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ESCAPE OF MASS IN HOMOGENEOUS DYNAMICS IN POSITIVE CHARACTERISTIC

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Abstract. We show that in positive characteristic the homogeneous probability measure supported on a periodic orbit of the diagonal group in the space of 2-lattices, when varied along rays of Hecke trees, may behave in sharp contrast to the zero characteristic analogue: For a large set of rays, the measures fail to converge to the uniform probability measure on the space of 2-lattices. More precisely, we prove that when the ray is rational there is uniform escape of mass, that there are uncountably many rays giving rise to escape of mass, and that there are rays along which the measures accumulate on measures which are not absolutely continuous with respect to the uniform measure on the space of 2-lattices.

1. Introduction

This paper deals with the study of the positive characteristic analogue (which turns out to have a surprisingly different outcome) of the discussion in [1], thus we start this introduction by briefly recalling it.

The space $X = \mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z})$ may be identified with the space of homothety classes of ($\mathbb{Z}$-)lattices in the plane $\mathbb{R}^2$. If we denote by A the diagonal subgroup of $\mathrm{PGL}_2(\mathbb{R})$, then the A-orbits in X constitute a very important object of study both from the dynamical point of view and because of their tight relation to geometry and number theory. Among these orbits, the periodic ones (recall that A is isomorphic to $\mathbb{R}$), may be considered as most important. Given a periodic orbit Ax in X, we denote by μ_x the unique A-invariant probability measure supported on it.

Fixing a prime $p \in \mathbb{Z}$ and a homothety class of a lattice $x \in X$, one looks at the countable subset $\mathscr{G}_p(x)$ of elements $y \in X$ such that there exist $k \in \mathbb{N}$ and $\Lambda_x \in x, \Lambda_y \in y$ with $\Lambda_y \subset \Lambda_x$ and $[\Lambda_x : \Lambda_y] = p^k$. Upon drawing an edge between $y, z \in \mathscr{G}_p(x)$ if and only if there exist representatives $\Lambda_y \in y$ and $\Lambda_z \in z$ such that $\Lambda_z \subset \Lambda_y$ and $[\Lambda_y : \Lambda_z] = p$, one endows $\mathscr{G}_p(x)$ with a graph structure which is in fact a $(p+1)$-regular tree known as the *p-Hecke tree* through x. The edge structure (which is of arithmetic origin) allows one to talk about *Hecke rays*

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starting from x in $\mathscr{G}_p(x)$. These are simply sequences of homothety classes of lattices $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{G}_p(x)$ with no repetitions such that $x_0 = x$ and such that there is an edge between x_n and x_{n+1} for all $n \in \mathbb{N}$.

It turns out that if Ax is periodic, then Ay is periodic for any $y \in \mathscr{G}_p(x)$ and thus it is natural to consider the possible weak-star limits of the periodic measures μ_{x_n}, where $(x_n)_{n\in\mathbb{N}}$ is a Hecke ray. Theorem 4.8 from [1] states that for all but potentially two special rays, the sequence $(\mu_{x_n})_{n\in\mathbb{N}}$ equidistributes in X; that is, it weak-star converges to the unique $\mathrm{PGL}_2(\mathbb{R})$-invariant probability measure on X. For the problematic two rays (which exist only when the prime p splits in a quadratic extension of $\mathbb{Q}$ which corresponds to the periodic orbit Ax), the collection of measures $(\mu_{x_n})_{n\in\mathbb{N}}$ is finite and, so, nothing interesting dynamically is happening. In particular, what we wish to stress for our analogy with the positive characteristic case studied here are the following facts: (i) the sequences $(\mu_{x_n})_{n\in\mathbb{N}}$ cannot exhibit escape of mass (i.e., they cannot accumulate on a measure giving the space total mass strictly less than 1); (ii) there is a natural notion of rationality of Hecke rays and since the (potentially existing) two Hecke rays which do not give rise to equidistribution are never rational, one has that rational rays always give rise to equidistribution.

In the current paper which deals with the positive characteristic analogue of the above discussion, the picture is in sharp contrast. As will be demonstrated, all rational rays exhibit a uniform amount of escape of mass (conjectured to be full), there are uncountably many rays which exhibit escape of mass, and further more, there are uncountably many rays for which the periodic measures of the ray accumulate on measures which are singular to the uniform measure.

We now abandon the above notation and present the notation and concepts necessary for stating our results. Let $\mathbb{F}_q$ be a finite field of order a positive power q of a prime p, and let $K = \mathbb{F}_q(Y)$ be the field of rational functions in one variable Y over $\mathbb{F}_q$. Let $R_\infty = \mathbb{F}_q[Y]$ be the ring of polynomials in Y over $\mathbb{F}_q$, let $K_\infty = \mathbb{F}_q((Y^{-1}))$ be the field of formal Laurent series in Y^{-1} over $\mathbb{F}_q$ and let $X_\infty = \mathrm{PGL}_2(K_\infty)/\mathrm{PGL}_2(R_\infty)$ be the space of homothety classes of R_∞-lattices in $K_\infty \times K_\infty$ (that is, of rank 2 free R_∞-submodules spanning the vector plane $K_\infty \times K_\infty$ over K_∞). A point $x \in X_\infty$ is called A_∞*-periodic* if its orbit under the diagonal subgroup A_∞ of $\mathrm{PGL}_2(K_\infty)$ is compact. This orbit $A_\infty x$ then carries a unique A_∞-invariant probability measure, denoted by μ_x. The aim of this paper is to study the asymptotic behavior of these measures μ_x (and in particular to prove unexpected escape of mass phenomena) as x varies in arithmetically defined subsets of A_∞-periodic points.

Recall that for every $x_0 \in X_\infty$ and every prime polynomial ν in R_∞, the *Hecke tree* $T_\nu(x_0)$ with root x_0 is the connected component of x_0 in the graph with vertex set X_∞, with an edge between the homothety classes of two R_∞-lattices Λ and Λ' when $\Lambda' \subset \Lambda$ and Λ/Λ' is isomorphic to $R_\infty/\nu R_\infty$ as an R_∞-module. The boundary at infinity $\Omega = \Omega_{x_0}$ of $T_\nu(x_0)$ identifies with the projective line $\mathbb{P}^1(K_\nu)$ over the completion K_ν of K associated with ν, and a point of Ω is called *rational* if it belongs to $\mathbb{P}^1(K)$. (Note that the identification of Ω with $\mathbb{P}^1(K_\nu)$ is not canonical, but the notion of rationality is well defined.) For every $\xi \in \Omega$, let $(x_n^{\xi})_{n\in\mathbb{N}}$ be the vertices along the geodesic ray (called a *Hecke ray*) in $T_v(x_0)$ from x_0 to ξ.

In what follows, we fix an A_∞-periodic point x_0 in X_∞. Note that the vertices of the Hecke-tree $T_v(x_0)$ then also have periodic A_∞-orbits. Our aim is to understand the possible sets Θ_ξ of weak-star accumulation points of the sequences of measures $(\mu_{x_n^\xi})_{n\in\mathbb{N}}$ on X_∞ associated with the vertices of the Hecke ray with endpoint ξ, when ξ varies in Ω. For all $\xi \in \Omega$ and $c > 0$, we say that

- ξ has *c-escape of mass* if there exists $\theta \in \Theta_\xi$ with $\theta(X_\infty) \le 1 - c$;
- ξ has *uniform c-escape of mass* if for every $\theta \in \Theta_\xi$ we have $\theta(X_\infty) \le 1 - c$.

Here is a summary of our results.

THEOREM 1. *There exists $c > 0$ such that any rational $\xi \in \Omega$ has uniform c-escape of mass.*

The following result also exhibits full espace of mass phenomena along Hecke rays.

THEOREM 2. *There exists (p, v, x_0) such that for every rational $\xi \in \Omega$, the zero measure belongs to Θ_ξ.*

The key approach to these results (proved in Section 4.1) is to use the geodesic flow on the quotient of the Bruhat-Tits tree of $(\mathrm{PGL}_2, K_\infty)$ (see for instance [27] and Section 2.3) by the lattice $\mathrm{PGL}_2(R_\infty)$.

Theorem 1 proves an escape of mass phenomenon along only countably many Hecke rays. Using the fact that the above constant c is independent of the rational Hecke ray, we can strengthen this in the next result (see Section 4.2).

THEOREM 3. *There exists $c > 0$ such that the set of $\xi \in \Omega$ having c-escape of mass is uncountable.*

As guided by the analogy with $\mathrm{PGL}_2(\mathbb{R})/\mathrm{PGL}_2(\mathbb{Z})$, we could still wonder if the part of the measure which does not go to infinity still equidistributes in X_∞, that is, converges to a measure proportional to the probability measure on X_∞ invariant under $\mathrm{PGL}_2(K_\infty)$. The next result proves that this is also not always the case.

THEOREM 4. *There exists $c' > 0$ such that for every A_∞-periodic point $x \in X_\infty$, there exist $\xi \in \Omega_x$ and $\theta \in \Theta_\xi$ such that $c'\mu_x \le \theta$. In particular, θ is not absolutely continuous with respect to the homogeneous measure on X_∞.*

We give explicit constants c, c' in the above statements. We will actually prove a stronger result, Theorem 18 in Section 4.4, which mixes the behaviors in Theorems 3 and 4. For this, the main tool (proved in Section 4.3) is an effective equidistribution result of sectors of Hecke spheres in positive characteristic, which we prove using the known exponential decay of matrix coefficients, see for instance [2]. We refer for instance to the works of Dani-Margulis [9], Clozel-Oh-Ullmo [7], Clozel-Ullmo [8], Eskin-Oh [13], Benoist-Oh [3] for equidistribution results of Hecke spheres in zero characteristic.

As we shall see in the main body of the text, Theorems 1, 3 and 4 are valid upon replacing K by any global function field (see also Section 5 for further extensions). In this more general case, there are several (albeit finitely many) ways to go to infinity in X_∞, and we will give more precise results towards which cusp of X_∞ the escape of mass occurs.

Going back to the comparison between the zero and positive characteristic cases, the underlying phenomenon which changes drastically is as follows. While in zero characteristic the size of the orbit $A_\infty x_n^\xi$ is exponential in n, in positive characteristic, it is linear in n due to the presence of the Frobenius automorphism (see Theorem 10). When this is combined with the fact that rational rays diverge in a linear speed, we get the results regarding the escape of mass.

Although the rigidity displayed in zero characteristic completely breaks down, as demonstrated by the above results, we still believe that the following conjecture holds. It implies in particular that the set of rays having uniform escape of mass (such as the rational rays) is a null set.

CONJECTURE 5. *For almost any $\xi \in \Omega$ (with respect to the natural probability measure), the averages $\frac{1}{N+1}\sum_{n=0}^{N} \mu_{x_n^\xi}$ converge to the homogeneous probability measure on X_∞.*

Conjecture 5 reflects our belief that the behaviour along rational rays is far from generic. In fact, after some computer experiments, we suggest the following.

CONJECTURE 6. *For any rational $\xi \in \Omega$, $\mu_{x_n^\xi}$ converges to the zero measure.*

This work raises many other natural questions which we plan on studying in subsequent works. A few examples are: Is the rationality of the ray characterized by a uniform (or full) escape of mass? Can we find irrational rays exhibiting an escape of mass in average? Do we have a criterion for the convergence (or convergence in average) towards (a multiple of) the homogeneous measure? What is the Hausdorff dimension of the set of ξ for which Theorem 3 holds?

Although we are working with the dynamics of rank 1 torus, it is interesting to compare our results with the huge corpus of works in dynamics on noncompact spaces, in particular locally homogeneous ones or moduli spaces, precisely devoted to prove that there is no escape of mass to infinity for nice sequences of probability measures on these spaces. This is in particular the case in homogeneous dynamics—with real Lie groups, thereby in zero characteristic—(see for instance [11, 4]) or in Teichmüller dynamics (see for instance [12, 15]).

Note that an escape of mass for the diagonal group is not a feature appearing only in positive characteristic. Over the reals, there are examples of escape of mass for the diagonal flow: for example, in [24, p. 232] the author arithmetically constructs a sequence of closed geodesics on the modular surface which converge to the zero measure (see also [28] for similar examples in higher dimensions). We stress, though, that these examples do not share the arithmetic relation between the measures along the sequence which is present in our results. Indeed, due to the results in [1], such an arithmetic relation cannot coexist with an escape of mass over the reals.

As another motivation for studying the limiting behaviour of $\mu_{x_n^\xi}$ (also originating from the analogy with [1]), let us indicate a relation with the distribution properties of the periods of the continued fraction expansion of certain sequences of quadratic irrationals. We refer for instance to the surveys [16, 25] for background.

We denote by $\mathcal{O}_\infty = \mathbb{F}_q[[Y^{-1}]]$ the local ring of K_∞ (consisting of power series in Y^{-1} over $\mathbb{F}_q$). Any element $f \in K_\infty$ may be uniquely written $f = [f] + \{f\}$ with $[f]$ in the polynomial ring $R_\infty = \mathbb{F}_q[Y]$ and $\{f\} \in Y^{-1}\mathcal{O}_\infty$. The *Artin map* $\Psi : Y^{-1}\mathcal{O}_\infty \smallsetminus \{0\} \to Y^{-1}\mathcal{O}_\infty$ is defined by $f \mapsto \{\frac{1}{f}\}$. Any $f \in K_\infty$ irrational (not in $K = \mathbb{F}_q(Y)$) has a unique continued fraction expansion

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

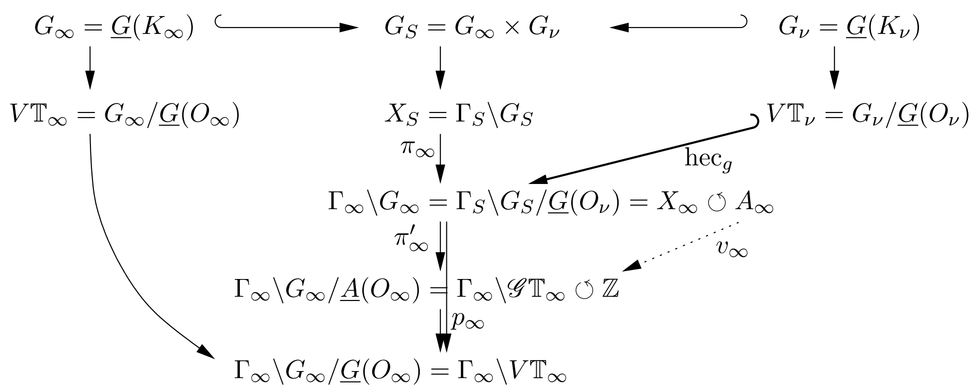
with $a_0 = [f] \in R_\infty$ and $a_n = \left[\frac{1}{\Psi^{n-1}(f-a_0)}\right]$ a non-constant polynomial, for $n \geq 1$.

Let $QI = \{f \in K_\infty \ : \ [K(f) : K] = 2\}$ be the set of quadratic irrationals over K in K_∞. Given an irrational $f \in Y^{-1}\mathcal{O}_\infty$, we have $f \in QI$ if and only if the continued fraction expansion of f is eventually periodic. We fix $f \in QI$. Assume for simplicity that the characteristic p is different from 2, that $f \in Y^{-1}\mathcal{O}_\infty$ has purely periodic continued fraction expansion and that the Galois conjugate f^σ of f over K belongs to $K_\infty \smallsetminus \mathcal{O}_\infty$. Let $g_f = \begin{bmatrix} f^\sigma & f \\ 1 & 1 \end{bmatrix} \in \mathrm{PGL}_2(K_\infty)$ and $x_f = g_f^{-1}\,\mathrm{PGL}_2(R_\infty) \in X_\infty$. It is then easy to prove that x_f is A_∞-periodic. Using the main results of [6], we may construct a natural cross-section C for the action of A_∞ on (a full-measure subset of) X_∞ and a natural map from C onto (a full-measure subset of) $Y^{-1}\mathcal{O}_\infty$, sending the intersection with C of the A_∞-orbit of x_f in X_∞ to the orbit of f under Ψ in $Y^{-1}\mathcal{O}_\infty$.

Given an irreducible polynomial $P \in \mathbb{F}_q[Y]$, our results imply strong statements regarding the asymptotics of the periods of the continued fraction expansions of the quadratic irrationals $P^n f$ as $n \to +\infty$ (in terms of length of the period and the degrees of the polynomials composing it), with a strikingly different outcome than the ones in [1]. The exact statements and the detailed analysis of this translation are too long to be included here, and we refer to the future note [20]. We only mention here, as indicated by the referee, that our results imply in particular that if $(a_{n,i})_{i\in\mathbb{N}}$ is the continued fraction expansion of $P^n f$ for all $n \in \mathbb{N}$, then $\sup_{n,i\in\mathbb{N}-\{0\}} \deg a_{n,i} = +\infty$, thus recovering a special case of [10, Theorem 4.5].

2. GLOBAL FUNCTION FIELDS AND BRUHAT-TITS TREES

This section introduces the notation and preliminary results used in this paper. We refer the reader to the following commutative diagram for a global view of this notation.



2.1. **Global function fields.** We refer for instance to [23, 26] for the content of this section.

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a positive power of a prime p. Let K be a *global function field* over $\mathbb{F}_q$, that is, the function field of a geometrically connected smooth projective curve $\mathbf{C}$ over $\mathbb{F}_q$, or equivalently an extension of $\mathbb{F}_q$ of transcendence degree 1, in which $\mathbb{F}_q$ is algebraically closed. The set $\mathscr{P}$ of *primes* of K is the set of closed points of $\mathbf{C}$, or equivalently the set of discrete valuations of K, trivial on $\mathbb{F}_q^\times$, with value group exactly $\mathbb{Z}$. We fix an element in $\mathscr{P}$ that we denote by ∞, and we denote by $\mathscr{P}_f$ the set $\mathscr{P}-\{\infty\}$.

For every $\omega \in \mathscr{P}$, we denote by R_ω the affine algebra of the affine curve $\mathbf{C}-\{\omega\}$ (which is a Dedekind ring), by v_ω the discrete valuation of K associated with ω (with the usual convention that $v_\omega(0) = +\infty$), by K_ω the associated completion of K (and again by v_ω the extension of v_ω to K_ω), by O_ω its local ring, by π_ω a uniformizer of O_ω, by k_ω its residual field (that we identify with its canonical lift in O_ω), and by $\deg(\omega)$ the degree of k_ω over $\mathbb{F}_q$. We assume, as we may using for instance the Riemann-Roch theorem, that π_ν belongs to R_∞ if $\nu \in \mathscr{P}_f$. Note that $R_\infty \subset O_\nu$ if $\nu \in \mathscr{P}_f$ (since an element in R_∞ has no pole at the closed point $\nu \neq \infty$ of $\mathbf{C}$), and that $R_\infty[\pi_\nu^{-1}] \cap O_\nu = R_\infty$.

We normalize the absolute value $|\cdot|_\omega$ associated to v_ω by $|x|_\omega = |k_\omega|^{-v_\omega(x)} = q^{-\deg\omega\, v_\omega(x)}$ for every $x \in K_\omega$. In particular, the product formula

$$\forall\, x \in K, \quad \prod_{\omega\in\mathscr{P}} |x|_\omega = 1$$

holds. Note that K_ω is the field $k_\omega((\pi_\omega))$ of Laurent series $f = \sum_{i\in\mathbb{Z}} f_i(\pi_\omega)^i$ in the variable π_ω over k_ω, where $f_i \in k_\omega$ is zero for $i \in \mathbb{Z}$ small enough. We have

$$|f|_\omega = |k_\omega|^{-\sup\{j\in\mathbb{Z}\,:\,\forall\, i<j,\ f_i=0\}}\,,$$

and $O_\omega = k_\omega[[\pi_\omega]]$ is the local ring of power series $f = \sum_{i\in\mathbb{N}} f_i(\pi_\omega)^i$ (where $f_i \in k_\omega$) in the variable π_ω over k_ω.

For every finite extension $\widetilde{K}_\omega$ of K_ω, we denote again by v_ω the unique extension of v_ω to a valuation on $\widetilde{K}_\omega$, and by $e(\widetilde{K}_\omega, K_\omega) = [v_\omega(\widetilde{K}_\omega^\times) : v_\omega({K_\omega}^\times)]$ its ramification index (see for instance [26, II §2]).

For instance, if $\mathbf{C}$ is the projective line $\mathbb{P}^1$ and if $\infty = [1:0]$ is its usual point at infinity, then $K = \mathbb{F}_q(Y)$, $\pi_\infty = Y^{-1}$, $K_\infty = \mathbb{F}_q((Y^{-1}))$, $O_\infty = \mathbb{F}_q[[Y^{-1}]]$, $k_\infty = \mathbb{F}_q$, $R_\infty = \mathbb{F}_q[Y]$ and the uniformizers π_v for $v \in \mathscr{P}_f$ may be taken to be the monic prime polynomials in R_∞, with $\deg v$ the degree of the polynomial π_v. This is the example considered in the introduction.

2.2. **The semi-simple group** PGL_2**.** In this section, we give the group-theoretic notation we are going to use in this paper, except in Section 5. Let K be as in Section 2.1. We fix $v \in \mathscr{P}_f$.

We denote by $\underline{G} = \mathrm{PGL}_2$ the (adjoint semi-simple absolutely simple) projective linear algebraic group over K in dimension 2. Whenever necessary, we embed PGL_2 in GL_3 by the adjoint representation on the vector space of traceless 2-by-2 matrices.

Let $\underline{A}$ be the *diagonal subgroup* of $\underline{G}$, that is, the algebraic subgroup of $\underline{G}$ consisting in the elements represented by diagonal matrices, which is a (split) maximal torus of $\underline{G}$ defined over K.

For every $\omega \in \mathscr{P}$ and every algebraic subgroup $\underline{H}$ of $\underline{G}$ defined over K_ω (for instance if $\underline{H}$ is defined over K), we set $H_\omega = \underline{H}(K_\omega)$, which is a non-Archimedean Lie group (and in particular a locally compact group).

We define $\Gamma_\infty = \underline{G}(R_\infty) = \mathrm{PGL}_2(R_\infty)$, which is a nonuniform lattice in $G_\infty = \mathrm{PGL}_2(K_\infty)$. For instance, when $\mathbf{C} = \mathbb{P}^1$, the lattice Γ_∞ is called *Nagao's lattice* [18] (or Weil's modular group [31]).

We denote by X_∞ the totally disconnected locally compact space $\Gamma_\infty\backslash G_\infty$ (contrarily to the introduction, we consider the left quotient, since it makes the connection with Bruhat-Tits theory easier). The space X_∞ is noncompact, and identifies by $\Gamma_\infty g \mapsto g^{-1}[R_\infty \times R_\infty]$ with the space of homothety classes $[\Lambda]$ under $K_\infty^\times$ of R_∞-lattices Λ in $K_\infty \times K_\infty$.

Let $S = \{\infty, v\}$ and let Γ_S be the S*-arithmetic group* $\underline{G}(R_\infty[{\pi_v}^{-1}])$, which embeds diagonally in the locally compact group $G_S = G_\infty \times G_v$ as a nonuniform lattice, and let $X_S = \Gamma_S\backslash G_S$. We identify G_∞ and G_v, hence any subgroup of them, with their images in G_S by the maps $x \mapsto (x, e)$ and $y \mapsto (e, y)$. Note that $\Gamma_S \cap \underline{G}(O_v) = \Gamma_\infty$ since $R_\infty[\pi_v^{-1}] \cap O_v = R_\infty$.

For every $\omega \in S$, we denote by

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the image in $G_\omega = \mathrm{PGL}_2(K_\omega)$ of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_\omega)$,
- v_ω the map from the abelian group $A_\omega = \underline{A}(K_\omega) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;:\; a, d \in K_\omega \right\}$ to $\mathbb{Z}$ defined by $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mapsto v_\omega(d/a)$, which is a group epimorphism with compact-open kernel $\underline{A}(O_\omega) = \underline{G}(O_\omega) \cap A_\omega$,

- $\alpha_\omega : K_\omega^\times \to A_\omega$ the group isomorphism $t \mapsto \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$ (whose inverse is the positive root of the torus $\underline{A}$ over K_ω) so that $v_\omega(\alpha_\omega(t)) = v_\omega(t)$, and $a_\omega = \alpha_\omega(\pi_\omega) = \begin{bmatrix} 1 & 0 \\ 0 & \pi_\omega \end{bmatrix}$ so that $v_\omega(a_\omega) = 1$.

2.3. **Bruhat-Tits trees.** Let $(K, v, \underline{G}, \underline{A})$ and the associated notation be as in Section 2.2.

Trees. Let T be a locally finite tree. Its set of vertices VT is endowed with the maximal distance for which two adjacent distinct vertices are at distance 1. A *geodesic ray* or *line* in T is an isometric map from $\mathbb{N}$ or $\mathbb{Z}$ to its set of vertices. The set of geodesic lines of T, endowed with the compact-open topology, is denoted by $\mathscr{G}T$.

An *end* of T is an equivalence class of geodesic rays, when two geodesic rays are equivalent if the intersection of their images is the image of a geodesic ray. The set of ends of T, endowed with the (compact, totally disconnected) quotient topology of the compact-open topology, is denoted by ∂T, and called the *boundary at infinity* of T.

The *translation length* of an isometry γ of T is

$$\ell_T(\gamma) = \min_{x \in VT} d(x, \gamma x) .$$

It is invariant under conjugation of γ in the isometry group of T. We will say that γ is *loxodromic* if $\ell_T(\gamma) > 0$, in which case there exists a unique image of a geodesic line in T on which γ translates a distance $\ell_T(\gamma)$, called the *translation axis* of γ.

The *geodesic flow* (with discrete times) $(\phi_m)_{m \in \mathbb{Z}}$ on the tree T is the right action $(\mathscr{G}T \times \mathbb{Z}) \to \mathscr{G}T$ of $\mathbb{Z}$ on $\mathscr{G}T$ by translations at the source, defined by

$$(\ell, m) \mapsto \{\phi_m \ell : n \mapsto \ell(n+m)\}$$

for all $m \in \mathbb{Z}$ and $\ell : \mathbb{Z} \to VT$ in $\mathscr{G}T$. Given a group Γ of automorphisms of T, the geodesic flow on T induces a right action of $\mathbb{Z}$ on $\Gamma\backslash\mathscr{G}T$, also called the *geodesic flow* of $\Gamma\backslash T$, and again denoted by $(\phi_m)_{m \in \mathbb{Z}}$.

The tree of PGL_2 over local fields. For $\omega \in S = \{\infty, v\}$, let $\mathbb{T}_\omega$ be the *Bruhat-Tits tree* of $(\underline{G}, K_\omega)$, see for instance [30]. We use its description given in [27].

Recall that an O_ω*-lattice* Λ in the K_ω-vector space $K_\omega \times K_\omega$ is a rank 2 free O_ω-submodule of $K_\omega \times K_\omega$, generating $K_\omega \times K_\omega$ as a vector space. The Bruhat-Tits tree $\mathbb{T}_\omega$ is the graph whose set of vertices $V\mathbb{T}_\omega$ is the set of homothety classes (under $K_\omega^\times$) $[\Lambda]$ of O_ω-lattices Λ in $K_\omega \times K_\omega$, and whose non-oriented edges are the pairs $\{x, x'\}$ of vertices such that there exist representatives Λ of x and Λ' of x' such that $\Lambda \subset \Lambda'$ and Λ'/Λ is isomorphic to $O_\omega/\pi_\omega O_\omega$. This graph is a regular tree of degree $|\mathbb{P}_1(k_\omega)| = |k_\omega| + 1$.

We denote by $*_\omega$ the homothety class of the O_ω-lattice $O_\omega \times O_\omega$ generated by the canonical basis of $K_\omega \times K_\omega$. The left linear action of $\mathrm{GL}_2(K_\omega)$ on $K_\omega \times K_\omega$ induces a faithful, transitive left action of G_ω on $V\mathbb{T}_\omega$. The stabilizer in G_ω of

$*_\omega$ is $\underline{G}(O_\omega)$. We will hence identify $G_\omega/\underline{G}(O_\omega)$ with $V\mathbb{T}_\omega$ by the map $g\,\underline{G}(O_\omega)\mapsto g*_\omega$.

We identify as usual the projective line $\mathbb{P}_1(K_\omega)$ with $K_\omega\cup\{\infty\}$ using the map $(x,y)\mapsto xy^{-1}$. There exists one and only one homeomorphism between the boundary at infinity $\partial\mathbb{T}_\omega$ of $\mathbb{T}_\omega$ and $\mathbb{P}_1(K_\omega)$ such that the (continuous) extension to $\partial\mathbb{T}_\omega$ of the isometric action of G_ω on $\mathbb{T}_\omega$ corresponds to the projective action of G_ω on $\mathbb{P}_1(K_\omega)$. From now on, we identify $\partial\mathbb{T}_\omega$ and $\mathbb{P}_1(K_\omega)$ by this homeomorphism.

The group G_ω hence acts simply transitively on the set of ordered triples of distinct points in $\partial\mathbb{T}_\omega$. In particular, the group G_ω acts transitively on the space $\mathscr{G}\mathbb{T}_\omega$ of geodesic lines in $\mathbb{T}_\omega$. The stabilizer under this action of the geodesic line

$$\ell_0 : n\mapsto [O_\omega\times\pi_\omega^n O_\omega] = a_\omega^n *_\omega$$

is the maximal compact-open subgroup $\underline{A}(O_\omega)$ of the diagonal group A_ω. We will hence identify $G_\omega/\underline{A}(O_\omega)$ with $\mathscr{G}\mathbb{T}_\omega$ by $g\underline{A}(O_\omega)\mapsto g\,\ell_0$. Furthermore, the stabilizer in G_ω of the ordered pair of endpoints $(\ell_0(-\infty)=0,\ell_0(+\infty)=\infty)$ of ℓ_0 in $\partial\mathbb{T}_\omega=\mathbb{P}_1(K_\omega)$ is $\underline{A}_\omega$. Therefore any element $\gamma_0\in G_\omega$ which is loxodromic on $\mathbb{T}_\omega$ is diagonalisable over K_ω. Besides, by [27, page 108], the translation length on $\mathbb{T}_\omega$ of $\gamma_0=\begin{bmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{bmatrix}$ is

$$\ell_{\mathbb{T}_\omega}(\gamma_0) = |v_\omega(\lambda_+)-v_\omega(\lambda_-)|\,. \tag{1}$$

Using the group morphism $v_\omega : A_\omega\to\mathbb{Z}$, the action by translations on the right of A_ω on $G_\omega/\underline{A}(O_\omega)$ corresponds to the geodesic flow on $\mathscr{G}\mathbb{T}_\omega$: for all $a\in A_\omega$ and $\ell\in\mathscr{G}\mathbb{T}_\omega=G_\omega/\underline{A}(O_\omega)$, we have

$$\ell\,a=\phi_{v_\omega(a)}\ell\,.$$

We denote by $\pi'_\infty : X_\infty=\Gamma_\infty\backslash G_\infty\to\Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty=\Gamma_\infty\backslash G_\infty/\underline{A}(O_\infty)$ the canonical projection (see the diagram at the beginning of Section 2). The previous equation proves that π'_∞ is equivariant with respect to the morphism $v_\infty : A_\infty\to\mathbb{Z}$, where A_∞ acts by translation on the right on X_∞ and $\mathbb{Z}$ by the (quotient) geodesic flow on $\Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty$: for all $x\in X_\infty$ and $a\in A$,

$$\pi'_\infty(x\,a)=\phi_{v_\infty(a)}\pi'_\infty(x)\,. \tag{2}$$

The principal bundle $\pi_\infty : X_S\to X_\infty$. Since Γ_S is irreducible, the group $\Gamma_\infty=\Gamma_S\cap\underline{G}(O_v)$ is dense in the stabiliser $\underline{G}(O_v)$ of the base point $*_v$ of the Bruhat-Tits tree $\mathbb{T}_v$. This stabilizer $\underline{G}(O_v)$ acts transitively on the set of geodesic rays in $\mathbb{T}_v$ starting from $*_v$. Thus Γ_∞ preserves and acts transitively on the sphere in $\mathbb{T}_v$ of any given radius centered at $*_v$. For every $g'\in G_v$, there hence exists $\gamma\in\Gamma_\infty$ and $n\in\mathbb{N}$ such that $\gamma^{-1}g'*_v=[O_v\times\pi_v^nO_v]=a_v^n*_v$. Therefore

$$G_v=\bigcup_{n\in\mathbb{N}}\Gamma_\infty\,a_v^n\,\underline{G}(O_v)\,. \tag{3}$$

In particular, $G_v=\Gamma_S\,\underline{G}(O_v)$.

Therefore, every element x of X_S may be written $\Gamma_S(g,g')$ with $g\in G_\infty$ and $g'\in\underline{G}(O_v)$. For all $g,h\in G_\infty$ and $g',h'\in\underline{G}(O_v)$, we have $\Gamma_S(g,g')=\Gamma_S(h,h')$ if and only if $gh^{-1} = g'(h')^{-1} \in \Gamma_S \cap \underline{G}(\mathcal{O}_v) = \Gamma_\infty$. Hence the map $\pi_\infty : X_S \to X_\infty$, where $\pi_\infty(x) = \Gamma_\infty g$ if $x = \Gamma_S(g, g')$ with $g' \in \underline{G}(\mathcal{O}_v)$, is well defined and continuous. The action of $\underline{G}(\mathcal{O}_v)$ by right translations on the second factor of $G_S = G_\infty \times G_v$ induces an action of $\underline{G}(\mathcal{O}_v)$ on $X_S = \Gamma_S \backslash G_S$, which is transitive and free on the fibers of π_∞. Hence $\pi_\infty : X_S \to X_\infty$ is a principal bundle under the group $\underline{G}(\mathcal{O}_v)$, which gives an identification between $X_\infty = \Gamma_\infty \backslash G_\infty$ and $X_S / \underline{G}(\mathcal{O}_v) = \Gamma_S \backslash G_S / \underline{G}(\mathcal{O}_v)$ (see the diagram at the beginning of Section 2).

Ends of the modular graph at the place ∞ and heights. The quotient graph $\Gamma_\infty \backslash \mathbb{T}_\infty$ will be called the *modular graph at* ∞ of K. By for instance [27], the *set of cusps* $\Gamma_\infty \backslash \mathbb{P}_1(K)$ is finite, and $\Gamma_\infty \backslash \mathbb{T}_\infty$ is the disjoint union of a finite connected subgraph containing $\Gamma_\infty *_\infty$ and of maximal open geodesic rays $h_z(]0, +\infty[)$, for $z = \Gamma_\infty \widetilde{z} \in \Gamma_\infty \backslash \mathbb{P}_1(K)$, where h_z (called a *cuspidal ray*) is the image by the canonical projection $\mathbb{T}_\infty \to \Gamma_\infty \backslash \mathbb{T}_\infty$ of a geodesic ray whose point at infinity in $\mathbb{P}_1(K) \subset \partial \mathbb{T}_\infty$ is equal to $\widetilde{z}$. Conversely, any geodesic ray whose point at infinity lies in $\mathbb{P}_1(K) \subset \partial \mathbb{T}_\infty$ contains a subray that maps injectively by the canonical projection $\mathbb{T}_\infty \to \Gamma_\infty \backslash \mathbb{T}_\infty$.

Let us denote by $\widehat{\Gamma_\infty \backslash \mathbb{T}_\infty} = (\Gamma_\infty \backslash \mathbb{T}_\infty) \sqcup \mathscr{E}_\infty$ Freudenthal's compactification (see [14]) of $\Gamma_\infty \backslash \mathbb{T}_\infty$ by its finite set of ends $\mathscr{E}_\infty$. This set of ends is indeed finite, in bijection with $\Gamma_\infty \backslash \mathbb{P}_1(K)$ by the map which associates to $z \in \Gamma_\infty \backslash \mathbb{P}_1(K)$ the end towards which the cuspidal ray h_z converges. See for instance [27] for a geometric interpretation of $\mathscr{E}_\infty$ in terms of the curve $\mathbf{C}$.

Let $\widehat{X_\infty} = X_\infty \sqcup \mathscr{E}_\infty$ and let $\widehat{p_\infty} : \widehat{X_\infty} \to \widehat{\Gamma_\infty \backslash \mathbb{T}_\infty}$ be the map equal to the identity map on $\mathscr{E}_\infty$ and to the canonical projection

$$p_\infty : X_\infty = \Gamma_\infty \backslash G_\infty \to \Gamma_\infty \backslash V\mathbb{T}_\infty = \Gamma_\infty \backslash G_\infty / \underline{G}(\mathcal{O}_\infty)$$

on X_∞ (see the diagram at the beginning of Section 2). Since p_∞ is a proper map, this defines a compactification of X_∞, by endowing $\widehat{X_\infty}$ with the compact metrisable topology generated by the open subsets of U and the sets $\widehat{p_\infty}^{-1}(U)$ with U an open neighborhood of a point in $\mathscr{E}_\infty$. We will say that $\mathscr{E}_\infty$ is the *set of cusps* of X_∞, and we will indicate towards which cusp of X_∞ the escape of mass occurs.

For every $x \in X_\infty$, define the *height* of x in X_∞ by

$$\mathrm{ht}_\infty(x) = d_{\Gamma_\infty \backslash \mathbb{T}_\infty}(p_\infty(x), \Gamma_\infty *_\infty) . \tag{4}$$

For every cusp $z \in \mathscr{E}_\infty$ of X_∞, define the *height of x in X_∞ relative to the cusp z* by $\mathrm{ht}_{\infty, z}(x) = 0$ if $p_\infty(x)$ does not belong to $h_z(]0, +\infty[)$, and

$$\mathrm{ht}_{\infty, z}(x) = d_{\Gamma_\infty \backslash \mathbb{T}_\infty}(p_\infty(x), h_z(0)) ,$$

otherwise.

LEMMA 7. *For all $g' \in G_\infty$ and $x \in X_\infty$, we have*

$$|\mathrm{ht}_\infty(x) - \mathrm{ht}_\infty(xg')| \le d_{\mathbb{T}_\infty}(*_\infty, g' *_\infty) ,$$

and $|\mathrm{ht}_{\infty, z}(x) - \mathrm{ht}_{\infty, z}(xg')| \le d_{\mathbb{T}_\infty}(*_\infty, g' *_\infty)$ *for every cusp $z \in \mathscr{E}_\infty$ of X_∞.*

Proof. Let $g\in G_\infty$ be such that $x=\Gamma_\infty g$. We have $p_\infty(x)=\Gamma_\infty\, g *_\infty$ and $p_\infty(xg')=\Gamma_\infty\, g\, g' *_\infty$. By the triangle inequality and since the projection map $\mathbb{T}_\infty \to \Gamma_\infty\backslash\mathbb{T}_\infty$ does not increase the distances, we have

$$\begin{aligned}|\operatorname{ht}_\infty(x)-\operatorname{ht}_\infty(xg')| &\le d_{\Gamma_\infty\backslash\mathbb{T}_\infty}(\Gamma_\infty\, g *_\infty, \Gamma_\infty\, g\, g' *_\infty)\\ &\le d_{\mathbb{T}_\infty}(g *_\infty, g\, g' *_\infty) = d_{\mathbb{T}_\infty}(*_\infty, g' *_\infty)\,.\end{aligned}$$

The second assertion follows if $p_\infty(x)$ and $p_\infty(xg')$ simultaneously belong or do not belong to (the image of) h_z. If for instance $p_\infty(x)$ belongs to h_z and $p_\infty(xg')$ does not belong to h_z, then

$$d_{\Gamma_\infty\backslash\mathbb{T}_\infty}(p_\infty(x), h_e(0)) \le d_{\Gamma_\infty\backslash\mathbb{T}_\infty}(p_\infty(x), p_\infty(xg'))$$

and the result holds as above. □

Example. Assume that **C** is the projective line over $\mathbb{F}_q$ and that ∞ is its usual point at infinity. Then the (image of the) geodesic ray in $\mathbb{T}_\infty$ starting from $*_\infty$ with point at infinity $\infty\in\mathbb{P}_1(K_\infty)$, which is

$$n\in\mathbb{N}\mapsto [\mathcal{O}_\infty\times\pi_\infty^n\,\mathcal{O}_\infty] = a_\infty^n *_\infty \in V\mathbb{T}_\infty\,,$$

is a (weak) fundamental domain for the action of Γ_∞ on $V\mathbb{T}_\infty$: it injects onto $\Gamma_\infty\backslash V\mathbb{T}_\infty$ by the canonical map $\mathbb{T}_\infty\to\Gamma_\infty\backslash\mathbb{T}_\infty$.

Hence $G_\infty = \coprod_{n\in\mathbb{N}}\Gamma_\infty\, a_\infty^n\, \underline{G}(\mathcal{O}_\infty)$. For every $g\in G_\infty$, the height of $x=\Gamma_\infty g$ is the unique $n\in\mathbb{N}$ such that $g\in\Gamma_\infty\, a_\infty^n\,\underline{G}(\mathcal{O}_\infty)$. Note that if one writes g in the Cartan decomposition of G_∞ as $g\in\underline{G}(\mathcal{O}_\infty)\, a_\infty^m\,\underline{G}(\mathcal{O}_\infty)$ for some $m\in\mathbb{N}$, then $m=d_{\mathbb{T}_\infty}(*_\infty, g *_\infty)\ge \operatorname{ht}_\infty(x)$, with usually strict inequality.

The quotient graph of finite groups $\Gamma_\infty\backslash\backslash\mathbb{T}_\infty$, whose underlying graph is the geodesic ray $\Gamma_\infty\backslash\mathbb{T}_\infty$, is called the *modular ray*. With $F_0=\underline{G}(k_\infty)$, $F_0'=F_0\cap F_1$ and $F_n=\left\{\begin{bmatrix} a & b\\ 0 & d\end{bmatrix}\in\Gamma_\infty \;:\; v_\infty(b)\ge -n\right\}$, the modular ray $\Gamma_\infty\backslash\backslash\mathbb{T}_\infty$ (which has only one end) is given by Figure 1.

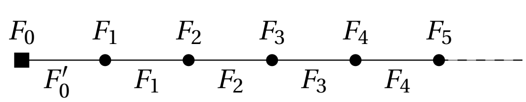


FIGURE 1. The modular ray $\mathrm{PGL}_2(k_\infty[Y])\backslash\backslash\mathbb{T}_\infty$

The full-down property in the modular graph (see for instance [27, 17]). If ρ is a geodesic ray in $\mathbb{T}_\infty$ whose image is a cuspidal ray in $\Gamma_\infty\backslash\mathbb{T}_\infty$, the stabilizers of the vertices of ρ different from the origin of ρ are strictly increasing along the ray. Hence the image in $\Gamma_\infty\backslash\mathbb{T}_\infty$ of a geodesic ray in $\mathbb{T}_\infty$ satisfies the following *full-down property*: if it starts to go down along the image of a cuspidal ray h_z for some $z\in\mathscr{E}_\infty$, then it needs to go all the way down to $h_z(0)$.

As explained in [27, 19], this full-down property has the following consequence: the image by the canonical map $\mathbb{T}_\infty\to\Gamma_\infty\backslash\mathbb{T}_\infty$ of a geodesic ray ρ

in $\mathbb{T}_\infty$ starting from $*_\infty$ either is an infinite sequence $a_0 b_0 a_1 b_1 a_2 b_2 \ldots$ of concatenations of paths a_i (possibly reduced to points) in the finite graph $\Gamma_\infty\backslash\mathbb{T}_\infty - \bigcup_{z\in\mathscr{E}_\infty} h_z(]0,+\infty[)$ and back and forth paths b_i (of even lengths at least 2) from the origin $h_{z_i}(0)$ of the cuspidal ray h_{z_i} to itself inside this ray, if ρ ends in an irrational point at infinity (that is, in $\mathbb{P}_1(K_\infty) - \mathbb{P}_1(K)$), or starts by such a finite sequence and then follows some cuspidal ray to infinity, otherwise.

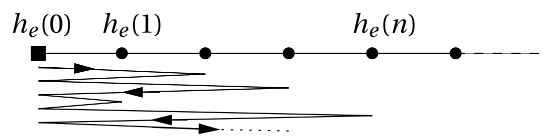


FIGURE 2. Back and forth paths in cuspidal rays

2.4. **A_∞-periodic orbits in X_∞.** Let $(K,\nu,\underline{G},\underline{A})$ and the associated notation be as in Section 2.2.

Let us give a description of the compact orbits for the action by translations on the right of the subgroup A_∞ on $X_\infty = \Gamma_\infty\backslash G_\infty$.

PROPOSITION 8. *For every $g \in G_\infty$, the following assertions are equivalent, where $x = \Gamma_\infty g \in X_\infty$:*

(1) *there exists a unique A_∞-invariant probability measure on the orbit xA_∞;*
(2) *the subgroup $A_\infty \cap g^{-1}\Gamma_\infty g$ is a (uniform) lattice in A_∞;*
(3) *the orbit of $\pi'_\infty(x)$ under the geodesic flow $(\phi_n)_{n\in\mathbb{Z}}$ on $\Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty$ is periodic;*
(4) *there exists $\gamma_0 \in \Gamma_\infty$ and $t_0 \in K_\infty^\times$ with $v_\infty(t_0)$ positive and minimal such that $\gamma_0\, g = g\,\alpha_\infty(t_0)$.*

If one of these conditions is satisfied, we say that x is *A_∞-periodic*, and the unique A_∞-invariant probability measure on xA_∞ is denoted by μ_x.

The elements $\gamma_0 \in \Gamma_\infty$ and $t_0 \in K_\infty^\times$ are said to be *associated with* g. Note that they depend on the choice of the representative g of x: if γ_0 is associated with g, then $\gamma^{-1}\gamma_0\gamma$ is associated with γg for every $\gamma \in \Gamma_\infty$. Furthermore, γ_0 is primitive (not a proper power of an element of Γ_∞) and loxodromic on $\mathbb{T}_\infty$. The period of $\pi'_\infty(x)$ under the geodesic flow $(\phi_n)_{n\in\mathbb{Z}}$ is the translation length of γ_0 on $\mathbb{T}_\infty$, which is equal to $v_\infty(t_0)$, and depends only on x.

Proof. The equivalence of (1) and (2) is well-known.

The equivalence of (2) and (3) follows from the equivariance of the canonical projection $\pi'_\infty : X_\infty = \Gamma_\infty\backslash G_\infty \to \Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty = \Gamma_\infty\backslash G_\infty/\underline{A}(O_\infty)$ with respect to the morphism $v_\infty : A_\infty \to \mathbb{Z}$ (see equation (2)).

The image $\Gamma_\infty\,\ell = \Gamma_\infty\, g\,\underline{A}(O_\infty)$ in $\Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty$ of the geodesic line $\ell = g\underline{A}(O_\infty) \in \mathscr{G}\mathbb{T}_\infty = G_\infty/\underline{A}(O_\infty)$ is periodic under the geodesic flow if and only if there exist $n > 0$ and $\gamma_0 \in \Gamma_\infty$ such that $\gamma_0\ell = \phi^n(\ell) = \ell\,\alpha_\infty(\pi_\infty^n)$, hence, since $\alpha_\infty : O_\infty^\times \to \underline{A}(O_\infty)$ is an isomorphism, if and only if there exist $n > 0$, $u_0 \in O_\infty^\times$ and $\gamma_0 \in \Gamma_\infty$ such that $\gamma_0 g = g\,\alpha_\infty(\pi_\infty^n)\alpha_\infty(u_0)$. With $t_0 = \pi_\infty^n u_0$ so that $v_\infty(t_0) = n > 0$, this proves the equivalence of (3) and (4). □

Let us now prove the additional properties of (γ_0, t_0) and discuss its uniqueness. Assume that n in the above proof is minimal. Then γ_0 is primitive and loxodromic, with translation axis the image of ℓ, translation length n, which is the period of $\Gamma_\infty \ell$ under the geodesic flow. Assume that $(\gamma'_0, t'_0) \in \Gamma_\infty \times K_\infty^\times$ satisfies $\gamma'_0 g = g\,\alpha_\infty(t'_0)$ with $n' = v_\infty(t'_0)$ positive and minimal. Then $n' = n$ and $\gamma'_0 \ell = \phi_n(\ell)$. Hence $\gamma_0^{-1}\gamma'_0$ belongs to the pointwise stabilizer in Γ_∞ of the image of ℓ, which is the finite group $g\underline{A}(O_\infty)g^{-1} \cap \Gamma_\infty$. Therefore, there exists $u'_0 \in {\alpha_\infty}^{-1}(g^{-1}\Gamma_\infty g \cap \underline{A}(O_\infty)) \subset O_\infty^\times$ such that $\gamma'_0 = \gamma_0 g \alpha_\infty(u'_0) g^{-1}$ and $t'_0 = t_0 u'_0$.

2.5. **Hecke trees.** Let $(K, v, \underline{G}, \underline{A})$ and the associated notation be as in Section 2.2.

The set X_∞ of homothety classes of R_∞-lattices in $K_\infty \times K_\infty$ is the set of vertices of a graph, whose non-oriented edges are the pairs $\{x, x'\}$ of vertices such that there exist representatives Λ of x and Λ' of x' such that $\Lambda \subset \Lambda'$ and Λ'/Λ is isomorphic to $R_\infty/\pi_v R_\infty$. The action of G_∞ on X_∞ extends to an (isometric) action by graph automorphisms on this graph.

For every $x \in X_\infty$, the connected component of the vertex x in this graph is a $(|k_v|+1)$-regular tree, called the *(v-)Hecke tree* of x, and denoted by $T_v(x)$. We have $T_v(x)g = T_v(xg)$ for all $x \in X_\infty$ and $g \in G_\infty$. A *(v-)Hecke ray* from x is a geodesic ray in the Hecke tree $T_v(x)$ starting from x.

The following description of the v-Hecke trees in X_∞ is well known, and is given, besides in order to fix the notation, only for the sake of completeness.

LEMMA 9. *Let $g \in G_\infty$ and $x = \Gamma_\infty g$ its image in X_∞. The map from G_v to X_∞ defined by $g' \mapsto \pi_\infty(\Gamma_S(g, g'))$ induces an isometric map* hec_g *from the vertex set $V\mathbb{T}_v = G_v/\underline{G}(O_v)$ of the Bruhat-Tits tree $\mathbb{T}_v$ onto the vertex set $VT_v(x)$ of the Hecke tree $T_v(x)$, sending $*_v$ to x. For every $\gamma_0 \in \Gamma_\infty$, the map* hec_g *conjugates the action of γ_0 on $\mathbb{T}_v$ to the right action of $g^{-1}\gamma_0 g \in \Gamma_\infty$ on $VT_v(x)$: for every $y \in V\mathbb{T}_v$, we have*

$$\mathrm{hec}_g(\gamma_0\, y) = \mathrm{hec}_g(y)\, g^{-1}\gamma_0 g\,. \tag{5}$$

For all $h \in G_\infty$ such that $\Gamma_\infty h = x$, we have $\mathrm{hec}_g = \mathrm{hec}_h$ *if and only if $g = h$; furthermore, the following diagram commutes:*

$$\begin{array}{ccc} V\mathbb{T}_v & \xrightarrow{gh^{-1}} & V\mathbb{T}_v \\ & {}_{\mathrm{hec}_h}\searrow \quad \swarrow_{\mathrm{hec}_g} & \\ & VT_v(x) & \end{array} \quad . \tag{6}$$

Note that hec_g depends on g and not only on x. We will denote again by hec_g the (continuous) extension $\partial\mathbb{T}_v \to \partial T_v(x)$ of hec_g to the boundaries at infinity of the Bruhat-Tits and Hecke trees.

Proof. Since the action by translations on the right of $\underline{G}(O_v)$ on X_S preserves the fibers of the bundle map $\pi_\infty : X_S \to X_\infty$, the map $g' \mapsto \pi_\infty(\Gamma_S(g, g'))$ does induce a map $\mathrm{hec}_g : V\mathbb{T}_v = G_v/\underline{G}(O_v) \to X_\infty$.

By definition of the Hecke tree $T_v(x)$ of $x = \Gamma_\infty g = g^{-1}[R_\infty \times R_\infty]$, its vertices are the points $g^{-1}\gamma[R_\infty \times \pi_v^n R_\infty]$ where $\gamma \in \Gamma_\infty$ and $n \in \mathbb{N}$. By equation (3), any element in G_v may be written $\gamma\, a_v^n\, g'$ for some $\gamma \in \Gamma_\infty$, $n \in \mathbb{N}$ and $g' \in$

$\underline{G}(O_v)$. Hence, the elements in $\mathrm{hec}_g(V\mathbb{T}_v)$ are the points $\pi_\infty(\Gamma_S(g, \gamma\, a_v^n\, g')) = \Gamma_\infty\, a_v^{-n}\gamma^{-1} g$ where $g' \in \underline{G}(O_v)$, $\gamma \in \Gamma_\infty$ and $n \in \mathbb{N}$. Therefore $\mathrm{hec}_g(V\mathbb{T}_v) = V\,T_v(x)$.

If $y, y' \in V\mathbb{T}_v$ are joined by an edge in $\mathbb{T}_v$, then again by density of Γ_∞ in $\underline{G}(O_v)$, there exists an element in Γ_∞ mapping the edge between y and y' into the geodesic ray with vertices $(a_v^n *_v)_{n\in\mathbb{N}}$. Up to exchanging y and y', there exists $n \in \mathbb{N}$ and $\gamma \in \Gamma_\infty$ such that $\gamma^{-1} y = a_v^n *_v$ and $\gamma^{-1} y' = a_v^{n+1} *_v$. In particular, $\mathrm{hec}_g(y) = \Gamma_\infty\, a_v^{-n}\gamma^{-1} g$ is joined by an edge to $\mathrm{hec}_g(y') = \Gamma_\infty\, a_v^{-n-1}\gamma^{-1} g$ in the Hecke tree $T_v(x)$. Hence hec_g induces a surjective graph morphism between the trees $\mathbb{T}_v$ and $T_v(x)$. Since both trees are regular of degree $|k_v|+1$, the map hec_g is an isomorphism of trees.

Equation (5) follows by writing $y \in V\mathbb{T}_v = G_v/\underline{G}(O_v)$ as $y = g'\underline{G}(O_v)$ for some $g' \in \Gamma_S$ (see the line following equation (3)), and by using the following equalities:

$$\begin{aligned}\pi_\infty(\Gamma_S(g,g'))\,g^{-1}\gamma_0^{-1} g &= \pi_\infty(\Gamma_S(g'^{-1} g, e))\,g^{-1}\gamma_0^{-1} g = \Gamma_\infty(g'^{-1} g)\; g^{-1}\gamma_0^{-1} g\\ &= \pi_\infty(\Gamma_S(g, \gamma_0 g'))\,.\end{aligned}$$

Let h be another element in G_∞ such that $\Gamma_\infty h = x$. Since $G_v = \Gamma_S\,\underline{G}(O_v)$ and by the definition of π_∞, we have $\mathrm{hec}_g = \mathrm{hec}_h$ if and only if $\Gamma_\infty\gamma^{-1} g = \Gamma_\infty\gamma^{-1} h$ for every $\gamma \in \Gamma_S$, that is $\gamma^{-1}(gh^{-1})\gamma \in \Gamma_\infty$ for every $\gamma \in \Gamma_S$. Writing $gh^{-1} = \begin{bmatrix} a & b\\ c & d\end{bmatrix}$ and using $\gamma = e$, we may take $a, b, c, d \in R_\infty$. Since the order of vanishing at a point of $\mathbf{C} - \{\infty\}$ of the element π_v of R_∞ is nonnegative and $v_v(\pi_v) = 1$, we have $v_\infty(\pi_v) \neq 0$ by the product formula. Taking $\gamma = \begin{bmatrix} \pi_v^n & 0\\ 0 & 1\end{bmatrix}$ gives $\pi_v^n c, \pi_v^{-n} b \in R_\infty$ for every $n \in \mathbb{Z}$, that is $c = b = 0$. Taking $\gamma = \begin{bmatrix} 1 & \pi_v^n\\ 0 & 1\end{bmatrix}$ gives $\pi_v^n(a-d) \in R_\infty$ for every $n \in \mathbb{Z}$, that is $a = d$. Hence $\mathrm{hec}_g = \mathrm{hec}_h$ if and only if gh^{-1} is the identity element in $\Gamma_\infty = \underline{G}(R_\infty)$.

The other claims are left to the reader. □

3. DYNAMICS OF THE MODULAR GROUP AT THE INFINITE PLACE ON THE BRUHAT-TITS TREE AT A FINITE PLACE

Let $(K, v, \underline{G}, \underline{A})$ and the associated notation be as in Section 2.2.

In this section, we study the dynamics of Γ_∞ on the Bruhat-Tits tree $\mathbb{T}_v$ of $(\underline{G}, K_v)$. Since $R_\infty \subset O_v$, the lattice $\Gamma_\infty = \underline{G}(R_\infty)$ is contained in the stabilizer $\underline{G}(O_v)$ in G_v of the base point $*_v$ in $\mathbb{T}_v$. Hence Γ_∞ does act on $\mathbb{T}_v$, and for every $n \in \mathbb{N}$, every $\gamma_0 \in \Gamma_\infty$ preserves the sphere

$$S_v(n) = S_{\mathbb{T}_v}(*_v, n)$$

of center $*_v$ and radius n in $\mathbb{T}_v$. Since $S_v(n)$ is finite, every orbit in $S_v(n)$ of the cyclic group ${\gamma_0}^{\mathbb{Z}}$ generated by γ_0 is periodic. The following linear growth property of these periodic orbits is a remarkable feature of the positive characteristic.

THEOREM 10. *Let γ_0 be an element in Γ_∞ which is loxodromic on $\mathbb{T}_\infty$. Let $\widetilde{K}_v = \widetilde{K}_v(\gamma_0)$ be the splitting field of γ_0 over K_v, with local ring $\widetilde{O}_v$, uniformizer $\widetilde{\pi}_v$ and residual field $\widetilde{k}_v$. Let $e_v = e_v(\gamma_0)$ be the ramification index $e(\widetilde{K}_v, K_v)$ of $\widetilde{K}_v$ over K_v. Let $d_v = d_v(\gamma_0)$ be the smallest positive integer such that the image of ${\gamma_0}^{d_v}$ in $\underline{G}(\widetilde{k}_v)$ (by reduction modulo $\widetilde{\pi}_v \widetilde{O}_v$) is the identity. Let $r_v = r_v(\gamma_0)$ be the biggest positive integer such that the image of ${\gamma_0}^{d_v}$ in $\underline{G}(\widetilde{O}_v / \widetilde{\pi}_v^{r_v+1} \widetilde{O}_v)$ is not the identity. Then there exists a constant $\kappa_v = \kappa_v(\gamma_0) \in \mathbb{N}$ such that for every big enough $n \in \mathbb{N}$, the maximal cardinality $m_n = m_n(\gamma_0)$ of an orbit of ${\gamma_0}^{\mathbb{Z}}$ in $S_v(n)$ satisfies*

$$m_n \le d_v \, p^{\lceil \log_p \frac{e_v n + \kappa_v}{r_v} \rceil} .$$

This result implies that the sequence $(m_n)_{n \in \mathbb{N}}$ has linear growth: for every $n \in \mathbb{N}$ big enough, we have

$$m_n \le \frac{d_v \, p}{r_v} \, (e_v \, n + \kappa_v) \, , \tag{7}$$

and that if γ_0 is diagonalisable over K_v, then for every $k \in \mathbb{N}$ big enough

$$m_{r_v p^k - \kappa_v} \le d_v \, p^k .$$

Proof. We start the proof by the following lemma on the growth of the valuations of the powers of the elements of O_v with their constant terms removed, which concentrates the positive characteristic feature.

LEMMA 11. *Let $a \in k_v^\times$, $\lambda \in a + \pi_v O_v$ and $n \in \mathbb{N}$. Define $m_n(\lambda) = \min\{k \in \mathbb{N} - \{0\} : \lambda^k \in a^k + \pi_v^n O_v\}$ and $r_\lambda = v_v(\lambda - a) > 0$. Then for every $n > r_\lambda$,*

$$m_n(\lambda) = p^{\lceil \log_p \frac{n}{r_\lambda} \rceil} .$$

In particular, $m_n(\lambda) < \frac{p}{r_\lambda} \, n$ for every $n > r_\lambda$ and $m_{r_\lambda p^k}(\lambda) = p^k$ for every $k \in \mathbb{N} - \{0\}$.

Proof. Up to replacing λ by $\frac{\lambda}{a}$, we may assume that $a = 1$. To simplify the notation, let $r = r_\lambda$. For every $k \in \mathbb{N} - \{0\}$, consider the expansion of k in base p given by $k = \sum_{i=0}^{s} a_i p^i$ where $s \in \mathbb{N}$ and $a_i \in \{0, \ldots, p-1\}$. Let

$$v_p(k) = \inf\{i \in \mathbb{N} : \forall \; j < i, \;\; a_j = 0\,\}$$

be the p-adic valuation of k. Then, using the Frobenius automorphism, and the fact that a_i is invertible in the characteristic subfield $\mathbb{F}_p$, hence in O_v, if and only if a_i is nonzero, we have

$$(1 + {\pi_v}^r O_v^\times)^k \subset \prod_{i=0}^{s} (1 + {\pi_v}^{r p^i} O_v^\times)^{a_i} \subset \prod_{0 \le i \le s, \; a_i \ne 0} (1 + {\pi_v}^{r p^i} O_v^\times) \subset 1 + {\pi_v}^{r p^{v_p(k)}} O_v^\times \, .$$

Hence for every $n \in \mathbb{N}$, we have $\lambda^k \in 1 + \pi_v^n O_v$ if and only if $r p^{v_p(k)} \ge n$. Therefore, for every $n > r$, if $r p^{m-1} < n \le r p^m$ (that is, if $m = \lceil \log_p \frac{n}{r} \rceil$), we have the equalities $m_n(\lambda) = \min\{k \in \mathbb{N} - \{0\} \, : \, v_p(k) = m\} = p^m$. The result follows. □

Now, let $\gamma_0 \in \Gamma_\infty$ be loxodromic on $\mathbb{T}_\infty$. Note that the constant d_v is well defined since $R_\infty \subset O_v \subset \widetilde{O}_v$. As we have seen in Section 2.3, there exist $\lambda_\pm$ in a finite extension of K such that the element γ_0 is conjugated to $\begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$ and $\widetilde{K_v} = K_v(\frac{\lambda_+}{\lambda_-})$. Note that λ_- and λ_+ are distinct since γ_0 is not the identity element.

Let $\widetilde{\mathbb{T}}_v$ be the Bruhat-Tits tree of $(\underline{G}, \widetilde{K}_v)$, and $\widetilde{*}_v = e\,\underline{G}(\widetilde{O}_v)$ its standard base point in $V\widetilde{\mathbb{T}}_v = \underline{G}(\widetilde{K}_v)/\underline{G}(\widetilde{O}_v)$. The value group of (the unique extension of) the valuation v_v on $\widetilde{K}_v^\times$ contains the value group $\mathbb{Z}$ of the valuation v_v on $K_v^\times$ with index e_v. By the correspondence between the action on the right of $\underline{A}(\widetilde{K}_\omega)$ on $\underline{G}(\widetilde{K}_\omega)/\underline{A}(\widetilde{O}_\omega)$ and the action of the geodesic flow on the geodesic lines in $\widetilde{\mathbb{T}}_v$, the sphere $S_v(n)$ of center $*_v$ and radius n in $\mathbb{T}_v$ is naturally contained in the sphere $S_{\widetilde{\mathbb{T}}_v}(\widetilde{*}_v, e_v\,n)$ of center $\widetilde{*}_v$ and radius $e_v\,n$ in $\widetilde{\mathbb{T}}_v$, for every $n \in \mathbb{N}$. Therefore, up to replacing K_v by $\widetilde{K}_v$, we may assume that γ_0 is diagonalisable over K_v, and we prove that the cardinality of every orbit of ${\gamma_0}^{\mathbb{Z}}$ in $S_v(n)$ is at most $d_v\, p^{\lceil \log_p \frac{n+\kappa_v}{r_v}\rceil}$ for every $n \in \mathbb{N}$, for some $\kappa_v \in \mathbb{N}$.

Note that the coefficients $\lambda_\pm$ have absolute value 1 in K_v. Indeed, $\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$ may be chosen to be conjugated to a representative of γ_0 in $\mathrm{GL}_2(R_\infty)$. Hence $\lambda_\pm$ satisfy an equation $P(\lambda_\pm) = 0$ with P a monic quadratic polynomial with coefficients in $R_\infty \subset O_v$. Therefore $|\lambda_\pm|_v{}^2 \le \max\{|\lambda_\pm|_v, 1\}$, so that $|\lambda_\pm|_v \le 1$, and equality holds by replacing γ_0 by its inverse. Hence $\lambda_\pm \in a_\pm + \pi_v O_v$ with $a_\pm \in k_v^\times$.

By the finiteness of $k_v^\times$, there exists a smallest $d_v \in \mathbb{N} - \{0\}$ such that ${a_-}^{d_v} = {a_+}^{d_v}$. Note that d_v coincides with the notation introduced in the statement of Theorem 10. Let

$$r_v = v_v\Big(\Big(\frac{\lambda_+}{\lambda_-}\Big)^{d_v} - 1\Big)\,. \tag{8}$$

Since γ_0 is loxodromic on $\mathbb{T}_\infty$, no power of γ_0 is the identity, hence $r_v > 0$. Note that r_v coincides with the notation introduced in the statement of Theorem 10. Up to replacing γ_0 by ${\gamma_0}^{d_v}$, to modify $\lambda_\pm$ by a common multiple by an element of $k_v^\times$, and to proving that $m_n(\gamma_0) \le p^{\lceil \log_p \frac{n+\kappa_v}{r_v}\rceil}$ for some $\kappa_v \in \mathbb{N}$ and for n big enough, we may assume that the constant terms in $k_v^\times$ of $\lambda_\pm$ are equal to 1, so that $d_v = 1$.

Since γ_0 is diagonalisable over K_v, there exists $h \in G_v$ such that

$$\gamma_0 = h \begin{bmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{bmatrix} h^{-1}.$$

Since $\lambda_- \neq \lambda_+$, the centralizer $Z_{G_v}(\gamma_0)$ of γ_0 in G_v is the abelian group $h\,A_v h^{-1}$. Note that h is well defined modulo multiplication on the right by an element of A_v.

Let $\ell_0 : n \mapsto a_v^n *_v$ be the geodesic line in $\mathbb{T}_v$ from $0 \in \partial\mathbb{T}_v$ to $\infty \in \partial\mathbb{T}_v$, through $*_v$ at time $n = 0$, which is pointwise fixed by $\underline{A}(O_v)$. The group A_v preserves $\ell_0(\mathbb{Z})$ and acts transitively on it. Note that the projective action of $\underline{A}(O_v)$ on $\mathbb{P}^1(K_v)$ fixes 0 and ∞, and acts transitively on $\pi_v^{-k} O_v^\times \subset \mathbb{P}^1(K_v)$ for every $k \in \mathbb{Z}$.

The geodesic line $\ell = h\,\ell_0$ is pointwise fixed by $h\,\underline{A}(\mathcal{O}_v)\,h^{-1}$. Up to multiplying h on the right by an element of A_v, we may assume that the closest point to $*_v$ on (the image of) ℓ is $h *_v = \ell(0)$. Let $s_v = s_v(\gamma_0) \in \mathbb{N}$ be the distance between $*_v$ and $h *_v$ in $\mathbb{T}_v$ (see Figure 3).

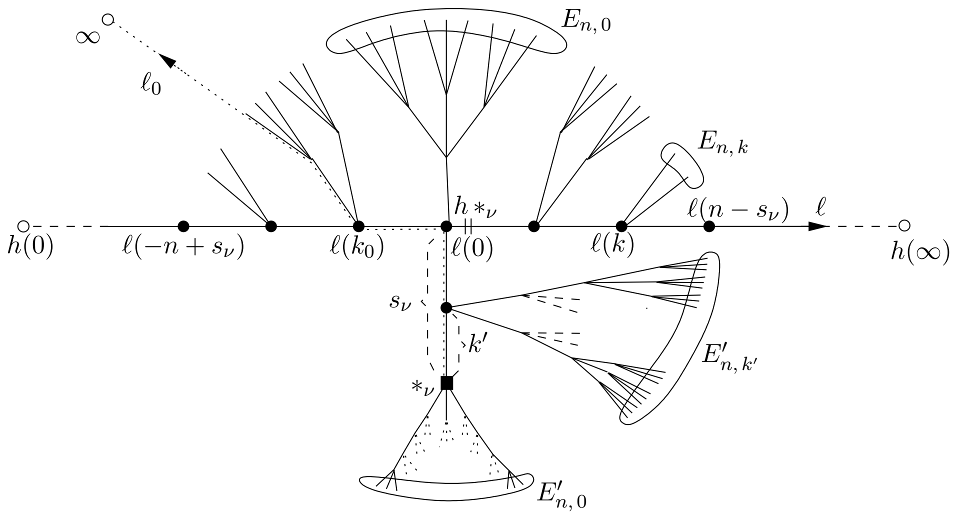


FIGURE 3. A partition of the Hecke sphere $S_v(n)$

Let $\mathrm{pr}_\ell : V\mathbb{T}_v \to \ell(\mathbb{Z})$ be the closest point map on the geodesic line ℓ. For all $n \in \mathbb{N}$ and $k \in \mathbb{N} - \{0\}$, define (see Figure 3)

$$E_{n,k} = \{x \in S_v(n) \,:\, \mathrm{pr}_\ell(x) = \ell(k)\}$$

and

$$E_{n,0} = \big\{x \in S_v(n) \,:\, \mathrm{pr}_\ell(x) = \ell(0), \quad [h *_v, *_v] \cap [h *_v, x] = \{h *_v\}\big\}\,.$$

For all $n, k' \in \mathbb{N}$ with $0 \le k' < s_v$, let $E'_{n,k'}$ be the set of $x \in S_v(n)$ such that the length of the common segment $[*_v, h *_v] \cap [*_v, x]$ is equal to k'. Then we have a partition

$$S_v(n) = \bigcup_{0 \le k' < s_v} E'_{n,k'} \cup \bigcup_{-n \le k \le n} E_{n,k}\,.$$

Since γ_0 fixes $*_v$, $h(0)$ and $h(\infty)$, it pointwise fixes $\ell(\mathbb{Z}) \cup [*_v, h *_v]$. Hence the above partition of $S_v(n)$ is invariant under γ_0.

Note that $E_{n,k}$ is contained in the set of points at distance $n - |k| - s_v$ from $h a_v^k *_v = \ell(k)$ on a geodesic ray from $h a_v^k *_v$ to a point in $h(\pi_v^{-k} \mathcal{O}_v^\times) \subset \mathbb{P}^1(K_v)$. Hence for any two points in $E_{n,k}$ (with n, k fixed), there exists an element in the centralizer of γ_0 mapping one to the other. In particular, the cardinality $c_{n,k} = \mathrm{Card}(\gamma_0^{\mathbb{Z}} y)$ is independent of $y \in E_{n,k}$.

Since $h a_v^{k-1} h^{-1}$ centralizes γ_0 and $h a_v^{-k+1} h^{-1} E_{n,k} \subset E_{n-|k|+1,1}$, we have $c_{n,k} = c_{n-|k|+1,1}$. For every $n' \in \mathbb{N}$, we have $c_{n',1} \le c_{n'+1,1}$, since the closest point map $E_{n'+1,1} \to E_{n',1}$ is onto and equivariant under γ_0.

Every point of $E'_{n,k'}$ is at distance $n+s_v-2k'$ from $h*_v$. Hence $ha_vh^{-1}(E'_{n,k'}) \subset E_{n+2s_v-2k'+1,1}$. Therefore $c'_{n,k'} = \mathrm{Card}(\gamma_0^{\mathbb{Z}}y)$ is independent of $y \in E'_{n,k'}$ and satisfies $c'_{n,k'} = c_{n+2s_v-2k'+1,1}$.

In particular, for every $n > s_v$, we have

$$m_n(\gamma_0) = \max\big\{ \max_{0\le k'<s_v} c'_{n,k'}, \max_{|k|\le n} c_{n,k}\big\} = c_{n+2s_v+1,1}\,.$$

Note that $h(0)$ and $h(\infty)$ do not belong to $\mathbb{P}_1(K)$, since γ_0, being loxodromic on $\mathbb{T}_\infty$, fixes no point of $\mathbb{P}_1(K)$. The positive subray of ℓ_0 hence has no subray whose image is entirely contained in the image of ℓ. Therefore $\ell_0(\mathbb{N})\cap\ell(\mathbb{Z})$ is either empty or the set of vertices of a compact interval $[\ell(0),\ell(k_0)]$ for some $k_0\in\mathbb{Z}$.

Assume first that $\ell_0(\mathbb{N})\cap\ell(\mathbb{Z})$ is empty. Then the segment $[*_v, h*_v]\cap[*_v,\infty[$ has length $k'_0 \in [0,s_v[\,\cap\mathbb{N}$. Define $\kappa = 2k'_0\in\mathbb{N}$. Since the point $a_v^{n'}*_v$ belongs to E'_{n',k'_0} for every $n'\ge k'_0$, the number $m_n(\gamma_0) = c_{n+2s_v+1,1} = c'_{n+2k'_0,k'_0}$ is the cardinality of the orbit under $\gamma_0^{\mathbb{Z}}$ of $a_v^{n+\kappa}*_v = [O_v\times{\pi_v}^{n+\kappa}O_v]$, if n is big enough.

Assume now that $\ell_0(\mathbb{N})\cap\ell(\mathbb{Z}) = [\ell(0),\ell(k_0)]\cap V\mathbb{T}_v$ for some $k_0\in\mathbb{Z}$ (see Figure 3). Define $\kappa = |k_0|+2s_v\in\mathbb{N}$. Since the point $a_v^{n'}*_v$ belongs to E_{n',k_0} for every $n'\ge|k_0|+s_v$, the number $m_n(\gamma_0) = c_{n+2s_v+1,1} = c_{n+|k_0|+2s_v,k_0}$ is the cardinality of the orbit under $\gamma_0^{\mathbb{Z}}$ of $a_v^{n+\kappa}*_v = [O_v\times{\pi_v}^{n+\kappa}O_v]$, if n is big enough.

For all $n\in\mathbb{N}$, an element of $\mathrm{GL}_2(O_v)$ fixes $[O_v\times\pi_v^nO_v]$ if and only if its $(2,1)$-coefficient vanishes modulo π_v^n, that is, if it belongs to the Hecke congruence subgroup of $\mathrm{GL}_2(O_v)$ modulo π_v^n. Let $\Gamma_\infty(\pi_v^n)$ be the kernel of the morphism $\Gamma_\infty\to\underline{G}(R_\infty/\pi_v^nR_\infty)$ of reduction modulo ${\pi_v}^n$. Thus for every $k\in\mathbb{N}$, if γ_0^k belongs to $\Gamma_\infty({\pi_v}^{n+\kappa})$, then it fixes $\ell_0(n+\kappa)$. Therefore, by the proof of Lemma 11 applied with $\lambda=\frac{\lambda_+}{\lambda_-}$, since the constant r_λ of Lemma 11 is equal to r_v by equation (8) since $d_v=1$, we have, if n is big enough,

$$\begin{aligned}
m_n(\gamma_0) &= \min\{k\in\mathbb{N}-\{0\}:\ {\gamma_0}^k\ell_0(n+\kappa)=\ell_0(n+\kappa)\}\\
&\le \min\{k\in\mathbb{N}-\{0\}:\ \gamma_0^k\in\Gamma_\infty({\pi_v}^{n+\kappa})\}\\
&\le \min\{k\in\mathbb{N}-\{0\}:\ \big(\frac{\lambda_+}{\lambda_-}\big)^k\in 1+{\pi_v}^{n+\kappa}O_v\}\\
&= \min\{k\in\mathbb{N}-\{0\}:\ v_p(k)\ge\log_p\frac{n+\kappa}{r_v}\} = p^{\lceil\log_p\frac{n+\kappa}{r_v}\rceil}\,.
\end{aligned}$$

This concludes the proof of Theorem 10. □

4. ESCAPE OF MASS ALONG HECKE RAYS OF A_∞-PERIODIC POINTS

Let $(K,v,\underline{G},\underline{A})$ and the associated notation be as in Section 2.2. We fix from now on an A_∞-periodic point x_0 in $X_\infty=\Gamma_\infty\backslash G_\infty$, as well as a representative g_0 of x_0 in Γ_∞, so that $x_0=\Gamma_\infty g_0$. In this section, we prove our main results on the asymptotic behavior of the A_∞-invariant probability measures μ_x supported on the A_∞-orbits in X_∞ of the vertices x of the v-Hecke tree $T_v(x_0)$ of x_0, as x tends to infinity in this tree along rays. We will recall below a proof that every vertex of $T_v(x_0)$ is indeed A_∞-periodic.

We denote by $\mathscr{P}(\widehat{X_\infty})$ the space of probability measures on the compactification $\widehat{X_\infty} = X_\infty \cup \mathscr{E}_\infty$ of X_∞ by its finite set of cusps $\mathscr{E}_\infty = \Gamma_\infty \backslash \mathbb{P}_1(K)$ (see Section 2.3). Let $\xi \in \partial T_v(x_0)$ be an end of the v-Hecke tree of x_0. Let Θ_ξ be the subset of $\mathscr{P}(\widehat{X_\infty})$ consisting of the weak-star accumulation points of the sequence $(\mu_{x_n^\xi})_{n\in\mathbb{N}}$ of A_∞-invariant probability measures on the vertices $(x_n^\xi)_{n\in\mathbb{N}}$ along the geodesic ray in $T_v(x_0)$ from x_0 to ξ.

For all $c > 0$ and $z \in \mathscr{E}_\infty$, we say that

- ξ has *c-escape of mass* if there exists $\theta \in \Theta_\xi$ with $\theta(\mathscr{E}_\infty) \geq c$.
- ξ has *c-escape of mass towards the cusp z* if there exists $\theta \in \Theta_\xi$ with $\theta(\{z\}) \geq c$.
- ξ has *uniform c-escape of mass* if for every $\theta \in \Theta_\xi$ we have $\theta(\mathscr{E}_\infty) \geq c$.
- ξ has *uniform c-escape of mass towards the cusp z* if for every $\theta \in \Theta_\xi$ we have $\theta(\{z\}) \geq c$.

4.1. **Uniform escape of mass along rational Hecke rays.** We start this section by defining the *rational Hecke rays* in the v-Hecke tree $T_v(x_0)$ of x_0, and we will then prove Theorem 12, a uniform escape of mass phenomenon for the A_∞-invariant probability measures μ_x, as x tends to infinity along these rays.

The group $\underline{G}(K)$ acts transitively on $\mathbb{P}_1(K)$, but its subgroups $\Gamma_\infty = \underline{G}(R_\infty)$ and $\Gamma_S = \underline{G}(R_\infty[\pi_v^{-1}])$ do not in general. The sets $\mathscr{E}_\infty = \Gamma_\infty \backslash \mathbb{P}_1(K)$ (with order at most the class number of R_∞) and $\Gamma_S \backslash \mathbb{P}_1(K)$ are finite and both canonical maps $\Gamma_\infty \backslash \mathbb{P}_1(K) \to \Gamma_S \backslash \mathbb{P}_1(K) \to \underline{G}(K) \backslash \mathbb{P}_1(K)$ may be non-injective. Note that for instance when $\mathbf{C}$ is the projective line over $\mathbb{F}_q$ and ∞ its usual point at infinity, then R_∞ is principal, and Γ_∞ does act transitively on $\mathbb{P}_1(K)$.

Since Γ_∞ preserves $\mathbb{P}_1(K)$ and by the commutativity of the diagram (6), the image $\mathrm{hec}_{g_0}(\mathbb{P}_1(K)) \subset \partial T_v(x_0)$ by hec_{g_0} of the set $\mathbb{P}_1(K)$ of rational points of $\partial \mathbb{T}_v = \mathbb{P}_1(K_v)$ does not depend on the choice of the representative g_0 of x_0, nor does the image by hec_{g_0} of the orbit of ∞ by any subgroup of $\underline{G}(K)$ containing Γ_∞, as for instance $\mathrm{hec}_{g_0}(\Gamma_S \infty)$.

A Hecke ray in $T_v(x_0)$, as well as its point at infinity, is said to be *rational* if its point at infinity belongs to $\mathrm{hec}_{g_0}(\mathbb{P}_1(K))$, and *$S$-rational* if its point at infinity belongs to $\mathrm{hec}_{g_0}(\Gamma_S \infty)$. In particular when Γ_∞ acts transitively on $\mathbb{P}_1(K)$ (that is, when the graph $\Gamma_\infty \backslash \mathbb{T}_\infty$ has only one end, as for instance when $\mathbf{C}$ is the projective line over $\mathbb{F}_q$ and ∞ its usual point at infinity), these two notions coincides. But there are examples of function fields when not all rational ends of $T_v(x_0)$ are S-rational (the two inclusions $\Gamma_\infty \infty \subset \Gamma_S \infty \subset \mathbb{P}_1(K)$ may be strict).

If ξ is a rational end of $T_v(x_0)$, the *cusp of X_∞ associated with* ξ is $z_\xi = \Gamma_\infty \gamma \infty \in \mathscr{E}_\infty$, where $\gamma \in \underline{G}(K)$ is such that $\xi = \mathrm{hec}_{g_0}(\gamma \infty)$. Note that z_ξ does not depend on the choices of g_0 or γ. If ξ is S-rational, we say that z_ξ is an *S-cusp* of X_∞.

THEOREM 12. *There exists $c = c(x_0) > 0$ such that every rational end ξ of the Hecke tree of x_0 has uniform c-escape of mass, and if furthermore ξ is S-rational, then ξ has uniform c-escape of mass towards the cusp of X_∞ associated with ∞.*

Proof. We start the proof by giving some notation. Let us fix elements $\gamma_0 \in \Gamma_\infty$ and $t_0 \in K_\infty^\times$ associated with the chosen representative g_0 of x_0 (see Proposition 8 and its following comment): we have

$$\gamma_0\, g_0 = g_0\, \alpha_\infty(t_0)$$

and $\rho_0 = v_\infty(t_0) > 0$ is the translation distance of γ_0 on $\mathbb{T}_\infty$.

Since $\underline{G}(K)$ acts transitively on $\mathbb{P}_1(K)$ and $\mathscr{E}_\infty = \Gamma_\infty \backslash \mathbb{P}_1(K)$ is finite, there exists a finite subset F_1 of $\underline{G}(K)$ such that $\mathbb{P}_1(K) = \Gamma_\infty F_1 \infty$, and we may assume that $\Gamma_S\infty = \Gamma_\infty(F_1 \cap \Gamma_S)\infty$.

Since $\underline{G}(K)$ commensurates Γ_S, there exists a finite subset F_2 of $\underline{G}(K)$ such that for all $\gamma \in F_1$ and $n \in \mathbb{N}$, there exists $b_{\gamma,n}$ in F_2 such that

$$\gamma\, a_\nu^n\, \gamma^{-1} \in \Gamma_S\, b_{\gamma,n}\,. \tag{9}$$

We assume that $1 \in F_2$ and $b_{\gamma,n} = 1$ if $\gamma \in \Gamma_S$.

For every $b \in \underline{G}(K)$, let $\overline{b} \in \Gamma_S$ be such that $b \in \overline{b}\,\underline{G}(O_\nu)$, which exists by Equation (3). We assume that $\overline{b} = b$ if $b \in \Gamma_S$.

Now that this notation has been given, we consider the rational ends ξ of the Hecke tree $T_\nu(x_0)$. Let $\gamma' = \gamma'_\xi \in \Gamma_\infty$ and $\gamma = \gamma_\xi \in F_1$ be such that $\xi = \mathrm{hec}_{g_0}(\gamma'\gamma\infty)$. We assume that $\gamma \in \Gamma_S$ if ξ is S-rational.

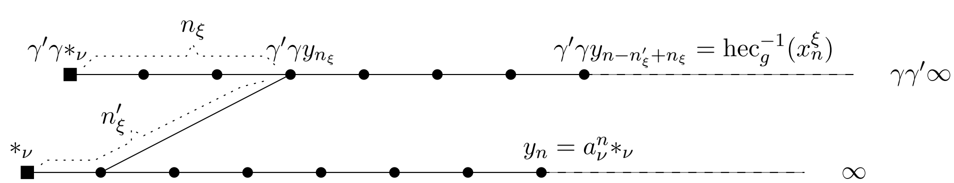


FIGURE 4. Rational Bruhat-Tits rays

For every $n \in \mathbb{N}$, let $y_n = a_\nu^n *_\nu$, so that for every rational end ξ of $T_\nu(x_0)$, the point at infinity of the image by hec_{g_0} of the geodesic ray $n \mapsto \gamma'\gamma y_n$ is ξ. Let $n'_\xi \in \mathbb{N}$ be the distance from $*_\nu$ to this ray, and let $n_\xi \in \mathbb{N}$ be such that

$$[*_\nu, \gamma'\gamma\infty[\, \cap\, [\gamma'\gamma *_\nu, \gamma'\gamma\infty[\, = [\gamma'\gamma y_{n_\xi}, \gamma'\gamma\infty[\,.$$

Let $r_\xi = n_\xi - n'_\xi \in \mathbb{Z}$. Denote by $(x_n = x_n^\xi)_{n\in\mathbb{N}}$ the geodesic ray in the Hecke tree $T_\nu(x_0)$ from x_0 to ξ. Using in the following sequence of equalities

- the definition of hec_{g_0} for the third equality and
- the definition of π_∞ (since $\gamma\, a_\nu^n\, \gamma^{-1} b_{\gamma,n}^{-1} \in \Gamma_S$ by equation (9)) for the fifth one,

we have, for every $n \in \mathbb{N}$ with $n \geq n_\xi$,

$$\begin{aligned} x_{n-r_\xi} &= \mathrm{hec}_{g_0}(\gamma'\gamma y_n) = \mathrm{hec}_{g_0}(\gamma'\gamma\, a_\nu^n *_\nu) = \pi_\infty(\Gamma_S(g_0, \gamma'\gamma\, a_\nu^n)) \\ &= \pi_\infty(\Gamma_S(g_0, \gamma'\gamma\, a_\nu^n\, \gamma^{-1}\, b_{\gamma,n}^{-1}\, b_{\gamma,n}\gamma)) = \Gamma_\infty\, (\gamma'\gamma\, a_\nu^n\, \gamma^{-1} b_{\gamma,n}^{-1}\, \overline{b_{\gamma,n}\gamma})^{-1} g_0 \\ &= \Gamma_\infty\, ((\overline{b_{\gamma,n}\gamma})^{-1} b_{\gamma,n}\gamma)\, a_\nu^{-n}\, (\gamma\gamma')^{-1} g_0\,. \end{aligned} \tag{10}$$

Let $\mathscr{E}_\infty(\xi)$ be the subset of $\mathscr{E}_\infty$ consisting of the elements

$$z_n = \Gamma_\infty (\overline{b_{\gamma,n}\gamma})^{-1} b_{\gamma,n}\gamma 0$$

as n varies. When ξ is S-rational, we have $x_{n-r_\xi} = \Gamma_\infty\, a_v^{-n}(\gamma'\gamma)^{-1} g_0$, and $\mathscr{E}_\infty(\xi)$ is the singleton of the cusp $z_\infty = \Gamma_\infty 0 = \Gamma_\infty\infty$ associated with ∞.

Let $\pi'_\infty : X_\infty = \Gamma_\infty\backslash G_\infty \to \Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty = \Gamma_\infty\backslash G_\infty/\underline{A}(O_\infty)$ be the canonical projection, which is equivariant under $v_\infty : A_\infty \to \mathbb{Z}$ (see Section 2.4). It extends to a continuous map from $\widehat{X_\infty} = X_\infty \sqcup \mathscr{E}_\infty$ to Freudenthal's compactification $\widehat{\Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty} = (\Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty)\sqcup\mathscr{E}_\infty$, by the identity on $\mathscr{E}_\infty$.

By equation (5), since $\gamma_0\, g_0 = g_0\,\alpha_\infty(t_0)$ and by equation (2), for every $k\in\mathbb{N}$ and $y \in V\mathbb{T}_v$, we have

$$\begin{aligned}\pi'_\infty(\mathrm{hec}_{g_0}(\gamma_0{}^k y)) &= \pi'_\infty(\mathrm{hec}_{g_0}(y) g_0{}^{-1}\gamma_0{}^k g_0) = \pi'_\infty(\mathrm{hec}_{g_0}(y)\alpha_\infty(t_0{}^k))\\ &= \phi_{\rho_0 k}(\pi'_\infty(\mathrm{hec}_{g_0}(y)))\,.\end{aligned}$$

In particular, since the orbits of γ_0 on $V\mathbb{T}_v$ are finite, every $x\in VT_v(x_0)$ is also A_∞-periodic and the A_∞-invariant probability measure μ_x on the compact orbit xA_∞ is well defined. Furthermore, with the notation of Theorem 10, for every $n \geq n'_\xi$, the orbit under the geodesic flow of $\pi'_\infty(x_n) = \pi'_\infty(\mathrm{hec}_{g_0}(\gamma'\gamma y_{n+r_\xi}))$ is periodic, with period λ_n bounded as follows:

$$\lambda_n \leq \rho_0 \min\{k\in\mathbb{N}-\{0\} : \gamma_0{}^k\gamma'\gamma y_{n+r_\xi} = \gamma'\gamma y_{n+r_\xi}\} \leq \rho_0\, m_n(\gamma_0)\,. \tag{11}$$

Let d be the distance in the graph $\Gamma_\infty\backslash\mathbb{T}_\infty$. Recall that $p_\infty : X_\infty \to \Gamma_\infty\backslash\mathbb{T}_\infty$ is the map $\Gamma_\infty g \mapsto \Gamma_\infty g *_\infty$ (see the diagram at the beginning of Section 2). Using

- Lemma 7 with $\kappa = \kappa_\xi = d_{\mathbb{T}_v}(*_v, (\gamma'\gamma)^{-1} g_0 *_v)$ for the first inequality,
- the definition of the height (see equation (4)) and equation (10) with the notation $\beta_n = (\overline{b_{\gamma,n+r_\xi}\gamma})^{-1} b_{\gamma,n+r_\xi}\gamma$ for the second equality,

we have, for all $n\in\mathbb{N}$,

$$\begin{aligned}\mathrm{ht}_\infty(x_n) &\geq \mathrm{ht}_\infty(x_n(\gamma^{-1}\gamma'^{-1} g_0)^{-1}) - \kappa\\ &= d(p_\infty(\Gamma_\infty\beta_n\, a_v^{-n-r_\xi}), \Gamma_\infty *_\infty) - \kappa\\ &= d(\Gamma_\infty\beta_n\, a_v^{-n-r_\xi} *_\infty, \Gamma_\infty *_\infty) - \kappa\,.\end{aligned} \tag{12}$$

Recall that β_n belongs to $\underline{G}(K)$, hence preserves the set of geodesic rays in $\mathbb{T}_\infty$ ending in $\mathbb{P}_1(K)\subset\partial\mathbb{T}_\infty$, and takes finitely many values as n varies. Let

$$\kappa' = \kappa'_\xi = \max_{n\in\mathbb{N}} d(\Gamma_\infty\beta_n *_\infty, \Gamma_\infty *_\infty) + \kappa\,.$$

Recall that any geodesic ray in $\mathbb{T}_\infty$ ending in $\mathbb{P}_1(K)$ has a subray that isometrically injects into $\Gamma_\infty\backslash\mathbb{T}_\infty$. Hence, using equation (12) and the triangle inequality, there exist constants $n''_\xi \geq n'_\xi$ and $\kappa''_\xi, \kappa'''_\xi \geq 0$ such that for every integer $n \geq n''_\xi$,

$$\begin{aligned}\mathrm{ht}_\infty(x_n) &\geq d(\Gamma_\infty\beta_n\, a_v^{-n-r_\xi} *_\infty, \Gamma_\infty\beta_n *_\infty) - \kappa'\\ &\geq d_{\mathbb{T}_\infty}(a_v^{-n-r_\xi} *_\infty, *_\infty) - \kappa''_\xi\\ &= (n+r_\xi)\,|v_\infty(\pi_v)| - \kappa''_\xi = n\,|v_\infty(\pi_v)| - \kappa'''_\xi\,.\end{aligned} \tag{13}$$

Since $a_v^{-m} *_v = [\pi_v^m O_v \times O_v]$ tends to $0 \in \mathbb{P}^1(K)$ as $m \to +\infty$, this argument in fact proves that $\mathrm{ht}_{\infty, z_n}(x_n) \geq n|v_\infty(\pi_v)| - \kappa_\xi'''$ for n big enough, where z_n is the cusp defined above.

For every $n \in \mathbb{N}$, let $\mu'_n = (\pi'_\infty)_* \mu_{x_n}$, which is the equiprobability on the finite orbit of $\pi'_\infty(x_n)$ under the geodesic flow on $\Gamma_\infty \backslash \mathscr{G}\mathbb{T}_\infty$. Recall that the pushforwards of measures by proper continuous maps preserve the total mass, and are weak-star continuous. The map π'_∞ is a fibration with compact fiber, hence a proper map. Therefore ξ has uniform c-escape of mass (respectively uniform c-escape of mass towards the cusp $z_\infty = \Gamma_\infty \infty$) if and only if for every weak-star accumulation point θ' of $(\mu'_n)_{n\in\mathbb{N}}$ in the space of probability measures on $\widehat{\Gamma_\infty \backslash \mathscr{G}\mathbb{T}_\infty}$, we have $\theta'(\mathscr{E}_\infty) \geq c$ (respectively $\theta'(\{z_\infty\}) \geq c$).

Let $o : \Gamma_\infty \backslash \mathscr{G}\mathbb{T}_\infty \to \Gamma_\infty \backslash V\mathbb{T}_\infty$ be the origin map $\Gamma_\infty \ell \mapsto \Gamma_\infty \ell(0)$, which is a proper map. For all $N \in \mathbb{N}$, let

$$K_N = o^{-1}\Big(\big\{x \in \Gamma_\infty \backslash V\mathbb{T}_\infty : \; x \in \bigcup_{z \in \mathscr{E}_\infty(\xi)} h_z([0,+\infty[), \;\; d(x, \Gamma_\infty *_\infty) \geq N\big\}\Big),$$

which are open subsets of $\Gamma_\infty \backslash \mathscr{G}\mathbb{T}_\infty$, which accumulate as $N \to +\infty$ exactly to $\mathscr{E}_\infty(\xi) \subset \widehat{\Gamma_\infty \backslash \mathscr{G}\mathbb{T}_\infty}$. By the full-down property (see Section 2.3), the orbit under the geodesic flow of $\pi'_\infty(x_n)$ passes at a distance from $\Gamma_\infty *_\infty$ which is bounded by the diameter N_0 of the finite graph $\Gamma_\infty \backslash \mathbb{T}_\infty - \bigcup_{z \in \mathscr{E}_\infty} h_z(]0+\infty[)$. Recall that this orbit is periodic, of period denoted by λ_n. Hence, if $N \geq N_0$ and if $\mathrm{ht}_\infty(x_n) \geq N$, the origins of $\phi_i(\pi'_\infty(x_n))$ for $0 \leq i \leq \lambda_n$ needs to range twice over all points at distance between N and $\mathrm{ht}_\infty(x_n)$ on a geodesic ray in $\Gamma_\infty \backslash \mathbb{T}_\infty$ between $\Gamma_\infty *_\infty$ and $o(\rho_\infty(x_n))$. Hence, if n is big enough, by the comment following equation (13) and by equation (11), we have

$$\mu'_n(K_N) \geq \frac{2(\mathrm{ht}_\infty(x_n) - N)}{\lambda_n} \geq \frac{2n|v_\infty(\pi_v)| - 2\kappa_\xi''' - 2N}{\rho_0 \, m_n(\gamma_0)} . \tag{14}$$

By the linear growth property of $(m_n(\gamma_0))_{n\in\mathbb{N}}$ (see equation (7) and the notation of Theorem 10), the right hand side of Equation (14) has a limit as $n \to +\infty$ at least

$$c = \frac{2\, r_v(\gamma_0)\, |v_\infty(\pi_v)|}{\rho_0 \, e_v(\gamma_0) \, d_v(\gamma_0) \, p} .$$

Hence for every weak-star accumulation point θ' of $(\mu'_n)_{n\in\mathbb{N}}$, we have $\theta'(\mathscr{E}_\infty(\xi)) \geq c$. This proves the result. □

Remark. The aim of this remark is to give some estimations on the constant c appearing in this proof, and to give examples of full escape of mass along rational Hecke rays.

As above, let $\gamma_0 \in \Gamma_\infty$ be a (primitive loxodromic on $\mathbb{T}_\infty$) element associated with x_0, and let us fix $\widetilde{\gamma_0} \in \mathrm{GL}_2(R_\infty)$ whose image in $\Gamma_\infty = \mathrm{PGL}_2(R_\infty)$ is γ_0. Note that $\det \widetilde{\gamma_0} \in R_\infty^\times = k_\infty^\times$, hence $v_\infty(\det \widetilde{\gamma_0}) = 0$. We may denote by $\lambda_\pm$ the eigenvalues of $\widetilde{\gamma_0}$ with $v_\infty(\lambda_+) > 0$, so that $v_\infty(\lambda_-) = -v_\infty(\lambda_+) < v_\infty(\lambda_+)$ and, by equation (1),

$$\rho_0 = 2|v_\infty(\lambda_-)| = 2|v_\infty(\mathrm{tr}(\widetilde{\gamma_0}))| .$$

With the notation of Theorem 10, let us define

$$\mathrm{LOM}(\gamma_0) = \frac{r_v(\gamma_0)\,|v_\infty(\pi_v)|}{|v_\infty(\mathrm{tr}\,\widetilde{\gamma_0})|\,e_v(\gamma_0)\,d_v(\gamma_0)}\,,$$

so that we chose $c = \mathrm{LOM}(\gamma_0)/p$ in the above proof.

Let us consider $n_k = r_v(\gamma_0)\,p^k - \lceil\frac{\kappa_v(\gamma_0)}{e_v(\gamma_0)}\rceil$ for $k \in \mathbb{N}$ big enough (again with the notation of Theorem 10), so that $m_{n_k}(\gamma_0) \le e_v(\gamma_0)\,d_v(\gamma_0)\,p^k$ by Theorem 10. Using this majoration on the denominator in equation (14), the above proof gives moreover that every weak-star accumulation point θ' of $(\mu'_{n_k})_{k\in\mathbb{N}}$ satisfies $\theta'(\mathscr{E}_\infty(\xi)) \ge \mathrm{LOM}(\gamma_0)$. In particular, the sequence $(\mu_{x_{n_k}})_{k\in\mathbb{N}}$ weak-star converges to the 0 measure on X_∞ if $\mathrm{LOM}(\gamma_0) = 1$. Let us give an example of this when $\mathbf{C}$ is the projective line and ∞ its usual point at infinity. Let $d = d_v(\gamma_0)$, $e = e_v(\gamma_0)$ and $r = r_v(\gamma_0)$, so that $\mathrm{LOM}(\gamma_0) = \frac{r\,|v_\infty(\pi_v)|}{e\,d\,|v_\infty(\mathrm{tr}\,\widetilde{\gamma_0})|}$. Let $\widetilde{k_v}$ be the residual field of the splitting field $\widetilde{K}_v$ of $\widetilde{\gamma_0}$ over K_v.

LEMMA 13. *Assume that the discriminant* $\Delta = (\mathrm{tr}\,\widetilde{\gamma_0})^2 - 4\det\widetilde{\gamma_0}$ *of* $\widetilde{\gamma_0}$ *is irreducible over* $\mathbb{F}_q$*, and let* $\pi_v = \Delta$*. Then* $\mathrm{LOM}(\gamma_0) = 1$.

This assumption is satisfied, for instance, if -1 is not a square modulo p (as for $p = 3$), if $p = q$ and if $\widetilde{\gamma}_0 = \begin{pmatrix} Y & 1 \\ 1 & 0 \end{pmatrix}$, where $Y = \pi_\infty^{-1}$ is the indeterminate in $K = \mathbb{F}_q(Y)$, since $\Delta = Y^2 + 4$. By the previous arguments, for every rational end $\xi \in \Omega$, there exists an element $\theta' \in \Theta_\xi$ which vanishes on X_∞. This proves Theorem 2 in the introduction. The above proof also gives a speed of escape of mass when $\mathrm{LOM}(\gamma_0) = 1$: for every compact subset C of X_∞, we have $\mu_{x_{n_k}}(C) = \mathrm{O}(\frac{1}{n_k})$ when $n_k = r_v(\gamma_0)\,p^k - \lceil\frac{\kappa_v(\gamma_0)}{e_v(\gamma_0)}\rceil$.

Proof. Since Δ is irreducible, we have $p \neq 2$. In particular, the roots of $\widetilde{\gamma}_0$ are $\lambda_\pm = \frac{1}{2}\big(\mathrm{tr}\,\widetilde{\gamma_0} \pm \sqrt{\pi_v}\big)$. We have $\widetilde{K}_v = K_v(\sqrt{\pi_v})$, and $\widetilde{k_v} = k_v$. In particular, the ramification index of the splitting field of $\widetilde{\gamma}_0$ over K_v is $e = 2$. Since the constant terms in $\widetilde{k_v}$ (modulo $\sqrt{\pi_v}$) of $\lambda_\pm$ are equal, we have $d = 1$. Since $\deg(\det\widetilde{\gamma_0}) = 0$, we have

$$|v_\infty(\pi_v)| = \deg\big((\mathrm{tr}\,\widetilde{\gamma_0})^2 - 4\det\widetilde{\gamma_0}\big) = 2\deg(\mathrm{tr}\,\widetilde{\gamma_0}) = 2\,|v_\infty(\mathrm{tr}\,\widetilde{\gamma_0})|\,.$$

Since $\widetilde{\gamma}_0$ is not congruent to the identity modulo $\sqrt{\pi_v}^{\,2} = \pi_v$, we have $r = 1$. Hence $\mathrm{LOM}(\gamma_0) = 1$. □

Let us give one more estimation on the constant $\mathrm{LOM}(\gamma_0)$ when $p \neq 2$ and $v_v(\mathrm{tr}\,\widetilde{\gamma_0}) > 0$. We then have

$$\lambda_\pm = \frac{1}{2}\Big(\mathrm{tr}\,\widetilde{\gamma_0} \pm \sqrt{(\mathrm{tr}\,\widetilde{\gamma_0})^2 - 4\det\widetilde{\gamma_0}}\Big).$$

Since $\det\widetilde{\gamma_0} \in k_\infty^\times \subset O_v^\times$, we have $v_v(-4\det\widetilde{\gamma_0}) = 0$, hence $e = 1$ (and $[\widetilde{k_v} : k_v] = 1$ if $-\det\widetilde{\gamma_0}$ is a square and 2 otherwise). The constant terms $a_\pm = \pm\sqrt{-4\det\widetilde{\gamma_0}} \in$

$\widetilde{k_v}^\times$ of $\lambda_\pm$ are opposite (and nonzero), hence $d=2$. By equation (8), we have $r=v_v\big(\frac{\lambda_+}{\lambda_+}-1\big)=v_v(\operatorname{tr}\widetilde{\gamma_0})$. Furthermore

$$|v_\infty(\operatorname{tr}\widetilde{\gamma_0})|=\deg(\operatorname{tr}\widetilde{\gamma_0})\geq v_v(\operatorname{tr}\widetilde{\gamma_0})\deg\pi_v=r\,|v_\infty(\pi_v)|\,.$$

Hence $\mathrm{LOM}(\gamma_0)\leq\frac{1}{2}$, with equality if and only if $\operatorname{tr}\widetilde{\gamma_0}$ is a constant multiple of a power of π_v, as, for instance, when $\pi_v=Y$ and $\widetilde{\gamma}_0=\begin{pmatrix}Y&1\\1&0\end{pmatrix}$. For these elements where equality holds, at least half the mass escapes to infinity along subsequences of every rational Hecke ray.

4.2. **Escape of mass along uncountably many Hecke rays.** In the previous section, we proved escape of mass phenomena along countably many Hecke rays, the rational ones. In this section, we use the uniformity of the escape of mass in Theorem 12 in order to prove that an escape of mass (towards a prescribed cusp of X_∞) actually occurs along uncountably many Hecke rays. We first introduce some notation that we will use from now on in this paper.

We denote by $\Omega=\partial T_v(x_0)$ the boundary at infinity of the Hecke tree $T_v(x_0)$ of x_0. For every $\xi\in\Omega$, we denote by $[x_0,\xi[$ the geodesic ray in $T_v(x_0)$ starting from x_0 and converging to ξ. We denote by $(x_n^\xi)_{n\in\mathbb{N}}$ the sequence of vertices of $[x_0,\xi[$, in this order along this ray. In particular, $x_0^\xi=x_0$ and $d(x_k^\xi,x_n^\xi)=|k-n|$.

Let $x\in VT_v(x_0)$. We define the *sector* of x by

$$\Omega_x=\big\{\xi\in\Omega\,:\,x\in[x_0,\xi[\big\}\,,$$

the *cone* of x by

$$C_x=\big\{y\in VT_v(x_0)\,:\,\exists\,\xi\in\Omega_x,\;\; y\in[x,\xi[\big\}\,,$$

and, for every $n\in\mathbb{N}$, the *sector-sphere* of x of radius n by

$$S_x^n=C_x\cap S_{T_v(x_0)}(x_0,n)\,.$$

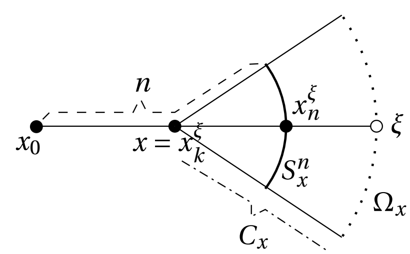


FIGURE 5. Sector-spheres in Hecke trees

The *depth* of the cone C_x or of the sector Ω_x of x is defined to be the distance in the Hecke tree $T_v(x_0)$ from x to x_0. The sector-sphere S_x^n is nonempty if and only if n is at least this depth. For every $\xi\in\Omega$, the sequences $(C_{x_n^\xi})_{n\in\mathbb{N}}$ and $(\Omega_{x_n^\xi})_{n\in\mathbb{N}}$ are strictly decreasing, with $\Omega_{x_0}=\Omega$, $C_{x_0}=VT_v(x_0)$, $\bigcap_{n\in\mathbb{N}}C_{x_n^\xi}=\emptyset$ and $\bigcap_{n\in\mathbb{N}}\Omega_{x_n^\xi}=\{\xi\}$.

Note that if two cones (or sectors) intersect nontrivially, then one of them is contained in the other. Also, sectors are nonempty compact-open sets in Ω and in particular contain infinitely many rational ends, and even infinitely many S-rational ends.

THEOREM 14. *There exists $c = c(x_0) > 0$ such that the set of $\xi \in \Omega$ having c-escape of mass towards the cusp $\Gamma_\infty\infty$ is uncountable.*

In particular, the set of $\xi \in \Omega$ having c-escape of mass is uncountable. Theorem 3 in the introduction follows immediately, being the case when $\mathbf{C}$ is the projective line, in which case X_∞ has only one cusp.

Proof. Let $c = c(x_0) \in]0,1]$ be the constant introduced in Theorem 12. Let $z = \Gamma_\infty\infty \in \mathscr{E}_\infty$. We fix a fundamental system $(V_n)_{n\in\mathbb{N}}$ of open neighborhoods of the cusp z in $\widehat{X_\infty} = X_\infty \cup \mathscr{E}_\infty$, so that $\{z\} = \bigcap_{n\in\mathbb{N}} V_n$. For all $n \in \mathbb{N}$, let $\Sigma_n = \{0,1\}^n$ be the set of words of length n in 0 and 1. Let $\Sigma = \bigcup_{n\in\mathbb{N}} \Sigma_n$ be the set of finite words in 0 and 1.

We are going to define a map $\psi : \Sigma \to VT_v(x_0)$ with the following properties: For all $n \in \mathbb{N}$ and $\alpha \in \Sigma_n$,

(1) if β is an initial subword of α, then $\Omega_{\psi(\alpha)} \subset \Omega_{\psi(\beta)}$,
(2) if β is an initial subword of α with $\beta \neq \alpha$, then the intersection $\Omega_{\psi(\beta 0)} \cap \Omega_{\psi(\beta 1)}$ is empty,
(3) the depth of the sector $\Omega_{\psi(\alpha)}$ is at least n,
(4) we have $\mu_{\psi(\alpha)}(V_n) \geq c - \frac{1}{n+1}$.

Assume for the moment that such a map ψ is constructed. Let $\Sigma_\infty = \{0,1\}^{\mathbb{N}}$, which is uncountable. For every $w \in \Sigma_\infty$, let w_n be the initial subword of length n of w. Note that by properties (1) and (3), for every $w \in \Sigma_\infty$, the sequence of sectors $(\Omega_{\psi(w_n)})_{n\in\mathbb{N}}$ is strictly nested, and its intersection contains a single point, denoted by ξ_w. Furthermore, for every $n \in \mathbb{N}$, we have $w_n \in [x_0, \xi_w[$. Note that by property (2), the map $w \mapsto \xi_w$ from Σ_∞ to Ω is injective. By property (4), for every $w \in \Sigma_\infty$, if θ_w is a weak-star accumulation point of $(\mu_{\psi(w_n)})_{n\in\mathbb{N}}$ in the space $\mathscr{P}(\widehat{X_\infty})$ of probability measures on the compact space X_∞, then $\theta_w(\{z\}) \geq c$. Hence ξ_w has c-escape of mass towards the cusp z. This proves Theorem 14.

We now build $\psi_{|\Sigma_n}$ by induction on $n \in \mathbb{N}$. Note that Σ_0 is reduced to the empty word $\emptyset$, and define $\psi(\emptyset) = x_0$. Note that properties (1)–(4) with $n = 0$ are then satisfied. Let $n \in \mathbb{N}$, assume that $\psi_{|\Sigma_n}$ is constructed, satisfying properties (1)–(4) for every $\alpha \in \Sigma_n$. For every $\alpha \in \Sigma_n$ and $j \in \{0,1\}$, let us now define $\psi(\alpha j)$.

By density, there exist distinct points ξ_0 and ξ_1 in $\Omega_{\psi(\alpha)}$ which are rational and whose associated cusps z_{ξ_0} and z_{ξ_1} of X_∞ respectively are both equal to z. By Theorem 12, ξ_0 and ξ_1 both have uniform c-escape of mass towards the cusp z.

For all $j \in \{0,1\}$ and $m \geq d(x_0, \psi(\alpha)) + 1$, the sector $\Omega_{x_m^{\xi_j}}$ is strictly contained in $\Omega_{\psi(\alpha)}$ and has depth at least $n+1$ by induction. Since $\xi_0 \neq \xi_1$, there exists

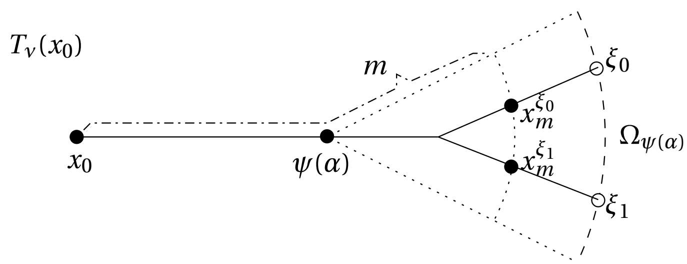


FIGURE 6. Iterated construction of nested sectors

$m_0 \in \mathbb{N}$ such that $x^{\xi_0}_{m_0} \neq x^{\xi_1}_{m_0}$, so that for every $m, m' \geq m_0$, the sectors $\Omega_{x^{\xi_0}_m}$ and $\Omega_{x^{\xi_1}_{m'}}$ are disjoint.

Let $j \in \{0,1\}$. We claim that there exists $n_j \geq m_0$ such that $\mu_{x^{\xi_j}_{n_j}}(V_{n+1}) \geq c - \frac{1}{n+2}$. Otherwise, for every accumulation point θ of $\big(\mu_{x^{\xi_j}_m}\big)_{m\in\mathbb{N}}$, we have $\theta(\{z\}) \leq c - \frac{1}{n+2}$, which contradicts the fact that ξ_j has uniform c-escape of mass towards the cusp z.

Defining $\psi(\alpha 0) = x^{\xi_0}_{n_0}$ and $\psi(\alpha 1) = x^{\xi_1}_{n_1}$ gives the result. □

4.3. **Effective equidistribution of sector-spheres.** The aim of this section is to prove an effective statement regarding the equidistribution in X_∞ of the sector-spheres of the vertices of the Hecke tree of x_0, Theorem 15, by using the effective decay of matrix coefficients for the action of G_S on $\mathbb{L}^2(X_S)$. This sectorial effective equidistribution result will be the main tool used in Section 4.4 in order to prove Theorem 4 and its improvements. We first introduce some notation.

We denote by $|E|$ the cardinality of any finite set E and by Δ_x the unit Dirac mass at any point x of any measurable space. For all $x \in VT_v(x_0)$ and $n \in \mathbb{N}$ with $n \geq k$ where $k = d_{T_v(x_0)}(x_0, x)$ is the depth of the sector C_x, let $\eta_{n,x}$ be the uniform probability measure on the (finite nonempty) sector-sphere S^n_x:

$$\eta_{n,x} = \frac{1}{|S^n_x|} \sum_{y \in S^n_x} \Delta_y\,,$$

that we consider as a probability measure on the locally compact space X_∞ with support S^n_x. Since the v-Hecke tree of x_0 (as is the Bruhat-Tits tree $\mathbb{T}_v$) is $|\mathbb{P}^1(k_v)|$-regular, note that $|S^n_x| = |k_v|^{n-k}$ if $x \neq x_0$ and $n \geq k$, and that $|S^n_x| = (|k_v|+1)|k_v|^{n-1}$ if $x = x_0$ and $n > 0$.

For every place $\omega \in \mathscr{P}$, we define $W_\omega = \underline{G}(O_\omega)$, which is a maximal compact-open subgroup of G_ω, and $W_S = W_\infty \times W_v \subset G_\infty \times G_v = G_S$, which is a maximal compact-open subgroup of G_S.

We denote by m_∞ (respectively m_S) the Haar measure on G_∞ (respectively G_S), normalized so that $m_\infty(W_\infty) = 1$ (respectively $m_S(W_S) = 1$). We again denote by m_∞ (respectively m_S) the measure on X_∞ (respectively X_S) such that the covering map $G_\infty \to X_\infty = \Gamma_\infty \backslash G_\infty$ (respectively $G_S \to X_S = \Gamma_S \backslash G_S$) locally preserves the measures. Note that this measure on X_∞ (respectively X_S) is nonzero and finite, but is not necessarily a probability measure, the above normalisation of the Haar measures will turn out to be more convenient. For every $k \in [1, +\infty]$, we define $\mathbb{L}^k(X_\infty) = \mathbb{L}^k(X_\infty, m_\infty)$ (respectively $\mathbb{L}^k(X_S) = \mathbb{L}^k(X_S, m_S)$).

The group $G = G_\infty$ (respectively $G = G_S$) acts (on the left) on the complex vector space of maps ψ from $X = X_\infty$ (respectively $X = X_S$) to $\mathbb{C}$, by right translation on the source: For every $g \in G$, if $R_g : X \to X$ is the right translation $x \mapsto xg$, then $g\psi = \psi \circ R_g : x \mapsto \psi(xg)$.

A map ψ from X to $\mathbb{C}$ is *locally constant* if there exists a compact-open subgroup U of $W = W_\infty$ (respectively $W = W_S$) which leaves ψ invariant:

$$\forall\, g \in U, \quad g\psi = \psi\,,$$

or equivalently, if ψ is constant on each orbit of U under the right action of G on X. Note that ψ is continuous, since the orbits of U are compact-open subsets. We define

$$d_\psi = \dim(\mathrm{Vect}_{\mathbb{C}}\, W\psi)$$

as the dimension of the complex vector space generated by the images of ψ under the elements of W, which is finite, and even satisfies $d_\psi \le [W : U]$. We define the *lc-norm* of every bounded locally constant map $\psi : X \to \mathbb{C}$ by

$$\|\psi\|_{lc} = \sqrt{d_\psi}\, \|\psi\|_\infty\,.$$

Though the lc-norm does not satisfy the triangle inequality, we have $\|\lambda\psi\|_{lc} = |\lambda|\, \|\psi\|_{lc}$ for every $\lambda \in \mathbb{C}$. We denote by $lc(X)$ the vector space of bounded locally constant maps ψ from X to $\mathbb{C}$.

Finally, given a set A and maps $f, g : A \to [0, +\infty[$, we will write $f \ll g$ if there exists a constant $c' > 0$ such that $f(a) \le c' g(a)$ for all $a \in A$. If f and g depend on a parameter p, we write $f \ll_p g$ if there exists a constant $c' > 0$, possibly depending on the parameter p, such that $f(a) \le c' g(a)$ for all $a \in A$.

The following result strenghtens the well-known result of equidistribution of full Hecke spheres (see for instance the works of Dani-Margulis [9], Clozel-Oh-Ullmo [7], Clozel-Ullmo [8], Eskin-Oh [13], Benoist-Oh [3] in characteristic 0), to an equidistribution result of sector-spheres, which is furthermore effective. Taking $x = x_0$ gives as a particular case an effective equidistribution result of the full Hecke spheres.

THEOREM 15. *There exists $\delta > 0$ such that for every $x \in VT_v(x_0)$, we have*

$$\left| \frac{m_\infty(\psi)}{m_\infty(X_\infty)} - \eta_{n,x}(\psi) \right| \ll \|\psi\|_{lc}\, e^{-\delta n} \tag{15}$$

for all $n \gg_x 1$ and $\psi \in lc(X_\infty)$.

Proof. Let us fix $x \in VT_v(x_0)$ and $\xi = \xi_x \in \Omega_x$, so that $x = x_k^\xi$ for some fixed $k = k_x \in \mathbb{N}$ (see Figure 5).

Step 1: Thickening the sector-spheres. Note that the sector-spheres are measure zero subsets of X_∞. In order to be able to apply (effective) mixing arguments, we have to replace them by (regular) bump functions around them. In this step, we will define nice compact-open neighborhoods of the sector-spheres, whose characteristic functions will be our bump functions. By the construction of the sector-spheres, it is more natural to lift the sector-spheres in X_S and to work in the bundle X_S over X_∞.

We will hence use a lot the W_v-bundle map π_∞ (see Section 2.3) from $X_S = \Gamma_S \backslash G_S$ to $X_\infty = \Gamma_\infty \backslash G_\infty$, defined by $\Gamma_S(g,h) \mapsto \Gamma_\infty g$ whenever $h \in W_v$. Recall (see Section 2.5) that the map hec_{g_0} from the Bruhat-Tits tree $\mathbb{T}_v$ to the Hecke tree $T_v(x_0)$, defined on $V\mathbb{T}_v = G_v/W_v$ by $hW_v \mapsto \pi_\infty(\Gamma_S(g_0,h))$ is an isomorphism of trees, and we identify $\partial \mathbb{T}_v = \mathbb{P}_1(K_v)$ and Ω by (the extension to the boundary at infinity of) this map. We endow $T_v(x_0)$ with the (left) action of G_v making hec_{g_0} equivariant. Since $W_v = \underline{G}(O_v)$ acts transitively on $\Omega = \mathbb{P}_1(O_v)$, we also fix $w = w_x \in W_v$ such that $w\infty = \xi$, where $\infty = [1:0]$.

For all $n \in \mathbb{N}$, we denote by B_v^n the stabiliser in W_v of the point x_n^∞ at distance n from x_0 on the geodesic ray $[x_0, \infty[$ in the Hecke tree $\mathbb{T}_v(x_0)$. The group $B_v = B_v^k$ acts transitively on the sector-spheres $S^n_{x_k^\infty}$ of x_k^∞ for all $n \in \mathbb{N}$. As we have already seen, for all $n \in \mathbb{N}$, we have

$$x_n^\infty = \mathrm{hec}_{g_0}(a_v^n *_v) = \pi_\infty(\Gamma_S(g_0, a_v^n))\,.$$

Note that $x_n^\xi = w x_n^\infty$ for all $n \in \mathbb{N}$. In particular, $x = w x_k^\infty$, hence $wB_v w^{-1}$ is the stabilizer in W_v of x. It acts transitively on the sector-spheres S_x^n of x for all $n \in \mathbb{N}$, with stabilizer of x_n^ξ equal to $wB_v^n w^{-1}$. Therefore, for all $n \in \mathbb{N}$,

$$S_x^n = wB_v w^{-1} x_n^\xi = wB_v x_n^\infty = \pi_\infty(\Gamma_S(g_0, wB_v a_v^n))\,. \tag{16}$$

Now that we have this nice description of the sector-spheres, let us define nice neighborhoods of them.

LEMMA 16. *There exist $\sigma_1, \sigma_2 > 0$ and a nondecreasing family $(B_\infty^\epsilon)_{\epsilon > 0}$ of compact-open subgroups of W_∞, which is a fundamental system of neighborhoods of the identity element in W_∞, and which satisfies*

$$\forall\, \epsilon > 0, \quad \sigma_1\, \epsilon^{-1} \le [W_\infty : B_\infty^\epsilon] \le \epsilon^{-1}\,, \tag{17}$$

and

$$\forall\, a \in A_\infty, \quad a^{-1} B_\infty^\epsilon a \subset B_\infty^{\epsilon\, e^{\sigma_2 |v_\infty(a)|}}\,. \tag{18}$$

Proof. For every $n \in \mathbb{N}$, let Z_n be the kernel of the reduction modulo π_∞^{n+1} map from $W_\infty = \underline{G}(O_\infty)$ to the finite group $\underline{G}(O_\infty/\pi_\infty^{n+1} O_\infty)$. Note that $Z_{n+1} \subset Z_n$. Let us consider $B_\infty^\epsilon = Z_{n_\epsilon}$ with $n_\epsilon = \lfloor \frac{-\log(\epsilon [W_\infty : Z_0])}{\log [Z_0 : Z_1]} \rfloor$ and $\sigma_1 = \frac{1}{[Z_0 : Z_1]}$. Then B_∞^ϵ is a compact-open subgroup of W_∞, we have $B_\infty^\epsilon \subset B_\infty^{\epsilon'}$ if $\epsilon \le \epsilon'$ and $\bigcap_{\epsilon > 0} B_\infty^\epsilon = \{1\}$. Equation (17) follows since the index $[Z_n : Z_{n+1}]$ is constant, hence $[W_\infty : Z_n] = [W_\infty : Z_0][Z_0 : Z_1]^n$.

For all $a, b, c, d \in K_\infty$ and $t \in K_\infty^\times$, we have

$$\begin{pmatrix}1 & 0\\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} = \begin{pmatrix}a & t\,b\\ t^{-1}c & d\end{pmatrix}.$$

Hence, using the isomorphism $\alpha_\infty : K_\infty^\times \to A_\infty$ defined in Section 2.2, we have $a^{-1} Z_n a \subset Z_{n-|v_\infty(a)|}$ for all $a \in A_\infty$ and $n \geq |v_\infty(a)|$ in $\mathbb{N}$. Equation (18) (which will only be used in Section 4.4) follows with $\sigma_2 = \log[Z_0 : Z_1]$. □

For every $\epsilon > 0$, we finally define the following compact-open subset of X_S

$$U_\epsilon = \Gamma_S(g_0 B_\infty^\epsilon, w B_v)\,,$$

so that, for all $n \in \mathbb{N}$, the image $\pi_\infty(U_\epsilon a_v^n)$ of its translate by a_v^n is a (small when ϵ is small) neighborhood of the sector-sphere S_x^n in X_∞, by equation (16).

Step 2: Using the decay of matrix coefficients. In this step, we use the following theorem about effective decay of matrix coefficients for the action of G_S on $\mathbb{L}^2(X_S)$ (see for instance [2]). For every $g = (g_\infty, g_v) \in G_S = G_\infty \times G_v$, we denote by $|g|_S$ the maximum of the norms of the adjoint representations of g_∞, g_v (for the operator norm on the 3×3 matrices with entries in K_∞, K_v).

THEOREM 17. *There exists $\delta_1 > 0$ such that*

$$\Big| m_S(\widetilde{\psi}\,\widetilde{\varphi}\circ R_g) - \frac{1}{m_S(X_S)}\, m_S(\widetilde{\psi})\, m_S(\widetilde{\varphi})\Big| \ll \sqrt{d_{\widetilde{\varphi}}\, d_{\widetilde{\psi}}}\; \|\widetilde{\varphi}\|_2\, \|\widetilde{\psi}\|_2\, |g|_S^{-\delta_1} \tag{19}$$

for all locally constant maps $\widetilde{\varphi}, \widetilde{\psi} \in \mathbb{L}^2(X_S)$ and for every $g \in G_S$. □

Now, let us fix $\psi \in lc(X_\infty)$. We denote by $\widetilde{\psi} = \psi\circ\pi_\infty$ its lift to X_S, which is constant on each right W_v-orbit, hence is locally constant (since invariant under $U \times W_v$ if ψ is invariant under U). Note that $\widetilde{\psi} \in \mathbb{L}^2(X_S)$ since m_S is finite and $\widetilde{\psi}$ is bounded. By the normalization of the Haar measures, we have

$$m_S(\widetilde{\psi}) = m_\infty(\psi) \quad\text{and}\quad m_S(X_S) = m_\infty(X_\infty)\,.$$

Since $\sqrt{d_{\widetilde{\psi}}} = \sqrt{d_\psi}$ and $\|\widetilde{\psi}\|_2 \leq \sqrt{m_S(X_S)}\, \|\widetilde{\psi}\|_\infty$, we have

$$\sqrt{d_{\widetilde{\psi}}}\, \|\widetilde{\psi}\|_2 \ll \|\psi\|_{lc}\,.$$

For every $\epsilon > 0$, let $\varphi_\epsilon = \frac{1}{m_S(U_\epsilon)} \mathbb{1}_{U_\epsilon}$ be the normalized characteristic function of U_ϵ, so that $m_S(\varphi_\epsilon) = 1$. The map $\varphi_\epsilon : X_S \to \mathbb{C}$ is locally constant, since it is invariant under the right action of the compact-open subgroup $B_\infty^\epsilon \times B_v$ of W_S. We have

$$d_{\varphi_\epsilon} = \dim \mathrm{Vect}_{\mathbb{C}}\, W_S\, \varphi_\epsilon \leq [W_S : B_\infty^\epsilon \times B_v] = [W_\infty : B_\infty^\epsilon][W_v : B_v]\,.$$

Since W_v is compact and acts freely on each of its orbits on X_S, there exists $\epsilon_0 = \epsilon_0(x) > 0$ such that if $\epsilon \in]0, \epsilon_0]$, the map from $B_\infty^\epsilon \times B_v$ to X_S defined by $(g, h) \mapsto \Gamma_S(g_0 g, w h)$ is injective, and measure preserving with image U_ϵ. Hence, by the normalization of the Haar measures, we have, for every $\epsilon \in]0, \epsilon_0]$,

$$\|\varphi_\epsilon\|_2 = m_S(U_\epsilon)^{-\frac{1}{2}} = (m_\infty(B_\infty^\epsilon)\, m_v(B_v))^{-\frac{1}{2}} = ([W_\infty : B_\infty^\epsilon]\,[W_v : B_v])^{-\frac{1}{2}}\,. \tag{20}$$

We therefore have $\sqrt{d_{\varphi_\epsilon}}\,\|\varphi_\epsilon\|_2 \le 1$. Note that for every $n\in\mathbb{N}$,

$$|a_v^{-n}|_S = \max\{|a_v^{-n}|_\infty, |a_v^{-n}|_v\} = \max\{|\pi_v^{\pm n}|_\infty, |\pi_v^{\pm n}|_v\} = \max\{|k_\infty|^{n|v_\infty(\pi_v)|}, |k_v|^n\}\,.$$

Applying equation (19) to the functions $\widetilde{\psi}$, $\widetilde{\varphi}=\varphi_\epsilon$ and taking $g=a_v^{-n}$, we hence have, with $\delta_2=\delta_1\max\{|v_\infty(\pi_v)|\log|k_\infty|, \log|k_v|\}>0$, for every $\epsilon\in\,]0,\epsilon_0]$,

$$\Big|\frac{1}{m_S(U_\epsilon)}\int_{U_\epsilon a_v^n}\widetilde{\psi}\,dm_S - \frac{m_\infty(\psi)}{m_\infty(X_\infty)}\Big| \ll \|\psi\|_{lc}\, e^{-\delta_2 n}\,. \tag{21}$$

Let us now relate, for ϵ small enough, the above quantity $\frac{1}{m_S(U_\epsilon)}\int_{U_\epsilon a_v^n}\widetilde{\psi}\,dm_S$ to the average

$$\eta_{n,x}(\psi) = \frac{1}{|S_x^n|}\sum_{y\in S_x^n}\psi(y)$$

of ψ on the sector-sphere S_x^n.

Let $w_1,\dots,w_\ell$ be representatives of the right cosets in B_v/B_v^n, so that B_v is a disjoint union $B_v=\coprod_{i=1}^{\ell} w_iB_v^n$ and $m_v(B_v)=[B_v:B_v^n]\,m_v(B_v^n)$. By the transitivity properties seen in Step 1, the map from B_v/B_v^n to S_x^n defined by $[h]\mapsto whw^{-1}x_n^\xi$ is a bijection. For every $y\in S_x^n$, let $i_y\in\{1,\dots,\ell\}$ be such that

$$y = ww_{i_y}w^{-1}x_n^\xi = ww_{i_y}x_n^\infty = \mathrm{hec}_{g_0}(ww_{i_y}a_v^n *_v) = \pi_\infty(g_0, ww_{i_y}a_v^n)\,.$$

Let

$$V_y = \Gamma_S(g_0B_\infty^\epsilon, ww_{i_y}B_v^n)$$

so that $V_y\,a_v^n = \Gamma_S(g_0B_\infty^\epsilon, ww_{i_y}a_v^n(a_v^{-n}B_v^na_v^n))$. Note that $a_v^{-n}B_v^na_v^n$ is contained in W_v, since B_v^n stabilizes $x_n^\infty = a_v^nx_0$, hence the restriction of π_∞ to $V_y\,a_v^n$ has image yB_∞^ϵ and its fibers are orbits of $a_v^{-n}B_v^na_v^n$. For every $\epsilon\in\,]0,\epsilon_0]$, since the map $(g,h)\mapsto\Gamma_S(g_0g,wh)$ from $B_\infty^\epsilon\times B_v$ to X_S is injective, we hence have

$$U_\epsilon = \coprod_{i=1}^{\ell}\Gamma_S(g_0B_\infty^\epsilon, ww_iB_v^n) = \coprod_{y\in S_x^n} V_y\,.$$

Therefore, for every $\epsilon\in\,]0,\epsilon_0]$, using equation (20) and by desintegration of m_S, we have

$$\begin{aligned}\frac{1}{m_S(U_\epsilon)}\int_{U_\epsilon a_v^n}\widetilde{\psi}\,dm_S &= \frac{1}{m_\infty(B_\infty^\epsilon)\,m_v(B_v)}\sum_{y\in S_x^n}\int_{V_ya_v^n}\widetilde{\psi}\,dm_S\\ &= \frac{1}{m_\infty(B_\infty^\epsilon)\,m_v(B_v^n)\,|S_x^n|}\sum_{y\in S_x^n}\int_{\pi_\infty(V_ya_v^n)} m_v(a_v^{-n}B_v^na_v^n)\,\psi\,dm_\infty\\ &= \frac{1}{|S_x^n|}\sum_{y\in S_x^n}\frac{1}{m_\infty(B_\infty^\epsilon)}\int_{yB_\infty^\epsilon}\psi\,dm_\infty\,. \end{aligned}\tag{22}$$

Define the ϵ-*thin part* X_∞^ϵ of X_∞ as the set of points $z\in X_\infty$ such that the map from B_∞^ϵ to X_∞ defined by $h\mapsto zh$ is not injective. Since ψ is locally constant, there exists $\epsilon_1=\epsilon_1(\psi)>0$ such that if $\epsilon\in\,]0,\epsilon_1]$, then ψ is B_∞^ϵ-invariant. If $y\in S_x^n-(S_x^n\cap X_\infty^\epsilon)$ and if $\epsilon\in\,]0,\epsilon_1]$, then

$$\frac{1}{m_\infty(B_\infty^\epsilon)}\int_{yB_\infty^\epsilon}\psi\,dm_\infty = \psi(y)\,. \tag{23}$$

A trivial majoration gives

$$\Big|\frac{1}{|S_x^n|}\sum_{y\in S_x^n\cap X_\infty^\epsilon}\Big(\psi(y)-\frac{1}{m_\infty(B_\infty^\epsilon)}\int_{B_\infty^\epsilon}\psi\, dm_\infty\Big)\Big|\le 2\,\|\psi\|_\infty\frac{|S_x^n\cap X_\infty^\epsilon|}{|S_x^n|}\,. \tag{24}$$

Separating the summation over S_x^n on one hand over $S_x^n\cap X_\infty^\epsilon$ and on the other hand over $S_x^n-(S_x^n\cap X_\infty^\epsilon)$, for every $\epsilon\in\,]0,\min\{\epsilon_0,\epsilon_1\}]$, by equations (21), (22), (23) and (24), we hence have

$$\Big|\eta_{n,x}(\psi)-\frac{m_\infty(\psi)}{m_\infty(X_\infty)}\Big|\ll\|\psi\|_{lc}\,e^{-\delta_2 n}+\|\psi\|_\infty\frac{|S_x^n\cap X_\infty^\epsilon|}{|S_x^n|}\,. \tag{25}$$

Step 3: Estimating the thin part of sector-spheres. The aim of this step is to prove that the part of the sector-spheres contained in the thin part of X_∞ is negligible, if ϵ is well-chosen. More precisely, let us prove that there exists $\delta_4>0$ such that for every $n\gg_x 1$, if $\epsilon=e^{-\delta_2 n}$, then

$$\frac{|S_x^n\cap X_\infty^\epsilon|}{|S_x^n|}\ll e^{-\delta_4 n} \tag{26}$$

for every $n\in\mathbb{N}$ with $n>k$.

For this, we will apply the arguments of Step 2 to a particular map $\psi=\psi_\epsilon$, where ψ_ϵ is, for every $\epsilon>0$, the characteristic function of the ϵ-thick part $X_\infty-X_\infty^\epsilon$ of X_∞. Note that ψ_ϵ is invariant under B_∞^ϵ, hence ψ_ϵ is bounded and locally constant, and $\epsilon_1(\psi_\epsilon)=+\infty$. Denoting by $\widetilde{\psi}_\epsilon$ the lift of ψ_ϵ to X_S, by equations (22) and (23) applied with $\psi=\psi_\epsilon$, for every $\epsilon\in\,]0,\epsilon_0]$, we have

$$\frac{1}{m_S(U_\epsilon)}\int_{U_\epsilon a_v^n}\widetilde{\psi}_\epsilon\,dm_S=\frac{1}{|S_x^n|}\sum_{y\in S_x^n}\psi_\epsilon(y)=\frac{|S_x^n-(S_x^n\cap X_\infty^\epsilon)|}{|S_x^n|}\,. \tag{27}$$

By equation (17), we have

$$\|\psi_\epsilon\|_{lc}=\sqrt{d_{\psi_\epsilon}}\,\|\psi_\epsilon\|_\infty\le\sqrt{[W_\infty:B_\infty^\epsilon]}\le\epsilon^{-\frac12}\,. \tag{28}$$

By the exponential decay of the volumes in the cusps of the graph of groups $\Gamma_\infty\backslash\backslash\mathbb{T}_\infty$, hence in X_∞, there exists $\delta_3>0$ such that

$$m_\infty(X_\infty^\epsilon)\ll\epsilon^{\delta_3} \tag{29}$$

for every $\epsilon>0$.

For every $n\in\mathbb{N}$, define $\epsilon=e^{-\delta_2 n}$. Note that if $n\gg_x 1$, then $\epsilon\le\epsilon_0$ (recall that ϵ_0 depends on x). Therefore, using

- equation (29) for the second inequality,
- equation (27) and the definition of ψ_ϵ for the third line,
- equation (21) with $\psi=\psi_\epsilon$ for the fourth inequality,
- equation (28) for the fifth inequality,
- the definition of ϵ and the constant $\delta_4=\delta_2\min\{\delta_3,\frac12\}>0$ for the last inequality,

we have, for all $n \gg_x 1$,

$$\begin{aligned}\frac{|S_x^n \cap X_\infty^\epsilon|}{|S_x^n|} &\le \Big|\frac{|S_x^n \cap X_\infty^\epsilon|}{|S_x^n|} - \frac{m_\infty(X_\infty^\epsilon)}{m_\infty(X_\infty)}\Big| + \frac{m_\infty(X_\infty^\epsilon)}{m_\infty(X_\infty)}\\ &\ll \Big|\frac{|S_x^n - (S_x^n \cap X_\infty^\epsilon)|}{|S_x^n|} - \frac{m_\infty(X_\infty - X_\infty^\epsilon)}{m_\infty(X_\infty)}\Big| + \epsilon^{\delta_3}\\ &= \Big|\frac{1}{m_S(U_\epsilon)}\int_{U_\epsilon a_v^n} \widetilde{\psi}_\epsilon \, dm_S - \frac{m_\infty(\psi_\epsilon)}{m_\infty(X_\infty)}\Big| + \epsilon^{\delta_3}\\ &\ll \|\psi_\epsilon\|_{lc}\, e^{-\delta_2 n} + \epsilon^{\delta_3} \le \epsilon^{-\frac{1}{2}} e^{-\delta_2 n} + \epsilon^{\delta_3} \le 2\, e^{-\delta_4 n}.\end{aligned}$$

This proves claim (26) of Step 3.

Step 4: Conclusion. Since $\|\psi\|_\infty \le \|\psi\|_{lc}$, Theorem 15 now follows from equations (25) and (26), with $\delta = \min\{\delta_2, \delta_4\}$. □

4.4. **Exotic behavior of A_∞-periodic measures along Hecke rays.** In this final section, we use the tools introduced in Sections 4.1 and 4.3 to construct even more exotic asymptotic behaviors of the A_∞-periodic measures μ_x as x varies along geodesic rays in the Hecke tree of x_0.

THEOREM 18. *Let $(\mu_i)_{i\in\mathbb{N}}$ be an enumeration of all periodic A_∞-invariant probability measures on X_∞. There exist $c, c' > 0$ such that the set of $\xi \in \Omega$ having c-escape of mass towards the cusp of X_∞ associated with ∞ and verifying*

$$\forall\, i \in \mathbb{N}, \exists\, \theta_i \in \Theta_\xi, \quad c' \mu_i \le \theta_i$$

is uncountable. In particular, $\{\xi \in \Omega : |\Theta_\xi| = \infty\}$ is uncountable.

Note that there are indeed only countably many periodic A_∞-orbits, and that this result immediately implies Theorem 4.

The proof of Theorem 18 relies on the following two lemmas. We consider again the family $(B_\infty^\epsilon)_{\epsilon>0}$ of compact-open subgroups of W_∞ constructed in Lemma 16.

LEMMA 19. *There exists $\delta' > 0$ such that for every $x \in VT_v(x_0)$, for every A_∞-periodic point $y_0 \in X_\infty$, and for every $n \gg_{x,y_0} 1$, the intersection $S_x^n \cap (y_0 A_\infty B_\infty^{e^{-\delta' n}})$ is nonempty.*

Proof. Let $\delta' = \frac{\delta}{3}$ where δ is the constant given by our effective equidistribution result of sector-spheres, Theorem 15. For every $\epsilon > 0$, let $\psi_\epsilon = \mathbb{1}_{y_0 A_\infty B_\infty^\epsilon}$ be the characteristic function of the B_∞^ϵ-thickening of the periodic orbit $y_0 A_\infty$, which is bounded and locally constant. We are going to use Theorem 15 applied to $\psi = \psi_\epsilon$ for a suitably chosen ϵ.

There exists $\epsilon_2 = \epsilon_2(y_0) > 0$ such that if $\epsilon \in\,]0, \epsilon_2]$, then the orbit map $B_\infty^\epsilon \to y_0 B_\infty^\epsilon$ is injective. Hence by the normalisation of the Haar measure and by equation (17), we have, for every $\epsilon \in\,]0, \epsilon_2]$,

$$m_\infty(\psi_\epsilon) = m_\infty(y_0 A_\infty B_\infty^\epsilon) \ge m_\infty(y_0 B_\infty^\epsilon) = m_\infty(B_\infty^\epsilon) = \frac{1}{[W_\infty : B_\infty^\epsilon]}.$$

Furthermore,

$$\|\psi_\epsilon\|_{lc} = \sqrt{d_{\psi_\epsilon}}\ \|\psi_\epsilon\|_\infty \le \sqrt{[W : B^\epsilon_\infty]} \le \epsilon^{-\frac{1}{2}}\,.$$

By Theorem 15, there exists $\kappa > 0$ such that if $n \gg_x 1$, we have, for every $\epsilon \in\ ]0, \epsilon_2]$,

$$\eta_{n,x}(\psi_\epsilon) \ge \frac{m_\infty(\psi_\epsilon)}{m_\infty(X_\infty)} - \kappa\,\|\psi_\epsilon\|_{lc}\, e^{-\delta\, n} \ge \frac{\epsilon}{m_\infty(X_\infty)} - \kappa\,\epsilon^{-\frac{1}{2}}\, e^{-\delta\, n}\,.$$

Let us now consider $\epsilon = 2(\kappa\, m_\infty(X_\infty)\, e^{-\delta\, n})^{\frac{2}{3}}$. By the definition of δ', we have $\epsilon \le e^{-\delta' n}$ if $n \gg 1$. If $n \gg_{y_0} 1$, then ϵ belongs to $]0, \epsilon_2]$. The previous centered formula then gives $\eta_{n,x}(\psi_\epsilon) > 0$ if $n \gg_{x,y_0} 1$. Hence if $n \gg_{x,y_0} 1$, the support of the measure $\eta_{n,x}$, which is S^n_x, meets the support of the function ψ_ϵ, which is $y_0 A_\infty B^\epsilon_\infty \subset y_0 A_\infty B^{e^{-\delta' n}}_\infty$, as wanted. □

LEMMA 20. *Let y and x_k, for $k \in \mathbb{N}$, be A_∞-periodic points in X_∞. Suppose that there exist $\sigma, \delta' > 0$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that:*

(1) *For every $k \in \mathbb{N}$, the period, under the geodesic flow in $\Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty$, of $\pi'_\infty(x_k)$ is at most $\sigma\, n_k$.*
(2) *There are infinitely many $k \in \mathbb{N}$ such that $x_k \in y A_\infty B^{e^{-\delta' n_k}}_\infty$.*

Then there exists a weak-star accumulation point θ of $(\mu_{x_k})_{k\in\mathbb{N}}$ such that

$$\frac{\delta'}{2\,\sigma_2\,\sigma}\,\mu_y \le \theta\,.$$

Proof. Recall that σ_2 is given by Lemma 16. Up to extracting a subsequence, we may assume that $x_k \in y A_\infty B^{e^{-\delta' n_k}}_\infty$ for every $k \in \mathbb{N}$.

By assumption (1) and by the equivariance of the canonical bundle map $\pi'_\infty : X_\infty = \Gamma_\infty\backslash G_\infty \to \Gamma_\infty\backslash\mathscr{G}\mathbb{T}_\infty = \Gamma_\infty\backslash G_\infty/\underline{A}(O_\infty)$ with respect to the epimorphism $v_\infty : A_\infty \to \mathbb{Z}$ where $\mathbb{Z}$ acts by the geodesic flow (see Section 2.3 and in particular equation (2)), we have

$$x_k A_\infty = \{x_k a\ :\ a \in A_\infty \text{ and } |v_\infty(a)| \le \sigma\, n_k\}\,.$$

For every $a \in A_\infty$ such that $|v_\infty(a)| \le \frac{\delta'}{2\sigma_2}\, n_k$, we have, by assumption (2) and by equation (18),

$$x_k a \in y A_\infty a^{-1} B^{e^{-\delta' n_k}}_\infty a \subset y A_\infty B^{e^{-\delta' n_k} e^{\sigma_2 |v_\infty(a)|}}_\infty \subset y A_\infty B^{e^{-\frac{\delta'}{2} n_k}}_\infty\,.$$

If m_{A_∞} is the Haar measure of A_∞ normalized so that $m_{A_\infty}(\underline{A}(O_\infty)) = 1$, then the pushforward of m_{A_∞} by v_∞ is the counting measure of $\mathbb{Z}$ and μ_{x_n} is the measure on $x_k A_\infty$ induced by m_{A_∞}, normalized to be a probability measure. Hence $\{x_k a : |v_\infty(a)| \le \frac{\delta'}{2\sigma_2}\, n_k\}$ occupies at least $\frac{\delta'}{2\sigma_2\,\sigma}$ of the total mass of $x_k A_\infty$, and accumulates on $y A_\infty$. Hence at least $\frac{\delta'}{2\sigma_2\,\sigma}$ of the total mass of any weak-star accumulation point of $(\mu_{x_k})_{k\in\mathbb{N}}$ is accumulated on μ_y. □

Proof of Theorem 18. Let $(\mu_i)_{i\in\mathbb{N}}$ be as in this statement and let $z \in \mathscr{E}_\infty$ be the cusp of X_∞ associated with ∞.

Let us denote by $(\eta_n)_{n\in\mathbb{N}}$ a sequence of measures on X_∞ which contains the zero measure as well as all the A_∞-invariant probability measures of the A_∞-periodic points of X_∞, in such a way that each measure appears infinitely many times. Using Lemma 19 and arguing similarly to the proof of Theorem 14, with $(V_n)_{n\in\mathbb{N}}$ a fundamental system of open neighborhoods of z in $\widehat{X_\infty} = X_\infty \cup \mathscr{E}_\infty$, we can build inductively uncountably many sequences $(x_k)_{k\in\mathbb{N}}$ in $V T_v(x_0)$ such that the following holds.

(1) The sequence of cones $(C_{x_k})_{k\in\mathbb{N}}$ is strictly nested, so that if $\bigcap_{k\in\mathbb{N}} \Omega_{x_k} = \{\xi\}$, then $(x_k)_{k\in\mathbb{N}}$ is a subsequence of the sequence $(x_n^\xi)_{n\in\mathbb{N}}$ of vertices of the Hecke ray from x_0 to ξ.

(2) If $(x_k)_{k\in\mathbb{N}} \neq (x'_k)_{k\in\mathbb{N}}$ are two of these sequences, then the sectors Ω_{x_k} and $\Omega_{x'_k}$ are disjoint for k big enough. In particular, the map $(x_k)_{k\in\mathbb{N}} \mapsto \xi = \lim_{k\to\infty} x_k$ is injective, and there are uncountably many such ξ's.

(3) For every $k \in \mathbb{N}$, denoting by n_k the depth of x_k which we may assume to be at least 1,
   (a) if $\eta_k = 0$ then $\mu_{x_k}(V_k) \geq c - \frac{1}{k+1}$, where $c = c(x_0) > 0$ is the constant introduced in Theorem 12,
   (b) if η_k is the A_∞-invariant probability measure on the orbit of an A_∞-periodic point y'_k, then $x_k \in y'_k A_\infty B_\infty^{e^{-\delta' n_k}}$.

Since case (a) occurs infinitely many times, the set Θ_ξ contains a weak-star accumulation point θ of $(\mu_{x_k})_{k\in\mathbb{N}}$ such that $\theta(\{z\}) \geq c$.

Let $i \in \mathbb{N}$, and let y_i be in the support of μ_i. By case (b), since there are infinitely many $k \in \mathbb{N}$ such that $\eta_k = \mu_i$, there are infinitely many $k \in \mathbb{N}$ such that $x_k \in y_i A_\infty B_\infty^{e^{-\delta' n_k}}$. With the terminology of Section 2.4, we use Theorem 10 applied to a loxodromic element γ_0 associated with the chosen representative g_0 of the A_∞-periodic point x_0. This result gives that the period, under the geodesic flow in $\Gamma_\infty \backslash \mathscr{G}\mathbb{T}_\infty$, of $\pi'_\infty(x_k)$ (since C_{x_k} has depth $n_k \geq 1$ in $T_v(x_0)$) is at most $\sigma\, n_k$ for some $\sigma > 0$ (depending only on p, γ_0, v). Applying Lemma 20 with $y = y_i$, the set Θ_ξ contains a weak-star accumulation point θ_i of $(\mu_{x_k})_{k\in\mathbb{N}}$ such that $\frac{\delta'}{2\sigma\sigma_2}\, \mu_i \leq \theta_i$.

This proves the result, with $c' = \frac{\delta'}{2\sigma\sigma_2}$ (which does not depend on i). □

5. GENERALISATION TO RANK-ONE SEMI-SIMPLE GROUPS

The aim of this final section is to explain to which rank-one groups the tools introduced in this paper are applying besides PGL_2. For the readability of this paper, we had restricted ourselves to the case of PGL_2 in Section 2.2. We refer for instance to [29, 30] for the already known content of this section.

Let K be as in Section 2.1. Let $\underline{G}$ be a connected semi-simple linear algebraic group defined over K, with K_∞-rank one. We fix an embedding $\underline{G} \to \mathrm{GL}_N$ for some $N \in \mathbb{N}$. The example considered before Section 5 (and in particular in the introduction) is $\underline{G} = \mathrm{PGL}_2$ (which is adjoint and absolutely simple).

For every $\omega \in \mathscr{P}$ and every algebraic group $\underline{H}$ defined over K_ω (for instance if $\underline{H}$ is defined over K), we set $H_\omega = \underline{H}(K_\omega)$, which is a non-Archimedean Lie group.

Note that when $v \in \mathscr{P}_f$, the K_v-rank of $\underline{G}$ may be 1 (as in the case $\underline{G} = \mathrm{PGL}_2$) or not. For instance, let D be a (finite dimensional) central simple algebra over K which is *ramified at* ∞ (that is, $D_\infty = D \otimes_K K_\infty$ is a division algebra). Then the algebraic group $\underline{G}$ with $\underline{G}(L) = \mathrm{PGL}_2(D \otimes_K L)$ for every K-algebra L is an (adjoint absolutely quasi-simple) connected semi-simple linear algebraic group defined over K, with K_∞-rank one. For all $v \in \mathscr{P}_f$, the group $\underline{G}$ has K_v-rank 1 if and only if D ramifies at v (that is, when $D_v = D \otimes_K K_v$ is a division algebra).

The next two results will be used in Remark 23 to explain the restrictions on the considered algebraic groups. The first one follows from a well-known argument of weak approximation.

LEMMA 21. *Let $v \in \mathscr{P}_f$, if the K_v-rank of $\underline{G}$ is 1, then there exist tori $\underline{A}$ in $\underline{G}$ defined over K, which splits over both K_∞ and K_v (hence is a maximal K_∞-split and K_v-split torus).*

Proof. By [21, Theorem 2] applied to the semisimple connected algebraic group $\underline{G}$ defined over the infinite field K, there exists $m \in \mathbb{N} - \{0\}$ such that the closure of the image of the diagonal embedding of $\underline{G}(K)$ in $G_\infty \times G_v$ contains the subgroup of $G_\infty \times G_v$ generated by m-th powers. Let γ_∞ and γ_v be nontrivial elements in $\underline{G}(K)$ which split over K_∞ and K_v respectively. There hence exists an element in $\underline{G}(K)$ arbitrarily close to both γ_∞^m and γ_v^m, which therefore splits simultaneously over K_∞ and K_v. □

PROPOSITION 22. *Let $\underline{H}$ be an adjoint, absolutely quasi-simple, connected, semi-simple algebraic group over a local field F of F-rank one. Let $\underline{T}$ be a maximal F-split torus, $\underline{Z}$ its centralizer, $\underline{P}$ a minimal parabolic subgroup of $\underline{H}$ over F, and $\underline{U}$ its unipotent radical. If $\underline{H}$ is isomorphic over F to the algebraic group $L \mapsto \mathrm{PGL}_2(D \otimes_F L)$ for every F-algebra L, where D is a central division algebra over F, then $\underline{Z}(F)$ acts transitively on $\underline{U}(F) - \{0\}$ by conjugation. If $\underline{H}$ is isomorphic over F to the algebraic group $L \mapsto \mathrm{PU}_{1,1}(D \otimes_F L)$ for every F-algebra L, where D is a quaternion division algebra over F, then $\underline{Z}(F)$ acts with finitely many orbits on $\underline{U}(F) - \{0\}$ by conjugation. Otherwise, $\underline{Z}(F)$ acts with infinitely many orbits on $\underline{U}(F) - \{0\}$ by conjugation.*

In particular, by the classification theorem [29], if furthermore $F = K_v$ for some $v \in \mathscr{P}$ and $\underline{H}$ is defined and isotropic over K, then $\underline{Z}(F)$ acts transitively on $\underline{U}(F) - \{0\}$ by conjugation if $\underline{H}$ is isomorphic over K to $\mathrm{PGL}_2(\underline{D})$, where $\underline{D}$ is a central division algebra over K, and $\underline{Z}(F)$ acts with finitely many orbits on $\underline{U}(F) - \{0\}$ by conjugation if $\underline{H}$ is isomorphic over K to $\mathrm{PU}_{1,1}(\underline{D})$, where $\underline{D}$ is a quaternion division algebra over K.

Proof. Let $H = \underline{H}(F)$, $T = \underline{T}(F)$, $Z = \underline{Z}(F)$ and $U = \underline{U}(F)$. Let us denote by $[a_{ij}]$ the image in PGL_2 of a matrix (a_{ij}) in GL_2.

If $\underline{U}$ is non-abelian (or equivalently if the (relative) root system of $\underline{H}$ is not reduced), it is easy to see that the action of $\underline{Z}(F)$ on $\underline{U}(F)-\{0\}$ by conjugation has infinitely many orbits. Conversely, assume that $\underline{U}$ is abelian. When $F=\mathbb{C}$ (then $H=\mathrm{PGL}_2(\mathbb{C})$) or $F=\mathbb{R}$ (then $H=\mathrm{PO}(1,n)$), the action of $\underline{Z}(F)$ on $\underline{U}(F)-\{0\}$ by conjugation is transitive. Hence assume that F is non-Archimedean. By the classification theorem [30], up to isomorphism, H is either $\mathrm{PGL}_2(D)$ for a central division algebra D over F, or $\mathrm{PU}_{1,1}(D)$ for a quaternion division algebra D over F and the Hermitian form $h(z_1,z_2)=\overline{z_1}z_2+\overline{z_2}z_1$.

In the first of the above two cases, we may take

$$T=\left\{\begin{bmatrix}a&0\\0&d\end{bmatrix}\,:\,a,d\in F^\times\right\},\ Z=\left\{\begin{bmatrix}a&0\\0&d\end{bmatrix}\,:\,a,d\in D^\times\right\},\ U=\left\{\begin{bmatrix}1&b\\0&1\end{bmatrix}\,:\,b\in D\right\}.$$

The transitivity of the action by conjugation of Z on $U-\{0\}$ follows hence from the transitivity of the action of $D^\times\times D^\times$ on $D-\{0\}$ by $(a,d)\cdot b=abd^{-1}$, which is immediate.

In the second case, we denote by $z\mapsto\overline{z}$ the canonical involution in the quaternion division algebra D over F, by $N:x\mapsto x\overline{x}$ and $\mathrm{Tr}:x\mapsto x+\overline{x}$ its (reduced) norm and trace, and by $(1,i,j,k)$ a standard basis of D over F. Recall that $F^\times/(F^\times)^2$ is finite and nontrivial. Indeed, this group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})\times\big(f^\times/(f^\times)^2\big)$ where f is the (finite) residue field of F, since if $\mathcal{O}_F$ is the local ring and π_F is a uniformizer in F, the map $(n,x)\mapsto\pi_F^n x$ from $\mathbb{Z}\times\mathcal{O}_F^\times$ to $F^\times$ is an isomorphism.

Let $\mathrm{Im}\,D=\{x\in D:\ \mathrm{Tr}(x)=0\}$ be the K-vector space of purely imaginary elements of D, endowed with the action of the orthogonal group $\mathrm{O}(N_{|\mathrm{Im}\,D})$ of the restriction to $\mathrm{Im}\,D$ of the norm. Since $F^\times/(F^\times)^2$ is finite and $N(F^\times)=(F^\times)^2$, there exists a finite subset A of $F^\times$ such that every line in $\mathrm{Im}\,D$ contains a vector whose norm lies in A. By Witt's theorem, the group $\mathrm{O}(N_{|\mathrm{Im}\,D})$ hence acts with finitely many orbits on the lines of $\mathrm{Im}\,D$.

The group $\mathrm{SL}_2(D)$ acts by $g\cdot M={}^t\overline{g}\,Mg$ on the 6-dimensional F-vector space $E=\left\{M=\begin{pmatrix}a&b\\\overline{b}&d\end{pmatrix}:\ a,d\in F,b\in D\right\}$, by preserving the Dieudonné determinant $\det M=ad-N(b)$, which is a quadratic form Q on E. Let $M_0=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, which is a Q-anisotropic element of E. The group $\mathrm{SU}_{1,1}(D)$ is the stabilizer of M_0 in $\mathrm{SL}_2(D)$ for the above action. Let $M_0^\perp$ be the 5-dimensional orthogonal of M_0 in E for Q, which is invariant under $\mathrm{SU}_{1,1}(D)$, and note that the restriction $Q_{|M_0^\perp}$ is non-degenerate. We consider the basis

$$\left(e_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},e_2=\begin{pmatrix}0&i\\\overline{i}&0\end{pmatrix},e_3=\begin{pmatrix}0&j\\\overline{j}&0\end{pmatrix},e_4=\begin{pmatrix}0&k\\\overline{k}&0\end{pmatrix},e_5=\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$$

of $M_0^\perp$, and we sometimes write matrices by blocks in the decomposition

$$(e_1,(e_2,e_3,e_4),e_5)\,.$$

The group $T = \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} : \lambda \in F^\times \right\}$ is a maximal F-split torus in $\mathrm{PO}(Q_{|M_0^\perp})$, whose centralizer Z contains $Z' = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix} : A \in \mathrm{O}(N_{|\mathrm{Im}\, D}) \right\}$. The projective upper triangular subgroup P of $\mathrm{PO}(Q_{|M_0^\perp})$ is a minimal F-parabolic subgroup of $\mathrm{PO}(Q_{|M_0^\perp})$, whose unipotent radical is, by an easy computation,

$$U = \left\{ \begin{bmatrix} 1 & xN(i) & yN(j) & zN(k) & N(xi+yj+zk) \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & y \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} : x, y, z \in F \right\}.$$

The action of Z' on U by conjugation thus identifies with the linear action of $\mathrm{O}(N_{|\mathrm{Im}\, D})$ on $\mathrm{Im}\, D$. The natural map $\mathrm{SU}_{1,1}(D) \to \mathrm{SO}(Q_{|M_0^\perp})$ induced by the action of $\mathrm{SU}_{1,1}(D)$ on $M_0^\perp$ is an isogeny, by the semi-simplicity of $\mathrm{SU}_{1,1}(D)$. Hence the adjoint groups $\mathrm{PU}_{1,1}(D)$ and $\mathrm{PO}(Q_{|M_0^\perp})$ are isomorphic.

For every $\lambda \in F$, the action by conjugation of $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \in T$ on each line in U is the multiplication by λ. Hence the action of T on each line in U is transitive on its nonzero vectors. The fact that the action of $\underline{Z}(F)$ on $\underline{U}(F) - \{0\}$ by conjugation has finitely many orbits hence follows from the fact that the action of $\mathrm{O}(N_{|\mathrm{Im}\, D})$ on the lines of $\mathrm{Im}\, D$ has finitely many orbits. □

REMARK 23. An appropriate version of this paper (including loss of mass phenomena of the homogeneous probability measures on the periodic orbits of the points along appropriate rays of the Hecke tree of any given periodic point of X_∞) is valid when we replace $\underline{G}$ by the linear algebraic group over K defined

- either by $\underline{G}(L) = \mathrm{PGL}_2(D \otimes_K L)$ for every K-algebra L, where D is a (finite dimensional) central division algebra over K which ramifies at the places ∞ and v, and we endow the algebraic group $\underline{G}$ with a R_∞-structure such that $\underline{G}(R_\infty) = \mathrm{PGL}_2(\mathscr{R}_\infty)$ where $\mathscr{R}_\infty$ is a R_∞-order in D (see [22] for any information on orders),
- or by $\underline{G}(L) = \mathrm{PU}_{1,1}(D \otimes_K L)$ for every K-algebra L, where D is a quaternion algebra over K (and the underlying Hermitian form is $(z_1, z_2) \mapsto \overline{z_1} z_2 + \overline{z_2} z_1$),

allowing, thanks to the transitivity properties described in Proposition 22, to prove a modified version of Theorem 10, when we replace Γ_∞ by a congruence subgroup and when we replace $\underline{A}$ by any torus over K in $\underline{G}$ which splits over both K_∞ and K_v (which exists by Lemma 21).

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