilde M}$ is any Patterson density of dimension $\delta_{\Gamma,\,F} = \delta_{\Gamma\circ\iota,\,F}$ for $(\Gamma, F\circ\iota)$, which exists by Proposition 3.9. For every nonempty $U \in \mathscr{B}$, let $\mathscr{M}(U)$ be the set of $v \in T^1\widetilde M$ for which there exist sequences $(\gamma_i)_{i\in\mathbb N}$ in $\Gamma$ and $(t_i)_{i\in\mathbb N}$ in $\mathbb R$ tending to $+\infty$ such that $\phi_{t_n}(\gamma_n v) \in U$. Then $\mathscr{M}(U)$ is a $\Gamma$-invariant measurable subset of $T^1\widetilde M$. Since $(T^1 M, (\phi_t)_{t\in\mathbb R}, m_F)$ is conservative (see Theorem 5.4), the subset $U$ is contained in $\mathscr{M}(U)$. Since $\widetilde m_F(U) > 0$ and by ergodicity (see Theorem 5.4), the set $\mathscr{M}(U)$ has full measure. Since $\Lambda_{\rm Myr}\Gamma = \{v_+ \;:\; v \in \bigcap_{U\in\mathscr{B}} \mathscr{M}(U)\}$ by definition (see Chapter 2), and since by the quasi-product structure of $\widetilde m_F$, the preimage by $v \mapsto v_+$ of a subset of positive measure has positive measure, we have that $\mu_{F,\,x}(\Lambda\Gamma - \Lambda_{\rm Myr}\Gamma) = 0$. □

**Proposition 5.13** *Assume that $\delta_{\Gamma,\,F} < +\infty$ and that $(\Gamma, F)$ is of divergence type. Let $(\mu_{F,\,x})_{x\in\widetilde M}$ be a Patterson density of dimension $\delta_{\Gamma,\,F}$ for $(\Gamma, F)$, and let $x, y \in \widetilde M$.*

*(1) The measure $\mu_{F,\,x}$ is atomless.*

*(2) As $n \to +\infty$, we have*

$$\max_{\gamma\in\Gamma,\; n-1<d(x,\gamma y)\le n} e^{\int_x^{\gamma y}\widetilde F} = \mathrm{o}\,(e^{\delta_{\Gamma,\,F}\, n})\,.$$

**Proof.** (1) Assume for a contradiction that $\xi \in \partial_\infty \widetilde{M}$ is an atom of $\mu_{F,\,x}$. Since $\mu_{F,\,x}$ gives full measure to the conical limit set, we have $\xi \in \Lambda_c\Gamma$, that is, there exists a sequence of elements $(\gamma_i)_{i\in\mathbb{N}}$ in $\Gamma$ such that $(\gamma_i\, x)_{i\in\mathbb{N}}$ converges to $\xi$ while staying at bounded distance from the geodesic ray $[x, \xi[$, which implies in particular that $\xi \in \mathscr{O}_x B(\gamma_i x, R)$ for any $R > 0$ large enough. Up to increasing $R$, by Mohsen's shadow lemma 3.10 with $K = \{x\}$, there exists $C > 0$ such that $\mu_{F,\,x}(\mathscr{O}_x B(\gamma_i x, R)) \le C\, e^{\int_x^{\gamma_i x}(\widetilde{F}-\delta_{\Gamma,\,F})}$ for all $i \in \mathbb{N}$. Hence there exists $\epsilon > 0$ such that for all $i \in \mathbb{N}$, using Equation (16),

$$e^{\int_x^{\gamma_i^{-1}x}(\widetilde{F}\circ\iota-\delta_{\Gamma,\,F})} = e^{\int_x^{\gamma_i x}(\widetilde{F}-\delta_{\Gamma,\,F})} \ge \epsilon\,.$$

Again by Mohsen's shadow lemma 3.10 applied to a Patterson density $(\mu_{F\circ\iota,\,x})_{x\in\widetilde{M}}$ of dimension $\delta_{\Gamma,\,F\circ\iota} = \delta_{\Gamma,\,F}$ for $(\Gamma, F\circ\iota)$, there exists $\epsilon' > 0$ such that $\mu_{F\circ\iota,\,x}(\mathscr{O}_x B(\gamma_i^{-1}x, R)) \ge \epsilon'$ for all $i \in \mathbb{N}$. Up to a subsequence, the sequence $(\gamma_i^{-1}x)_{i\in\mathbb{N}}$ converges to $\eta \in \partial_\infty\widetilde{M}$, with $\mu_{F\circ\iota,\,x}(\{\eta\}) > 0$. In particular, the measure $\mu_{F\circ\iota,\,x} \otimes \mu_{F,\,x}$ has an atom at $(\eta, \xi)$, a contradiction to Theorem 5.4 (iv) and Proposition 5.5 (c).

(2) Assume for a contradiction that there exists $c > 0$ and a sequence $(\gamma_i)_{i\in\mathbb{N}}$ in $\Gamma$ such that $\lim_{i\to+\infty} d(x, \gamma_i y) = +\infty$ and $e^{\int_x^{\gamma_i y}\widetilde{F}} \ge c\; e^{\delta_{\Gamma,\,F}\; d(x,\gamma_i y)}$, hence $e^{\int_x^{\gamma_i y}(\widetilde{F}-\delta_{\Gamma,\,F})} \ge c$. Up to extraction, the sequence $(\gamma_i\, x)_{i\in\mathbb{N}}$ converges to $\xi \in \partial_\infty\widetilde{M}$. By Mohsen's shadow lemma 3.10, $\xi$ is an atom of $\mu_{F,\,x}$, a contradiction to (1). □

**Corollary 5.14** *Let $\sigma \in \mathbb{R}$. There exists a Patterson density of dimension $\sigma$ for $(\Gamma, F)$ whose support contains strictly $\Lambda\Gamma$ if and only if $\Lambda\Gamma \ne \partial_\infty\widetilde{M}$ and the Poincaré series $\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$ converges.*

**Proof.** Assume first that $\Lambda\Gamma \ne \partial_\infty\widetilde{M}$ and that the Poincaré series $\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$ converges. For every $\xi \in \partial_\infty\widetilde{M} - \Lambda\Gamma$, the construction of a Patterson density $(\nu_x^\xi)_{x\in\widetilde{M}}$ of dimension $\sigma$ for $(\Gamma, F)$ whose support contains $\xi$, hence contains strictly $\Lambda\Gamma$, has been seen in the Remark following the statement of Theorem 5.3.

Conversely, assume that there exists a Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\sigma$ for $(\Gamma, F)$ whose support contains strictly $\Lambda\Gamma$. In particular, $\Lambda\Gamma \ne \partial_\infty\widetilde{M}$. We have $\delta_{\Gamma,\,F} \le \sigma$ by Corollary 3.11, and in particular $\delta_{\Gamma,\,F} < +\infty$. If the Poincaré series $\sum_{\gamma\in\Gamma} e^{\int_x^{\gamma x}(\widetilde{F}-\sigma)}$ diverges, then $\sigma = \delta_{\Gamma,\,F}$, and $(\Gamma, F)$ is of divergence type. Hence by Corollary 5.12, the support of $\mu_x$ is equal to $\Lambda\Gamma$, a contradiction. □

**Corollary 5.15** *Let $\sigma \in \mathbb{R}$. If there exists a Gibbs measure of dimension $\sigma$ for $(\Gamma, F)$ which is finite on $T^1M$, then $\sigma = \delta_{\Gamma,\,F}$, the pair $(\Gamma, F)$ is of divergence type, there exists a unique (up to a scalar multiple) Gibbs measure $m_F$ with potential $F$ on $T^1M$, its support is $\Omega\Gamma$, the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M$ is ergodic and conservative with respect to $m_F$, and the union of the set of fixed points in $T^1\widetilde{M}$ of the elements of $\Gamma$ which do not pointwise fix $\Lambda\Gamma$ has measure 0 for $\widetilde{m}_F$.*

It follows in particular from the first statement that if $\Gamma$ is convex-cocompact, then there exists no Patterson density of dimension $\sigma > \delta_{\Gamma,\,F}$ (see also Proposition 5.11).

**Proof.** Let $m$ be a finite Gibbs measure of dimension $\sigma$ for $(\Gamma, F)$. Up to adding a constant to $F$, we may assume that $\sigma > 0$. Since $(\phi_t)_{t\in\mathbb{R}}$ preserves $m$, Poincaré's recurrence theorem (see for instance [Kre, page 16]) says that $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is conservative. Hence by Theorem 5.3, the Poincaré series of $(\Gamma, F)$ diverges at $\sigma$. Since there are no Patterson densities of dimension strictly less than $\delta_{\Gamma,\,F}$ (by Corollary 3.11 (2)), and since $Q_{\Gamma,\,F}(\sigma) < +\infty$ if $\sigma > \delta_{\Gamma,\,F}$, this implies that $\sigma = \delta_{\Gamma,\,F}$. Hence $(\Gamma, F)$ is of divergence type. By Theorem 5.4, $(\phi_t)_{t\in\mathbb{R}}$ is ergodic for $m$. The uniqueness property of the Gibbs measure $m_F$ of potential $F$ then follows by its construction from Corollary 5.12. Since the (topological) non-wandering set $\Omega\Gamma$ (see Subsection 2.4) is the image in $T^1M$ of the set of the elements $v \in T^1\widetilde{M}$ such that $v_-, v_+ \in \Lambda\Gamma$, the claim on the support of $m_F$ follows also from Corollary 5.12.

The union $A$ of the fixed points sets in $\widetilde{\Omega}\Gamma$ of the elements of $\Gamma$ which do not pointwise fix $\Lambda\Gamma$ is invariant under the geodesic flow. It is $\Gamma$-invariant, closed and properly contained in $\widetilde{\Omega}\Gamma$, by Lemma 2.2. Note that $\widetilde{\Omega}\Gamma$ is the support of $\widetilde{m}_F$ (see Subsection 3.7). Hence by ergodicity, the measure of $A$ with respect to $\widetilde{m}_F$ is 0. □

**Remark.** If $\delta_{\Gamma,\,F} < +\infty$ and $(\Gamma, F)$ is of divergence type (which implies that $(\Gamma, F\circ\iota)$ also is, by Equation (19) in Lemma 3.3), it follows from the uniqueness statement in Corollary 5.12 that if we normalise the Patterson density families for $(\Gamma, F)$ and $(\Gamma, F\circ\iota)$ given by Corollary 5.12 to have total mass 1 for a given point $x_0 \in \widetilde{M}$, and if we consider the associated Gibbs measures $\widetilde{m}_F$ and $\widetilde{m}_{F\circ\iota}$, then

- $(\mu_{F,\,x})_{x\in\widetilde{M}} = (\mu_{F+s,\,x})_{x\in\widetilde{M}}$ for every $s$ in $\mathbb{R}$ (by Equation (18));
- $\iota_*\widetilde{m}_F = \widetilde{m}_{F\circ\iota}$ and $\iota_* m_F = m_{F\circ\iota}$ (by Equation (30)); in particular, when $\widetilde{F}$ is reversible, we have $(\mu_{F\circ\iota,\,x})_{x\in\widetilde{M}} = (\mu_{F,\,x})_{x\in\widetilde{M}}$ and $\iota_*\widetilde{m}_F = \widetilde{m}_F$.
- for every $s \in \mathbb{R}$, if $F'$ is a potential cohomologous to $F$, then $m_{F'+s} = m_F$ (by the remarks at the end of Subsection 3.6 and Subsection 3.7).

In the next chapters, we will often assume that there exists a finite Gibbs measure (of dimension $\delta_{\Gamma,\,F} < +\infty$) on $T^1M$ with potential $F$. It is then ergodic and unique up to normalisation, by Corollary 5.15. In particular, $T^1M$, endowed with a given potential $F$, does not carry simultaneously a finite Gibbs measure and an infinite one.

# 6 Thermodynamic formalism and equilibrium states

Let $\widetilde{M}, \Gamma, \widetilde{F}$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$; $\Gamma$ is a non-elementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map. In this chapter, we denote by $\phi = (\phi_t)_{t\in\mathbb{R}}$ the geodesic flow, both on $T^1\widetilde{M}$ and on $T^1M = \Gamma\backslash T^1\widetilde{M}$.

Given a measurable map $H : T^1M \to \mathbb{R}$, we denote by

$$H_- : v \mapsto \max\{0, -H(v)\}$$

the *negative part* of $H$. Note that for every positive Borel measure $m$ on $T^1M$ for which the negative part of $H$ is integrable, the integral $\int_{T^1M} H\,dm$ is well defined in $\mathbb{R}\cup\{+\infty\}$ as

$$\int_{T^1M} H\,dm = \int_{T^1M} \max\{0, H\}\,dm - \int_{T^1M} H_-\,dm\,.$$

Note that if $m, m_1, m_2$ are positive Borel measures on $T^1M$ and if $m = tm_1 + (1-t)m_2$ for some $t \in \,]0,1[\,$, then the negative part of $F$ is integrable with respect to $m$ if and only if it is integrable with respect to both $m_1$ and $m_2$.

Let $\mathscr{M}_F(T^1M)$ be the space of $\phi$-invariant Borel probability measures $m$ on $T^1M$ such that the negative part of $F$ is $m$-integrable, endowed with the weak-star topology. Note that the support of any $\phi$-invariant probability measure is contained in the (topological) non-wandering set $\Omega\Gamma$. Note that $\mathscr{M}_F(T^1M)$ is convex, but in general not closed in the convex weak-star compact space of all $\phi$-invariant Borel probability measures on $T^1M$.

For every $m \in \mathscr{M}_F(T^1M)$, the *(metric) pressure* of the potential $F$ with respect to $m$ is the element of $\mathbb{R} \cup \{+\infty\}$ defined by

$$P_{\Gamma,\,F}(m) = h_m(\phi) + \int_{T^1M} F\,dm\ ,$$

where $h_m(\phi)$ is the (metric) entropy of the geodesic flow $\phi$ with respect to $m$ (see [KatH] or Subsection 6.1 for the properties we will need on the entropy). The *(topological) pressure* of the potential $F$ is the upper bound of its metric (or measure-theoretic) pressures, that is, the element of $\mathbb{R} \cup \{+\infty\}$ defined by

$$P(\Gamma, F) = \sup_{m \in \mathscr{M}_F(T^1M)} P_{\Gamma,\,F}(m)\ .$$

An element $m \in \mathscr{M}_F(T^1M)$ realising the upper bound, that is, such that $P(\Gamma, F) = P_{\Gamma,\,F}(m)$, is called an *equilibrium state* for $(\Gamma, F)$. Using the convexity of $\mathscr{M}_F(T^1M)$, the affine property of the metric entropy (hence of the metric pressure), and the ergodic decomposition of probability measures invariant under the geodesic flow, the pressure $P(\Gamma, F)$ of $F$ is also the upper bound of the metric pressures of $F$ with respect to the $\phi$-ergodic elements in $\mathscr{M}_F(T^1M)$. Furthermore, by the Krein-Milman theorem, any equilibrium state is an average of $\phi$-ergodic ones.

Note that for every $m \in \mathscr{M}_F(T^1M)$, we have $\iota_* m \in \mathscr{M}_{F\circ\iota}(T^1M)$ and $P_{\Gamma,\,F\circ\iota}(\iota_* m) = P_{\Gamma,\,F}(m)$. Hence

$$P(\Gamma, F) = P(\Gamma, F \circ \iota)\ .$$

For all $\kappa \in \mathbb{R}$ and $m \in \mathscr{M}_F(T^1M)$, we have $P_{\Gamma,\,F+\kappa}(m) = P_{\Gamma,\,F}(m) + \kappa$ and $P(\Gamma, F + \kappa) = P(\Gamma, F) + \kappa$; furthermore, $m$ is an equilibrium state for $(\Gamma, F+\kappa)$ if and only if $m$ is an equilibrium state for $(\Gamma, F)$.

The aim of this Chapter 6 is, following the case $F \equiv 0$ in [OtP], to prove the following result, saying in particular that a finite Gibbs state, once renormalised to be a probability measure, is the unique equilibrium state of a given potential.

**Theorem 6.1** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Let $\Gamma$ be a non-elementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map such that $\delta_{\Gamma,\,F} < +\infty$.*

*(1) We have*

$$P(\Gamma, F) = \delta_{\Gamma,\,F}\ . \tag{87}$$

*(2) If there exists a finite Gibbs measure $m_F$ for $(\Gamma, F)$ such that the negative part of $F$ is $m_F$-integrable, then $m^F = \frac{m_F}{\|m_F\|}$ is the unique equilibrium state for $(\Gamma, F)$. Otherwise, there exists no equilibrium state for $(\Gamma, F)$.*

It follows in particular from Equation (87) that the topological pressure $P(\Gamma, F)$ is finite if the critical exponent $\delta_{\Gamma,\,F}$ is finite.

**Remark.** When $\Gamma$ is torsion free and convex-cocompact, Theorem 6.1 has been proved in [Sch3]. In this case, the critical exponent $\delta_\Gamma$ of $\Gamma$ is the topological entropy $h$ of the geodesic flow of $M$ restricted to the (topological) non-wandering set $\Omega\Gamma$ (see [Mann]), and $\delta_{\Gamma,\,F}$ is the topological pressure $P(F)$ of the potential function $F : T^1M \to \mathbb{R}$ induced by $\widetilde{F}$ (see [Rue2]), that is

$$P(F) = \max_{m \in \mathscr{M}(T^1M)} \Big( \, h_m(\phi) + \int_{T^1M} F \; dm \, \Big)$$

where $\mathscr{M}(T^1M)$ is the set of all Borel probability measures on $T^1M$ invariant under the geodesic flow. Note that the reference [Sch3] uses the multiplicative convention for the Poincaré series as mentioned in the beginning of Chapter 3, but the uniqueness property ensures that the two constructions of the Gibbs measures coincide in this case. Note that any Gibbs measure is finite in the convex-cocompact case, and uniqueness follows from general argument of [BoR], hence the proof (using the Shadow lemma and entropy computations of [Kai]) is much easier and shorter in this particular case.

An immediate consequence of the equality $P(\Gamma, F) = \delta_{\Gamma,\,F}$ is that the periods of the normalised potential are nonpositive.

**Corollary 6.2** *For every periodic orbit $g$ of the geodesic flow on $T^1M$, we have*

$$\int_g (F - \delta_{\Gamma,\,F}) \; \le 0 \; .$$

**Proof.** Let $\mathscr{L}_g$ be the Lebesgue measure along a periodic $g$ with length $\ell(g)$ (see the beginning of Subsection 4.1). The measure $\frac{\mathscr{L}_g}{\ell(g)}$ is a probability measure (with compact support) on $T^1M$ invariant by the geodesic flow, whose metric entropy is 0 (since rotations of the circle have entropy zero). Hence

$$\delta_{\Gamma,\,F} = P(\Gamma, F) \ge P_{\Gamma,\,F}(m) = \int_{T^1M} F \; dm = \frac{\mathscr{L}_g(F)}{\ell(g)} \; ,$$

which proves the result. □

In particular, for every compact subset $K$ of $T^1M$, for every measure $\mu$ on $T^1M$ which belongs to the weak-star closure of the set of linear combinations (with nonpositive coefficients) of the Lebesgue measures along the periodic orbits contained in $K$, we have

$$\int_{T^1M} (F - \delta_{\Gamma,\,F}) \; d\mu \; \le 0 \; .$$

## 6.1 Measurable partitions and entropy

We refer to [Roh, Par1, KatH] for the generalities (at the bare minimum for what follows) on measurable partitions and entropy recalled in this subsection, that the knowledgeable reader may skip.

Recall that a *partition* $\zeta$ of a set $X$ is a set of pairwise disjoint nonempty subsets of $X$ whose union is $X$. We will denote by $\pi_\zeta : X \to \zeta$ the canonical projection map which maps every $x \in X$ to the (unique) element $z = \zeta(x)$ of $\zeta$ containing $x$. Given two partitions $\zeta$ and $\zeta'$ of $X$, we define their *refinement* as the partition

$$\zeta \vee \zeta' = \{z \cap z' \ : \ z \in \zeta, z' \in \zeta', z \cap z' \neq \emptyset\} ,$$

and similarly for finitely many partitions.

Recall that two probability spaces with complete $\sigma$-algebras are *isomorphic* if there exists a measure-preserving bimeasurable bijection between full measure subsets of them. A probability space is *standard* if its $\sigma$-algebra is complete and if it is isomorphic to a probability space with underlying set the disjoint union of a compact interval with a measurable countable set, where the restriction to the interval is the Lebesgue $\sigma$-algebra and the Lebesgue measure, and where the restriction to the countable set is the discrete $\sigma$-algebra.

Fix a standard probability space $(X, \mathscr{A}, m)$. The following result that we state without proof is well known (see for instance [OtP, Lemme 7]).

**Lemma 6.3** *If $(\phi_t)_{t\in\mathbb{R}}$ is a $m$-ergodic measure preserving flow on $X$, then there exists a countable subset $D$ of $\mathbb{R}$ such that $\phi_\tau$ is $m$-ergodic for every $\tau \in \mathbb{R} - D$.*

For every partition $\zeta$ of the set $X$, denote by $\widehat{\zeta}$ the sub-$\sigma$-algebra of elements of $\mathscr{A}$ which are unions of elements of $\zeta$. Recall that $\zeta$ is *$m$-measurable* (or *measurable* if $(\mathscr{A}, m)$ is implicit) if there exist a full measure subset $Y$ in $X$ and a sequence $(A_n)_{n\in\mathbb{N}}$ in $\widehat{\zeta}$, satisfying the following separation property: for all $z \neq z'$ in $\zeta$, there exists $n \in \mathbb{N}$ such that either we have $z \cap Y \subset A_n$ and $z' \cap Y \subset X - A_n$ or we have $z' \cap Y \subset A_n$ and $z \cap Y \subset X - A_n$. For instance, a finite partition of $X$ is measurable if and only if each element of the partition is measurable. Given two $m$-measurable partitions $\zeta, \zeta'$, we write $\zeta \preceq \zeta'$ (the conventions differ amongst the references) if $\zeta'(x) \subset \zeta(x)$ for $m$-almost every $x \in X$, and we say that $\zeta$ and $\zeta'$ are *$m$-equivalent* if $\zeta \preceq \zeta'$ and $\zeta' \preceq \zeta$. Given a sequence of $m$-measurable partitions $(\zeta_n)_{n\in\mathbb{N}}$, there exists a $m$-measurable partition, denoted by $\zeta = \bigvee_{n\in\mathbb{N}} \zeta_n$ and unique up to $m$-equivalence, such that $\zeta_n \preceq \zeta$ for every $n \in \mathbb{N}$, and if $\zeta'$ is a $m$-measurable partition such that $\zeta_n \preceq \zeta'$ for every $n \in \mathbb{N}$, then $\zeta \preceq \zeta'$.

Fix a measurable partition $\zeta$ of $(X, \mathscr{A}, m)$. The triple

$$\big(\ \zeta,\ \ \mathscr{A}_\zeta = \pi_\zeta(\widehat{\zeta}) = \{\pi_\zeta(A) \ : \ A \in \widehat{\zeta}\},\ \ m_\zeta = (\pi_\zeta)_* m : A \mapsto m(\pi_\zeta^{-1}(A))\ \big)$$

is a standard probability space, called the *factor space* of $(X, \mathscr{A}, m)$ by $\zeta$. In particular, saying that for $m_\zeta$-almost every $z \in \zeta$, the set $z$ satisfies some property and for $m$-almost every $x \in X$, the set $\zeta(x)$ satisfies this property are equivalent.

There exists a family $(\mathscr{A}_z, m_z)_{z\in\zeta}$ (sometimes considered as defined only for $m_\zeta$-almost every $z$ in $\zeta$), where $(\mathscr{A}_z, m_z)$ is a standard probability measure on $z$, such that:

(1) for every $\mathscr{A}$-measurable map $f : X \to \mathbb{C}$, the map $f_{|z}$ is $\mathscr{A}_z$-measurable for $m_\zeta$-almost every $z$ in $\zeta$;

(2) the measure $m$ disintegrates with respect to the projection $\pi_\zeta$, with family of *conditional measures* on the fibres the family $(m_z)_{z\in\zeta}$: for every $\mathscr{A}$-integrable map $f : X \to \mathbb{C}$, the map $z \mapsto \int_z f_{|z}\, dm_z$ is $m_\zeta$-integrable and

$$\int_X f\, dm = \int_{z\in\zeta} \int_z f_{|z}\, dm_z\, dm_\zeta(z)\ .$$

This last integral is equal to $\int_{x\in X} \int_{\zeta(x)} f_{|\zeta(x)}\, dm_{\zeta(x)}\, dm(x)$. If two such families $(\mathscr{A}_z, m_z)_{z\in\zeta}$ and $(\mathscr{A}'_z, m'_z)_{z\in\zeta}$ satisfy properties (1) and (2), then $m_z = m'_z$ for $m_\zeta$-almost every $z \in \zeta$. The properties (1) and (2) of such a family $(\mathscr{A}_z, m_z)_{z\in\zeta}$ are satisfied if they are satisfied for $f$ the characteristic functions of elements of $\mathscr{A}$.

If $\zeta$ is finite or countable, then for every $z \in \zeta$ such that $m(z) \neq 0$, we have $m_z = \frac{1}{m(z)}\, m_{|z}$.

Fix a measure-preserving transformation $\varphi : X \to X$. We denote by $\varphi^{-1}\zeta$ the $m$-measurable partition of $X$ by the preimages of the elements of $\zeta$ by $\varphi$. We say that $\zeta$ is *generating* under $\varphi$ if the $m$-measurable partition $\bigvee_{k\in\mathbb{N}} \varphi^{-k}\zeta$ is $m$-equivalent to the partition by points. For all measurable partitions $\zeta$ and $\zeta'$ of $X$, we define the *entropy of* $\zeta'$ *relative to* $\zeta$ by

$$H_m(\zeta' \,|\, \zeta) = \int_{x\in X} -\log m_{\zeta(x)}(\zeta'(x) \cap \zeta(x))\, dm(x)\ ,$$

with the convention that $-\log 0 = +\infty$. Note that if $\zeta$ and $\zeta'$ are finite or countable, then

$$H_m(\zeta' \,|\, \zeta) = \sum_{z\in\zeta,\, z'\in\zeta',\, m(z)\neq 0} -m(z' \cap z) \log \frac{m(z' \cap z)}{m(z)}\ ,$$

as more usual. If $\zeta \preceq \varphi^{-1}\zeta$, we define the *entropy of* $\varphi$ *relative to* $\zeta$ by

$$h_m(\varphi, \zeta) = H_m(\varphi^{-1}\zeta \,|\, \zeta)\ .$$

The *(metric) entropy* $h_m(\varphi)$ of $\varphi$ with respect to $m$ is the upper bound of $h_m(\varphi, \zeta)$ for $\zeta$ a measurable partition such that $\zeta \preceq \varphi^{-1}\zeta$. The *(metric) entropy* $h_m(\phi)$ of a flow $\phi = (\phi_t)_{t\in\mathbb{R}}$ of measure-preserving transformations of $X$ is $h_m(\phi) = h_m(\phi_1)$. We have $h_m(\phi_t) = |t| h_m(\phi)$ for every $t \neq 0$.

### 6.2 Proof of the Variational Principle

The notation for this proof is the following one. Let $\delta = \delta_{\Gamma,\,F}$ and let $\widetilde{m}_F$ be a Gibbs measure on $T^1\widetilde{M}$ for $\Gamma$ with potential $F$: there exist Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\delta$ for $(\Gamma, F\circ\iota)$ and $(\Gamma, F)$ such that, using the Hopf parametrisation $T^1\widetilde{M} \to \partial^2_\infty\widetilde{M} \times \mathbb{R}$ identifying $v$ and $(v_-, v_+, t)$,

$$d\,\widetilde{m}(v) = e^{C_{F\circ\iota-\delta,\, v_-}(x,\pi(v)) + C_{F-\delta,\, v_+}(x,\pi(v))} d\mu^\iota_x(v_-)\, d\mu_x(v_+)\, dt \tag{88}$$

for any $x \in \widetilde{M}$. Let $m_F$ be the induced measure on $T^1M$ (see Subsection 3.7). Denote by $(\mu_{W^{\rm su}(v)})_{v\in T^1\widetilde{M}}$ and $(\mu_{W^{\rm su}(v)})_{v\in T^1M}$ the families of measures on the strong unstable leaves of $T^1\widetilde{M}$ and $T^1M$ associated with the Patterson density $(\mu_x)_{x\in\widetilde{M}}$ as in Subsection 3.9.

Before beginning the proof of Theorem 6.1, we introduce another cocycle. For all $v, w$ in the same strong unstable leaf in $T^1M$, we define

$$c_F(v,w) = \lim_{t\to+\infty} \int_0^t \big(F(\phi_{-s}v) - F(\phi_{-s}w)\big)\; ds\;, \tag{89}$$

which exists since $F$ is Hölder-continuous and $d(\phi_{-t}v, \phi_{-t}w)$ tends exponentially fast to 0. Note that $c_F$ is unchanged by adding a constant to $F$, and that it satisfies the cocycle property

$$c_F(v,v') + c_F(v',v'') = c_F(v,v'')$$

for all $v, v', v''$ in the same strong unstable leaf in $T^1M$. For all $t \in \mathbb{R}$ and $v, w$ in the same strong unstable leaf in $T^1M$, we have

$$c_F(\phi_t v, \phi_t w) = c_F(v,w) + \int_0^t \big(F(\phi_s v) - F(\phi_s w)\big)\; ds\;. \tag{90}$$

The relation between this cocycle and the Gibbs cocycle is the following. For all $v, w$ in the same strong unstable leaf in $T^1M$, for all lifts $\widetilde{v}$ and $\widetilde{w}$ of $v$ and $w$, respectively, in the same strong unstable leaf in $T^1\widetilde{M}$, we have, for every $\sigma \in \mathbb{R}$,

$$c_F(v,w) = -\,C_{F\circ\iota-\sigma,\,\widetilde{v}_-}(\pi(\widetilde{v}), \pi(\widetilde{w}))\;. \tag{91}$$

This follows from the definition of the Gibbs cocycle, since $x = \pi(\widetilde{v})$ and $y = \pi(\widetilde{w})$ being in the same horosphere centred at $\widetilde{v}_-$ implies that $C_{F\circ\iota-\sigma,\,\widetilde{v}_-}(x,y) = C_{F\circ\iota,\,\widetilde{v}_-}(x,y)$, and since the map $t \mapsto \phi_t(-\widetilde{v}) = -\,\phi_{-t}(\widetilde{v})$ from $[0,+\infty[$ to $\widetilde{M}$ is the geodesic ray from $x$ to $\widetilde{v}_-$.

**Step 1.** The first step of the proof of Theorem 6.1 is a construction of measurable partitions with nice geometric properties which allow entropy computations.

Here is what we mean by "nice geometric properties", exactly as in [OtP]. Given a probability measure $m$ on $T^1M$, a partition $\zeta$ of $T^1M$ is called *$m$-subordinated* to the strong unstable partition $\mathscr{W}^{\rm su}$ of $T^1M$ if for $m$-almost every $v$ in $T^1M$, the set $\zeta(v)$ is a relatively compact neighbourhood of $v$ in $W^{\rm su}(v)$.

The next result constructs a measurable partition, subordinated to the strong unstable foliation and realising the upper bound in the definition of the entropy, for every ergodic invariant probability measure on $T^1M$. It is due to [LeS] for Anosov transformations of compact manifolds, and the adaptation we will use to our geodesic flows is the content of the propositions 1 and 4 of [OtP].

**Proposition 6.4 (Otal-Peigné)** *Let $m$ be a probability measure on $T^1M$ invariant under the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$, and let $\tau > 0$ be such that $\phi_\tau$ is ergodic for $m$. Then there exists a $m$-measurable partition $\zeta$ of $T^1M$, which is generating under $\phi_\tau$ and subordinated to $\mathscr{W}^{\rm su}$, such that $(\phi_\tau^{-1}\zeta)(v)$ is contained in $\zeta(v)$ for $m$-almost every $v$, and such that*

$$h_m(\phi_\tau) = h_m(\phi_\tau, \zeta)\;.$$

**Remark.** The assumptions of [OtP] and ours are the same except that we allow $\Gamma$ to have fixed points on $\widetilde{M}$. Here are a few comments. The proof of Proposition 1 in [OtP] starts by fixing $u \in T^1M$ in the support of the measure $m$. For every $r > 0$, the authors consider the standard dynamical neighbourhood $U_r$ of $u$ defined by

$$U_r = \bigcup_{|t|<r} \phi_t\Big(\bigcup_{v\in\mathscr{B}^{\rm ss}(u,\,r)} \mathscr{B}^{\rm su}(v,r)\Big)\;,$$

where $\mathscr{B}^{\rm ss}(u,r)$ is the open $r$-neighbourhood of $u$ in its strong stable leaf for the induced Riemannian metric, and similarly for $\mathscr{B}^{\rm su}(v,r)$. Noting that $U_r$ is the domain of a local chart of the foliation $\mathscr{W}^{\rm su}$ if $r$ is small enough, they define $\zeta$ as $\bigvee_{n=0}^{+\infty}\phi_{n\tau}\widehat{\zeta'}$ where $\zeta'$ is the partition of $T^1M$ whose elements are $T^1M-U_r$ and the local leaves of $\mathscr{W}^{\rm su}$ in $U_r$. They then prove that for Lebesgue-almost every $r>0$ such that $4r$ is strictly less than the injectivity radius of $M$ at the origin $\pi(u)$ of $u$, the partition $\zeta$ is as required.

In our case, starting with the same $u$, we first consider a lift $\widetilde{u}$ of $u$ in $T^1\widetilde{M}$. The neighbourhood $\widetilde{U}_r$ of $\widetilde{u}$ constructed as above is then invariant under the stabiliser $\Gamma_{\widetilde{u}}$ of $\widetilde{u}$ in $\Gamma$. We take $r$ small enough such that the images of $\widetilde{U}_{4r}$ by the elements of the finite set $\Gamma_{\pi(\widetilde{u})}-\Gamma_{\widetilde{u}}$ are pairwise disjoint (where $\Gamma_{\pi(\widetilde{u})}$ is the stabiliser of $\pi(\widetilde{u})$ in $\Gamma$) and such that the restriction to $\widetilde{U}_{4r}$ of the canonical projection from $T^1\widetilde{M}$ to $T^1M$ induces a homeomorphism from $\Gamma_{\widetilde{u}}\backslash\widetilde{U}_{4r}$ onto an open neighbourhood of $u$. The construction in [OtP] then provides the above result.

Furthermore, the assertion in Proposition 6.4 that $(\phi_\tau^{-1}\zeta)(v)$ is contained in $\zeta(v)$ for $m$-almost every $v$ is not stated in [OtP, Prop. 1] but is proved there. It implies that $\zeta\preceq\phi_{-\tau}\zeta$ (as needed for the computation of the relative entropy), but we will need an actual containment and not a containment up to a set of $m$-measure 0 in the third step of the proof.

**Step 2.** In order to apply Proposition 6.4 and the formula giving the entropy relative to a measurable partition, we need to be able to compute the conditional measures. In this second step of the proof of Theorem 6.1, we show that a family of conditional measures $(m_z)_{z\in\zeta}$ of a finite Gibbs measure (normalised to be a probability measure) with respect to a subordinated measurable partition is a family of measures absolutely continuous with respect to (the restrictions to the elements of the partition of) the measures $\mu_{W^{\rm su}(v)}$ on the strong unstable leaves, constructed using the above cocycle. The following lemma is a generalisation (up to appropriate normalisations) of Proposition 3.18, when the partition was by the whole strong unstable leaves.

**Lemma 6.5** *Assume that $m_F$ is finite and let $m^F=\frac{m_F}{\|m_F\|}$. Let $\zeta$ be a $m^F$-measurable partition of $T^1M$ which is $m^F$-subordinated to $\mathscr{W}^{\rm su}$. Then a family of conditional measures $(m^F_z)_{z\in\zeta}$ of $m^F$ with respect to $\zeta$ is given by*

$$dm^F_z(w)=\frac{\mathbb{1}_z(w)}{\int_z e^{-c_F(w,u)}\,d\mu_{W^{\rm su}(v)}(u)}\,d\mu_{W^{\rm su}(v)}(w)\;,\tag{92}$$

*where $w\in z\in\zeta$, for any $v\in T^1M$ such that $z\subset W^{\rm su}(v)$.*

Note that when $F$ is constant (and in particular when $m_F$ is the Bowen-Margulis measure), $m^F_z$ is just the normalised restriction to $z$ of $\mu_{W^{\rm su}(v)}$.

**Proof.** For every relatively compact Borel subset $z$ of a strong unstable leaf $W^{\rm su}(v)$ in $T^1M$, such that the support of $\mu_{W^{\rm su}(v)}$ meets the interior of $z$, let us define a finite measure $m'_z$ on $z$ by

$$dm'_z(w)=\frac{\mathbb{1}_z(w)\,d\mu_{W^{\rm su}(v)}(w)}{\int_z e^{-c_F(w,u)}\,d\mu_{W^{\rm su}(v)}(u)}\;,$$

noting that the denominator is positive and finite and that the numerator is a finite measure. Furthermore, $m'_z$ is a Borel probability measure on $z$, since by the cocycle property of $c_F$,

$$\|m'_z\| = \frac{\int_z e^{-c_F(v,w)}\, d\mu_{W^{\rm su}(v)}(w)}{\int_z e^{-c_F(v,u)}\, d\mu_{W^{\rm su}(v)}(u)} = 1\ .$$

Let us check the two properties for $(m'_z)_{z\in\zeta}$ to be a family of conditional measures for $m^F$ with respect to $\zeta$.

In order to check the measurability property, let $f : T^1M \to \mathbb{C}$ be a measurable map, that we may assume to be bounded with compact support. For every $n \in \mathbb{N}$, let $X_n$ be the Borel subset of points of $T^1\widetilde{M}$ whose stabiliser in $\Gamma$ has cardinality $n$, which is $\Gamma$-invariant and $\phi$-invariant. Since $m_F$ is finite, hence ergodic by Corollary 5.15, we fix $n_0 \in \mathbb{N}$ such that $X_{n_0}$ has full measure for $\widetilde{m}_F$. We may hence assume that $f$ vanishes outside the image of $X_{n_0}$ in $T^1M$. Up to multiplying by $n_0$ the maps and measures when lifting them to $T^1\widetilde{M}$ (see Subsection 2.6), we may assume that $n_0 = 1$. For $m^F$-almost every $v \in T^1M$, since $z = \zeta(v)$ is a measurable subset of $T^1M$, the restriction of $f$ to $z$ is measurable, and

$$v \mapsto \int_z f_{|z}\, dm'_z = \int_{w\in T^1M} \frac{\mathbb{1}_{\zeta(v)}(w) f(w)}{\int_{u\in T^1M} \mathbb{1}_{\zeta(v)}(u)\, e^{-c_F(w,u)}\, d\mu_{W^{\rm su}(v)}(u)}\, d\mu_{W^{\rm su}(v)}(w)$$

is clearly measurable, since $v \mapsto \mu_{W^{\rm su}(v)}$ is weak-star continuous and $\mathbb{1}_{\zeta(v)}(w) = \mathbb{1}_{\zeta(w)}(v)$.

In order to check the disintegration property, we give a sequence of equalities, whose guide (to be read simultaneously with the equalities below) is the following:

- for the first equality, we use the definition of the measures $(m'_z)_{z\in\zeta}$ and the topological properties of the elements of the partition $\zeta$;
- for the second equality, we use a lifting by the canonical projection $T\widetilde{p} : T^1\widetilde{M} \to T^1M$; we denote by $\mathscr{F}_\Gamma$ a measurable (strict) fundamental domain for the action of $\Gamma$ on $T^1\widetilde{M}$ and by $\widetilde{f} = f \circ T\widetilde{p}$ the lift of $f$ to $T^1\widetilde{M}$; for every $v \in T^1\widetilde{M}$, let

$$\widetilde{\zeta}(v) = W^{\rm su}(v) \cap (T\widetilde{p})^{-1}\big(\zeta(T\widetilde{p}(v))\big)\ ;$$

note that for $\widetilde{m}_F$-almost every $v \in T^1\widetilde{M}$, the restriction $T\widetilde{p}_{|\widetilde{\zeta}(v)} : \widetilde{\zeta}(v) \to \zeta(T\widetilde{p}(v))$ is a bi-measurable bijection: otherwise, there would exist $\gamma \in \Gamma$ sending an element in $W^{\rm su}(v)$ to a distinct element in $W^{\rm su}(v)$, thus (we assumed $\Gamma$ to be torsion-free) $\gamma$ is a parabolic element fixing the parabolic fixed point $v_-$, and since $\widetilde{m}_F$ is finite and the set of parabolic fixed points is countable, the set of $v \in T^1\widetilde{M}$ such that $v_-$ is parabolic has measure 0 for $\widetilde{m}_F$ by Proposition 5.13 (1) and Corollary 5.15; we also use Equation (91); note that unless the function which is integrated is 0, we have $w_- = v_-$;
- for the third equality, we use the fact that $\sum_{\gamma\in\Gamma} \mathbb{1}_{\mathscr{F}_\Gamma}(\gamma w) = 1$ by the definition of a fundamental domain;
- for the fourth equality, we use the change of variables $v \mapsto \gamma v$ and $w \mapsto \gamma w$, the invariance under $\Gamma$ of $\widetilde{f}$, of the Gibbs cocycle and of the Gibbs measure $\widetilde{m}_F$, the equivariance properties under $\Gamma$ of the strong stable leaves and their measures, as well as the equivariance property $\widetilde{\zeta}(\gamma v) = \gamma\widetilde{\zeta}(v)$ for $\widetilde{m}_F$-almost every $v \in T^1\widetilde{M}$ which follows from the construction of the lift $\widetilde{\zeta}(v)$ of $\zeta(v)$;
- for the fifth equality, we first simplify by $\sum_{\gamma\in\Gamma} \mathbb{1}_{\mathscr{F}_\Gamma}(\gamma^{-1}v) = 1$, and then, fixing $x \in \widetilde{M}$, we apply the definition of the measures $\mu_{W^{\rm su}(v)}$ (see Equation (48)) and $\widetilde{m}_F$ (see Equation (88)); we use the Hopf parametrisation $v = (v_-, v_+, t)$ with respect to $x$ and the homeomorphism $w \mapsto w_+$ from $W^{\rm su}(v)$ to $\partial_\infty\widetilde{M} - \{v_-\}$; we note that, since $m_F$ is

finite, the diagonal has measure 0 for $\mu^\iota_x \otimes \mu_x$ (see just after Lemma 5.9) and the Patterson density $(\mu_x)_{x\in\widetilde{M}}$ is atomless (see Proposition 5.13 and Corollary 5.15);

- for the sixth equality, we use the cocycle equality

$$C_{F\circ\iota-\delta,\,v_-}(x,\pi(v)) = C_{F\circ\iota-\delta,\,v_-}(x,\pi(w)) + C_{F\circ\iota-\delta,\,v_-}(\pi(w),\pi(v))\ ;$$

we also use that since the relations $w \in W^{\rm su}(v)$ and $T\widetilde{p}(w) \in \zeta(T\widetilde{p}(v))$ between elements $w, v \in T^1\widetilde{M}$ are symmetric (as belonging to the same element of $\zeta$ is a symmetric relation between two elements in $T^1M$), we have $w \in \widetilde{\zeta}(v)$ if and only if $v \in \widetilde{\zeta}(w)$, and this implies that $\widetilde{\zeta}(v) = \widetilde{\zeta}(w)$; furthermore, unless the function which is integrated is 0, we have $w_- = v_-$ and $W^{\rm su}(v) = W^{\rm su}(w)$, and we denote $w = (v_-, w_+, t)$;

- for the seventh equality, we apply Fubini's theorem (exchanging the integrations on $v_+$ and $w_+$);

- for the last ones, again since the map $v \mapsto v_+$ from $W^{\rm su}(w)$ to $\partial_\infty\widetilde{M} - \{w_-\}$ is a homeomorphism, we use the Hopf parametrisation $w = (w_- = v_-, w_+, t)$ with respect to $x$ and the fact that some ratio of two integrals is 1:

$$\int_{v\in T^1M}\int_{w\in\zeta(v)} f(w)\,dm'_{\zeta(v)}(w)dm^F(v)$$

$$=\int_{v\in T^1M}\int_{w\in W^{\rm su}(v)} \frac{\mathbb{1}_{\zeta(v)}(w)f(w)\;d\mu_{W^{\rm su}(v)}(w)\;dm_F(v)}{\int_{u\in W^{\rm su}(v)}\mathbb{1}_{\zeta(v)}(u)\,e^{-c_F(w,\,u)}\;d\mu_{W^{\rm su}(v)}(u)\;\|m_F\|}$$

$$=\int_{v\in T^1\widetilde M}\int_{w\in W^{\rm su}(v)} \mathbb{1}_{\mathscr F_\Gamma}(v)\,\frac{\mathbb{1}_{\widetilde\zeta(v)}(w)\,\widetilde f(w)\;d\mu_{W^{\rm su}(v)}(w)\;d\widetilde m_F(v)}{\int_{u\in W^{\rm su}(v)}\mathbb{1}_{\widetilde\zeta(v)}(u)\,e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))}\;d\mu_{W^{\rm su}(v)}(u)\;\|m_F\|}$$

$$=\int_{v\in T^1\widetilde M}\int_{w\in W^{\rm su}(v)} \sum_{\gamma\in\Gamma}\mathbb{1}_{\mathscr F_\Gamma}(v)\,\mathbb{1}_{\mathscr F_\Gamma}(\gamma w)$$

$$\frac{\mathbb{1}_{\widetilde\zeta(v)}(w)\,\widetilde f(w)\;d\mu_{W^{\rm su}(v)}(w)\;d\widetilde m_F(v)}{\int_{u\in W^{\rm su}(v)}\mathbb{1}_{\widetilde\zeta(v)}(u)\,e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))}\;d\mu_{W^{\rm su}(v)}(u)\;\|m_F\|}$$

$$=\int_{v\in T^1\widetilde M}\int_{w\in W^{\rm su}(v)}\;\sum_{\gamma\in\Gamma}\mathbb{1}_{\mathscr F_\Gamma}(\gamma^{-1}v)\,\mathbb{1}_{\mathscr F_\Gamma}(w)$$

$$\frac{\mathbb{1}_{\widetilde\zeta(v)}(w)\,\widetilde f(w)\;d\mu_{W^{\rm su}(v)}(w)\;d\widetilde m_F(v)}{\int_{u\in W^{\rm su}(v)}\mathbb{1}_{\widetilde\zeta(v)}(u)\,e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))}\;d\mu_{W^{\rm su}(v)}(u)\;\|m_F\|}$$

$$=\int_{\substack{t\,\in\,\mathbb R\\ v_-\,\in\,\partial_\infty\widetilde M\\ v_+\,\in\,\partial_\infty\widetilde M\\ v_+\,\neq\, v_-}}\int_{\substack{w_+\,\in\,\partial_\infty\widetilde M\\ w_+\,\neq\, v_-}}\;\frac{\mathbb{1}_{\mathscr F_\Gamma}(w)\,\mathbb{1}_{\widetilde\zeta(v)}(w)\,\widetilde f(w)}{\int_{u\in W^{\rm su}(v)}\mathbb{1}_{\widetilde\zeta(v)}(u)\,e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))+C_{F-\delta,\,u_+}(x,\,\pi(u))}\;d\mu_x(u_+)}$$

$$e^{C_{F-\delta,\,w_+}(x,\,\pi(w))+C_{F\circ\iota-\delta,\,v_-}(x,\,\pi(v))+C_{F-\delta,\,v_+}(x,\,\pi(v))}\;\frac{d\mu_x(w_+)d\mu^\iota_x(v_-)d\mu_x(v_+)dt}{\|m_F\|}$$

$$=\int_{\substack{t\,\in\,\mathbb R\\ v_-\,\in\,\partial_\infty\widetilde M\\ v_+\,\in\,\partial_\infty\widetilde M\\ v_+\,\neq\, v_-}}\int_{\substack{w_+\,\in\,\partial_\infty\widetilde M\\ w_+\,\neq\, v_-}}\;\frac{\mathbb{1}_{\mathscr F_\Gamma}(w)\,\mathbb{1}_{\widetilde\zeta(w)}(v)\,\widetilde f(w)\,e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(v))+C_{F-\delta,\,v_+}(x,\,\pi(v))}}{\int_{u\in W^{\rm su}(w)}\mathbb{1}_{\widetilde\zeta(w)}(u)\,e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))+C_{F-\delta,\,u_+}(x,\,\pi(u))}\;d\mu_x(u_+)}$$

$$\frac{e^{C_{F-\delta,\,w_+}(x,\,\pi(w))+C_{F\circ\iota-\delta,\,v_-}(x,\,\pi(w))}\;d\mu_x(w_+)d\mu^\iota_x(v_-)d\mu_x(v_+)dt}{\|m_F\|}$$

$$=\int_{\substack{t\,\in\,\mathbb R\\ v_-\,\in\,\partial_\infty\widetilde M\\ w_+\,\in\,\partial_\infty\widetilde M\\ w_+\,\neq\, v_-}}\;\frac{\int_{\substack{v_+\,\in\,\partial_\infty\widetilde M\\ v_+\,\neq\, w_-}}\mathbb{1}_{\widetilde\zeta(w)}(v)\,e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(v))+C_{F-\delta,\,v_+}(x,\,\pi(v))}\;d\mu_x(v_+)}{\int_{u\in W^{\rm su}(w)}\mathbb{1}_{\widetilde\zeta(w)}(u)\,e^{C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(u))+C_{F-\delta,\,u_+}(x,\,\pi(u))}\;d\mu_x(u_+)}$$

$$\mathbb{1}_{\mathscr F_\Gamma}(w)\,\widetilde f(w)\;\frac{e^{C_{F-\delta,\,w_+}(x,\,\pi(w))+C_{F\circ\iota-\delta,\,v_-}(x,\,\pi(w))}\;d\mu_x(w_+)d\mu^\iota_x(v_-)dt}{\|m_F\|}$$

$$=\int_{w\in T^1\widetilde M}\mathbb{1}_{\mathscr F_\Gamma}(w)\,\widetilde f(w)\,\frac{dm_F(w)}{\|m_F\|}=\int_{T^1M} f\,dm^F\;.$$

This is exactly the required disintegration property, and ends the proof of Lemma 6.5. □

**Step 3.** In this penultimate step, we prove that for every $\phi$-ergodic $\phi$-invariant probability measure $m$ on $T^1M$ for which the negative part of $F$ is integrable, its metric pressure $P_{\Gamma,\,F}(m)$ is at most equal to the critical exponent $\delta=\delta_{\Gamma,\,F}$, with equality if and only if the given Gibbs measure $m_F$ is finite, in which case $m$ is the normalisation of $m_F$ to a probability measure. The interplay between these two measures $m_F$ and $m$ is crucial in the following result.

**Lemma 6.6** *Let $m$ be a $\phi$-ergodic $\phi$-invariant probability measure on $T^1M$ such that $\int_{T^1M} F_-\,dm < +\infty$. Let $\tau > 0$ be such that $\phi_\tau$ is ergodic with respect to $m$, and let $\zeta$ be a partition associated with $(m,\tau)$ by Proposition 6.4.*

*(1) The real number*

$$G(v) = -\log \int_{\zeta(v)} e^{-c_F(v,\,w)} d\mu_{W^{\rm su}(v)}(w)$$

*and the measure $m^F_{\zeta(v)}$ given by Equation (92) are defined for $m$-almost every $v$ in $T^1M$, and we have, for $m$-almost every $v$ in $T^1M$,*

$$-\log m^F_{\zeta(v)}((\phi_\tau^{-1}\zeta)(v)) = \tau\delta - \int_0^\tau F(\phi_t v)\;dt\; + G(\phi_\tau v) - G(v)\;. \tag{93}$$

*(2) We have*

$$\int_{T^1M} -\log m^F_{\zeta(v)}((\phi_\tau^{-1}\zeta)(v))\;dm(v) = \tau\delta - \tau\int_{T^1M} F\;dm\;. \tag{94}$$

*(3) If $\int_{T^1M} F_-\;dm_F < +\infty$ and if $m_F$ is finite, then, with $m^F = \frac{m_F}{\|m_F\|}$, we have*

$$P_{\Gamma,\,F}(m^F) = \delta\;.$$

*(4) We have $P_{\Gamma,\,F}(m) \le \delta$, hence*

$$P(\Gamma, F) \le \delta\;.$$

*(5) If $P_{\Gamma,\,F}(m) = \delta$, then $m_F$ is finite and $m = m^F$, where $m^F = \frac{m_F}{\|m_F\|}$.*

**Proof.** We will follow the same scheme of proof as the one for $F \equiv 0$ in [OtP].

(1) Note that the measure $\mu_{W^{\rm su}(v)}$ is defined for every $v \in T^1M$. By the disintegration properties of $\widetilde{m}$ proved in Subsection 3.9, for $m$-almost every $v$, the element $v$ belongs to the support of $\mu_{W^{\rm su}(v)}$. Furthermore, $\zeta(v)$ is a relatively compact neighbourhood of $v$ in $W^{\rm su}(v)$ for $m$-almost every $v$, since $\zeta$ is $m$-subordinated to the strong unstable foliation. Hence the measure

$$dm^F_{\zeta(v)}(w) = \frac{\mathbb{1}_{\zeta(v)}(w)}{\int_{\zeta(v)} e^{-c_F(w,\,u)}\,d\mu_{W^{\rm su}(v)}(u)}\;d\mu_{W^{\rm su}(v)}(w)$$

is well defined for $m$-almost every $v$. (Note that if $m_F$ is finite and if $\zeta$ is also $m^F$-measurable and $m^F$-subordinated to $\mathscr{W}^{\rm su}$, then $m^F_{\zeta(v)}$ is, by Lemma 6.5, the conditional measure of $m^F$ on $\zeta(v)$, but we are not assuming $m_F$ to be finite in Assertion (1), (2) or (4).)

Recall that $(\phi_\tau^{-1}\zeta)(v)$ is contained in $\zeta(v)$ for $m$-almost every $v$ by Proposition 6.4, and that by definition

$$(\phi_\tau^{-1}\zeta)(v) = \phi_{-\tau}(\zeta(\phi_\tau v))\;.$$

Let $w \in \phi_\tau W^{\rm su}(v)$ and $u \in W^{\rm su}(v)$. By Equation (90), we have

$$c_F(\phi_{-\tau} w, v) = c_F(w, \phi_\tau v) + \int_0^{-\tau} \big(F(\phi_t w) - F(\phi_{t+\tau} v)\big)\, dt\ .$$

By Equation (51), we have

$$d\mu_{W^{\rm su}(v)}(\phi_{-\tau} w) = e^{\int_0^\tau (F(\phi_{s-\tau} w)-\delta)\,dt}\, d\mu_{W^{\rm su}(\phi_\tau v)}(w) = e^{\int_{-\tau}^0 (F(\phi_t w)-\delta)\,dt}\, d\mu_{W^{\rm su}(\phi_\tau v)}(w)\ .$$

By the cocycle property, we have $c_F(\phi_{-\tau} w, u) = c_F(\phi_{-\tau} w, v) + c_F(v, u)$ and $c_F(w, \phi_\tau v) = -c_F(\phi_\tau v, w)$. Hence for $m$-almost every $v$, we have

$$\begin{aligned}
& m^F_{\zeta(v)}((\phi_\tau^{-1})\zeta(v)) \\
&= \int_{\phi_{-\tau}(\zeta(\phi_\tau(v)))} \frac{d\mu_{W^{\rm su}(v)}(w')}{\int_{\zeta(v)} e^{-c_F(w',\,u)}\, d\mu_{W^{\rm su}(v)}(u)} \\
&= \int_{\zeta(\phi_\tau(v))} \frac{e^{c_F(\phi_{-\tau} w,\, v)}\, d\mu_{W^{\rm su}(v)}(\phi_{-\tau} w)}{\int_{\zeta(v)} e^{-c_F(v,u)}\, d\mu_{W^{\rm su}(v)}(u)} \\
&= \frac{\int_{\zeta(\phi_\tau(v))} e^{c_F(w,\,\phi_\tau v) + \int_0^{-\tau}(F(\phi_t w) - F(\phi_{t+\tau} v))\,dt + \int_{-\tau}^0 (F(\phi_t w)-\delta)\,dt}\, d\mu_{W^{\rm su}(\phi_\tau v)}(w)}{\int_{\zeta(v)} e^{-c_F(v,\,u)}\, d\mu_{W^{\rm su}(v)}(u)} \\
&= e^{-\tau\delta - \int_0^{-\tau} F(\phi_{t+\tau} v)\,dt\, - G(\phi_\tau v) + G(v)}\ .
\end{aligned}$$

This proves Assertion (1).

(2) We are going to use the following quite classical result that we state without proof (see for instance [OtP, Lemme 8]).

**Claim.** *Let $T$ be a measure-preserving transformation of a probability space $(X, \mathscr{A}, \mu)$, let $H : X \to \mathbb{R}$ be a measurable map such that $h = H \circ T - H$ has integrable negative part. Then $h$ is integrable and $\int_X h\, dm = 0$.*

By Assertion (1), for $m$-almost every $v \in T^1 M$, since $m^F_{\zeta(v)}$ is a probability measure, we have

$$G \circ \phi_\tau(v) - G(v) \geq -\tau\delta + \int_0^\tau F(\phi_t v)\, dt\ .$$

Since the negative part of $F$ is $m$-integrable, by Fubini's theorem and the $\phi$-invariance of $m$, the map $g = G \circ \phi_\tau - G$ hence has $m$-integrable negative part. Applying the above claim, we may integrate Equation (93) over $v \in T^1 M$ for the measure $m$. The vanishing of $\int_{T^1 M} g\, dm$ then yields Equation (94).

(3) If $m_F$ is finite, then its normalised measure $m^F$ is an ergodic probability measure on $T^1 M$ (see Corollary 5.15). Hence there exists $\tau > 0$ such that $\phi_\tau$ is ergodic for $m^F$ (see Lemma 6.3). If $\int_{T^1 M} F_-\, dm_F < +\infty$, applying Equation (94) to $m = m^F$ and $\zeta = \zeta'$ a $m^F$-measurable partition associated with $(m^F, \tau)$ by Proposition 6.4, we have, by the definition of the entropy of $\phi_\tau$ relative to $\zeta'$ and since $(\phi_\tau^{-1}\zeta')(v)$ is contained in $\zeta'(v)$ for $m^F$-almost every $v$,

$$h_{m^F}(\phi_\tau, \zeta') = \int_{v \in T^1 M} - \log m^F_{\zeta'(v)}((\phi_\tau^{-1}\zeta')(v))\, dm^F(v) = \tau\delta - \tau \int_{T^1 M} F\, dm^F\ .$$

By the definition of the entropy of $\phi$ with respect to $m^F$ and by Proposition 6.4, we have

$$h_{m^F}(\phi) = \frac{1}{\tau} h_{m^F}(\phi_\tau) = \frac{1}{\tau} h_{m^F}(\phi_\tau, \zeta') = \delta - \int_{T^1 M} F \, dm^F .$$

This gives $P_{\Gamma,\,F}(m^F) = \delta$.

(4) For $m$-almost every $v \in T^1 M$, by Assertion (1), let us define $\psi(v) \in [0, +\infty]$ by

$$\psi(v) = \frac{m^F_{\zeta(v)}\big((\phi_\tau^{-1}\zeta)(v)\big)}{m_{\zeta(v)}\big((\phi_\tau^{-1}\zeta)(v)\big)} \quad \text{if} \quad m_{\zeta(v)}\big((\phi_\tau^{-1}\zeta)(v)\big) > 0\,, \quad \text{and} \quad \psi(v) = +\infty \quad \text{otherwise}\,.$$

**Key claim. (Otal-Peigné)** *The map $\psi$ is $m$-measurable, the maps $\psi$ and $\log \psi$ are $m$-integrable, and $\int_X \psi \, dm \le 1$.*

**Proof.** The proof when $F = 0$ in the key result [OtP, Fait 9] extends immediately, by replacing $\mu_v^+$ in that reference by $\mu_{W^{\rm su}(v)}$, and by noting, to prove the measurability of $\psi$ as indicated, that $v \mapsto \mu_{W^{\rm su}(v)}$ is weak-star continuous as already mentioned, hence $v \mapsto \mu_{W^{\rm su}(v)}(A)$ is measurable for every Borel subset $A$ of $T^1M$. □

By the definition of the (metric) entropy and by Proposition 6.4, we have as above

$$\tau h_m(\phi) = h_m(\phi_\tau) = h_m(\phi_\tau, \zeta) = \int_{T^1 M} - \log m_{\zeta(v)}((\phi_\tau^{-1}\zeta)(v)) \, dm(v) .$$

Hence by Equation (94),

$$\int_{T^1 M} \log \psi \, dm = -\tau\delta + \tau \int_{T^1 M} F \, dm + \tau h_m(\phi) .$$

By Jensen's inequality and by the above Claim, we have

$$\int_{T^1 M} \log \psi \, dm \le \log \Big( \int_{T^1 M} \psi \, dm \Big) \le 0 .$$

Hence

$$P_{\Gamma,\,F}(m) = h_m(\phi) + \int_{T^1 M} F \, dm \le \delta ,$$

as required.

The last claim of Assertion (4) follows by the definition of $P(\Gamma, F)$ (with the restriction to ergodic probability measures) since $m$ is arbitrary and the existence of $\tau > 0$ such that $\phi_\tau$ is ergodic for $m$ is guaranteed by Lemma 6.3.

(5) The equality case in Jensen's inequality implies that $\psi(v) = 1$ for $m$-almost every $v \in T^1 M$. Hence the measure $m^F_{\zeta(v)}$ and the conditional measure $m_{\zeta(v)}$ coincide on the $\sigma$-algebra generated by the restriction to $\zeta(v)$ of $\phi_\tau^{-1}\zeta$ for $m$-almost every $v \in T^1M$. Similarly by replacing $\phi_\tau$ by $\phi_{k\tau}$, these measures coincide on the $\sigma$-algebra generated by the restriction to $\zeta(v)$ of $\phi_\tau^{-k}\zeta$ for $m$-almost every $v \in T^1M$. Since $\zeta$ is generating under $\phi_\tau$ (see Proposition 6.4), these measures are equal, for $m$-almost every $v \in T^1M$.

First assume that $m_F$ is finite. In particular the normalised measure $m^F$ is ergodic by Corollary 5.15. To prove that $m$ and $m^F$ coincide, let us show, using Hopf's argument, that

$m(f) = m^F(f)$ for every $f : T^1M \to \mathbb{R}$ continuous with compact support. Let $A_f$ be the set of $v \in T^1M$ such that the limit $\lim_{t\to+\infty} \frac{1}{t}\int_0^t f(\phi_s v)\, ds$ exists and is equal to $m^F(f)$. By Birkhoff's ergodic theorem, $A_f$ has full measure with respect to $m^F$. By the uniform continuity of $f$, the set $A_f$ is saturated by the stable foliation. By the quasi-product structure (see the comment after Equation (63)), for every $v \in T^1M$, the intersection $A_f \cap W^{\rm su}(v)$ has full measure for $\mu_{W^{\rm su}(v)}$. By the definition of $m^F_{\zeta(v)}$, this implies that the intersection $A_f \cap \zeta(v)$ has full measure with respect to $m^F_{\zeta(v)}$. By the above equality of measures, we have that $A_f \cap \zeta(v)$ has full measure with respect to $m_{\zeta(v)}$, for $m$-almost every $v$. By definition of the conditional measures of $m$, this implies that $A_f$ has full measure with respect to $m$. By Birkhoff's ergodic theorem applied this time to $m$, we have $m(f) = m^F(f)$, as required.

Let us prove that $m_F$ is indeed finite. By the Hopf-Tsuji-Sullivan-Roblin theorems 5.3 and 5.4, the dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m_F)$ is either completely dissipative or conservative.

Assume that the first case holds. Let $f : T^1M \to \mathbb{R}$ be Lipschitz continuous with compact support. Then for $m_F$-almost every $v \in T^1M$, we have $\lim_{t\to+\infty} f(\phi_s v)\, ds = 0$. Considering the set

$$A'_f = \{v \in T^1M \;:\; \lim_{t\to+\infty} \frac{1}{t}\int_0^t f(\phi_s v)\, ds = 0\}\;,$$

which has full measure with respect to $m_F$, we have as above that $m(f) = 0$ for every such $f : T^1M \to \mathbb{R}$. This contradicts the fact that $m$ is a probability measure.

Assume that the second case holds. Then there exists a positive Lipschitz continuous map $\rho : T^1M \to \mathbb{R}$ with $\|\rho\|_{\mathbb{L}^1(m_F)} = 1$ such that, for $m_F$-almost every $v \in T^1M$, we have

$$\int_0^\infty \rho\circ\phi_{-t}(v)dt = \int_0^\infty \rho\circ\phi_t(v)dt = +\infty$$

(see for instance the proof of Assertion (h) of Proposition 5.5). The same argument as above, this time using Hopf's ratio ergodic theorem, gives that $\frac{m(f)}{m(\rho)} = \frac{m_F(f)}{m_F(\rho)}$. This implies that $m_F$ is finite, since $m$ is a probability measure. $\square$

**Step 4.** In this last step of the proof of Theorem 6.1, we prove the converse inequality $\delta \le P(\Gamma, F)$. Assertion (4) of Lemma 6.6 then implies that $\delta = P(\Gamma, F)$. If $m_F$ is finite and $\int_{T^1M} F_-\, dm_F < +\infty$, then the assertions (3) and (5) of Lemma 6.6 imply that the normalised Gibbs measure $m^F$ is the unique equilibrium state. Otherwise, Assertion (5) of Lemma 6.6 also implies that there is no equilibrium state (since $m = m^F$ implies that the negative part of the potential $F$ is integrable with respect to $m^F$ if it is with respect to $m$). Therefore Theorem 6.1 follows from the following result.

**Lemma 6.7** *We have $\delta_{\Gamma,F} \le P(\Gamma, F)$.*

**Proof.** We start the proof by recalling an alternative definition of the topological pressure of a flow on a compact metric space.

Let $X$ be a compact metric space whose distance is denoted by $d$, let $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow of homeomorphisms of $X$, and let $G : X \to \mathbb{R}$ be a continuous map.

For all $T \geq 0$ and $\epsilon > 0$, a subset $E$ of $X$ is $(\epsilon, T)$-*separated* if for all distinct points $x$ and $y$ in $E$, there exists $t \in [0, T]$ such that $d(\varphi_t x, \varphi_t y) \geq \epsilon$. Define $P_{d,\,\epsilon,\,T,\,G}$ as the upper bound of

$$\sum_{x \in E} e^{\int_0^T G(\varphi_t x)\, dt}$$

on all $(\epsilon, T)$-separated subsets $E$ of $X$, and the *combinatorial pressure* of $(G, \varphi)$ as

$$P_X(G, \varphi) = \sup_{\epsilon > 0}\ \limsup_{T \to +\infty}\ \frac{1}{T}\ \log P_{d,\,\epsilon,\,T,\,G}\ .$$

The following Variational Principle says that the combinatorial pressure and the topological pressure coincide. We denote by $\mathscr{M}(X, \varphi)$ the weak-star compact convex set of probability measures on $X$ invariant by the flow $\varphi$.

**Theorem 6.8** *(see for instance [Wal, Th 9.10]) We have*

$$P_X(G, \varphi) = \sup_{m \in \mathscr{M}(X,\,\varphi)} \Big( h_m(\varphi) + \int_X G\, dm \Big)\ .$$

Note that this result is stated for transformations in [Wal, Th 9.10], but its proof extends to flows.

Let us resume the proof of Lemma 6.7. We may assume that $\delta = \delta_{\Gamma,\,F} > 0$. We are going to construct, using Subsection 4.4, for every $\delta'' \in \,]0, \delta[$ , a compact subset $K$ of $T^1 M$ invariant by the geodesic flow, such that the combinatorial pressure of $(F_{|K}, \phi_{|K})$ on $K$ satisfies

$$P_K(F_{|K}, \phi_{|K}) \geq \delta''\ .$$

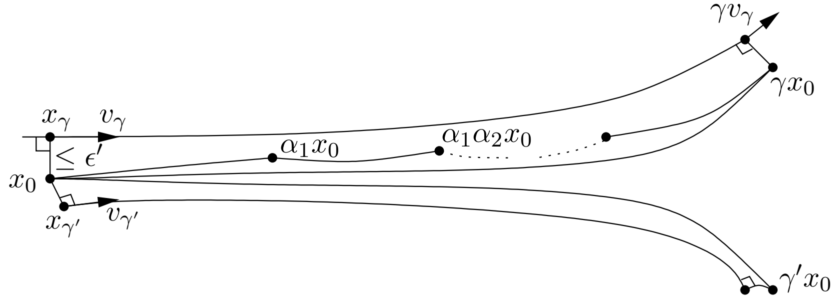
(We endow $K$ with the distance $d$ which is the restriction to $K$ of the quotient of the Riemannian distance of $T^1 \widetilde{M}$ (for the Sasaki metric).) Since $\mathscr{M}(K, \varphi)$ is contained in $\mathscr{M}_F(T^1 M)$ (by considering measures on $K$ as measures on $T^1 M$ with support in $K$), this implies that $P(\Gamma, F) \geq \delta_{\Gamma,\,F}$ as required in Lemma 6.7.

Let us now define the constants and objects we will use in the proof of Lemma 6.7. Let $\delta' \in \,]\delta'', \delta[$ and $s \in \,]\delta'', \delta'[$ . Let $L \geq 5$ (which will be chosen large enough) and $\epsilon' \in \,]0, \frac{1}{12}[$ (which will be chosen small enough). Let $\epsilon \in \,]0, \frac{1}{16}]$ (depending only on $\epsilon'$ and tending to 0 as $\epsilon' \to 0$) be given by Lemma 2.8 with $\ell = 3$. Let $\theta > 0$ (depending only on $\epsilon$) be defined by Proposition 4.9. Let $C = 6$. Let $x_0 \in \widetilde{M}$, $v_0 \in T^1_{x_0} \widetilde{M}$, $N > L$ and $S$ a finite subset of $\Gamma$ (depending on $\delta', \theta, C, L$) be given by Proposition 4.10. Let $G$ be the subsemigroup of $\Gamma$ generated by $S$.

By Proposition 4.9 (whose assumptions are satisfied by the assertions of Proposition 4.10), the semigroup $G$ is free on $S$ and for every $\gamma \in G$ of nonzero length $\ell$ as a word in the elements of $S$, we have

$$d(\gamma x_0, [x_0, \gamma^2 x_0]) \leq \epsilon \quad \text{and} \quad d(x_0, \gamma x_0) \geq (N - 2\epsilon)\ell \geq 3\ .$$

Hence by Lemma 2.8, every nontrivial element $\gamma$ of $G$ is loxodromic and if $x_\gamma$ is the closest point to $x_0$ on the translation axis $\mathrm{Axe}_\gamma$ of $\gamma$, then $d(x_0, x_\gamma) \leq \epsilon'$ (see the picture above Lemma 2.8). Let $v_\gamma$ be the unit tangent vector at $x_\gamma$ pointing towards $\gamma x_\gamma$, which is tangent to the translation axis $\mathrm{Axe}_\gamma$ (see the picture below).




Let $T\widetilde{p} : T^1\widetilde{M} \to T^1M = \Gamma\backslash T^1\widetilde{M}$ be the canonical projection. For every $\eta > 0$, denote by $\overline{\mathscr{N}}_\eta(A)$ the closed $\eta$-neighbourhood of a subset $A$ of $\widetilde{M}$. Consider the $\Gamma$-invariant subset $\widetilde{K}$ of $\widetilde{M}$ defined by

$$\widetilde{K} = \big\{ v \in T^1\widetilde{M} \;:\; \forall\, t \in \mathbb{R},\;\; \pi(\phi_t v) \in \Gamma \bigcup_{\alpha\in S} \overline{\mathscr{N}}_{\epsilon+\epsilon'}([x_0, \alpha x_0]) \;\big\} \,.$$

Its image $K = T\widetilde{p}(\widetilde{K})$, which consists in the elements of $T^1M$ whose orbit under the geodesic flow stays at distance at most $\epsilon + \epsilon'$ of $\bigcup_{\alpha\in S} T\widetilde{p}([x_0, \alpha x_0])$, is compact, since $S$ is finite. Note that $K$ (which depends on $L$ and $\epsilon'$) is invariant under the geodesic flow, by construction.

Now, to finish the proof, we are going to construct, for a fixed $\epsilon'' > 0$ small enough, an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and for every $k \in \mathbb{N}$, a $(\epsilon'', n_k)$-separated subset of $K$ with a good control of the integral of the potential on its flow lines.

For every nontrivial element $\gamma \in G$, let $\ell \in \mathbb{N} - \{0\}$ and $\alpha_1, \dots, \alpha_\ell \in S$ be such that $\gamma = \alpha_1 \dots \alpha_k$. Note that $[x_0, \gamma x_0]$ is contained in

$$\overline{\mathscr{N}}_\epsilon([x_0, \alpha_1 x_0] \cup [\alpha_1 x_0, \alpha_1\alpha_2 x_0] \cup \dots \cup [\alpha_1 \dots \alpha_{k-1} x_0, \gamma x_0])$$

by Proposition 4.9.

We hence have, for every $t \in \mathbb{R}$,

$$\pi(\phi_t v_\gamma) \in \mathrm{Axe}_\gamma = \gamma^{\mathbb{Z}}[x_\gamma, \gamma x_\gamma] \subset \Gamma\, \overline{\mathscr{N}}_{\epsilon'}([x_0, \gamma x_0]) \subset \Gamma \bigcup_{\alpha\in S} \overline{\mathscr{N}}_{\epsilon+\epsilon'}([x_0, \alpha x_0]) \,.$$

This proves that $v_\gamma$ belongs to $\widetilde{K}$ for every nontrivial $\gamma$ in $G$.

If $\epsilon'$ (hence $\epsilon$) is small enough and $L$ is large enough, the proof of Theorem 4.11 says that the critical exponent $\delta_{G,\,F}$ of the subsemigroup $G$ is at least $s$. In particular, since $\delta'' < s$ and by the definition of $\delta_{G,\,F}$, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that for every $k \in \mathbb{N}$, we have

$$\sum_{\gamma\in G,\; n_k-1<d(x_0,\gamma x_0)\leq n_k} e^{\int_{x_0}^{\gamma x_0} \widetilde{F}} \geq e^{\delta''\, n_k} \,. \tag{95}$$

Let us define

$$A_k = \{\gamma \in G,\;\, n_k - 1 < d(x_0, \gamma x_0) \leq n_k\} \,.$$

Furthermore, if $\epsilon'$ (hence $\epsilon$) is small enough, by Lemma 3.2, there exists $c \geq 0$ (depending only on $\epsilon$, on the Hölder constants of $\widetilde{F}$ and on $\max_{\pi^{-1}(B(x_0,\,1))}\; |\widetilde{F}|$) such that for every nontrivial element $\gamma \in G$,

$$\Big| \int_{x_\gamma}^{\gamma x_\gamma} \widetilde{F} \;-\; \int_{x_0}^{\gamma x_0} \widetilde{F} \;\Big| \leq c \,. \tag{96}$$

We will need the following two lemmas.

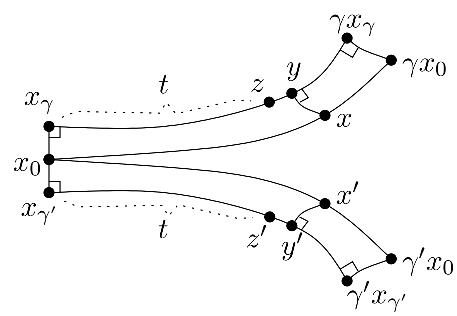
**Lemma 6.9** *For every $k \in \mathbb{N}$, for all distinct $\gamma, \gamma'$ in $A_k$, the element $\gamma$ is not an initial subword of $\gamma'$.*

**Proof.** Assume for a contradiction that $\gamma \in G$ is an initial subword of $\gamma' \in G$, different from $\gamma'$, and that the distances from $\gamma x_0$ and $\gamma' x_0$ to $x_0$ belong to $]n_k - 1, n_k]$. Let $\gamma_1$ be a nontrivial word in $S$ such that $\gamma' = \gamma\gamma_1$. By Proposition 4.9, the point $\gamma x_0$ is at distance at most $\epsilon$ from the geodesic segment $[x_0, \gamma' x_0]$. Hence by the triangle inequality

$$\begin{aligned} d(x_0, \gamma_1 x_0) = d(\gamma x_0, \gamma\gamma_1 x_0) &\leq d(x_0, \gamma' x_0) - d(x_0, \gamma x_0) + 2\,\epsilon \\ &\leq n_k - (n_k - 1) + 2\,\epsilon = 1 + 2\,\epsilon\,. \end{aligned}$$

By Proposition 4.9, since the word-length $\ell$ of $\gamma_1$ is at least 1, we have $d(x_0, \gamma_1 x_0) \geq N - 2\epsilon$. Hence $L < N \leq 1 + 4\epsilon \leq 5$, a contradiction. $\square$

**Lemma 6.10** *For every $\epsilon'' \in \,]0, \frac{1}{2} - 6\epsilon'[\,$, for all $k \in \mathbb{N}$ and $\gamma \neq \gamma'$ in $A_k$, there exists $t \in [0, n_k]$ such that $d(\pi(\phi_t v_\gamma), \pi(\phi_t v_{\gamma'})) \geq \epsilon''$.*




**Proof.** Since $\epsilon \leq \frac{1}{16}$, by the last claim of Proposition 4.9 and by the previous Lemma 6.9, there exists $x \in [x_0, \gamma x_0]$ and $x' \in [x_0, \gamma' x_0]$ such that $d(x_0, x) = d(x_0, x')$ and $d(x, x') \geq \frac{1}{2}$. Let $y$ and $y'$ be the closest points to $x$ and $x'$ on $[x_\gamma, \gamma x_\gamma]$ and $[x_{\gamma'}, \gamma' x_{\gamma'}]$ respectively, which satisfy $d(x, y), d(x', y') \leq \epsilon'$ by convexity. Let $t = d(x_0, x) - 2\epsilon'$ and $z = \pi(\phi_t v_\gamma)$ and $z' = \pi(\phi_t v_{\gamma'})$.

We have $t \geq 0$ since otherwise $\frac{1}{2} \leq d(x, x') \leq d(x, x_0) + d(x_0, x') \leq 4\,\epsilon'$, which contradicts the fact that $\epsilon' \leq \frac{1}{12}$. We have $t \leq d(x_0, x) \leq d(x_0, \gamma x_0) \leq n_k$.

By the triangle inequality, and since closest point maps do not increase the distances, we have

$$d(x_\gamma, z) = t = d(x_0, x) - 2\epsilon' \leq d(x_\gamma, y) \leq d(x_0, x)\,.$$

Hence

$$\begin{aligned} d(z, x) &\leq d(z, y) + d(y, x) = (d(x_\gamma, y) - d(x_\gamma, z)) + d(y, x) \\ &\leq d(x_0, x) - (d(x_0, x) - 2\epsilon') + \epsilon' = 3\epsilon'\,. \end{aligned}$$

Similarly, $d(z', x') \leq 3\epsilon'$, hence by the triangle inequality

$$d(z, z') \geq d(x, x') - d(x, z) - d(x', z') \geq \frac{1}{2} - 6\epsilon' \geq \epsilon''\,. \quad \square$$

Since $d_S(\phi_t v_\gamma, \phi_t v_{\gamma'}) \geq d(\pi(\phi_t v_\gamma), \pi(\phi_t v_{\gamma'}))$ by Equation (4), this lemma implies that the set $\widetilde{E}_k = \{v_\gamma \;:\; \gamma \in A_k\}$ is $(\epsilon'', n_k)$-separated.

Let $C_K = \inf_{\widetilde{K}} \widetilde{F}$, which is finite since $F$ is continuous on the compact subset $K$. With $\ell(\gamma) = d(x_\gamma, \gamma x_\gamma)$ the translation length of $\gamma \in A_k$, we have

$$n_k - 1 - 2\epsilon' \leq d(x_0, \gamma x_0) - 2\epsilon' \leq \ell(\gamma) \leq d(x_0, \gamma x_0) \leq n_k\,.$$

Since $v_\gamma \in \widetilde{K}$ and $\widetilde{K}$ is invariant under the geodesic flow, we hence have, for all $k \in \mathbb{N}$ and $\gamma \in A_k$,

$$\int_{\ell(\gamma)}^{n_k} \widetilde{F}(\phi_t v_\gamma)\, dt \geq C_K(1+2\epsilon')\,.$$

Since $K$ is compact and since the isometric action of $\Gamma$ on $\widetilde{M}$ is proper, there exists $\epsilon'' \in \,]0, \frac{1}{2} - 6\epsilon'[$ (this interval is nonempty since $\epsilon' < \frac{1}{12}$) and $N_K > 0$ (when $\Gamma$ is torsion free, we may take $N_K = 1$ and $\epsilon'' > 0$ less than the lower bound on $\pi(K)$ of the injectivity radius of $M$) such that $\widetilde{E}_k$ may be subdivided in at most $N_K$ subsets that the map $T\widetilde{p} : T^1\widetilde{M} \to T^1M$ inject into $(\epsilon'', n_k)$-separated subsets. For at least one of these sets, let us call it $E_k$, we have, by Equations (96) and (95)

$$\begin{aligned}\sum_{v\in E_k} e^{\int_0^{n_k} F(\phi_t v)\, dt} &\geq \frac{1}{N_K}\sum_{\gamma\in A_k} e^{\int_0^{n_k} \widetilde{F}(\phi_t v_\gamma)\, dt} \geq \frac{e^{C_K(1+2\epsilon')}}{N_K}\sum_{\gamma\in A_k} e^{\int_{x_\gamma}^{\gamma x_\gamma} \widetilde{F}} \\ &\geq \frac{e^{C_K(1+2\epsilon')-c}}{N_K}\sum_{\gamma\in A_k} e^{\int_{x_0}^{\gamma x_0} \widetilde{F}} \geq \frac{e^{C_K(1+2\epsilon')-c}}{N_K}\, e^{\delta''\, n_k}\,.\end{aligned}$$

Taking the logarithm, dividing by $n_k$ and letting $k \to +\infty$, we have

$$P_K(F_{|K}, \phi_{|K}) \geq \delta''\,.$$

This is what we wanted to prove. $\square$

# 7 The Liouville measure as a Gibbs measure

Let $(\widetilde{M}, \Gamma)$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$ and $\Gamma$ is a non-elementary discrete group of isometries of $\widetilde{M}$.

The aim of this chapter is to investigate when the Liouville measure of the geodesic flow on $T^1M = \Gamma\backslash T^1\widetilde{M}$ coincides (up to a multiplicative constant) with a Gibbs measure associated to some Hölder potential. This result is known when $M$ is compact, but we provide a complete short proof in Subsection 7.3. Finding a necessary and sufficient condition in the noncompact case is still open. The approach that we follow in Subsection 7.4 actually allows us to get more than what was previously known on the subject. In particular, we obtain that if $\widetilde{M}$ admits a discrete and cocompact group of isometries, then the Liouville measure on $T^1M$ is a Gibbs measure as soon as it is conservative (see Theorem 7.2 and its following comment), but not necessarily finite.

Recall that the *Liouville measure* $\mathrm{vol}_{T^1\widetilde{M}}$ on $T^1\widetilde{M}$ is the Riemannian volume of the Sasaki metric on $T^1\widetilde{M}$ (see Subsection 2.3 for a definition). Equivalently, it disintegrates with respect to the projection $\pi : T^1\widetilde{M} \to \widetilde{M}$, with measure on the basis $\widetilde{M}$ the Riemannian volume $\mathrm{vol}_{\widetilde{M}}$ of $\widetilde{M}$ and conditional measures on the fibres $T^1_x\widetilde{M}$ the Riemannian spherical measure $\mathrm{vol}_{T^1_x\widetilde{M}}$ of the unit sphere of the Euclidean space $T_x\widetilde{M}$: for every $f \in \mathscr{C}_c(T^1\widetilde{M};\mathbb{R})$, we have

$$\int_{v\in T^1\widetilde{M}} f(v)\, d\,\mathrm{vol}_{T^1\widetilde{M}}(v) = \int_{x\in\widetilde{M}}\int_{v\in T^1_x\widetilde{M}} f(v)\, d\,\mathrm{vol}_{T^1_x\widetilde{M}}(v)\, d\,\mathrm{vol}_{\widetilde{M}}(x)\,.$$

It is invariant under the action of the geodesic flow (and the action of $\mathbb{Z}/2\mathbb{Z}$ by time reversal). It induces (see Subsection 2.6) a measure $\operatorname{vol}_{T^1M}$ on the Riemannian orbifold $T^1M = \Gamma\backslash T^1\widetilde{M}$, called the *Liouville measure* on $T^1M$, invariant under the geodesic flow (and time reversal), which disintegrates similarly over the Riemannian orbifold $M = \Gamma\backslash\widetilde{M}$ with fibres the Riemannian orbifolds $T^1_xM = \Gamma_{\widetilde{x}}\backslash T^1_{\widetilde{x}}\widetilde{M}$ where $\widetilde{x}$ is a lift of $x$, whose stabiliser in $\Gamma$ is denoted by $\Gamma_{\widetilde{x}}$:

$$d\operatorname{vol}_{T^1M}(v) = \int_{x\in M} d\operatorname{vol}_{T^1_xM}(v)\; d\operatorname{vol}_M(x)\;.$$

In particular, the Liouville measure $\operatorname{vol}_{T^1M}$ is finite if and only if $M$ has finite volume: since the fixed point set of a Riemannian isometry different from id of a connected Riemannian manifold has Riemannian measure 0, if $M$ has dimension $m$, then

$$\operatorname{Vol}(T^1M) = \operatorname{Vol}(\mathbb{S}^{m-1})\;\operatorname{Vol}(M)\;.$$

It can happen that $\operatorname{vol}_{T^1M}$ is infinite and ergodic, see for example [Ree], where she proves that the Liouville measure on a Galois cover with covering group $\mathbb{Z}^d$ of a compact hyperbolic manifold is ergodic with respect to the geodesic flow if and only if $d \leq 2$.

Let us now define the potential on $T^1\widetilde{M}$ with respect to which the Liouville measure is a Gibbs measure in all known cases. Let $v \in T^1\widetilde{M}$ and $t \in \mathbb{R}$. Denote by $E^{\rm su}(v)$ (respectively $E^{\rm ss}(v)$) the tangent space $T_vW^{\rm su}(v)$ at $v$ to $W^{\rm su}(v)$ (respectively $T_vW^{\rm ss}(v)$ at $v$ to $W^{\rm ss}(v)$) and let $E^0(v) = \mathbb{R}\frac{d}{dt}_{|t=0}(\phi_tv)$ be the tangent space at $v$ of its geodesic flow line. We have

$$T_vT^1\widetilde{M} = E^{\rm su}(v)\;\oplus\;E^0(v)\;\oplus\;E^{\rm ss}(v)\;.$$

Let us also define

$$E^{\rm u}(v) = E^{\rm su}(v)\;\oplus\;E^0(v)\quad\text{and}\quad E^{\rm s}(v) = E^0(v)\;\oplus\;E^{\rm ss}(v)\;.$$

For every $v \in T^1M$, we again denote by $E^{\rm su}(v)$, $E^0(v)$, $E^{\rm ss}(v)$ the images of $E^{\rm su}(\widetilde{v})$, $E^0(\widetilde{v})$, $E^{\rm ss}(\widetilde{v})$, respectively, by the canonical map $TT^1\widetilde{M} \to TT^1M = \Gamma\backslash TT^1\widetilde{M}$, where $\widetilde{v}$ is any lift of $v$ by the canonical projection $T^1\widetilde{M} \to T^1M$.

Recall that the *Jacobian* of a smooth map $f : N \to N'$ between two Riemannian manifolds is the map $\operatorname{Jac} f : N \to [0,+\infty[$ such that $\operatorname{Jac} f(x)$ is the absolute value of the determinant (of the matrix in any orthonormal bases) of the linear map $T_xf$ between the two Euclidean spaces $T_xN$ and $T_{f(x)}N'$, for every $x \in N$. For all $t \in \mathbb{R}$ and $v \in T^1\widetilde{M}$, the map $\phi_t$ on $T^1\widetilde{M}$ sends the smooth submanifold $W^{\rm su}(v)$ to the smooth submanifold $W^{\rm su}(\phi_tv)$. We endow each leaf of $\mathscr{W}^{\rm su}$ with the Riemannian metric induced by the Sasaki metric on $T^1\widetilde{M}$. We can therefore define the Jacobian of the restriction of $\phi_t$ to the smooth submanifold $W^{\rm su}(v)$ of $T^1\widetilde{M}$ at the point $v$, and we denote it by $\widetilde{J}^{\rm su}(v,t) > 0$. It is smooth in $t$ and continuous in $(v,t)$ (and studying its regularity in $v$ will be one of the main points below). The flow property of the geodesic flow and the invariance of the strong unstable foliation under the geodesic flow give the following cocycle relation: for all $s,t \in \mathbb{R}$ and $v \in T^1\widetilde{M}$,

$$\widetilde{J}^{\rm su}(v,0) = 1\quad\text{and}\quad\widetilde{J}^{\rm su}(v,s+t) = \widetilde{J}^{\rm su}(\phi_tv,s)\;\widetilde{J}^{\rm su}(v,t)\;. \tag{97}$$

Now, we define the following map

$$\widetilde{F}^{\rm su}(v) = -\frac{d}{dt}_{|t=0}\log\widetilde{J}^{\rm su}(v,t) = \frac{d}{dt}_{|t=0}\log\widetilde{J}^{\rm su}(\phi_tv,-t)\,. \tag{98}$$

The maps $v \mapsto \widetilde{J}^{\,\rm su}(v,t)$ and $v \mapsto \widetilde{F}^{\,\rm su}(v)$, being $\Gamma$-invariant, define maps $v \mapsto J^{\rm su}(v,t)$ and $v \mapsto F^{\,\rm su}(v)$ on $T^1M$ (which, when $\Gamma$ is torsion free, may be defined directly on $T^1M$). We have $\widetilde{F}^{\,\rm su} \leq 0$ by the dilation properties of the geodesic flow on the unstable foliation. An estimation of the gap map $D_{F^{\rm su}} : \widetilde{M} \times \partial^2_\infty \widetilde{M} \to \mathbb{R}$ (see Subsection 3.4) of the potential $F^{\,\rm su}$ would be interesting.

Note that $\widetilde{J}^{\,\rm su}(v,t) = e^{(n-1)t}$ for all $v \in T^1\widetilde{M}$ and $t \in \mathbb{R}$ if $M$ has dimension $n$ and constant sectional curvature $-1$, so that $\widetilde{F}^{\,\rm su}$ is then the constant map with value $-(n-1)$.

We will prove in this chapter the following results.

**Theorem 7.1** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded. Then $\widetilde{F}^{\rm su}$ is Hölder-continuous, bounded, reversible and invariant under the full isometry group of $\widetilde{M}$ .*

Note that the new hypothesis of having bounded derivatives of sectional curvatures is satisfied if there exists a compact set in $\widetilde{M}$ whose images by the group $\operatorname{Isom}(\widetilde{M})$ cover $\widetilde{M}$, for instance if there exists a cocompact discrete group of isometries of $\widetilde{M}$.

Since $\widetilde{F}^{\,\rm su}$ is bounded under the hypotheses of Theorem 7.1, the critical exponent $\delta_{\Gamma,\,F^{\rm su}}$ is finite (see Lemma 3.3 (iv)). We will then denote by $\widetilde{m}_{F^{\rm su}}$ and $m_{F^{\rm su}}$ the Gibbs measures on $T^1\widetilde{M}$ and $T^1M$ associated with a fixed pair of Patterson densities of dimension $\delta_{\Gamma,\,F^{\rm su}}$ for $(\Gamma, F^{\,\rm su}\circ\iota)$ and $(\Gamma, F^{\,\rm su})$.

**Theorem 7.2** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded.*

*If the geodesic flow of $M$ is conservative with respect to the Liouville measure and if $\delta_{\Gamma,\,F^{\rm su}} \leq 0$, then the Liouville measure is proportional to the Gibbs measure $m_{F^{\rm su}}$ of the potential $F^{\rm su}$. Furthermore, $(\Gamma, F^{\,\rm su})$ is of divergence type and*

$$P(\Gamma, F^{\,\rm su}) = \delta_{\Gamma,\,F^{\rm su}} = P_{Gur}(\Gamma, F^{\,\rm su}) = 0\ .$$

The results of the article of Bowen-Ruelle [BoR] for Axiom A flows imply, in the case of the geodesic flow on the unit tangent bundle of a compact negatively curved Riemannian manifold, that its Liouville measure coincides with a multiple of its Gibbs measure $m_{F^{\rm su}}$. See in particular Remark 5.2 in loc. cit., and Subsection 7.3.

**Remarks.** (1) The inequality $\delta_{\Gamma,\,F^{\rm su}} \leq 0$ is satisfied when $M$ is compact, by Ruelle's inequality (see Equation (102) below, using the fact that $\delta_{\Gamma,\,F^{\rm su}} = P(\Gamma, F^{\rm su})$ by the Variational Principle theorem 6.1 (1)). Hence if $\widetilde{M}$ admits a discrete and cocompact group of isometries, then $\delta_{\Gamma,\,F^{\rm su}} \leq 0$, by Lemma 3.3 (viii) and the fact that $\widetilde{F}^{\,\rm su}$ is invariant under any isometry group of $\widetilde{M}$.

It would be interesting to know whether this inequality remains true for any non-elementary discrete group of isometries $\Gamma$ of $\widetilde{M}$.

(2) The assumptions that the derivatives of the sectional curvature are uniformly bounded and that $\delta_{\Gamma,\,F^{\rm su}} \leq 0$ are satisfied if $M$ is a Riemannian cover of a compact manifold, by the previous remark. Hence Theorem 1.2 in the introduction follows from Theorem 7.2.

## 7.1 The Hölder-continuity of the (un)stable Jacobian

We will prove in this subsection the following Hölder regularity result stated in Theorem 7.3. As we make no compactness assumption, the choice of distances is important.

As explained in Subsection 2.4, we endow $T\widetilde{M}$ and then $TT\widetilde{M}$ with their Sasaki Riemannian metric, and $T^1\widetilde{M}$ and then $TT^1\widetilde{M}$ with their induced Riemannian metric. We denote by $\|\cdot\|$ their Riemannian norms. Note that the distance on $T^1\widetilde{M}$ defined by Equation (5) defines the same Hölder structure as the Riemannian distance, by Lemma 2.3.

Given a Euclidean space $E$ with finite dimension $n$ and $k \in \{1, \cdots, n-1\}$, we endow the *Grassmannian manifold* $\mathrm{Gr}_k(E)$ of $k$-dimensional vector subspaces of $E$ with the distance which is the Hausdorff distance between their unit spheres

$$d_E(A, A') = \max_{w\in A,\ w'\in A',\ \|w\|=\|w'\|=1} \left\{ \min_{v'\in A',\ \|v'\|=1} \|w - v'\|,\ \min_{v\in A,\ \|v\|=1} \|w' - v\| \right\}.$$

Let $m$ be the dimension of $\widetilde{M}$. Let $E \to T^1\widetilde{M}$ be a vector bundle over $T^1\widetilde{M}$ of rank $r \geq 1$, endowed with a Riemannian bundle metric and with a length distance inducing the Euclidean distance on each fibre. We will be interested in two such Riemannian vector bundles, the tangent bundle of $T^1\widetilde{M}$ endowed with the Sasaki metric and its Riemannian distance, and the fibre bundle over $T^1\widetilde{M}$ whose fibre over $v \in T^1\widetilde{M}$ is the product Euclidean space $T_vT^1\widetilde{M} \times T_{\phi_1 v}T^1\widetilde{M}$, with the distance induced by the Riemannian distance on $TT^1\widetilde{M} \times TT^1\widetilde{M}$. For $k \in \mathbb{N}$, denote by $\mathrm{Gr}_k(E) \to T^1\widetilde{M}$ the *Grassmannian bundle* of rank $k$ of $E \to T^1\widetilde{M}$, whose fibre over $v \in T^1\widetilde{M}$ is the Grassmannian manifold $\mathrm{Gr}_k(E_v)$ of $k$-dimensional vector subspaces of the Tiber $E_v$ of $E \to T^1\widetilde{M}$ over $v$. For instance, $v \mapsto E^{\mathrm{ss}}(v)$ and $v \mapsto E^{\mathrm{su}}(v)$ are continuous sections of $\mathrm{Gr}_{m-1}(TT^1\widetilde{M}) \to T^1\widetilde{M}$. Upon identifying a linear map and its graph, the map $v \mapsto T_v\phi_1$ is a continuous section of the Grassmannian bundle of rank $2m-1$ of the second Riemannian vector bundle above. Since the unit spheres of the elements of $\mathrm{Gr}_k(E)$ are nonempty compact subsets of the metric space $E$ if $1 \leq k \leq r$, we endow $\mathrm{Gr}_k(E)$ with the distance $d$, where $d(A, A')$ is the Hausdorff distance between the unit spheres of the $k$-dimensional Euclidean spaces $A, A' \in \mathrm{Gr}_k(E)$. The restriction of $d$ to each fibre $\mathrm{Gr}_k(E_v)$ is the distance $d_{E_v}$ defined above.

**Theorem 7.3** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded. Then the maps $v \mapsto E^{\mathrm{su}}(v)$ and $v \mapsto E^{\mathrm{ss}}(v)$ defined on $T^1\widetilde{M}$ are (globally) Hölder-continuous.*

The assumption that the derivatives of the sectional curvature are uniformly bounded is necessary: Ballmann-Brin-Burns [BaBB] have constructed a finite volume complete Riemannian surface with pinched negative sectional curvature whose strong stable foliation is not Hölder-continuous.

Since the distributions are Lipschitz (uniformly smooth) in the direction of the flow lines, note that the maps $v \mapsto E^{\mathrm{u}}(v)$, $v \mapsto E^{\mathrm{s}}(v)$ are also (uniformly locally) Hölder-continuous, and that the regularity of $\widetilde{F}^{\mathrm{su}}$ claimed in Theorem 7.1 holds. The reversibility of the potential $F^{\mathrm{su}}$ follows from the fact that the Liouville measure is invariant under the geodesic flow. The boundedness of $\widetilde{F}^{\mathrm{su}}$ follows from the bounds on the sectional curvature and its tangent map.

We will only prove the result for $v \mapsto E^{\rm ss}(v)$, giving furthermore an explicit estimate on the Hölder exponent. The strong unstable case is similar, by time reversal.

Let us give a few historical comments on Theorem 7.3. When $\widetilde{M}$ has dimension 2 and has a cocompact discrete isometry group, the strong stable and unstable foliations are $\mathrm{C}^1$, hence Hölder regular by compactness (see [Hop2]). Furthermore, Hurder and Katok [HuK, Theo. 3.1, Coro. 3.5] have proved that these sub-bundles are $\mathrm{C}^{1,\alpha}$ for every $\alpha \in \,]0,1[$ (see also [HiP]), and that if they are $\mathrm{C}^{1,1}$, then they are $\mathrm{C}^\infty$, in which case by a theorem of Ghys [Ghy, p. 267], the geodesic flow is $\mathrm{C}^\infty$-conjugated to the geodesic flow of a hyperbolic surface.

When $\widetilde{M}$ has dimension at least 3 and has a cocompact discrete isometry group, this result of Hölder regularity of the strong stable and unstable foliations is due to Anosov [Ano]. We follow Brin's and Brin-Stuck's proofs in [Bri], which modernise Anosov's original proof under the same assumptions, and [BrS, §6.1] in the case of Anosov diffeomorphism (not flows) on compact manifolds. Here, we provide details to explain why this proof works under the assumptions of Theorem 7.3.

**Proof of Theorem 7.3.** Recall that the geodesic flow of $\widetilde{M}$ is an *Anosov flow* for the Sasaki metric: there exist $c > 0$ and $\lambda \in \,]0,1[$ (and one may take $\lambda = e^{-1}$ since the sectional curvature of $\widetilde{M}$ is normalised to have upper bound $-1$, by standard arguments of Jacobi fields) such that for all $v \in T^1\widetilde{M}$, $V^{\rm ss} \in E^{\rm ss}(v)$, $V^{\rm su} \in E^{\rm su}(v)$ and $t \geq 0$, we have

$$\|T\phi_t(V^{\rm ss})\| \leq c\,\lambda^t\,\|V^{\rm ss}\| \quad\text{and}\quad \|T\phi_{-t}(V^{\rm su})\| \leq c\,\lambda^t\,\|V^{\rm su}\|\;,$$

and for all $v \in T^1\widetilde{M}$, $V^{\rm s} \in E^{\rm s}(v)$, $V^{\rm u} \in E^{\rm u}(v)$ and $t \geq 0$, we have

$$\|T\phi_{-t}(V^{\rm s})\| \geq c^{-1}\|V^{\rm s}\| \quad\text{and}\quad \|T\phi_t(V^{\rm u})\| \geq c^{-1}\|V^{\rm u}\|\;.$$

We start by giving two lemmas, the second one, that we state without proof, involving only linear algebra.

**Lemma 7.4** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Then the angles between $E^{\rm ss}(v)$ and any one of $E^{\rm su}(v)$, $E^0(v)$ or $E^{\rm u}(v)$ are uniformly bounded from below by a positive constant.*

**Proof.** We prove that the angle between $E^{\rm ss}(v)$ and $E^{\rm u}(v)$ is bounded from below by a positive constant, the other cases are analogous.

Using the operator norm on linear maps, let $C = \sup_{v\in T^1\widetilde{M}} \|T_v\phi_1\|$, which is finite since $M$ has pinched negative curvature (see for instance [Bal, page 65]). Let $c > 0$ and $\lambda \in \,]0,1[$ be as above. Let $N \in \mathbb{N}$ be large enough so that $c^{-1} \geq 2c\lambda^N$. For all $v \in T^1\widetilde{M}$, $V^{\rm ss} \in E^{\rm ss}(v)$ and $V^{\rm u} \in E^{\rm u}(v)$ with $\|V^{\rm ss}\| = \|V^{\rm u}\| = 1$, we have

$$\begin{aligned}C^N\|V^{\rm u} - V^{\rm ss}\| &\geq \|T\phi_N(V^{\rm u} - V^{\rm ss})\| \geq \|T\phi_N(V^{\rm u})\| - \|T\phi_N(V^{\rm ss})\|\\ &\geq c^{-1}\|V^{\rm u}\| - c\lambda^N\|V^{\rm ss}\| \geq c\lambda^N\;.\end{aligned}$$

The result follows. $\Box$

**Lemma 7.5** *(Brin-Stuck [BrS, Lem. 6.1.1]) Let $c > 0$, $\lambda \in \,]0,1[$, $\delta \in [0,1[$ and $D \geq 1$, let $E$ be a finite dimensional Euclidean vector space, let $A^1, A^2$ be two vector subspaces of $E$ of* *the same dimension, and let* $(L^1_n)_{n\in\mathbb{N}}, (L^2_n)_{n\in\mathbb{N}}$ *be two sequences of linear endomorphisms of* $E$ *such that, for every* $n \in \mathbb{N}$*, for* $i = 1, 2$*, for all* $v \in A^i$ *and* $w \in (A^i)^\perp$*, we have*

$$\|L^1_n - L^2_n\| \le \delta D^n, \quad \|L^i_n(v)\| \le c\,\lambda^n\,\|v\|, \quad \text{and} \quad \|L^i_n(w)\| \ge c^{-1}\|w\|\ .$$

*Then* $d_E(A^1, A^2) \le 3\,\frac{c^2}{\lambda}\,\delta^{\,\frac{-\log\lambda}{\log D - \log\lambda}}$.

Up to replacing the Sasaki metric by the equivalent (by Lemma 7.4) Riemannian metric whose norm is

$$\|V\|' = \sqrt{\|\,V^{\rm su}\,\|^2 + \|\,V^0\,\|^2 + \|\,V^{\rm ss}\,\|^2} \tag{99}$$

for every $V = V^{\rm su} + V^0 + V^{\rm ss}$ where $V^{\rm su} \in E^{\rm su}(v)$, $V^0 \in E^0(v)$ and $V^{\rm ss} \in E^{\rm ss}(v)$, we assume that the subspaces $E^{\rm su}(v), E^0(v)$ and $E^{\rm ss}(v)$ are pairwise orthogonal for every $v \in T^1\widetilde{M}$. For all $v, w \in T^1\widetilde{M}$ and $W \in T_wT^1M$, we denote by $\|^v_w\, W$ the parallel transport of $W$ along any fixed length-minimising geodesic segment from $w$ to $v$. Note that the Sasaki metric on $T^1\widetilde{M}$ has a positive lower bound on its injectivity radius at all points, and this remains true with the modified metric. In particular, there exists $\kappa > 0$ such that for all $v, w \in T^1\widetilde{M}$, if $d(v, w) < \kappa$, there exists a unique geodesic segment from $w$ to $v$ (recall that $T^1\widetilde{M}$ is not nonpositively curved), and we will use the parallel transports along these geodesic. Let $D \ge 4$ be a constant such that, for all $v, w \in T^1\widetilde{M}$,

$$d(\phi_1 v, \phi_1 w) \le \sqrt{D}\, d(v, w), \quad \|T_v\phi_1\| \le \sqrt{D}$$
$$\text{and} \quad \big\|\, T_v\phi_1 - \|^{\phi_1 v}_{\phi_1 w} \circ T_w\phi_1 \circ \|^w_v\, \big\| \le \sqrt{D}\, d(v, w)\ ,$$

which exists since $v \mapsto T_v\phi_1$ is bounded and Lipschitz, since the derivatives of the sectional curvature are uniformly bounded (see for instance [Bal, page 64]). For all $v, w \in T^1\widetilde{M}$, let

$$a_n(v) = \|T_v\phi_n\| \quad \text{and} \quad b_n(v, w) = \big\|\, T_v\phi_n - \|^{\phi_n v}_{\phi_n w} \circ T_w\phi_n \circ \|^w_v\, \big\|\ .$$

Then $a_{n+1}(v) = \|T_{\phi_n v}\,\phi_1 \circ T_v\phi_n\| \le a_1(\phi_n v)\, a_n(v)$ and

$$\begin{aligned} b_{n+1}(v, w) =& \|T_{\phi_n v}\,\phi_1 \circ T_v\phi_n - \|^{\phi_1\phi_n v}_{\phi_1\phi_n w} \circ T_{\phi_n w}\phi_1 \circ \|^{\phi_n w}_{\phi_n v}\, \big(\, \|^{\phi_n v}_{\phi_n w} \circ T_w\phi_n \circ \|^w_v\,\big)\| \\ &\le a_1(\phi_n v)\, b_n(v, w) + b_1(\phi_n v, \phi_n w)\, a_n(w)\ . \end{aligned}$$

Hence by induction and a geometric series argument, $a_n(v) \le D^{\frac{n}{2}}$ and

$$b_n(v, w) \le D^{\frac{n}{2}} \sum_{k=0}^{n-1} d(\phi_k v, \phi_k w) \le D^{\frac{n}{2}}\, \frac{D^{\frac{n}{2}} - 1}{\sqrt{D} - 1}\, d(v, w) \le D^n\, d(v, w)\ .$$

Now, let $v, w \in T^1\widetilde{M}$ be such that $d(v, w) < \kappa$. For every $n \in \mathbb{N}$, let $f_{v,n} : T_{\phi_n v}T^1\widetilde{M} \to T_vT^1\widetilde{M}$ be an isometric map. We are going to apply Lemma 7.5 with $c, \lambda$ given by Anosov's flow property, with $D$ as above, with $\delta = d(v, w)$, with $E$ the Euclidean space $T_vT^1\widetilde{M}$, and with

$$A^1 = E^{\rm ss}(v), \quad A^2 = \|^v_w\, E^{\rm ss}(w), \quad L^1_n = f_{v,n} \circ T_v\phi_n, \quad L^2_n = f_{v,n} \circ \|^{\phi_n v}_{\phi_n w} \circ T_w\phi_n \circ \|^w_v\ ,$$

for every $n \in \mathbb{N}$. The hypotheses of Lemma 7.5 are easily checked (since $(A^1)^\perp = E^{\rm u}(v)$ and the parallel transport is isometric), and we hence have

$$d(E^{\rm ss}(v), \|^v_w\, E^{\rm ss}(w)) \le 3\, \frac{c^2}{\lambda}\, d(v, w)^{\frac{-\log\lambda}{\log D - \log\lambda}}\ .$$

Since $d(\,\|^v_w\, E^{\rm ss}(w), E^{\rm ss}(w)) \le d(v, w)$ by Equation (3), the result hence follows by the triangle inequality. □

## 7.2 Absolute continuity of the strong unstable foliation

In order to prove Theorem 7.2, it will be useful to write the Liouville measure as the local product (up to a density) of the Lebesgue measures on the unstable manifolds and the Lebesgue measures on the strong stable manifolds. This is due to Anosov and Sinai in the compact case, but we follow closely Brin's proof [Bri], adapting it to the noncompact case.

Let us first recall some definitions. Let $N$ be a smooth Riemannian manifold, and let $\mathscr{F}$ be a continuous foliation of $N$ with smooth leaves. We denote by $\mathscr{F}(v)$ the leaf of $v \in N$. By a *transversal* to $\mathscr{F}$, we mean in this subsection a smooth submanifold $T$ of $N$ such that for every $v \in T$, we have $T_vT \oplus T_v\mathscr{F}_v = T_vN$. We denote by $\operatorname{vol}_T$ the induced Riemannian measure on $T$. For instance, every stable leaf is a transversal to the strong unstable foliation $\mathscr{F} = \mathscr{W}^{\rm su}$ of $N = T^1\widetilde{M}$ endowed with the Sasaki metric.

If $U$ is the domain of a foliated chart of $\mathscr{F}$, the *local leaf* $\mathscr{F}_U(v)$ of $v \in U$ in $U$ is the connected component of $v$ in $\mathscr{F}(v) \cap U$. We denote by $\operatorname{vol}_{\mathscr{F}_U(v)}$ the induced Riemannian measure on $\mathscr{F}_U(v)$. We denote by $\mathscr{U}\mathscr{T}$ the set of pairs $(U,T)$ where $U$ is the domain of a foliated chart of $\mathscr{F}$ and $T$ is a transversal to $\mathscr{F}$ contained in $U$ such that $U = \bigcup_{v\in T} \mathscr{F}_U(v)$ and $\mathscr{F}_U(v) \cap \mathscr{F}_U(w) = \emptyset$ for all $v \neq w$ in $T$. In particular, we have a continuous fibration $U \to T$, with fibres the local leaves. For every $v \in N$, there exists $(U,T) \in \mathscr{U}\mathscr{T}$ such that $v \in T$. A *holonomy map* of $\mathscr{F}$ is a homeomorphism $h : T \to T'$ between two transversals to $\mathscr{F}$ contained in the domain $U$ of a foliated chart of $\mathscr{F}$, such that $h(v) \in \mathscr{F}_U(v)$ for every $v \in T$ and $\mathscr{F}_U(v) \cap \mathscr{F}_U(w) = \emptyset$ for all $v \neq w$ in $T$. Note that these holonomy maps are in general only continuous, and a priori not absolutely continuous with respect to the Lebesgue measures. When $\mathscr{F} = \mathscr{W}^{\rm su}$ and $N = T^1\widetilde{M}$, they are Hölder-continuous by Subsection 7.1 if the sectional curvature $M$ has bounded derivatives. But this is not sufficient to imply that they are absolutely continuous with respect to the Lebesgue measures.

The foliation $\mathscr{F}$ is *transversally absolutely continuous* if its holonomy maps preserve the Lebesgue measure classes, that is, if for every holonomy map $h : T \to T'$, the measure $h_* \operatorname{vol}_T$ is absolutely continuous with respect to $\operatorname{vol}_{T'}$. For every $\epsilon > 0$, a transversal $T$ to $\mathscr{F}$ is *$\epsilon$-orthogonal* to $\mathscr{F}$ if for every $v \in T$, the Riemannian angle between a unit tangent vector at $v$ to $T$ and a unit tangent vector at $v$ to $\mathscr{F}(v)$ is at least $\epsilon$. For instance, if $N = T^1\widetilde{M}$, then since $\widetilde{M}$ has pinched negative curvature, there exists $\epsilon_0 > 0$ such that every stable leaf is an $\epsilon_0$-orthogonal transversal to the strong unstable foliation $\mathscr{F} = \mathscr{W}^{\rm su}$. The foliation $\mathscr{F}$ is said to be *transversally absolutely continuous with locally uniformly bounded Jacobians* if furthermore, for every $\epsilon > 0$, for every compact set $K$ in $N$ and for every holonomy map $h : T \to T'$ of $\mathscr{F}$ with $\epsilon$-orthogonal transversals $T, T'$ contained in $K$, the *Jacobian* $\frac{d(h^{-1})_* \operatorname{vol}_T}{d \operatorname{vol}_{T'}}$ of the absolutely continuous map $h$ is (almost everywhere) bounded by a constant depending only on $K$ and $\epsilon$. By the change of variable formula, if $\mathscr{F}$ is smooth, then $\mathscr{F}$ is transversally absolutely continuous, and the above Jacobians of the holonomy maps are their Jacobians as smooth maps.

The foliation $\mathscr{F}$ is *absolutely continuous* if for every $(U,T) \in \mathscr{U}\mathscr{T}$, the restriction to $U$ of the Riemannian measure of $N$ disintegrates with respect to the fibration $U \to T$ over the induced Riemannian measure $\operatorname{vol}_T$ on $T$, with conditional measures absolutely continuous with respect to the induced Riemannian measures $\operatorname{vol}_{\mathscr{F}_U(v)}$ on the local leaves $\mathscr{F}_U(v)$ for ($\operatorname{vol}_T$-almost) every $v \in T$: There exists a measurable family $(\delta_v)_{v\in T}$ of positive measurable functions $\delta_v : \mathscr{F}_U(v) \to \mathbb{R}$ such that for every measurable subset $A$ of $U$, we have

$$\operatorname{vol}_{T^1\widetilde{M}}(A)=\int_{v\in T}\int_{w\in\mathscr{F}_U(v)}\mathbb{1}_A(w)\delta_v(w)\,d\operatorname{vol}_{\mathscr{F}_U(v)}(w)\,d\operatorname{vol}_T(v)\ .$$

It is *absolutely continuous with locally bounded conditional densities* if furthermore, for every $\epsilon>0$ and every compact subset $K$ in $N$, there exists $c_{K,\epsilon}\geq 0$ such that, for every $(U,T)\in\mathscr{U}\mathscr{T}$ with $U\subset K$ and $T$ $\epsilon$-orthogonal, the above densities $\delta_v$ are bounded for every $v\in T$ by the constant $c_{K,\epsilon}$.

By [Bri, Prop. 3.5] and [Bri, Rem. 3.9], if $\mathscr{F}$ is transversally absolutely continuous (respectively transversally absolutely continuous with locally uniformly bounded Jacobians), then $\mathscr{F}$ is absolutely continuous (respectively absolutely continuous with locally bounded conditional densities), but the converse is not always true.

**Theorem 7.6** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded. Then the strong stable, strong unstable, stable and unstable foliations are transversally absolutely continuous with locally uniformly bounded Jacobians, hence are absolutely continuous with locally bounded conditional densities.*

**Proof.** We follow closely Brin's proof [Bri, Theo. 5.1], only giving the arguments for the extension to the noncompact case. By the smoothness of the geodesic foliation, and by a time reversal argument, we only consider the case of the strong unstable foliation.

Let $K$ be a compact subset of $T^1\widetilde{M}$, let $h:A\to A'$ be a holonomy map contained in the domain $U$ of a foliated chart of $\mathscr{W}^{\rm su}$, with $U\subset K$, such that the transversals $A$ and $A'$ ar $(2\epsilon)$-orthogonal to $\mathscr{W}^{\rm su}$, for some fixed $\epsilon>0$. For every $n\in\mathbb{N}$, let $\mathscr{F}_n$ be the smooth foliation of $T^1\widetilde{M}$, whose leaf $\mathscr{F}_n(v)$ through every $v\in T^1\widetilde{M}$ is the set of outer unit normal vector to the sphere $S(\pi(\phi_{-n}v),n)$ of radius $n$ centred at $\pi(\phi_{-n}v)$. As $n\to+\infty$, the foliation $\mathscr{F}_n$ converges to $\mathscr{W}^{\rm su}$ uniformly on compact sets: for every $\epsilon>0$, if $n$ is large enough, for all $v,w\in K$ such that $w\in W^{\rm su}(v)$, if $w_n$ is the outer unit normal vector to $S(\pi(\phi_{-n}v),n)$ at its closest point to $\pi(w)$, we have $d(w,w_n)\leq\epsilon$. Hence, up to shrinking $A,A'$ to neighbourhoods of given points $v\in A, v'\in A'$ respectively (and then using finite covering arguments), for $n$ large enough, $A$ and $A'$ are $\epsilon$-orthogonal transversals to the foliation $\mathscr{F}_n$, and there exists a (smooth) holonomy map $h_n:A\to A'$ for $\mathscr{F}_n$, which converges uniformly to $h$ as $n\to+\infty$.

By [Bri, Lem. 2.7], in order to prove Theorem 7.6, it is sufficient to prove that the (smooth) Jacobian $\operatorname{Jac}h_n$ of $h_n$ is uniformly bounded on $U$, by a constant depending only on $K$ and $\epsilon$. Let us write

$$h_n=\phi_{-n}\circ H_n\circ\phi_n\ ,$$

where $H_n:\phi_n(A)\to\phi_n(A')$ is a holonomy map of the foliation with leaves the fibres of the fibration $T^1\widetilde{M}\to\widetilde{M}$. For $0\leq k\leq n-1$, we denote by $J_k, J'_k$ the Jacobian of $\phi_1$ at $\phi_k(v),\phi_k(v')$, respectively. We have by the chain rule

$$\operatorname{Jac}h_n(v)=\prod_{k=0}^{n-1}\frac{1}{J'_k}\ \ \operatorname{Jac}H_n(\phi_n v)\ \prod_{k=0}^{n-1}J_k=\operatorname{Jac}H_n(\phi_n v)\ \prod_{k=0}^{n-1}\frac{J_k}{J'_k}\ .$$

Exactly as in the compact case (see [Bri, page 94]), the Hölder continuity of the strong unstable distribution and uniform transversality arguments prove that there exist two constants $c, c' > 0$ (depending only on $K$ and $\epsilon$) such that, for all $k \geq 0$ and $v \in A$, we have $|J_k - J'_k| \leq c\, e^{-c'\,k}$. The fact that the sectional curvature is pinched between two negative constants implies that there exists a constant $c'' > 0$ such that, for all $k \geq 0$ and $v \in A$, we have $\frac{1}{c''} \leq J_k \leq c''$. Since $1 + x \leq e^x$ for all $x \geq 0$, we therefore have

$$\prod_{k=0}^{n-1} \frac{J_k}{J'_k} \leq \prod_{k=0}^{n-1} (1 + c\, c''\, e^{-c'\,k}) \leq e^{c\,c'' \sum_{k=0}^{+\infty} e^{-c'\,k}} < +\infty\,.$$

Exactly as in the compact case (see [Bri, Lem. 4.2]), the Anosov property of the geodesic flow in pinched negative sectionnal curvature implies that at all of their points $v$, the transversals $\phi_n(A)$ and $\phi_n(A')$ are uniformly (in $n$ for $n$ large enough and in $v$) transverse to the unit tangent spheres $T_{\pi(v)}\widetilde{M}$. Since $H_n$ is a holonomy map of a smooth Riemannian foliation of a Riemannian manifold, between two transversals whose angles with the leaves of the foliation are uniformly bounded from below, the Jacobian $\mathrm{Jac}\, H_n$ is uniformly bounded. Hence the Jacobian $\mathrm{Jac}\, h_n$ is uniformly bounded, as required. $\square$

### 7.3 The Liouville measure as an equilibrium state

In this subsection, we recall the definition of Lyapounov's exponents and the main results about them (Oseledets's theorem, Ruelle's inequality, the Pesin formula, see for instance [Man2, Led1, Man1, KatM]), in order to prove that the Liouville measure, once normalised, is the equilibrium state of an appropriate potential, under compactness assumptions.

We denote here by $\phi = \phi_1$ the time one map of the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M$. We fix a probability measure $\mu$ on $T^1M$ invariant under $(\phi_t)_{t\in\mathbb{R}}$. Recall that $\log^+ t = \max\{\log t, 0\}$ for every $t > 0$, and that $T^1\widetilde{M}$ is endowed with Sasaki's Riemannian metric.

*Oseledets's theorem* (see for instance [Ose], [KatM, Theo. S.2.9 page 665]) asserts that if $\int_{T^1M} \log^+ \|T\phi^{\pm 1}\|\, d\mu < \infty$ (in particular if $M$ is compact), then there exists a Borel subset $\mathrm{Reg}$ of $T^1M$, invariant under $(\phi_t)_{t\in\mathbb{R}}$ and with full measure with respect to $\mu$, such that for every $v \in \mathrm{Reg}$, there exists a unique direct sum decomposition (of closed convex cones when $\Gamma$ has torsion), called the *Oseledets decomposition* at $v$,

$$T_vT^1M = \oplus_{i=1}^{s(v)} E_i(v)$$

and unique real numbers $\chi_1(v) < \cdots < \chi_{s(v)}(v)$, called the *Lyapounov exponents* of $v$, such that

(1) the map $v \mapsto s(v)$ from $\mathrm{Reg}$ to $\mathbb{N}$ is measurable; for all $i, r \in \mathbb{N}$, the map $v \mapsto E_i(v)$ from $\{v \in \mathrm{Reg}\ :\ s(v) \geq i, \dim E_i(v) = r\}$ to the space of closed subsets of $TT^1M$ (endowed with Chabauty's topology) is measurable; and for all $v \in \mathrm{Reg}$ and $i \in \{1, \dots, s(v)\}$, we have $T_v\phi(E_i(v)) = E_i(\phi v)$ and $\chi_i(\phi v) = \chi_i(v)$;

(2) for all $v \in \mathrm{Reg}$, $i \in \{1, \dots, s(v)\}$ and $V \in E_i(v) - \{0\}$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|T_v\phi^n(V)\| = \chi_i(v)\ ;$$

(3) for all $v \in \mathrm{Reg}$ and $i \in \{1, \dots, s(v)\}$, if $\dim E_i(v) = k_i(v)$, then

$$\lim_{n\to+\infty} \frac{1}{n} \log J^{E_i}\phi^n(v) = k_i(v)\, \chi_i(v)\,,$$

where $J^{E_i}\phi^n(v)$ denotes the Jacobian of the restriction of $T_v\phi^n$ to $E_i(v)$;

(4) the angle between two subspaces of the Oseledets decomposition has at most subexponential decay: for all $v$ in Reg, $i \neq j$ in $\{1, \dots, s(v)\}$, $V \in E_i(v) - \{0\}$ and $W \in E_j(v) - \{0\}$, we have

$$\lim_{n \to \pm\infty} \frac{1}{|n|} \log \big| \sin \angle(T_v \phi^n(V), T_v \phi^n(W)) \big| = 0 \,.$$

In particular, since $E^{\rm su}(v) = \oplus_{1 \leq i \leq s(v),\ \chi_i(v) > 0} E_i(v)$ for every $v \in$ Reg, we deduce from the last two points that for $\mu$-almost every $v \in T^1M$, we have

$$\lim_{n \to +\infty} \frac{1}{n} \int_0^n F^{\rm su}(\phi_t v)\, dt = - \lim_{n \to +\infty} \frac{1}{n} \log J^{\rm su}(v, n) = - \sum_{1 \leq i \leq s(v),\ \chi_i(v) > 0} k_i(v)\, \chi_i(v) \,.$$

Note that when $\mu$ is ergodic for $(\phi_t)_{t \in \mathbb{R}}$, by Birkhoff's ergodic theorem, for $\mu$-almost every $v \in T^1M$, this equality becomes

$$\int_{T^1M} F^{\rm su}\, d\mu = - \sum_{1 \leq i \leq s(v),\ \chi_i(v) > 0} k_i(v)\, \chi_i(v) \,. \tag{100}$$

The *Ruelle inequality* (see [Rue1]) asserts that if $M$ is compact, then

$$h_\mu(\phi) \leq \int_{v \in T^1M} \sum_{1 \leq i \leq s(v),\ \chi_i(v) > 0} k_i(v)\, \chi_i(v)\, d\mu(v) \,. \tag{101}$$

It is unclear if it is possible to extend the proof of Ruelle's inequality in the case where $M$ is noncompact, with perhaps additional geometric assumptions besides the pinched negative sectional curvature.

When $M$ is compact and $\mu$ is ergodic, by Equation (100) and Equation (101), we have $h_\mu(\phi) + \int_{T^1M} F^{\rm su}\, d\mu \leq 0$. Since the upper bound defining the topological pressure of a potential (see Chapter 6) may be taken on the ergodic probability measures invariant under $(\phi_t)_{t \in \mathbb{R}}$ by the convexity properties of the metric entropy, we hence have

$$P(\Gamma, F^{\rm su}) \leq 0 \,. \tag{102}$$

The *Pesin formula* (see for instance [Pes, Man1]) asserts that if $\mu$ is absolutely continuous with respect to the Lebesgue measure class, then we have equality in Equation (101):

$$h_\mu(\phi) = \int_{v \in T^1M} \sum_{1 \leq i \leq s(v),\ \chi_i(v) > 0} k_i(v)\, \chi_i(v)\, d\mu(v) \,. \tag{103}$$

If $M$ is noncompact, it is unclear when the Pesin formula remains valid.

In particular, if $M$ is compact, then the Liouville measure ${\rm vol}_{T^1M}$ is ergodic (see for instance [Bri, page 95]), and the probability measure $L$ proportional to the Liouville measure satisfies

$$P(\Gamma, F^{\rm su}) \leq 0 = h_L(\phi) + \int_{T^1M} F^{\rm su}\, dL \,.$$

Hence $L$ is an equilibrium state for the potential $F^{\rm su}$. The Variational Principle (Theorem 6.1) implies that $L = m_{F^{\rm su}}$, which proves the following expected result.

**Theorem 7.7** *If $M$ is compact, then the normalised Liouville measure $\frac{\operatorname{vol}_{T^1M}}{\operatorname{Vol}(T^1M)}$ coincides with the normalised Gibbs measure $\frac{m_{F^{\rm su}}}{\|m_{F^{\rm su}}\|}$ of the potential $F^{\rm su}$.* $\square$

**Remark.** Assume in this remark that $M$ has finite volume with constant curvature. Then $F^{\rm su}$ is constant, hence the Gibbs measure of the potential $F^{\rm su}$ and the Bowen-Margulis measure (which is the Gibbs measure of the potential 0) coincide. The Patterson-Sullivan measure of $\Gamma$ at the origin of the ball model of the real hyperbolic $n$-space $\mathbb{H}^n_{\mathbb{R}}$ may be taken to be the standard Riemannian measure of the unit sphere $\mathbb{S}^{n-1} = \partial_\infty \mathbb{H}^n_{\mathbb{R}}$. The exact ratio between the total masses of $T^1M$ for the Liouville measure and for the Gibbs measure $m_{F^{\rm su}}$ is computed in [PaP3, Sect. 7], yielding

$$\frac{\|m_{F^{\rm su}}\|}{\operatorname{Vol}(T^1M)} = 2^{n-1}\operatorname{Vol}(\mathbb{S}^{n-1})\,.$$

### 7.4 The Liouville measure satisfies the Gibbs property

In this subsection, we prove that, under the previous bounded geometry assumption, the Liouville measure satisfies a Gibbs property as defined in Subsection 3.8, as well as a leafwise version of it.

In the proof of this Gibbs property, as well as in the coming Subsection 7.5 in order to prove Theorem 7.2, we will use the following version of the Gibbs property for the Liouville measure in restriction to the strong unstable and strong stable foliations. Recall that $B^{\rm su}(v,r)$ is the open ball of centre $v$ and radius $r$ for the Hamenstädt distance $d_{W^{\rm su}(v)}$ on $W^{\rm su}(v)$, and similarly for $B^{\rm ss}(v,r)$ (see Subsection 2.4).

**Lemma 7.8** *For every compact subset $K$ in $T^1\widetilde{M}$ and for every $r>0$, there exists a constant $c'_{K,r}>0$ such that for all $v\in\Gamma K$ and $T\geq 0$ verifying $\phi_T v\in\Gamma K$, we have*

$$\frac{1}{c'_{K,r}}\, e^{\int_0^T \widetilde{F}^{\rm su}(\phi_t v)\,dt} \leq \operatorname{vol}_{W^{\rm su}(v)}(\phi_{-T}B^{\rm su}(\phi_T v, r)) \leq c'_{K,r}\, e^{\int_0^T \widetilde{F}^{\rm su}(\phi_t v)\,dt}\,,$$

*and for all $v\in\Gamma K$ and $T\geq 0$ verifying $\phi_{-T} v\in\Gamma K$, we have*

$$\frac{1}{c'_{K,r}}\, e^{\int_{-T}^0 \widetilde{F}^{\rm su}\circ\iota(\phi_t v)\,dt} \leq \operatorname{vol}_{W^{\rm ss}(v)}(\phi_{T}B^{\rm ss}(\phi_{-T} v, r)) \leq c'_{K,r}\, e^{\int_{-T}^0 \widetilde{F}^{\rm su}\circ\iota(\phi_t v)\,dt}\,.$$

**Proof.** Fix $K,r$ as in the statement. In order to prove the first claim, observe that, for $v,T$ as in its statement,

$$\begin{aligned}\operatorname{vol}_{W^{\rm su}(v)}(\phi_{-T}B^{\rm su}(\phi_T v, r)) &= \int_{B^{\rm su}(\phi_T v,\, r)} d\big((\phi_T)_*\operatorname{vol}_{W^{\rm su}(v)}\big)\\ &= \int_{B^{\rm su}(\phi_T v,\, r)} \operatorname{Jac}\big((\phi_{-T})_{|W^{\rm su}(\phi_T v)}\big)\, d\operatorname{vol}_{W^{\rm su}(\phi_T v)}\,.\end{aligned}$$

The Hölder-continuity of the strong unstable distribution and the compactness of $K$ imply that there exists a constant $c$ depending only on $K$ and $r$ such that, for $v,T$ as in the statement of the first claim and for every $w\in B^{\rm su}(\phi_T v,\, r)$, we have

$$\frac{1}{c}\operatorname{Jac}\big((\phi_{-T})_{|W^{\rm su}(\phi_T v)}\big)(\phi_T v) \leq \operatorname{Jac}\big((\phi_{-T})_{|W^{\rm su}(\phi_T v)}\big)(w) \leq c\,\operatorname{Jac}\big((\phi_{-T})_{|W^{\rm su}(\phi_T v)}\big)(\phi_T v)\,.$$

By the definition of the unstable Jacobian $\widetilde{J}^{\rm su}$, we have $\operatorname{Jac}(\phi_{-T|W^{\rm su}(\phi_T v)})(\phi_T v) = \widetilde{J}^{\rm su}(\phi_T v, -T)$. Hence

$$\begin{aligned} \frac{1}{c}\,\widetilde{J}^{\rm su}(\phi_T v,-T)\,&\operatorname{vol}_{W^{\rm su}(\phi_T v)}(B^{\rm su}(\phi_T v,\,r)) \\ &\leq \operatorname{vol}_{W^{\rm su}(v)}(\phi_{-T}B^{\rm su}(\phi_T v,r)) \\ &\leq c\,\widetilde{J}^{\rm su}(\phi_T v,-T)\,\operatorname{vol}_{W^{\rm su}(\phi_T v)}(B^{\rm su}(\phi_T v,\,r))\,. \end{aligned} \tag{104}$$

Since $K$ is compact, the Hamenstädt ball of radius $r$ and centre $u\in K$ in its strong unstable leaf $W^{\rm su}(u)$ is contained in the Riemannian ball of radius $r'$ and centre $u$ in $W^{\rm su}(u)$, and contains the Riemannian ball of radius $r''$ and centre $u$ in $W^{\rm su}(u)$, for some $r'$ and $r''$ depending only on $r$ and $K$. For instance by the standard comparison theorems of volumes of Riemannian balls and by the compactness of $K$, the volume $\operatorname{vol}_{W^{\rm su}(w)}(B^{\rm su}(w,r))$ for $w\in\Gamma K$ is hence uniformly bounded from above and from below by positive constants depending only on $K$ and $r$.

By the cocycle formula (97), for all $s,t\in\mathbb{R}$, we have $\log\widetilde{J}^{\rm su}(\phi_T v,0)=0$ and

$$\log\widetilde{J}^{\rm su}(\phi_T v,s+t)=\log\widetilde{J}^{\rm su}(\phi_t\phi_T v,s)+\log\widetilde{J}^{\rm su}(\phi_T v,t)\ ,$$

hence, by the definition of the potential $\widetilde{F}^{\rm su}$,

$$\begin{aligned} \int_0^T \widetilde{F}^{\rm su}(\phi_t v)\,dt &= \int_{-T}^0 \widetilde{F}^{\rm su}(\phi_t\phi_T v)\,dt = \int_{-T}^0 -\frac{d}{ds}_{|s=0}\log\widetilde{J}^{\rm su}(\phi_t\phi_T v,s)\,dt \\ &= \int_{-T}^0 -\frac{d}{dt}\log\widetilde{J}^{\rm su}(\phi_T v,t)\,dt = \log\widetilde{J}^{\rm su}(\phi_T v,-T)\ . \end{aligned}$$

The first claim hence follows from Equation (104).

In order to prove the second claim, observe that the map $\iota: v\mapsto -v$ is an isometry of Sasaki's metric on $T^1\widetilde{M}$, anti-commuting with the geodesic flow, exchanging the strong stable and strong unstable balls (see Equation (15)), so that, for $v,T$ as in the statement of the second claim,

$$\operatorname{vol}_{W^{\rm ss}(v)}(\phi_T B^{\rm ss}(\phi_{-T}v,r))=\operatorname{vol}_{W^{\rm su}(-v)}(\phi_{-T}B^{\rm su}(\phi_T(-v),r))\ .$$

Thus, since $-v$ and $\phi_T(-v)$ belong to $\Gamma\iota(K)$ (and $\iota(K)$ is compact), the first claim gives, for some constant $c'>0$ depending only on $r$ and $K$,

$$\frac{1}{c'}\,e^{\int_0^T\widetilde{F}^{\rm su}(\phi_s(-v))\,ds}\leq\operatorname{vol}_{W^{\rm ss}(v)}(\phi_T B^{\rm ss}(\phi_{-T}v,r))\leq c'\,e^{\int_0^T\widetilde{F}^{\rm su}(\phi_s(-v))\,ds}\,.$$

The equality

$$\int_0^T\widetilde{F}^{\rm su}(\phi_s(-v))\,ds=\int_0^T\widetilde{F}^{\rm su}\circ\iota(\phi_{-s}v)\,ds=\int_{-T}^0\widetilde{F}^{\rm su}\circ\iota(\phi_s v)\,ds$$

then gives the second claim. □

**Proposition 7.9** *Assume that the derivatives of the sectional curvature are uniformly bounded. Then the Liouville measure on $T^1M$ satisfies the Gibbs property for the potential $F^{\rm su}$ and the constant $c(F^{\rm su})=0$.*

In the compact case, this was proved for hyperbolic diffeomorphisms in Bowen-Ruelle in [BoR, Lem. 4.2], and later in Katok-Hasselblatt in [KatH, Lem. 20.4.2]. It is possible to adapt their arguments to flows in the noncompact case. However, it is shorter to use the absolute continuity properties of the strong unstable foliation, proved in Section 7.2. We follow this approach here.

**Proof.** Note that $F^{\rm su}$ is Hölder-continuous and bounded by Theorem 7.1.

Note that by the definition of the dynamical balls in Subsection 3.8, for every compact subset $K$ of $T^1\widetilde{M}$, for all $r \in \,]0,1]$, $v \in K$ and $T \geq 1$, the set $B(v;T,0,r)$ is contained in the compact subset $\pi^{-1}(\mathcal{N}_1(\pi(K)))$ of $T^1\widetilde{M}$. Hence the multiplicity of the restriction to $B(v;T,0,r)$ of the map $T^1\widetilde{M} \to T^1M$ is bounded by a constant depending only on $K$, by the discreteness of the action of $\Gamma$ on $T^1\widetilde{M}$.

Since the Liouville measure on $T^1\widetilde{M}$ is invariant under the geodesic flow and by Remark (2) following the definition 3.15 (in Subsection 3.8) of the Gibbs property, we only have to prove that for every compact subset $K$ of $T^1\widetilde{M}$, there exist $r \in \,]0,1]$, $C_{K,r} \geq 1$ and $T_0 \geq 0$ such that for all $v \in K$ and $T \geq T_0$ with $\phi_T v \in \Gamma K$, we have

$$\frac{1}{C_{K,r}}\, e^{\int_0^T \widetilde{F}^{\rm su}(\phi_t v)\,dt} \leq \operatorname{vol}_{T^1\widetilde{M}}(B(v;T,0,r)) \leq C_{K,r}\, e^{\int_0^T \widetilde{F}^{\rm su}(\phi_t v)\,dt}\,.$$

We first replace the dynamical balls by sets better adapted to the local product structure. For all $r > 0$, $v \in T^1\widetilde{M}$ and $T \geq 0$, consider the sets

$$C(v;T,r) = \bigcup_{w\in B^{\rm ss}(v,\,r)}\ \bigcup_{|s|<r} B^{\rm su}(\phi_s w, r e^{-T})\,.$$

Note that $\gamma C(v;T,r) = C(\gamma v;T,r)$ for every $\gamma \in \Gamma$.

**Lemma 7.10** *For every compact subset $K$ in $T^1\widetilde{M}$ and every $r > 0$, there exists $r' = r'_{K,r} > 0$ such that for all $v \in \Gamma K$ and $T \geq 3r$ satisfying $\phi_T v \in \Gamma K$, we have*

$$C(v;T,\frac{r}{3}) \subset B(v;T,0,r) \subset C(v;T,r')\,.$$

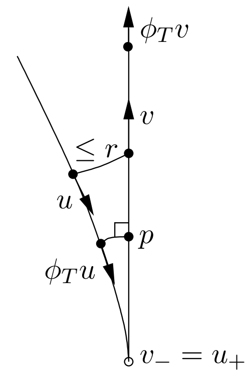
**Proof.** Let $v \in T^1\widetilde{M}$, $T \geq 0$, $r > 0$ and $u \in C(v;T,\frac{r}{3})$. By the definition of $C(v;T,\frac{r}{3})$, let $s \in \,]-\frac{r}{3},\frac{r}{3}[$ and $w \in B^{\rm ss}(v,\frac{r}{3})$ be such that $u \in B^{\rm su}(\phi_s w, \frac{r}{3}\,e^{-T})$. Then for every $t \in [0,T]$, by the triangle inequality, by Equation (9), and by the contraction properties (8) and (11) of the Hamenstädt distances, we have

$$\begin{aligned}
d(\pi(\phi_t u),\pi(\phi_t v)) &\leq d(\pi(\phi_t u),\pi(\phi_{t+s}w)) + d(\pi(\phi_{t+s}w),\pi(\phi_t w)) + d(\pi(\phi_t w),\pi(\phi_t v))\\
&\leq d_{W^{\rm su}(\phi_{t+s}w)}(\phi_t u,\phi_{t+s}w) + |s| + d_{W^{\rm ss}(\phi_t w)}(\phi_t w,\phi_t v)\\
&= e^t\, d_{W^{\rm su}(\phi_s w)}(u,\phi_s w) + |s| + e^{-t}\, d_{W^{\rm ss}(w)}(w,v)\\
&< e^t\,\frac{r}{3}\,e^{-T} + \frac{r}{3} + e^{-t}\,\frac{r}{3} \leq r\,.
\end{aligned}$$

Hence $u \in B(v;T,0,r)$ by the definition of these dynamical balls. This proves the inclusion on the left hand side.

To prove the other inclusion, we start by the following remark.

For all $u, v \in T^1\widetilde{M}$, $T \geq 0$ and $r > 0$ such that $d(\pi(u), \pi(v)) \leq r$ and $d(\pi(\phi_T u), \pi(\phi_T v)) \leq r$, if $T > 2r$ then $u_+ \neq v_-$. Otherwise, let $p$ be the closest point to $\pi(\phi_T u)$ on the geodesic line defined by $v$. Since $T > 2r$ and by convexity, we have $p \in \,]v_-, \pi(v)]$, $d(\pi(\phi_T u), p) \leq r$ and $d(p, \pi(v)) \geq T - 2r$. Hence




$$\begin{aligned} d(\pi(\phi_T v), \pi(\phi_T u)) &\geq d(\pi(\phi_T v), p) - r \\ &= d(\pi(\phi_T v), \pi(v)) + d(\pi(v), p) - r \\ &\geq T + (T - 2r) - r > r\ , \end{aligned}$$

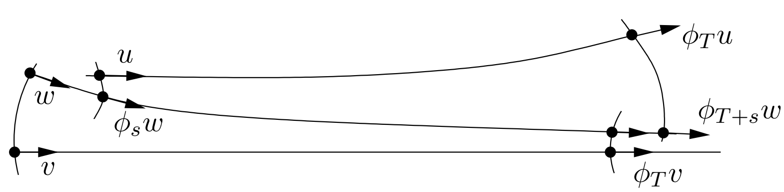
a contradiction.

For every $\xi \in \partial_\infty \widetilde{M}$, let (see Section 3.9)

$$U_\xi = \{u \in T^1\widetilde{M}\ :\ u_+ \neq \xi\}\ .$$

A similar proof shows that if furthermore $T \geq 3r$ then $u$ stays in a compact subset of $U_{v_-}$.

Now, let $K, r$ be as in the statement. Let $v \in K$, $T \geq 3r$ and $u \in T^1\widetilde{M}$ be such that $\phi_T v \in \Gamma K$ and $u \in B(v; T, 0, r)$, so that $d(\pi(u), \pi(v)) \leq r$ and $d(\pi(\phi_T u), \pi(\phi_T v)) \leq r$. Then $d(u, v) \leq r + \pi$ and $d(\phi_T u, \phi_T v) \leq r + \pi$, by the properties of Sasaki's metric. Recall that the map from $U_{v_-}$ to $W^{\rm su}(v)$, sending $u \in U_{v_-}$ to the unique $w \in W^{\rm su}(v)$ such that $u_+ = w_+$, is a trivialisable continuous fibration, with fiber over $w \in W^{\rm su}(v)$ the stable leaf $W^{\rm s}(w)$. The Hamenstädt distances are proper on the strong stable and strong unstable leaves, and vary continuously with the leaves. Every compact subset $K'$ of $U_{v'_-}$ is contained in $U_{v''_-}$ if $v''$ is close enough to $v'$, and hence in $C(v''; 0, r)$ for $r > 0$ (depending only on $K'$ and $v'$) big enough. By the compactness of $K$ (with a finite covering argument), and by equivariance, there hence exists $r' > 0$ depending only on $K$ and $r$ such that $u \in C(v; 0, r')$ and $\phi_T u \in C(\phi_T v; 0, r')$. By the definition of the sets $C(v_0; 0, r_0)$, let $s, s' \in \,]-r', r'[\,$, and let $w \in B^{\rm ss}(v, r')$ and $w' \in B^{\rm ss}(\phi_T v, r')$ be such that $u \in B^{\rm su}(\phi_s w, r')$ and $\phi_T u \in B^{\rm su}(\phi_{s'} w', r')$.




By the uniqueness of the decomposition, we have $w' = \phi_T w$ and $\phi_{s'} w' = \phi_T \phi_s w$. Hence

$$d_{W^{\rm su}(\phi_s w)}(u, \phi_s w) = e^{-T}\ d_{W^{\rm su}(\phi_T \phi_s w)}(\phi_T u, \phi_T \phi_s w) \leq e^{-T}\, r'\ .$$

Hence $u \in C(v; T, r')$, which proves the inclusion on the right hand side. $\square$

By this lemma, we hence only have to prove that for every compact subset $K$ in $T^1\widetilde{M}$ and every $r > 0$, there exists $C'_{K,r} \geq 1$ such that for all $v \in K$ and $T \geq 0$ with $\phi_T v \in \Gamma K$, we have

$$\frac{1}{C'_{K,r}}\ e^{\int_0^T \widetilde{F}^{\rm su}(\phi_t v)\, dt} \leq \operatorname{vol}_{T^1\widetilde{M}}(C(v; T, r)) \leq C'_{K,r}\ e^{\int_0^T \widetilde{F}^{\rm su}(\phi_t v)\, dt}\ . \tag{105}$$

Let us fix $K, r$ as above. Recall that the foliations $\widetilde{\mathscr{W}}^{\rm su}$ and $\widetilde{\mathscr{W}}^{\rm ss}$ are absolutely continuous with locally bounded conditional densities (see Theorem 7.6), and that each leaf of one of these foliations is an $\epsilon_0$-orthogonal transversal to the other one, for some fixed $\epsilon_0 > 0$. Hence, there exists $c \geq 1$ (depending only on $K, r, \epsilon_0$) such that for all $v \in K$ and $T \geq 0$, we have

$$\frac{1}{c}\int_{B^{\rm ss}(v,\,r)}\int_{-r}^{r} \operatorname{vol}_{W^{\rm su}(w)}(B^{\rm su}(\phi_s w, re^{-T}))\, ds\, d\operatorname{vol}_{W^{\rm ss}(v)}(w) \leq \operatorname{vol}_{T^1\widetilde{M}}(C(v;T,r))$$
$$\leq c \int_{B^{\rm ss}(v,\,r)}\int_{-r}^{r} \operatorname{vol}_{W^{\rm su}(w)}(B^{\rm su}(\phi_s w, re^{-T}))\, ds\, d\operatorname{vol}_{W^{\rm ss}(v)}(w)\ . \qquad (106)$$

There exists a compact subset $K'$ in $T^1\widetilde{M}$ (depending only on $K$ and $r$) such that for all $v \in K, T \geq 0$, $s \in [-r, r]$ and $w \in B^{\rm ss}(v,\,r)$ with $\phi_T v \in \Gamma K$, we have $\phi_s w \in K'$ and $\phi_T(\phi_s w) \in \Gamma K'$. We have $B^{\rm su}(\phi_s w, re^{-T}) = \phi_{-T} B^{\rm su}(\phi_T(\phi_s w), r)$ by Equation (13). By Lemma 7.8, for all $v, T, s, w$ as above, we hence have

$$\frac{1}{c'_{K',\,r}}\; e^{\int_0^T \widetilde{F}^{\rm su}(\phi_{t+s}w)\,dt} \leq \operatorname{vol}_{W^{\rm su}(w)}(B^{\rm su}(\phi_s w, re^{-T})) \leq c'_{K',\,r}\; e^{\int_0^T \widetilde{F}^{\rm su}(\phi_{t+s}w)\,dt}\ .$$

Furthermore, by Equation (9) for the strong stable leaves, we have

$$d(\pi(v), \pi(\phi_s w)) \leq |s| + d(\pi(v), \pi(w)) \leq r + d_{W^{\rm ss}(v)}(v, w) \leq 2r$$

and

$$d(\pi(\phi_T v), \pi(\phi_{T+s} w)) \leq |s| + d(\pi(\phi_T v), \pi(\phi_T w)) \leq r + e^{-T} d_{W^{\rm ss}(v)}(v, w) \leq 2r\ .$$

Since the potential $\widetilde{F}^{\rm su}$ is Hölder-continuous and bounded, by Lemma 3.2 applied twice with $r_0 = 2r$, there exists a constant $c' > 0$ (depending only on $\|\widetilde{F}^{\rm su}\|_\infty$ and $r$) such that for all $v \in K$, $s \in [-r, r]$ and $w \in B^{\rm ss}(v,\,r)$, we have

$$\Big|\int_0^T \widetilde{F}^{\rm su}(\phi_{t+s}w)\,dt - \int_0^T \widetilde{F}^{\rm su}(\phi_t v)\,dt\Big| = \Big|\int_{\pi(\phi_s w)}^{\pi(\phi_{T+s}w)} \widetilde{F}^{\rm su} - \int_{\pi(v)}^{\pi(\phi_T v)} \widetilde{F}^{\rm su}\Big| \leq c'\ .$$

Moreover, by arguments already seen in the proof of Lemma 7.8 for the case of the strong unstable foliation, there exists $c'' \geq 1$ (depending only on $K$ and $r$) such that for every $v \in K$,

$$\frac{1}{c''} \leq \operatorname{vol}_{W^{\rm ss}(v)}(B^{\rm ss}(v, r)) \leq c''\ .$$

By Equation (106), Equation (105) therefore holds with $C'_{K,\,r} = c\,c'' e^{c'}\, c'_{K',\,r} \max\{2r, \frac{1}{2r}\}$. □

### 7.5 Conservative Liouville measures are Gibbs measures

The aim of this subsection is to give a proof of Theorem 7.2. We hence assume now that its hypotheses are satisfied.

By Theorem 7.1 and the assumptions of Theorem 7.2, the potential $\widetilde{F}^{\rm su}$ is Hölder-continuous and bounded. Thus, $\delta = \delta_{\Gamma,\,F^{\rm su}}$ is finite (and furthermore assumed to be nonpositive) and there exists indeed a Gibbs measure $\widetilde{m}_{F^{\rm su}}$ on $T^1\widetilde{M}$ for the potential $\widetilde{F}^{\rm su}$

(see Subsection 3.7). We denote respectively by $(\mu_{W^{\rm su}(v)})_{v\in T^1\widetilde M}$ and $(\mu^\iota_{W^{\rm ss}(v)})_{v\in T^1\widetilde M}$ the conditional measures of $\widetilde m_{F^{\rm su}}$ on the strong unstable and strong stable foliations defined in Subsection 3.9.

We have the following result for these conditional measures, analogous to Lemma 7.8 for the leafwise Riemannian measures. It may also be considered as a leafwise version of the Gibbs property of the Gibbs measure $\widetilde m_{F^{\rm su}}$ that has been proved in Proposition 3.16.

**Lemma 7.11** *For every compact subset $K$ in $T^1\widetilde M$ and for every $r>0$, there exists a constant $c''_{K,r}>0$ such that for all $v\in\Gamma K$ and $T\geq 0$ such that $\phi_T v\in\Gamma K$, we have*

$$\frac{1}{c''_{K,r}}\;e^{\int_0^T(\widetilde F^{\rm su}(\phi_t v)-\delta)\,dt}\leq \mu_{W^{\rm su}(v)}(\phi_{-T}B^{\rm su}(\phi_T v,r))\leq c''_{K,r}\;e^{\int_0^T(\widetilde F^{\rm su}(\phi_t v)-\delta)\,dt}\,,$$

*and for all $v\in\Gamma K$ and $T\geq 0$ such that $\phi_{-T}v\in\Gamma K$, we have*

$$\frac{1}{c''_{K,r}}\;e^{\int_{-T}^0(\widetilde F^{\rm su}\circ\iota(\phi_t v)-\delta)\,dt}\leq \mu^\iota_{W^{\rm ss}(v)}(\phi_T B^{\rm ss}(\phi_{-T} v,r))\leq c''_{K,r}\;e^{\int_{-T}^0(\widetilde F^{\rm su}\circ\iota(\phi_t v)-\delta)\,dt}\,.$$

**Proof.** Let us fix $K,r$ as in the statement. In order to prove the first claim, observe that, by Equation (51), for all $v\in T^1\widetilde M$, $r>0$ and $T\geq0$,

$$\begin{aligned}\mu_{W^{\rm su}(v)}(\phi_{-T}B^{\rm su}(\phi_T v,r))&=\int_{\phi_T w\in B^{\rm su}(\phi_T v,r)}\frac{d(\phi_T)_*\mu_{W^{\rm su}(v)}}{d\mu_{W^{\rm su}(\phi_T v)}}(\phi_T w)\,d\mu_{W^{\rm su}(\phi_T v)}(\phi_T w)\\&=\int_{\phi_T w\in B^{\rm su}(\phi_T v,r)}e^{\int_0^T(\widetilde F(\phi_t w)-\delta)\,dt}\,d\mu_{W^{\rm su}(\phi_T v)}(\phi_T w)\,. \qquad (107)\end{aligned}$$

By convexity and by Equation (9), we have

$$d(\pi(v),\pi(w))\leq d(\pi(\phi_T v),\pi(\phi_T w))\leq d_{W^{\rm su}(\phi_T v)}(\phi_T v,\phi_T w)\leq r$$

for all $v\in T^1\widetilde M$, $r>0$ and $w\in\phi_{-T}B^{\rm su}(\phi_T v,\,r)$. Hence, applying Lemma 3.2 with $r_0=r$ and by the equivariance of $\widetilde F^{\rm su}$, there exists two positive constants $c_1$ and $c_2$ such that for all $T\geq0$ and $v\in T^1\widetilde M$ such that $v,\phi_T v\in\Gamma K$, for every $w\in\phi_{-T}B^{\rm su}(\phi_T v,\,r)$, we have

$$\Big|\int_0^T(\widetilde F^{\rm su}(\phi_t w)-\delta)\,dt-\int_0^T(\widetilde F^{\rm su}(\phi_t v)-\delta)\,dt\Big|\leq 2c_1r^{c_2}+2r\max_{\pi^{-1}\big(\mathscr N_r(\pi(K))\big)}|\widetilde F^{\rm su}-\delta|\,.$$

The right hand side of this inequality is a finite constant which depends only $r$ and $K$. By the compactness of $K$ and the change of leaf property stated in Equation (52), the numbers $\mu_{W^{\rm su}(\phi_T v)}(B^{\rm su}(\phi_T v,\,r))$, for all $T\geq0$ and $v\in T^1\widetilde M$ such that $v,\phi_T v\in\Gamma K$, are bounded from above and from below by positive constants depending only on $K$ and $r$. The first claim of Lemma 7.11 hence follows from Equation (107).

The second claim is proved analogously. $\Box$

Theorem 7.2 will easily follow from the following result.

**Lemma 7.12** *Under the assumptions of Theorem 7.2, the Liouville measure $\mathrm{vol}_{T^1M}$ is absolutely continuous with respect to $m_{F^{\rm su}}$.*

**Proof.** The idea of the proof is to use the local product structure of both measures, and to prove that the conditional measures of the Liouville measure are absolutely continuous with the ones of the Gibbs measure, using the previous controls on leafwise balls and Vitali type arguments.

Since $T^1\widetilde{M}$ is $\sigma$-compact and by $\Gamma$-equivariance, we only have to prove that for every big enough compact subset $K$ of $T^1\widetilde{M}$, the restriction of $\operatorname{vol}_{T^1\widetilde{M}}$ to $K$ is absolutely continuous with respect to the restriction of $\widetilde{m}_{F^{\rm su}}$ to $K$. Let us fix a compact subset $K$ of $T^1\widetilde{M}$, containing the domain $U$ of a foliated chart of the strong unstable foliation $\widetilde{\mathscr{W}}^{\rm su}$. In particular, $K$ has positive Liouville measure. For every $v \in U$, we will denote by $W^{\rm su}_{\rm loc}(v)$ the local strong unstable leaf through $v$, that is the connected component of $v$ in the intersection $W^{\rm su}(v) \cap U$.

Let $U'$ be the set of $v \in U$ such that the (measurable) set of elements $w \in W^{\rm su}_{\rm loc}(v)$, for which there exists a sequence $(t_k)_{k\in\mathbb{N}}$ in $[0,+\infty[$ tending to $+\infty$ with $\phi_{t_k} w \in \Gamma K$ for every $k \in \mathbb{N}$, has full Riemannian measure in $W^{\rm su}_{\rm loc}(v)$. By the conservativity of the Liouville measure of $T^1M$, and therefore its recurrence, since $K$ has positive Liouville measure, and by the local product structure of the Liouville measure (see Theorem 7.6), the set $U'$ has full Liouville measure in $U$.

Let us prove that for every $v \in U'$, the restriction of $\operatorname{vol}_{W^{\rm su}(v)}$ to $W^{\rm su}_{\rm loc}(v)$ is bounded by a constant times the restriction of $\mu_{W^{\rm su}(v)}$ to $W^{\rm su}_{\rm loc}(v)$. By a similar proof, when restricted to the local strong stable leaf $W^{\rm ss}_{\rm loc}(v)$ through Liouville-almost every $v$, the measure $\operatorname{vol}_{W^{\rm ss}(v)}$ is bounded by a constant times $\mu^{\iota}_{W^{\rm ss}(v)}$. Since the Riemannian measure is absolutely continuous with respect to the product of the Riemannian measures along the strong stable leaves, strong unstable leaves and flow lines by Theorem 7.6, and by the disintegration property (see Proposition 3.18 (2)) of the Gibbs measure $\widetilde{m}_{F^{\rm su}}$ over the strong unstable measure $\mu_{W^{\rm su}(v)}$, with conditional measures $\mu^{\iota}_{W^{\rm s}(w)}$ which are by definition (see Equation (59)) absolutely continuous with respect to the products of the strong stable measures $\mu^{\iota}_{W^{\rm ss}(w)}$ (which varies absolutely continuously on $w \in W^{\rm su}_{\rm loc}(v)$ by Equation (62)) and the Lebesgue measure along the flow line, the result follows.

Let us fix $r \in \,]0,1]$. Let $A$ be a fixed measurable subset of $T^1\widetilde{M}$ whose closure is contained in $W^{\rm su}_{\rm loc}(v)$. Let $A' = A \cap U'$, which has full Riemannian measure in $A$, by the definition of $U'$. Let $V$ be any open neighbourhood of $A$ in $W^{\rm su}_{\rm loc}(v)$.

For every $w \in A$, there exists $T_w \geq 0$ such that if $t \geq T_w$, the set $\phi_{-t}B^{\rm su}(\phi_t w, r)$, which is equal to $B^{\rm su}(w, e^{-t}r)$ by Equation (13), is contained in $V$. By the definition of $U'$, for every $w \in A'$, we may choose $t_w \geq T_w$, such that $\phi_{t_w} w \in \Gamma K$. In particular, $B^{\rm su}(w, e^{-t_w} r)$ is contained in $V$ for every $w \in A'$. Up to enlarging slightly $K$, we can assume that $t_w \in \mathbb{N}$ for all $w \in A'$.

By a Vitali type of argument, let us now construct a family $(w_i)_{i\in I}$ in $A'$ indexed by a finite or countable initial segment $I$ in $\mathbb{N}$, such that, with $t_i = t_{w_i}$ to simplify the notation, the family $(B_i = B^{\rm su}(w_i, e^{-t_i}\, r))_{i\in I}$ has pairwise disjoint elements, and such that the family $(B'_i = B^{\rm su}(w_i, 2\, e^{-t_i}\, r))_{i\in I}$ covers $A'$.

We proceed by induction. If $A' = \emptyset$, take $I = \emptyset$. Otherwise, take $w_0 \in A'$ such that the integer $t_0 = t_{w_0}$ is minimal. Assume that $w_0, \dots, w_k$ are constructed. Let $w_{k+1} \in A'$ with $t_{k+1} = t_{w_{k+1}}$ minimal in the set of $w \in A'$ such that $B^{\rm su}(w, e^{-t_w}\, r)$ does not meet $\bigcup_{0\leq i\leq k} B_i$, if this set is nonempty, otherwise the construction stops at rank $k$, and we take $I = \{0, \dots, k\}$. Since $V$ is relatively compact, the sequence $(t_i)_{i\in I}$, if infinite, tends to $+\infty$. The construction then does give a finite or countable family, since $T^1\widetilde{M}$ is separable.

By construction, for every $w \in A'$, the ball $B^{\rm su}(w, e^{-t_w}\, r)$ meets $B_i$ for some $i \in I$ and $t_w \geq t_i$. Hence by the triangle inequality

$$d_{W^{\rm su}(w)}(w, w_i) \leq e^{-t_w}\, r + e^{-t_i}\, r \leq 2\, e^{-t_i}\, r\ .$$

Thus $w$ belongs to $B^{\rm su}(w_i, 2\, e^{-t_i}\, r)$. This proves the existence of $I$ as required.

Now, since the elements of $(B_i)_{i\in I}$ are contained in $V$ and are pairwise disjoint, by the $\sigma$-additivity of $\mu_{W^{\rm su}(v)}$, by Lemma 7.11, since $\delta \leq 0$, by Lemma 7.8, since the family $(B'_i)_{i\in I}$ covers $A'$, and since $A'$ has full Riemannian measure in $A$, we have

$$\begin{aligned}
\mu_{W^{\rm su}(v)}(V) &\geq \mu_{W^{\rm su}(v)}\Big(\bigcup_{i\in I} B_i\Big) = \sum_{i\in I} \mu_{W^{\rm su}(v)}(B_i) \geq \frac{1}{c''_{K,\,r}} \sum_{i\in I} e^{\int_0^{t_i} (\widetilde F^{\rm su}(\phi_s w_i) - \delta)\, ds} \\
&\geq \frac{1}{c''_{K,\,r}} \sum_{i\in I} e^{\int_0^{t_i} \widetilde F^{\rm su}(\phi_s w_i)\, ds} \geq \frac{1}{c''_{K,\,r}\, c'_{K,\,2r}} \sum_{i\in I} {\rm vol}_{W^{\rm su}(v)}(B'_i) \\
&\geq \frac{1}{c''_{K,\,r}\, c'_{K,\,2r}}\, {\rm vol}_{W^{\rm su}(v)}\Big(\bigcup_{i\in I} B'_i\Big) \\
&\geq \frac{1}{c''_{K,\,r}\, c'_{K,\,2r}}\, {\rm vol}_{W^{\rm su}(v)}(A') = \frac{1}{c''_{K,\,r}\, c'_{K,\,2r}}\, {\rm vol}_{W^{\rm su}(v)}(A)\ .
\end{aligned}$$

Since $V$ is any open neighbourhood of $A$ in $W^{\rm su}_{loc}(v)$, and as the constants $c''_{K,\,r}$ and $c'_{K,\,2r}$ do not depend on $V$, by the regularity of $\mu_{W^{\rm su}(v)}$, we have

$${\rm vol}_{W^{\rm su}(v)}(A) \leq c''_{K,\,r}\, c'_{K,\,2r}\, \mu_{W^{\rm su}(v)}(A)$$

for every Borel subset $A$ whose closure is contained in $W^{\rm su}_{loc}(v)$. This proves the result. □

Let us finally conclude the proof of Theorem 7.2. Recall that $\Omega_c\Gamma$ is the subset of elements of $T^1M$ which are positively and negatively recurrent under the geodesic flow. Since the Liouville measure on $T^1M$ is conservative, the measurable set $\Omega_c\Gamma$ has full Liouville measure, the limit set of $\Gamma$ is equal to $\partial_\infty \widetilde M$ and the (topological) non-wandering set $\Omega\Gamma$ of the geodesic flow is equal to $T^1M$.

Lemma 7.12 implies that $m_{F^{\rm su}}(\Omega_c\Gamma) > 0$, so that $m_{F^{\rm su}}$ is not completely dissipative. By the Hopf-Tsuji-Sullivan-Roblin theorems 5.3 and 5.4, this implies that $(\Gamma, F^{\rm su})$ is not of convergence type, hence is of divergence type, and that $m_{F^{\rm su}}$ is ergodic and conservative, hence also gives full measure to $\Omega_c\Gamma$. A standard ergodicity argument then implies that ${\rm vol}_{T^1M}$ and $m_{F^{\rm su}}$ are proportional, as required. Their constants for the Gibbs property have to be equal, which proves (by Theorem 4.7 and by Theorem 6.1 (1)) the final equalities stated in Theorem 7.2. □

**Remark.** The same proof shows that if the derivatives of the sectional curvature of $\widetilde M$ are uniformly bounded, if $\delta_{\Gamma,\, F^{\rm su}} = 0$, and if the geodesic flow of $M$ is recurrent with respect to the Gibbs measure $m_{F^{\rm su}}$ on $T^1M$, then $m_{F^{\rm su}}$ is proportional to the conservative part of the Liouville measure ${\rm vol}_{T^1M}$.

Note that the Ahlfors conjecture, proved by Calegari-Gabai [CG] (after works of Bonahon and Canary) says that if $\widetilde M = \mathbb{H}^3_{\mathbb{R}}$ is the real hyperbolic space of dimension 3, if $\Gamma$ is finitely generated, then either $\Lambda\Gamma$ has Lebesgue measure zero, or $\Lambda\Gamma = \partial_\infty \mathbb{H}^3_{\mathbb{R}}$. But when $\Gamma$ is not finitely generated, it could happen that $\Lambda\Gamma$ and $\partial_\infty \widetilde M - \Lambda\Gamma$ are both of positive Lebesgue measure in $\partial_\infty \mathbb{H}^3_{\mathbb{R}}$, so that the conservative and the dissipative part of the Liouville measure $\mathrm{vol}_{T^1M}$ would both be nontrivial. This explains how it could happen that $m_{F^{\rm su}}$ could be proportional to the conservative part of the Liouville measure, but not to the Liouville measure.

A similar proof (using the Furstenberg 2-recurrence property defined in the introduction and the global Gibbs property as in Proposition 7.9 instead of their foliated version as in Lemma 7.8) also shows the following fact, generalising known results when $M$ is compact (see for instance [KatH, Sect. 20.3]).

**Proposition 7.13** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$ and let $\Gamma$ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F}: T^1\widetilde{M} \to \mathbb{R}$ be a $\Gamma$-invariant Hölder-continuous potential and let $c \in \mathbb{R}$. Then there exists, up to a multiplicative constant, at most one locally finite (Borel, positive) measure on $T^1M$ invariant under the geodesic flow, ergodic and 2-recurrent, which satisfies the Gibbs property (see Definition 3.15) for the potential $F$ and the constant $c$.*

# 8 Finiteness and mixing of Gibbs states

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$; $\Gamma$ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F}: T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map.

Fix $x, y$ in $\widetilde{M}$. Assume that $\delta_{\Gamma,F} < +\infty$ (see Subsection 3.2 for comments on this condition, which is not so important but without which we can not even define Gibbs measures). Let $\widetilde{m}_F$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ for respectively $(\Gamma, F\circ\iota)$ and $(\Gamma, F)$ of (common) dimension $\delta_{\Gamma,F\circ\iota} = \delta_{\Gamma,F}$. Denote by $\|m_F\|$ the total mass of the measure $m_F$ on $T^1M$ induced by $\widetilde{m}_F$.

Compared to the previous ones, the sections 9, 10 and 11.7 will require stronger assumptions on the Gibbs measure $m_F$. We will assume that it is finite and mixing under the geodesic flow. These assumptions are already present in the case $F = 0$ considered by [Rob1]. But essentially, as explained in the coming Subsection 8.1, only the finiteness of $m_F$ is important (it is a nonempty assumption, even in the case $F = 0$, see for instance [DaOP]).

Indeed, we have proved in Corollary 5.15 that, when it is finite, the Gibbs measure $m_F$ is unique up to scaling and is ergodic (and the Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ are also unique up to scaling). We will give one finiteness criterion in Subsection 8.2. But we prefer to start by stating the main dynamical tool to be used in the coming chapters, saying that the mixing property is essentially always true as soon as the Gibbs measure $m_F$ is finite.

## 8.1 Babillot's mixing criterion for Gibbs states

Since any Gibbs measure is a quasi-product measure (see Subsection 3.7 and Subsection 3.9), we may apply Babillot's result [Bab2, Theo. 1] to obtain the mixing property of the geodesic flow. Recall that the *length spectrum* of $M$ is the subset of $\mathbb{R}$ consisting of the translation lengths of the elements of $\Gamma$, or, equivalently, of the lengths of the periodic orbits of the geodesic flow on $T^1M$. A continuous flow of homeomorphisms $(\varphi_t)_{t\in\mathbb{R}}$ of a topological space $X$ is *topologically mixing* if for all nonempty open subsets $U$ and $V$ of $X$, there exists $t_0 \in \mathbb{R}$ such that $\varphi_t(U) \cap V \neq \emptyset$ for every $t \geq t_0$.

**Theorem 8.1 (Babillot)** *If $\delta_{\Gamma,\,F} < +\infty$ and if $m_F$ is finite, then the following conditions are equivalent:*

*(1) the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M$ is mixing for the measure $m_F$;*

*(2) the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M$ is topologically mixing on its (topological) non-wandering set $\Omega\Gamma$;*

*(3) the length spectrum of $M$ is not contained in a discrete subgroup of $\mathbb{R}$.*

For instance, the last condition is satisfied if $\widetilde{M}$ is a symmetric space, or if $\Gamma$ contains a parabolic element, or if $M$ has dimension 2, or if $\Lambda\Gamma$ is not totally disconnected (see for instance [Dal1, Dal2] and their references).

Conjecturally, the non arithmeticity of the length spectrum (that is, the validity of the third assertion above) should be always true, hence the only assumption in this subsection should be the finiteness of the Gibbs measure.

Also note that, at least since Margulis's thesis [Marg], the mixing hypothesis is standard in obtaining precise counting results (see [EM] and the surveys [Bab3, Oh], amongst many references).

In Chapter 10, to prove Theorem 10.4, we will use the following consequence of the mixing property of the Gibbs measures. The forthcoming result of equidistribution of pieces of strong unstable leaves pushed by the geodesic flow, is due to [Rob1, Coro. 3.2] when $F = 0$ and to [Bab2, Theo. 3] under a more general quasi-product hypothesis, satisfied by our Gibbs measures (see Subsection 3.9).

For all $v, w$ in the same leaf of $\widetilde{\mathscr{W}}^{\rm su}$ in $T^1\widetilde{M}$, let us define

$$c_{\widetilde{F}}(w,v) = C_{F\circ\iota-\delta_{\Gamma,\,F},\,w_-}(\pi(v),\pi(w))\;, \tag{108}$$

which is equal to 0 if $\widetilde{F} = 0$ (see Subsection 6.2 for the cocycle $c_F$ associated to $\mathscr{W}^{\rm su}$ in $T^1M$, and Subsection 10.1 for more information on $c_{\widetilde{F}}$).

**Theorem 8.2 (Babillot)** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative curvature at most 1. Let $\Gamma$ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : \widetilde{M} \to \mathbb{R}$ be a $\Gamma$-invariant Hölder-continuous map. Assume that $\delta = \delta_{\Gamma,\,F}$ is finite, and that the Gibbs measure $m_F$ on $T^1M$ is finite and mixing under the geodesic flow.*

*Then for every $v \in T^1\widetilde{M}$ such that $v_- \in \Lambda\Gamma$, for every relatively compact Borel subset $B$ of $W^{\rm su}(v)$, for every uniformly continuous and $m_F$-integrable $\psi : T^1M \to \mathbb{R}$, if $\widetilde{\psi} = \psi\circ T\widetilde{p}$ where $T\widetilde{p} : T^1\widetilde{M} \to T^1M$ is the canonical projection, we have*

$$\lim_{t\to+\infty}\int_B \widetilde{\psi}\circ\phi_t(w)\; e^{c_{\widetilde{F}}(w,\,v)}\; d\mu_{W^{\rm su}(v)}(w) = \int_B e^{c_{\widetilde{F}}(w,\,v)}\; d\mu_{W^{\rm su}(v)}(w)\;\frac{1}{\|m_F\|}\int_{T^1M}\psi\; dm_F\;.$$

As usual, we may replace continuous functions with compact support by relatively compact Borel subsets with negligible boundary: Under the hypotheses of the above theorem, with $v$ and $B$ as above, for every relatively compact Borel subset $A$ of $T^1\widetilde{M}$ whose boundary has measure 0 for $\widetilde{m}_F$, we have

$$\lim_{t\to+\infty}\sum_{\gamma\in\Gamma}\int_{B\cap\phi_{-t}\gamma A} e^{c_{\widetilde{F}}(w,\,v)}\;d\mu_{W^{\rm su}(v)}(w)=\int_B e^{c_{\widetilde{F}}(w,\,v)}\;d\mu_{W^{\rm su}(v)}(w)\;\frac{\widetilde{m}_F(A)}{\|m_F\|}\;.$$

**Proof.** We only give this proof for completeness, following very closely Babillot's arguments.

Let $v$, $B$, $\psi$ and $\widetilde{\psi}$ be as in the statement. Since $\lim_{t\to+\infty}0=0$, we may assume that $B$ meets the support of $\mu_{W^{\rm su}(v)}$. By the linearity in $\psi$ of the formula to be established in Theorem 8.2, we may assume that $\psi\geq 0$. We want to compute the limit as $t\to+\infty$ of the following ratio, whose denominator is different from 0,

$$R_t=\frac{\displaystyle\int_B\widetilde{\psi}\circ\phi_t(w)\;e^{c_{\widetilde{F}}(w,\,v)}\;d\mu_{W^{\rm su}(v)}(w)}{\displaystyle\int_B e^{c_{\widetilde{F}}(w,\,v)}\;d\mu_{W^{\rm su}(v)}(w)}\geq 0\;.$$

We first thicken the subset $B$ of $W^{\rm su}(v)$ to a subset $\widetilde{\mathscr{B}}$ of $T^1\widetilde{M}$ in the following way. Let $A_+=\{w_+\;:\;w\in B\}$ be the Borel subset of $\partial_\infty\widetilde{M}$ consisting of the positive endpoints of the vectors of $B$. Let $A_-$ be a small enough compact neighbourhood of $v_-$, disjoint from the closure of $A_+$ (which exists since $A_+$ is relatively compact in $\partial_\infty\widetilde{M}-\{v_-\}$), and let $\eta>0$ be small enough. Note that $\mu^\iota_{x_0}(A_-)>0$ since $v_-$ belongs to $\Lambda\Gamma$, which is the support of $\mu^\iota_{x_0}$ by Corollary 5.12 (with the help of Corollary 5.15). Using the Hopf parametrisation $(w_-,w_+,t_w)\in\partial^2_\infty\widetilde{M}\times\mathbb{R}$ of $w\in T^1\widetilde{M}$ given in Remark (2) in Subsection 3.7 with respect to the base point $x_0=\pi(v)$, let us define now

$$\widetilde{\mathscr{B}}=\{w\in T^1\widetilde{M}\;:\;w_-\in A_-,\;\;w_+\in A_+,\;\;t_w-t_v\in[0,\eta]\}\;,$$

which is a relatively compact Borel subset of $T^1\widetilde{M}$. Up to subdividing $B$ into finitely many Borel subsets, if $A_-$ and $\eta$ are small enough, by the ergodicity of $m_F$ and by the finite additivity in $B$ of the formula to be established in Theorem 8.2, we may assume that for $\widetilde{m}_F$-almost every $x\in\widetilde{\mathscr{B}}$, exactly $N$ elements of $\widetilde{\mathscr{B}}$ have the same image as $x$ by the canonical projection $T^1\widetilde{M}\to T^1M=\Gamma\backslash T^1\widetilde{M}$, for some $N\in\mathbb{N}-\{0\}$. We denote by $\mathscr{B}$ the image of $\widetilde{\mathscr{B}}$ by this projection.

In these Hopf coordinates, the measure $\widetilde{m}_F$ can be written (see Remark (2) in Subsection 3.7) in the following quasi-product form

$$d\widetilde{m}_F(w)=f(w_-,w_+)\;d\mu^\iota_{x_0}(w_-)\,d\mu_{x_0}(w_+)\,dt_w\;,$$

where $f=(D_{F-\delta,\,x_0})^{-2}$ is a continuous and positive function on $\partial^2_\infty\widetilde{M}$. By Equation (43), for every $w\in T^1\widetilde{M}$, we have

$$f(w_-,w_+)=e^{C_{F\circ\iota-\delta,\,w_-}(x_0,\,\pi(w))+C_{F-\delta,\,w_+}(x_0,\,\pi(w))}\;.$$

By Equation (108), since $d\mu_{W^{\rm su}(v)}(w) = e^{C_{F-\delta,\, w_+}(x_0,\, \pi(w))}\, d\mu_{x_0}(w_+)$ by Equation (48), since $w_- = v_-$ and $t_w = t_v$ if $w \in W^{\rm su}(v)$, and since $\pi(v) = x_0$, we have

$$R_t = \frac{\displaystyle\int_{A_+} \widetilde{\psi} \circ \phi_t(v_-, w_+, t_v)\; f(v_-, w_+)\; d\mu_{x_0}(w_+)}{\displaystyle\int_{A_+} f(v_-, w_+)\; d\mu_{x_0}(w_+)}\;.$$

Now, let us fix $\epsilon > 0$. Note that $\widetilde{\psi}$ is uniformly continuous, that $f$ is uniformly continuous and minorated by a positive constant on compact sets, that every $w \in T^1\widetilde{M}$ such that $w_+ \neq v_-$ and the unit tangent vector with Hopf coordinates $(v_-, w_+, t_v)$ are in the same stable leaf, on which $\phi_t$ for $t \geq 0$ does not increase the distance $d'_{T^1\widetilde{M}}$, and that $\widetilde{\mathscr{B}}$ is relatively compact. Hence, if $A_+$ and $\eta$ are small enough, then for all $t \geq 0$, $r \in [0, \eta]$ and $w \in \mathscr{B}$, we have

- $e^{-\epsilon} \leq \frac{f(w_-,\, w_+)}{f(v_-,\, w_+)} \leq e^{\epsilon}$,
- $\Big|\widetilde{\psi} \circ \phi_t(w_-, w_+, t_v + r) - \widetilde{\psi} \circ \phi_t(v_-, w_+, t_v)\Big| \leq \epsilon\;.$

For every $t \geq 0$, let

$$R'_t = \frac{\displaystyle\int_{A_- \times A_+ \times [0,\, \eta]} \widetilde{\psi} \circ \phi_t(w_-, w_+, t_v + r)\; f(w_-, w_+)\; d\mu^{\iota}_{x_0}(w_-)\, d\mu_{x_0}(w_+)\, dr}{\displaystyle\int_{A_- \times A_+ \times [0,\, \eta]} f(w_-, w_+)\; d\mu^{\iota}_{x_0}(w_-)\, d\mu_{x_0}(w_+)\, dr}\;.$$

Using a simplification by $\eta\, \mu^{\iota}_{x_0}(A_-) > 0$, the above estimations show that

$$(R'_t - \epsilon)e^{-2\epsilon} \leq R_t \leq (R'_t + \epsilon)e^{2\epsilon}$$

for every $t \geq 0$. Using a simplification by $N$ for the last equality, we have

$$R'_t = \frac{\displaystyle\int_{\widetilde{\mathscr{B}}} \widetilde{\psi} \circ \phi_t\; d\widetilde{m}_F}{\widetilde{m}_F(\widetilde{\mathscr{B}})} = \frac{\displaystyle\int_{\mathscr{B}} \psi \circ \phi_t\; dm_F}{m_F(\mathscr{B})}\;.$$

Since $\psi$ and $\mathbb{1}_{\mathscr{B}}$ are $m_F$-integrable, the mixing property of the geodesic flow with respect to the normalised measure $\frac{1}{\|m_F\|} m_F$ implies that $R'_t$ converges, as $t \to +\infty$, towards $\frac{1}{\|m_F\|} \int_{T^1M} \psi\, dm_F$. The conclusion of the proof follows. □

The next subsection gives a finiteness criterion for the Gibbs measures, hence allowing us to use Babillot's theorems 8.1 and 8.2.

## 8.2 A finiteness criterion for Gibbs states

A point $p \in \partial_\infty \widetilde{M}$ is a *bounded parabolic fixed point* of $\Gamma$ if it is the fixed point of a parabolic element of $\Gamma$ and if its stabiliser $\Gamma_p$ in $\Gamma$ acts properly with compact quotient on $\Lambda\Gamma - \{p\}$. The discrete nonelementary group of isometries $\Gamma$ of $\widetilde{M}$ is said to be *geometrically finite* if every element of $\Lambda\Gamma$ is either a conical limit point or a bounded parabolic fixed point of $\Gamma$ (see for instance [Bowd]).

The following finiteness criterion for Gibbs measures is due to [DaOP] when $F = 0$. With the multiplicative approach explained at the beginning of Chapter 3, it is due to Coudène [Cou2].

**Theorem 8.3** *Assume that $\delta_{\Gamma,\,F}$ is finite, and that $\Gamma$ is geometrically finite with $(\Gamma, F)$ of divergence type. Then the Gibbs measure $m_F$ is finite if and only if for every parabolic fixed point $p$ of $\Gamma$, the series*

$$\sum_{\alpha\in\Gamma_p} d(x, \alpha y)\; e^{\int_x^{\alpha y}(\widetilde{F}-\delta_{\Gamma,\,F})}$$

*converges, where $\Gamma_p$ is the stabiliser of $p$ in $\Gamma$.*

**Proof.** We will follow the scheme of proof of [DaOP, Theo. B]. Note that the convergence of the above series depends neither on $x$ nor on $y$, and that the result is immediate if $\Gamma$ has no parabolic element, since then the support of the Gibbs measure, which is $\Omega\Gamma$, is compact. Let $\delta = \delta_{\Gamma,\,F}$.

Let $\mathrm{Par}_\Gamma$ be the set of parabolic fixed points of $\Gamma$. Since $\Gamma$ is geometrically finite (see for instance [Bowd]), the action of $\Gamma$ on $\mathrm{Par}_\Gamma$ has only finitely many orbits, and there exists a $\Gamma$-equivariant family $(\mathscr{H}_p)_{p\in\mathrm{Par}_\Gamma}$ of pairwise disjoint closed horoballs, with $\mathscr{H}_p$ centred at $p$, such that the quotient

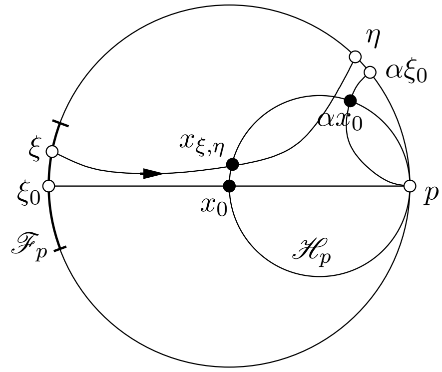
$$M_0 = \Gamma\backslash\big(\mathscr{C}\Lambda\Gamma - \bigcup_{p\in\mathrm{Par}_\Gamma}\mathscr{H}_p\big)$$

is compact. For every $p \in \mathrm{Par}_\Gamma$, we denote by $\Gamma_p$ the stabiliser of $p$ in $\Gamma$, and by $\mathscr{F}_p$ a relatively compact measurable strict fundamental domain for the action of $\Gamma_p$ on $\Lambda\Gamma - \{p\}$. The horoball $\mathscr{H}_p$ is *precisely invariant* under the stabiliser of $p$ in $\Gamma$: The inclusion $\mathscr{H}_p \subset \widetilde{M}$ induces an injection $(\Gamma_p\backslash\mathscr{H}_p) \to (\Gamma\backslash\widetilde{M})$ and we will identify $\Gamma_p\backslash\mathscr{H}_p$ with its image. Recall that $\pi : T^1M \to M$ is the canonical projection.

Since the measure $m_F$ on $T^1M$ is finite if and only if its push-forward measure $\pi_*m_F$ on $M$ is finite, since the support of $\pi_*m_F$ is contained in $\Gamma\backslash\mathscr{C}\Lambda\Gamma$, since $M_0$ is compact and since there are only finitely many orbits of parabolic fixed points, the Gibbs measure $m_F$ is finite if and only if for every parabolic fixed point $p$ of $\Gamma$, we have

$$\pi_*m_F(\Gamma_p\backslash\mathscr{H}_p) < +\infty\ .$$

Fix such a point $p$. For every $(\xi, \eta) \in \partial^2_\infty\widetilde{M}$, we will denote by $(\xi\eta)$ the geodesic line with endpoints $\xi$ and $\eta$ oriented from $\xi$ to $\eta$; if $\xi \neq p$ and $(\xi\eta)$ meets $\mathscr{H}_p$, we denote by $x_{\xi,\,\eta}$ the first intersection point of $(\xi\eta)$ with $\partial\mathscr{H}_p$. Note that since $\widetilde{M}$ is $\mathrm{CAT}(-1)$, if $\xi \neq p$ and if the geodesic lines $(\xi\eta)$ and $(\xi\eta')$ both meet $\mathscr{H}_p$, then $d(x_{\xi,\,\eta}, x_{\xi,\,\eta'}) \leq 2\log(1 + \sqrt{2})$ (see for instance [PaP1, Lemma 2.9]).




We define $x_0 = x_{\xi_0,\,p}$ for any fixed $\xi_0 \in \mathscr{F}_p$. In particular, since $\mathscr{F}_p$ is relatively compact in $\partial_\infty \widetilde{M} - \{p\}$, there exists $\kappa \geq 0$ such that for every $\xi \in \mathscr{F}_p$, for every $\eta \in \partial_\infty \widetilde{M} - \{\xi\}$ such that $(\xi\eta)$ meets $\mathscr{H}_p$, we have

$$d(x_0, x_{\xi,\,\eta}) \leq \kappa\ . \tag{109}$$

By the disjointness of the horoballs in the family $(\mathscr{H}_p)_{p\in \mathrm{Par}_\Gamma}$, a geodesic in the support of $\pi_* m_F$ meeting $\Gamma_p\backslash\mathscr{H}_p$ has a unique lift in $\widetilde{M}$ meeting $\mathscr{H}_p$ starting from the fundamental domain $\mathscr{F}_p$, unless it has a lift which starts from $p$; its other endpoint is in $\alpha\mathscr{F}_p$ for some $\alpha \in \Gamma_p$, unless it is equal to $p$. Since $\Gamma$ is of divergence type, the Patterson densities $\mu_{x_0}$ and $\mu^\iota_{x_0}$ are atomless (see Proposition 5.13) and the diagonal of $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ has zero measure with respect to $\mu^\iota_{x_0} \otimes \mu_{x_0}$ (see Assertion (c) of Proposition 5.5). Note that $\mu_{x_0}(\mathscr{F}_p)$ and $\mu^\iota_{x_0}(\mathscr{F}_p)$ are positive, otherwise $\{p\}$ would be the support of $\mu_{x_0}$ or $\mu^\iota_{x_0}$, hence would be $\Gamma$-invariant, contradicting the fact that $\Gamma$ is nonelementary.

Hence, by the very definition of the Gibbs measure $\widetilde{m}_F$, we have

$$\pi_* m_F(\Gamma_p\backslash\mathscr{H}_p) = \sum_{\alpha\in\Gamma_p} \int_{(\xi,\,\eta)\in\mathscr{F}_p\times\alpha\mathscr{F}_p} \mathrm{length}\big((\xi\eta)\cap\mathscr{H}_p\big)\ \frac{d\mu^\iota_{x_0}(\xi)\,d\mu_{x_0}(\eta)}{D_{F-\delta,\,x_0}(\xi,\eta)^2}\ .$$

By Equation (29), by Equation (109) and by Lemma 3.4 (1), there exists $c > 0$ (which depends only on $\kappa$, the constants in Lemma 3.4 and $\max_{\pi^{-1}(B(x_0,\kappa))}|\widetilde{F}|$) such that, for every $(\xi,\eta) \in \mathscr{F}_p \times \partial_\infty\widetilde{M}$ such that $\xi \neq \eta$ and $(\xi\eta)$ meets $\mathscr{H}_p$, we have

$$\frac{1}{c} \leq \frac{1}{D_{F-\delta,\,x_0}(\xi,\eta)^2} = e^{C_{F\circ\iota-\delta,\,\xi}(x_0,\,x_{\xi,\,\eta}) + C_{F-\delta,\,\eta}(x_0,\,x_{\xi,\,\eta})} \leq c\ .$$

Let $(\xi,\eta) \in \mathscr{F}_p \times \alpha\mathscr{F}_p$ be such that $\xi \neq \eta$ and $(\xi\eta)$ meets $\mathscr{H}_p$. The geodesic line from $\alpha^{-1}\eta \in \mathscr{F}_p$ to $\alpha^{-1}\xi$ also meets $\mathscr{H}_p$. The exiting point $y_{\xi,\,\eta}$ of $(\xi\eta)$ out of $\mathscr{H}_p$ is equal to $\alpha\, x_{\alpha^{-1}\eta,\,\alpha^{-1}\xi}$. Hence $d(\alpha x_0, y_{\xi,\,\eta}) = d(x_0, x_{\alpha^{-1}\eta,\,\alpha^{-1}\xi}) \leq \kappa$. Therefore, by the triangle inequality (see the picture above), we have

$$d(x_0, \alpha x_0) - 2\kappa \leq \mathrm{length}\big((\xi\eta)\cap\mathscr{H}_p\big) \leq d(x_0,\alpha x_0) + 2\kappa\ .$$

Hence

$$\frac{1}{c}\ \mu^\iota_{x_0}(\mathscr{F}_p) \sum_{\alpha\in\Gamma_p} \big(d(x_0,\alpha x_0) - 2\kappa\big)\mu_{x_0}(\alpha\mathscr{F}_p) \\ \leq \pi_* m_F(\Gamma_p\backslash\mathscr{H}_p) \leq c\ \mu^\iota_{x_0}(\mathscr{F}_p) \sum_{\alpha\in\Gamma_p} \big(d(x_0,\alpha x_0) + 2\kappa\big)\mu_{x_0}(\alpha\mathscr{F}_p)\ . \tag{110}$$

By the equations (38) and (39), for every $\alpha\in\Gamma_p$, we have

$$\mu_{x_0}(\alpha\mathscr{F}_p)=\mu_{\alpha^{-1}x_0}(\mathscr{F}_p)=\int_{\xi\in\mathscr{F}_p} e^{-C_{F-\delta,\,\xi}(\alpha^{-1}x_0,\,x_0)}\;d\mu_{x_0}(\xi)\;. \tag{111}$$

When $\alpha$ goes to infinity in the discrete group $\Gamma_p$, the point $\alpha^{-1}x_0$ converges to $p$. Hence, for every $\xi\in\mathscr{F}_p$, except for finitely many $\alpha$, the set $\mathscr{F}_p$ is contained in the shadow seen from $\alpha^{-1}x_0$ of $B(x_0,\kappa+1)$. Therefore, by Lemma 3.4 (2), there exists a constant $c'>0$ such that for all $\xi\in\mathscr{F}_p$ and $\alpha\in\Gamma_p$, we have

$$\Big|\;C_{F-\delta,\,\xi}(\alpha^{-1}x_0,x_0)+\int_{\alpha^{-1}x_0}^{x_0}(\widetilde{F}-\delta)\;\Big|\leq c'\;. \tag{112}$$

Hence by Equation (111) and by invariance, we have

$$e^{-c'}\mu_{x_0}(\mathscr{F}_p)\;e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\leq\mu_{x_0}(\alpha\mathscr{F}_p)\leq e^{c'}\,\mu_{x_0}(\mathscr{F}_p)\;e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\;. \tag{113}$$

Putting together the equations (110) and (113), there exists $C>0$ such that

$$\frac{1}{C}\sum_{\alpha\in\Gamma_p}\big(d(x_0,\alpha x_0)-C\big)e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\leq\pi_*m_F(\Gamma_p\backslash\mathscr{H}_p)$$
$$\leq C\sum_{\alpha\in\Gamma_p}\big(d(x_0,\alpha x_0)+C\big)e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\;.$$

This proves the result. □

A criterion for $(\Gamma,F)$ to be of divergence type, when $\Gamma$ is geometrically finite, is the following one, again due to [DaOP] when $F=0$.

**Theorem 8.4** *Assume that $\delta_{\Gamma,\,F}$ is finite and that $\Gamma$ is geometrically finite. If for every parabolic fixed point $p$ of $\Gamma$ with stabiliser $\Gamma_p$, we have $\delta_{\Gamma_p,\,F}<\delta_{\Gamma,\,F}$, then $(\Gamma,F)$ is of divergence type.*

**Proof.** Let $(\mu_x)_{x\in\widetilde{M}}$ be the Patterson density for $(\Gamma,F)$ of dimension $\delta_{\Gamma,\,F}$ constructed in Proposition 3.9: we use the notation $h,s_k,\overline{Q}_{z,\,y}(s),\mu_{z,\,s}$ for $k\in\mathbb{N},s>\delta_{\Gamma,\,F}$, $z\in\widetilde{M}$ introduced for that purpose. We fix an arbitrary $y\in\widetilde{M}$. We start with an independent lemma (which is necessary, by Proposition 5.13).

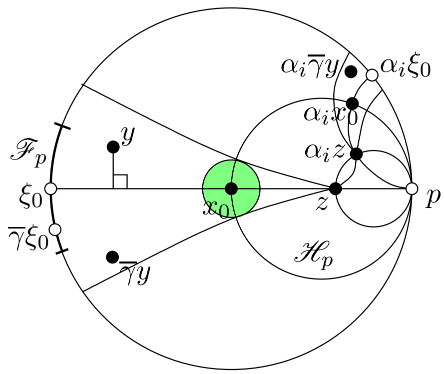
**Lemma 8.5** *Let $p$ be a bounded parabolic fixed point of $\Gamma$ such that $\delta_{\Gamma_p,\,F}<\delta_{\Gamma,\,F}$. Then $\mu_x(\{p\})=0$.*

**Proof.** Let $\mathscr{H}_p,\mathscr{F}_p,\kappa,\xi_0,x_0$ be as in the above proof of Theorem 8.3. We may assume that $\xi_0\notin\Gamma p$. Up to replacing $\mathscr{H}_p$ by a smaller horoball centred at $p$, we may assume that the closest point to $y$ on the geodesic line $]\xi_0,p[$ between $\xi_0$ and $p$ does not belong to the geodesic ray $[x_0,p[$. In particular, by the convexity of the horoballs and since $\gamma\mathscr{H}_p\cap\mathscr{H}_p$ is empty if $\gamma\notin\Gamma_p$, the orbit $\Gamma y$ does not meet $\mathscr{H}_p$.

For every $\gamma\in\Gamma$, choose a representative $\overline{\gamma}$ of the left coset of $\gamma$ in $\Gamma_p\backslash\Gamma$ such that $\overline{\gamma}\xi_0\in\mathscr{F}_p$ (which is possible since $\gamma\xi_0\in\Lambda\Gamma-\{p\}$ and $\mathscr{F}_p$ is a fundamental domain for the action of $\Gamma_p$ on $\Lambda\Gamma-\{p\}$).

Let $\epsilon > 0$ such that $\delta_{\Gamma,\,F} > \delta_{\Gamma_p,\,F} + \epsilon$. Let $r_\epsilon \geq 0$ such that $h(t + r) \leq e^{\epsilon t} h(r)$ for all $t \geq 0$ and $r \geq r_\epsilon$. There exists $\kappa' \geq \kappa$ large enough so that if $z$ is the point of $[x_0, p[x$ at distance $\max\{r_\epsilon, \kappa' + 2\}$ of $x_0$, then for every $\gamma \in \Gamma$, there exists a representative $\overline{\gamma}$ of the left coset of $\gamma$ in $\Gamma_p\backslash\Gamma$ such that $\overline{\gamma}\, y \in \mathscr{O}_z B(x_0, \kappa')$. In particular, $d(z, \Gamma y) \geq r_\epsilon$. Let $\epsilon' \in [0, 1]$ be such that

$$\mu_z\big(\partial(\mathscr{O}_z B(x_0, \kappa' + \epsilon'))\big) = 0 \ .$$




Let $X$ be the compact subset $\Gamma y \cup \Lambda\Gamma$ of $\widetilde{M} \cup \partial_\infty \widetilde{M}$, which contains the support of the measures $\mu_{z,\,s}$ and $\mu_z$ for all $z \in \widetilde{M}$ and $s > \delta_{\Gamma,\,F}$. Let

$$\mathscr{C} = X \cap \bigcup_{\xi \in \mathscr{O}_z B(x_0,\,\kappa' + \epsilon')} [z, \xi]$$

be the set of points of $X$ on a geodesic ray, including its point at infinity, from $z$ through the closed ball $B(x_0, \kappa' + \epsilon')$.

By Equation (109) and by convexity, the set $\mathscr{F}_p$ is contained in the shadow $\mathscr{O}_z B(x_0, \kappa)$, hence in $\mathscr{C}$. Since the point $\overline{\gamma}y$ belongs to $\mathscr{C}$ for every $\gamma \in \Gamma$, we have $X = \{p\} \cup \bigcup_{\alpha \in \Gamma_p} \alpha\mathscr{C}$.

Let $(\alpha_i)_{i\in\mathbb{N}}$ be an enumeration of $\Gamma_p$, and for every $n \in \mathbb{N}$, let $U_n = X - \bigcup_{0 \leq i \leq n} \alpha_i\mathscr{C}$. Then $(U_n)_{n\in\mathbb{N}}$ is a non-increasing sequence of neighbourhoods of $p$ in $X$, with intersection $\{p\}$. Hence

$$\mu_z(\{p\}) = \lim_{n\to+\infty} \mu_z(U_n) \ .$$

By the convexity of horoballs and a standard argument of quasi-geodesics, there exists a constant $c > 0$ such that for $i$ large enough and for every $\gamma \in \Gamma$, the point $\alpha_i z$ is at distance at most $c$ from the geodesic segment $[z, \alpha_i\overline{\gamma}y]$ (see the picture above). Hence, by (two applications of) Lemma 3.2, there exists a constant $c' > 0$ such that, for every large enough $i$, for every $\gamma \in \Gamma$, and for every $s \in [\delta_{\Gamma,\,F}, \delta_{\Gamma,\,F} + 1]$, we have

$$\Big|\ \int_z^{\alpha_i\overline{\gamma}y} (\widetilde{F} - s) - \int_z^{\alpha_i z} (\widetilde{F} - s)\ - \int_z^{\overline{\gamma}y} (\widetilde{F} - s)\ \Big| \leq c' \ .$$

Since $h$ is nondecreasing, by the triangle inequality and since $d(z, \Gamma y) \geq r_\epsilon$, we have

$$h(d(z, \alpha_i\overline{\gamma}y)) \leq h(d(z, \alpha_i z) + d(z, \overline{\gamma}y)) \leq e^{\epsilon\, d(z,\,\alpha_i z)}\ h(d(z, \overline{\gamma}y)) \ .$$

For every $s \in ]\delta_{\Gamma,\,F}, \delta_{\Gamma,\,F}+1]$, we hence have, for every $n$ large enough,

$$\begin{aligned}\mu_{z,\,s}(U_n) &\leq \sum_{i=n+1}^{+\infty} \sum_{\overline{\gamma} \in \Gamma_p \backslash \Gamma} \frac{1}{\overline{Q}_{y,\,y}(s)}\, h(d(z, \alpha_i \overline{\gamma} y))\, e^{\int_z^{\alpha_i \overline{\gamma} y} (\widetilde{F}-s)} \\ &\leq e^{c'} \sum_{i=n+1}^{+\infty} e^{\int_z^{\alpha_i z} (\widetilde{F}-(s-\epsilon))} \sum_{\overline{\gamma} \in \Gamma_p \backslash \Gamma} \frac{1}{\overline{Q}_{y,\,y}(s)}\, h(d(z, \overline{\gamma} y))\, e^{\int_z^{\overline{\gamma} y} (\widetilde{F}-s)} \\ &\leq e^{c'}\, \|\mu_{z,s}\| \sum_{i=n+1}^{+\infty} e^{\int_z^{\alpha_i z} (\widetilde{F}-(s-\epsilon))}\ .\end{aligned}$$

Taking $s = s_k$ for $k$ large enough, and letting $k$ tends to $+\infty$, we get (note that the constant function 1 on $\partial_\infty \widetilde{M}$ is continuous with compact support and that $\mu_z(\partial U_n) = 0$ for every $n \in \mathbb{N}$, by the choice of $\epsilon'$)

$$\mu_z(U_n) \leq e^{c'}\, \|\mu_z\| \sum_{i=n+1}^{+\infty} e^{\int_z^{\alpha_i z} (\widetilde{F}-(\delta_{\Gamma,\,F}-\epsilon))}\ .$$

Since $\delta_{\Gamma,\,F} - \epsilon > \delta_{\Gamma_p,\,F}$ and since the remainder of a convergent series tends to 0, we have

$$\mu_z(\{p\}) = \lim_{n \to +\infty} \mu_z(U_n) = 0\ .$$

Since $\mu_x$ and $\mu_z$ are absolutely continuous with respect to each other, Lemma 8.5 follows. □

Now, since the limit points of $\Gamma$ that are not conical ones are (countably many) bounded parabolic fixed points if $\Gamma$ is geometrically finite, this lemma implies that $\mu_x(\Lambda_c\Gamma) = \mu_x(\Lambda\Gamma) > 0$. Theorem 8.4 hence follows from Corollary 5.10. □

**Corollary 8.6** *Assume that $\delta_{\Gamma,\,F}$ is finite, that $\Gamma$ is geometrically finite, and that $\delta_{\Gamma_p,\,F} < \delta_{\Gamma,\,F}$ for every parabolic fixed point $p$, where $\Gamma_p$ is the stabiliser of $p$ in $\Gamma$. Then the Gibbs measure $m_F$ is finite.*

In particular, if $\Gamma$ is geometrically finite, if $\delta_{\Gamma_p} < \delta_\Gamma$ for every $p \in \mathrm{Par}_\Gamma$ (which is in particular the case when $\widetilde{M}$ is a symmetric space), and if $\|F\|_\infty$ is small enough (that is, if $\|F\|_\infty \leq \frac{1}{2}(\delta_\Gamma - \delta_{\Gamma_p})$ for every $p \in \mathrm{Par}_\Gamma$), then $m_F$ is finite, by Lemma 3.3 (iv).

**Proof.** This is immediate by Theorems 8.4 and 8.3. □

As in [Cou2], this allows us to construct finite (hence ergodic) Gibbs measures for any geometrically finite group $\Gamma$, by choosing an appropriate potential $F$ in the preimage in $T^1M$ of the cuspidal parts of $M$.

# 9 Growth and equidistribution of orbits and periods

We will give in this chapter precise asymptotic results as $t$ goes to $+\infty$ for the counting functions defined in Subsection 4.1, as corollaries of convergence results for suitable measures and equidistribution results.

Let $(\widetilde{M}, \Gamma, F)$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$; $\Gamma$ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map. Fix $x, y$ in $\widetilde{M}$. Assume that $\delta_{\Gamma,\,F} < +\infty$. Let $\widetilde{m}_F$ be the Gibbs measure on $T^1\widetilde{M}$ associated with a pair of Patterson densities $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ for respectively $(\Gamma, F \circ \iota)$ and $(\Gamma, F)$ of (common) dimension $\delta_{\Gamma,\,F\circ\iota} = \delta_{\Gamma,\,F}$. Denote by $\|m_F\|$ the total mass of the measure $m_F$ on $T^1M = \Gamma\backslash T^1\widetilde{M}$ induced by $\widetilde{m}_F$.

## 9.1 Convergence of measures on the square product

The aim of this subsection is to prove the following weak-star convergence result, whose consequences will be discussed in the next subsections.

**Theorem 9.1** *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of $(\Gamma, F)$ is finite and positive, and that the Gibbs measure $m_F$ is finite and mixing under the geodesic flow on $T^1M$. As $t$ goes to $+\infty$, the measures*

$$\nu_{x,\,y,\,F,\,t} = \delta_{\Gamma,\,F}\;\|m_F\|\;e^{-\delta_{\Gamma,\,F}\,t} \sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde{F}}\;\mathscr{D}_{\gamma^{-1}x}\otimes\mathscr{D}_{\gamma y}$$

*converge to the product measure $\mu^\iota_y \otimes \mu_x$ with respect to the weak-star convergence of measures on $(\widetilde{M} \cup \partial_\infty\widetilde{M})^2$.*

We will follow closely the proof for the case $F = 0$ given in [Rob1, Chap. 4]. It relies on the next technical proposition, improved in Proposition 9.3. To simplify the notation, we set $\nu_T = \nu_{x,\,y,\,F,\,T}$ for every $T \ge 0$. We recall the notation $\mathscr{C}^\pm_r(z, Z)$, $K^\pm(z, r, Z)$, $K(z, r)$, $v_{\xi,\,\eta,\,z}$, $L_r(z, w)$ and $\mathscr{O}^\pm_r(z, w)$ of Subsection 5.1, as well as the notation of the antipodal map $\iota_z : \partial_\infty\widetilde{M} \to \partial_\infty\widetilde{M}$ with respect to any $z \in \widetilde{M}$ defined in Subsection 2.3.

**Proposition 9.2** *Assume that $\delta_{\Gamma,\,F}$ is finite and positive, and that $m_F$ is finite and mixing under the geodesic flow on $T^1M$. For every $\epsilon > 0$, for all $\xi_0$ and $\eta_0$ in $\partial_\infty\widetilde{M}$ such that $\iota_x(\xi_0) \in \Lambda\Gamma$ and $\iota_y(\eta_0) \in \Lambda\Gamma$, there exist open neighbourhoods $V$ and $W$ of respectively $\xi_0$ and $\eta_0$ in $\partial_\infty\widetilde{M}$ such that for all Borel subsets $A \subset V$ and $B \subset W$,*

$$\limsup_{T\to+\infty}\;\nu_T\big(\mathscr{C}^-_1(y, B) \times \mathscr{C}^-_1(x, A)\big) \;\le\; e^{\epsilon}\,\mu^\iota_y(B)\,\mu_x(A)$$

$$\text{and}\quad \liminf_{T\to+\infty}\;\nu_T\big(\mathscr{C}^+_1(y, B) \times \mathscr{C}^+_1(x, A)\big) \;\ge\; e^{-\epsilon}\,\mu^\iota_y(B)\,\mu_x(A)\,.$$

**Proof.** To simplify the notation, let $\delta = \delta_{\Gamma,\,F} > 0$. We will denote by $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ positive functions of $\epsilon$ (depending only on $\delta$, on $\max_{\pi^{-1}(B(z,2))}|\widetilde{F}|$ for $z = x, y$, and on the constants $c_1, c_2, c_3, c_4 > 0$ appearing in Lemma 3.4 for $\widetilde{F}$ and $\widetilde{F}\circ\iota$), that converge to 0 as $\epsilon$ goes to 0, and whose exact computation is, though possible, unnecessary.

Let $\epsilon, \xi_0, \eta_0$ be as in the statement of Proposition 9.2. We first define the neighbourhoods $V$ and $W$ required in the statement of Proposition 9.2.

Let $r$ be in $]0, \min\{1, \epsilon\}[$ possibly outside a countable subset. By the finiteness of the Patterson densities, we may assume that

$$\mu^\iota_x(\partial\mathscr{O}_{\xi_0}B(x, r)) = \mu_y(\partial\mathscr{O}_{\eta_0}B(y, r)) = 0\,. \tag{114}$$

Since the support of the Patterson densities is $\Lambda\Gamma$ (since $m_F$ is finite, see Corollary 5.12 and 5.15), and since $\iota_x(\xi_0)$ and $\iota_y(\eta_0)$ belong to $\Lambda\Gamma$, we have

$$C_r = \mu^\iota_x(\mathscr{O}_{\xi_0}B(x,r))\;\mu_y(\mathscr{O}_{\eta_0}B(y,r)) > 0\;. \tag{115}$$

By Equation (114) and the convergence property of $\mathscr{O}^\pm_r(z,r)$ to $\mathscr{O}_{\xi_0}B(x,r)$ as $z\to\xi_0$ described in Subsection 5.1, there exist open neighbourhoods $\widehat{V},\widehat{W}$ of (respectively) $\xi_0,\eta_0$ in $\widetilde{M}\cup\partial_\infty\widetilde{M}$ such that for every $r$ as above, for all $z\in\widehat{V}$ and $w\in\widehat{W}$,

$$e^{-\epsilon}\mu^\iota_x(\mathscr{O}_{\xi_0}B(x,r))\le\mu^\iota_x(\mathscr{O}^\pm_r(z,x))\le e^{\epsilon}\mu^\iota_x(\mathscr{O}_{\xi_0}B(x,r)) \tag{116}$$

and

$$e^{-\epsilon}\mu_y(\mathscr{O}_{\eta_0}B(y,r))\le\mu_y(\mathscr{O}^\pm_r(w,y))\le e^{\epsilon}\mu_y(\mathscr{O}_{\eta_0}B(y,r))\;. \tag{117}$$

Finally, let $V,W$ be open neighbourhoods of $\xi_0,\eta_0$ in $\partial_\infty\widetilde{M}$ whose closures are contained in $\widehat{V},\widehat{W}$ respectively.

For all Borel subsets $A$ in $V$ and $B$ in $W$, we defined (see Subsection 5.1)

$$K^+=K^+(x,r,A)\quad\text{and}\quad K^-=K^-(y,r,B)\;.$$

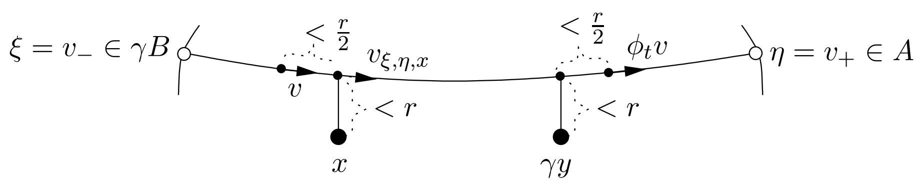
The heart of the proof is to give two pairs of upper and lower bounds (respectively in Equation (119) and Equation (121)) of

$$I_\pm(T)=\int_0^{T\pm3r} e^{\delta\,t}\sum_{\gamma\in\Gamma}\widetilde{m}_F(K^+\cap\phi_{-t}\gamma K^-)\;dt\;.$$

Both of them use the following estimate: For every $(\xi,\eta)\in\partial^2_\infty\widetilde{M}$ such that the orthogonal projection $w$ of $x$ on the geodesic line between $\xi$ and $\eta$ satisfies $d(w,x)\le r\le\min\{1,\epsilon\}$, and in particular if $(\xi,\eta)\in L_r(x,\gamma y)$, we have, by Equation (29) and Lemma 3.4 (1),

$$e^{-\epsilon_1}\le D_{F-\delta,\,x}(\xi,\eta)=e^{-\frac12(C_{F-\delta,\,\eta}(x,\,w)+C_{F\circ\iota-\delta,\,\xi}(x,\,w))}\le e^{\epsilon_1}\;. \tag{118}$$

**First upper and lower bound.** We use the definition of the Gibbs measure to estimate individually $\widetilde{m}_F(K^+\cap\phi_{-t}\gamma K^-)$.




For every $\gamma\in\Gamma$ such that $d(x,\gamma y)>2r$ and for every $t\ge0$, we have, by the definition of the various geometric sets in Subsection 5.1, that $K^+\cap\phi_{-t}\gamma K^-$ is the set of $v\in T^1\widetilde{M}$ such that $v$ is $r$-close to $x$, $v_+\in A$, $v_-\in\gamma B$ and $\phi_t v$ is $r$-close to $\gamma y$ (see the picture above). Furthermore, by the definition of $\widetilde{m}_F$,

$$\widetilde{m}_F(K^+\cap\phi_{-t}\gamma K^-)=$$

$$\int_{(\xi,\,\eta)\in L_r(x,\,\gamma y)\cap(\gamma B\times A)}\frac{d\mu^\iota_x(\xi)\;d\mu_y(\eta)}{D_{F-\delta,\,x}(\xi,\eta)^2}\int_{-\frac r2}^{\frac r2}\mathbb{1}_{K(\gamma y,\,r)}(\phi_{t+s}v_{\xi,\,\eta,\,x})\;ds\;,$$

where $\mathbb{1}_Z$ is the characteristic function of a subset $Z$.

As in [Rob1, pages 59, 60] which only uses an estimate as in Equation (118) and geometric arguments, there exists a constant $c_1 > 0$ such that for every $T > 3r$, if

$$J_\pm(T) = \sum_{\substack{\gamma \in \Gamma \;:\; d(x,\gamma y) \leq T, \\ \gamma^{-1}x \in \mathscr{C}_1^\pm(y,B) \cap \widehat{W}, \\ \gamma y \in \mathscr{C}_1^\pm(x,A) \cap \widehat{V}}} \mu_x^\iota\big(\mathscr{O}_r^\pm(\gamma y, x)\big)\; \mu_x\big(\mathscr{O}_r^\pm(x,\gamma y)\big) e^{\delta\, d(x,\gamma y)} \;,$$

then the following two inequalities hold

$$I_-(T) \leq \; e^{\epsilon_2}\; r^2\; J_+(T) \; + c_1 \quad \text{and} \quad I_+(T) \geq \; e^{-\epsilon_2}\; r^2\; J_-(T) \quad - c_1 \;. \tag{119}$$

**Second upper and lower bound.** Now, we use the mixing property of the geodesic flow to estimate the sum $\sum_{\gamma\in\Gamma} \widetilde{m}_F(K^+ \cap \phi_{-t}\gamma K^-)$.

Since the geodesic flow is mixing (the definitions in Section 2.6 take care of the possible presence of torsion in $\Gamma$), we have, for $t$ large enough,

$$e^{-\epsilon}\; \widetilde{m}_F(K^+)\; \widetilde{m}_F(K^-) \;\leq\; \|m_F\| \sum_{\gamma\in\Gamma} \widetilde{m}_F(K^+ \cap \phi_{-t}\gamma K^-) \;\leq\; e^{\epsilon}\; \widetilde{m}_F(K^+)\; \widetilde{m}_F(K^-) \;. \tag{120}$$

By the definition of $K^+ = K^+(x,r,A)$, we have

$$\widetilde{m}_F(K^+) = r \int_{\eta\in A} d\mu_x(\eta) \int_{\zeta\in\mathscr{O}_\eta B(x,r)} D_{F-\delta,\,x}(\zeta,\eta)^{-2}\; d\mu_x^\iota(\zeta) \;.$$

By Equation (118) and by Equation (116) applied to $z \in A$ since $A \subset \widehat{V}$, we have

$$e^{-\epsilon-2\epsilon_1}\; r\; \mu_x(A)\; \mu_x^\iota(\mathscr{O}_{\xi_0} B(x,r)) \;\leq \widetilde{m}_F(K^+) \leq\; e^{\epsilon+2\epsilon_1}\; r\; \mu_x(A)\; \mu_x^\iota(\mathscr{O}_{\xi_0} B(x,r)) \;.$$

Similarly

$$e^{-\epsilon-2\epsilon_1}\; r\; \mu_y^\iota(B)\; \mu_y(\mathscr{O}_{\eta_0} B(y,r)) \;\leq \widetilde{m}_F(K^-) \leq\; e^{\epsilon+2\epsilon_1}\; r\; \mu_y^\iota(B)\; \mu_y(\mathscr{O}_{\eta_0} B(y,r)) \;.$$

By taking the product of these inequalities, by multiplying Equation (120) by $e^{\delta t}$ and integrating it over $t \in [0, T \pm 3r]$, we have since $\delta > 0$ and $r \leq \epsilon$, for some constant $c_2$ independent of $T$ (coming from the fact that the estimations above are valid only for $t$ large enough, hence we have to cut the integral over $t \in [0, T \pm 3r]$ in two), and by the definition of $C_r$ in Equation (115), for $T$ large enough,

$$\delta\; \|m_F\|\; I_-(T) \geq e^{-\epsilon_3}\; r^2\; C_r\; \mu_y^\iota(B)\; \mu_x(A)\; e^{\delta T} \quad - c_2 \tag{121}$$

$$\text{and} \quad \delta\; \|m_F\|\; I_+(T) \leq e^{\epsilon_3}\; r^2\; C_r\; \mu_y^\iota(B)\; \mu_x(A)\; e^{\delta T} \quad + c_2 \;.$$

**Estimate on $C_r$.** Before showing how these two upper and lower bounds of $I_\pm(T)$ prove the result, let us give a lower and upper estimate on $C_r$.

By the defining properties of the Patterson densities (see the equations (38) and (39)), we have

$$\mu_y\big(\mathscr{O}_r^\pm(\gamma^{-1}x, y)\big) = \mu_{\gamma y}\big(\mathscr{O}_r^\pm(x,\gamma y)\big) = \int_{\eta\in\mathscr{O}_r^\pm(x,\gamma y)} e^{C_{F-\delta,\,\eta}(x,\gamma y)}\; d\mu_x(\eta) \;.$$

Note that $\mathscr{O}_r^{\pm}(x,\gamma y)\subset \mathscr{O}_x B(\gamma y,2r)$ by the convexity of the distance. Hence, by Lemma 3.4 (2) applied to $\widetilde{F}-\delta$ and $y'=\gamma y$, since $\max_{\pi^{-1}(B(\gamma y,\,2r))}|\widetilde{F}|\leq \max_{\pi^{-1}(B(y,2))}|\widetilde{F}|$ by invariance, and since $r\leq\epsilon$, we have

$$e^{-\epsilon_4}\,\mu_y\big(\mathscr{O}_r^{\pm}(\gamma^{-1}x,y)\big)\;\leq\;\mu_x\big(\mathscr{O}_r^{\pm}(x,\gamma y)\big)\,e^{-\int_x^{\gamma y}(\widetilde{F}-\delta)}\;\leq\;e^{\epsilon_4}\,\mu_y\big(\mathscr{O}_r^{\pm}(\gamma^{-1}x,y)\big)\;.$$

If $(\gamma y,\gamma^{-1}x)\in\widehat{V}\times\widehat{W}$, we have, respectively by the definition of $C_r$ (see Equation (115)), the definitions of $\widehat{V}$ and $\widehat{W}$ (see the equations (116) and (117)), and the previous inequality,

$$\begin{aligned}C_r&=\mu_x^{\iota}(\mathscr{O}_{\xi_0}B(x,r))\,\mu_y(\mathscr{O}_{\eta_0}B(y,r))\\&\geq e^{-2\epsilon}\,\mu_x^{\iota}(\mathscr{O}_r^{+}(\gamma y,x))\,\mu_y(\mathscr{O}_r^{+}(\gamma^{-1}x,y))\\&\geq e^{-2\epsilon-\epsilon_4}\,\mu_x^{\iota}(\mathscr{O}_r^{+}(\gamma y,x))\,\mu_x\big(\mathscr{O}_r^{+}(x,\gamma y)\big)\,e^{-\int_x^{\gamma y}(\widetilde{F}-\delta)}\;.\end{aligned}$$

Similarly,

$$C_r\leq e^{2\epsilon+\epsilon_4}\,\mu_x^{\iota}(\mathscr{O}_r^{-}(\gamma y,x))\,\mu_x\big(\mathscr{O}_r^{-}(x,\gamma y)\big)\,e^{-\int_x^{\gamma y}(\widetilde{F}-\delta)}\;.$$

**Conclusion of the proof.** Now, respectively by the definition of the measure $\nu_T=\nu_{x,\,y,\,F,\,T}$, by using some constant $c_3>0$ independent of $T$, by the previous lower bound on $C_r$ and the definition of $J_{\pm}(T)$, by Equation (119), and by Equation (121) with some constant $c_4$ independent of $T$, we have

$$\begin{aligned}C_r\,\nu_T\big(\mathscr{C}_1^{+}(y,B)\times\mathscr{C}_1^{+}(x,A)\big)&=\delta\,\|m_F\|\,e^{-\delta T}\sum_{\substack{\gamma\in\Gamma\;:\;d(x,\gamma y)\leq T,\\(\gamma^{-1}x,\gamma y)\in\mathscr{C}_1^{+}(y,B)\times\mathscr{C}_1^{+}(x,A)}}C_r\,e^{\int_x^{\gamma y}\widetilde{F}}\\&\geq\delta\,\|m_F\|\,e^{-\delta T}\sum_{\substack{\gamma\in\Gamma\;:\;d(x,\gamma y)\leq T,\\ \gamma^{-1}x\in\mathscr{C}_1^{+}(y,B)\cap\widehat{W},\\ \gamma y\in\mathscr{C}_1^{+}(x,A)\cap\widehat{V}}}C_r\,e^{\int_x^{\gamma y}\widetilde{F}}\;-c_3\,e^{-\delta T}\\&\geq\delta\,\|m_F\|\,e^{-\delta T}\,e^{-2\epsilon-\epsilon_4}\,J_{+}(T)-c_3\,e^{-\delta T}\\&\geq\delta\,\|m_F\|\,e^{-\delta T}\,e^{-\epsilon_2-2\epsilon-\epsilon_4}\,r^{-2}\big(I_{-}(T)-c_1\big)-c_3\,e^{-\delta T}\\&\geq e^{-\epsilon_5}\,C_r\,\mu_y^{\iota}(B)\,\mu_x(A)-c_4\,e^{-\delta T}\;.\end{aligned}$$

Similarly,

$$C_r\,\nu_T\big(\mathscr{C}_1^{-}(y,B)\times\mathscr{C}_1^{-}(x,A)\big)\leq e^{\epsilon_5}\,C_r\,\mu_y^{\iota}(B)\,\mu_x(A)+c_4\,e^{-\delta T}\;.$$

Dividing by $C_r>0$ (independent of $T$), the result follows. □

The next result is a small improvement of Proposition 9.2 which gets rid of the assumption that $\iota_x(\xi_0),\iota_y(\eta_0)\in\Lambda\Gamma$, at the price of replacing 1-thickened/1-thinned cones by $R$-thickened/$R$-thinned ones, for some $R>0$ large enough. Note that when $\Lambda\Gamma=\partial_\infty\widetilde{M}$, then this step is not necessary (that is $R=1$ works). In particular the reader interested only in the case when $\Gamma$ is cocompact may skip this proposition.

**Proposition 9.3** *Assume that $\delta_{\Gamma,\,F}$ is finite and positive, and that $m_F$ is finite and mixing under the geodesic flow on $T^1M$. For every $\epsilon'>0$, for all $\xi_0$ and $\eta_0$ in $\partial_\infty\widetilde{M}$, there exist $R>0$ and open neighbourhoods $V$ and $W$ of respectively $\xi_0$ and $\eta_0$ in $\partial_\infty\widetilde{M}$ such that for all Borel subsets $A\subset V$ and $B\subset W$,*

$$\limsup_{T\to+\infty}\;\nu_T\big(\mathscr{C}_R^{-}(y,B)\times\mathscr{C}_R^{-}(x,A)\big)\;\leq\;e^{\epsilon'}\,\mu_x^{\iota}(B)\,\mu_x(A)$$

$$\text{and}\quad \liminf_{T\to+\infty}\ \nu_T\big(\mathscr{C}_R^+(y,B)\times\mathscr{C}_R^+(x,A)\big)\ \geq\ e^{-\epsilon'}\,\mu_x^\iota(B)\,\mu_x(A)\;.$$

**Proof.** The idea of the proof is that two cones over the same subset of $\partial_\infty\widetilde{M}$, with vertices two distinct points in $\widetilde{M}$, are asymptotic near infinity, and that we have an appropriate change of base point formula for the Patterson measures.

Let $\epsilon>0$ and $\xi_0,\eta_0\in\partial_\infty\widetilde{M}$. Fix $\zeta_0\in\Lambda\Gamma-\{\xi_0,\eta_0\}$ and $x_0$ (respectively $y_0$) a point on the geodesic line between $\zeta_0$ and $\xi_0$ (respectively $\zeta_0$ and $\eta_0$). Let

$$R=1+\max\{d(x,x_0),\ d(y,y_0)\}$$

and, to simplify the notation, let $\delta=\delta_{\Gamma,\,F}$.

Let us now define the neighbourhoods $V$ and $W$ required by the statement. Let $V_0,W_0$ be small enough open neighbourhoods of $\xi_0,\eta_0$ respectively in $\partial_\infty\widetilde{M}$, so that Proposition 9.2 holds with $x_0,y_0,V_0,W_0$ replacing $x,y,V,W$ respectively. Let $\widehat{V_0},\widehat{W_0}$ be open neighbourhoods of $\xi_0,\eta_0$ in $\widetilde{M}\cup\partial_\infty\widetilde{M}$ such that $V_0$ contains $\widehat{V_0}\cap\partial_\infty\widetilde{M}$ and $W_0$ contains $\widehat{W_0}\cap\partial_\infty\widetilde{M}$, respectively, and such that for all $z\in\widehat{V_0}\cap\widetilde{M}$, $w\in\widehat{W_0}\cap\widetilde{M}$,

$$|\,d(x_0,z)-d(x,z)-\beta_{\xi_0}(x_0,x)\,|\leq\epsilon\;,\quad |\,d(y_0,w)-d(y,w)-\beta_{\eta_0}(y_0,y)\,|\leq\epsilon\;,\tag{122}$$

$$\Big|\int_x^z\widetilde{F}-\int_{x_0}^z\widetilde{F}+C_{F,\,\xi_0}(x,x_0)\Big|\leq\epsilon\quad\text{and}\quad\Big|\int_y^w\widetilde{F}\circ\iota-\int_{y_0}^w\widetilde{F}\circ\iota+C_{F\circ\iota,\,\eta_0}(y,y_0)\Big|\leq\epsilon\;.\tag{123}$$

Let us consider neighbourhoods $V$ and $W$ of $\xi_0$ and $\eta_0$, respectively, in $\partial_\infty\widetilde{M}$ whose closures are contained in $\widehat{V_0}$ and $\widehat{W_0}$, respectively.

Now, we start the proof with some computations. If $\gamma y_0,\gamma y\in\widehat{V_0}$ and $\gamma^{-1}x_0,\gamma^{-1}x\in\widehat{W_0}$, the formulae (122) and (123) imply that

$$\begin{aligned}d(x_0,\gamma y_0)\leq d(x,\gamma y_0)+\beta_{\xi_0}(x_0,x)+\epsilon&=d(y_0,\gamma^{-1}x)+\beta_{\xi_0}(x_0,x)+\epsilon\\&\leq d(y,\gamma^{-1}x)+\beta_{\eta_0}(y_0,y)+\beta_{\xi_0}(x_0,x)+2\epsilon\\&=d(x,\gamma y)-\beta_{\xi_0}(x,x_0)-\beta_{\eta_0}(y,y_0)+2\epsilon\;,\end{aligned}$$

and that, by Equation (16),

$$\begin{aligned}e^{\int_x^{\gamma y}\widetilde{F}}&\leq e^{\int_{x_0}^{\gamma y}\widetilde{F}-C_{F,\,\xi_0}(x,x_0)+\epsilon}=e^{\int_y^{\gamma^{-1}x_0}\widetilde{F}\circ\iota-C_{F,\,\xi_0}(x,x_0)+\epsilon}\\&\leq e^{\int_{y_0}^{\gamma^{-1}x_0}\widetilde{F}\circ\iota-C_{F\circ\iota,\,\eta_0}(y,y_0)-C_{F,\,\xi_0}(x,x_0)+2\epsilon}=e^{\int_{x_0}^{\gamma y_0}\widetilde{F}}\;e^{-C_{F\circ\iota,\,\eta_0}(y,y_0)-C_{F,\,\xi_0}(x,x_0)+2\epsilon}\;.\end{aligned}$$

Let $\widehat{V_{-R}}=\{z\in\widetilde{M}\ :\ B(z,R)\subset\widehat{V_0}\}$ and $\widehat{W_{-R}}=\{z\in\widetilde{M}\ :\ B(z,R)\subset\widehat{W_0}\}$, and let $A$ and $B$ be Borel subsets of $V$ and $W$ respectively. Note that if

$$(\gamma^{-1}x,\gamma y)\in\big(\mathscr{C}_R^-(y,B)\times\mathscr{C}_R^-(x,A)\big)\cap\big(\widehat{W_{-R}}\times\widehat{V_{-R}}\big)\;,$$

then

$$(\gamma^{-1}x_0,\gamma y_0)\in\big(\mathscr{C}_1^-(y_0,B)\times\mathscr{C}_1^-(x_0,A)\big)\cap\big(\widehat{W_0}\times\widehat{V_0}\big)$$

by the choice of $R$ and the definition of the $r$-thickened/$r$-thinned cones $\mathscr{C}_r^\pm(z,Z)$, and that $\big(\mathscr{C}_R^-(y,B)\times\mathscr{C}_R^-(x,A)\big)\setminus\big(\widehat{W_{-R}}\times\widehat{V_{-R}}\big)$ is a bounded subset of $\widetilde{M}\times\widetilde{M}$.

Now, by the definition of the measure $\nu_T = \nu_{x,\,y,\,F,\,T}$, and by the previous computations, for some constant $c_1' > 0$ independent of $T$, we have

$$\begin{aligned}
&\nu_T\big(\mathscr{C}_R^-(y,B)\times\mathscr{C}_R^-(x,A)\big)\\
&\qquad = \delta\,\|m_F\|\,e^{-\delta T}\sum_{\substack{\gamma\in\Gamma\;:\;d(x,\gamma y)\,\leq\, T,\\ (\gamma^{-1}x,\,\gamma y)\,\in\,\mathscr{C}_R^-(y,B)\times\mathscr{C}_R^-(x,A)}} e^{\int_x^{\gamma y}\widetilde{F}}\\
&\qquad \leq \nu_{x_0,\,y_0,\,F,\,T-\beta_{\xi_0}(x,x_0)-\beta_{\eta_0}(y,y_0)+2\epsilon}\big(\mathscr{C}_1^-(y_0,B)\times\mathscr{C}_1^-(x_0,A)\big)\\
&\qquad\quad\times\; e^{-C_{F\circ\iota,\,\eta_0}(y,\,y_0)-C_{F,\,\xi_0}(x,\,x_0)+2\epsilon}\;e^{-\delta\beta_{\xi_0}(x,\,x_0)-\delta\beta_{\eta_0}(y,\,y_0)+2\epsilon}\;+c_1'e^{-\delta T}\;.
\end{aligned}$$

By Equation (24) and Proposition 9.2, we then have

$$\begin{aligned}
&\limsup_{T\to+\infty}\;\nu_T\big(\mathscr{C}_R^-(y,B)\times\mathscr{C}_R^-(x,A)\big)\\
&\qquad\qquad\leq e^{5\epsilon}\;\mu^\iota_{y_0}(B)\;\mu_{x_0}(A)\;e^{-C_{F\circ\iota-\delta,\,\eta_0}(y,\,y_0)}\;e^{-C_{F-\delta,\,\xi_0}(x,\,x_0)}\;.
\end{aligned}$$

By continuity in the equations (122) and (123), since $A\subset V\subset\widehat{V_0}$ and $B\subset W\subset\widehat{W_0}$, we have, for all $\xi\in A$ and $\eta\in B$,

$$|C_{F-\delta,\,\xi}(x,x_0)-C_{F-\delta,\,\xi_0}(x,x_0)|\leq(1+\delta)\epsilon$$

and

$$|C_{F\circ\iota-\delta,\,\eta}(y,y_0)-C_{F\circ\iota-\delta,\,\eta_0}(y,y_0)|\leq(1+\delta)\epsilon\;.$$

By Equation (39) for the Patterson densities $(\mu_x)_{x\in\widetilde{M}}$ and $(\mu^\iota_x)_{x\in\widetilde{M}}$, we hence have

$$\mu_{x_0}(A)\;e^{-C_{F-\delta,\,\xi_0}(x,x_0)}\leq\mu_x(A)\;e^{(1+\delta)\epsilon}$$

and

$$\mu^\iota_{y_0}(B)\;e^{-C_{F\circ\iota-\delta,\,\eta_0}(y,y_0)}\leq\mu^\iota_y(B)\;e^{(1+\delta)\epsilon}\;.$$

Therefore

$$\limsup_{T\to+\infty}\;\nu_T\big(\mathscr{C}_R^-(y,B)\times\mathscr{C}_R^-(x,A)\big)\;\leq\;e^{(7+2\delta)\epsilon}\;\mu^\iota_y(B)\;\mu_x(A)\;.$$

The analogous lower bound is proved similarly, and Proposition 9.3 follows. □

**Proof of Theorem 9.1.** The end of the proof of Theorem 9.1 follows from Proposition 9.3 exactly as in the "Troisième étape : conclusion" in [Rob1, page 62-63], by replacing $\mu_x$ there by $\mu_x$, as well as $\mu_y$ by $\mu^\iota_y$, $\nu^t_{x,y}$ by $\nu_{x,\,y,\,F,\,t}$, and $r$ by $R$. □

As mentioned by the referee (who provided its proof), we have the following version of Theorem 9.1 when the critical exponent is possibly nonpositive. We use the convention that $\frac{1-e^{-s}}{s}=1$ if $s=0$, and denote by $\overset{*}{\rightharpoonup}$ the weak-star convergence of measures.

**Theorem 9.4** *Assume that $\delta_{\Gamma,\,F}<+\infty$ and that the Gibbs measure $m_F$ is finite and mixing under the geodesic flow on $T^1M$. For every $c>0$, as $t$ goes to $+\infty$, we have*

$$\frac{\|m_F\|\;\delta_{\Gamma,\,F}}{1-e^{-c\delta_{\Gamma,\,F}}}\;e^{-\delta_{\Gamma,\,F}\;t}\sum_{\gamma\in\Gamma\;:\;t-c<d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}}\;\mathscr{D}_{\gamma^{-1}x}\otimes\mathscr{D}_{\gamma y}\;\overset{*}{\rightharpoonup}\;\mu^\iota_y\otimes\mu_x\;.$$

Note that Theorem 9.1 follows from this result by an easy geometric series argument.

**Proof.** The proof of Theorem 9.1 could be adapted to prove this version. But by applying Theorem 9.1 by replacing $F$ by $F+\kappa$ for $\kappa$ large enough so that $\delta_{\Gamma,\,F+\kappa} = \delta_{\Gamma,\,F} + \kappa > 0$, since proving the weak-star convergence of measures can be checked by evaluation on nonnegative continuous maps with compact support having positive integral for the limit measure, Theorem 9.4 is also a consequence of the following classical lemma.

**Lemma 9.5** *Let $I$ be a discrete set and $f, g : I \to [0, +\infty[$ be maps with $f$ proper. Assume that there exist $\delta, \kappa \in \mathbb{R}$ with $\delta + \kappa > 0$ such that, as $t \to +\infty$,*

$$\sum_{i\in I,\ f(i)\leq t} e^{\kappa f(i)}\, g(i) \ \sim\ \frac{e^{(\delta+\kappa)t}}{\delta+\kappa}\,.$$

*Then for every $c > 0$, we have, as $t \to +\infty$,*

$$\sum_{i\in I,\ t-c<f(i)\leq t} g(i) \ \sim\ \frac{1-e^{-c\,\delta}}{\delta}\ e^{\delta\, t}\,.$$

**Proof.** For every $\epsilon > 0$ small enough, if $s$ is large enough, we have

$$e^{-\epsilon^2}\ \frac{e^{(\delta+\kappa)s}}{\delta+\kappa} \leq \sum_{i\in I,\ f(i)\leq s} e^{\kappa f(i)}\, g(i) \leq e^{\epsilon^2}\ \frac{e^{(\delta+\kappa)s}}{\delta+\kappa}\,.$$

Applying this at $s$ and $s-\epsilon$ and subtracting, we have, for some $A > 0$ large enough independent of $s$ and $\epsilon$,

$$e^{-A\epsilon}\ \epsilon\ e^{\delta\, s} \leq \sum_{i\in I,\ s-\epsilon<f(i)\leq s} g(i) \leq e^{A\epsilon}\ \epsilon\ e^{\delta\, s}\,.$$

Let $n \in \mathbb{N}$ and $t$ be large enough. Applying this with $\epsilon = \frac{c}{n}$ and $s = t - \frac{ck}{n}$ for $k = 0, \dots, n-1$ and summing over $k$, a geometric series argument gives

$$e^{-A\frac{c}{n}}\ \frac{c}{n}\ \frac{1-e^{-c\,\delta}}{1-e^{-\frac{c\delta}{n}}}\ e^{\delta\, t} \leq \sum_{i\in I,\ t-c<f(i)\leq t} g(i) \leq e^{A\frac{c}{n}}\ \frac{c}{n}\ \frac{1-e^{-c\,\delta}}{1-e^{-\frac{c\delta}{n}}}\ e^{\delta\, t}\,,$$

say when $\delta \neq 0$, but with the conventions obvious by continuity, this also holds when $\delta = 0$. Since $\frac{c(1-e^{-c\,\delta})}{n(1-e^{-\frac{c\delta}{n}})} \to \frac{1-e^{-c\,\delta}}{\delta}$ as $n$ goes to $+\infty$, the result follows. □ □

## 9.2 Counting orbit points of discrete groups

We deduce in this subsection some corollaries of Theorem 9.1, keeping the notation of the beginning of Chapter 9.

**Corollary 9.6** *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of $(\Gamma, F)$ is finite and positive, and that the Gibbs measure $m_F$ is finite and mixing under the geodesic flow in $T^1M$. As $t$ goes to $+\infty$, the measures*

$$\delta_{\Gamma,\,F}\ \|m_F\|\ e^{-\delta_{\Gamma,\,F}\ t} \sum_{\gamma\in\Gamma\ :\ d(x,\gamma y)\leq t} e^{\int_x^{\gamma y} \widetilde{F}}\ \mathscr{D}_{\gamma y}$$

*converge to the measure* $\|\mu^\iota_y\|\,\mu_x$ *for the weak-star convergence of measures on* $\widetilde{M}\cup\partial_\infty\widetilde{M}$*, and the measures*

$$\delta_{\Gamma,\,F}\,\|m_F\|\;e^{-\delta_{\Gamma,\,F}\;t}\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}}\;\mathscr{D}_{\gamma^{-1}x}$$

*similarly converge to the measure* $\|\mu_x\|\,\mu^\iota_y$*.*

**Proof.** Since the push-forward of measures by a continuous map is linear and weak-star continuous, the result follows from Theorem 9.1 using the projections $(\widetilde{M}\cup\partial_\infty\widetilde{M})^2\to(\widetilde{M}\cup\partial_\infty\widetilde{M})$ on the first factor and on the second factor. $\Box$

The second assertion of Corollary 1.4 may also be obtained by exchanging $x$ and $y$ and $F$ and $F\circ\iota$ (see Equation (16) and the last remark of Subsection 5.3).

**Corollary 9.7** *Assume that the critical exponent* $\delta_{\Gamma,\,F}$ *of* $(\Gamma,F)$ *is finite and positive, and that the Gibbs measure* $m_F$ *is finite and mixing under the geodesic flow on* $T^1M$*. As* $t$ *goes to* $+\infty$*,*

$$\sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}}\;\sim\;\frac{\|\mu^\iota_y\|\;\|\mu_x\|}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{\delta_{\Gamma,\,F}\;t}\;.$$

**Proof.** This follows by taking the total mass of the above measures on the compact space $\widetilde{M}\cup\partial_\infty\widetilde{M}$. $\Box$

As another immediate corollary of Corollary 9.6, we obtain the following sharp asymptotic property on the sectorial orbital counting function $G_{\Gamma,\,F,\,x,\,y,\,U}$ (introduced in Subsection 4.1), improving Corollary 4.5 (1).

**Corollary 9.8** *Assume that the critical exponent* $\delta_{\Gamma,\,F}$ *of* $(\Gamma,F)$ *is finite and positive, and that the Gibbs measure* $m_F$ *is finite and mixing under the geodesic flow on* $T^1M$*. Let* $U$ *be an open subset of* $\partial_\infty\widetilde{M}$ *meeting* $\Lambda\Gamma$ *such that* $\mu_x(\partial U)=0$*. Then as* $t$ *goes to* $+\infty$*,*

$$G_{\Gamma,\,F,\,x,\,y,\,U}(t)\;\sim\;\frac{\|\mu^\iota_y\|\;\mu_x(U)}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{\delta_{\Gamma,\,F}\;t}\;.$$

**Proof.** This follows by considering the characteristic functions of cones $\mathscr{C}_xU$ on open subsets $U$ of $\partial_\infty\widetilde{M}$ (or by taking $V=\partial_\infty\widetilde{M}$ in the next corollary). $\Box$

Similarly, we obtain the following sharp asymptotic property on the bisectorial orbital counting function $G_{\Gamma,\,F,\,x,\,y,\,U,\,V}$ (introduced in Subsection 4.1), improving Corollary 4.5 (2).

**Corollary 9.9** *Assume that the critical exponent* $\delta_{\Gamma,\,F}$ *of* $(\Gamma,F)$ *is finite and positive, and that the Gibbs measure* $m_F$ *is finite and mixing under the geodesic flow on* $T^1M$*. Let* $U$ *and* $V$ *be two open subsets of* $\partial_\infty\widetilde{M}$ *meeting* $\Lambda\Gamma$ *such that* $\mu_x(\partial U)=\mu^\iota_y(\partial V)=0$*. Then as* $t$ *goes to* $+\infty$*,*

$$G_{\Gamma,\,F,\,x,\,y,\,U,\,V}(t)\;\sim\;\frac{\mu_x(U)\;\mu^\iota_y(V)}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{\delta_{\Gamma,\,F}\;t}\;.$$

**Proof.** Use Theorem 9.1 and consider the product of the characteristic functions of the cones $\mathscr{C}_x U$ and $\mathscr{C}_y V$ on the open subsets $U$ and $V$ of $\partial_\infty \widetilde{M}$. $\Box$

In the same way Theorem 9.4 has been deduced from Theorem 9.1, the following result, which no longer assumes $\delta_{\Gamma,\,F} > 0$, can be deduced from Corollaries 9.6, 9.7, 9.8 and 9.9.

**Corollary 9.10** *Assume that $\delta_{\Gamma,\,F} < +\infty$, and that the Gibbs measure $m_F$ is finite and mixing under the geodesic flow on $T^1M$. Let $U$ and $V$ be two open subsets of $\partial_\infty \widetilde{M}$ meeting $\Lambda\Gamma$ such that $\mu_x(\partial U) = \mu^\iota_x(\partial V) = 0$. Then for every $c > 0$, as $t$ goes to $+\infty$,*

$$\frac{\delta_{\Gamma,\,F}\;\|m_F\|}{1-e^{-c\,\delta_{\Gamma,\,F}}}\;e^{-\delta_{\Gamma,\,F}\;t}\sum_{\gamma\in\Gamma\;:\;t-c<d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}}\;\mathscr{D}_{\gamma y}\;\overset{*}{\rightharpoonup}\;\|\mu^\iota_y\|\;\mu_x\,;$$

$$\frac{\delta_{\Gamma,\,F}\;\|m_F\|}{1-e^{-c\,\delta_{\Gamma,\,F}}}\;e^{-\delta_{\Gamma,\,F}\;t}\sum_{\gamma\in\Gamma\;:\;t-c<d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}}\;\mathscr{D}_{\gamma^{-1} x}\;\overset{*}{\rightharpoonup}\;\|\mu_x\|\;\mu^\iota_y\,;$$

$$\sum_{\gamma\in\Gamma\;:\;t-c<d(x,\gamma y)\leq t} e^{\int_x^{\gamma y}\widetilde{F}}\;\sim\;\frac{\|\mu^\iota_y\|\;\|\mu_x\|\;(1-e^{-c\,\delta_{\Gamma,\,F}})}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{\delta_{\Gamma,\,F}\;t}\,;$$

$$G_{\Gamma,\,F,\,x,\,y,\,U,\,c}(t)\;\sim\;\frac{\|\mu^\iota_y\|\;\mu_x(U)\;(1-e^{-c\,\delta_{\Gamma,\,F}})}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{\delta_{\Gamma,\,F}\;t}\,;$$

$$G_{\Gamma,\,F,\,x,\,y,\,U,\,V,\,c}(t)\;\sim\;\frac{\mu_x(U)\;\mu^\iota_y(V)\;(1-e^{-c\,\delta_{\Gamma,\,F}})}{\delta_{\Gamma,\,F}\;\|m_F\|}\;e^{\delta_{\Gamma,\,F}\;t}\,.$$

### 9.3 Equidistribution and counting of periodic orbits of the geodesic flow

The aim of this subsection is to use Theorem 9.1 to prove that the periodic orbits of the geodesic flow in $T^1M$, appropriately weighted by their periods with respect to the potential, equidistribute with respect to the Gibbs measure. When $F = 0$ and $M$ is convex-cocompact, the result is due to Bowen [Bow1, Bow2, Bow3], see also [Par2]. We follow Roblin's proof when $F = 0$ in [Rob1].

Recall (see Subsection 4.1) that for every $t \in \mathbb{R}$ and for every periodic (we will only consider primitive ones in this subsection) orbit $g$ of length $\ell(g)$ of the geodesic flow on $T^1M$, we denote by $\mathscr{L}_g$ the Lebesgue measure along $g$, by $\int_g F = \mathscr{L}_g(F)$ the period of $g$ for the potential $F$, by $\mathscr{P}\mathrm{er}'(t)$ the set of primitive periodic orbits of the geodesic flow on $T^1M$ with length at most $t \in \mathbb{R}$.

As in Bowen's two equidistribution results in the convex-cocompact case, the first assertion of the theorem below claims the equidistribution of the Lebesgue measures of the closed orbits when weighted by the exponential of their periods for the potential, the second one claims the equidistribution of their Lebesgue means.

**Theorem 9.11** *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of $(\Gamma, F)$ is finite and positive, and that the Gibbs measure $m_F$ is finite and mixing under the geodesic flow of $T^1M$.*

*(1) As $t$ goes to $+\infty$, the measures*

$$\delta_{\Gamma,\,F}\;\;e^{-\delta_{\Gamma,\,F}\;t}\sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\;\mathscr{L}_g$$

*converge to the normalised Gibbs measure $\frac{m_F}{\|m_F\|}$ with respect to the weak-star convergence of measures on $T^1M$.*

*(2) As $t$ goes to $+\infty$, the measures*

$$\delta_{\Gamma,\,F}\; t\;\; e^{-\delta_{\Gamma,\,F}\;t} \sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\;\frac{\mathscr{L}_g}{\ell(g)}$$

*weak-star converge to $\frac{m_F}{\|m_F\|}$.*

**Proof.** Let $\delta=\delta_{\Gamma,\,F}$. Let us denote by $\mathscr{H}_\Gamma$ the set of loxodromic elements of $\Gamma$. For every $\gamma\in\mathscr{H}_\Gamma$, let $\ell(\gamma)$ be its translation length, let $\mathrm{Axe}_\gamma$ be its translation axis, oriented from the repulsive fixed point $\gamma_-$ to the attractive one $\gamma_+$, and let $\mathscr{L}_\gamma$ be the measure on $T^1\widetilde{M}$ which is the lift to $T^1\widetilde{M}$ of the Lebesgue measure along the translation axis of $\gamma$: For every map $f:T^1\widetilde{M}\to\mathbb{R}$ which is continuous with compact support, we have

$$\mathscr{L}_\gamma(f)=\int_{t\in\mathbb{R}} f(\phi_t v)\;dt$$

where $v$ is any unit tangent vector to the oriented geodesic line $\mathrm{Axe}_\gamma$. Recall that the period of $\gamma$ for $F$ is $\mathrm{Per}_F(\gamma)=\int_0^{\ell(\gamma)}\widetilde{F}(\phi_t v)\;dt$ with $v$ as above. Note that $\alpha_*\mathscr{L}_\gamma=\mathscr{L}_{\alpha\gamma\alpha^{-1}}$ for every $\alpha\in\Gamma$. The set $\mathscr{H}_{\Gamma,\,t}$ of loxodromic elements of $\Gamma$ whose translation length is at most $t$, and its subset $\mathscr{H}'_{\Gamma,\,t}$ of primitive elements, are invariant under conjugation by elements of $\Gamma$.

(1) Let us prove the first statement of Theorem 9.11. Every primitive periodic orbit $g$ of the geodesic flow in $T^1M$ is the image by $T^1\widetilde{M}$ of the lift by $\pi:T^1\widetilde{M}\to\widetilde{M}$ of the (oriented) translation axis of a primitive loxodromic element of $\Gamma$, whose translation length is the length of $g$, and the number of such loxodromic elements with given translation axis is equal to the cardinality of the stabiliser of any tangent vector to this translation axis. By the definition (and continuity) of the induced measure in $T^1M$ of a $\Gamma$-invariant measure in $T^1\widetilde{M}$ by the branched cover $T^1\widetilde{M}\to T^1M$, we only have to show that, as $t$ goes to $+\infty$, the measures

$$\nu'_t=\delta\;\; e^{-\delta\,t}\sum_{\gamma\in\mathscr{H}'_{\Gamma,\,t}} e^{\mathrm{Per}_F(\gamma)}\;\mathscr{L}_\gamma$$

converge to $\frac{\widetilde{m}_F}{\|m_F\|}$ with respect to the weak-star convergence of measures on $T^1\widetilde{M}$.

**Step 1.** Let us first prove that, as $t$ goes to $+\infty$, the measures

$$\nu''_t=\delta\;\; e^{-\delta\,t}\sum_{\gamma\in\mathscr{H}_{\Gamma,\,t}} e^{\mathrm{Per}_F(\gamma)}\;\mathscr{L}_\gamma$$

converge to $\frac{\widetilde{m}_F}{\|m_F\|}$ with respect to the weak-star convergence of measures on $T^1\widetilde{M}$.

By the definition of the weak-star convergence, for every fixed compact subset $K$ of $\widetilde{M}$, we only need to prove this convergence when the measures are restricted to the compact subset $\pi^{-1}(K)$ of $T^1\widetilde{M}$. Let $\epsilon\in\,]0,\frac{1}{2}]$. As in the proof of Proposition 9.2, we will denote by $\epsilon_1,\epsilon_2,\epsilon_3$ positive functions of $\epsilon$ (depending only on $\delta>0$, on $\max_{\pi^{-1}(\mathscr{N}_1(K))}|\widetilde{F}|$, and on the constants $c_1,c_2,c_3,c_4>0$ appearing in Lemma 3.4 for $\widetilde{F}$ as well as for $\widetilde{F}\circ\iota$), that converge to $0$ as $\epsilon$ goes to $0$.

For every $x \in K$, let $V(x,\epsilon)$ be the set of pairs $(\xi,\eta)$ of distinct elements in $\widetilde{M} \cup \partial_\infty \widetilde{M}$ such that the geodesic segment, ray or line between $\xi$ and $\eta$ meets $B(x,\epsilon)$. As in Equation (118), by Equation (29) and Lemma 3.4 (1), for every $(\xi,\eta) \in V(x,\epsilon) \cap \partial^2_\infty \widetilde{M}$, we have

$$e^{-\epsilon_1} \le D_{F-\delta,\,x}(\xi,\eta) \le e^{\epsilon_1} \;. \tag{124}$$

For all $x \in K$ and $t > 0$, let $\nu$ and $\nu_t$ be the measures on $(\widetilde{M} \cup \partial_\infty \widetilde{M})^2$ defined by:

$$d\nu(\xi,\eta) = \frac{d\mu^\iota_x(\xi) \otimes d\mu_x(\eta)}{D_{F-\delta,\,x}(\xi,\eta)^2}\;, \qquad \nu_t = \delta\, \|m_F\|\, e^{-\delta t} \sum_{\gamma\in\Gamma,\; d(x,\gamma x)\le t} e^{\int_x^{\gamma x} \widetilde{F}}\; \mathscr{D}_{\gamma^{-1}x} \otimes \mathscr{D}_{\gamma x} \;.$$

By Theorem 9.1 (taking $y = x$), we know that $\nu_t$ weak-star converges to $\mu^\iota_x \otimes \mu_x$ as $t \to +\infty$. Let $\psi$ be a continuous nonnegative map with support in $V(x,\epsilon)$. By Equation (124), we hence have

$$e^{-2\,\epsilon_1}\;\nu(\psi) \le \liminf_{t\to+\infty} \nu_t(\psi) \le \limsup_{t\to+\infty} \nu_t(\psi) \le e^{2\,\epsilon_1}\;\nu(\psi) \;.$$

Let

$$\nu'''_t = \delta\, \|m_F\|\, e^{-\delta t} \sum_{\gamma\in\mathscr{H}_{\Gamma,\,t}} e^{\mathrm{Per}_F(\gamma)}\;\; \mathscr{D}_{\gamma_-} \otimes \mathscr{D}_{\gamma_+} \;.$$

Note that by Gromov's hyperbolicity criterion (see also Lemma 2.8), for every $\gamma \in \Gamma$, if $(\gamma^{-1}x, \gamma x) \in V(x,\epsilon)$ and $d(x,\gamma x)$ is large enough, then $\gamma$ is loxodromic and $x$ is at distance at most $2\epsilon$ from the translation axis $\mathrm{Axe}_\gamma$ of $\gamma$. In particular

$$\ell(\gamma) \le d(x,\gamma x) \le \ell(\gamma) + 4\epsilon \;.$$

Furthermore, $\gamma^{\pm 1}x$ is close to $\gamma_\pm$ uniformly in $\gamma$, hence so is $\mathscr{D}_{\gamma^{\pm1}x}$ to $\mathscr{D}_{\gamma_\pm}$. By Lemma 3.2 and the remark following it, if $p$ is the closest point on $\mathrm{Axe}_\gamma$ to $x$, we hence have

$$\begin{aligned}\Big|\int_x^{\gamma x} \widetilde{F} - \mathrm{Per}_F(\gamma)\Big| &= \Big|\int_x^{\gamma x} \widetilde{F} - \int_p^{\gamma p} \widetilde{F}\,\Big| \\ &\le 2\, c_3 (2\epsilon)^{c_4} + 2\,\epsilon\,\Big(\max_{\pi^{-1}(B(x,\,2\epsilon))} |\widetilde{F}| + \max_{\pi^{-1}(B(\gamma x,\,2\epsilon))} |\widetilde{F}|\Big) \;.\end{aligned}$$

The right hand side of this inequality, which we will denote by $\eta(2\epsilon)$ for future use, tends to 0 as $\epsilon$ goes to 0 uniformly in $\gamma$ by the $\Gamma$-invariance of $\widetilde{F}$. Hence if $t$ is large enough, then

$$e^{-\epsilon_2}\nu_t(\psi) \le \nu'''_t(\psi) \le e^{\epsilon_2}\nu_{t+4\epsilon}(\psi) \;.$$

Identifying $T^1\widetilde{M}$ with $\partial^2_\infty \widetilde{M} \times \mathbb{R}$ by the Hopf parametrisation with respect to the base point $x$, note that

$$\frac{1}{\|m_F\|}\;\nu'''_t \otimes ds = \nu''_t \;.$$

Therefore, for every continuous map $\psi' : T^1\widetilde{M} \to \mathbb{R}$ with compact support in $V(x,\epsilon) \times \mathbb{R}$, which is a product of two continuous maps on each of the two variables of this product, we have

$$e^{-\epsilon_3}\;\frac{m_F(\psi')}{\|m_F\|} \le \liminf_{t\to+\infty} \nu''_t(\psi') \le \limsup_{t\to+\infty} \nu''_t(\psi') \le e^{\epsilon_3}\;\frac{m_F(\psi')}{\|m_F\|} \;.$$

Approximating uniformly continuous maps with compact support in $V(x,\epsilon)\times\mathbb{R}$ by finite linear combinations of product maps, covering $\pi^{-1}(K)$ by finitely many sets $V(x,\epsilon)\times\mathbb{R}$, using a partition of unity and letting $\epsilon$ goes to 0, the first step follows.

**Step 2.** Let us now prove that $\nu''_t-\nu'_t$ weak-star converges to 0 as $t\to+\infty$, which implies the first assertion of Theorem 9.11.

We start by proving the following lemma, provided by the referee, which says that the contribution of the non-primitive periodic orbits is negligible.

Let $x\in\widetilde{M}$, $\epsilon\in\,]0,\frac{1}{2}]$ and $U_-,U_+$ be small enough neighbourhoods in $\partial_\infty\widetilde{M}$ of two distinct points in $\Lambda\Gamma$ (with boundary of measure 0 for the Patterson measures $\mu^\iota_x,\mu_x$ respectively) so that the geodesic line between any point in $U_-$ and any point in $U_+$ passes at distance at most $\epsilon$ from $x$. In particular, for every loxodromic element $\gamma\in\Gamma$ with $\gamma_\pm\in U_\pm$, its translation axis passes at distance at most $\epsilon$ from $x$, so that, as seen above,

$$\ell(\gamma)\le d(x,\gamma x)\le \ell(\gamma)+2\epsilon\,, \tag{125}$$

and, with the above notation $\eta(\cdot)$,

$$\Big|\int_x^{\gamma x}\widetilde{F}-\mathrm{Per}_F(\gamma)\Big|\le\eta(\epsilon)\,. \tag{126}$$

For every $k\in\mathbb{N}-\{0\}$, let $H_k$ be the set of loxodromic elements $\gamma\in\Gamma$ such that $\gamma_\pm\in U_\pm$ and there exists $\alpha\in\Gamma$ with $\gamma=\alpha^k$. Note that $H_k=\{\alpha^k\;:\;\alpha\in H_1\}$ and $H_k\subset H_1$.

**Lemma 9.12** *For every $\epsilon'>0$, there exists $C>0$ and $\rho\in\,]0,1[$ such that, for all $n,k\in\mathbb{N}$ with $k\ge 2$, we have*

$$e^{-\delta n}\sum_{\gamma\in H_k,\;n-1<d(x,\gamma x)\le n} e^{\int_x^{\gamma x}\widetilde{F}}\le C\,\rho^n+\frac{\epsilon'}{2^k}\,.$$

**Proof.** In order to obtain the two terms in the right hand side of this inequality, we will separate the sum on its left hand side into the elements of $H_k$ which are $k$-th powers of elements with not too large translation length, and the others.

Let us fix $\epsilon''>0$. For every $\alpha\in H_1$, we have

$$e^{\mathrm{Per}_F(\alpha)}<e^{\delta\,\ell(\alpha)}\,,$$

otherwise for every $n\in\mathbb{N}$, we have $e^{\mathrm{Per}_F(\alpha^n)}\ge e^{\delta\,\ell(\alpha^n)}$, hence $e^{\int_x^{\alpha^n x}\widetilde{F}}\ge e^{-\eta(\epsilon)-2\epsilon}\,e^{\delta\,d(x,,\alpha^n x)}$ by Equations (125) and (126), a contradiction to Proposition 5.13 (2). By discreteness, there are only finitely many loxodromic elements in $\Gamma$, whose translation axis passes at distance at most $\epsilon$ from $x$, with translation length as most some constant. Hence, again by Proposition 5.13 (2), there exists $t_0=t_0(\epsilon'')>0$ such that if $\alpha\in H_1$ satisfies $\ell(\alpha)\ge t_0$, then

$$e^{\int_x^{\alpha x}\widetilde{F}}\le\epsilon''\,e^{\delta\,d(x,\,\alpha x)}\,.$$

Since $\sum_{\alpha'\in H_1,\;\ell(\alpha')\le t}e^{\mathrm{Per}_F(\alpha')}=\nu''_t(U_-\times U_+)$ and as seen above, there exists $t_1,c>0$ such that, for all $t\ge t_1$,

$$\sum_{\alpha'\in H_1,\;t-1<\ell(\alpha')\le t}e^{\int_x^{\alpha' x}\widetilde{F}}\ge e^{-\eta(\epsilon)}\sum_{\alpha'\in H_1,\;t-1<\ell(\alpha')\le t}e^{\mathrm{Per}_F(\alpha')}\ge\frac{1}{c}\,e^{\delta\,t}\,. \tag{127}$$

Let $t_2 = \max\{t_0, t_1\}$. We define $\rho = \max_{\alpha\in H_1,\, \ell(\alpha)\leq t_2} e^{\frac{\mathrm{Per}_F(\alpha)}{\ell(\alpha)}-\delta} \in\, ]0,1[$ and $C = e^{\eta(\epsilon)}\,\rho^{-1-2\,\epsilon}\,\mathrm{Card}\{\alpha\in H_1\;:\;\ell(\alpha)\leq t_2\}$ (which both depend on $\epsilon''$).

Now, as indicated at the beginning, we write, for every $n,k\in\mathbb{N}$ with $k\geq 2$, using Equations (125) and (126),

$$\sum_{\substack{\gamma\in H_k\\ n-1<d(x,\gamma x)\leq n}} e^{\int_x^{\gamma x}\widetilde{F}} = \sum_{\substack{\alpha\in H_1\\ n-1<\,d(x,\alpha^k x)\leq n}} e^{\int_x^{\alpha^k x}\widetilde{F}} \leq \sum_{\substack{\alpha\in H_1\\ n-1-2\epsilon\,<\ell(\alpha^k)\leq n}} e^{\eta(\epsilon)} e^{\mathrm{Per}_F(\alpha^k)}$$
$$\leq \sum_{\substack{\alpha\in H_1,\ \ell(\alpha)\leq t_2\\ n-1-2\epsilon\,<k\,\ell(\alpha)\leq n}} e^{\eta(\epsilon)} e^{k\,\mathrm{Per}_F(\alpha)} + \sum_{\substack{\alpha\in H_1,\ \ell(\alpha)\geq t_2\\ \frac{n}{k}-1<\ell(\alpha)\leq\frac{n}{k}}} e^{\eta(\epsilon)} e^{k\,\mathrm{Per}_F(\alpha)}\,.$$

Let us denote by $I$ and $J$ the first and second sums in the last line above. We have, by the definition of $\rho$ and $C$,

$$I\leq e^{\eta(\epsilon)} \sum_{\substack{\alpha\in H_1,\ \ell(\alpha)\leq t_2\\ n-1-2\epsilon\,<k\,\ell(\alpha)\leq n}} \rho^{k\,\ell(\alpha)} e^{k\,\delta\,\ell(\alpha)} \leq e^{\eta(\epsilon)} \sum_{\alpha\in H_1,\ \ell(\alpha)\leq t_2} \rho^{n-1-2\epsilon} e^{\delta\, n} \leq C\,\rho^n\, e^{\delta\, n}\,.$$

By Equation (127), for every $\alpha\in H_1$ such that $t_0\leq\ell(\alpha)\leq\frac{n}{k}$, we have

$$e^{\int_x^{\alpha x}\widetilde{F}} \leq \epsilon''\, e^{\delta\, d(x,\alpha x)} \leq \epsilon''\, e^{\delta\,(\ell(\alpha)+2\,\epsilon)} \leq \epsilon''\, e^{\delta\,(\frac{n}{k}+2\,\epsilon)} \leq c\,\epsilon''\, e^{2\,\epsilon\,\delta} \sum_{\alpha'\in H_1,\ \frac{n}{k}-1<\ell(\alpha')\leq\frac{n}{k}} e^{\int_x^{\alpha' x}\widetilde{F}}\,.$$

We hence have

$$J\leq e^{(1+k)\eta(\epsilon)} \sum_{\substack{\alpha\in H_1,\ \ell(\alpha)\geq t_0\\ \frac{n}{k}-1<\ell(\alpha)\leq\frac{n}{k}}} e^{k\,\int_x^{\alpha x}\widetilde{F}} = e^{(1+k)\eta(\epsilon)} \sum_{\substack{\alpha\in H_1,\ \ell(\alpha)\geq t_0\\ \frac{n}{k}-1<\ell(\alpha)\leq\frac{n}{k}}} e^{\int_x^{\alpha x}\widetilde{F}}\big(e^{\int_x^{\alpha x}\widetilde{F}}\big)^{k-1}$$
$$\leq e^{(1+k)\eta(\epsilon)}\,(c\,\epsilon''\,e^{2\,\epsilon\,\delta})^{k-1} \sum_{\substack{\alpha\in H_1,\ \ell(\alpha)\geq t_0\\ \frac{n}{k}-1<\ell(\alpha)\leq\frac{n}{k}}} e^{\int_x^{\alpha x}\widetilde{F}} \Big(\sum_{\alpha'\in H_1,\ \frac{n}{k}-1<\ell(\alpha')\leq\frac{n}{k}} e^{\int_x^{\alpha' x}\widetilde{F}}\Big)^{k-1}\,.$$

By the triangle inequality, for all $\alpha_1,\dots,\alpha_k\in H_1$ with $\ell(\alpha_i)\leq\frac{n}{k}$, we have

$$d(x,\alpha_1\dots\alpha_k x)\leq\sum_{i=1}^k d(x,\alpha_i x)\leq\sum_{i=1}^k \ell(\alpha_i)+2\,k\,\epsilon\leq n+2\,k\,\epsilon\,.$$

By a quasi-geodesic argument as in Proposition 4.9 and by Lemma 3.2, there exists $c'>0$ such that if $\epsilon, U_\pm$ are small enough and $t$ big enough, for every $k\geq 2$, the product elements $\alpha_1\dots\alpha_k$ in $\Gamma$, where $\alpha_1,\dots,\alpha_k\in H_1$ with $t-1<d(x,\alpha_i x)\leq t+2\epsilon$ and $d(\alpha_i x,\alpha_j x)\geq 6$ if $\alpha_i\neq\alpha_j$ are pairwise distinct and

$$\Big|\int_x^{\alpha_1\dots\alpha_k x}\widetilde{F}-\sum_{i=1}^k\int_x^{\alpha_i x}\widetilde{F}\,\Big|\leq k\,c'\,.$$

Hence, up to increasing $t_0$, by developing the above $(k-1)$-th power and by a definite proportion argument to make the relevant orbit points 6-separated, there exists $c''>0$ (independent of $\epsilon''$) such that

$$J\leq(c''\epsilon'')^k\sum_{\gamma\in\Gamma,\ d(x,\gamma x)\leq n+2\,k\,\epsilon} e^{\int_x^{\gamma x}\widetilde{F}}\,.$$

By Corollary 4.1, there exists $c''' > 0$ such that this sum is at most $c''' e^{\delta\,(n+2\,k\,\epsilon)}$. Hence, if $\epsilon'' > 0$ is small enough, we have $J \leq \frac{\epsilon'}{2^k}\, e^{\delta\, n}$, as required. $\Box$

Let us now conclude the proof of Step 2. Since the translation lengths of the loxodromic elements of $\Gamma$ whose translation axis meets a given compact subset of $\widetilde{M}$ have a positive lower bound by discreteness, there exists $a > 0$ such that for all $k \in \mathbb{N} - \{0\}$ and $\gamma \in H_k$ with $d(x, \gamma x) \leq n$, we have $k \leq a\, n$. By Equations (125) and (126), and by a finite summation, we hence have that

$$\sum_{\gamma \in \bigcup_{k\geq 2} H_k,\; n-1<\ell(\gamma)\leq n} e^{\mathrm{Per}_F(\gamma)}$$

tends to 0 as $n \to +\infty$. By a standard covering argument by small open sets (of the form $U_- \times U_+ \times\, ]-\eta, +\eta[$ in Hopf's coordinates, with $U_\pm$ as above and $\eta > 0$ small enough) of the support of a continuous map $\psi'$ with compact support on $T^1\widetilde{M}$, and since $|\mathscr{L}_\gamma(\psi')|$ is uniformly bounded (by the product of the maximum of $|\psi'|$ and the diameter of its support) for every loxodromic element $\gamma \in \Gamma$, this concludes the proof of Step 2.

(2) The deduction of the second assertion from the first one is standard. Consider the measures

$$m'_t = \delta e^{-\delta t} \sum_{g \in \mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\, \mathscr{L}_g \quad \text{and} \quad m''_t = \delta t e^{-\delta t} \sum_{g \in \mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g}{\ell(g)}$$

on $T^1M$. Fix a continuous map $\psi : T^1M \to [0, +\infty[$ with compact support. For every $\epsilon > 0$, for every $t > 0$, we have

$$\begin{aligned} m''_t(\psi) \geq m'_t(\psi) &\geq \delta e^{-\delta t} \sum_{g \in \mathscr{P}\mathrm{er}'(t) - \mathscr{P}\mathrm{er}'(e^{-\epsilon} t)} e^{\mathscr{L}_g(F)}\, \mathscr{L}_g(\psi) \\ &\geq e^{-\epsilon} \delta t e^{-\delta t} \sum_{g \in \mathscr{P}\mathrm{er}'(t) - \mathscr{P}\mathrm{er}'(e^{-\epsilon} t)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g(\psi)}{\ell(g)} \\ &= e^{-\epsilon} m''_t(\psi) - e^{-\epsilon} \delta t\, e^{-\delta t} \sum_{g \in \mathscr{P}\mathrm{er}'(e^{-\epsilon} t)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g(\psi)}{\ell(g)} \;. \end{aligned}$$

Since the closed orbits meeting the support of $\psi$ have a positive lower bound on their lengths, and by the first assertion of Theorem 9.11, there exists a constant $c > 0$ such that the second term of the above difference is at most $c\, e^{-\delta t} e^{\delta\, e^{-\epsilon} t}$, which tends to 0 as $t$ tends to infinity. Hence by applying twice the first assertion of Theorem 9.11, we have

$$\frac{m_F(\psi)}{\|m_F\|} = \lim_{t \to +\infty} m'_t(\psi) \leq \liminf_{t \to +\infty} m''_t(\psi) \leq \limsup_{t \to +\infty} m''_t(\psi) \leq \lim_{t \to +\infty} e^{\epsilon} m'_t(\psi) = e^{\epsilon} \frac{m_F(\psi)}{\|m_F\|} \;,$$

and the result follows by letting $\epsilon$ go to 0 (and writing any continuous map $\psi : T^1M \to \mathbb{R}$ with compact support into the sum of its positive and negative parts). $\Box$

**Remark 9.13** Step 2 of the above proof shows that the Gurevich pressure, if it is finite and if $m_F$ is finite and mixing, may be defined by considering only primitive periodic orbits: if $W$ is any relatively compact open subset of $T^1M$ meeting $\Omega\Gamma$, for every $c > 0$ large enough, we have

$$P_{Gur}(\Gamma, F) = \lim_{t\to+\infty} \frac{1}{t} \log \sum_{g\in\mathscr{P}\mathrm{er}'(t)-\mathscr{P}\mathrm{er}'(t-c),\ g\cap W\neq\emptyset} e^{\int_g F}\ ,$$

and if $P_{Gur}(\Gamma, F) > 0$, then

$$P_{Gur}(\Gamma, F) = \lim_{t\to+\infty} \frac{1}{t} \log \sum_{g\in\mathscr{P}\mathrm{er}'(t),\ g\cap W\neq\emptyset} e^{\int_g F}\ .$$

In the same way Theorem 9.4 has been deduced from Theorem 9.1, the first claim of the following result, which no longer assumes $\delta_{\Gamma,\,F} > 0$, can be deduced from Theorem 9.11. The second claim follows immediately.

**Theorem 9.14** *Assume that $\delta_{\Gamma,\,F} < +\infty$, and that the Gibbs measure $m_F$ is finite and mixing under the geodesic flow on $T^1M$. Let $U$ and $V$ be two open subsets of $\partial_\infty\widetilde{M}$ meeting $\Lambda\Gamma$ such that $\mu_x(\partial U) = \mu^\iota_x(\partial V) = 0$. Then for every $c > 0$, as $t$ goes to $+\infty$,*

$$\frac{\delta_{\Gamma,\,F}}{1-e^{-c\,\delta_{\Gamma,\,F}}}\ e^{-\delta_{\Gamma,\,F}\ t} \sum_{g\in\mathscr{P}\mathrm{er}'(t)-\mathscr{P}\mathrm{er}'(t-c)} e^{\mathscr{L}_g(F)}\ \mathscr{L}_g \ \overset{*}{\rightharpoonup}\ \frac{m_F}{\|m_F\|}\ ,$$

$$\frac{\delta_{\Gamma,\,F}}{1-e^{-c\,\delta_{\Gamma,\,F}}}\ t\ e^{-\delta_{\Gamma,\,F}\ t} \sum_{g\in\mathscr{P}\mathrm{er}'(t)-\mathscr{P}\mathrm{er}'(t-c)} e^{\mathscr{L}_g(F)}\ \frac{\mathscr{L}_g}{\ell(g)} \ \overset{*}{\rightharpoonup}\ \frac{m_F}{\|m_F\|}\ .$$

Applying the equidistribution result 9.11 to characteristic functions, for every relatively compact open subset $U$ of $T^1M$ whose boundary has Gibbs measure 0, we have that, as $t \to +\infty$,

$$\sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\mathrm{Per}_F(g)}\,\frac{\ell(g\cap U)}{\ell(g)} \ \sim\ \frac{e^{\delta_{\Gamma,\,F}t}}{\delta_{\Gamma,\,F}t}\,\frac{m_F(U)}{\|m_F\|}\ .$$

As there exists a compact subset of $T^1M$ containing all closed orbits of the geodesic flow if $\Gamma$ is convex-cocompact, the following corollary holds.

**Corollary 9.15** *Assume that $\Gamma$ is convex-cocompact, that $(\phi_t)_{t\in\mathbb{R}}$ is topologically mixing and that $\delta_{\Gamma,\,F} < +\infty$. Then for every $c > 0$, as $t \to +\infty$,*

$$\sum_{g\in\mathscr{P}\mathrm{er}'(t)-\mathscr{P}\mathrm{er}'(t-c)} e^{\mathrm{Per}_F(g)} \sim \frac{1-e^{c\,\delta_{\Gamma,\,F}}}{\delta_{\Gamma,\,F}}\ \frac{e^{\delta_{\Gamma,\,F}\ t}}{t}\ .$$

*If furthermore $\delta_{\Gamma,\,F} > 0$, then as $t \to +\infty$,*

$$\sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\mathrm{Per}_F(g)} \sim \frac{e^{\delta_{\Gamma,\,F}\ t}}{\delta_{\Gamma,\,F}\ t}\ .$$

Let $\mathscr{C}_b(T^1M)^*$ be the dual topological space of the Banach space of real bounded continuous functions on $T^1M$. By the *narrow convergence* of finite measures, we mean as usual the convergence for the weak-star topology in $\mathscr{C}_b(T^1M)^*$. Corollary 9.15 can be improved: it holds when "convex-cocompact" is replaced by "geometrically finite with finite Gibbs measure", though the extension requires more work. For this, we improve, under the geometrically finiteness condition, Theorem 9.11 from weak-star convergence to narrow convergence, as in [Rob1] for the case $F = 0$. The main point is to prove that there is no loss of mass in the cuspidal parts during the convergence process.

**Theorem 9.16** *Assume that the critical exponent $\delta_{\Gamma,\,F}$ of $(\Gamma, F)$ is finite and positive, that the Gibbs measure $m_F$ is finite and mixing under the geodesic flow on $T^1M$, and that $\Gamma$ is geometrically finite.*

*(1) As $t$ goes to $+\infty$, the measures*

$$\delta_{\Gamma,\,F}\ e^{-\delta_{\Gamma,\,F}\,t} \sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\ \mathscr{L}_g$$

*converge to the normalised Gibbs measure $\frac{m_F}{\|m_F\|}$ with respect to the narrow convergence.*

*(2) As $t$ goes to $+\infty$,*

$$\delta_{\Gamma,\,F}\ t\ e^{-\delta_{\Gamma,\,F}\,t} \sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\ \frac{\mathscr{L}_g}{\ell(g)}$$

*converges to $\frac{m_F}{\|m_F\|}$ with respect to the narrow convergence.*

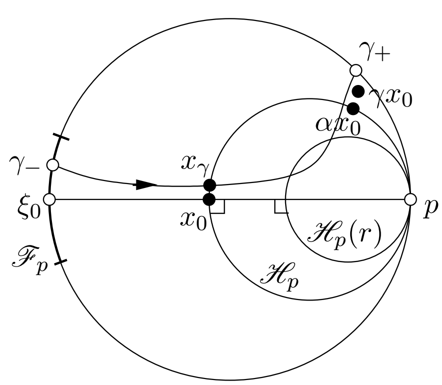
**Proof.** The deduction of Assertion (2) from Assertion (1) proceeds as in the proof of Theorem 9.11, using the fact that since $\Gamma$ is geometrically finite, there exists a compact set of $M$ meeting (though not necessarily containing) all closed geodesics, hence there exists $c' > 0$ such that every closed geodesic of $M$ has length at least $c'$.

Let us prove Assertion (1). Let $\delta = \delta_{\Gamma,\,F}$. For $t \geq 0$, consider the measure

$$m_t = \delta\, e^{-\delta t} \sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\ \mathscr{L}_g$$

on $T^1M$ and let $\pi_* m_t$ be its push-forward on $M$. We will use the notation of the proof of Theorem 8.3, in particular $\mathrm{Par}_\Gamma, \mathscr{H}_p, \Gamma_p, \mathscr{F}_p, x_0, \kappa$ for every parabolic fixed point $p$ of $\Gamma$.

For every $r \geq 0$ and for every parabolic fixed point $p$ of $\Gamma$, let $\mathscr{H}_p(r)$ be the horoball centred at $p$, contained in $\mathscr{H}_p$ such that the distance between the horospheres $\partial\mathscr{H}_p(r)$ and $\partial\mathscr{H}_p$ is $r$.



By the finiteness of the number of orbits of parabolic fixed points under $\Gamma$ and by the compactness of the quotient $\Gamma\backslash\big(\mathscr{C}\Lambda\Gamma - \bigcup_{p\in\mathrm{Par}_\Gamma}\mathscr{H}_p(r)\big)$, we only have to prove that for every parabolic fixed point $p$ of $\Gamma$,

$$\lim_{r\to+\infty}\ \limsup_{t\to+\infty} \pi_* m_t(\Gamma_p\backslash\mathscr{H}_p(r)) = 0\ .$$

Fix such a point $p$. A closed geodesic in $M$ meeting $\Gamma_p\backslash\mathscr{H}_p$ has a unique lift in $\widetilde{M}$ meeting $\mathscr{H}_p$ starting from the fundamental domain $\mathscr{F}_p$ for the action of $\Gamma_p$ on $\Lambda\Gamma - \{p\}$; its other endpoint is in $\alpha\mathscr{F}_p$ for some $\alpha\in\Gamma_p$. For all $t\geq 0$ and $\alpha\in\Gamma_p$, let $\Gamma(t,\alpha)$ be the set of loxodromic elements $\gamma$ of $\Gamma$ with repulsive fixed point $\gamma_-$ in $\mathscr{F}_p$, attractive fixed point $\gamma_+$ in $\alpha\mathscr{F}_p$ and translation length $\ell(\gamma)$ at most $t$. Then (the inequality being an equality if the elements of $\Gamma(t,\alpha)$ were being assumed to be primitive)

$$\pi_* m_t(\Gamma_p\backslash\mathscr{H}_p(r)) \leq \delta e^{-\delta t}\sum_{\alpha\in\Gamma_p}\ \sum_{\gamma\in\Gamma(t,\,\alpha)} e^{\mathrm{Per}_F(\gamma)}\ \mathrm{length}(\mathrm{Axe}_\gamma\cap\mathscr{H}_p(r))\ . \tag{128}$$

Let $\gamma\in\Gamma(t,\alpha)$ be such that its translation axis $\mathrm{Axe}_\gamma$ meets $\mathscr{H}_p(r)$ (note that for the others, we have $\mathrm{length}(\mathrm{Axe}_\gamma\cap\mathscr{H}_p(r))=0$). We orient $\mathrm{Axe}_\gamma$ from $\gamma_-$ to $\gamma_+$, and we denote by $x_\gamma$ the entrance point of $\mathrm{Axe}_\gamma$ in $\mathscr{H}_p$. By Equation (109), we have

$$d(x_0,x_\gamma)\leq\kappa\ .$$

Since the distance between any point of $\partial\mathscr{H}_p$ and any point of $\mathscr{H}_p(r)$ is at least $r$, we have (see the above picture),

$$\mathrm{length}(\mathrm{Axe}_\gamma\cap\mathscr{H}_p(r))\leq d(x_0,\alpha x_0)-2r+2\kappa\ . \tag{129}$$

In particular,

$$d(x_0,\alpha x_0)\geq 2r-2\kappa\ . \tag{130}$$

Consider the constants $R$ and $C$ given by Mohsen's shadow lemma 3.10 applied to $(\mu_x)_{x\in\widetilde{M}}$ as above and $K=\{x_0\}$, and let $\mathscr{O}_\gamma=\mathscr{O}_{x_0}B(\gamma x_0,R)$. The point $\alpha x_0$ is at distance uniformly bounded from $\mathrm{Axe}_\gamma$. Since $\mathscr{H}_p$ is precisely invariant (see the beginning of the proof of Theorem 8.3) and $x_0\in\partial\mathscr{H}_p$, the point $\gamma x_0$ does not belong to the interior of $\mathscr{H}_p$. Hence $\alpha x_0$ is at distance uniformly bounded from $[x_0,\gamma x_0]$. Therefore $x_0=\alpha^{-1}\alpha x_0$ is at distance uniformly bounded from $[\alpha^{-1}x_0,\alpha^{-1}\gamma x_0]$. If $r$ is large enough, then $d(x_0,\alpha^{-1}x_0)$ is large, hence $\alpha^{-1}x_0$ is close to $p$. Therefore, there exists a compact subset $K$ (independent of $t,r,\alpha,\gamma$) of $\partial_\infty\widetilde{M}-\{p\}$ such that the set $\alpha^{-1}\mathscr{O}_\gamma=\mathscr{O}_{\alpha^{-1}x_0}B(\alpha^{-1}\gamma x_0,R)$ is contained in $K$ for $r$ large enough.

Since $d(x_0,x_\gamma)\leq\kappa$ and by (two applications of) Lemma 3.2, there exists $c_1\geq 0$ such that

$$\Big|\,\mathrm{Per}_F(\gamma)-\int_{x_0}^{\gamma x_0}\widetilde{F}\,\Big| = \Big|\int_{x_\gamma}^{\gamma x_\gamma}\widetilde{F}\ -\int_{x_0}^{\gamma x_0}\widetilde{F}\,\Big|\leq c_1\ .$$

Since $x_0$ is at distance at most $\kappa$ from $\mathrm{Axe}_\gamma$, we have

$$d(x_0,\gamma x_0)\leq\ell(\gamma)+2\kappa\leq t+2\kappa\ ,$$

and if furthermore $\gamma\notin\Gamma(t-1,\alpha)$, then

$$t-1\leq\ell(\gamma)\leq d(x_0,\gamma x_0)\ .$$

Hence, by Mohsen's shadow lemma 3.10,

$$e^{\mathrm{Per}_F(\gamma)} \le e^{c_1} e^{\int_{x_0}^{\gamma x_0} \widetilde{F}} \le e^{c_1+2\delta\kappa} e^{\delta t} e^{\int_{x_0}^{\gamma x_0}(\widetilde{F}-\delta)} \le C\, e^{c_1+2\delta\kappa} e^{\delta t} \mu_{x_0}(\mathscr{O}_\gamma)\ .$$

By the discreteness of $\Gamma x_0$, there exists $N > 0$ (independent of $t$) such that a point $\xi \in \partial_\infty \widetilde{M}$ belongs to at most $N$ shadows $\mathscr{O}_\gamma$ for $\gamma \in \Gamma$ with $t-1 \le d(x_0, \gamma x_0) \le t+2\kappa$. Since the shadow $\mathscr{O}_\gamma$ is contained in $\alpha K$ if $\mathrm{Axe}_\gamma$ meets $\mathscr{H}_p(r)$ for $r$ large enough, we hence have, for $r$ large enough,

$$\sum_{\gamma \in \Gamma(t,\,\alpha)-\Gamma(t-1,\,\alpha)\,:\;\mathrm{Axe}_\gamma \cap \mathscr{H}_p(r) \neq \emptyset} e^{\mathrm{Per}_F(\gamma)} \le CN\, e^{c_1+2\delta\kappa}\, e^{\delta t} \mu_{x_0}(\alpha K)\ .$$

By a summation, there exists therefore a constant $c_2 > 0$ such that

$$\sum_{\gamma \in \Gamma(t,\,\alpha)\,:\;\mathrm{Axe}_\gamma \cap \mathscr{H}_p(r) \neq \emptyset} e^{\mathrm{Per}_F(\gamma)} \le c_2\, e^{\delta t} \mu_{x_0}(\alpha K)\ . \tag{131}$$

By the equations (38), (39) and (112), we have, for some constant $c_3 > 0$,

$$\mu_{x_0}(\alpha K) = \mu_{\alpha^{-1}x_0}(K) = \int_{\xi \in K} e^{-C_{F-\delta,\,\xi}(\alpha^{-1}x_0,\, x_0)}\, d\mu_{x_0}(\xi) \le c_3 \mu_{x_0}(K)\, e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\ . \tag{132}$$

By the equations (128), (129), (130), (131) and (132), for $r$ large enough and for every $t \ge 0$, we hence have

$$\begin{aligned}
&\pi_* m_t(\Gamma_p \backslash \mathscr{H}_p(r)) \\
&\le \delta c_2 c_3 \mu_{x_0}(K) \sum_{\alpha \in \Gamma_p\,:\; d(x_0,\,\alpha x_0) \ge 2r-2\kappa} (d(x_0, \alpha x_0) - 2r + 2\kappa)\, e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}-\delta)}\ .
\end{aligned}$$

The result then follows from Theorem 8.3, since we assume $m_F$ to be finite (which implies that $(\Gamma, F)$ is of divergence type by Corollary 5.15). □

In a similar way to that where Theorem 9.4 has been deduced from Theorem 9.1, the first claim of the following result, which no longer assumes $\delta_{\Gamma,\,F} > 0$, can be deduced from Theorem 9.16. The second claim follows immediately.

**Theorem 9.17** *Assume that $\delta_{\Gamma,\,F} < +\infty$, that $m_F$ is finite and mixing under the geodesic flow on $T^1M$, and that $\Gamma$ is geometrically finite. Then for every $c > 0$, as $t \to +\infty$, the measures*

$$\frac{\delta_{\Gamma,\,F}}{1-e^{-c\,\delta_{\Gamma,\,F}}}\ e^{-\delta_{\Gamma,\,F}\, t} \sum_{g \in \mathscr{P}\mathrm{er}'(t)-\mathscr{P}\mathrm{er}'(t-c)} e^{\mathscr{L}_g(F)}\ \mathscr{L}_g$$

*and*

$$\frac{\delta_{\Gamma,\,F}}{1-e^{-c\,\delta_{\Gamma,\,F}}}\ t\ e^{-\delta_{\Gamma,\,F}\, t} \sum_{g \in \mathscr{P}\mathrm{er}'(t)-\mathscr{P}\mathrm{er}'(t-c)} e^{\mathscr{L}_g(F)}\ \frac{\mathscr{L}_g}{\ell(g)}$$

*converge to $\frac{m_F}{\|m_F\|}$ with respect to the narrow convergence.*

The next result is immediate, applying the second assertions of Theorems 9.16 and 9.17 above to the (bounded) constant function 1. By Subsection 8.2, the finiteness of $m_F$ (which was automatic in the convex-cocompact case but no longer is in the geometrically finite one) follows if $\delta_{\Gamma_p,\,F} < \delta_{\Gamma,\,F}$ for every parabolic fixed point $p$ of $\Gamma$.

**Corollary 9.18** *Assume that $\Gamma$ is geometrically finite, $(\phi_t)_{t\in\mathbb{R}}$ is topologically mixing, $\delta_{\Gamma,\,F} < +\infty$ and $m_F$ is finite. Then for every $c > 0$, as $t \to +\infty$, we have*

$$\sum_{g\in\mathscr{P}\mathrm{er}'(t)-\mathscr{P}\mathrm{er}'(t-c)} e^{\mathrm{Per}_F(g)} \;\sim\; \frac{1-e^{-c\,\delta_{\Gamma,\,F}}}{\delta_{\Gamma,\,F}}\;\frac{e^{\delta_{\Gamma,\,F}\;t}}{t}\;.$$

*If furthermore $\delta_{\Gamma,\,F} > 0$, then, as $t \to +\infty$, we have*

$$\sum_{g\in\mathscr{P}\mathrm{er}'(t)} e^{\mathrm{Per}_F(g)} \sim \frac{e^{\delta_{\Gamma,\,F}\;t}}{\delta_{\Gamma,\,F}\;t}\;.$$

## 9.4 The case of infinite Gibbs measure

In this subsection, with the notation of the beginning of Chapter 9, we still assume that $\delta_{\Gamma,\,F} < +\infty$, but we now assume that the Gibbs measure $m_F$ on $T^1M$ is infinite. We state (weak) analogs of the previous asymptotic results in this chapter, generalising the case $F = 0$ in [Rob1], leaving the adaptation of the proofs to the reader.

Babillot's theorem [Bab2, Theo. 1], saying that the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M$ is mixing for the measure $m_F$, is also valid if $m_F$ is infinite, with the standard definition of the mixing property of infinite measures.

**Theorem 9.19 (Babillot)** *Assume that $\delta_{\Gamma,\,F} < +\infty$, that the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M$ is topologically mixing on its (topological) non-wandering set $\Omega\Gamma$ (or equivalently that the length spectrum of $M$ is not contained in a discrete subgroup of $\mathbb{R}$), and that $m_F$ is infinite. For all bounded Borel subsets $A$ and $B$ in $T^1M$, we have*

$$\lim_{t\to+\infty} m_F(A\cap\phi_t B) = 0\;.$$

Equivalently, the mixing property may be written as follows: for all bounded Borel subsets $A$ and $B$ in $T^1\widetilde{M}$,

$$\lim_{t\to+\infty}\sum_{\gamma\in\Gamma}\widetilde{m}_F(A\cap\phi_t\gamma B) = 0\;,$$

which is the appropriate replacement for the use of the mixing property in the proof of Proposition 9.2.

Corollaries 9.7 and 9.10 on the asymptotic of the orbital counting function become the following result.

**Theorem 9.20** *Assume that $\delta_{\Gamma,\,F} < +\infty$, that the geodesic flow on $T^1M$ is topologically mixing, and that $m_F$ is infinite. Then, as $t \to +\infty$, for every $c > 0$,*

$$\sum_{\gamma\in\Gamma,\;t-c<d(x,\,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde{F}} \;=\; \mathrm{o}\,(e^{\delta_{\Gamma,\,F}\;t})\,,$$

*and if $\delta_{\Gamma,\,F} > 0$, then*

$$\sum_{\gamma\in\Gamma,\;d(x,\,\gamma y)\le t} e^{\int_x^{\gamma y}\widetilde{F}} \;=\; \mathrm{o}\,(e^{\delta_{\Gamma,\,F}\;t})\,.$$

Theorems 9.11 and 9.14 on the asymptotic of the period counting function become the following result.

**Theorem 9.21** *Assume that $\delta = \delta_{\Gamma,\,F} < +\infty$, that the geodesic flow on $T^1M$ is topologically mixing, and that $m_F$ is infinite. Then, as $t \to +\infty$, for every $c > 0$,*

$$e^{-\delta\, t} \sum_{g \in \mathscr{P}\mathrm{er}'(t) - \mathscr{P}\mathrm{er}'(t-c)} e^{\mathscr{L}_g(F)}\, \mathscr{L}_g \overset{*}{\rightharpoonup} 0\,, \quad t\, e^{-\delta\, t} \sum_{g \in \mathscr{P}\mathrm{er}'(t) - \mathscr{P}\mathrm{er}'(t-c)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g}{\ell(g)} \overset{*}{\rightharpoonup} 0\,,$$

*and if $\delta > 0$, then*

$$e^{-\delta\, t} \sum_{g \in \mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\, \mathscr{L}_g \overset{*}{\rightharpoonup} 0\,, \quad t\, e^{-\delta\, t} \sum_{g \in \mathscr{P}\mathrm{er}'(t)} e^{\mathscr{L}_g(F)}\, \frac{\mathscr{L}_g}{\ell(g)} \overset{*}{\rightharpoonup} 0\,.$$

# 10 The ergodic theory of the strong unstable foliation

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$; $\Gamma$ is a nonelementary discrete group of isometries of $\widetilde{M}$; and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map. Let $x_0 \in \widetilde{M}$ and $\delta = \delta_{\Gamma,\,F}$. The aim of this chapter is to prove a result of unique ergodicity for the strong unstable foliation of $T^1M = \Gamma\backslash T^1\widetilde{M}$ endowed with the conditional measures of Gibbs measures. We refer for instance to [Wal, BoM] for generalities on unique ergodicity of measurable dynamical systems and foliations, though in our case we will (and have to, in order to take into account the potential) consider quasi-invariant measures instead of invariant ones.

## 10.1 Quasi-invariant transverse measures

We first recall some definitions. Let $N$ be a smooth manifold of dimension $n$ endowed with a smooth action of a discrete group $G$ and a $\mathrm{C}^0$ foliation $\mathscr{F}$ of dimension $k$ invariant by $G$. A *transversal* to $\mathscr{F}$ is a $\mathrm{C}^0$ submanifold $T$ of $N$ such that for every $x \in T$, there exists a foliated chart $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ at $x$, sending $x$ to $0$ and $T \cap U$ to $\{0\} \times \mathbb{R}^{n-k}$. Let $\mathscr{T}(\mathscr{F})$ be the set of transversals to $\mathscr{F}$. A *holonomy map* for $\mathscr{F}$ is a homeomorphism $f : T \to T'$ between two transversals to $\mathscr{F}$, such that $f(x)$ is in the same leaf of $\mathscr{F}$ as $x$ for every $x \in T$. A *$G$-invariant cocycle* for the foliation $\mathscr{F}$ is a continuous map $c$ defined on the subspace of $N \times N$ of pairs of points in a same leaf of $\mathscr{F}$, with real values, satisfying $c(u,v) + c(v,w) = c(u,w)$ and $c(u,v) = c(\gamma u, \gamma v)$ for all triples $(u,v,w)$ of points in a same leaf of $\mathscr{F}$ and every $\gamma \in G$. Given a $G$-invariant cocycle $c$ for $\mathscr{F}$, a *$G$-equivariant $c$-quasi-invariant transverse measure for $\mathscr{F}$* is a family $\nu = (\nu_T)_{T \in \mathscr{T}(\mathscr{F})}$, where $\nu_T$ is a (not necessarily finite) locally finite (positive Borel) measure on the transversal $T$, nonzero for at least one $T$, satisfying the following properties for all $T, T' \in \mathscr{T}(\mathscr{F})$:

(i) if $T' \subset T$, then $(\nu_T)_{|T'} = \nu_{T'}$,

(ii) $\gamma_* \nu_T = \nu_{\gamma T}$ for every $\gamma \in G$,

(iii) for all holonomy maps $f : T \to T'$, the measures $\nu_{T'}$ and $f_* \nu_T$ on $T'$ are equivalent, and their Radon-Nikodym derivative satisfies, for $\nu_T$-almost every $x \in T$,

$$\frac{d\, \nu_{T'}}{d\, f_* \nu_T}(f(x)) = e^{c(f(x),\, x)}\,.$$

When $c = 0$, such a family $\nu = (\nu_T)_{T\in\mathscr{T}(\mathscr{F})}$ is said to be *invariant under holonomy*: for all holonomy maps $f : T \to T'$, we have $f_*\nu_T = \nu_{T'}$. Such a family $\nu = (\nu_T)_{T\in\mathscr{T}(\mathscr{F})}$ is *ergodic* if, given a $G$-invariant subset $A$ of $N$, which is a union of leaves of $\mathscr{F}$, whose intersection with any transversal is measurable, then either for all transversals $T$ we have $\nu_T(A \cap T) = 0$, or for all transversals $T$ we have $\nu_T({}^cA \cap T) = 0$.

We will apply these definitions with $N = T^1\widetilde{M}$, $G = \Gamma$, $\mathscr{F}$ the strong unstable foliation $\widetilde{\mathscr{W}}^{\rm su}$ of $T^1\widetilde{M}$ (defined in Subsection 2.4) and with the ($\Gamma$-invariant) cocycle $c = c_{\widetilde{F}}$ defined as follows: for all $v, w$ in the same leaf of $\widetilde{\mathscr{W}}^{\rm su}$,

$$c_{\widetilde{F}}(v,w) = \lim_{t\to+\infty}\int_0^t \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}w)\,dt = -C_{F\circ\iota-\delta,\,v_-}(\pi(v),\pi(w))\ . \tag{133}$$

Note that when $F$ is constant, this cocycle is equal to 0. Note that $c_{\widetilde{F}} = c_{\widetilde{F}+\kappa}$ for every $\kappa\in\mathbb{R}$, hence the property of being a $c_{\widetilde{F}}$-quasi-invariant transverse measure is unchanged by replacing $\widetilde{F}$ with $\widetilde{F}+\kappa$, for every $\kappa\in\mathbb{R}$.

We will say that a $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu$ for $\widetilde{\mathscr{W}}^{\rm su}$ *gives full measure to the negatively recurrent set* if for all transversals $T$ to $\widetilde{\mathscr{W}}^{\rm su}$, the set of $v \in T$ such that $v_-$ belongs to the conical limit set $\Lambda_c\Gamma$ has full measure with respect to $\nu_T$, where $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$.

**Remark (1).** When $\Gamma$ is torsion free, the image of the strong unstable foliation $\widetilde{\mathscr{W}}^{\rm su}$ of $T^1\widetilde{M}$ by the covering map $T^1\widetilde{M} \to T^1M$ defines a foliation denoted by $\mathscr{W}^{\rm su}$ on $T^1M$, the cocycle $c_{\widetilde{F}}$ defines by passing to the quotient a cocycle $c_F$ for this foliation (already defined in Subsection 6.2, see Equation (89) and Equation (91)), hence we may work directly with $c_F$-quasi-invariant transverse measures for the foliation $\mathscr{W}^{\rm su}$ on $T^1M$ (defined as above by forgetting the equivariance property (ii)). The case with torsion (useful when working with arithmetic groups, for instance) makes it easier to work equivariantly on $T^1\widetilde{M}$ than to work with singular foliations on $T^1M$.

**Remark (2).** As seen in Subsection 3.9, for every $v \in T^1M$, the stable submanifold $T = W^{\rm s}(v)$ of $v$ in $T^1\widetilde{M}$ is a transversal to the strong unstable foliation $\widetilde{\mathscr{W}}^{\rm su}$ of $T^1\widetilde{M}$, and every strong unstable leaf $W^{\rm su}(w)$ with $w_- \neq v_+$ meets this transversal in one and only one point $w'$. If $U_T = \{w \in T^1\widetilde{M}\ :\ w_- \neq v_+\}$, then the map $\psi_T : U_T \to T$ defined by $w \mapsto w'$ is a continuous fibration, whose fibre above $w' \in T$ is the strong unstable leaf of $w'$.

In particular, any small enough transversal $T'$ to the strong unstable foliation $\widetilde{\mathscr{W}}^{\rm su}$ of $T^1\widetilde{M}$ admits a holonomy map $f : T' \to T$ where $T$ is a submanifold of a stable submanifold. Hence by the above properties (i) and (iii), a $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ for $\widetilde{\mathscr{W}}^{\rm su}$ is uniquely determined by the measures $\nu_T$ where $T$ is a stable submanifold of $T^1\widetilde{M}$. We will use this remark without further comment in what follows.

**Remark (3).** Let us give another justification of our terminology of Gibbs measure, by further developing the analogy with symbolic dynamics. We refer to Subsection 3.8 for the notation of a countable alphabet $A$, the distance $d$ on $\Sigma = A^{\mathbb{Z}}$, the full shift $\sigma : \Sigma \to \Sigma$, the cylinders $[a_{-n'}, \dots, a_n]$ where $n, n' \in \mathbb{N}$ and $a_{-n'}, \dots, a_n \in A$, a Hölder-continuous map $F : \Sigma \to \mathbb{R}$, and the definition of a Gibbs measure for the potential $F$ on $\Sigma$ (see Equation (46)).

For every $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma$, we define the *strong unstable leaf* of $x$ by

$$W^{\rm su}(x) = \{y = (y_i)_{i\in\mathbb{Z}} \in \Sigma \;:\; \exists\, n \in \mathbb{Z},\ \forall\, i \leq n, \quad y_i = x_i\}\ .$$

Note that these sets $W^{\rm su}(x)$ for all $x \in \Sigma$ are either equal or disjoint. We define the *unstable Gibbs cocycle* of $(\Sigma, \sigma, F)$ (compare with Equation (133), with [Rue3, Zin, Kel], and in particular with [HaR]) as the map $c_F$ which associates, to all $x, y$ in the same strong unstable leaf, the real number

$$c_F(x, y) = \sum_{i=1}^{+\infty} F(\sigma^{-i}x) - F(\sigma^{-i}y)\ ,$$

which exists by the Hölder-continuity of $F$, and satisfies the cocycle properties $c_F(x, y) + c_F(y, z) = c_F(x, z)$ and $c_F(x, y) = -c_F(y, x)$ if $x, y, z$ are in the same strong unstable leaf. Similarly, let us define the *stable leaf* of $x$ by

$$W^{\rm s}(x) = \{y = (y_i)_{i\in\mathbb{Z}} \in \Sigma \;:\; \exists\, n, k \in \mathbb{Z},\ \forall\, i \geq n, \quad y_i = x_{i+k}\}\ .$$

We denote respectively by $\mathscr{W}^{\rm su}$ and $\mathscr{W}^{\rm s}$ the partitions of $\Sigma$ in strong unstable leaves and in stable leaves.

The projection map $p : \Sigma \to \Sigma^+ = A^{\mathbb{N}}$ defined by $(x_i)_{i\in\mathbb{Z}} \mapsto (x_i)_{i\in\mathbb{N}}$ is a 1-Lipschitz fibration, whose fibres are contained in (strong) stable leaves. For all $z = (z_i)_{i\in\mathbb{N}}$ and $z' = (z'_i)_{i\in\mathbb{N}}$ in $\Sigma^+$, the map $f_{z',\,z} : p^{-1}(z) \to p^{-1}(z')$, uniquely defined by $(x_i)_{i\in\mathbb{Z}} \mapsto (y_i)_{i\in\mathbb{Z}}$ where $y_i = x_i$ for every $i < 0$, is a Lipschitz homeomorphism. Note that $x$ and $f_{z',\,z}(x)$ are in the same strong unstable leaf, for every $x \in p^{-1}(z)$. Hence we will consider the family $(p^{-1}(z))_{z\in\Sigma^+}$ of fibres of $p$ as a family of transversals to the partition into strong unstable leaves of $\Sigma$, and $(f_{z',\,z}))_{z,\,z'\in\Sigma^+}$ as a family of holonomy maps along the strong unstable leaves of $\Sigma$.

We define a *$c_F$-quasi-invariant transverse measure* for $\mathscr{W}^{\rm su}$ to be a family $(\nu_z)_{z\in\Sigma^+}$, where $\nu_z$ is a measure on $p^{-1}(z)$ for every $z \in \Sigma^+$, such that there exists a constant $c > 0$ such that for all $z, z'$ in $\Sigma^+$, with $f = f_{z',\,z}$, the measures $\nu_{z'}$ and $f_*\nu_z$ have the same measure class, and their Radon-Nikodym derivative satisfies, for $\nu_z$-almost every $x \in p^{-1}(z)$,

$$\frac{1}{c}\; e^{c_F(f(x),\,x)} \leq \frac{d\,\nu_{z'}}{d\,f_*\nu_z}(f(x)) \leq c\; e^{c_F(f(x),\,x)}\ .$$

We will also consider $\nu_z$ as a measure on $\Sigma$ with support in $p^{-1}(z)$.

By the cocycle property of the Radon-Nikodym derivatives, note that we may take $c = 1$, up to replacing the cocycle $c_F$ by a cohomologous one.

**Proposition 10.1** *Let $m_F$ be a probability Gibbs measure for the potential $F$ on $\Sigma$. Then there exists a $c_F$-quasi-invariant transverse measure $(\nu_z)_{z\in\Sigma^+}$ for $\mathscr{W}^{\rm su}$ such that $m_F$ disintegrates, by the fibration $p$, with family of conditional measures $(\nu_z)_{z\in\Sigma^+}$.*

When the alphabet $A$ is finite, there exists in fact a unique, up a multiplicative constant, $c_F$-quasi-invariant transverse measure for $\mathscr{W}^{\rm su}$, see [BoM, Prop. 3.3] when $F = 0$ (compare with [PoS, Prop. 6 (i)]) and [BaL, Prop. 4.4] for general $F$. Analogous uniqueness results for geodesic flows are the main goal of this chapter (see Remark (1) after Theorem 10.4).

**Proof.** We refer for instance to Subsection 6.1 and [LeG] for basics on conditional measures and martingales.

Let $(\nu_z)_{z\in\Sigma^+}$ be a family of conditional measures of $m_F$ for the fibration $p$. By definition, $\nu_z$ is a probability measure on $p^{-1}(z)$ such that for all Borel subsets $B$ in $\Sigma$, the map $z\mapsto \nu_z(B)$ is measurable and we have

$$\int_{x\in\Sigma} \mathbb{1}_B(x)\, dm_F(x) = \int_{z\in\Sigma^+} \nu_z(B)\, d(p_*m_F)(z)\ . \tag{134}$$

Note that $(\nu_z)_{z\in\Sigma^+}$ is well defined up to a set of $p_*m_F$-measure zero of $z\in\Sigma^+$.

**Lemma 10.2** *For $p_*m_F$-almost every $z=(z_i)_{i\in\mathbb{N}}\in\Sigma^+$, for all $a_{-n'},\dots,a_{-1}\in A$, we have*

$$\nu_z([a_{-n'},\dots,a_{-1}]) = \lim_{k\to+\infty}\frac{m_F([a_{-n'},\dots,a_{-1}]\cap[z_0,\dots,z_k])}{m_F([z_0,\dots,z_k])}\ .$$

**Proof.** For every $k\in\mathbb{N}$, let $\mathscr{F}_k$ be the $\sigma$-algebra generated by the cylinders $[a_0,\dots,a_k]$ in $\Sigma$, where $a_0,\dots,a_k\in A$. Then $(\mathscr{F}_k)_{k\in\mathbb{N}}$ is a filtration in the Borel $\sigma$-algebra $\mathscr{B}$ of $\Sigma$, and a map $f:\Sigma\to\mathbb{R}$ is $\mathscr{F}_\infty$-measurable for $\mathscr{F}_\infty=\bigvee_{k\in\mathbb{N}}\mathscr{F}_k$ if and only if $f(x)$ depends only on $p(x)$ for all $x\in\Sigma$ and $z\mapsto f(p^{-1}(z)):\Sigma^+\to\mathbb{R}$ is measurable. For every Borel subset $B$ in the probability space $(\Sigma,m_F)$, the sequence $(X_k=E[\mathbb{1}_B\mid\mathscr{F}_k])_{k\in\mathbb{N}}$ is a (basic example of a) martingale adapted to the filtration $(\mathscr{F}_k)_{k\in\mathbb{N}}$: we have $E[X_{k+1}\mid\mathscr{F}_k]=X_k$ for every $k\in\mathbb{N}$. Since $\mathbb{1}_B$ is $m_F$-integrable, by the $\mathbb{L}^1$ martingale convergence theorem, the random variable $X_n$ converges almost surely (and in $\mathbb{L}^1$) to $X_\infty=E[\mathbb{1}_B\mid\mathscr{F}_\infty]$. By the definition of the conditional expectation and by Equation (134), for $m_F$-almost every $x\in\Sigma$, we have $E[\mathbb{1}_B\mid\mathscr{F}_\infty](x)=\nu_{p(x)}(B)$. Since a Gibbs measure gives a nonzero mass to any cylinder (see Equation (46)) and since a map $f:\Sigma\to\mathbb{R}$ is $\mathscr{F}_k$-measurable if and only if $f(x)$ depends only on the components $x_0,\dots,x_k$ of $x$, for $m_F$-almost every $x=(x_i)_{i\in\mathbb{Z}}\in\Sigma$, we have $E[\mathbb{1}_B\mid\mathscr{F}_k](x)=\frac{m_F(B\cap[x_0,\dots,x_k])}{m_F([x_0,\dots,x_k])}$. The result follows. $\Box$

For all sets $X$ and maps $g,g':X\to\mathbb{R}$, let us write $g\asymp g'$ if there exists $c>0$ such that $\frac{1}{c}\,g(x)\le g'(x)\le c\,g(x)$ for every $x\in X$. The above lemma and Equation (46) imply that there exists $\delta'_F\in\mathbb{R}$ such that for $m_F$-almost every $x=(x_i)_{i\in\mathbb{Z}}\in\Sigma$, we have

$$\nu_{p(x)}([x_{-n'},\dots,x_{-1}])\asymp\lim_{n\to+\infty}\frac{e^{\sum_{i=-n'}^{n}F(\sigma^i x)-(n+n'+1)\delta'_F}}{e^{\sum_{i=0}^{n}F(\sigma^i x)-(n+1)\delta'_F}} = e^{\sum_{i=1}^{n'}F(\sigma^{-i}x)-n'\,\delta'_F}\ .$$

Hence for $p_*m_F$-almost all $z,z'\in\Sigma^+$ and for $\nu_z$-almost every $x=(x_i)_{i\in\mathbb{Z}}\in p^{-1}(z)$, if $f=f_{z',z}$, then

$$\frac{\nu_{z'}([x_{-n'},\dots,x_{-1}])}{\nu_z([x_{-n'},\dots,x_{-1}])}\asymp e^{\sum_{i=1}^{n'}F(\sigma^{-i}f(x))-F(\sigma^{-i}x)}\ .$$

Proposition 10.1 now follows by taking limits and modifying the family $(\nu_z)_{z\in\Sigma^+}$ on a set of $p_*m_F$-measure zero. $\Box$

## 10.2 Quasi-invariant measures on the space of horospheres

Let us start this subsection by giving a few definitions. Given a topological space $X$ endowed with a continuous action of a discrete group $G$, a *real cocycle* on $X$ is a continuous map $\check{c}:G\times X\to\mathbb{R}$ such that, for all $x\in X$ and $\gamma,\gamma'\in G$, we have

$$\check{c}\,(\gamma\gamma',x)=\check{c}\,(\gamma,\gamma' x)+\check{c}\,(\gamma',x)\ .$$

A real cocycle $\check{c}'$ on $X$ is *cohomologous* to $\check{c}$ via a continuous map $f : X \to \mathbb{R}$ if $\check{c}'(\gamma, x) - \check{c}(\gamma, x) = f(\gamma x) - f(x)$ for all $x \in X$ and $\gamma \in G$. Given a real cocycle $\check{c}$ on $X$, a *$\check{c}$-quasi-invariant measure* on $X$ is a (not necessarily finite) locally finite nonzero (positive Borel) measure $\check{\nu}$ on $X$ such that for every $\gamma \in G$, the measures $\gamma_*\check{\nu}$ and $\check{\nu}$ are equivalent, with, for $\check{\nu}$-almost every $x \in X$,

$$\frac{d\gamma_*\check{\nu}}{d\check{\nu}}(x) = e^{\check{c}\,(\gamma^{-1},\,x)} \;.$$

We will be interested in the case when $X$ is the space $\mathscr{H}_{\widetilde{M}}$ of horospheres of $\widetilde{M}$, endowed with the topology of Hausdorff convergence on compact subsets and the action of $G = \Gamma$ on subsets of $\widetilde{M}$, and when $\check{c}$ is the real cocycle

$$\check{c}_{\widetilde{F}} = \check{c}_{\widetilde{F},\,x_0} : (\gamma, H) \mapsto C_{F\circ\iota,\,H_\infty}(x_0, \gamma^{-1}x_0) \;,$$

where $H$ is a horosphere centred at $H_\infty$ and $\gamma \in \Gamma$.

Note that $\check{c}_{\widetilde{F}} = 0$ if $\widetilde{F} = 0$, that $\check{c}_{\widetilde{F}} = \check{c}_{\widetilde{F},\,x_0}$ does depend on $x_0$, but that $\check{c}_{\widetilde{F},\,x'_0}$ is cohomologous to $\check{c}_{\widetilde{F},\,x_0}$ via the map $f : H \mapsto C_{F\circ\iota,\,H_\infty}(x'_0, x_0)$ for every $x'_0 \in \widetilde{M}$. If two potentials $\widetilde{F}^*$ and $\widetilde{F}$ are cohomologous via the map $\widetilde{G}$, then $\widetilde{F}^*\circ\iota$ and $\widetilde{F}\circ\iota$ are cohomologous via the map $-\widetilde{G}\circ\iota$, hence, by Equation (27), the real cocycle $\check{c}_{\widetilde{F}^*}$ is cohomologous to $\check{c}_{\widetilde{F}}$ via the map $f : H \mapsto \widetilde{G}\circ\iota(v_{x_0\,H_\infty})$, with the notation of Remark (1) just before Lemma 3.4.

As explained for instance in [Sch1, Sch3], there exists a correspondence between $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measures $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ for $\widetilde{\mathscr{W}}^{\rm su}$ and $\check{c}_{\widetilde{F}}$-quasi-invariant measures $\check{\nu}$ on $\mathscr{H}_{\widetilde{M}}$, see Proposition 10.3 below. This second viewpoint is the one used in [Rob1] when $F = 0$, in which case the $\check{c}_{\widetilde{F},\,x_0}$-quasi-invariant measures on $\mathscr{H}_{\widetilde{M}}$ are just the $\Gamma$-equivariant measures, which explains why it is easier to work with the second viewpoint then. The first viewpoint turns out to be a bit more convenient when $F$ is not constant.

Let $\mathscr{H}^{\rm su}$ be the space of strong unstable leaves of $T^1\widetilde{M}$, that is, the quotient topological space of $T^1\widetilde{M}$ by the equivalence relation "being in the same strong unstable leaf", with the quotient action of $\Gamma$. We denote by $p^{\rm su} : T^1\widetilde{M} \to \mathscr{H}^{\rm su}$ the canonical projection.

Let $\mathscr{H}_{\widetilde{M}}$ be the space of horospheres in $\widetilde{M}$, endowed with the Chabauty topology on closed subsets of $\widetilde{M}$ (induced by the Hausdorff distance on compact subsets, see for instance [Pau2, §1.2]).

Note that the map which associates to a horosphere in $\widetilde{M}$ its outer unit normal bundle is a $\Gamma$-equivariant homeomorphism between $\mathscr{H}_{\widetilde{M}}$ and $\mathscr{H}^{\rm su}$, whose inverse is the map (also denoted by $\pi$) which associates to a strong unstable leaf $W$ the horosphere $\pi(W)$. For every $v \in T^1\widetilde{M}$, the restriction of $\pi\circ p^{\rm su}$ to the transversal $T = W^{\rm s}(v)$ to the foliation $\widetilde{\mathscr{W}}^{\rm su}$ in $T^1\widetilde{M}$ is a homeomorphism onto its image in $\mathscr{H}_{\widetilde{M}}$, and we will identify $T$ with its image by this map.

Let us endow $\partial_\infty\widetilde{M}\times\mathbb{R}$ with the action of $\Gamma$ defined as follows: for all $\gamma \in \Gamma$, $\xi \in \partial_\infty\widetilde{M}$, and $t \in \mathbb{R}$,

$$\gamma(\xi, t) = (\gamma\xi, t + \beta_\xi(\gamma^{-1}x_0, x_0)) \;.$$

Note that for all horospheres $H$ with centre $H_\infty$ and $u \in H$, the real number $\beta_{H_\infty}(x_0, u)$ does not depend on $u \in H$, it will be denoted by $\beta_{H_\infty}(x_0, H)$. The map from $\mathscr{H}_{\widetilde{M}}$ to $\partial_\infty\widetilde{M}\times\mathbb{R}$, defined by

$$H \mapsto \big(H_\infty, \beta_{H_\infty}(x_0, H)\big) \;,$$

is a $\Gamma$-equivariant homeomorphism. It depends on $x_0$, but only up to a translation (depending on the first factor) in the second factor $\mathbb{R}$.

**Proposition 10.3** *For every $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ for $\widetilde{\mathscr{W}}^{\rm su}$, there exists one and only one $\breve{c}_{\widetilde{F}}$-quasi-invariant measure $\breve{\nu} = \breve{\nu}_{x_0}$ on $\mathscr{H}_{\widetilde{M}}$ such that for every $v \in T^1\widetilde{M}$, if $T = W^{\rm s}(v)$, then for all $w \in T$ and $H = \pi(W^{\rm su}(w))$, we have*

$$d\breve{\nu}(H) = e^{C_{F\circ\iota,\, w_-}(\pi(w),\, x_0)}\; d\nu_T(w)\ . \tag{135}$$

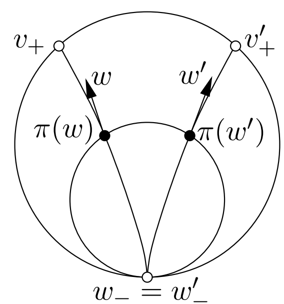
*Conversely, for every $\breve{c}_{\widetilde{F}}$-quasi-invariant measure $\breve{\nu}$ on $\mathscr{H}_{\widetilde{M}}$, there exists one and only one $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ for $\widetilde{\mathscr{W}}^{\rm su}$ such that Equation (135) holds for every stable leaf $T$.*

Note that the measure $\breve{\nu} = \breve{\nu}_{x_0}$ does depend on the base point $x_0$, hence the aforementioned correspondence is not canonical.

Note that the above measure $\breve{\nu}$ on $\mathscr{H}_{\widetilde{M}}$ gives full measure to the set of horospheres centred at conical limit points if and only if the transverse measure $\nu$ for $\widetilde{\mathscr{W}}^{\rm su}$ gives full measure to the negatively recurrent set.

**Proof.** The open subsets $T = W^{\rm s}(v)$ of $\mathscr{H}_{\widetilde{M}}$ cover $\mathscr{H}_{\widetilde{M}}$ as $v$ ranges over $T^1\widetilde{M}$. Let us prove that the measures on these open subsets defined by the member on the right of Equation (135) glue together to define a measure on $\mathscr{H}_{\widetilde{M}}$, that is, that two of them coincide on the intersection of their domain.

Let $v, v' \in T^1\widetilde{M}$, and let $T = W^{\rm s}(v)$ and $T' = W^{\rm s}(v')$. The map from $\{w \in T \;:\; w_- \neq v'_+\}$ to $\{w' \in T' \;:\; w'_- \neq v_+\}$ which sends $w$ to the unique element $w' \in W^{\rm su}(w)$ such that $w'_+ = v'_+$ is a holonomy map for the foliation $\widetilde{\mathscr{W}}^{\rm su}$. Note that $w'_- = w_-$ and that $\pi(w)$ and $\pi(w')$ lie on the same horosphere centred at $w'_- = w_-$. In particular, $\beta_{w_-}(\pi(w), \pi(w') = 0$.




By the properties (i) and (iii) of $\nu$, we hence have, for every $w \in \{w \in T \;:\; w_- \neq v'_+\}$,

$$\begin{aligned}
e^{C_{F\circ\iota,\, w'_-}(\pi(w'),\, x_0)}\; d\nu_{T'}(w') &= e^{C_{F\circ\iota,\, w'_-}(\pi(w'),\, x_0)}\; e^{c_{\widetilde{F}}(w',\, w)}\; d\nu_T(w)\\
&= e^{C_{F\circ\iota,\, w_-}(\pi(w'),\, x_0) - C_{F\circ\iota,\, w_-}(\pi(w'),\, \pi(w))}\; d\nu_T(w)\\
&= e^{C_{F\circ\iota,\, w_-}(\pi(w),\, x_0)}\; d\nu_T(w)\ .
\end{aligned} \tag{136}$$

This proves the existence and uniqueness of a nonzero measure $\breve{\nu}$ on $\mathscr{H}_{\widetilde{M}}$ satisfying Equation (135) for every stable leaf $T$.

Let us now prove that $\breve{\nu}$ is $\breve{c}_{\widetilde{F}}$-quasi-invariant. Let $\gamma \in \Gamma$, $v \in T^1\widetilde{M}$, and $T = W^{\rm s}(v)$, so that $\gamma T = W^{\rm s}(\gamma v)$. Let $w \in \{u \in \gamma T \;:\; u_- \neq v_+\}$ and $H = \pi(W^{\rm ss}(w))$. Using respectively

- the definition of $\breve{\nu}$ for the second equality,
- the invariance of the Gibbs cocycle (see Equation (26)) and the property (ii) of $\nu$ for the third equality,

• the property (iii) of $\nu$ for the holonomy map from $\{u \in T \;:\; u_- \neq \gamma v_+\}$ to $\{u \in \gamma T \;:\; u_- \neq v_+\}$ for the foliation $\widetilde{\mathscr{W}}^{\rm su}$ sending $w'$ to the unique element $w \in W^{\rm su}(w')$ such that $w_+ = \gamma v_+$ for the fourth equality,

• the fact that $w'_- = w_-$ and the definition of the cocycle $c_{\widetilde{F}}$ for the fifth equality,

• the cocycle properties (see Equation (25)) of the Gibbs cocycle for the sixth equality,

• the definition of the cocycle $\breve{c}_{\widetilde{F}}$ and the facts that $\pi(W^{\rm su}(w')) = \pi(W^{\rm su}(w)) = H$ and $H_\infty = w'_-$ for the seventh equality, we have

$$\begin{aligned} d\gamma_*\breve{\nu}(H) &= d\breve{\nu}(\gamma^{-1}H) = e^{C_{F\circ\iota,\,(\gamma^{-1}w)_-}(\pi(\gamma^{-1}w),\,x_0)}\;d\nu_T(\gamma^{-1}w)\\ &= e^{C_{F\circ\iota,\,w_-}(\pi(w),\,\gamma x_0)}\;d\nu_{\gamma T}(w) = e^{C_{F\circ\iota,\,w_-}(\pi(w),\,\gamma x_0)}\;e^{c_{\widetilde{F}}(w,\,w')}\;d\nu_T(w')\\ &= e^{C_{F\circ\iota,\,w'_-}(\pi(w),\,\gamma x_0)-C_{F\circ\iota,\,w'_-}(\pi(w),\,\pi(w'))}\;d\nu_T(w')\\ &= e^{C_{F\circ\iota,\,w'_-}(x_0,\,\gamma x_0)}\;e^{C_{F\circ\iota,\,w'_-}(\pi(w'),\,x_0)}\;d\nu_T(w')\\ &= e^{\breve{c}_{\widetilde{F}}(\gamma^{-1},\,H)}\;d\breve{\nu}(H)\;. \end{aligned} \tag{137}$$

This proves that $\breve{\nu}$ is $\breve{c}_{\widetilde{F}}$-quasi-invariant.

Conversely, let $\breve{\nu}$ be a $\breve{c}_{\widetilde{F}}$-quasi-invariant nonzero measure on $\mathscr{H}_{\widetilde{M}}$. For every stable leaf $T = W^{\rm s}(v)$, let us define

$$d\nu_T(w) = e^{-C_{F\circ\iota,\,w_-}(\pi(w),\,x_0)}\;d\breve{\nu}\big(\pi(W^{\rm su}(w))\big)\;.$$

The computations (136) and (137) show that it is possible to define, using restrictions and holonomy maps, a measure $\nu_T$ for any transversal $T$ to $\widetilde{\mathscr{W}}^{\rm su}$ such that the family $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ satisfies the properties (i), (ii) and (iii). $\Box$

### 10.3 Classification of quasi-invariant transverse measures for $\mathscr{W}^{\rm su}$

Let us now state our unique ergodicity result of the strong unstable foliation, starting by defining the transverse measure which will play the central role.

By Subsection 3.9, if $\delta = \delta_{\Gamma,\,F} < +\infty$, to every Patterson density $(\mu^\iota_x)_{x\in\widetilde{M}}$ of dimension $\delta$ for $(\Gamma, F\circ\iota)$ is associated, for any $v \in T^1\widetilde{M}$ with stable leaf $T = W^{\rm s}(v)$, a nonzero measure $\mu^\iota_T$ on $T$, defined (independently of $x_0$), using the homeomorphism from $T$ to $(\partial_\infty\widetilde{M} - \{v_+\}) \times \mathbb{R}$ defined by $w' \mapsto \big(w'_-, t = \beta_{v_+}(\pi(v), \pi(w'))\big)$, by (see Equation (60))

$$d\mu^\iota_T(w') = e^{C_{F\circ\iota-\delta,\,w'_-}(x_0,\,\pi(w'))}\;d\mu^\iota_{x_0}(w'_-)\,dt\;. \tag{138}$$

This measure is finite on the compact subsets of $T$, and if $\mu^\iota_{x_0}({}^c\Lambda_c\Gamma) = 0$, then the set of $w' \in T$ such that $w'_-$ belongs to $\Lambda_c\Gamma$ has full measure with respect to $\mu^\iota_T$. By Remark (2) in Subsection 10.1, by Equation (61) and Equation (62), there exists hence a unique $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $(\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ for $\widetilde{\mathscr{W}}^{\rm su}$, giving full measure to the negatively recurrent set, such that if $T = W^{\rm s}(v)$ for some $v \in T^1\widetilde{M}$, then $\mu^\iota_T$ is the above measure. Note that $(\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ is unchanged by replacing $F$ with $F+\kappa$, for any $\kappa\in\mathbb{R}$.

The next result proves that such a transverse measure is unique, under our usual assumptions (see Chapter 8, in particular Theorem 8.1, for situations where they are satisfied).

**Theorem 10.4** *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold, with dimension at least 2 and pinched negative curvature at most $-1$. Let $\Gamma$ be a nonelementary discrete group of isometries of $\widetilde{M}$. Let $\widetilde{F} : \widetilde{M} \to \mathbb{R}$ be a $\Gamma$-invariant Hölder-continuous map. Assume that $\delta_{\Gamma,\,F} < +\infty$ and that the Gibbs measure $m_F$ on $T^1M$ is finite and mixing under the geodesic flow.*

*Then the family $(\mu^{\iota}_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ is, up to a multiplicative constant, the unique $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure, giving full measure to the negatively recurrent set, for the strong unstable foliation on $T^1\widetilde{M}$. Furthermore, $(\mu^{\iota}_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ is ergodic.*

**Remark (1).** If $F = 0$, this theorem (with the formulation of Corollary 10.5 in the next remark if $M$ has dimension 2) is due to Furstenberg [Fur] when $M$ is a compact hyperbolic surface, to Dani [Dan], Dani-Smillie [DaS], Ratner [Rat3] when $M$ is a finite volume hyperbolic surface, to Bowen-Marcus [BoM] when $M$ is a compact or convex-cocompact manifold, via a coding argument (see also Coudène [Cou1] for a very short dynamical argument in the case of finite volume surfaces), and to Roblin [Rob1, Chap. 6] in full generality. When $F = 0$, $\widetilde{M}$ is a higher rank symmetric space of noncompact type, $G$ is the identity component of the isometry group of $\widetilde{M}$, and $\Gamma$ is a lattice in $G$, we refer to Ratner's classification theorem of the measures on $\Gamma\backslash G$ invariant under a unipotent subgroup of $G$ (see [Rat1, Rat2, MaT, Mor]). It would be interesting to study the case $F$ non-constant in higher rank.

For general potentials $F$ (with the slightly different definition of Gibbs measure mentioned in the beginning of Chapter 3), this theorem was proved in Babillot-Ledrappier [BaL] in a symbolic framework or when $M$ is an abelian Riemannian cover of a compact manifold, and in Schapira [Sch3] when $M$ is compact or convex-cocompact. This theorem (with $\Gamma$ torsion free and $\widetilde{F}$ bounded) is stated, and its proof is sketched, in [Sch1, Chap. 8.2.2] following Roblin's arguments in [Rob1, Chap. 6]. In the proof below, we will also follow closely Roblin's strategy of proof, though the presence of potentials requires many adjustments and new lemmas.

**Remark (2).** When $\widetilde{M}$ is a surface, it is well-known since Furstenberg's work (see for instance [Sch2]) that this theorem corresponds to a unique ergodicity result for horocyclic flows. When $\widetilde{M}$ is a symmetric space, this theorem corresponds to a unique ergodicity result for some unipotent group actions.

More precisely, assume first that $\widetilde{M}$ is a surface, fix one of the two orientations of $\widetilde{M}$ and assume that $\Gamma$ preserves the orientation. Then each horosphere is naturally oriented, say anticlockwise. A *horocyclic flow* on $T^1\widetilde{M}$ is a continuous one-parameter group of homeomorphisms $(h_s)_{s\in\mathbb{R}}$ of $T^1\widetilde{M}$, such that, for every $v \in T^1\widetilde{M}$, the map $s \mapsto h_s v$ is an orientation preserving homeomorphism from $\mathbb{R}$ to $W^{\rm su}(v)$ and for every $s \in \mathbb{R}$, the homeomorphism $v \mapsto h_s v$ of $T^1\widetilde{M}$ is $\Gamma$-equivariant. By passing to the quotient, it induces a continuous one-parameter group of homeomorphisms of $T^1M$, called the *horocyclic flow* on $T^1M$ and also denoted by $(h_s)_{s\in\mathbb{R}}$. One way to define $h_s v$ is as the element of $W^{\rm su}(v)$ on the right of $v$ if $s \geq 0$, and on its left if $s \leq 0$, such that the Riemannian length of the arc of the horocycle $\pi(W^{\rm su}(v))$ from $\pi(v)$ to $\pi(h_s(v))$ is $s$. In constant curvature $-1$, this horocycle flow satisfies furthermore that $\phi_t \circ h_s = h_{s\,e^t} \circ \phi_t$ for all $s, t \in \mathbb{R}$. But in variable curvature, this is no longer always true, and the Riemannian parametrisation is not always the appropriate one. See for instance [Marc] which constructs, when $M$ is compact, a horocycle flow satisfying, besides more precise regularity properties, that

$\phi_t \circ h_s = h_{s\, e^{\delta_\Gamma t}} \circ \phi_t$ for all $s, t \in \mathbb{R}$.

A locally finite (positive Borel) measure $\widetilde{m}$ on $T^1\widetilde{M}$ is *$c_{\widetilde{F}}$-quasi-invariant under the horocyclic flow* $(h_s)_{s\in\mathbb{R}}$ if for every $s \in \mathbb{R}$, the measures $\widetilde{m}$ and $(h_s)_*\widetilde{m}$ are mutually absolutely continuous, and satisfy, for $\widetilde{m}$-almost every $v \in T^1\widetilde{M}$,

$$\frac{d\,(h_s)_*\widetilde{m}}{d\,\widetilde{m}}(h_s v) = e^{c_{\widetilde{F}}(v,\, h_s v)}\ .$$

More generally, assume that there exists a continuous action (on the left) of a unimodular real Lie group $\mathbb{U}$ on $T^1\widetilde{M}$, commuting with the action of $\Gamma$, such that for every $v \in T^1\widetilde{M}$ the map $u \mapsto u \cdot v$ is a homeomorphism from $\mathbb{U}$ to $W^{\rm su}(v)$. Such an action exists for instance when $\widetilde{M}$ is a symmetric space, that is, a real, complex, quaternionic or octonionic hyperbolic space (see for instance [Park]), since a nilpotent Lie group is unimodular. We say that a locally finite measure $\widetilde{m}$ on $T^1\widetilde{M}$ is *$c_{\widetilde{F}}$-quasi-invariant under the action of* $\mathbb{U}$ if for every $u \in \mathbb{U}$, the measures $\widetilde{m}$ and $u_*\widetilde{m}$ are absolutely continuous with respect to each other, and satisfy, for $\widetilde{m}$-almost every $v \in T^1\widetilde{M}$,

$$\frac{d\,u_*\widetilde{m}}{d\,\widetilde{m}}(u\cdot v) = e^{c_{\widetilde{F}}(v,\, u\cdot v)}\ .$$

**Corollary 10.5** *Under the assumptions of Theorem 10.4, there exists, up to a multiplicative constant, a unique locally finite $\Gamma$-invariant measure on $T^1\widetilde{M}$, giving full measure to $\{w \in T^1\widetilde{M}\ :\ w_- \in \Lambda_c\Gamma\}$, which is $c_{\widetilde{F}}$-quasi-invariant under the action of $\mathbb{U}$.*

The special case when $\mathbb{U} = \mathbb{R}$ gives that if $\widetilde{M}$ has dimension 2 and if $(h_s)_{s\in\mathbb{R}}$ is a horocyclic flow, then under the assumptions of Theorem 10.4, there exists, up to a multiplicative constant, a unique locally finite $\Gamma$-invariant measure on $T^1\widetilde{M}$, giving full measure to $\{w \in T^1\widetilde{M}\ :\ w_- \in \Lambda_c\Gamma\}$, which is $c_{\widetilde{F}}$-quasi-invariant under $(h_s)_{s\in\mathbb{R}}$.

**Proof.** We start the proof by a construction which, when $\mathbb{U} = \mathbb{R}$, is due to Burger [Bur] (see more generally [Rob1, §1C]) when $F = 0$, and to the last author [Sch2, Sch3] (see also Lemma 10.7 below) when $\Gamma$ is torsion free, with a local approach.

Recall (see Subsection 3.9 before Proposition 3.18, exchanging the stable and unstable foliations) that, for every $v \in T^1\widetilde{M}$, with $T = W^{\rm s}(v)$, the map $\psi_T$ from the open set $U_T = \{w \in T^1\widetilde{M}\ :\ w_- \neq v_+\}$ to $T$, sending $w \in U_T$ to the unique element $w'$ in $W^{\rm su}(w) \cap T$, is a continuous fibration over $T$, whose fibre over $w' \in T$ is the strong unstable leaf of $w'$. (Note that $U_T$ is invariant under $\mathbb{U}$, and this fibration is a principal fibration under $\mathbb{U}$.)

Let $\lambda$ be a Haar measure on $\mathbb{U}$ (say the usual Lebesgue measure when $\mathbb{U} = \mathbb{R}$) and, for every $v \in T^1\widetilde{M}$, let $\lambda_{W^{\rm su}(v)}$ be the push-forward measure of $\lambda$ by the map $u \mapsto u \cdot v$, whose support is $W^{\rm su}(v)$. Since $\mathbb{U}$ acts transitively on $W^{\rm su}(v)$ and since $\lambda$ is invariant under right translations, the measure $\lambda_{W^{\rm su}(v)}$ indeed depends only on the strong unstable leaf of $v$. Since $\lambda$ is invariant under left translations, we have $u_*\lambda_{W^{\rm su}(v)} = \lambda_{W^{\rm su}(v)}$ for every $u \in \mathbb{U}$. Since the action of $\Gamma$ on $T^1\widetilde{M}$ commutes with the action of $\mathbb{U}$, for every $\gamma \in \Gamma$, we have $\gamma_*\lambda_{W^{\rm su}(v)} = \lambda_{W^{\rm su}(\gamma v)}$. Note that since the action is continuous, for every $v \in T^1\widetilde{M}$, with $T = W^{\rm s}(v)$, for every $f \in \mathscr{C}_c(T^1\widetilde{M};\mathbb{R})$ with support contained in $U_T$, the map from $T$ to $\mathbb{R}$ defined by

$$w' \mapsto \int_{w\in W^{\rm su}(w')} f(w)\ e^{c_{\widetilde{F}}(w,\, w')}\ d\lambda_{W^{\rm su}(w')}(w) = \int_{u\in\mathbb{U}} f(u\cdot w')\ e^{c_{\widetilde{F}}(u\cdot w',\, w')}\ d\lambda(u)$$

is continuous with compact support.

Let $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ be a $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{\rm su}$. We claim that there exists a unique (positive Borel) measure $\widetilde{m}^{\nu,\,\lambda}$ on $T^1\widetilde{M}$ such that for every $v\in T^1\widetilde{M}$, with $T=W^{\rm s}(v)$, the restriction to $U_T$ of the measure $\widetilde{m}^{\nu,\,\lambda}$ disintegrates by the fibration $\psi_T$ over the measure $\nu_T$, with conditional measure on the fibre $W^{\rm su}(w')$ of $w'\in T$ the measure $e^{c_{\widetilde{F}}(w,\,w')}\;d\lambda_{W^{\rm su}(w')}(w)$: for every $f\in\mathscr{C}_c(T^1\widetilde{M};\mathbb{R})$ with support contained in $U_T$, we have

$$\int_{w\in T^1\widetilde{M}} f(w)\;d\widetilde{m}^{\nu,\,\lambda}(w)=\int_{w'\in T}\int_{u\in\mathbb{U}} f(u\cdot w')\;e^{c_{\widetilde{F}}(u\cdot w',\,w')}\;d\lambda(u)\;d\nu_T(w')\;.$$

We refer to the proof of Lemma 10.7 for a proof of this claim.

**Lemma 10.6** *The map $\nu\mapsto\widetilde{m}^{\nu,\,\lambda}$ is a bijection from the set of $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measures for $\widetilde{\mathscr{W}}^{\rm su}$, which give full measure to the negatively recurrent set, to the set of $\Gamma$-invariant locally finite measures on $T^1\widetilde{M}$ which are $c_{\widetilde{F}}$-quasi-invariant under the action of $\mathbb{U}$ and give full measure to $\{w\in T^1\widetilde{M}\;:\;w_-\in\Lambda_c\Gamma\}$.*

**Proof.** By construction, $\nu$ is $\Gamma$-equivariant if and only if $\widetilde{m}^{\nu,\,\lambda}$ is $\Gamma$-invariant, and $\nu_T$ is locally finite for every $T$ if and only if $\widetilde{m}^{\nu,\,\lambda}$ is locally finite. For every $v\in T^1\widetilde{M}$, with $T=W^{\rm s}(v)$, since the support of $\lambda_{W^{\rm su}(v)}$ is equal to $W^{\rm su}(v)$, the measure $\nu_T$ gives full measure to $\{w'\in T\;:\;w'_-\in\Lambda_c\Gamma\}$ if and only if the restriction of $\widetilde{m}^{\nu,\,\lambda}$ to $U_T$ gives full measure to $\{w\in U_T\;:\;w_-\in\Lambda_c\Gamma\}$.

For all $v\in T^1\widetilde{M}$, $w\in T=W^{\rm s}(v)$ and $u\in\mathbb{U}$, since $\lambda_{W^{\rm su}(w')}$ is invariant under $\mathbb{U}$ and by the cocycle property of $c_{\widetilde{F}}$, we have

$$d\widetilde{m}^{\nu,\,\lambda}(u\cdot w)=\int_{w'\in T} e^{c_{\widetilde{F}}(u\cdot w,\,w')}\;d\lambda_{W^{\rm su}(w')}(u\cdot w)\;d\nu_T(w')=e^{c_{\widetilde{F}}(u\cdot w,\,w)}\;d\widetilde{m}^{\nu,\,\lambda}(w)\;.$$

Hence $\widetilde{m}^{\nu,\,\lambda}$ is $c_{\widetilde{F}}$-quasi-invariant under the action of $\mathbb{U}$.

Conversely, let $\widetilde{m}$ be a $\Gamma$-invariant locally finite measure on $T^1\widetilde{M}$ which is $c_{\widetilde{F}}$-quasi-invariant under the action of $\mathbb{U}$. Then given any $v\in T^1\widetilde{M}$, with $T=W^{\rm s}(v)$, for the principal fibration $U_T\to T$ with group $\mathbb{U}$, the conditional measures of $m_{|U_T}$ on the fibre $W^{\rm su}(w')=\mathbb{U}\cdot w'$ of almost every $w'\in T$ are invariant under $\mathbb{U}$. This conditional measure may be taken to be equal to $\lambda_{W^{\rm su}(w')}$, by the uniqueness property of Haar measures, up to multiplying by a measurable function the (locally finite) measure $\nu_T$ on $T$ over which $\widetilde{m}$ disintegrates. Then the family $\nu=(\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ is clearly a $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{\rm su}$. □

Since for every $t>0$, we have $\widetilde{m}^{t\nu,\,\lambda}=t\,\widetilde{m}^{\nu,\,\lambda}$, Corollary 10.5 indeed follows from Theorem 10.4. □

**Proof of Theorem 10.4.** Up to adding a large enough constant to $F$, which does not change the statement, we assume that $\delta=\delta_{\Gamma,\,F}$ is positive. Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ be the Patterson densities of the same dimension $\delta$ for $(\Gamma,F\circ\iota)$ and $(\Gamma,F)$, such that $m_F$ is the Gibbs measure on $T^1M$ (induced by the Gibbs measure $\widetilde{m}_F$ on $T^1\widetilde{M}$) associated with this pair of Patterson densities. Note that they are uniquely defined up to a scalar multiple, and give full measure to the conical limit set, by Corollary 5.15 and Corollary 5.12. Hence the family $\mu^\iota = (\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ associated to $(\mu^\iota_x)_{x\in\widetilde{M}}$ by Equation (138) is indeed a $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{\rm su}$, giving full measure to the negatively recurrent set.

The scheme of the eight steps proof of Theorem 10.4 is the following one. Given a second such family $\nu$, we first construct a measure $m^{\nu,\,\mu}$ which is a quasi-product measure of the measures defined by $\nu$ on the stable leaves and the measures defined by $m_F$ on the strong unstable leaves, so that $m_F = m^{\mu^\iota,\,\mu}$. We then introduce diffusion operators $(I_r)_{r>0}$ along the strong unstable leaves acting on the locally integrable functions on $T^1M$. We use the mixing property of the geodesic flow (see Subsection 8.1) combined with the action of these diffusion operators to prove that $m_F = m^{\mu^\iota,\,\mu}$ is absolutely continuous with respect to $m^{\nu,\,\mu}$ (this is the most technical part, which will require several steps). We then conclude that $\nu$ and $\mu^\iota$ are homothetic by an easy argument of disintegration and ergodicity.

**Step 1.** Let $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ be a $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{\rm su}$, giving full measure to the negatively recurrent set. In this first step, we construct a measure $\widetilde{m}^{\nu,\,\mu}$ on $T^1\widetilde{M}$ which, for every stable leaf $T$, disintegrates, with conditional measures in the class of the strong unstable measures $\mu_{W^{\rm su}(w')}$ (as does the Gibbs measure), over the measure $\nu_T$ (instead of over the measure $\mu^\iota_T$ as does the Gibbs measure).

Recall (see Subsection 3.9 before Proposition 3.18, exchanging the stable and unstable foliations) that, for every $v\in T^1\widetilde{M}$, with $T = W^{\rm s}(v)$, the map $\psi_T$ from the open set $U_T = \{w\in T^1\widetilde{M}\;:\;w_-\neq v_+\}$ to $T$, sending $w$ to the unique element $w'$ in $W^{\rm su}(w)\cap T$, is a continuous fibration over $T$, whose fibre over $w'\in T$ is the strong unstable leaf of $w'$.

**Lemma 10.7** *There exists a unique (positive Borel) measure $\widetilde{m}^{\nu,\,\mu}$ on $T^1\widetilde{M}$ such that for every $v\in T^1\widetilde{M}$, with $T = W^{\rm s}(v)$, the restriction to $U_T$ of the measure $\widetilde{m}^{\nu,\,\mu}$ disintegrates by the fibration $\psi_T$ over the measure $\nu_T$, with conditional measure $e^{c_{\widetilde{F}}(w,\,w')}\,d\mu_{W^{\rm su}(w')}(w)$ on the fibre $W^{\rm su}(w')$ of $w'\in T$: for every $w\in T^1\widetilde{M}$ such that $w_-\neq v_+$, we have*

$$d\widetilde{m}^{\nu,\,\mu}(w) = \int_{w'\in T} e^{c_{\widetilde{F}}(w,\,w')}\;d\mu_{W^{\rm su}(w')}(w)\;d\nu_T(w')\;. \tag{139}$$

*Furthermore, $\widetilde{m}^{\nu,\,\mu}$ is $\Gamma$-invariant and locally finite, and the set $\widetilde{\Omega}_c\Gamma$ of elements $v\in T^1\widetilde{M}$ such that $v_-, v_+\in\Lambda_c\Gamma$ has full measure for $\widetilde{m}^{\nu,\,\mu}$.*

Hence the measure $\widetilde{m}^{\nu,\,\mu}$ defines a locally finite measure $m^{\nu,\,\mu}$ on $T^1M$, which gives full measure to the set of two-sided recurrent elements.

Note that by the disintegration of the Gibbs measure (see Proposition 3.18 (1), exchanging the stable and unstable foliations), we have

$$\widetilde{m}_F = \widetilde{m}^{\mu^\iota,\,\mu}\quad\text{and}\quad m_F = m^{\mu^\iota,\,\mu}\;.$$

**Proof.** Let $v, v'\in T^1\widetilde{M}$, and $T = W^{\rm s}(v)$, $T' = W^{\rm s}(v')$. The map from $U_{T'}\cap T = \{w'\in T\;:\;w'_-\neq v'_+\}$ to $U_T\cap T' = \{w''\in T'\;:\;w''_-\neq v_+\}$ sending $w'$ to the unique element $w''$ of $W^{\rm su}(w')\cap T'$ is a holonomy map for the strong unstable foliation. Hence $d\nu_{T'}(w'') = e^{c_{\widetilde{F}}(w'',\,w')}d\nu_T(w')$ by the $c_{\widetilde{F}}$-quasi-invariance property of $\nu$. Since $W^{\rm su}(w') = W^{\rm su}(w'')$ and by the cocycle property of $c_{\widetilde{F}}$, for every $w\in U_T\cap U_{T'}$, we therefore have

$$\int_{w'\in T} e^{c_{\widetilde{F}}(w,\,w')}\;d\mu_{W^{\rm su}(w')}(w)\;d\nu_T(w') = \int_{w''\in T'} e^{c_{\widetilde{F}}(w,\,w'')}\;d\mu_{W^{\rm su}(w'')}(w)\;d\nu_{T'}(w'')\;.$$

Hence the measures on the sets $U_T$ defined by the right hand side of Equation (139) glue together to define a measure $\widetilde{m}^{\nu,\,\mu}$ on $T^1\widetilde{M}$. The uniqueness follows from the fact that the open sets $U_T$ cover $T^1\widetilde{M}$.

The measure $\widetilde{m}^{\nu,\,\mu}$ is locally finite since $\nu_T$ and $\mu_{W^{\rm su}(w')}$ are locally finite. It is $\Gamma$-invariant since the families $\nu$ and $(\mu_{W^{\rm su}(w')})_{w'\in T^1\widetilde{M}}$ are $\Gamma$-equivariant and the cocycle $c_{\widetilde{F}}$ is $\Gamma$-invariant. The last claim of the lemma follows from the fact that $\nu_T$ gives full measure to the set of $w'\in T$ such that $w'_-\in\Lambda_c\Gamma$ and $\mu_{W^{\rm su}(w')}$ to the elements $w\in W^{\rm su}(w')$ such that $w_+\in\Lambda_c\Gamma$. □

Note that the measure $m^{\nu,\,\mu}$ is a priori not finite (and could be infinite if $m_F$ was not assumed to be finite, see for instance [Bur]). We will turn it into a probability measure by multiplying it by a convenient bump function.

We fix a nonnegative (continuous) $\rho\in\mathscr{C}_c(T^1M;\mathbb{R})$ with large enough (compact) support, so that

$$\int_{T^1M}\rho\,dm_F>0\quad\text{and}\quad\int_{T^1M}\rho\,dm^{\nu,\,\mu}=1\,.$$

With $T\widetilde{p}:T^1\widetilde{M}\to T^1M$ the canonical projection, let $\widetilde{\rho}=\rho\circ T\widetilde{p}$ be the lift of $\rho$ to $T^1\widetilde{M}$. Consider the measure $\widetilde{\Pi}=\widetilde{\rho}\,\widetilde{m}^{\nu,\,\mu}$ on $T^1\widetilde{M}$, which induces a probability measure

$$\Pi=\rho\,m^{\nu,\,\mu}$$

on $T^1M$.

**Step 2.** In this step, we construct two families $(I_r)_{r>0}$ and $(J_r)_{r>0}$ of diffusion operators along the strong unstable leaves, acting on the locally integrable functions on $T^1M$, and prove an adjointness property for them which will be crucial for the proof of Theorem 10.4.

Recall (see Subsection 2.4) that $d_{W^{\rm su}(v)}$ is the Hamenstädt distance on the strong unstable leaf $W^{\rm su}(v)$ of $v\in T^1\widetilde{M}$. Recall that we denote by $B^{\rm su}(v,r)$ the open ball of centre $v$ and radius $r$ for the distance $d_{W^{\rm su}(v)}$.

Recall that $T\widetilde{p}:T^1\widetilde{M}\to T^1M$ is the canonical projection. For every $r>0$ and for all measurable nonnegative maps $\psi:T^1M\to\mathbb{R}$, let $\widetilde{\psi}=\psi\circ T\widetilde{p}$ be the *lift* of $\psi$ to $T^1\widetilde{M}$. Let us define $\widetilde{I}_r\widetilde{\psi}:T^1\widetilde{M}\to\mathbb{R}$ as an integral of $\widetilde{\psi}$ on the balls of radius $r$ of the strong unstable leaves:

$$\forall\,u\in T^1\widetilde{M},\quad \widetilde{I}_r\widetilde{\psi}\,(u)=\int_{v\in B^{\rm su}(u,\,r)}\widetilde{\psi}(v)\;e^{c_{\widetilde{F}}(v,\,u)}\;d\mu_{W^{\rm su}(u)}(v)\,. \tag{140}$$

We define similarly

$$\forall\,u\in T^1\widetilde{M},\quad \widetilde{J}_r\widetilde{\psi}\,(u)=\int_{v\in B^{\rm su}(u,\,r)}\widetilde{\psi}(v)\;d\mu_{W^{\rm su}(u)}(v)\,. \tag{141}$$

The maps $\widetilde{I}_r\widetilde{\psi}$ and $\widetilde{J}_r\widetilde{\psi}$ are $\Gamma$-invariant, by Equation (49) and the $\Gamma$-invariance of $c_{\widetilde{F}}$ and $\widetilde{\psi}$. Hence they define measurable nonnegative maps $I_r\psi, J_r\psi:T^1M\to[0,+\infty]$ whose lifts are $\widetilde{I}_r\widetilde{\psi},\widetilde{J}_r\widetilde{\psi}$. Note that $I_r\psi=J_r\psi$ if $F=0$.

Since $\widetilde{\psi}$ is nonnegative, then for every $u\in T^1\widetilde{M}$, the maps $r\mapsto\widetilde{I}_r\widetilde{\psi}\,(u)$ and $r\mapsto\widetilde{J}_r\widetilde{\psi}\,(u)$ are nondecreasing. The operators $\widetilde{I}_r$, $\widetilde{J}_r$, $I_r$ and $J_r$ are positive: For instance, if $\psi\le\psi'$ then $I_r\psi\le I_r\psi'$.

If $\widetilde{\psi}$ is continuous and positive at an element $u \in T^1\widetilde{M}$ belonging to the support of $\mu_{W^{\rm su}(u)}$ (which is the set of vectors $v \in W^{\rm su}(u)$ such that $v_+ \in \Lambda\Gamma$), then $\widetilde{I}_r\widetilde{\psi}\,(u) > 0$. In particular, we have $\widetilde{I}_r\widetilde{\rho}\,(u) > 0$ if $\widetilde{\rho}(u) > 0$ and $u_+ \in \Lambda\Gamma$. Hence the $\Gamma$-invariant map $\frac{\widetilde{\rho}}{\widetilde{I}_r\widetilde{\rho}}$, defined to have value 0 at $u \in T^1\widetilde{M}$ if $\widetilde{\rho}(u) = 0$ or $u_+ \notin \Lambda\Gamma$, induces a measurable nonnegative map $\frac{\rho}{I_r\rho}$ on $T^1M$.

Let us denote by $\mathbb{1}$ the constant map with value 1 on $T^1M$ and on $T^1\widetilde{M}$, which satisfies $\widetilde{I}_r\mathbb{1}(u) > 0$ if $u_+ \in \Lambda\Gamma$. We will apply the mixing property of Theorem 8.2 when $B$ is a Hamenstädt ball $B^{\rm su}(u, r)$, giving that, for every $u \in T^1M$ such that $u_- \in \Lambda\Gamma$ and every nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$,

$$\lim_{t\to+\infty} I_r(\psi\circ\phi_t)(u) = \frac{I_r\mathbb{1}\,(u)}{\|m_F\|}\int_{T^1M}\psi\;dm_F\;. \tag{142}$$

The diffusion operators $I_r$ on the strong unstable balls have the following commutation property with the geodesic flow.

**Lemma 10.8** *For all $t \in \mathbb{R}$, $r > 0$, $u \in T^1M$ and $\psi : T^1M \to \mathbb{R}$ measurable nonnegative, we have*

$$I_{re^t}(\psi\circ\phi_{-t})(\phi_t u) = e^{\delta\, t - \int_0^t F(\phi_s u)\,ds}\; I_r\psi\,(u)\;.$$

**Proof.** By the equations (51), (23) and (25), for all $t \in \mathbb{R}$, $u \in T^1\widetilde{M}$ and $w \in W^{\rm su}(u)$, we have

$$\begin{aligned} d\,\mu_{W^{\rm su}(\phi_t u)}(\phi_t w) &= e^{-C_{F-\delta,\,w_+}(\pi(\phi_t w),\,\pi(w))}\;d\,\mu_{W^{\rm su}(u)}(w)\\ &= e^{-C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(\phi_t w))}\;d\,\mu_{W^{\rm su}(u)}(w)\;. \end{aligned}$$

Hence for all $t \in \mathbb{R}$, $r > 0$, $u \in T^1\widetilde{M}$ and $\widetilde{\psi} : T^1M \to \mathbb{R}$ measurable nonnegative, by the definition of the operators $\widetilde{I}_r$ and by Equation (13), using the change of variables $v = \phi_t w$ and the definition of the cocycle $c_{\widetilde{F}}$ in Equation (133), we have

$$\begin{aligned} \widetilde{I}_{re^t}(\widetilde{\psi}\circ\phi_{-t})(\phi_t u) &= \int_{v\in B^{\rm su}(\phi_t u,\, re^t)} \widetilde{\psi}(\phi_{-t}v)\;e^{c_{\widetilde{F}}(v,\,\phi_t u)}\;d\mu_{W^{\rm su}(\phi_t u)}(v)\\ &= \int_{w\in B^{\rm su}(u,\, r)} \widetilde{\psi}(w)\;e^{c_{\widetilde{F}}(\phi_t w,\,\phi_t u)} e^{-C_{F\circ\iota-\delta,\,w_-}(\pi(w),\,\pi(\phi_t w))}\;d\,\mu_{W^{\rm su}(u)}(w)\\ &= e^{-C_{F\circ\iota-\delta,\,u_-}(\pi(u),\,\pi(\phi_t u))}\;\widetilde{I}_r\widetilde{\psi}(u) = e^{-\int_0^t(\widetilde{F}(\phi_s u)-\delta)\,ds}\;\widetilde{I}_r\widetilde{\psi}(u)\;. \end{aligned} \tag{143}$$

The result follows. □

The operators $I_r$ and $J_r$ satisfy the following crucial adjointness property for the scalar product of $\mathbb{L}^2(m^{\nu,\,\mu})$.

**Lemma 10.9** *For all $r > 0$ and $\psi, \varphi : T^1M \to \mathbb{R}$ measurable nonnegative, we have*

$$\int_{T^1M} I_r\psi\;\varphi\;dm^{\nu,\,\mu} = \int_{T^1M}\psi\;J_r\varphi\;dm^{\nu,\,\mu}\;.$$

**Proof.** We assume that $\Gamma$ is torsion free: the general case follows by proving, for every $k \in \mathbb{N} - \{0\}$, an analogous statement for the restriction of $m^{\nu,\mu}$ to the image in $T^1M$ of the $\Gamma$-invariant Borel subset of points in $T^1\widetilde{M}$ whose stabiliser in $\Gamma$ has order $k$, and by summation (see Subsection 2.6).

By [Rob1, §1C], there exists hence a weak fundamental domain in $T^1\widetilde{M}$ with $\widetilde{m}^{\nu,\mu}$-negligible boundary, that is an open subset $\mathscr{D}$ in $T^1\widetilde{M}$ such that $\gamma\mathscr{D}\cap\mathscr{D}$ is empty for every $\gamma \in \Gamma - \{\mathrm{id}\}$ and $\bigcup_{\gamma\in\Gamma}\gamma\,\overline{\mathscr{D}} = T^1\widetilde{M}$, and such that $\widetilde{m}^{\nu,\mu}(\partial\mathscr{D}) = 0$.

Let $T = W^{\mathrm{s}}(v)$ be a stable leaf in $T^1\widetilde{M}$. Recall that $v_+$ is not an atom of $\mu_x$, since $m_F$ is finite, hence $(\Gamma, F)$ is of divergence type (see Lemma 5.13). In particular, $U_T = \{w \in T^1\widetilde{M} \;:\; w_- \neq v_+\}$ has full measure for $\widetilde{m}^{\nu,\mu}$. Recall that $\mathbb{1}_A$ is the characteristic function of a subset $A$. Using

- Lemma 10.7 and Equation (140) for the second equality,
- the equality $W^{\mathrm{su}}(w) = W^{\mathrm{su}}(w')$ and the cocycle property of $c_{\widetilde{F}}$ for the third one,
- an elementary argument using $\Gamma$ and $\mathscr{D}$, the fact that $w \in B^{\mathrm{su}}(u, r)$ if and only if $u \in B^{\mathrm{su}}(w, r)$ and Fubini's theorem for positive functions for the fourth one, we have

$$\begin{aligned}
\int_{T^1M} I_r\psi\;\varphi\;dm^{\nu,\mu} &= \int_{T^1\widetilde{M}} \widetilde{I_r\psi}\;\;\widetilde{\varphi}\;\mathbb{1}_{\mathscr{D}}\;d\widetilde{m}^{\nu,\mu}\\
&= \int_{w'\in T}\int_{w\in W^{\mathrm{su}}(w')}\Big(\int_{u\in B^{\mathrm{su}}(w,r)}\widetilde{\psi}(u)\;e^{c_{\widetilde{F}}(u,w)}\;d\mu_{W^{\mathrm{su}}(w)}(u)\Big)\\
&\qquad\qquad \widetilde{\varphi}(w)\;\mathbb{1}_{\mathscr{D}}(w)\;e^{c_{\widetilde{F}}(w,w')}\;d\mu_{W^{\mathrm{su}}(w')}(w)\;d\nu_T(w')\\
&= \int_{w'\in T}\int_{w\in W^{\mathrm{su}}(w')}\int_{u\in W^{\mathrm{su}}(w')}\widetilde{\varphi}(w)\;\widetilde{\psi}(u)\;\mathbb{1}_{\mathscr{D}}(w)\;\mathbb{1}_{B^{\mathrm{su}}(w,r)}(u)\\
&\qquad\qquad e^{c_{\widetilde{F}}(u,w')}\;d\mu_{W^{\mathrm{su}}(w')}(u)\;d\mu_{W^{\mathrm{su}}(w')}(w)\;d\nu_T(w')\\
&= \int_{w'\in T}\int_{u\in W^{\mathrm{su}}(w')}\int_{w\in W^{\mathrm{su}}(w')}\widetilde{\varphi}(w)\;\widetilde{\psi}(u)\;\mathbb{1}_{\mathscr{D}}(u)\;\mathbb{1}_{B^{\mathrm{su}}(u,r)}(w)\\
&\qquad\qquad e^{c_{\widetilde{F}}(u,w')}\;d\mu_{W^{\mathrm{su}}(w')}(w)\;d\mu_{W^{\mathrm{su}}(w')}(u)\;d\nu_T(w')\\
&= \int_{T^1M}\psi\;J_r\varphi\;dm^{\nu,\mu}\;,
\end{aligned}$$

as required. $\square$

From now on, we consider a nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$. For every $r > 0$, applying the above lemma with $\varphi = \frac{\rho}{I_r(\rho)}$, we have

$$\int_{T^1M}\frac{I_r\psi}{I_r\rho}\;\rho\;dm^{\nu,\mu} = \int_{T^1M}\psi\;J_r\big(\frac{\rho}{I_r\rho}\big)\;dm^{\nu,\mu}\;. \tag{144}$$

In the next three steps, we will prove that, up to positive multiplicative constants, the right hand side of this equality is bounded from above by $\int_{T^1M}\psi\;dm^{\nu,\mu}$, and that its left hand side is bounded from below by $\int_{T^1M}\psi\;dm_F$ if $r$ is large enough. Since this is valid for any nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$, this will imply that $m_F$ is absolutely continuous with respect to $m^{\nu,\mu}$.

**Step 3.** In this step, let us prove that the map $J_r(\frac{\rho}{I_r\rho})$ is bounded from above on a subset of $T^1M$ of full measure with respect to $m^{\nu,\mu}$, which implies that the right hand side of Equation (144) is bounded from above by a constant times $\int_{w\in T^1M}\psi\;dm^{\nu,\mu}$.

Recalling that $\widetilde{M}$ has pinched negative curvature, the following result of polynomial growth of horospheres is due to Roblin [Rob1, Prop. 6.3 (a)] (see also [Bowd]). Recall that $\widetilde{\Omega}_c\Gamma$ is the set of $v \in T^1\widetilde{M}$ such that $v_-, v_+ \in \Lambda_c\Gamma$, and that $\Omega_c\Gamma$ is its image in $T^1 M$.

**Lemma 10.10 (Roblin)** *There exists $N \in \mathbb{N} - \{0\}$ such that for all $w \in \widetilde{\Omega}_c\Gamma$ and $r > 0$, there exist $w_1, \dots, w_N \in W^{\rm su}(w)$ such that*

$$B^{\rm su}(w,r) \;\cap\; \overline{\widetilde{\Omega}_c\Gamma} \;\subset\; \bigcup_{i=1}^{N} B^{\rm su}\Big(w_i, \frac{r}{2}\Big)\;.$$

**Lemma 10.11** *For all $r > 0$ and $w \in \Omega_c\Gamma$, we have*

$$J_r\Big(\frac{\rho}{I_r\rho}\Big)(w) \le N\;.$$

**Proof.** Let us fix $r > 0$ and $w \in \widetilde{\Omega}_c\Gamma$, and let $w_1, \dots, w_N \in T^1\widetilde{M}$ be as in Lemma 10.10. For all $i \in \{1, \dots, N\}$ and $u \in B^{\rm su}(w_i, \frac{r}{2})$, since $B^{\rm su}(u,r)$ then contains $B^{\rm su}(w_i, \frac{r}{2})$, we have

$$\begin{aligned}\widetilde{I}_r\widetilde{\rho}\,(u) &= \int_{v\in B^{\rm su}(u,\,r)} \widetilde{\rho}\,(v)\; e^{c_{\widetilde{F}}(v,\,u)}\; d\mu_{W^{\rm su}(u)}(v)\\ &\ge e^{c_{\widetilde{F}}(w,\,u)} \int_{v\in B^{\rm su}(w_i,\,\frac{r}{2})} \widetilde{\rho}\,(v)\; e^{c_{\widetilde{F}}(v,\,w)}\; d\mu_{W^{\rm su}(w)}(v)\;.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{u\in B^{\rm su}(w_i,\,\frac{r}{2})} \frac{\widetilde{\rho}\,(u)}{\widetilde{I}_r\widetilde{\rho}\,(u)}\; d\mu_{W^{\rm su}(w)}(u)\\ \le &\int_{u\in B^{\rm su}(w_i,\,\frac{r}{2})} \frac{\widetilde{\rho}\,(u)\; e^{c_{\widetilde{F}}(u,\,w)}}{\int_{v\in B^{\rm su}(w_i,\,\frac{r}{2})} \widetilde{\rho}\,(v)\; e^{c_{\widetilde{F}}(v,\,w)}\; d\mu_{W^{\rm su}(w)}(v)}\; d\mu_{W^{\rm su}(w)}(u) = 1\;.\end{aligned}$$

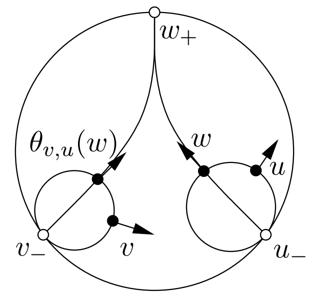
Using Lemma 10.10, since the support of the measure $\mu_{W^{\rm su}(w)}$ is contained in the closed set $\{u \in T^1\widetilde{M} \;:\; u_- = w_-,\; u_+ \in \Lambda\Gamma\}$, which is contained in $\overline{\widetilde{\Omega}_c\Gamma}$ by the density of $\Lambda_c\Gamma$ in $\Lambda\Gamma$, the result follows. □

As required, since $\Omega_c\Gamma$ has full $m^{\nu,\mu}$-measure by Lemma 10.7, this lemma implies that

$$\int_{T^1M} \psi\; J_r\big(\frac{\rho}{I_r\rho}\big)\; dm^{\nu,\,\mu} \le N \int_{T^1M} \psi\; dm^{\nu,\,\mu}\;. \tag{145}$$

**Step 4.** In this most technical step, after giving the useful notation, we prove some continuity properties of the diffusion operators $I_r\psi\,(u)$ on the strong unstable balls, as functions of $u$. For this, we will adapt the technical lemmas of [Rob1, Sect. 1H] to the presence of a possibly unbounded potential, and add some more lemmas using the Hölder regularity of $F$ needed to control the Gibbs cycles.

For all $u, v \in T^1\widetilde{M}$ such that $u_- \neq v_-$, recall (see Subsection 3.9 before Equation (52)) that the map $\theta_{v,u} = \theta^{su}_{v,u} : \{w \in W^{\rm su}(u) \ : \ w_+ \neq v_-\} \to \{w \in W^{\rm su}(v) \ : \ w_+ \neq u_-\}$ is the homeomorphism sending $w$ to the unique element in $W^{\rm su}(v) \cap W^{\rm s}(w)$. Note that $\theta_{v',u'} = \theta_{v,u}$ if $u' \in W^{\rm su}(u)$ and $v' \in W^{\rm su}(v)$. We have $\theta_{u,v} = {\theta_{v,u}}^{-1}$ and, whenever defined, $\theta_{w,u} = \theta_{w,v} \circ \theta_{v,u}$. For every $t \in \mathbb{R}$, we have




$$\phi_t \circ \theta_{v,u} = \theta_{\phi_t v,u} \quad \text{and} \quad \theta_{v,u} \circ \phi_t = \theta_{v,\phi_{-t}u} \ . \qquad (146)$$

We recall the definition of the canonical neighbourhoods (called *dynamical cells*) of the unit tangent vectors, adapted to variable curvature. For all $u \in T^1\widetilde{M}$ and $r_1, r_2, r_3 > 0$,

$$\mathscr{V}_{r_1,r_2,r_3}(u) = \bigcup_{|s|<r_3} \phi_s\Big(\bigcup_{v \in B^{\rm ss}(u,r_1)} \theta_{v,u}(B^{\rm su}(u,r_2))\Big) \ . \qquad (147)$$

The sets $\mathscr{V}_{r_1,r_2,r_3}(u)$ are nondecreasing in $r_1, r_2, r_3$ and form, as $r_1, r_2, r_3$ range over $]0, +\infty[$, a neighbourhood basis of the vector $u$ in $T^1\widetilde{M}$. Furthermore, for every $t \in \mathbb{R}$, by the equations (146), (13) and (14), we have

$$\phi_t\big(\mathscr{V}_{r_1,r_2,r_3}(u)\big) = \mathscr{V}_{e^{-t}r_1,e^t r_2,r_3}(\phi_t u) \ . \qquad (148)$$

For every $u \in T^1\widetilde{M}$, let

$$u_\epsilon = \bigcup_{|s|<\epsilon} \phi_s B^{\rm ss}(u,\epsilon) \ .$$

The sets $u_\epsilon$ are nondecreasing in $\epsilon$ and form, as $\epsilon$ ranges over $]0,1]$, a neighbourhood basis of the vector $u$ in its stable leaf $W^{\rm s}(u)$. Furthermore, for all $\gamma \in \Gamma$ and $t \geq 0$, we have $\gamma(u_\epsilon) = (\gamma u)_\epsilon$ and, by Equation (14),

$$\phi_t(u_\epsilon) \subset (\phi_t u)_\epsilon \quad \text{and} \quad (\phi_{-t}u)_\epsilon \subset \phi_{-t}(u_\epsilon) \ . \qquad (149)$$

The following result contains the elementary properties of the notation introduced above. It says that if $v$ is in a small dynamical cell around $u$, then the change of strong unstable leaf map $\theta_{v,u}$ does not move points too much. The important feature of this lemma is the uniformity of the constants, which will be useful to control the variations of the potential function and of the Gibbs cocycle.

**Lemma 10.12** *For every $\epsilon > 0$, if $r_1, r_2, r_3 > 0$ are small enough, then for every $u \in T^1\widetilde{M}$, we have*

*(1) for all $v \in \mathscr{V}_{r_1,r_2,r_3}(u)$ and $w \in B^{\rm su}(u,2)$, we have*

$$\theta_{v,u}(w) \in w_\epsilon \ ,$$

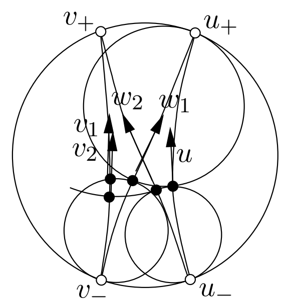
*(2) for all $r \in [\frac{1}{2}, 2]$ and $v \in \mathscr{V}_{r_1,r_2,r_3}(u)$, we have*

$$B^{\rm su}(v, r\,e^{-\epsilon}) \subset \theta_{v,u}(B^{\rm su}(u,r)) \subset B^{\rm su}(v, re^{\epsilon}) \ .$$

**Proof.** (1) Let us recall the statement [Rob1, Lem. 1.13]: for every $\epsilon' > 0$, there exists $r(\epsilon') > 0$ such that for all $u' \in T^1\widetilde{M}$, $v' \in B^{\rm ss}(u', r(\epsilon'))$ and $w' \in B^{\rm su}(u', 3)$ (the statement is given in loc. cit. with the number 2 instead of this number 3, but the proof is the same), we have
$$\theta_{v',u'}(w') \in (w')_{\epsilon'}\ .$$

Now, for every $\epsilon > 0$, let $\epsilon' = \min\{r(\frac{\epsilon}{3}), \frac{\epsilon}{3}\} > 0$. Let $r_1 \in\, ]0, r(\epsilon')]$, $r_2 \in\, ]0, 1]$ and $r_3 \in\, ]0, \frac{\epsilon}{3}]$.

Let $u \in T^1\widetilde{M}$, $w \in B^{\rm su}(u, 2)$ and $v \in \mathscr{V}_{r_1,r_2,r_3}(u)$. By the definition of $\mathscr{V}_{r_1,r_2,r_3}(u)$, there exist $s_1 \in\, ]-r_3, r_3[\,$, $w_1 \in B^{\rm ss}(u, r_1)$ and $w_2 \in B^{\rm su}(u, r_2)$ such that if $v_1 = \theta_{w_1,\,u}(w_2)$, then $v = \phi_{s_1}v_1$. By [Rob1, Lem. 1.13] as recalled above, since $r_2 \le 3$ and $r_1 \le r(\epsilon')$, there exist $s_2 \in\, ]-\epsilon', \epsilon'[$ and $v_2 \in B^{\rm ss}(w_2, \epsilon')$ such that $v_1 = \phi_{s_2}v_2$, so that in particular $v = \phi_{s_1+s_2}v_2$.




By the triangle inequality, we have $w \in B^{\rm su}(u, 2) \subset B^{\rm su}(w_2, 2+r_2) \subset B^{\rm su}(w_2, 3)$. Since $W^{\rm su}(w) = W^{\rm su}(u) = W^{\rm su}(w_2)$ and $v_2 \in B^{\rm ss}(w_2, r(\frac{\epsilon}{3}))$, applying again [Rob1, Lem. 1.13] as recalled above, there exist $s_3 \in\, ]-\frac{\epsilon}{3}, \frac{\epsilon}{3}[$ and $v_3 \in B^{\rm ss}(w, \frac{\epsilon}{3})$ such that $\theta_{v_2,u}(w) = \theta_{v_2,w_2}(w) = \phi_{s_3}v_3$. Hence we have by Equation (146) that
$$\theta_{v,\,u}(w) = \theta_{\phi_{s_1}v_1,\,u}(w) = \theta_{\phi_{s_1+s_2}v_2,\,u}(w) = \phi_{s_1+s_2}\theta_{v_2,\,u}(w) = \phi_{s_1+s_2+s_3}v_3 \in w_\epsilon\ .$$

(2) By [Rob1, Coro. 1.14], for every $\epsilon > 0$, there exist $r'_1, r'_2, r'_3 > 0$ such that for all $u' \in T^1\widetilde{M}$ and $v' \in \mathscr{V}_{r'_1,r'_2,r'_3}(u')$, we have
$$B^{\rm su}(v', e^{-\epsilon}) \subset \theta_{v',u'}(B^{\rm su}(u', 1)) \subset B^{\rm su}(v', e^{\epsilon})\ . \tag{150}$$

Assume that $r_1 \in\, ]0, \frac{r'_1}{2}]$, $r_2 \in\, ]0, \frac{r'_2}{2}]$ and $r_3 \le r'_3$. For every $r \in [\frac{1}{2}, 2]$, let $t = \log r$. For every $v \in \mathscr{V}_{r_1,r_2,r_3}(u)$, let $u' = \phi_{-t}u$ and $v' = \phi_{-t}v$. Then
$$v' \in \phi_{-t}\mathscr{V}_{r_1,\,r_2,\,r_3}(u) = \mathscr{V}_{e^t r_1,\,e^{-t}r_2,\,r_3}(u') \subset \mathscr{V}_{r'_1,\,r'_2,\,r'_3}(u')\ ,$$

and the result follows by applying the map $\phi_t$ to the inclusions (150), while using the equations (13) and (146). □

We will need the following consequence of the Hölder regularity of $\widetilde{F}$ (which is empty if $\widetilde{F} = 0$).

**Lemma 10.13** *For every $\epsilon > 0$, if $\epsilon' > 0$ is small enough, then for all $w \in T^1\widetilde{M}$ and $w' \in w_{\epsilon'}$, we have*
$$|\,C_{F-\delta,\,w_+}(\pi(w), \pi(w'))\,| \le \epsilon\Big(1 + \sup_{\pi^{-1}(B(\pi(w),\,1))} |\widetilde{F}|\Big)\ .$$

**Proof.** First observe that
$$\forall\ v \in u_\epsilon, \quad d(\pi(v), \pi(u)) < 2\epsilon\ . \tag{151}$$

Indeed, if $v = \phi_s w$ where $|s| < \epsilon$ and $w \in B^{\rm ss}(u, \epsilon)$, then $d(\pi(v), \pi(w)) = |s| < \epsilon$ and by Equation (9) (exchanging the stable and unstable foliations), we have

$$d(\pi(u), \pi(w)) \leq d_{W^{\rm ss}(u)}(u, w) < \epsilon \;,$$

so that Equation (151) holds by the triangle inequality.

Now by Equation (151) and by Lemma 3.4, there exist two constants $c_1, c_2$ such that for every $w' \in w_{\epsilon'}$,

$$|\, C_{F-\delta,\, w_+}(\pi(w), \pi(w'))\,| \leq c_1 (2\epsilon')^{c_2} + (2\epsilon')\Big(\max_{\pi^{-1}(B(\pi(w),\, 2\epsilon'))} |\widetilde{F}| + \delta\Big) \;,$$

which proves the result. □

A second consequence of the Hölder regularity of $\widetilde{F}$ that we will need is the following result (again empty if $F = 0$).

**Lemma 10.14** *For all $\epsilon > 0$ and $T \geq 0$, if $r_1$, $r_2$, $r_3 > 0$ are small enough, then for all $u \in T^1\widetilde{M}$ and $v \in \mathscr{V}_{r_1, r_2, r_3}(u)$, we have*

$$\Big|\int_0^T \widetilde{F}(\phi_t u) - \widetilde{F}(\phi_t v)\; dt\,\Big| \leq \epsilon \;.$$

**Proof.** This follows from the uniform continuity of the geodesic flow on compact interval of times and the fact that the sets $\mathscr{V}_{r_1, r_2, r_3}(u)$ for $r_1$, $r_2$, $r_3 > 0$ form a neighbourhood basis of $u \in T^1\widetilde{M}$. □

The following lemma (which is still empty if $F = 0$), giving a uniform continuity property of the cocycle $c_{\widetilde{F}}$, will be useful to prove the continuity properties of $u \mapsto \widetilde{I}_r \widetilde{\varphi}(u)$.

**Lemma 10.15** *For every $\epsilon > 0$, if $\epsilon', r'_1, r'_2, r'_3 > 0$ are small enough, then for all $u \in T^1\widetilde{M}$, $w \in B^{\rm su}(u, 2)$, $v \in \mathscr{V}_{r'_1, r'_2, r'_3}(u)$ and $w' \in w_{\epsilon'} \cap W^{\rm su}(v)$, we have*

$$|\, c_{\widetilde{F}}(w',\, v) - c_{\widetilde{F}}(w,\, u)\,| \leq \epsilon \;.$$

A less precise version of this result follows quite directly from the Hölder-continuity of $c_{\widetilde{F}}$. The proof shows that this result also holds if $w' \in \mathscr{V}_{r'_1, r'_2, r'_3}(u) \cap W^{\rm su}(v)$. But we will use the explicit form given above (with $\frac{\epsilon}{2}$ instead of $\epsilon$!).

**Proof.** Let us fix $\epsilon > 0$. Let $c > 0$ and $\alpha \in \,]0, 1]$ be as in the beginning of the proof of Lemma 10.13. By Equation (9) and Equation (8), there exists a constant $c' > 0$ such that, for all $t \geq 0$, $u' \in T^1\widetilde{M}$ and $u'' \in B^{\rm su}(u', 3)$, we have

$$d(\phi_{-t}u', \phi_{-t}u'') \leq c'\, d_{W^{\rm su}(\phi_{-t}u')}(\phi_{-t}u', \phi_{-t}u'') \leq c'\, e^{-t}\, d_{W^{\rm su}(u')}(u', u'') \leq 3\, c'\, e^{-t} \;.$$

Let $T = \max\{0, -\log(\frac{1}{3c'}\sqrt[\alpha]{\frac{\epsilon\alpha}{4c}}\,)\,\}$. By the Hölder regularity of $\widetilde{F}$, for every $w \in B^{\rm su}(u, 2) \subset B^{\rm su}(u, 3)$, we have

$$\Big|\int_T^{+\infty} \widetilde{F}(\phi_{-t}w) - \widetilde{F}(\phi_{-t}u)\; dt\,\Big| \leq \int_T^{+\infty} c\,(3\,c'\,e^{-t})^\alpha = \frac{c}{\alpha}\,(3\,c'\,e^{-T})^\alpha \leq \frac{\epsilon}{4} \;.$$

If $\epsilon', r'_1, r'_2, r'_3$ are small enough, then for all $v \in \mathscr{V}_{r'_1, r'_2, r'_3}(u)$ and $w' \in w_{\epsilon'} \cap W^{\rm su}(v)$, we have $w' \in B^{\rm su}(v, 3)$ since $w \in B^{\rm su}(u, 2)$, hence similarly

$$\Big| \int_T^{+\infty} \widetilde{F}(\phi_{-t}w') - \widetilde{F}(\phi_{-t}v)\, dt \Big| \leq \frac{\epsilon}{4}\,.$$

Since $w' \in w_{\epsilon'}$, there exist $s' \in \,]-\epsilon', \epsilon'[$ and $w'' \in B^{\rm ss}(w, \epsilon')$ such that

$$w' = \phi_{s'} w'' = \phi_{s'}\, \theta_{w'',\, w}(w)\,,$$

this last equality holding since $w'' \in W^{\rm s}(w)$. Since $w \in B^{\rm su}(w, \epsilon')$, we hence have

$$w' \in \mathscr{V}_{\epsilon',\, \epsilon',\, \epsilon'}(w)\,.$$

Let $r_1, r_2, r_3 > 0$ be the constants associated by Lemma 10.14 to $\frac{\epsilon}{4}$ and $T$. If $\epsilon', r'_1$, $r'_2$, $r'_3 > 0$ are small enough, then $\phi_{-T}v \in \mathscr{V}_{r_1, r_2, r_3}(\phi_{-T}u)$ and $\phi_{-T}w' \in \mathscr{V}_{r_1, r_2, r_3}(\phi_{-T}w)$ by Equation (148). Hence by Lemma 10.14, using a change of variable $t' = T - t$, we have

$$\Big| \int_0^{T} \widetilde{F}(\phi_{-t}u) - \widetilde{F}(\phi_{-t}v)\, dt \Big| = \Big| \int_0^{T} \widetilde{F}(\phi_{t'}(\phi_{-T}u)) - \widetilde{F}(\phi_{t'}(\phi_{-T}v))\, dt' \Big| \leq \frac{\epsilon}{4}$$

and similarly

$$\Big| \int_0^{T} \widetilde{F}(\phi_{-t}w) - \widetilde{F}(\phi_{-t}w')\, dt \Big| \leq \frac{\epsilon}{4}\,.$$

Now, by the definition of the cocycle $c_{\widetilde{F}}$, we have

$$\begin{aligned}
&|\, c_{\widetilde{F}}(w',\, v) - c_{\widetilde{F}}(w,\, u)\,| \\
&= \Big| \int_0^{+\infty} \widetilde{F}(\phi_{-t}w') - \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}w) + \widetilde{F}(\phi_{-t}u)\, dt \Big| \\
&\leq \Big| \int_T^{+\infty} \widetilde{F}(\phi_{-t}w') - \widetilde{F}(\phi_{-t}v)\, dt \Big| + \Big| \int_T^{+\infty} \widetilde{F}(\phi_{-t}w) - \widetilde{F}(\phi_{-t}u)\, dt \Big| \\
&\quad + \Big| \int_0^{T} \widetilde{F}(\phi_{-t}u) - \widetilde{F}(\phi_{-t}v)\, dt \Big| + \Big| \int_0^{T} \widetilde{F}(\phi_{-t}w) - \widetilde{F}(\phi_{-t}w')\, dt \Big| \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon\,.
\end{aligned}$$

This proves the result. □

Let us now introduce the functional version of the notation $u_\epsilon$. For every $\epsilon > 0$ and every nonnegative measurable map $\varphi : T^1M \to \mathbb{R}$, whose lift to $T^1\widetilde{M}$ is denoted by $\widetilde{\varphi}$, let us define

$$\widetilde{\varphi}_{\overline{\epsilon}} : u \mapsto \sup_{v \in u_\epsilon} \widetilde{\varphi}(v) \quad \text{and} \quad \widetilde{\varphi}_{\underline{\epsilon}} : u \mapsto \inf_{v \in u_\epsilon} \widetilde{\varphi}(v)\,.$$

Note that $\widetilde{\varphi}_{\overline{\epsilon'}} \leq \widetilde{\varphi}_{\overline{\epsilon}}$ and $\widetilde{\varphi}_{\underline{\epsilon'}} \geq \widetilde{\varphi}_{\underline{\epsilon}}$ if $\epsilon' \leq \epsilon$. The maps $\widetilde{\varphi}_{\overline{\epsilon}}$ and $\widetilde{\varphi}_{\underline{\epsilon}}$ are $\Gamma$-invariant, hence define nonnegative measurable maps $\varphi_{\overline{\epsilon}}$ and $\varphi_{\underline{\epsilon}}$ from $T^1M$ to $\mathbb{R}$. For every $t \geq 0$, by Equation (149), we have

$$(\varphi \circ \phi_t)_{\overline{\epsilon}} \leq \varphi_{\overline{\epsilon}} \circ \phi_t \quad \text{and} \quad (\varphi \circ \phi_t)_{\underline{\epsilon}} \geq \varphi_{\underline{\epsilon}} \circ \phi_t\,. \tag{152}$$

The following result is the regularity property in $u$ of $\widetilde{I}_r\widetilde{\varphi}\,(u)$ that will be needed in the next steps.

**Proposition 10.16** *Let $K$ be a compact subset of $T^1\widetilde{M}$. For all $\epsilon > 0$ and $r \geq 1$, if $r_1, r_2, r_3 > 0$ are small enough, then for all $u \in \Gamma K$ and $v \in \mathscr{V}_{r_1, r_2, r_3}(u)$ and for every nonnegative measurable map $\varphi : T^1M \to \mathbb{R}$, we have*

$$e^{-\epsilon}\widetilde{I}_{re^{-\epsilon}}(\widetilde{\varphi}_{\underline{\epsilon}})\ (u) \leq \widetilde{I}_r\widetilde{\varphi}\ (v) \leq e^{\epsilon}\widetilde{I}_{re^{\epsilon}}(\widetilde{\varphi}_{\overline{\epsilon}})\ (u)\ . \tag{153}$$

**Proof.** Let us first prove that we may assume without loss of generality that $r = 1$.

Assume that the result is true for $r = 1$. Let $\epsilon > 0$, $r \geq 1$ and $t = -\log r \leq 0$. For all $r_1, r_2, r_3 > 0$, $u \in T^1\widetilde{M}$ and $v \in \mathscr{V}_{e^t r_1,\, e^{-t} r_2,\, r_3}(u)$, we have $\phi_t v \in \mathscr{V}_{r_1, r_2, r_3}(\phi_t u)$ by Equation (148). Hence by the case $r = 1$ of this proposition applied to $\phi_t K$, $\frac{\epsilon}{2}$ and $\widetilde{\varphi} \circ \phi_{-t}$, we have, if $r_1, r_2, r_3$ are small enough,

$$e^{-\frac{\epsilon}{2}}\ \widetilde{I}_{e^{-\frac{\epsilon}{2}}}((\widetilde{\varphi} \circ \phi_{-t})_{\underline{\frac{\epsilon}{2}}})\ (\phi_t u) \leq \widetilde{I}_1(\widetilde{\varphi} \circ \phi_{-t})\ (\phi_t v) \leq e^{\frac{\epsilon}{2}}\ \widetilde{I}_{e^{\frac{\epsilon}{2}}}((\widetilde{\varphi} \circ \phi_{-t})_{\overline{\frac{\epsilon}{2}}})\ (\phi_t u)\ .$$

By the monotonicity properties of the maps $s \mapsto \widetilde{I}_s\widetilde{\varphi}'(u')$, $\epsilon' \mapsto \widetilde{\varphi}'_{\overline{\epsilon'}}(u')$ and $\epsilon' \mapsto \widetilde{\varphi}'_{\underline{\epsilon'}}(u')$ for every $u' \in T^1\widetilde{M}$ when $\varphi' : T^1\widetilde{M} \to \mathbb{R}$ is nonnegative, and by Equation (152) since $-t \geq 0$, we have

$$e^{-\frac{\epsilon}{2}}\ \widetilde{I}_{e^{-\epsilon}}(\widetilde{\varphi}_{\underline{\epsilon}} \circ \phi_{-t})\ (\phi_t u) \leq \widetilde{I}_1(\widetilde{\varphi} \circ \phi_{-t})\ (\phi_t v) \leq e^{\frac{\epsilon}{2}}\ \widetilde{I}_{e^{\epsilon}}(\widetilde{\varphi}_{\overline{\epsilon}} \circ \phi_{-t})\ (\phi_t u)\ .$$

By Lemma 10.8, we have

$$\begin{aligned} &e^{-\frac{\epsilon}{2}}\ e^{\delta t - \int_0^t \widetilde{F}(\phi_s u)\, ds}\ \widetilde{I}_{r\, e^{-\epsilon}}(\widetilde{\varphi}_{\underline{\epsilon}})\ (u) \\ &\qquad\qquad \leq e^{\delta t - \int_0^t \widetilde{F}(\phi_s v)\, ds}\ \widetilde{I}_r(\widetilde{\varphi})\ (v) \leq e^{\frac{\epsilon}{2}}\ e^{\delta t - \int_0^t \widetilde{F}(\phi_s u)\, ds}\ \widetilde{I}_{r\, e^{\epsilon}}(\widetilde{\varphi}_{\overline{\epsilon}})\ (u)\ . \end{aligned}$$

Dividing by $e^{\delta t}$ and applying Lemma 10.14 (with $\frac{\epsilon}{2}$ instead of $\epsilon$), the general case of Proposition 10.16 follows.

Let us now prove Proposition 10.16 with $r = 1$. Let $\epsilon, r_1, r_2, r_3 > 0$, $u \in T^1\widetilde{M}$, $v \in \mathscr{V}_{r_1, r_2, r_3}(u)$ and let $\varphi : T^1M \to \mathbb{R}$ be a nonnegative measurable map. Since $e^{\epsilon}\widetilde{I}_{e^{\epsilon}}(\widetilde{\varphi}_{\overline{\epsilon}})\ (u)$ is nondecreasing in $\epsilon$ and since $e^{-\epsilon}\widetilde{I}_{e^{-\epsilon}}(\widetilde{\varphi}_{\underline{\epsilon}})\ (u)$ is nonincreasing in $\epsilon$, we may assume that $\epsilon \leq \log 2$.

Let $\epsilon' \in\ ]0, \epsilon[$ and $w \in B^{\rm su}(u, e^{\pm\epsilon})$. By Lemma 10.12 (1), if $r_1, r_2, r_3 > 0$ are small enough (depending only on $\epsilon'$), we have

$$\theta_{v,\,u}(w) \in w_{\epsilon'} \subset w_{\epsilon}\ . \tag{154}$$

By Equation (52), using the change of variables $w' = \theta_{v,\,u}(w)$, we have

$$\begin{aligned} &\int_{w' \in\, \theta_{v,\,u}(B^{\rm su}(u,\, e^{\pm\epsilon}))} \widetilde{\varphi}(w')\ e^{c_{\widetilde{F}}(w',\, v)}\ d\mu_{W^{\rm su}(v)}(w') \\ &= \int_{w \in B^{\rm su}(u,\, e^{\pm\epsilon})} \widetilde{\varphi}(\theta_{v,u}(w))\ e^{c_{\widetilde{F}}(\theta_{v,u}(w),\, v) - c_{\widetilde{F}}(w,\, u) - C_{F-\delta,\, w_+}(\pi(\theta_{v,u}(w)),\, \pi(w))} \\ &\qquad\qquad\qquad\qquad\qquad\qquad e^{c_{\widetilde{F}}(w,\, u)}\ d\mu_{W^{\rm su}(u)}(w)\ . \end{aligned} \tag{155}$$

By Lemma 10.12 (2) with respectively $r = e^{-\epsilon}$ and $r = e^{\epsilon}$, we have

$$\theta_{v,\,u}(B^{\rm su}(u, e^{-\epsilon})) \subset B^{\rm su}(v, 1) \subset \theta_{v,\,u}(B^{\rm su}(u, e^{\epsilon}))\ .$$

Note that $\widetilde{F}$ is bounded on $\bigcup_{w\in\Gamma K}\pi^{-1}(B(\pi(w),1))$ if $K$ is compact, since $\widetilde{F}$ is $\Gamma$-invariant and continuous. By Equation (154), by Lemma 10.15 and by Lemma 10.13, the result follows from Equation (155). $\Box$

**Step 5.** In this step, in order to work in the direction of giving a lower bound of the left hand side $\int_{T^1M}\frac{I_r\psi}{I_r\rho}\,\rho\,dm^{\nu,\mu}$ of Equation (144) by $\int_{T^1M}\psi\,dm_F$ if $r$ is large enough, up to a positive multiplicative constant, we give a lower bound of $\frac{I_r\psi}{I_r\rho}(u)$ for some $r$ depending on $u$ in a full measure subset for $\Pi=\rho\,dm^{\nu,\mu}$. This lower bound will be improved in the next step.

We fix a point $x_0$ in $\widetilde{M}$. For every $R>0$, let $K_R$ be the closure of $\pi^{-1}(B(x_0,R))\cap\widetilde{\Omega}_c\Gamma$, which is a compact subset of $T^1\widetilde{M}$. Let $\widetilde{\Omega}_R$ be the $\Gamma$-invariant set of elements $v\in T^1\widetilde{M}$ such that there exists a sequence $(t_i)_{i\in\mathbb{N}}$ in $[0,+\infty[$ converging to $+\infty$ such that $\phi_{-t_i}v\in\Gamma K_R$ for every $i\in\mathbb{N}$. Let $\Omega_R$ be the image of $\widetilde{\Omega}_R$ in $T^1M$. Since the set of elements of $T^1\widetilde{M}$ whose image in $T^1M$ is negatively recurrent under the geodesic flow is the increasing union of the family $(\Omega_R)_{R>0}$, and since $\nu$ gives full measure to the negatively recurrent vectors, we fix $R>0$ large enough so that

$$\Pi(\Omega_R)>1-\frac{1}{16N}\;,$$

where $\Pi=\rho\,dm^{\nu,\mu}$ has been defined at the end of Step 1, and $N$ is given by Lemma 10.10.

**Lemma 10.17** *There exist $c_1,c_2>0$ and a $\Gamma$-invariant Borel map $r:\widetilde{\Omega}_R\to[0,+\infty[$ such that for every nonnegative $\psi'\in\mathscr{C}_c(T^1M;\mathbb{R})$ and for every $u'\in\widetilde{\Omega}_R$,*

$$\widetilde{I}_{r(u')}\widetilde{\psi}'\,(u')\geq c_1\Big(\int_{T^1M}\psi'\,dm_F\Big)\,\widetilde{I}_{r(u')}\widetilde{\rho}\,(u')$$

*and*

$$0<\widetilde{I}_{3\,r(u')}\widetilde{\rho}\,(u')\leq c_2\,\widetilde{I}_{r(u')}\widetilde{\rho}\,(u')\;.$$

**Proof.** Let $c=\frac{1}{8}\inf_{v\in K_R}\;\widetilde{I}_1 1\,(v)$ and $c'=8\sup_{v\in K_R}\;\widetilde{I}_{12}1\,(v)$, which satisfy $c\leq c'$. Since $K_R$ is contained in the support of $\widetilde{\Pi}=\widetilde{\rho}\,\widetilde{m}^{\nu,\mu}$, for every $u\in K_R$, we have $\widetilde{I}_{\frac{1}{2}}1\,(u)>0$ and $\widetilde{I}_{24}1\,(u)<+\infty$. By Proposition 10.16 applied with $K=K_R$, $\epsilon=\log 2$, $r=1$ or $r=12$, and $\widetilde{\varphi}=1$, the maps $v\mapsto\widetilde{I}_1 1\,(v)$ and $v\mapsto\widetilde{I}_{12}1\,(v)$ are hence locally with positive lower bound and locally with finite upper bound on $K_R$. Since $K_R$ is compact, we have $c>0$ and $c'<+\infty$. Define

$$c_1=\frac{c}{c'\int_{T^1M}\rho\,dm_F}\quad\text{and}\quad c_2=\frac{c'}{c}\;.$$

Let $\varphi=\rho$ (as fixed in the end of Step 1, in particular $\int_{T^1M}\rho\,dm_F>0$) or $\varphi=\psi'$ (as in the statement). By uniform continuity, let us fix $\epsilon\in\,]0,\log 2[$ small enough so that, with the notation introduced before Proposition 10.16, we have

$$\int_{T^1M}\varphi_{\underline{\epsilon}}\,dm_F\geq\frac{1}{2}\int_{T^1M}\varphi\,dm_F\quad\text{and}\quad\int_{T^1M}\rho_{\overline{\epsilon}}\,dm_F\leq 2\int_{T^1M}\rho\,dm_F\;.\tag{156}$$

Proposition 10.16, applied with this $\epsilon$, with $r=1$ or $r=3$, and with $K=K_R$, gives us $r_1,r_2,r_3>0$ such that its conclusion holds.

By compactness, let $S$ be a finite subset of $K_R$ such that $K_R$ is covered by the dynamical cells $\mathscr{V}_{r_1,r_2,r_3}(u)$ as $u$ ranges over $S$.

In the following sequence of inequalities, we use respectively

- Proposition 10.16 applied to the function $\widetilde{\varphi}\circ\phi_t$;
- Equation (152) and monotonicity properties;
- Equation (142) (which is the consequence of the mixing property we are using) with $r=1$, since $S$ is finite and since for all $u\in\Gamma S$, we have $u_-\in\Lambda\Gamma$ as $\Gamma S\subset\Gamma K_R\subset\widetilde{\Omega}\Gamma$;
- and Equation (156):

There exists $t_0\geq 0$ such that for every $t\geq t_0$, for all $u\in\Gamma S$ and $v\in\mathscr{V}_{r_1,r_2,r_3}(u)$,

$$\begin{aligned}\widetilde{I}_2(\widetilde{\varphi}\circ\phi_t)\,(v) &\geq e^{-\epsilon}\;\widetilde{I}_{2e^{-\epsilon}}(\widetilde{\varphi}\circ\phi_t)_{\underline{\epsilon}}\,(u)\geq\frac{1}{2}\;\widetilde{I}_1(\widetilde{\varphi}_{\underline{\epsilon}}\circ\phi_t)\,(u)\\ &\geq\frac{\widetilde{I}_1 1\,(u)}{4\,\|m_F\|}\int_{T^1M}\varphi_{\underline{\epsilon}}\;dm_F\geq\frac{c}{\|m_F\|}\int_{T^1M}\varphi\;dm_F\;.\end{aligned}\tag{157}$$

In particular, $\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)\,(v)>0$. Similarly, an upper bound is given by

$$\begin{aligned}\widetilde{I}_6(\widetilde{\rho}\circ\phi_t)\,(v) &\leq e^{\epsilon}\;\widetilde{I}_{6e^{\epsilon}}(\widetilde{\rho}\circ\phi_t)_{\overline{\epsilon}}\,(u)\leq 2\;\widetilde{I}_{12}(\widetilde{\rho}_{\overline{\epsilon}}\circ\phi_t)\,(u)\\ &\leq\frac{4\,\widetilde{I}_{12}1\,(u)}{\|m_F\|}\int_{T^1M}\rho_{\overline{\epsilon}}\;dm_F\leq\frac{c'}{\|m_F\|}\int_{T^1M}\rho\;dm_F\;.\end{aligned}\tag{158}$$

Since $\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)\,(v)\leq\widetilde{I}_6(\widetilde{\rho}\circ\phi_t)\,(v)\leq\frac{c'}{\|m_F\|}\int_{T^1M}\rho\;dm_F$, we have, by Equation (157) with $\varphi=\psi'$,

$$\begin{aligned}\widetilde{I}_2(\widetilde{\psi'}\circ\phi_t)\,(v) &\geq\frac{c}{c'\int_{T^1M}\rho\;dm_F}\int_{T^1M}\psi'\;dm_F\;\;\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)\,(v)\\ &=c_1\int_{T^1M}\psi'\;dm_F\;\;\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)(v)\;.\end{aligned}$$

By Lemma 10.8 and a cancellation argument, we hence have

$$\widetilde{I}_{2e^t}\widetilde{\psi}\,(\phi_t v)\geq c_1\int_{T^1M}\psi\;dm_F\;\;\widetilde{I}_{2e^t}\widetilde{\rho}\,(\phi_t v)\;.$$

Similarly, since $0<\widetilde{I}_6(\widetilde{\rho}\circ\phi_t)\,(v)\leq\frac{c'}{c}\;\widetilde{I}_2(\widetilde{\rho}\circ\phi_t)\,(v)$ by Equation (157) and Equation (158), we have

$$0<\widetilde{I}_{6\,e^t}\widetilde{\rho}\,(\phi_t v)\leq c_2\,\widetilde{I}_{2e^t}\widetilde{\rho}\,(\phi_t v)\;.$$

Since for every $u'\in\widetilde{\Omega}_R$ there exists $t(u')\geq t_0\geq 0$ (which may be taken to be constant on orbits of $\Gamma$ and measurable) such that $u'\in\phi_{t(u')}\Gamma K_R$, this concludes the proof of Lemma 10.17, with $r(u')=2\,e^{t(u')}$ (note that the map $r$ depends only on $R$). □

**Step 6.** In this step, we start to improve the previous one by giving a lower bound of $\frac{I_r\psi}{I_r\rho}(u)$ for $u$ in some good subset $\widetilde{G}_r$ of $T^1\widetilde{M}$, that will be shown in the next step to be large enough for our purpose, if $r$ is large enough.

For every $r>0$, we first define the good set $\widetilde{G}_r$ (depending on the nonnegative $\psi\in\mathscr{C}_c(T^1M;\mathbb{R})$ that has been fixed at the end of Step 2) on which the lower bound will take place.

For every $s' > 0$, let $\widetilde{\mathscr{E}}_{s'}$ be the $\Gamma$-invariant Borel set of $u \in \widetilde{\Omega}_R$ such that $r(u) \leq s'$, where the map $r$ is given by Lemma 10.17 in the previous step. The set $\widetilde{\Omega}_R$ is the increasing union of the family $(\widetilde{\mathscr{E}}_{s'})_{s'>0}$. Note that $\Pi(\Omega_R) > 1 - \frac{1}{16N}$ by the choice of $R$ in the previous step. Hence there exist $\sigma > 0$ and a $\Gamma$-invariant closed subset $\widetilde{\mathscr{E}}$ of $\widetilde{\mathscr{E}}_\sigma$ whose image $\mathscr{E}$ in $T^1M$ satisfies

$$\Pi(\mathscr{E}) > 1 - \frac{1}{8N} \; .$$

For every $r > 0$, let $\widetilde{\Delta}_r$ be the $\Gamma$-invariant Borel set of $u \in \widetilde{\Omega}_R$ such that

$$\widetilde{I}_r(\widetilde{\rho}\,\mathbb{1}_{\widetilde{\mathscr{E}}})\,(u) \leq \frac{1}{2}\,\widetilde{I}_r\widetilde{\rho}\,(u) \; . \tag{159}$$

For every $r > 0$, let $\widetilde{Z}_r$ be the $\Gamma$-invariant Borel set of $u \in \widetilde{\Omega}_R$ such that

$$\widetilde{I}_{r+\sigma}\widetilde{\psi}\,(u) > 2\,\widetilde{I}_r\widetilde{\psi}\,(u) \; . \tag{160}$$

Define

$$\widetilde{G}_r = \widetilde{\Omega}_c\Gamma - (\widetilde{\Delta}_r \cup \widetilde{Z}_r) \; ,$$

which is a $\Gamma$-invariant Borel subset of $T^1\widetilde{M}$ (depending on $\psi$). Let $\Delta_r, Z_r, G_r$ be the images of $\widetilde{\Delta}_r, \widetilde{Z}_r, \widetilde{G}_r$ in $T^1M$.

The next result now gives the important property of the set $\widetilde{G}_r$ introduced above, saying that it is indeed a good set concerning the problem of finding a lower bound of $\frac{\widetilde{I}_r\widetilde{\psi}}{\widetilde{I}_r\widetilde{\rho}}(u)$ by a constant times $\int_{T^1M}\psi\,dm_F$.

**Lemma 10.18** *There exists $c_3 > 0$ such that for every nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$, for all $r > 0$ and $u \in \widetilde{G}_r$, we have*

$$\widetilde{I}_r\widetilde{\psi}\,(u) \geq c_3 \int_{T^1M} \psi\,dm_F \;\; \widetilde{I}_r\widetilde{\rho}\,(u)$$

**Proof.** Let $r > 0$ and $u \in \widetilde{G}_r$.

As pioneered in [Rud] to study the ergodic theory of the strong unstable foliation, and as in [Rob1, page 89], we start the proof by a Vitali covering argument. Consider the measure $\mathbb{M}$ on $W^{\rm su}(u)$ defined by

$$d\mathbb{M}(w) = \widetilde{\rho}(w)\,e^{c_{\widetilde{F}}(w,\,u)}\,d\mu_{W^{\rm su}(u)}(w) \; .$$

For all $v \in B^{\rm su}(u,r) \cap \widetilde{\mathscr{E}}$, we have $v \in \widetilde{\Omega}_R$ and, by the second property of Lemma 10.17,

$$\mathbb{M}\big(B^{\rm su}(v, 2\,r(v))\big) \leq c_2\,\mathbb{M}\big(B^{\rm su}(v, r(v))\big) \; .$$

Let us construct a finite or countable sequence of pairwise disjoint balls $(B_i)_{1\leq i<i_*}$ where $i_* \in (\mathbb{N} - \{0,1\}) \cup \{+\infty\}$ amongst the open balls $B^{\rm su}(v, r(v))$ with $v \in B^{\rm su}(u,r) \cap \widetilde{\mathscr{E}}$, as follows. For the first one, choose a ball $B_1 = B^{\rm su}(v_1, r(v_1))$ with maximal radius $r(v_1)$. Assume that the first $n$ balls $(B_i)_{1\leq i\leq n}$ are constructed. Let $i_* = n+1$ if for every $v \in B^{\rm su}(u,r) \cap \widetilde{\mathscr{E}}$, the ball $B^{\rm su}(v, r(v))$ meets some ball $B_i$ with $1 \leq i \leq n$. Otherwise, choose a ball $B_{n+1} = B^{\rm su}(v_{n+1}, r(v_{n+1}))$ disjoint from all balls $B_i$ with $1 \leq i \leq n$, with maximal radius $r(v_{n+1})$ amongst them. If the process does not stop, let $i_* = +\infty$. Note that

$(r(v_i))_{1\leq i<i_*}$ is nonincreasing by the maximality property at each step. Since $B^{\rm su}(u,r)\cap\widetilde{\mathscr{E}}$ is relatively compact, for every $\rho>0$, there are only finitely many pairwise disjoint balls centred at a point $v\in B^{\rm su}(u,r)\cap\widetilde{\mathscr{E}}$ of radius $r(v)$ at least $\rho$. Hence if $i_*=+\infty$, then $r(v_i)$ tends to 0 as $i\to+\infty$.

Let us prove that $B^{\rm su}(u,r)\cap\widetilde{\mathscr{E}}$ is contained in $\bigcup_{1\leq i<i_*}B^{\rm su}(v_i,2\,r(v_i))$. Indeed, let $v\in B^{\rm su}(u,r)\cap\widetilde{\mathscr{E}}$ and let $k=\max\{i\in[1,i_*[\cap\mathbb{N}\;:\;r(v_i)\geq r(v)\}$, which exists by the maximality of $r(v_1)$ and since $\lim_{i\to+\infty}r(v_i)=0$ if $i_*=+\infty$. Then either $k+1=i_*$ or $k+1<i_*$. In the first case, by construction, the open ball $B^{\rm su}(v,r(v))$ meets $B_i$ for some $i\in\{1,\dots,k\}$, say at a point $v'$. Hence, by the triangle inequality,

$$d(v,v_i)\leq d(v,v')+d(v',v_i)<r(v)+r(v_i)\leq 2\,r(v_i)\;.$$

Therefore $v\in B^{\rm su}(v_i,2\,r(v_i))$. If we have $k+1<i_*$, then $r(v)>r(v_{k+1})$ by the maximality of $k$. Hence, by the maximality property at the step $k+1$, the ball $B^{\rm su}(v,r(v))$ meets $B_i$ for some $i\in\{1,\dots,k\}$, and hence as above $v\in B^{\rm su}(v_i,2\,r(v_i))$.

This family of balls hence satisfies

$$\begin{aligned}\sum_{1\leq i<i_*}\mathbb{M}(B_i)&\geq\frac{1}{c_2}\sum_{1\leq i<i_*}\mathbb{M}\big(B^{\rm su}(v_i,2\,r(v_i))\big)\geq\frac{1}{c_2}\,\mathbb{M}\Big(\bigcup_{1\leq i<i_*}B^{\rm su}(v_i,2\,r(v_i))\Big)\\&\geq\frac{1}{c_2}\,\mathbb{M}(B^{\rm su}(u,r)\cap\widetilde{\mathscr{E}})\;.\end{aligned}\tag{161}$$

Now, using respectively

- the fact that $u$ does not belong to $\widetilde{Z}_r$ (see Equation (160)),
- the fact that the balls $B_i$ for $1\leq i<i_*$ are pairwise disjoint and contained in $B^{\rm su}(u,r+\sigma)$ (since $v_i\in\widetilde{\mathscr{E}}$ implies that $r(v_i)\leq\sigma$ by the definition of $\widetilde{\mathscr{E}}$),
- the equality $W^{\rm su}(v_i)=W^{\rm su}(u)$ and the cocycle property of $c_{\widetilde{F}}$,
- Lemma 10.17 in Step 5 since $\widetilde{\mathscr{E}}\subset\widetilde{\Omega}_R$,
- the cocycle property of $c_{\widetilde{F}}$,
- the conclusion (161) of the above Vitali argument,
- the fact that $u$ does not belong to $\widetilde{\Delta}_r$ (see Equation (159)),

we have

$$\begin{aligned}\widetilde{I}_r\widetilde{\psi}\,(u)&\geq\frac12\,\widetilde{I}_{r+\sigma}\widetilde{\psi}\,(u)\geq\frac12\sum_{1\leq i<i_*}\int_{w\in B_i}\widetilde{\psi}(w)\;e^{c_{\widetilde{F}}(w,\,u)}\;d\mu_{W^{\rm su}(u)}(w)\\&=\frac12\sum_{1\leq i<i_*}\widetilde{I}_{r(v_i)}\widetilde{\psi}\,(v_i)\;e^{c_{\widetilde{F}}(v_i,u)}\\&\geq\frac{c_1}{2}\int_{T^1M}\psi\;dm_F\sum_{1\leq i<i_*}\widetilde{I}_{r(v_i)}\widetilde{\rho}\,(v_i)\;e^{c_{\widetilde{F}}(v_i,u)}\\&=\frac{c_1}{2}\int_{T^1M}\psi\;dm_F\sum_{1\leq i<i_*}\int_{w\in B_i}\widetilde{\rho}(w)\;e^{c_{\widetilde{F}}(w,\,u)}\;d\mu_{W^{\rm su}(u)}(w)\\&\geq\frac{c_1}{2\,c_2}\int_{T^1M}\psi\;dm_F\int_{w\in B^{\rm su}(u,\,r)\cap\widetilde{\mathscr{E}}}\widetilde{\rho}(w)\;e^{c_{\widetilde{F}}(w,\,u)}\;d\mu_{W^{\rm su}(u)}(w)\\&\geq\frac{c_1}{4\,c_2}\int_{T^1M}\psi\;dm_F\;\;\widetilde{I}_r\widetilde{\rho}\,(u)\;.\end{aligned}$$

This proves Lemma 10.18 with $c_3 = \frac{c_1}{4\,c_2}$. $\Box$

**Step 7.** In this penultimate step, we conclude the search of a lower bound of $\frac{\widetilde{I}_r\widetilde{\psi}}{\widetilde{I}_r\widetilde{\rho}}(u)$ for $r$ large enough, on a large enough subset of $u \in T^1\widetilde{M}$, in order to obtain a lower bound of the left hand side $\int_{T^1M} \frac{I_r\psi}{I_r\rho}\ \rho\ dm^{\nu,\,\mu}$ of Equation (144) by a positive constant times $\int_{T^1M} \psi\, dm_F$.

We start by proving that the good set $G_r$ has large measure for some $r$ large enough, or equivalently that the bad sets $Z_r$ and $\Delta_r$ have small measures.

**Lemma 10.19** *For every $r \geq 1$, we have $\Pi(\Delta_r) \leq \frac{1}{4}$.*

**Proof.** By the crucial adjointness property of the operators $I_r$ and $J_r$ (see Lemma 10.9 in Step 2), we have

$$\int_{T^1M} I_r(\rho\ \mathbb{1}_{\,^c\mathscr{E}})\ \frac{\rho}{I_r\rho}\ \mathbb{1}_{\,\Delta_r}\ dm^{\nu,\,\mu} = \int_{T^1M} \rho\ \mathbb{1}_{\,^c\mathscr{E}}\ J_r\Big(\frac{\rho}{I_r\rho}\ \mathbb{1}_{\,\Delta_r}\Big)\ dm^{\nu,\,\mu}\ . \tag{162}$$

Since $\mathbb{1}_{\,^c\mathscr{E}} = 1 - \mathbb{1}_{\mathscr{E}}$, the definition of $\Delta_r$ (see Equation (159)) implies that for every $u \in \Delta_r$, we have

$$\frac{I_r(\rho\ \mathbb{1}_{\,^c\mathscr{E}})}{I_r\rho}(u) = 1 - \frac{I_r(\rho\ \mathbb{1}_{\mathscr{E}})}{I_r\rho}(u) \geq \frac{1}{2}\ .$$

Hence the left hand side of Equation (162) is bounded from below by $\frac{1}{2}\ \int_{T^1M} \rho\ \mathbb{1}_{\,\Delta_r}\ dm^{\nu,\,\mu} = \frac{1}{2}\ \Pi(\Delta_r)$. Since $\mathbb{1}_{\,\Delta_r} \leq 1$, since $J_r$ is a positive operator, and by Lemma 10.11 in Step 3, the right hand side of Equation (162) is bounded from above by $N\,\Pi(^c\mathscr{E})$, which is at most $\frac{1}{8}$ by the definition of $\mathscr{E}$ in the beginning of Step 6. This proves the result. $\Box$

Before proving that the bad set $Z_r$ also has small measure for some $r$ large enough, we start by showing that the map $r \mapsto \widetilde{I}_r\widetilde{\varphi}\,(u)$, defined by integrating nonnegative Borel maps $\widetilde{\varphi}$ (for the strong unstable measure weighted by the cocycle $c_{\widetilde{F}}$) on strong unstable balls, has subexponential growth in the radius of the ball. When $\widetilde{F}$ is bounded, the proof below even proves its polynomial growth, recovering with a different presentation the result of Roblin when $\widetilde{F} = 0$.

**Lemma 10.20** *For every $u \in T^1M$ and for every bounded nonnegative measurable map $\varphi : T^1M \to \mathbb{R}$, there exists $c_u > 0$ such that, as $r$ tends to $+\infty$,*

$$I_r\varphi\,(u) = \mathrm{O}(e^{c_u\,(\log r)^2})\ .$$

**Proof.** We may assume that $\varphi$ is constant with value 1. We fix $u \in T^1\widetilde{M}$ and we allow $v$ to vary in $W^{\rm su}(u)$. Define $x = \pi(u)$ and $y = \pi(v)$. By the definition of $\mu_{W^{\rm su}(u)}$ (see Equation (48)), we have

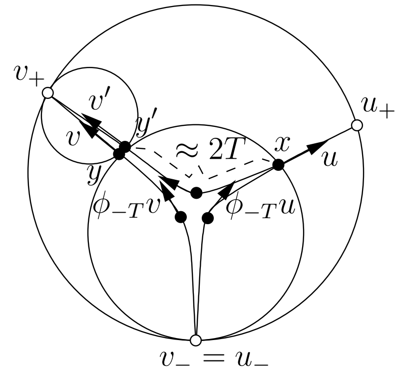
$$e^{c_{\widetilde{F}}(v,\,u)}\ d\mu_{W^{\rm su}(u)}(v) = e^{c_{\widetilde{F}}(v,\,u)} e^{C_{F-\delta,\,v_+}(x,\,y)}\ d\mu_x(v_+)\ .$$

Since the Patterson measure $\mu_x$ is finite, we hence only have to prove that, as $v$ goes to infinity in $W^{\rm su}(u)$, we have

$$c_{\widetilde{F}}(v,u) + C_{F-\delta,\,v_+}(x,y) = \mathrm{O}\left((\log d_{W^{\rm su}(u)}(u,v))^2\right)\ .$$

By the convexity of horoballs, there exists $v' \in W^{\rm ss}(v)$ which is tangent to the geodesic ray from $x$ to $v_+$. Let $y' = \pi(v')$. By the properties of geodesic triangles in CAT$(-1)$ spaces (or the techniques of approximation by trees), there exists a universal constant $c'' \geq 0$ such that if $T = \log d_{W^{\rm su}(u)}(u, v)$ is large enough, then

(i) $d_{W^{\rm ss}(\phi_{-T}v)}(\phi_{-T}v, \phi_{-T}v') \leq c''$,
(ii) $d_{W^{\rm su}(\phi_{-T}v)}(\phi_{-T}v, \phi_{-T}u) \leq c''$,
(iii) $|\, d(y', x) - 2T \,| \leq c''$.




By the definition of the cocycles, we have

$$\begin{aligned}
c_{\widetilde{F}}(v,u) + C_{F-\delta,\, v_+}(x,y) &= \int_0^{+\infty} \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}u)\; dt + \int_0^{+\infty} \widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t v')\; dt \\
&\quad - \int_{-d(y',\,x)}^{0} \big(\widetilde{F}(\phi_t v') - \delta\big)\; dt \\
&= \int_T^{+\infty} \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}u)\; dt + \int_{-T}^{+\infty} \widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t v')\; dt \\
&\quad + \delta\, d(y',x) - \int_{-d(y',\,x)}^{-T} \widetilde{F}(\phi_t v')\; dt - \int_0^T \widetilde{F}(\phi_{-t}u)\; dt\,.
\end{aligned}$$

Let $c \geq 0$ and $\alpha \in \,]0,1]$ be the Hölder constants of $\widetilde{F}$, as in the beginning of the proof of Lemma 10.13. By Equation (9) and Equation (8), there exists $c' > 0$ such that

$$d(\phi_{-t}v, \phi_{-t}u) \leq c'\, d_{W^{\rm su}(\phi_{-t}v)}(\phi_{-t}v, \phi_{-t}u) = c' e^{T-t} d_{W^{\rm su}(\phi_{-T}v)}(\phi_{-T}v, \phi_{-T}u)\,.$$

By (ii), we hence have

$$\int_T^{+\infty} \widetilde{F}(\phi_{-t}v) - \widetilde{F}(\phi_{-t}u)\; dt \leq \frac{c}{\alpha}(c'c'')^\alpha\,.$$

Similarly by (i), we have

$$\int_{-T}^{+\infty} \widetilde{F}(\phi_t v) - \widetilde{F}(\phi_t v')\; dt \leq \frac{c}{\alpha}(c'c'')^\alpha\,.$$

As mentioned in the remark at the end of Subsection 2.1, by the Hölder-continuity of $\widetilde{F}$, we have

$$|\, \widetilde{F}(w) - \widetilde{F}(w') \,| \leq 3\, c\, d(w,w') + c$$

for all $w, w' \in T^1\widetilde{M}$. Hence by (iii) and since $d(\phi_t w, w) = |t|$ for every $w \in T^1\widetilde{M}$, we have

$$\begin{aligned}
&|\, c_{\widetilde{F}}(v,u) + C_{F-\delta,\, v_+}(x,y) \,| \\
\leq\;& 2\frac{c}{\alpha}(c'c'')^\alpha + \delta\,(2T + c'') + 2\,(T + c'')\Big(\max_{\pi^{-1}(\{x\})} |\widetilde{F}| + 3c\,(T + c'') + c\Big)\,.
\end{aligned}$$

The result follows. $\square$

Let us now prove that the bad set $Z_r$ also has small measure for some $r$ large enough.

**Lemma 10.21** *For every $r_0 \geq 1$, there exists $r \geq r_0$ such that $\Pi(Z_r) \leq \frac{1}{4}$.*

**Proof.** If the interior of the support of $\psi$ does not meet $\Omega_c\Gamma$, then $Z_r$ is empty, and the result is true. We hence assume that the interior of the support of $\psi$ meets $\Omega_c\Gamma$.

Let us fix $u \in \widetilde{\Omega}_c\Gamma$. For every $t \geq 0$, let $n(t)$ be the number of integers $k$ in $[0, \lfloor \frac{t}{\sigma} \rfloor]$ such that there exists $r \in [k\sigma, (k+1)\sigma[$ with $u \in \widetilde{Z}_r$. Hence the period of time spent by $u$ in the sets $\widetilde{Z}_r$ as $r$ ranges from 0 to $t$, that is $\int_0^t \mathbb{1}_{\widetilde{Z}_r}(u)\, dr$, is at most $n(t)\,\sigma$. To prove Lemma 10.21, we will first prove that $n(t)$ grows slowly in $t$.

Recall without proof the following result of [Dal2] (generalising, to our (almost no) assumptions on $\widetilde{M}$ and $\Gamma$, results of Hedlund and others), which is valid since the geodesic flow is mixing, hence topologically mixing on $\Omega\Gamma$ by Babillot's Theorem 8.1.

**Proposition 10.22 (Dal'Bo)** *For every $v \in \widetilde{\Omega}_c\Gamma$, the intersection with $\widetilde{\Omega}_c\Gamma$ of the orbit $\Gamma W^{\rm su}(v)$ under $\Gamma$ of the strong unstable leaf of $v$ is dense in $\widetilde{\Omega}_c\Gamma$.*

Hence $W^{\rm su}(u)$ meets the (nonempty) intersection with $\widetilde{\Omega}_c\Gamma$ of the interior of the support of $\widetilde{\psi}$. In particular, there exists $t_0 \geq 1$ such that $\widetilde{I}_{t_0}\widetilde{\psi}\,(u) > 0$. As the hinted proof in [Rob1, page 91] is slightly incorrect, let us prove the following claim.

**Claim.** There exists $c > 0$ (depending on $u$) such that for every $t \geq t_0$,

$$\widetilde{I}_{t+2\sigma}\widetilde{\psi}\,(u) \geq c\, 2^{\frac{n(t)}{2}}\ .$$

**Proof.** To simplify the notation, let $f$ be the nondecreasing map $r \mapsto \widetilde{I}_r\widetilde{\psi}\,(u)$, and for every $t \geq t_0$, let $n = n(t)$. Let $0 \leq k_1 < k_2 < \cdots < k_n \leq \lfloor \frac{t}{\sigma} \rfloor$ and $r_i \in [k_i\sigma, (k_i+1)\sigma[$ such that $u \in \widetilde{Z}_{r_i}$. Note that $r_{i+2} \geq r_i + \sigma$ (but $r_{i+1} - r_i$ could be strictly less than $\sigma$). Let $p = \lfloor \frac{t_0}{\sigma} \rfloor + 2$ and $q = \lfloor \frac{n-p}{2} \rfloor$. Note that $n - 2q - 1 \geq p - 1 \geq 0$ and

$$r_{n-2q} \geq k_{n-2q}\,\sigma \geq (n-2q-1)\,\sigma \geq (p-1)\,\sigma \geq t_0\ .$$

By applying $q+1$ times the definition of the sets $\widetilde{Z}_r$ (see Equation (160)) and by monotonicity, we have

$$\begin{aligned} f(t+2\sigma) &\geq f(r_n+\sigma) \geq 2\,f(r_n) \geq 2\,f(r_{n-2}+\sigma) \geq \cdots \geq 2^q f(r_{n-2q}+\sigma) \\ &\geq 2^{q+1} f(r_{n-2q}) \geq 2^{\frac{n-p}{2}}\, f(t_0)\ . \end{aligned}$$

Hence the result follows with $c = 2^{-\frac{p}{2}}\, f(t_0) > 0$. □

This claim and Lemma 10.20 imply the following slow growth property of $n$: we have $n(t) = \mathrm{O}((\log t)^2)$ as $t \to +\infty$. Therefore, for every $u \in \widetilde{\Omega}_c\Gamma$, we have, by the definition of the map $t \mapsto n(t)$,

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \mathbb{1}_{\widetilde{Z}_r}(u)\, dr = 0\ .$$

Since $\Omega_c\Gamma$ has full measure with respect to the probability measure $\Pi$, and by Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \Pi(Z_r)\, dr = 0\ .$$

Lemma 10.21 now follows immediately. $\square$

Let us now conclude the proof that the left hand side $\int_{T^1M} \frac{I_r\psi}{I_r\rho}\,\rho\,dm^{\nu,\mu}$ of Equation (144) is at least a positive constant times $\int_{T^1M}\psi\,dm_F$.

By Lemma 10.21 and Lemma 10.19, since $G_r = \Omega_c\Gamma - (\Delta_r \cup Z_r)$ and since $\Omega_c\Gamma$ has full measure with respect to $\Pi = \rho\, m^{\nu,\mu}$ by Lemma 10.7, there exists $r \geq 1$ large enough such that

$$\Pi(G_r) \geq 1 - \Pi(Z_r) - \Pi(\Delta_r) \geq \frac{1}{2}\,.$$

Hence, using Lemma 10.18, we have

$$\begin{aligned}\int_{T^1M} \frac{I_r\psi}{I_r\rho}\,\rho\,dm^{\nu,\mu} &= \int_{\Omega_c\Gamma} \frac{I_r\psi}{I_r\rho}\,d\Pi \geq \int_{G_r} \frac{I_r\psi}{I_r\rho}\,d\Pi \\ &\geq c_3\,\Pi(G_r)\int_{T^1M}\psi\,dm_F \geq \frac{c_3}{2}\int_{T^1M}\psi\,dm_F\,, \end{aligned}\tag{163}$$

which is the lower bound we were looking for.

**Step 8.** This step, which is an easy ergodicity argument, is the final one to conclude the proof of Theorem 10.4: We prove now that $\nu$ (as given in Step 1) is proportional to $\mu^\iota$, which is ergodic.

By Equation (144), Equation (163) and Equation (145), with $c_4 = \frac{2\,N}{c_3}$, we have, for every nonnegative $\psi \in \mathscr{C}_c(T^1M;\mathbb{R})$,

$$\int_{T^1M}\psi\,dm_F \leq c_4\int_{T^1M}\psi\,dm^{\nu,\mu}\,.$$

Hence $m_F \leq c_4\, m^{\nu,\mu}$, and $m_F = m^{\mu^\iota,\mu}$ is absolutely continuous with respect to $m^{\nu,\mu}$. By disintegration (see Lemma 10.7 in Step 1), for every transversal $T$ to the strong unstable foliation, the measure $\mu^\iota_T$ is hence absolutely continuous with respect to $\nu_T$. Note that this is true for any $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\nu$ which gives full support to the negatively recurrent set.

This implies in particular that the $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure $\mu^\iota = (\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ is ergodic. Indeed, let $A$ be a $\Gamma$-invariant $\widetilde{\mathscr{W}}^{\rm su}$-saturated (that is, which is a union of strong unstable leaves) subset of $T^1\widetilde{M}$, whose intersection with any transversal is measurable, and assume that $\mu^\iota_T(A\cap T) > 0$ for at least one transversal $T$. Then $(\mathbb{1}_{A\cap T}\,\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ is also a $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure for the strong unstable foliation. By the above absolute continuity claim applied to this family, the measure $\mu^\iota_T$ is absolutely continuous with respect to $\mathbb{1}_{A\cap T}\,\mu^\iota_T$, hence $\mu^\iota_T({}^cA\cap T) = 0$ for all transversals $T$.

Now, to prove Step 8, we may assume without loss of generality that $\nu$ is ergodic. Let $T$ be a transversal to the strong unstable foliation, such that $\nu_T(T) > 0$ (for the other transversals $T'$, we have by absolute continuity $\mu^\iota_{T'} = \nu_{T'} = 0$). Let $\mathscr{G}_T$ be the pseudo-group of holonomy maps between transversals to $\mathscr{W}^{\rm su}$ contained in $T$. Since $\mu^\iota_T$ is absolutely continuous with respect to $\nu_T$, there exists a measurable map $f_T = \frac{d\mu^\iota_T}{d\nu_T} : T \to [0,+\infty[$ (well defined $\nu_T$-almost everywhere) such that $\mu^\iota_T = f_T\,\nu_T$. Since $\mu^\iota$ and $\nu$ are quasi-invariant under holonomy with respect to the same cocycle $c_{\widetilde{F}}$, the map $f_T$ is $\nu_T$-almost everywhere invariant under $\mathscr{G}_T$. By ergodicity, the map $f_T$ is hence $\nu_T$-almost everywhere equal to a constant $c_T$. By the density property of strong unstable leaves recalled in Proposition 10.22, and since $T \cap \widetilde{\Omega}_c\Gamma$ has full measure with respect to both $\mu^\iota_T$ and $\nu_T$, the constant $c_T$ is independent of $T$. Hence $\mu^\iota$ and $\nu$ are proportional, which concludes the proof of Theorem 10.4. $\square$

To conclude Chapter 10, we obtain a complete classification of the $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measures for the strong unstable foliation of $T^1\widetilde{M}$, under the assumption that $\Gamma$ is geometrically finite. We refer to the beginning of Subsection 8.2 for a definition of a geometrically finite group of isometries of $\widetilde{M}$.

The additional measures are the following ones. Let $\widetilde{W}$ be a strong unstable leaf in $T^1\widetilde{M}$ such that the family $(\gamma\widetilde{W})_{\gamma\in\Gamma}$ is locally finite (that is, for every compact subset $K$ of $T^1\widetilde{M}$, the set of $\gamma \in \Gamma$ such that $\gamma\widetilde{W}$ meets $K$ is finite), or, equivalently, such that the image of $\widetilde{W}$ in $T^1M$ is a closed subset of $T^1M$. Fix $v \in \widetilde{W}$ and let $\Gamma_{\widetilde{W}}$ be the stabiliser of $\widetilde{W}$ in $\Gamma$. We denote by $\mathscr{D}_z$ the unit Dirac mass at a point $z \in T^1\widetilde{M}$. For every transversal $T$ to $\widetilde{\mathscr{W}}^{\rm su}$, the measure

$$\nu_T^{\widetilde{W}} = \sum_{\gamma\in\Gamma/\Gamma_{\widetilde{W}},\; w\in T\cap\gamma\widetilde{W}} e^{c_{\widetilde{F}}(w,\,v)}\;\mathscr{D}_w$$

is locally finite, and $(\nu_T^{\widetilde{W}})_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ is a nonzero $\Gamma$-equivariant family of locally finite measures on transversals to the strong unstable foliation, which is stable by restrictions, and satisfies the property (iii) of $c_{\widetilde{F}}$-quasi-invariant transverse measures. Note that replacing $v$ by another element of $\widetilde{W}$ changes this family only by a multiplicative constant. Note that, since $\widetilde{W} = W^{\rm su}(v)$, the (discrete, countable) support of $\nu_T^{\widetilde{W}}$ is the set $\{w \in T \;:\; w_- \in \Gamma v_-\}$. When $\Gamma$ is torsion free, if $T$ is a transversal to the strong unstable foliation $\mathscr{W}^{\rm su}$ on $T^1M$ and if $W$ is a leaf of $\mathscr{W}^{\rm su}$ which is a closed subset of $T^1M$, the associated measure on $T$ is

$$\sum_{w\in T\cap W} e^{c_F(w,\,v)}\;\mathscr{D}_w$$

where $c_F(w,v) = \int_0^{+\infty} F(\phi_{-t}w) - F(\phi_{-t}v)\;dt$ (see Equation (89)).

**Corollary 10.23** *Under the assumptions of Theorem 10.4, assume furthermore that $\Gamma$ is geometrically finite. Then, up to a multiplicative constant, the only ergodic $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measures for the strong unstable foliation on $T^1\widetilde{M}$ are*

- *the family $(\mu^\iota_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ or*
- *the families $(\nu_T^{\widetilde{W}})_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ where $\widetilde{W}$ is a strong unstable leaf of $T^1\widetilde{M}$ whose image in $T^1M$ is a closed subset.*

**Proof.** Since $\Gamma$ is geometrically finite, a point $\xi$ in $\partial_\infty\widetilde{M} - \Lambda_c\Gamma$ is either a point in $\partial_\infty\widetilde{M} - \Lambda\Gamma$ (in which case its stabiliser in $\Gamma$ is finite) or a (bounded) parabolic limit point (in which case its stabiliser in $\Gamma$ is infinite). In both cases, the image in $M$ of any horosphere centred at $\xi$ is a closed subset: This follows from the fact that there exists, in both cases, a horoball centred at $\xi$ which is precisely invariant under the stabiliser of $\xi$ in $\Gamma$ (see the beginning of Subsection 8.2 for the second case). In particular, for every $w \in T^1\widetilde{M}$ such that $w_- \notin \Lambda_c\Gamma$, the image in $T^1M$ of the strong unstable leaf $W^{\rm su}(w)$ is a closed subset (see also [Dal2] for further information on the topology of the strong unstable manifolds in $M$).

Let $\nu = (\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ be an ergodic $\Gamma$-equivariant $c_{\widetilde{F}}$-quasi-invariant transverse measure for $\widetilde{\mathscr{W}}^{\rm su}$.

Assume first that there exists a transversal $T$ to $\widetilde{\mathscr{W}}^{\rm su}$ such that the measurable set $A = \{w \in T \;:\; w_- \notin \Lambda_c\Gamma\}$ has positive measure with respect to $\nu_T$. Then by ergodicity, the set $A$, which is saturated by (the intersections with $T$ of) the leaves of $\widetilde{\mathscr{W}}^{\rm su}$, has full measure. Furthermore, there exists $w \in A$ such that the support of $\nu_T$ is the closure in $T$ of the intersection with $T$ of $\Gamma W^{\rm su}(w)$. Since $w_- \notin \Lambda_c\Gamma$, this intersection is already closed, hence by quasi-invariance under holonomy of $\nu$, with $\widetilde{W} = W^{\rm su}(w)$, we have that $(\nu_T)_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$ is a multiple of $(\nu_T^{\widetilde{W}})_{T\in\mathscr{T}(\widetilde{\mathscr{W}}^{\rm su})}$.

Assume now on the contrary that $\nu$ gives full measure to the negatively recurrent set. Then the result follows from Theorem 10.4. $\Box$

# 11 Gibbs states on Galois covers

Let $(\widetilde{M}, \Gamma, \widetilde{F})$ be as in the beginning of Chapter 2: $\widetilde{M}$ is a complete simply connected Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most $-1$, $\Gamma$ is a nonelementary discrete group of isometries of $\widetilde{M}$, and $\widetilde{F} : T^1\widetilde{M} \to \mathbb{R}$ is a Hölder-continuous $\Gamma$-invariant map. Fix $x_0 \in \widetilde{M}$.

A subgroup $\Gamma_0$ of the isometry group $\mathrm{Isom}(\widetilde{M})$ of $\widetilde{M}$ is said *to normalise* $\Gamma$ if it is contained in the *normaliser*

$$N(\Gamma) = \{\gamma \in \mathrm{Isom}(\widetilde{M}) \;:\; \gamma\Gamma\gamma^{-1} = \Gamma\}$$

of $\Gamma$ in $\mathrm{Isom}(\widetilde{M})$.

We study in this chapter the behaviour of the critical exponents, the Patterson densities and the Gibbs measures associated to normal subgroups of $\Gamma$. The main point is that this gives a natural framework in which one can study precisely these objects when the Gibbs measure is not finite. We prove results analogous to those in the chapters 3, 4, 5, 8, 10, underlining the fact that, contrarily to most of these chapters, the Gibbs measures may be infinite. Most of the results are extensions of those when $F = 0$ in [Rob3], with similar proofs, and we will concentrate on the new features. This paper of Roblin furthermore mentions in its introduction that its contents remain valid for Patterson densities with potentials (referring to the multiplicative approach of Ledrappier-Coudène, see the beginning of Chapter 3 for an explanation of the differences with the additive convention considered in this text).

The structure of this section is the following one. The first two subsections give the tools used in the others, an extension of Mohsen's shadow lemma and an extension of the Fatou-Roblin radial convergence theorem, concerning a super-group $\Gamma_0$ normalising $\Gamma$ (Subsection 11.1 is used (at least) in the proofs of Subsections 11.2, 11.3, 11.4, 11.5, 11.6, though Subsection 11.2 is only used in the proofs of Subsections 11.4 and 11.7). The next three subsections 11.3, 11.4, 11.5 concern extensions of Chapters 3 and 5, where the effect of the potential $F$ is perturbed by a character $\chi$ of $\Gamma$. The last two subsections give our results concerning the normal subgroups $\Gamma'$ of $\Gamma$, in particular the equality of its critical exponent with potential to that of $\Gamma$, and the ergodic theory of its Patterson densities with potential.

## 11.1 Improving Mohsen's shadow lemma

Mohsen's shadow lemma 3.10 for a Patterson density of $(\Gamma, F)$, as its original version by Sullivan [Sul1] when $F = 0$ (see [Rob1, p. 10]), requires the balls whose shadows are considered to be centred at special points. The following result, due to [Rob3, Théo. 1.1.1] when $F = 0$, improves the range of validity of Mohsen's shadow lemma, in a uniform way. It will be useful in order to study Patterson densities on Galois covers of $M$. Its proof follows those of the above works of Sullivan, Mohsen and Roblin.

**Proposition 11.1** *Let $\Gamma_0$ be a subgroup of $\operatorname{Isom}(\widetilde{M})$, which contains and normalises $\Gamma$, such that $\widetilde{F}$ is $\Gamma_0$-invariant. For every compact subset $K$ of $\widetilde{M}$, there exist nondecreasing maps $R = R_K$ and $C = C_K$ from $[\delta_{\Gamma,F}, +\infty[$ to $]0, +\infty[$ such that for every $\sigma \geq \delta_{\Gamma,F}$, for every Patterson density $(\mu_x)_{x\in\widetilde{M}}$ for $(\Gamma, F)$ of dimension $\sigma$, for all $x, y \in \Gamma_0 K$, we have*

$$\frac{1}{C(\sigma)}\, \|\mu_y\|\, e^{\int_x^y (\widetilde{F}-\sigma)} \leq \mu_x\big(\mathscr{O}_x B(y, R(\sigma))\big) \leq C(\sigma)\, \|\mu_y\|\, e^{\int_x^y (\widetilde{F}-\sigma)}\ . \tag{164}$$

**Proof.** We start by the following key observation, which will be used several times in this chapter.

**Lemma 11.2** *For every $\alpha \in N(\Gamma)$, if $(\mu_x)_{x\in\widetilde{M}}$ is a Patterson density for $(\Gamma, F)$ of dimension $\sigma$, then so is $((\alpha^{-1})_*\mu_{\alpha x})_{x\in\widetilde{M}}$.*

**Proof.** For all $\gamma \in \Gamma$ and $x \in \widetilde{M}$, since $\alpha\gamma\alpha^{-1} \in \Gamma$, we have

$$\gamma_*(\alpha^{-1})_*\mu_{\alpha x} = (\alpha^{-1})_*(\alpha\gamma\alpha^{-1})_*\mu_{\alpha x} = (\alpha^{-1})_*\mu_{\alpha\gamma x}\ ,$$

and for all $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty \widetilde{M}$, we have

$$\frac{d(\alpha^{-1})_*\mu_{\alpha x}}{d(\alpha^{-1})_*\mu_{\alpha y}}(\xi) = \frac{d\mu_{\alpha x}}{d\mu_{\alpha y}}(\alpha\xi) = e^{-C_{F-\sigma,\,\alpha\xi}(\alpha x, \alpha y)} = e^{-C_{F-\sigma,\,\xi}(x,y)}\ ,$$

since $\widetilde{F}$ is $\Gamma_0$-invariant. □

Now, let $K$ be a fixed compact subset of $\widetilde{M}$. Since the statement is empty if $\delta_{\Gamma,F} = +\infty$, we assume that $\delta_{\Gamma,F} < +\infty$. By the above observation, we only have to prove Equation (164) when $x \in K$, since the validity of Equation (164) when $x \in \alpha K$ for any $\alpha \in \Gamma_0$ is obtained by replacing $(\mu_x)_{x\in\widetilde{M}}$ by $((\alpha^{-1})_*\mu_{\alpha x})_{x\in\widetilde{M}}$ and $y$ by $\alpha^{-1}y$.

Let us prove that there exist non-decreasing maps $R', C' : [\delta_{\Gamma,F}, +\infty[\ \to\ ]0, +\infty[$ such that for every $\sigma \geq \delta_{\Gamma,F}$, for every Patterson density $(\mu_x)_{x\in\widetilde{M}}$ for $(\Gamma, F)$ of dimension $\sigma$, for all $x \in K$ and $y \in \Gamma_0 K$, we have

$$\frac{1}{C'(\sigma)}\, \|\mu_y\| \leq \mu_y(\mathscr{O}_x B(y, R'(\sigma))) \leq C'(\sigma)\, \|\mu_y\|\ . \tag{165}$$

To prove that Equation (165) implies Equation (164), note that we have, by Equation (39),

$$\mu_x(\mathscr{O}_x B(y, R'(\sigma))) = \int_{\xi\in\mathscr{O}_x B(y,\, R'(\sigma))} e^{-C_{F-\sigma,\,\xi}(x,y)}\, d\mu_y(\xi)\ .$$

By Lemma 3.4 (2) applied with $r = r_0 = R'(\sigma)$, since $\widetilde{F}$ is $\Gamma_0$-invariant, hence is uniformly bounded on $\pi^{-1}(B(y, R'(\sigma)))$ for $y \in \Gamma_0 K$, there exists $C''(\sigma) > 0$ (depending only on $r$, and that may be taken to be nondecreasing in $\sigma$) such that for all $x \in K$ and $\xi \in \mathscr{O}_x B(y, R'(\sigma))$,

$$\Big|\, C_{F-\sigma,\,\xi}(x,y) + \int_x^y (\widetilde{F} - \sigma)\,\Big| \leq C''(\sigma)\;.$$

Hence

$$\begin{aligned} e^{-C''(\sigma)}\; e^{\int_x^y (\widetilde{F}-\sigma)}\; \mu_y(\mathscr{O}_x B(y, R'(\sigma))) &\leq \mu_x(\mathscr{O}_x B(y, R'(\sigma))) \\ &\leq e^{C''(\sigma)}\; e^{\int_x^y (\widetilde{F}-\sigma)}\; \mu_y(\mathscr{O}_x B(y, R'(\sigma)))\;. \end{aligned}$$

Therefore the result follows with $R = R'$ and $C = C' e^{C''}$.

Let us now prove the claim involving Equation (165).

The upper bound in this equation is clear with $C'(\sigma) = 1$, whatever $R'(\sigma)$ is. For future use, this proves that for every $R > 0$, for every compact subset $K$ of $\widetilde{M}$, there exists $c > 0$ (depending only on $K$ and $R$) such that, for all $x, y \in \Gamma_0 K$, we have

$$\mu_x(\mathscr{O}_x B(y, R)) \leq c\; \|\mu_y\|\; e^{\int_x^y (\widetilde{F}-\sigma)}\;. \tag{166}$$

To prove the lower bound in Equation (165), we assume for a contradiction that there exist sequences $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ in $K$, $(R_i)_{i\in\mathbb{N}}$ in $]0, +\infty[$ converging to $+\infty$, $(\alpha_i)_{i\in\mathbb{N}}$ in $\Gamma_0$ and $(\mu^i)_{i\in\mathbb{N}}$, where $\mu^i$ is a Patterson density for $(\Gamma, F)$ with bounded dimension $\sigma_i$ such that

$$\lim_{i\to+\infty}\; \frac{1}{\|\mu^i_{\alpha_i x_i}\|}\; \mu^i_{\alpha_i x_i}(\mathscr{O}_{x'_i} B(\alpha_i x_i, R_i)) \;= 0\;. \tag{167}$$

Up to extracting a subsequence, we have $\lim x_i = x \in K$, $\lim x'_i = x' \in K$, $\lim \sigma_i = \sigma \geq \delta_{\Gamma,\,F}$ and $\lim \alpha_i^{-1} x'_i = \xi \in \widetilde{M} \cup \partial_\infty \widetilde{M}$. By Equation (167), the shadow $\mathscr{O}_{x'_i} B(\alpha_i x_i, R_i)$ is different from $\partial_\infty \widetilde{M}$, hence $d(x'_i, \alpha_i x_i) \geq R_i$, which tends to $+\infty$ as $i \to +\infty$. Therefore $\xi \in \partial_\infty \widetilde{M}$. Let

$$\nu^i = \;\frac{1}{\|\mu^i_{\alpha_i x_i}\|}\; (\alpha_i^{-1})_* \mu^i_{\alpha_i x_i}\;,$$

which is a probability measure on the compact metrisable space $\partial_\infty \widetilde{M}$. By Banach-Alaoglu's theorem, up to extraction, the sequence $(\nu^i)_{i\in\mathbb{N}}$ weak-star converges to a probability measure $\nu_x$. Define, for every $z \in \widetilde{M}$, a measure $\nu_z$ on $\partial_\infty \widetilde{M}$ by

$$d\nu_z(\xi') = e^{-C_{F-\sigma,\,\xi}(z,x)}\; d\nu_x(\xi')\;.$$

By the continuity of the Gibbs cocycle and by taking limits, the family $(\nu_z)_{z\in\widetilde{M}}$ is a Patterson density for $(\Gamma, F)$ of dimension $\sigma$.

Let $V$ be a relatively compact open subset of $\partial_\infty \widetilde{M} - \{\xi\}$. Then $V$ is contained in $\mathscr{O}_{\alpha_i^{-1} x'_i} B(x_i, R_i)$ for $i$ large enough, since $\lim R_i = +\infty$ and $(x_i)_{i\in\mathbb{N}}$ stays in the compact subset $K$. Hence

$$\nu_x(V) \leq \limsup_{i\to+\infty} \nu^i(\mathscr{O}_{\alpha_i^{-1} x'_i} B(x_i, R_i)) = 0$$

by Equation (167). Therefore the measure $\nu_x$ is supported in $\{\xi\}$. Since the support of a Patterson density of $(\Gamma, F)$ is $\Gamma$-invariant, this implies that $\xi$ is fixed by $\Gamma$. This is a contradiction since $\Gamma$ is nonelementary. □

The following consequence is due to [Rob3, Lem. 1.2.4] when $F = 0$.

**Corollary 11.3** *Let $\Gamma_0$ be a discrete subgroup of* $\operatorname{Isom}(\widetilde{M})$*, which contains and normalises $\Gamma$, such that $\widetilde{F}$ is $\Gamma_0$-invariant. Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density for $(\Gamma, F)$ of dimension $\sigma$.*

*Then the critical exponent of the series $Q'(s) = \sum_{y\in\Gamma_0 x_0} \|\mu_y\|\, e^{\int_{x_0}^{y}(\widetilde{F}-s)}$ is at most $\sigma$. If $\mu_{x_0}$ gives positive measure to $\Lambda_c\Gamma_0$, then this critical exponent is equal to $\sigma$, and this series diverges at $s = \sigma$.*

**Proof.** Note that $\sigma \geq \delta_{\Gamma,F}$ by Corollary 3.11 (2). Let $R = R_{\{x_0\}}(\sigma)$ and $C = C_{\{x_0\}}(\sigma)$ be given by Proposition 11.1 for $K = \{x_0\}$. For every $n \in \mathbb{N}$, let

$$E_n = \{y \in \Gamma_0 x_0 \;:\; n \leq d(x_0, y) < n+1\}\,.$$

As seen in the proof of Corollary 3.11, since $\Gamma_0$ is discrete, there exists $\kappa \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, any element of $\partial_\infty\widetilde{M}$ belongs to at most $\kappa$ elements of the family $\big(\mathscr{O}_{x_0}B(y,R)\big)_{y\in E_n}$.

By the lower bound in Equation (164) in Proposition 11.1, for all $n \in \mathbb{N}$, we have

$$\|\mu_{x_0}\| \geq \frac{1}{\kappa}\sum_{y\in E_n}\mu_{x_0}(\mathscr{O}_{x_0}B(y,R)) \geq \frac{1}{C\kappa}\sum_{y\in E_n}\|\mu_y\|\, e^{\int_{x_0}^{y}(\widetilde{F}-\sigma)}\,.$$

Hence if $s > \sigma$, then the series $Q'(s) = \sum_{y\in\Gamma_0 x_0}\|\mu_y\|\, e^{\int_{x_0}^{y}(\widetilde{F}-s)}$ converges.

If the series $Q'(\sigma)$ converges, let us prove that $\mu_{x_0}(\Lambda_c\Gamma_0) = 0$, which gives the last assertion by contraposition. For every $n \in \mathbb{N}$, let us define

$$A_n = \bigcap_{k\in\mathbb{N}}\;\bigcup_{y\in\Gamma_0 x_0 - B(x_0,k)}\mathscr{O}_{x_0}B(y,n)\,.$$

For every $n \in \mathbb{N}$, by Equation (166) in the proof of Proposition 11.1 (with $K = \{x_0\}$) when $n$ is large enough, there exists $c_n > 0$ such that, for every $k \in \mathbb{N}$,

$$\mu_{x_0}(A_n) \leq \sum_{y\in\Gamma_0 x_0 - B(x_0,k)}\mu_{x_0}(\mathscr{O}_{x_0}B(y,n)) \leq c_n\sum_{y\in\Gamma_0 x_0 - B(x_0,k)}\|\mu_y\|\, e^{\int_{x_0}^{y}(\widetilde{F}-\sigma)}\,.$$

Since the sum on the right hand side tends to $0$ as $k \to +\infty$, we have $\mu_{x_0}(A_n) = 0$ for every $n \in \mathbb{N}$. Since $\Lambda_c\Gamma_0 = \bigcup_{n\in\mathbb{N}} A_n$ by the definition of the conical limit set, we have $\mu_{x_0}(\Lambda_c\Gamma_0) = 0$, as required. □

Here is another consequence of Proposition 11.1, due to [Rob3, Lem. 1.2.3] when $F = 0$. For every $r > 0$ and every nonelementary discrete group $\Gamma_0$ of isometries of $\widetilde{M}$, let $\Lambda_{c,r}\Gamma_0$ be the set of elements $\xi \in \partial_\infty\widetilde{M}$ such that there exist $\rho < r$ and $(\gamma_n)_{n\in\mathbb{N}}$ in $\Gamma_0$ such that $(\gamma_n x_0)_{n\in\mathbb{N}}$ converges to $\xi$ and $d(\gamma_n x_0, [x_0,\xi[) < \rho$ (or equivalently $\xi \in \mathscr{O}_{x_0}B(\gamma_n x_0, \rho)$). Note that the sets $\Lambda_{c,r}\Gamma_0$ are $\Gamma_0$-invariant (by the properties of asymptotic geodesic rays), are nondecreasing in $r$, and satisfy $\Lambda_c\Gamma_0 = \bigcup_{r>0}\Lambda_{c,r}\Gamma_0$.

**Corollary 11.4** *Let $\Gamma_0$ be a discrete subgroup of* $\operatorname{Isom}(\widetilde{M})$*, which contains and normalises $\Gamma$, such that $\widetilde{F}$ is $\Gamma_0$-invariant. Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density for $(\Gamma, F)$ of dimension $\sigma$. Then for every $r > R_{\{x_0\}}(\sigma)$, we have*

$$\mu_{x_0}(\Lambda_c\Gamma_0 - \Lambda_{c,r}\Gamma_0) = 0\,.$$

**Proof.** Let $\rho = R_{\{x_0\}}(\sigma)$ and $r > \rho$. First assume that the quasi-invariant measure $\mu_{x_0}$ is ergodic for the action of $\Gamma$. If $\mu_{x_0}(\Lambda_c\Gamma_0) = 0$, the result is clear. Otherwise, $\mu_{x_0}({}^c\Lambda_c\Gamma_0) = 0$ by ergodicity, and $\mu_{x_0}(\Lambda_{c,\,R}\Gamma_0) > 0$ for $R > \rho$ large enough, so that again by ergodicity $\mu_{x_0}({}^c\Lambda_{c,\,R}\Gamma_0) = 0$.

Recall without proof the following Vitali covering argument.

**Lemma 11.5** *(Roblin [Rob3, Lem. 1.2.1]) For every $s > 0$ and for every discrete subset $Z$ of $\widetilde{M}$, there exists a subset $Z^*$ of $Z$ such that the shadows $\mathscr{O}_{x_0}B(z,s)$ for $z \in Z^*$ are pairwise disjoint and $\bigcup_{z\in Z}\mathscr{O}_{x_0}B(z,s) \subset \bigcup_{z\in Z^*}\mathscr{O}_{x_0}B(z,5s)$.*

For every $n \in \mathbb{N}$, applying this lemma with $s = R$ and $Z_n = \Gamma_0 x_0 - B(x_0,n)$ gives a subset $Z_n^*$ of $Z_n$. Since $R > \rho$, the shadows $\mathscr{O}_{x_0}B(z,\rho)$ for $z \in Z_n^*$ are pairwise disjoint. Define

$$A_n = \bigcup_{z\in Z_n^*}\mathscr{O}_{x_0}B(z,\rho)\;.$$

We have $\sigma \geq \delta_{\Gamma,\,F}$ by Corollary 3.11 (2). By Proposition 11.1 and the definition of $\rho$, by Equation (166), there exist $c, C > 0$ such that, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mu_{x_0}(A_n) &= \sum_{z\in Z_n^*}\mu_{x_0}\big(\mathscr{O}_{x_0}B(z,\rho)\big) \geq \frac{1}{C}\sum_{z\in Z_n^*}\|\mu_z\|\; e^{\int_{x_0}^{z}(\widetilde{F}-\sigma)}\\
&\geq \frac{1}{c\,C}\sum_{z\in Z_n^*}\mu_{x_0}\big(\mathscr{O}_{x_0}B(z,5R)\big) \geq \frac{1}{c\,C}\mu_{x_0}\Big(\bigcup_{z\in Z_n}\mathscr{O}_{x_0}B(z,R)\Big)\\
&\geq \frac{1}{c\,C}\mu_{x_0}(\Lambda_{c,\,R}\Gamma_0) = \frac{\|\mu_{x_0}\|}{c\,C}\;.
\end{aligned}$$

Since $r > \rho$, the set $\Lambda_{c,\,r}\Gamma_0$ contains the nonincreasing intersection of $\bigcup_{z\in Z_n}\mathscr{O}_{x_0}B(z,\rho)$ (which contains $A_n$). Hence $\mu_{x_0}(\Lambda_{c,\,r}\Gamma_0) \geq \frac{\|\mu_{x_0}\|}{c\,C(\sigma)} > 0$, and again by ergodicity, we have $\mu_{x_0}({}^c\Lambda_{c,\,r}\Gamma_0) = 0$.

By a Krein-Milman type of argument, since the map $(\mu_x)_{x\in\widetilde{M}} \mapsto \mu_{x_0}$ is a bijection from the set of Patterson densities for $(\Gamma, F)$ of dimension $\sigma$ to a convex cone of finite quasi-invariant measures on $\partial_\infty\widetilde{M}$, whose extremal points are the ergodic ones, the result follows. $\square$

## 11.2 The Fatou-Roblin radial convergence theorem

We state in this subsection the main tool for the classification of Patterson densities on nilpotent covers of $M$, to be given in Subsection 11.7. It is (an immediate extension of) one of the main results of [Rob3], which we will call the *Fatou-Roblin radial convergence* theorem. Recall that a sequence of points $(y_i)_{i\in\mathbb{N}}$ in $\widetilde{M}$ converges *radially* to a point $\xi \in \partial_\infty\widetilde{M}$ if it converges to $\xi$ while staying at bounded distance from a geodesic ray.

Recall that Fatou's (ratio) radial convergence theorem says that given $g$ and $h$ two positive harmonic functions on the open unit disc $\mathbb{D}^2$ in $\mathbb{R}^2$, for almost every point $\zeta$ on the unit circle $\mathbb{S}^1$, the ratio $g(z)/h(z)$ has a limit when $z \in \mathbb{D}^2$ radially converges to $\zeta$. Consider a nonelementary Fuchsian group $\Gamma'$ that is, when $\mathbb{D}^2$ is endowed with Poincaré's metric (and hence is a real hyperbolic plane), a non virtually cyclic discrete subgroup of isometries of $\mathbb{D}^2$. If $(\mu_x)_{x\in\mathbb{D}^2}$ is a Patterson density for $\Gamma'$ (without potential), then Sullivan has proved that the map $x \mapsto \|\mu_x\|$ is harmonic. Hence the ratio of the total masses of two such Patterson densities has radial limits almost everywhere.

The next result extends this, and we refer to [Rob3] for more motivation.

Given two (Borel positive) measures $\mu$ and $\nu$ on $\partial_\infty \widetilde{M}$, if $\mu = \mu' + \mu''$ is the unique decomposition of $\mu$ as the sum of two (Borel positive) measures $\mu', \mu''$ with $\mu' \ll \nu$ and $\mu'' \perp \nu$, we denote by $\frac{d\mu}{d\nu}$ the ($\nu$-almost everywhere well defined) Radon-Nikodym derivative of the part $\mu'$ of $\mu$ which is absolutely continuous with respect to $\nu$, so that

$$\mu \geq \frac{d\mu}{d\nu}\, \nu\ . \tag{168}$$

**Theorem 11.6 (Fatou-Roblin radial convergence theorem)** *Let $\Gamma_0$ be a discrete subgroup of $\operatorname{Isom}(\widetilde{M})$, which contains and normalises $\Gamma$, such that $\widetilde{F}$ is $\Gamma_0$-invariant. Let $\sigma \in \mathbb{R}$ and let $(\mu_x)_{x\in\widetilde{M}}$ and $(\nu_x)_{x\in\widetilde{M}}$ be two Patterson densities of dimension $\sigma$ for $(\Gamma, F)$. Then for $\nu_{x_0}$-almost every $\xi \in \Lambda_c\Gamma_0$, if $(y_i)_{i\in\mathbb{N}}$ is a sequence in $\Gamma_0 x_0$ converging radially to $\xi$, then*

$$\lim_{i\to+\infty} \frac{\|\mu_{y_i}\|}{\|\nu_{y_i}\|} \;=\; \frac{d\,\mu_{x_0}}{d\,\nu_{x_0}}(\xi)\ .$$

Note that this theorem is only interesting when $\nu_{x_0}(\Lambda_c\Gamma_0) > 0$ and $\nu_{x_0}(\Lambda_c\Gamma) = 0$, that is, by the Hopf-Tsuji-Sullivan-Roblin theorem 5.3, when $(\Gamma_0, F)$ is of divergence type and when $(\Gamma, F)$ is of convergence type. In particular, it is trivial if $\Gamma_0 = \Gamma$.

Indeed, the statement is empty if $\nu_{x_0}(\Lambda_c\Gamma_0) = 0$. And if $\nu_{x_0}(\Lambda_c\Gamma) > 0$, then the Hopf-Tsuji-Sullivan-Roblin theorem 5.4 and the uniqueness property in Corollary 5.12 imply that $\Lambda_c\Gamma$ has full measure both for $\nu_{x_0}$ and $\mu_{x_0}$, and that $\nu_{x_0}$ and $\mu_{x_0}$ are proportional, so that the result is immediate.

**Proof.** The proof of [Rob3, Théo. 1.2.2] extends almost immediately, except the first of its five steps, that needs some adaptation.

We endow any infinite subset of $\Gamma_0 x_0$ with its *Fréchet filter* of the complementary sets of its finite subsets. Note that by the discreteness of the orbits and the definition of the topology on $\widetilde{M} \cup \partial_\infty \widetilde{M}$, for all $\rho \geq 0$ and $\xi \in \partial_\infty \widetilde{M}$, a sequence $(y_i)_{i\in\mathbb{N}}$ in $\Gamma_0 x_0$, which goes out of every finite subset of $\Gamma_0 x_0$, and satisfies $d(y_i,\, [x_0,\, \xi[) \leq \rho$ for all $i \in \mathbb{N}$, converges to $\xi$.

Let $R = R_{\{x_0\}}(\sigma)$ and $C = C_{\{x_0\}}(\sigma)$ be given by Proposition 11.1 with $K = \{x_0\}$. Let us fix $r \in [R, +\infty[$. For every $\xi \in \Lambda_{c,\,r}\Gamma_0$, let

$$D_r^+(\xi) = \lim_{\rho\to r^-}\ \limsup_{y\in\Gamma_0 x_0,\ d(y,\,[x_0,\,\xi[)\leq\rho} \frac{\|\mu_y\|}{\|\nu_y\|}\ .$$

This limit does exist, since the function of $\rho$ is nondecreasing, and $D_r^+ : \Lambda_{c,\,r}\Gamma_0 \to \mathbb{R}\cup\{+\infty\}$ is a Borel $\Gamma$-invariant map, by the properties of asymptotic geodesic rays and the $\Gamma$-equivariance property of $(\mu_x)_{x\in\widetilde{M}}$ and $(\nu_x)_{x\in\widetilde{M}}$.

We claim that if $D_r^+(\xi) \geq 1$ for $\nu_{x_0}$-almost every $\xi \in \Lambda_{c,\,r}\Gamma_0$, then there exists $c' > 0$ such that $\mu_{x_0} \geq c'\nu_{x_0}$ on $\Lambda_{c,\,r}\Gamma_0$.

Given this claim, the remainder of the proof of [Rob3, Théo. 1.2.2] yields without modification the proof of Theorem 11.6.

To prove this claim, for every Borel subset $B$ of $\partial_\infty \widetilde{M}$, and every open neighbourhood $V$ of $B$, let

$$Z = \{z \in \Gamma_0 x_0 \;:\; \mathscr{O}_{x_0} B(z,r) \subset V \;\text{ and }\; \|\mu_z\| \geq \frac{1}{2}\|\nu_z\|\}\;,$$

and let $Z^*$ be the subset of $Z$ given by Lemma 11.5 with $s = r$. The assumption of the above claim implies that, up to a set of zero $\nu_{x_0}$-measure, the set $B \cap \Lambda_{c,\,r}\Gamma_0$ is contained in $\bigcup_{z\in Z} \mathscr{O}_{x_0} B(z,r)$. Hence, by the properties of $Z^*$, for the constant $c > 0$ given by Equation (166), which depends only on $5r$ and $K = \{x_0\}$, and by Proposition 11.1, we have

$$\begin{aligned}\nu_{x_0}(B \cap \Lambda_{c,\,r}\Gamma_0) &\leq \sum_{z\in Z^*} \nu_{x_0}\big(\mathscr{O}_{x_0} B(z,5r)\big) \leq c \sum_{z\in Z^*} \|\nu_z\|\; e^{\int_{x_0}^{z}(\widetilde{F}-\sigma)}\\ &\leq 2c \sum_{z\in Z^*} \|\mu_z\|\; e^{\int_{x_0}^{z}(\widetilde{F}-\sigma)} \leq 2c\,C \sum_{z\in Z^*} \mu_{x_0}\big(\mathscr{O}_{x_0} B(z,r)\big) \leq 2c\,C\;\mu_{x_0}(V)\;.\end{aligned}$$

Since $V$ is an arbitrary open neighbourhood of $B$, we have $\nu_{x_0}(B\cap\Lambda_{c,\,r}\Gamma_0) \leq 2c\,C\;\mu_{x_0}(B)$, which proves the claim. $\square$

The next result, especially interesting when $\Gamma_0$ is convex-cocompact, gives a criterion for a Patterson density of $(\Gamma, F)$ to be determined by its total masses at given points.

**Corollary 11.7** *Let $\Gamma_0$ be a discrete subgroup of $\operatorname{Isom}(\widetilde{M})$, which contains and normalises $\Gamma$, such that $\widetilde{F}$ is $\Gamma_0$-invariant. Let $\sigma, \sigma' \in \mathbb{R}$. Let $(\mu_x)_{x\in\widetilde{M}}$ and $(\nu_x)_{x\in\widetilde{M}}$ be two Patterson densities for $(\Gamma, F)$ of dimension $\sigma$ and $\sigma'$ respectively, giving full measure to $\Lambda_c\Gamma_0$. If for every $y \in \Gamma_0 x_0$ we have $\|\mu_y\| = \|\nu_y\|$, then $(\mu_x)_{x\in\widetilde{M}} = (\nu_x)_{x\in\widetilde{M}}$ (and in particular $\sigma = \sigma'$).*

**Proof.** The proof is the same as when $F = 0$ (see [Rob3, Coro. 1.2.5]). $\square$

### 11.3 Characters and critical exponents

Recall that a (real) *character* of a group $G$ is a group morphism from $G$ to the additive group $\mathbb{R}$.

Note that the kernel $\operatorname{Ker}\chi$ of a character $\chi$ of $\Gamma$ is a nonelementary discrete group of isometries of $\widetilde{M}$, since otherwise $\Gamma$ is the extension of an abelian group by a virtually nilpotent group, hence is virtually solvable, hence is elementary.

In this subsection, we fix a character $\chi$ of $\Gamma$. Define the *(twisted) Poincaré series of* $(\Gamma, F, \chi)$ as

$$Q_{\Gamma,\,F,\,\chi}(s) = Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s) = \sum_{\gamma\in\Gamma} e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-s)}\;.$$

The *(twisted) critical exponent* of $(\Gamma, F, \chi)$ is the element $\delta_{\Gamma,\,F,\,\chi}$ in $[-\infty, +\infty]$ defined by

$$\delta_{\Gamma,\,F,\,\chi} = \limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma,\; n-1<d(x,\gamma y)\leq n} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}\;.$$

If $\delta_{\Gamma,\,F,\,\chi} < +\infty$ (we will prove in the following Proposition 11.8 that $\delta_{\Gamma,\,F,\,\chi} > -\infty$), we say that $(\Gamma, F, \chi)$ is *of divergence type* if the series $Q_{\Gamma,\,F,\,\chi}(\delta_{\Gamma,\,F,\,\chi})$ diverges, and *of convergence type* otherwise.

Since $\widetilde{F}$ is Hölder-continuous, the critical exponent $\delta_{\Gamma,\,F,\,\chi}$ of $(\Gamma, F, \chi)$ does not depend on $x, y$, and we will prove below that the above upper limit is a limit if $\delta_{\Gamma,\,F,\,\chi} > 0$. When

$F = 0$, these objects were introduced in [Rob3], and we will extend in this chapter the results of this reference, which itself extends results of [BaL].

For all $s \geq 0$ and $c > 0$, for all $x, y \in \widetilde{M}$ and for all open subsets $U$ and $V$ of $\partial_\infty \widetilde{M}$, let

$$G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V}(t) = \sum_{\gamma\in\Gamma\,:\,d(x,\gamma y)\leq t\,,\ \gamma y\in\mathscr{C}_x U\,,\ \gamma^{-1}x\in\mathscr{C}_y V} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}\;,$$

and

$$G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V,\,c}(t) = \sum_{\gamma\in\Gamma\,:\,t-c<d(x,\gamma y)\leq t\,,\ \gamma y\in\mathscr{C}_x U\,,\ \gamma^{-1}x\in\mathscr{C}_y V} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}\;.$$

The map $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V}(s)$ will be called the *(twisted) bisectorial orbital counting function* of $(\Gamma, F, U, V)$ and $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V,\,c}(s)$ the *(twisted) annular bisectorial orbital counting function* of $(\Gamma, F, U, V)$. When $V = \partial_\infty \widetilde{M}$, we denote them by $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U}(s)$ and $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,c}(s)$, and call them the *(twisted) sectorial orbital counting function* and *(twisted) annular sectorial orbital counting function* of $(\Gamma, F, \chi, U)$. When $U = V = \partial_\infty \widetilde{M}$, we denote them by $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y}(s)$ and $s \mapsto G_{\Gamma,\,F,\,\chi,\,x,\,y,\,c}(s)$, and we call them the *(twisted) orbital counting function* and *(twisted) annular orbital counting function* of $(\Gamma, F, \chi)$.

Let $g$ be a periodic orbit of the geodesic flow on $T^1M$. Choose $\gamma_g$ one of the (finitely many up to conjugation) loxodromic elements of $\Gamma$ such that if $x$ is a point of the translation axis of $\gamma_g$, then $g$ is obtained by first lifting to $T^1\widetilde{M}$ (by the unit tangent vectors) the geodesic segment $[x, \gamma_g x]$ oriented from $x$ to $\gamma_g x$ and then by taking the image by $T^1\widetilde{M} \to T^1M$. Define $\chi(g) = \chi(\gamma_g)$, which does not depend on the choice of $\gamma_g$, since for all $\gamma, \gamma' \in \Gamma$, we have $\chi(\gamma) = \chi(\gamma')$ if $\gamma$ and $\gamma'$ are conjugated or if $\gamma'\gamma^{-1}$ has finite order. For every relatively compact open subset $W$ of $T^1M$ meeting the (topological) non-wandering set $\Omega\Gamma$, we define the *(twisted) period counting series* of $(\Gamma, F, \chi, W)$ (see Subsection 4.1 when $\chi = 0$) as

$$Z_{\Gamma,\,F,\,\chi,\,W}(s) = \sum_{g\in\mathscr{P}\mathrm{er}(s),\ g\cap W\neq\emptyset} e^{\chi(g)+\int_g F}\;;$$

We also define, for every $c > 0$, the *(twisted) annular period counting series* of $(\Gamma, F, \chi, W)$ as

$$Z_{\Gamma,\,F,\,\chi,\,W,\,c}(s) = \sum_{g\in\mathscr{P}\mathrm{er}(s)-\mathscr{P}\mathrm{er}(s-c),\ g\cap W\neq\emptyset} e^{\chi(g)+\int_g F}\;;$$

The *(twisted) Gurevich pressure* of $(\Gamma, F, \chi)$ is

$$P_{Gur}(\Gamma,\ F,\ \chi) = \limsup_{s\to+\infty}\ \frac{1}{s}\log Z_{\Gamma,\,F,\,\chi,\,W,\,c}(s)\;. \tag{169}$$

We will prove below that the twisted Gurevich pressure depends neither on $W$ nor on $c > 0$ and that the above upper limit is a limit if $c$ is large enough.

**Remark.** Assume in this remark that $\Gamma$ is torsion free and that every closed differential 1-form on $M$ is cohomologous to a (uniformly locally) Hölder-continuous one (this second assumption is for instance satisfied if $M$ is compact). As explained for instance in [Bab1] in a particular case, the interpretation of the characters of $\Gamma$ as closed 1-forms on $M$ allows us to reduce the above objects twisted by a character to ones associated to another potential, as we shall now see.

Given a differential 1-form $\omega$, which is a smooth map from $TM$ to $\mathbb{R}$ (linear on each fibre of $\pi$) that we assume to be Hölder-continuous, its restriction to $T^1M$, which will also be denoted by $\omega$, is a (smooth) potential on $T^1M$. Note that there are potentials which do not come from differential 1-forms.

If two Hölder-continuous differential 1-forms $\omega$ and $\omega'$ are cohomologous (as differential 1-forms), and if $f : M \to \mathbb{R}$ is a smooth map such that $\omega' - \omega = df$, then their associated potentials $\omega$ and $\omega'$ are cohomologous (as potentials) via the map $G = f \circ \pi : T^1M \to \mathbb{R}$.

Let $H^1(M;\mathbb{R})$ be the space of cohomology classes of closed differential 1-forms on $M$. Consider the map

$$[\omega] \mapsto \Big(\gamma \mapsto \int_\gamma \omega = \int_x^{\gamma x} \widetilde{\omega}\Big) ,$$

where $x$ is any point of $\widetilde{M}$ and $\widetilde{\omega}$ is the potential associated as explained above to the lift $\widetilde{p}^*\omega$ of $\omega$ to $\widetilde{M}$ by the canonical projection $\widetilde{p} : \widetilde{M} \to M$. By Hurewicz's theorem, this map from $H^1(M;\mathbb{R})$ to $\mathrm{Hom}(\Gamma,\mathbb{R})$ is a linear isomorphism. For every $\chi \in \mathrm{Hom}(\Gamma,\mathbb{R})$, let $\omega_\chi$ be a Hölder-continuous differential 1-form on $M$ such that $[\omega_\chi]$ maps to $\chi$, and let $\widetilde{\omega}_\chi$ be its lift to $\widetilde{M}$.

Under the hypotheses of this remark, we then immediately have

$$Q_{\Gamma,\,F,\,\chi,\,x,\,x}(s) = Q_{\Gamma,\,F+\omega_\chi,\,x,\,x}(s) ,$$

$$\delta_{\Gamma,\,F,\,\chi} = \delta_{\Gamma,\,F+\omega_\chi} ,$$

and

$$P_{Gur}(\Gamma,F,\chi) = P_{Gur}(\Gamma,F+\omega_\chi) .$$

When $F = 0$, we recover the Poincaré series associated with a cohomology class of $\Gamma$ introduced in [Bab1], and the twisted critical exponent is then called the *cohomological pressure*.

After this remark, let us give elementary properties satisfied by the twisted critical exponents.

**Proposition 11.8** *Let $\chi$ be a character of $\Gamma$.*

*(i) The Poincaré series of $(\Gamma,F,\chi)$ converges if $s > \delta_{\Gamma,\,F,\,\chi}$ and diverges if $s < \delta_{\Gamma,\,F,\,\chi}$. The Poincaré series of $(\Gamma,F,\chi)$ diverges at $\delta_{\Gamma,\,F,\,\chi}$ if and only if the Poincaré series of $(\Gamma,F+\kappa,\chi)$ diverges at $\delta_{\Gamma,\,F+\kappa,\,\chi}$, for every $\kappa \in \mathbb{R}$, and in particular*

$$\forall\, \kappa \in \mathbb{R}, \quad \delta_{\Gamma,\,F+\kappa,\,\chi} = \delta_{\Gamma,\,F,\,\chi} + \kappa .$$

*(ii) We have*

$$\forall\, s \in \mathbb{R},\ Q_{\Gamma,\,F\circ\iota,\,-\chi,\,x,\,y}(s) = Q_{\Gamma,\,F,\,\chi,\,y,\,x}(s) \quad \text{and} \quad \delta_{\Gamma,\,F\circ\iota,\,-\chi} = \delta_{\Gamma,\,F,\,\chi} .$$

*In particular, $(\Gamma,F\circ\iota,-\chi)$ is of divergence type if and only if $(\Gamma,F,\chi)$ is of divergence type.*

*(iii) If $\Gamma'$ is a nonelementary subgroup of $\Gamma$, denoting by $F' : \Gamma'\backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$, and by $\chi'$ the restriction to $\Gamma'$ of $\chi$, we have*

$$\delta_{\Gamma',\,F',\,\chi'} \leq \delta_{\Gamma,\,F,\,\chi} .$$

*(iv) We have*

$$\forall\, s\in\mathbb{R},\ Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s)+Q_{\Gamma,\,F\circ\iota,\,\chi,\,y,\,x}(s)\geq Q_{\Gamma,\,F,\,x,\,y}(s) \tag{170}$$

*and hence*

$$\max\{\,\delta_{\Gamma,\,F,\,\chi},\ \delta_{\Gamma,\,F\circ\iota,\,\chi}\}\geq\delta_{\Gamma,\,F}\ .$$

*(v) If there exists $c\geq 0$ such that $|\chi(\gamma)|\leq c\, d(x_0,\gamma x_0)$ for every $\gamma\in\Gamma$, then*

$$\delta_{\Gamma,\,F}-c\ \leq\ \delta_{\Gamma,\,F,\,\chi}\ \leq\ \delta_{\Gamma,\,F}+c\ .$$

*(vi) We have $\delta_{\Gamma,\,F,\,\chi}>-\infty$.*

*(vii) The map $(F,\chi)\mapsto\delta_{\Gamma,\,F,\,\chi}$ is convex, that is, if $\chi^*$ is another character of $\Gamma$, if $\widetilde{F^*}:T^1\widetilde{M}\rightarrow\mathbb{R}$ is another Hölder-continuous $\Gamma$-invariant map, inducing $F^*:\Gamma\backslash T^1\widetilde{M}\rightarrow\mathbb{R}$, if $\delta_{\Gamma,\,F,\,\chi}$ and $\delta_{\Gamma,\,F^*,\,\chi^*}$ are finite, then for every $t\in[0,1]$, we have*

$$\delta_{\Gamma,\,tF+(1-t)F^*,\,t\chi+(1-t)\chi^*}\leq t\,\delta_{\Gamma,\,F,\,\chi}+(1-t)\,\delta_{\Gamma,\,F^*,\,\chi^*}\ .$$

*(viii) For every $c>0$, we have*

$$\delta_{\Gamma,\,F,\,\chi}=\limsup_{n\rightarrow+\infty}\ \frac{1}{n}\log\sum_{\gamma\in\Gamma,\ n-c<d(x,\,\gamma y)\leq n}e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}\ ,$$

*and if $\delta_{\Gamma,\,F,\,\chi}\geq 0$, then*

$$\delta_{\Gamma,\,F,\,\chi}=\limsup_{n\rightarrow+\infty}\ \frac{1}{n}\log\sum_{\gamma\in\Gamma,\ d(x,\,\gamma y)\leq n}e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}\ .$$

*(ix) If $\widetilde{F^*}:T^1\widetilde{M}\rightarrow\mathbb{R}$ is another Hölder-continuous $\Gamma$-invariant map, which is cohomologous to $\widetilde{F}$, and if $F^*:T^1M\rightarrow\mathbb{R}$ is the induced map, then*

$$\delta_{\Gamma,\,F,\,\chi}=\delta_{\Gamma,\,F^*,\,\chi}\ ,$$

*and $(\Gamma,F^*,-\chi)$ is of divergence type if and only if $(\Gamma,F,\chi)$ is of divergence type.*

Before giving a proof, here are a few immediate consequences. It follows from Assertion (iv) that if $F$ is reversible, then

$$\delta_{\Gamma,\,F,\,-\chi}=\delta_{\Gamma,\,F,\,\chi}\geq\delta_{\Gamma,\,F}\ . \tag{171}$$

We claim that the assumption of Assertion (v) is satisfied for instance if $\Gamma$ is convex-cocompact. More generally, assume that $\Gamma$ is finitely generated (which is the case if it is convex-cocompact), and let $S$ be a finite generating set of $\Gamma$. For every $\gamma\in\Gamma$, let $\|\gamma\|_S$ be the smallest length of a word in $S\cup S^{-1}$ representing $\gamma$. For every character $\chi$ of $\Gamma$, we have

$$|\chi(\gamma)|\leq\|\gamma\|_S\ \max_{s\in S}|\chi(s)|\ .$$

Assume also that the map $\gamma\mapsto\gamma x_0$ from $\Gamma$ (endowed with the word distance $(\gamma,\gamma')\mapsto\|\gamma^{-1}\gamma'\|_S$) to $\widetilde{M}$ is quasi-isometric, that is, assume that there exists $c>0$ such that for every $\gamma\in\Gamma$, we have

$$d(x_0,\gamma x_0)\geq c\|\gamma\|_S\ .$$

This property does not depend on $x_0$ and is satisfied for instance if $\Gamma$ is convex-cocompact. Then the assumption of Assertion (v) is indeed satisfied.

If $\Gamma$ satisfies the hypothesis of Assertion (v) and if $\delta_{\Gamma,\,F}$ is finite (by Lemma 3.3 (iv), this is for instance the case if $\widetilde{F}$ is bounded on $\pi^{-1}(\mathscr{C}\Lambda\Gamma)$, which itself is the case if $\Gamma$ is convex-cocompact), then $\delta_{\Gamma,\,F,\,\chi}$ is finite.

In particular, if $\Gamma$ is convex-cocompact, then $\delta_{\Gamma,\,F,\,\chi}$ is finite for every character $\chi$ of $\Gamma$.

**Proof.** The proof is an immediate adaptation of the proof of Lemma 3.3. For instance, to prove Equation (170), we use

$$\begin{aligned}\sum_{\gamma\in\Gamma}\; e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-s)}+\sum_{\gamma\in\Gamma}\; e^{\chi(\gamma^{-1})+\int_y^{\gamma^{-1}x}(\widetilde{F}\circ\iota-s)} &= \sum_{\gamma\in\Gamma}\;\big(e^{\chi(\gamma)}+e^{-\chi(\gamma)}\big)\;e^{\int_x^{\gamma y}(\widetilde{F}-s)}\\ &\geq \sum_{\gamma\in\Gamma}\; e^{\int_x^{\gamma y}(\widetilde{F}-s)}\;. \quad \Box\end{aligned}$$

The following results concerning the twisted critical exponent and twisted Gurevich pressure have proofs similar to those when $\chi=0$ seen in Chapter 4.

**Theorem 11.9** *(1) For every $c>0$, we have $G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V,\,c}(t)=O(e^{\delta_{\Gamma,\,F,\,\chi}\,t})$ as $t$ goes to $+\infty$. If $\delta_{\Gamma,\,F,\,\chi}>0$, we have $G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V}(t)=O(e^{\delta_{\Gamma,\,F,\,\chi}\,t})$ as $t$ goes to $+\infty$.*

*(2) For every $c>0$ large enough, we have*

$$\delta_{\Gamma,\,F,\,\chi}=\lim_{n\to+\infty}\frac{1}{n}\log\sum_{\gamma\in\Gamma,\;n-c<d(x,\gamma y)\leq n} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}=\lim_{s\to+\infty}\frac{1}{s}\log\;G_{\Gamma,\,F,\,\chi,\,x,\,y,\,c}(s)\;.$$

*If $\delta_{\Gamma,\,F,\,\chi}>0$, then*

$$\delta_{\Gamma,\,F,\,\chi}=\lim_{n\to+\infty}\frac{1}{n}\log\sum_{\gamma\in\Gamma,\;d(x,\gamma y)\leq n} e^{\chi(\gamma)+\int_x^{\gamma y}\widetilde{F}}=\lim_{s\to+\infty}\frac{1}{s}\log\;G_{\Gamma,\,F,\,\chi,\,x,\,y}(s)\;.$$

*(3) Let $U$ be an open subset of $\partial_\infty\widetilde{M}$ meeting $\Lambda\Gamma$. For every $c>0$ large enough, we have*

$$\lim_{t\to+\infty}\frac{1}{t}\log G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,c}(t)=\delta_{\Gamma,\,F,\,\chi}\,,$$

*and if $\delta_{\Gamma,\,F,\,\chi}>0$, then*

$$\lim_{t\to+\infty}\frac{1}{t}\log G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U}(t)=\delta_{\Gamma,\,F,\,\chi}\,.$$

*(4) Let $U$ and $V$ be any two open subsets of $\partial_\infty\widetilde{M}$ meeting the limit set $\Lambda\Gamma$. For every $c>0$ large enough, we have*

$$\lim_{t\to+\infty}\frac{1}{t}\log G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V,\,c}(t)=\delta_{\Gamma,\,F,\,\chi}\,,$$

*and if $\delta_{\Gamma,\,F,\,\chi}>0$, then*

$$\lim_{t\to+\infty}\frac{1}{t}\log G_{\Gamma,\,F,\,\chi,\,x,\,y,\,U,\,V}(t)=\delta_{\Gamma,\,F,\,\chi}\,.$$

*(5) The definition of $P_{Gur}(\Gamma, F,\, \chi)$ in Equation (169) does not depend on $c > 0$ and we have*
$$P_{Gur}(\Gamma, F,\, \chi) = \delta_{\Gamma,\,F,\chi}\ .$$
*Let $W$ be a relatively compact open subset of $T^1M$ meeting the non-wandering set $\Omega\Gamma$, if $c > 0$ is large enough, then*
$$P_{Gur}(\Gamma, F,\, \chi) = \lim_{s\to+\infty}\ \frac{1}{s}\log Z_{\Gamma,\,F,\,\chi,\,W,\,c}(s)\ ,$$
*and if $P_{Gur}(\Gamma, F) > 0$, then*
$$P_{Gur}(\Gamma, F,\, \chi) = \lim_{s\to+\infty}\ \frac{1}{s}\log Z_{\Gamma,\,F,\,\chi,\,W}(s)\ .$$

**Proof.** (1) See the proof of Corollary 4.1 (using the twisted Patterson measure constructed in the coming Subsection 11.4 and the extension of Mohsen's shadow lemma in Proposition 11.1).

(2) See the proof of Theorems 4.3 and 4.2.

(3) See the proof of Corollaries 4.6 (1) and 4.5 (1), where Equation (66) should be replaced by
$$a_{t,\,U',\,z} = e^{-\chi(\gamma)}\ a_{t,\,\gamma U',\,\gamma z}\ ,$$
and the constant $c_2'$ in Equation (68) should be replaced by $c_2' + \max_{1\le i\le k}\ |\chi(\gamma_i)|$.

(4) See the proof of Corollaries 4.6 (2) and 4.5 (2), with adjustments similar to those above, in particular since now
$$b_{t,\,U',\,V',\,z,\,w} = e^{\chi(\alpha)}\ b_{t,\,U',\,\alpha V',\,z,\,\alpha w}\ .$$

(5) The fact that the twisted Gurevich pressure does not depend on $c > 0$ is proved as the first claim of Assertion (vii) of Lemma 3.3. For the other claims, see the proof of Theorem 4.7, using the fact that for all $\gamma, \gamma' \in \Gamma$, we have $\chi(\gamma) = \chi(\gamma')$ if $\gamma'\gamma^{-1}$ has finite order. □

## 11.4 Characters and Patterson densities

Let $\chi : \Gamma \to \mathbb{R}$ be a character of $\Gamma$.

For every $\sigma \in \mathbb{R}$, a *twisted Patterson density* of dimension $\sigma$ for $(\Gamma, F, \chi)$ is a family of finite nonzero (positive Borel) measures $(\mu_x)_{x\in\widetilde{M}}$ on $\partial_\infty\widetilde{M}$ such that, for every $\gamma \in \Gamma$, for all $x, y \in \widetilde{M}$, for every $\xi \in \partial_\infty\widetilde{M}$, we have
$$\gamma_*\mu_x = \ e^{-\chi(\gamma)}\ \mu_{\gamma x}\ , \tag{172}$$
$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_{F-\sigma,\,\xi}(x,y)}\ . \tag{173}$$
This definition is due to Babillot-Ledrappier [BaL] and Roblin [Rob3] when $F = 0$.

Note that if $\Gamma'$ is a nonelementary subgroup of $\operatorname{Ker}\chi$ (for instance $\Gamma' = \operatorname{Ker}\chi$), denoting by $F' : \Gamma'\backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$, then a twisted Patterson density of dimension

$\sigma$ for $(\Gamma, F, \chi)$ is a (standard) Patterson density of dimension $\sigma$ for $(\Gamma', F')$, and in particular $\sigma \geq \delta_{\Gamma', F'}$ by Corollary 3.11 (2).

Similarly as in Subsection 3.6, for every twisted Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\sigma$ for $(\Gamma, F, \chi)$, we have:

- $(\mu_x)_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension $\sigma + s$ for $(\Gamma, F + s, \chi)$, for every $s \in \mathbb{R}$.
- the support of $\mu_x$, which is independent of $x \in \widetilde{M}$, contains $\Lambda\Gamma$.

The twisted Patterson densities satisfy the following elementary properties, due to [Rob3] when $F = 0$.

**Proposition 11.10** *Let $\chi$ be a character of $\Gamma$.*

*(1) If $\delta_{\Gamma, F, \chi} < +\infty$, then there exists a twisted Patterson density of dimension $\delta_{\Gamma, F, \chi}$ for $(\Gamma, F, \chi)$, whose support is exactly $\Lambda\Gamma$.*

*(2) For every $\sigma \in \mathbb{R}$, if there exists a twisted Patterson density of dimension $\sigma$ for $(\Gamma, F, \chi)$, then $\sigma \geq \delta_{\Gamma, F, \chi}$.*

*(3) If $\Gamma'$ is a nonelementary normal subgroup of $\Gamma$, with $F' : \Gamma'\backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$, if $(\mu'_x)_{x\in\widetilde{M}}$ is a Patterson density of dimension $\sigma \in \mathbb{R}$ for $(\Gamma', F')$, which is ergodic for the action of $\Gamma'$ and quasi-invariant for the action of $\Gamma$, then there exists a character $\chi'$ of $\Gamma$, whose kernel contains $\Gamma'$, such that $(\mu'_x)_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension $\sigma$ for $(\Gamma, F, \chi')$.*

*(4) For every $\sigma \in \mathbb{R}$ such that there exists a twisted Patterson density of dimension $\sigma$ for $(\Gamma, F, \chi)$, and for all $x, y \in \widetilde{M}$, there exists $c > 0$ such that for every $n \in \mathbb{N}$, we have*

$$\sum_{\gamma\in\Gamma\,:\, n-1<d(x,\gamma y)\leq n} e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-\sigma)} \leq c\,.$$

*(5) Assume that $\delta_{\Gamma, F} < +\infty$, that $(\Gamma, F)$ is of divergence type and that $F$ is reversible. If $\delta_{\Gamma, F} = \delta_{\Gamma, F, \chi}$, then $\chi = 0$.*

*(6) Every twisted Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\sigma \in \mathbb{R}$ for $(\Gamma, F, \chi)$ satisfies the following "doubling property of shadows": For every compact subset $K$ of $\widetilde{M}$, for every $R > 0$ large enough, there exists $C = C(K, R) > 0$ such that for all $\gamma \in \Gamma$ and $x, y \in K$, we have*

$$\mu_x(\mathscr{O}_x B(\gamma y, 5R)) \leq C\, \mu_x(\mathscr{O}_x B(\gamma y, R))\,.$$

Note that the reversibility assumption of Assertion (5) is necessary, as pointed out by the referee: for instance if $\Gamma$ is torsion free, if $M$ is compact and if $\chi$ is a nontrivial character of $\Gamma$, let $\widetilde{F} = -\widetilde{\omega}_\chi$, with the notation of the remark above Proposition 11.8, which is not reversible since $F \circ \iota = -F$. Then $\delta_{\Gamma, F, 0} = \delta_{\Gamma, 0, -\chi}$ and $\delta_{\Gamma, F, 2\chi} = \delta_{\Gamma, 0, \chi}$ as seen in that remark, and $\delta_{\Gamma, 0, -\chi} = \delta_{\Gamma, 0, \chi}$ by Proposition 11.8 (ii). Hence $\delta_{\Gamma, F, 0} = \delta_{\Gamma, F, 2\chi}$, though $2\chi \neq 0$.

**Proof.** (1) As in [Rob3] when $F = 0$ and in [Moh] when $\chi = 0$, the construction is similar to Patterson's (see the proof of Proposition 3.9). For every $z \in \widetilde{M}$, let $\mathscr{D}_z$ be the unit Dirac mass at $z$. Let $h : [0, +\infty[\, \to\, ]0, +\infty[$ be a nondecreasing map such that

- for every $\epsilon > 0$, there exists $r_\epsilon \geq 0$ such that $h(t + r) \leq e^{\epsilon t} h(r)$ for all $t \geq 0$ and $r \geq r_\epsilon$;
- for all $x, y \in \widetilde{M}$, if $\overline{Q}_{x,y}(s) = \sum_{\gamma\in\Gamma} e^{\chi(\gamma)+\int_x^{\gamma y}(\widetilde{F}-s)}\, h(d(x, \gamma y))$, then $\overline{Q}_{x,y}(s)$ diverges if and only if $s \leq \delta_{\Gamma, F, \chi}$.

Such a map is easy to construct, by taking $\log h$ to be piecewise affine on $[0,+\infty[$, with positive slope tending to 0 near $+\infty$ (when the series $Q_{\Gamma,\,F,\,\chi,\,x,\,y}(s)$ diverges at $s=\delta_{\Gamma,\,F,\,\chi}$, we may take $h=1$ constant).

For $s>\delta=\delta_{\Gamma,\,F,\,\chi}$, let us define the measure

$$\mu_{x,\,s}=\frac{1}{\overline{Q}_{x_0,\,x_0}(s)}\sum_{\gamma\in\Gamma}\;e^{\chi(\gamma)+\int_x^{\gamma x_0}(\widetilde{F}-s)}\;h(d(x,x_0))\;\mathscr{D}_{\gamma x_0}$$

on $\widetilde{M}$. By the weak-star compactness of the space of probability measures on the compact space $\widetilde{M}\cup\partial_\infty\widetilde{M}$, there exists a sequence $(s_k)_{k\in\mathbb{N}}$ in $]\delta,+\infty[$ converging to $\delta$ such that the sequence of measures $(\mu_{x_0,\,s_k})_{k\in\mathbb{N}}$ weak-star converges to a measure $\mu_{x_0}$ on $\widetilde{M}\cup\partial_\infty\widetilde{M}$, with support contained in $\Lambda\Gamma$, hence equal to $\Lambda\Gamma$ by minimality. For every $x\in\widetilde{M}$, defining

$$d\mu_x(\xi)=e^{-C_{F-\delta,\,\xi}(x,\,x_0)}\;d\mu_{x_0}(\xi)\;,$$

it is easy to see that $(\mu_x)_{x\in\widetilde{M}}=\lim_{k\to+\infty}(\mu_{x,\,s_k})_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension $\delta$ for $(\Gamma,F,\chi)$, with support $\Lambda\Gamma$.

(2) Let $(\mu_x)_{x\in\widetilde{M}}$ be a twisted Patterson density of dimension $\sigma$ for $(\Gamma,F,\chi)$. Note that $\|\mu_{\gamma x_0}\|=e^{\chi(\gamma)}\|\mu_{x_0}\|$ by Equation (172). The result hence follows from the first claim of Corollary 11.3 applied by taking $(\Gamma_0,\Gamma)$ therein to be $(\Gamma,\operatorname{Ker}\chi)$.

(3) The proof is the same as the one when $F=0$ in [Rob3, Lem. 2.1.1 (c)]. Let $(\mu'_x)_{x\in\widetilde{M}}$ be as in the statement of Assertion (3). By Lemma 11.2 (applied with $\Gamma$ therein equal to $\Gamma'$), since $\Gamma$ normalises $\Gamma'$, for every $\alpha\in\Gamma$, the family $((\alpha^{-1})_*\mu'_{\alpha x})_{x\in\widetilde{M}}$ is also a Patterson density for $(\Gamma',F')$ of dimension $\sigma$. By assumption, for every $x\in\widetilde{M}$, the measure $(\alpha^{-1})_*\mu'_{\alpha x}$ is absolutely continuous with respect to $\mu'_x$. By ergodicity, the ($\mu'_x$-almost everywhere well defined) Radon-Nikodym derivative $\frac{d\,(\alpha^{-1})_*\mu'_{\alpha x}}{d\,\mu'_x}$ (which is $\Gamma'$-invariant and measurable) is $\mu'_x$-almost everywhere equal to a constant $c(\alpha)>0$. We have $c(\gamma')=1$ if $\gamma'\in\Gamma'$ and for all $\alpha,\beta\in\Gamma$, we have, for $\mu'_x$-almost every $\xi\in\partial_\infty\widetilde{M}$,

$$\begin{aligned}c(\alpha\beta)&=\frac{d((\alpha\beta)^{-1})_*\mu'_{\alpha\beta x}}{d\,\mu'_x}(\xi)=\frac{d(\alpha^{-1})_*\mu'_{\alpha\beta x}}{d\,\mu'_{\beta x}}(\beta\xi)\,\frac{d(\beta^{-1})_*\mu'_{\beta x}}{d\,\mu'_x}(\xi)\\&=\frac{d(\alpha^{-1})_*\mu'_{\alpha\beta x}}{d(\alpha^{-1})_*\mu'_{\alpha x}}(\beta\xi)\,\frac{d(\alpha^{-1})_*\mu'_{\alpha x}}{d\,\mu'_x}(\beta\xi)\,\frac{d\,\mu'_x}{d\,\mu'_{\beta x}}(\beta\xi)\;c(\beta)\\&=c(\alpha)c(\beta)\;,\end{aligned}$$

this last equality holding since the action of $\beta$ on $\partial_\infty\widetilde{M}$ preserves the measure class of $\mu'_x$ and since both $((\alpha^{-1})_*\mu'_{\alpha x})_{x\in\widetilde{M}}$ and $(\mu'_x)_{x\in\widetilde{M}}$ are Patterson densities for $(\Gamma',F')$ of dimension $\sigma$ (see Equation (173)).

Hence $\chi':\gamma\mapsto\log c(\gamma)$ is a character of $\Gamma$, trivial on $\Gamma'$, and $(\mu'_x)_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension $\sigma$ for $(\Gamma,F,\chi')$.

(4) The proof is similar to that when $\chi=0$ in Corollary 3.11 (1), using Proposition 11.1 instead of Mohsen's shadow lemma 3.10.

(5) The proof is similar to that when $F=0$ (see [Rob3, Lem. 2.1.2]). Let $\delta=\delta_{\Gamma,\,F}=\delta_{\Gamma,\,F,\,\chi}<+\infty$. Let $(\mu_x)_{x\in\widetilde{M}}$ be a twisted Patterson density of dimension $\delta=\delta_{\Gamma,\,F,\,\chi}$ for

$(\Gamma, F, \chi)$, which exists by Assertion (1). Let $(\mu^*_x)_{x\in\widetilde{M}}$ be a twisted Patterson density of dimension $\delta$ for $(\Gamma, F, -\chi)$, which exists by Assertion (1), since

$$\delta = \delta_{\Gamma,\,F,\,\chi} = \delta_{\Gamma,\,F\circ\iota,\,-\chi} = \delta_{\Gamma,\,F,\,-\chi}$$

using Proposition 11.8 (ii) for the second equality, and the reversibility of $F$ and Proposition 11.8 (ix) for the last equality. Let $(\mu^0_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\delta = \delta_{\Gamma,\,F}$ for $(\Gamma, F)$, which gives full measure to $\Lambda_c\Gamma$ by Corollary 5.12, since $(\Gamma, F)$ is of divergence type. Let $x_0 \in \widetilde{M}$, and let us normalise the above Patterson densities so that

$$\|\mu_{x_0}\| = \|\mu^*_{x_0}\| = \|\mu^0_{x_0}\| \;.$$

Then $\big(\nu_x = \frac{1}{2}(\mu_x + \mu^*_x)\big)_{x\in\widetilde{M}}$ is a Patterson density of dimension $\delta$ for $(\operatorname{Ker}\chi, F)$, such that, for every $\gamma\in\Gamma$,

$$\|\nu_{\gamma x_0}\| = \Big\|\frac{1}{2}\big(e^{\chi(\gamma)}\mu_{x_0} + e^{-\chi(\gamma)}\mu^*_{x_0}\big)\Big\| = \cosh\chi(\gamma)\;\|\mu^0_{x_0}\| \geq \|\mu^0_{x_0}\| = \|\mu^0_{\gamma x_0}\|\;.$$

By the Fatou-Roblin radial convergence theorem 11.6 applied with $(\Gamma_0, \Gamma)$ therein equal to $(\Gamma, \operatorname{Ker}\chi)$ and $\sigma = \delta$, and by Equation (168), we hence have $\nu_{x_0} \geq \frac{d\nu_{x_0}}{d\mu^0_{x_0}}\,\mu^0_{x_0} \geq \mu^0_{x_0}$. Since $\|\nu_{x_0}\| = \|\mu^0_{x_0}\|$, we therefore have $\nu_{x_0} = \mu^0_{x_0}$. Thus $\chi = 0$ by Equation (172) applied to $(\nu_x)_{x\in\widetilde{M}}$ and $x = x_0$, since $\gamma_*\mu^0_{x_0} = \mu^0_{\gamma x_0}$ for every $\gamma\in\Gamma$.

(6) The proof is similar to the proof of Proposition 3.12 (1), using Proposition 11.1 with $(\Gamma_0, \Gamma)$ therein replaced by $(\Gamma, \operatorname{Ker}\chi)$ (or more precisely its proof, which shows that $R(\sigma)$ may be taken as large as wanted, and that the constant $C(\sigma)$ appearing for the upper and lower bounds can be made independent, though depending on which $R(\sigma)$ is taken), instead of Mohsen's shadow lemma 3.10. $\Box$

**Remark.** Let $\widetilde{F}^* : T^1\widetilde{M} \to \mathbb{R}$ be a Hölder-continuous $\Gamma$-invariant map, which is cohomologous to $\widetilde{F}$ via the $\Gamma$-invariant map $\widetilde{G} : T^1\widetilde{M} \to \mathbb{R}$. Let $(\mu_x)_{x\in\widetilde{M}}$ be a twisted Patterson density of dimension $\sigma\in\mathbb{R}$ for $(\Gamma, F, \chi)$. Then the family of measures $(\mu^*_x)_{x\in\widetilde{M}}$ defined by, for every $\xi\in\partial_\infty\widetilde{M}$,

$$d\mu^*_x(\xi) = e^{\widetilde{G}(v_{x\xi})}\;d\mu_x(\xi)\;,$$

where $v_{x\xi}$ is the tangent vector at the origin of the geodesic ray from a point $x\in\widetilde{M}$ to $\xi$, is a twisted Patterson density of dimension also $\sigma$ for $(\Gamma, F^*, \chi)$.

## 11.5 Characters and the Hopf-Tsuji-Sullivan-Roblin theorem

Let $\chi$ be a character of $\Gamma$ and $\sigma\in\mathbb{R}$. Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ and $(\mu_x)_{x\in\widetilde{M}}$ be twisted Patterson densities of the same dimension $\sigma$ for $(\Gamma, F\circ\iota, -\chi)$ and $(\Gamma, F, \chi)$ respectively.

Using the Hopf parametrisation $v\mapsto(v_-, v_+, t)$ with respect to the base point $x_0$ and Equation (29), we define the *(twisted) Gibbs measure on $T^1\widetilde{M}$ associated with the pair of twisted Patterson densities* $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$ as the measure $\widetilde{m}$ on $T^1\widetilde{M}$ given by

$$\begin{aligned} d\,\widetilde{m}(v) &= \frac{d\mu^\iota_x(v_-)\,d\mu_x(v_+)\,dt}{D_{F-\sigma,\,x}(v_-, v_+)^2} \\ &= e^{C_{F\circ\iota-\sigma,\,v_-}(x,\,\pi(v)) + C_{F-\sigma,\,v_+}(x,\,\pi(v))}\;d\mu^\iota_x(v_-)\,d\mu_x(v_+)\,dt\;. \end{aligned}$$

The measure $\widetilde{m}$ on $T^1\widetilde{M}$ is independent of $x \in \widetilde{M}$ by the equations (173) and (25), hence is invariant under the action of $\Gamma$ by the equations (30) and (172) (the cancellation $e^{-\chi(\gamma)}e^{\chi(\gamma)} = 1$ is the reason why it is important that the Patterson density $(\mu^\iota_x)_{x\in\widetilde{M}}$ is twisted by the opposite character $-\chi$). It is invariant under the geodesic flow. Hence it defines a measure $m$ on $T^1M$ which is invariant under the quotient geodesic flow, called the *(twisted) Gibbs measure on $T^1M$ associated with the pair of twisted Patterson densities* $\big((\mu^\iota_x)_{x\in\widetilde{M}}, (\mu_x)_{x\in\widetilde{M}}\big)$.

The measure $\iota_*\widetilde{m}$ is the twisted Gibbs measure on $T^1\widetilde{M}$ associated with the switched pair of twisted Patterson densities $\big((\mu_x)_{x\in\widetilde{M}}, (\mu^\iota_x)_{x\in\widetilde{M}}\big)$ (and similarly on $T^1M$).

The following results have proofs similar to those of Theorem 5.3 and Theorem 5.4.

**Theorem 11.11** *The following assertions are equivalent.*

*(1) The twisted Poincaré series of $(\Gamma, F, \chi)$ at the point $\sigma$ converges: $Q_{\Gamma,F,\chi}(\sigma) < +\infty$.*

*(2) There exists $x \in \widetilde{M}$ such that $\mu_x(\Lambda_c\Gamma) = 0$.*

*(2') For every $x \in \widetilde{M}$, we have $\mu_x(\Lambda_c\Gamma) = 0$.*

*(3) There exists $x \in \widetilde{M}$ such that $\mu^\iota_x(\Lambda_c\Gamma) = 0$.*

*(3') For every $x \in \widetilde{M}$, we have $\mu^\iota_x(\Lambda_c\Gamma) = 0$.*

*(4) There exists $x \in \widetilde{M}$ such that the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is non-ergodic and completely dissipative.*

*(4') For every $x \in \widetilde{M}$, the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is non-ergodic and completely dissipative.*

*(5) The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is non-ergodic and completely dissipative.*

**Theorem 11.12** *The following assertions are equivalent.*

*(i) The twisted Poincaré series of $(\Gamma, F, \chi)$ at the point $\sigma$ diverges: $Q_{\Gamma,F,\chi}(\sigma) = +\infty$.*

*(ii) There exists $x \in \widetilde{M}$ such that $\mu_x({}^c\Lambda_c\Gamma) = 0$.*

*(ii') For every $x \in \widetilde{M}$, we have $\mu_x({}^c\Lambda_c\Gamma) = 0$.*

*(iii) There exists $x \in \widetilde{M}$ such that $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$.*

*(iii') For every $x \in \widetilde{M}$, we have $\mu^\iota_x({}^c\Lambda_c\Gamma) = 0$.*

*(iv) There exists $x \in \widetilde{M}$ such that the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is ergodic and conservative, and $\mu^\iota_x \otimes \mu_x$ is atomless.*

*(iv) For every $x \in \widetilde{M}$, the dynamical system $(\partial^2_\infty\widetilde{M}, \Gamma, (\mu^\iota_x \otimes \mu_x)_{|\partial^2_\infty\widetilde{M}})$ is ergodic and conservative, and $\mu^\iota_x \otimes \mu_x$ is atomless.*

*(v) The dynamical system $(T^1M, (\phi_t)_{t\in\mathbb{R}}, m)$ is ergodic and conservative.*

**Proof.** Let us give a few words on the proofs of these two results, besides referring to Subsection 5.2. We use Proposition 11.1 with $(\Gamma_0, \Gamma)$ therein replaced by $(\Gamma, \operatorname{Ker}\chi)$, coupled with Equation (166), instead of Mohsen's shadow lemma 3.10. In particular, using Proposition 11.10 (4) instead of Lemma 3.11 (1), Lemma 5.6 now becomes the following statement:

Let $x \in \widetilde{M}$, if $Q_{\Gamma,\,F,\,\chi}(\sigma) = +\infty$ and if $R$ is large enough, with $K = K(x, R)$, there exists $c_8' > 0$ such that for every sufficiently large $T > 0$

$$\int_0^T \int_0^T \Big( \sum_{\alpha,\,\beta\in\Gamma} \widetilde{m}(K \cap \phi_{-t}\alpha K \cap \phi_{-s-t}\beta K) \Big)\, ds\, dt \leq c_8' \Big( \sum_{\gamma\in\Gamma,\, d(x,\,\gamma x)\leq T} e^{\chi(\gamma)+\int_x^{\gamma x}(\widetilde{F}-\sigma)} \Big)^2 .$$

Similarly, Lemma 5.7 becomes:

Let $x \in \widetilde{M}$, if $Q_{\Gamma,\,F,\,\chi}(\sigma) = +\infty$ and if $R$ is large enough, with $K = K(x, R)$, there exists $c_9' > 0$ such that for every sufficiently large $T > 0$

$$\int_0^T \Big( \sum_{\gamma\in\Gamma} \widetilde{m}(K \cap \phi_{-t}\gamma K) \Big)\, dt \geq c_9' \sum_{\gamma\in\Gamma,\, d(x,\,\gamma x)\leq T} e^{\chi(\gamma)+\int_x^{\gamma x}(\widetilde{F}-\sigma)} .$$

To prove the analog of Assertion (g) in Proposition 5.5, we first use Proposition 11.10 (6) instead of Proposition 3.12 (1). Then, since in the twisted case we have $\|\mu_{\gamma x}\| = e^{\chi(\gamma)} \|\mu_x\|$ for every $\gamma \in \Gamma$, Equation (83) becomes

$$\begin{aligned} \frac{e^{-\chi(\gamma_n)}}{c}\, \mu_{\gamma_n x}(B \cap \mathscr{O}_x B(\gamma_n x, R)) &\leq \frac{\mu_x(B \cap \mathscr{O}_x B(\gamma_n x, R))}{\mu_x(\mathscr{O}_x B(\gamma_n x, R))} \\ &\leq c\, e^{-\chi(\gamma_n)}\, \mu_{\gamma_n x}(B \cap \mathscr{O}_x B(\gamma_n x, R)) . \end{aligned}$$

and one concludes as in the proof of Assertion (g) using this time the property (172) of the twisted Patterson densities.

Now the deduction of Theorem 5.3 and Theorem 5.4 from Proposition 5.5 remains valid. $\square$

As when $\chi = 0$ in Subsection 5.3, the following results are extensions of respectively Corollary 5.10, Corollary 5.12, Proposition 5.13 and Corollary 5.15, the last claim being an immediate consequence of its assertions (1) and (2) a).

**Corollary 11.13** *(1) Assume that $\delta_{\Gamma,\,F,\,\chi} < +\infty$ and that there exists a twisted Patterson density $(\mu_x)_{x\in X}$ for $(\Gamma, F, \chi)$ of dimension $\sigma \geq \delta_{\Gamma,\,F,\,\chi}$ such that $\mu_{x_0}(\Lambda_c\Gamma) > 0$, then $\sigma = \delta_{\Gamma,\,F,\,\chi}$ and $(\Gamma, F, \chi)$ is of divergence type.*

*(2) Assume that $\delta_{\Gamma,\,F,\,\chi} < +\infty$ and that $(\Gamma, F, \chi)$ is of divergence type.*

*a) There exists a twisted Patterson density $(\mu_x)_{x\in X}$ for $(\Gamma, F, \chi)$ of dimension $\delta_{\Gamma,\,F,\,\chi}$, which is unique up to a scalar multiple. The measures $\mu_x$ are ergodic with respect to $\Gamma$, give full measure to $\Lambda_c\Gamma$, and are atomless. In particular their support is $\Lambda\Gamma$.*

*b) If $\mathscr{D}_z$ is the unit Dirac mass at a point $z \in \widetilde{M}$, then for every $x \in \widetilde{M}$, we have*

$$\frac{\mu_x}{\|\mu_x\|} = \lim_{s\to\delta_{\Gamma,\,F,\,\chi}^{\;+}} \frac{1}{Q_{\Gamma,\,F,\,\chi,\,x,\,x}(s)} \sum_{\gamma\in\Gamma} e^{\chi(\gamma)+\int_x^{\gamma x}(\widetilde{F}-s)}\, \mathscr{D}_{\gamma x} . \tag{174}$$

*c) We have $\mu_x(\Lambda\Gamma - \Lambda_{\rm Myr}\Gamma) = 0$.*

*(3) Let $\sigma \in \mathbb{R}$. If there exists a twisted Gibbs measure of dimension $\sigma$ for $(\Gamma, F, \chi)$ which is finite on $T^1M$, then $\sigma = \delta_{\Gamma,\,F,\,\chi}$, the triple $(\Gamma, F, \chi)$ is of divergence type, there exists a unique (up to a scalar multiple) twisted Gibbs measure with potential $F$ on $T^1M$, its support is $\Omega\Gamma$, and the geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1M$ is ergodic with respect to $m_F$.*

*(4) If $\chi$ is a character of $\Gamma$, two twisted Patterson densities for $(\Gamma, F, \chi)$, giving full measure to $\Lambda_c\Gamma$, are proportional by a positive constant, and in particular have the same dimension.*

**Proof.** We only prove in Assertion (2) a) the claim that $\mu_x$ is atomless, to indicate how the extensions proceed.

Let $\sigma = \delta_{\Gamma,\,F,\,\chi} < +\infty$, which is equal to $\delta_{\Gamma,\,F\circ\iota,\,-\chi}$ by Proposition 11.8 (ii). As seen in the beginning of Subsection 11.4, if $\Gamma' = \operatorname{Ker}\chi$, denoting by $F' : \Gamma'\backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$, we have $\sigma \geq \delta_{\Gamma',\,F'}$.

Assume for a contradiction that $\xi \in \partial_\infty\widetilde{M}$ is an atom of $\mu_x$. Since $\mu_x$ gives full measure to the conical limit set by previous claim of Assertion (2) a), we have $\xi \in \Lambda_c\Gamma$, that is, there exists a sequence of elements $(\gamma_i)_{i\in\mathbb{N}}$ in $\Gamma$ such that $(\gamma_i\,x)_{i\in\mathbb{N}}$ converges to $\xi$ while staying at bounded distance from the geodesic ray $[x, \xi[$, which implies in particular that $\xi \in \mathscr{O}_x B(\gamma_i x, R)$ for any $R > 0$ large enough. Up to increasing $R$, by Proposition 11.1 applied by taking $(\Gamma_0, \Gamma)$ therein to be $(\Gamma, \operatorname{Ker}\chi)$ and with $K = \{x\}$, there exists $C > 0$ such that

$$\mu_x(\mathscr{O}_x B(\gamma_i x, R)) \leq C\,\|\mu_{\gamma_i x}\|\; e^{\int_x^{\gamma_i x}(\widetilde{F}-\sigma)} = C\; e^{\chi(\gamma_i)}\;\|\mu_x\|\; e^{\int_x^{\gamma_i x}(\widetilde{F}-\sigma)}$$

for all $i \in \mathbb{N}$, using Equation (172). Hence there exists $\epsilon > 0$ such that for all $i \in \mathbb{N}$, using Equation (16),

$$e^{\int_x^{\gamma_i^{-1}x}(\widetilde{F}\circ\iota-\sigma)} = e^{\int_x^{\gamma_i x}(\widetilde{F}-\sigma)} \geq \epsilon\; e^{-\chi(\gamma_i)}\,.$$

Note that $(\Gamma, F\circ\iota, -\chi)$ is of divergence type by Proposition 11.8 (ii). Let $(\mu^\iota_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma = \delta_{\Gamma,\,F\circ\iota,\,-\chi}$ for $(\Gamma, F\circ\iota, -\chi)$, which exists by the previous claims of Assertion (2) a). Again by Proposition 11.1, we have, for all $i \in \mathbb{N}$,

$$\begin{aligned}\mu^\iota_x(\mathscr{O}_x B(\gamma_i^{-1}x, R)) &\geq \frac{1}{C}\;\|\mu^\iota_{\gamma_i^{-1}x}\|\; e^{\int_x^{\gamma_i^{-1}x}(\widetilde{F}\circ\iota-\sigma)} = \frac{1}{C}\; e^{-\chi(\gamma_i^{-1})}\;\|\mu^\iota_x\|\; e^{\int_x^{\gamma_i^{-1}x}(\widetilde{F}\circ\iota-\sigma)}\\ &\geq \frac{\epsilon}{C}\;\|\mu^\iota_x\| > 0\,.\end{aligned}$$

Up to extracting a subsequence, the sequence $(\gamma_i^{-1}x)_{i\in\mathbb{N}}$ converges to $\eta \in \partial_\infty\widetilde{M}$, with $\mu^\iota_x(\{\eta\}) > 0$. In particular, the measure $\mu^\iota_x \otimes \mu_x$ has an atom at $(\eta, \xi)$, a contradiction to Theorem 11.12 (iv'). $\Box$

## 11.6 Galois covers and critical exponents

Let $\Gamma'$ be a nonelementary subgroup of $\Gamma$, and let $F' : \Gamma'\backslash T^1\widetilde{M} \to \mathbb{R}$ be the map induced by $\widetilde{F}$.

The aim of this subsection is to compare the critical exponents of $(\Gamma, F)$ and $(\Gamma', F')$. We saw in Lemma 3.3 that

$$\delta_{\Gamma',\,F'} \leq \delta_{\Gamma,\,F}\,. \tag{175}$$

The case of equality in Equation (175) is related to the amenability properties of the left quotient $\Gamma'\backslash\Gamma$, in the following sense.

We endow the left quotient $\Gamma'\backslash\Gamma$ with the discrete topology, and with the (right) action of $\Gamma$ by translations on the right. We denote by $R_\alpha : [\gamma] \mapsto [\gamma\alpha]$ the action of $\alpha \in \Gamma$ on $\Gamma'\backslash\Gamma$. A *right-invariant mean* $\mathfrak{m}$ on the left quotient $\Gamma'\backslash\Gamma$ is a linear map $\mathfrak{m} : \ell^\infty(\Gamma'\backslash\Gamma) \to \mathbb{R}$ such that $\mathfrak{m}(1) = 1$, $\mathfrak{m}(f) \geq 0$ if $f \geq 0$ and $\mathfrak{m}(f \circ R_\alpha) = \mathfrak{m}(f)$ for every $\alpha \in \Gamma$. The left quotient $\Gamma'\backslash\Gamma$ is *amenable* if it has a right-invariant mean. When $\Gamma'$ is normal in $\Gamma$, there are many equivalent definitions, see for instance [Gre, Pate]. For instance, if $\Gamma'$ is normal in $\Gamma$ and if the group $\Gamma'\backslash\Gamma$ is virtually solvable, then the left quotient $\Gamma'\backslash\Gamma$ is amenable.

When $F = 0$, the following result is due to [Rob3, Théo. 2.2.2]. We suspect that the normality assumption on $\Gamma'$ might not be necessary.

**Theorem 11.14** *If $F$ is reversible, if $\Gamma'$ is a nonelementary normal subgroup of $\Gamma$ and if $\Gamma'\backslash\Gamma$ is amenable, then $\delta_{\Gamma',\,F'} = \delta_{\Gamma,\,F}$.*

**Proof.** Let $\mathfrak{m} : \ell^\infty(\Gamma'\backslash\Gamma)$ be a right-invariant mean on the left quotient $\Gamma'\backslash\Gamma$. We may assume that $\delta' = \delta_{\Gamma',\,F'} < +\infty$, otherwise the result follows from Equation (175).

Let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\delta'$ for $(\Gamma', F')$, which exists as seen in Proposition 3.9. For every $\alpha \in \Gamma$, we claim that there exists a map $f_\alpha \in \ell^\infty(\Gamma'\backslash\Gamma)$ such that, for every $[\gamma] \in \Gamma'\backslash\Gamma$,

$$f_\alpha([\gamma]) = \log\|\mu_{\gamma\alpha x_0}\| - \log\|\mu_{\gamma x_0}\| \,.$$

Since $(\mu_x)_{x\in\widetilde{M}}$ is $\Gamma'$-equivariant, for every $\gamma \in \Gamma$, the value of $f_\alpha([\gamma])$ indeed depends only on the class of $\gamma$ in $\Gamma'\backslash\Gamma$. By Lemma 3.4 (1), there exists $c_1 > 0$ such that

$$\begin{aligned}|\, f_\alpha([\gamma])\,| &\leq \max_{\xi\in\partial_\infty\widetilde{M}} \big|\, C_{F-\delta',\,\xi}(\gamma\alpha x_0, \gamma x_0)\,\big| \\ &\leq c_1\, e^{d(\alpha x_0,\, x_0)} \;+\; d(\alpha x_0, x_0) \max_{\pi^{-1}(B(\alpha x_0,\, d(\alpha x_0,\, x_0)))} |\widetilde{F} - \delta'| \,.\end{aligned}$$

Hence $f_\alpha$ indeed belongs to $\ell^\infty(\Gamma'\backslash\Gamma)$.

For all $\alpha, \beta, \gamma \in \Gamma$, we have

$$f_{\alpha\beta}([\gamma]) = \log\|\mu_{\gamma\alpha\beta x_0}\| - \log\|\mu_{\gamma\alpha x_0}\| + \log\|\mu_{\gamma\alpha x_0}\| - \log\|\mu_{\gamma x_0}\| \,.$$

Therefore $f_{\alpha\beta} = f_\beta \circ R_\alpha + f_\alpha$, and the map $\chi : \Gamma \to \mathbb{R}$ defined by $\chi(\alpha) = \mathfrak{m}(f_\alpha)$ is a character of $\Gamma$.

For every $x \in \widetilde{M}$ and every continuous map $\varphi : \partial_\infty\widetilde{M} \to \mathbb{R}$, let $f_{x,\,\varphi} \in \ell^\infty(\Gamma'\backslash\Gamma)$ be the map defined by

$$f_{x,\,\varphi}([\gamma]) = \frac{1}{\|\mu_{\gamma x_0}\|} \int \varphi \, d\,(\gamma^{-1})_*\mu_{\gamma x} \,.$$

This map is well defined, and bounded by an argument similar to the above one. By Riesz's theorem, there exists a (positive Borel) measure $\nu_x$ on $\partial_\infty\widetilde{M}$ such that

$$\int \varphi \, d\nu_x = \mathfrak{m}(f_{x,\,\varphi})$$

for every continuous map $\varphi : \partial_\infty\widetilde{M} \to \mathbb{R}$. For all $x \in \widetilde{M}$, $\gamma \in \Gamma$, $\gamma' \in \Gamma'$ and $\varphi : \partial_\infty\widetilde{M} \to \mathbb{R}$ continuous, using the fact that $\Gamma'$ is normal in $\Gamma$, we have

$$\|\mu_{\gamma\gamma' x_0}\| = \|\mu_{(\gamma\gamma'\gamma^{-1})\gamma x_0}\| = \|(\gamma\gamma'\gamma^{-1})_*\mu_{\gamma x_0}\| = \|\mu_{\gamma x_0}\| \quad \text{and} \quad f_{\gamma' x,\,\varphi} = f_{x,\,\varphi\circ\gamma'} \circ R_{\gamma'} \,.$$

Using this and Lemma 11.2, it is then easy to check that $(\nu_x)_{x\in\widetilde{M}}$ is also a Patterson density of dimension $\delta'$ for $(\Gamma', F')$.

Since the proof of Jensen's inequality does not require the measure to be $\sigma$-additive, by the convexity of the exponential map, for every $f \in \ell^\infty(\Gamma'\backslash\Gamma)$, we have $e^{\mathfrak{m}(f)} \leq \mathfrak{m}(e^f)$. In particular, for every $\alpha \in \Gamma$, we have

$$e^{\chi(\alpha)} = e^{\mathfrak{m}(f_\alpha)} \leq \mathfrak{m}(e^{f_\alpha}) = \mathfrak{m}(f_{\alpha x_0,\,1}) = \|\nu_{\alpha x_0}\| \ .$$

By the first assertion of Corollary 11.3 applied by taking $(\Gamma_0, \Gamma, (\mu_x)_{\in\widetilde{M}})$ therein equal to $(\Gamma, \Gamma', (\nu_x)_{\in\widetilde{M}})$, we have $\delta' \geq \delta_{\Gamma,\,F,\,\chi}$. Since $F$ is reversible, we have $\delta_{\Gamma,\,F,\,\chi} \geq \delta_{\Gamma,\,F}$ by Equation (171). Hence $\delta' \geq \delta_{\Gamma,\,F}$, and the result follows from Equation (175). □

**Remark 11.15** The reversibility assumption on $F$ in Theorem 11.14 is necessary.

**Proof.** Let us construct examples which prove that Theorem 11.14 fails without the reversibility assumption on $F$. Assume that $\Gamma$ is torsion free and cocompact, and that there exists a nontrivial character $\chi$ of $\Gamma$. Let $\omega_\chi$ be the 1-form on $M$ corresponding to $\chi$, see the remark above Proposition 11.8. Let $\gamma \in \Gamma$ be such that $\chi(\gamma) > 0$: In particular, $\gamma$ is loxodromic. Let $x$ be a point on the translation axis of $\gamma$.

For every $N \in \mathbb{N}$, let $\widetilde{F} = N\,\widetilde{\omega}_\chi$. The Poincaré series of $\Gamma$ satisfies, for all $s \geq 0$,

$$Q_{\Gamma,\,F,\,x,\,x}(s) \geq \sum_{k\in\mathbb{N}} e^{\int_x^{\gamma^k x}(\widetilde{F}-s)} = \sum_{k\in\mathbb{N}} e^{k\int_x^{\gamma x}(N\,\widetilde{\omega}_\chi - s)} = \sum_{k\in\mathbb{N}} e^{k(N\,\chi(\gamma) - s\,\ell(\gamma))} \ ,$$

which diverges if $s \leq \frac{N\chi(\gamma)}{\ell(\gamma)}$. Hence $\delta_{\Gamma,\,F} \geq N\,\frac{\chi(\gamma)}{\ell(\gamma)}$.

If $\Gamma'$ is the kernel of $\chi$ and $F' : \Gamma'\backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$, then (using for instance the equality between the critical exponent and the Gurevich pressure of $(\Gamma', F')$ in Theorem 4.7), we have $\delta_{\Gamma',\,F'} = \delta_{\Gamma',\,0}$, which is independent of $N$. Taking $N$ big enough, we then have $\delta_{\Gamma,\,F} > \delta_{\Gamma',\,F'}$. Since the finitely generated group $\Gamma'\backslash\Gamma$ is abelian, hence amenable, this proves the result. □

A nonelementary normal subgroup $\Gamma'$ of $\Gamma$ cannot be too small. For instance, we have seen that the inclusion $\Lambda\Gamma' \subset \Lambda\Gamma$ is an equality. Similarly, the default of the inequality (175) to be an equality cannot be too large, even if $\Gamma'\backslash\Gamma$ is non-amenable. If $\widetilde{M}$ is a real hyperbolic space $\mathbb{H}^n_\mathbb{R}$ with $n \geq 3$ or a complex hyperbolic space $\mathbb{H}^n_\mathbb{C}$ with $n \geq 2$, and if $\Gamma$ is a lattice in $\mathrm{Isom}(\widetilde{M})$, then Shalom [Sha, Theo. 1.4] has proved some results of this type. When $F = 0$, the following result is due to [Rob3, Théo. 2.2.1].

**Theorem 11.16** *(1) If $\Gamma'$ is any nonelementary normal subgroup of $\Gamma$, then*

$$\delta_{\Gamma',\,F'} \geq \frac{1}{2}\,\delta_{\Gamma,\,F+F\circ\iota} \ .$$

*(2) If $F$ is reversible, if $\delta_{\Gamma,\,2F} < 2\ \delta_{\Gamma,\,F}$ and if $(\Gamma, 2F)$ is of divergence type, then $\delta_{\Gamma',\,F'} > \frac{1}{2}\,\delta_{\Gamma,\,2F}$.*

Note that the condition that $\delta_{\Gamma,\,F+F\circ\iota} < 2\ \delta_{\Gamma,\,F}$ is satisfied for instance if the upper bound $\sup_{\pi^{-1}(\mathscr{C}\Lambda\Gamma)} |F|$ is small enough, by Lemma 3.3 (iv).

**Proof.** Let $\delta = \delta_{\Gamma,\,F+F\circ\iota}$ and $\delta' = \delta_{\Gamma',\,F'}$. If $F$ is reversible, then $F+F\circ\iota$ is cohomologous to $2F$, hence $\delta_{\Gamma,\,F+F\circ\iota} = \delta_{\Gamma,\,2F}$ and $(\Gamma, 2F)$ is of divergence type if and only if $(\Gamma, F+F\circ\iota)$ is of divergence type (see the remark at the end of Subsection 3.2).

We may assume that $\delta' < +\infty$. Let $(\mu'_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\delta'$ for $(\Gamma', F')$, such that the dynamical system $(\partial_\infty\widetilde{M}, \Gamma', \mu'_{x_0})$ is ergodic, which exists as seen in Subsection 3.6. Let $R' = R_{\{x_0\}}(\delta')$ and $C' = C_{\{x_0\}}(\delta')$ be given by Proposition 11.1 applied by taking $(\Gamma_0, \Gamma, (\mu_x)_{x\in\widetilde{M}})$ therein equal to $(\Gamma, \Gamma', (\mu'_x)_{x\in\widetilde{M}})$.

(1) For every $y \in \Gamma x_0$, Proposition 11.1 gives that

$$\|\mu'_y\| \geq \mu'_y\big(\mathscr{O}_y B(x_0, R')\big) \geq \frac{\|\mu'_{x_0}\|}{C'}\; e^{\int_y^{x_0}(\widetilde{F}-\delta')}\;. \tag{176}$$

Hence for every $s > \delta'$, we have

$$\sum_{\gamma\in\Gamma} e^{\int_{x_0}^{\gamma x_0}(\widetilde{F}+\widetilde{F}\circ\iota-\delta'-s)} \leq \frac{C'}{\|\mu'_{x_0}\|}\sum_{\gamma\in\Gamma}\|\mu'_{\gamma x_0}\|\; e^{\int_{x_0}^{\gamma x_0}(\widetilde{F}-s)}\;.$$

By the first claim of Corollary 11.3, the sum on the right hand side converges. This proves that $\delta \leq 2\,\delta'$.

(2) Assume for a contradiction that $\delta' = \frac{\delta}{2}$. Since $(\Gamma, F+F\circ\iota)$ is of divergence type, let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\delta$ for $(\Gamma, F+F\circ\iota)$, which gives full measure to $\Lambda_c\Gamma$ (see Corollary 5.12). By Corollary 11.4 applied with $\Gamma_0 = \Gamma$, let $r \geq R'$ be large enough so that $\mu_{x_0}$ gives full measure to $\Lambda_{c,\,r}\Gamma$.

Let us first prove that $\mu_{x_0}$ is absolutely continuous with respect to $\mu'_{x_0}$. Let $B$ be any Borel subset of $\partial_\infty\widetilde{M}$, and let $V$ be any open neighbourhood of $B$. Let

$$Z = \{z \in \Gamma x_0 \;:\; \mathscr{O}_{x_0}B(z,r) \subset V\}\;,$$

and let $Z^*$ be the subset of $Z$ constructed in Lemma 11.5 with $s = r$. Using respectively

- the fact that $\Lambda_{c,\,r}\Gamma$ has full measure with respect to $\mu_{x_0}$,
- the fact that $B\cap\Lambda_{c,\,r}\Gamma$ is contained in $\bigcup_{z\in Z}\mathscr{O}_{x_0}B(z,r)$ and Lemma 11.5,
- Equation (166), which provides a constant $c > 0$ depending only on $R = 5r$ and $K = \{x_0\}$,
- the $\Gamma$-equivariance of $(\mu_x)_{x\in\widetilde{M}}$, which implies that $\|\mu_{\gamma x_0}\| = \|\mu_{x_0}\|$ for every $\gamma\in\Gamma$,
- Equation (176),
- the equality $\delta = 2\delta'$,
- Proposition 11.1 applied by taking $(\Gamma_0, \Gamma, (\mu_x)_{\in\widetilde{M}})$ therein equal to $(\Gamma, \Gamma', (\mu'_x)_{\in\widetilde{M}})$, since $r \geq R'$, and setting $c' = \frac{c\,(C')^2\,\|\mu_{x_0}\|}{\|\mu'_{x_0}\|}$,
- the properties of $Z^*$ seen in Lemma 11.5 and the definition of $Z$,

we have

$$\begin{aligned}
\mu_{x_0}(B) = \mu_{x_0}(B\cap\Lambda_{c,\,r}\Gamma) &\le \sum_{z\in Z^*}\mu_{x_0}\big(\mathscr{O}_{x_0}B(z,5r)\big)\\
&\le \sum_{z\in Z^*} c\,\|\mu_z\|\;e^{\int_{x_0}^{z}(\widetilde F+\widetilde F\circ\iota-\delta)} = c\,\|\mu_{x_0}\|\sum_{z\in Z^*} e^{\int_{x_0}^{z}(\widetilde F+\widetilde F\circ\iota-\delta)}\\
&\le c\,\|\mu_{x_0}\|\sum_{z\in Z^*}\frac{C'\|\mu'_z\|}{\|\mu'_{x_0}\|}\;e^{-\int_{z}^{x_0}(\widetilde F-\delta')}\;e^{\int_{x_0}^{z}(\widetilde F+\widetilde F\circ\iota-\delta)}\\
&= \frac{c\,C'\,\|\mu_{x_0}\|}{\|\mu'_{x_0}\|}\sum_{z\in Z^*}\|\mu'_z\|\;e^{\int_{x_0}^{z}(\widetilde F-\delta')}\\
&\le c'\sum_{z\in Z^*}\mu'_{x_0}\big(\mathscr{O}_{x_0}B(z,r)\big)\le c'\;\mu'_{x_0}(V)\;.
\end{aligned}$$

Since $V$ is an arbitrary neighbourhood of $B$, we hence have $\mu_{x_0}\le c'\;\mu'_{x_0}$, as required.

Now, the Radon-Nikodym derivative $\frac{d\,\mu_{x_0}}{d\,\mu'_{x_0}}$ is quasi-invariant under $\Gamma'$ and the measure $\mu_{x_0}$ is not the zero measure. By ergodicity of $(\partial_\infty\widetilde M,\Gamma',\mu'_{x_0})$, the map $\frac{d\,\mu_{x_0}}{d\,\mu'_{x_0}}$ is hence positive $\mu'_{x_0}$-almost everywhere, so that $\mu_{x_0}$ and $\mu'_{x_0}$ are equivalent. Therefore $\mu'_{x_0}$ is quasi-invariant under $\Gamma$, as is $\mu_{x_0}$. By Proposition 11.10 (3), there exists a character $\chi$ of $\Gamma$ such that $(\mu'_x)_{x\in\widetilde M}$ is a twisted Patterson density of dimension $\delta'$ for $(\Gamma,F,\chi)$. Hence $\delta'\ge\delta_{\Gamma,\,F,\,\chi}$ by Proposition 11.10 (2). Since $F$ is reversible, we have $\delta_{\Gamma,\,F,\,\chi}\ge\delta_{\Gamma,\,F}$ by Equation (171). Since $\delta_{\Gamma,\,F}>\frac{\delta}{2}$ by assumption, we have $\delta'>\frac{\delta}{2}$, a contradiction. □

## 11.7 Classification of ergodic Patterson densities on nilpotent covers

Let $\Gamma'$ be a normal subgroup of $\Gamma$, and let $F':\Gamma'\backslash T^1\widetilde M\to\mathbb R$ be the map induced by $\widetilde F$.

We start by recalling the definition of a nilpotent group. Let $G$ be a group. If $H$ and $H'$ are subgroups of $G$, let $[H,H']$ be the subgroup of $G$ generated by the elements $[h,h']=hh'h^{-1}h'^{-1}$ with $h\in H$ and $h'\in H'$. The *lower central series* of a group $G$ is the sequence of normal subgroups $(G_n)_{n\in\mathbb N}$ defined by induction by

$$G_0=G\quad\text{and}\quad G_{n+1}=[G,G_n]\;\text{ for }\;n\in\mathbb N\,.$$

The group $G$ is *nilpotent* if there exists $n\in\mathbb N$ such that $G_n=\{e\}$. It is well known that a nilpotent group is in particular amenable.

Note that some fundamental groups $\Gamma$ of compact connected hyperbolic orbifolds $M$, as hyperbolic triangle groups, have no nontrivial character, but contain many (non convex-cocompact, so that Proposition 5.11 (2) applies) normal subgroups $\Gamma'$, hence some restriction on the quotient $\Gamma/\Gamma'$ in the following result is needed to be able to construct characters.

**Theorem 11.17** *Let $\Gamma'$ be a normal subgroup of $\Gamma$ such that $\Gamma/\Gamma'$ is nilpotent. Assume that $\delta_{\Gamma,\,F}<+\infty$. For every $\sigma\in[\delta_{\Gamma,\,F},+\infty[$, the set of Patterson densities of $(\Gamma',F')$ of dimension $\sigma$, giving full measure to $\Lambda_c\Gamma$ and ergodic with respect to the action of $\Gamma'$, is equal to the set of twisted Patterson densities of $(\Gamma,F,\chi)$ of dimension $\sigma$ for the characters $\chi$ of $\Gamma$ vanishing on $\Gamma'$ such that $\delta_{\Gamma,\,F,\,\chi}=\sigma$, giving full measure to $\Lambda_c\Gamma$.*

**Proof.** As usual with nilpotent groups, the proof proceeds by induction on the nilpotency order. By taking the preimages of the terms of the lower central series of $\Gamma/\Gamma'$, there exist $n \in \mathbb{N}$ and subgroups $\Gamma_k$ of $\Gamma$ for $k = 0, \dots, n$ such that $\Gamma_0 = \Gamma, \Gamma_n = \Gamma'$, $\Gamma_{k+1}$ is a normal subgroup of $\Gamma_k$ and $\Gamma_{k+1}/\Gamma' = [\Gamma/\Gamma', \Gamma_k/\Gamma']$ for $k = 0, \dots, n-1$. We denote by $F_k : \Gamma_k \backslash T^1\widetilde{M} \to \mathbb{R}$ the map induced by $\widetilde{F}$. Let us fix $\sigma \geq \delta_{\Gamma,\, F}$.

If $n = 0$, then $\Gamma = \Gamma'$ and the only character of $\Gamma$ vanishing on $\Gamma'$ is $\chi = 0$. The result is then immediate, as we have seen in Corollary 5.10 that if there exists a Patterson density $(\mu_x)_{x\in X}$ for $(\Gamma, F)$ of dimension $\sigma \geq \delta_{\Gamma,\, F}$ such that $\mu_x(\Lambda_c\Gamma) > 0$, then $\sigma = \delta_{\Gamma,\, F}$ and $(\Gamma, F)$ is of divergence type, hence the measures $\mu_x$ are ergodic for the action of $\Gamma$ by Theorem 5.4. Therefore we assume that $n \geq 1$.

**Step 1.** There exists $c > 0$ such that for every $k \geq 1$, for every Patterson density $(\mu_x)_{x\in\widetilde{M}}$ of dimension $\sigma$ for $(\Gamma_k, F_k)$, giving full measure to $\Lambda_c\Gamma$, and for all $\alpha \in \Gamma_{k-1}$, we have

$$(\alpha^{-1})_*\mu_{\alpha x_0} \geq c\, e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}\circ\iota - \sigma)}\ \mu_{x_0}\ .$$

**Proof.** Let $c = \frac{1}{C_{\{x_0\}}(\sigma)} > 0$, with the notation of Proposition 11.1. For all $\gamma \in \Gamma$ and $\alpha \in \Gamma_{k-1}$, since $[\alpha, \gamma] \in [\Gamma_{k-1}, \Gamma]$ belongs to $\Gamma_k$, we have, by this proposition,

$$\begin{aligned}\|(\alpha^{-1})_*\mu_{\alpha\gamma x_0}\| = \|\mu_{\alpha\gamma x_0}\| = \|\mu_{\gamma\alpha x_0}\| &\geq \frac{1}{C_{\{x_0\}}(\sigma)}\ \|\mu_{\gamma x_0}\|\ e^{\int_{\gamma\alpha x_0}^{\gamma x_0}(\widetilde{F}-\sigma)}\\ &= c\ \|\mu_{\gamma x_0}\|\ e^{\int_{x_0}^{\alpha x_0}(\widetilde{F}\circ\iota-\sigma)}\ .\end{aligned}$$

By Lemma 11.2 since $\Gamma_{k-1}$ normalises $\Gamma_k$, and by the Fatou-Roblin radial convergence theorem 11.6 applied by replacing $(\Gamma_0, \Gamma)$ therein by $(\Gamma, \Gamma_{k-1})$, this proves the first step. □

**Step 2.** Let $\mu = (\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension $\sigma$ for $(\Gamma', F')$, giving full measure to $\Lambda_c\Gamma$, and ergodic for the action of $\Gamma'$. Then there exists a character $\chi$ of $\Gamma$, vanishing on $\Gamma'$, such that $\mu$ is a twisted Patterson density of dimension $\sigma$ for $(\Gamma, F, \chi)$.

**Proof.** Let us first prove that, for $k \in \{2, \dots, n\}$, if $\mu$ is a Patterson density for $(\Gamma_k, F_k)$, then $\mu$ is a Patterson density for $(\Gamma_{k-1}, F_{k-1})$.

Since $\mu_{x_0}$ is ergodic under $\Gamma'$ hence under $\Gamma_k$, and since $\mu_{x_0}$ is absolutely continuous with respect to $(\alpha^{-1})_*\mu_{\alpha x_0}$ for every $\alpha \in \Gamma_{k-1}$ by Step 1, there exists (by Proposition 11.10 (3) with $(\Gamma, \Gamma')$ therein replaced by $(\Gamma_{k-1}, \Gamma_k)$) a character $\chi_{k-1}$ of $\Gamma_{k-1}$, trivial on $\Gamma_k$, such that $\mu_{x_0} = e^{-\chi_{k-1}(\alpha)}\ (\alpha^{-1})_*\mu_{\alpha x_0}$ for every $\alpha \in \Gamma_{k-1}$. For every $\alpha \in \Gamma_{k-1}$, we have

$$\chi_{k-1}(\alpha) = \log\|\mu_{\alpha x}\| - \log\|\mu_x\|$$

for every $x \in \widetilde{M}$ (by the definition of $\chi_{k-1}$ for $x = x_0$ and by Lemma 11.2 and Equation (39) to extend to any $x$).

Since $\Gamma/\Gamma'$ is amenable, let $\mathfrak{m}$ be a right-invariant mean on $\Gamma'\backslash\Gamma = \Gamma/\Gamma'$. As seen in the proof of Theorem 11.14, for every $\alpha \in \Gamma$, the map $f_\alpha$ which associates to $[\gamma] \in \Gamma/\Gamma'$ the real number $\log\|\mu_{\gamma\alpha x_0}\| - \log\|\mu_{\gamma x_0}\|$ is well defined and bounded, and the map $\chi : \Gamma \to \mathbb{R}$ defined by $\chi(\alpha) = \mathfrak{m}(f_\alpha)$ is a character of $\Gamma$. This character hence coincides with $\chi_{k-1}$ on $\Gamma_{k-1}$, by applying the above centred formula to $x = \gamma x_0$ for every $\gamma \in \Gamma$ and since, as seen in the proof of Step 1, $\|\mu_{\gamma\alpha x_0}\| = \|\mu_{\alpha\gamma x_0}\|$ for every $\gamma \in \Gamma$. Since $\Gamma_{k-1}/\Gamma' \subset [\Gamma/\Gamma', \Gamma_{k-2}/\Gamma']$, the additive character $\chi$ vanishes on $\Gamma_{k-1}$, hence $\chi_{k-1} = 0$. This proves the preliminary claim.

Now, by induction, we hence have that $\mu$ is a Patterson density for $(\Gamma_1, F_1)$. Applying again Step 1 gives that $\mu$ is quasi-invariant under $\Gamma$. Applying Proposition 11.10 (3) (using for this the ergodicity of $\mu_{x_0}$ under $\Gamma'$ hence under $\Gamma_1$), the second step follows. $\Box$

**Step 3.** If $\chi$ is a character of $\Gamma$ vanishing on $\Gamma'$, if $(\mu_x)_{x\in\widetilde{M}}$ is a twisted Patterson density of dimension $\sigma$ for $(\Gamma, F, \chi)$, giving full measure to $\Lambda_c\Gamma$, then $\sigma = \delta_{\Gamma,\,F,\,\chi}$ and $\mu_{x_0}$ is ergodic for the action of $\Gamma'$.

**Proof.** Let $A$ be a Borel subset of $\Lambda_c\Gamma$ invariant under $\Gamma'$, with nonzero $\mu_{x_0}$-measure, and let $\mathbb{1}_A$ be its characteristic function. Then $(\mathbb{1}_A\,\mu_x)_{x\in\widetilde{M}}$ is a Patterson density of dimension $\sigma$ for $(\Gamma', F')$, giving full measure to $\Lambda_c\Gamma$. The preliminary claim in the proof of Step 2 shows that $(\mathbb{1}_A\,\mu_x)_{x\in\widetilde{M}}$ is also a Patterson density of dimension $\sigma$ for $(\Gamma_1, F_1)$. By Step 1, $\mathbb{1}_A\,\mu_x$ is quasi-invariant under $\Gamma$, hence $A$ is invariant up to a set of measure 0 under $\Gamma$. In particular, $(\mathbb{1}_A\,\mu_x)_{x\in\widetilde{M}}$ is also a twisted Patterson density of dimension $\sigma$ for $(\Gamma, F, \chi)$, giving full measure to $\Lambda_c\Gamma$. By their uniqueness property seen in Corollary 11.13 (1) and (2), we have $\sigma = \delta_{\Gamma,\,F,\,\chi}$ and there exists $c > 0$ such that $\mathbb{1}_A\,\mu_{x_0} = c\,\mu_{x_0}$. In particular $\mu_{x_0}({}^c\!A) = 0$, which proves the ergodicity of $\mu_{x_0}$ under $\Gamma'$. $\Box$

Step 2 and Step 3 combine together to prove Theorem 11.17. $\Box$

The following result is then immediate. The assertions (2) and (3) of Theorem 1.10 in the introduction follow from it and from Corollary 11.13.

**Corollary 11.18** *Let $\Gamma'$ be a normal subgroup of $\Gamma$ such that $\Gamma/\Gamma'$ is nilpotent.*

*(1) Assume that $\delta_{\Gamma,\,F} < +\infty$, that $(\Gamma, F)$ is of divergence type and that $F$ is reversible. Then the unique (up to a scalar multiple) Patterson density $(\mu_{\Gamma,\,F,\,x})_{x\in\widetilde{M}}$ of dimension $\delta_{\Gamma,\,F}$ for $(\Gamma, F)$ is also the unique (up to a scalar multiple) Patterson density of dimension $\delta_{\Gamma,\,F}$ for $(\Gamma', F')$ giving full measure to $\Lambda_c\Gamma$.*

*(2) If $\Gamma$ is convex-cocompact, then the set of ergodic Patterson densities for $(\Gamma', F')$ with support $\Lambda\Gamma$ is the set of twisted Patterson densities of $(\Gamma, F, \chi)$ of dimension $\delta_{\Gamma,\,F,\,\chi}$ with support $\Lambda\Gamma$ for the characters $\chi$ of $\Gamma$ vanishing on $\Gamma'$.*

**Proof.** (1) By Corollary 5.12, there exists a unique (up to a scalar multiple) Patterson density $\mu = (\mu_{\Gamma,\,F,\,x})_{x\in\widetilde{M}}$ of dimension $\sigma = \delta_{\Gamma,\,F}$ for $(\Gamma, F)$, and it gives full measure to $\Lambda_c\Gamma$. By restriction, $\mu$ is also a Patterson density of dimension $\sigma$ for $(\Gamma', F')$ giving full measure to $\Lambda_c\Gamma$.

Let $\nu = (\nu_x)_{x\in\widetilde{M}}$ be another Patterson density of dimension $\sigma$ for $(\Gamma', F')$ giving full measure to $\Lambda_c\Gamma$. Assume furthermore that $\nu$ is ergodic with respect to the action of $\Gamma'$. By Theorem 11.17, there exists a character $\chi$ of $\Gamma$ such that $\nu$ is a twisted Patterson density for $(\Gamma, F, \chi)$ of dimension $\delta_{\Gamma,\,F,\,\chi} = \sigma$. By Proposition 11.10 (5) whose assumptions are satisfied, we have $\chi = 0$. Hence $\nu$ is a scalar multiple of $\mu$. By Krein-Milman's theorem, the ergodicity assumption on $\nu$ may be dropped. This proves Assertion (1).

(2) Since $\Lambda_c\Gamma = \Lambda\Gamma$ when $\Gamma$ is convex-cocompact, Assertion (2) follows immediately from Theorem 11.17. $\Box$

We refer to [Rob3] for possible improvements of the second assertion when $\Gamma$ is only assumed to be geometrically finite (with $\delta_{\Gamma,\,F} < +\infty$).

It follows from the first assertion that not only the dynamical system $(\partial_\infty \widetilde{M}, \Gamma, \mu_{\Gamma,\,F,\,x})$ is ergodic, but that the dynamical system $(\partial_\infty \widetilde{M}, \Gamma', \mu_{\Gamma,\,F,\,x})$ is also ergodic. This extends results of Babillot-Ledrappier [BaL] when $F = 0$ and $\Gamma'/\Gamma$ is isomorphic to $\mathbb{Z}^d$ for $d \in \mathbb{N}$, of Hamenstädt [Ham4] when $M$ is compact, and of Roblin [Rob3] when $F = 0$ .

**Remark.** Let $\Gamma'$ be a subgroup of $\Gamma$. Assume that $\delta_{\Gamma,\,F} < +\infty$ and that $(\Gamma, F)$ is of divergence type. Let $\widetilde{m}_F$ be the Gibbs measure of $(\Gamma,\, F)$ on $T^1\widetilde{M}$ (which is unique up to a scalar multiple). Let $\phi = (\phi_t)_{t\in\mathbb{R}}$ be the geodesic flow on the (orbifold) cover $M' = \Gamma'\backslash\widetilde{M}$ of $M$ defined by the group $\Gamma'$. Let $m'_F$ be the measure induced on $T^1M' = \Gamma'\backslash T^1\widetilde{M}$ by $\widetilde{m}_F$. Note that if $\Gamma'$ has infinite index in $\Gamma$, the Gibbs measure $m'_F$ is infinite.

Even if $\Gamma'$ is normal, if $\Gamma'\backslash\Gamma$ is nilpotent and if $m_F$ is finite, the dynamical system

$$(T^1M', \phi, m'_F)$$

is not necessarily ergodic: the ergodicity of $(\partial_\infty \widetilde{M}, \Gamma', \mu_{\Gamma,\,F,\,x})$ does not imply the ergodicity of $(T^1M', \phi, m'_F)$. See [Ree] for examples.

A necessary and sufficient condition on the left quotient $\Gamma'\backslash\Gamma$ for $(T^1M', \phi, m'_F)$ to be ergodic is still unknown (see the end of the introduction for more details on open problems).

## List of Symbols

$\lfloor x \rfloor$ integral part of $x \in \mathbb{R}$. 14

$[a, b, c, d]_F$ crossratio of pairwise distinct $a, b, c, d \in \widetilde{M}$ with respect to $F$. 41

$\mathbb{1}_A$ characteristic function of a subset $A$. 14

$\mathrm{Axe}_\gamma$ translation axis of an isometry $\gamma$ of $\widetilde{M}$. 16

$\beta_\xi(x, y)$ Busemann cocycle. 17

$B^{\rm ss}(v, r)$ Ball of center $v$ and radius $r$ for the Hamenstädt distance on $W^{\rm ss}(v)$. 23

$\mathscr{B}^{\rm ss}(v, r)$ Ball of center $v$ and radius $r$ for the induced Riemannian metric on $W^{\rm ss}(v)$. 23

$B^{\rm su}(v, r)$ Ball of center $v$ and radius $r$ for the Hamenstädt distance on $W^{\rm su}(v)$. 23

$\mathscr{B}^{\rm su}(v, r)$ Ball of center $v$ and radius $r$ for the induced Riemannian metric on $W^{\rm su}(v)$. 23

$C_{F,\xi}(x, y)$ Gibbs cocycle for the potential $F$. 34

${}^cA$ complementary set of a subset $A$. 14

$\mathscr{C}_x B$ cone on $B \subset \partial_\infty \widetilde{M}$ with vertex $x \in \widetilde{M} \cup \partial_\infty \widetilde{M}$. 16

$\mathscr{C}^+_r(z, Z)$ $r$-thickened cone over $Z \subset \partial_\infty \widetilde{M}$ with vertex $z \in \widetilde{M}$. 90

$\mathscr{C}^-_r(z, Z)$ $r$-thinned cone over $Z \subset \partial_\infty \widetilde{M}$ with vertex $z \in \widetilde{M}$. 90

$\mathscr{C}\Lambda\Gamma$ convex hull of the limit set of $\Gamma$. 16

$\delta_\Gamma$ critical exponent of $\Gamma$. 30

$\delta_{\Gamma,F}$ critical exponent of $(\Gamma, F)$. 30

$\delta_{\Gamma,F,\chi}$ (twisted) critical exponent of $(\Gamma, F, \chi)$. 8, 203

$\partial_\infty \widetilde{M}$ boundary at infinity of $\widetilde{M}$. 15

$\partial^2_\infty \widetilde{M}$ space of ordered pairs of distinct points of $\partial_\infty \widetilde{M}$. 18

$d$ distance on $\widetilde{M}$ and $T^1\widetilde{M}$, and on any metric space. 15, 19

$d'$ another distance on $T^1\widetilde{M}$. 19

$d_x$ visual distance on $\partial_\infty \widetilde{M}$ seen from $x \in M$. 15

$d_{W^{\rm ss}(w)}$ Hamenstädt's distance on the strong stable leaf of $w \in T^1\widetilde{M}$. 23

$d_{W^{\rm su}(w)}$ Hamenstädt's distance on the strong unstable leaf of $w \in T^1\widetilde{M}$. 21

$K^-(z,r,Z)$ set of $v \in T^1\widetilde{M}$ such that $v_- \in Z$ and $v$ is $r$-close to $z$. 90

$K^+(z,r,Z)$ set of $v \in T^1\widetilde{M}$ such that $v_+ \in Z$ and $v$ is $r$-close to $z$. 90

$\Lambda\Gamma$ limit set of $\Gamma$. 16

$\Lambda_c\Gamma$ conical (or radial) limit set of $\Gamma$. 17

$\Lambda_{c,\,r}\Gamma$ filtration of the conical limit set of $\Gamma$. 200

$\Lambda_{\rm Myr}\Gamma$ Myrberg limit set of $\Gamma$. 17

$\ell(\gamma)$ translation length of $\gamma \in \operatorname{Isom}(\widetilde{M})$. 16

$\mathscr{L}_g$ Lebesgue measure along a periodic orbit $g$. 65

$L_r(z,w)$ set of $(\xi,\eta) \in (\partial_\infty\widetilde{M})^2$ such that $]\xi,\eta[$ meets first $B(z,r)$ then $B(w,r)$. 90

log natural logarithm (with $\log(e) = 1$). 14

$\widetilde{M}$ pinched negatively curved complete simply connected Riemannian manifold. 14

$M$ Riemannian orbifold $M = \Gamma\backslash\widetilde{M}$. 18

$\widetilde{m}_F$ Gibbs measure on $T^1\widetilde{M}$ with potential $F$. 50

$m_F$ Gibbs measure on $T^1M$ with potential $F$. 50

$\mathscr{N}_r A$ open $r$-neighbourhood of $A$. 16

$\mathscr{N}_r^- A$ open $r$-interior of $A$. 16

$N(\Gamma)$ normaliser of $\Gamma$ in $\operatorname{Isom}(\widetilde{M})$. 197

$\widetilde{\Omega}\Gamma$ set of $v \in T^1\widetilde{M}$ such that $v_-, v_+ \in \Lambda\Gamma$. 18

$\widetilde{\Omega}_c\Gamma$ set of $v \in T^1\widetilde{M}$ such that $v_-, v_+ \in \Lambda_c\Gamma$. 18

$\Omega\Gamma$ nonwandering set of the geodesic flow in $T^1M$. 18

$\Omega_c\Gamma$ two-sided recurrent set of the geodesic flow in $T^1M$. 18

$\mathscr{O}_x A$ shadow of $A \subset \widetilde{M}$ seen form $x \in \widetilde{M} \cup \partial_\infty\widetilde{M}$. 16

$\phi_t$ geodesic flow at time $t \in \mathbb{R}$ on $T^1\widetilde{M}$ and on $T^1M$. 20

$\pi$ base point projections $T^1\widetilde{M} \to \widetilde{M}$ and $T^1M \to M$. 18

$P_{Gur}(\Gamma,\, F)$ Gurevich pressure of $(\Gamma, F)$. 66

$P_{Gur}(\Gamma,\, F,\, \chi)$ (twisted) Gurevich pressure of $(\Gamma, F, \chi)$. 204

$P_{\Gamma,\,F}(m)$ metric pressure of a measure $m$. 106

# Index

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