

On the nonarchimedean quadratic Lagrange spectra

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Abstract

We study Diophantine approximation in completions of functions fields over finite fields, and in particular in fields of formal Laurent series over finite fields. We introduce a Lagrange spectrum for the approximation by orbits of quadratic irrationals under the modular group. We give nonarchimedean analogs of various well known results in the real case: the closedness and boundedness of the Lagrange spectrum, the existence of a Hall ray, as well as computations of various Hurwitz constants. We use geometric methods of group actions on Bruhat-Tits trees. ¹

1 Introduction

Diophantine approximation in local fields of positive characteristic has been studied for many years, both from the classical viewpoint (see the works of de Mathan, Lasjaunias, Osgood, W. Schmidt, Thakur, Voloch for instance) and from the point of view of arithmetic geometry (see the works of Mahler, Manin, and many others), as well from an ergodic theory viewpoint (see for instance [BeN]). In this paper, we are interested in the approximation by quadratic irrationals: we define and study the *quadratic* Lagrange spectra in completions of function fields over finite fields with respect to the absolute values defined by discrete valuations. In this introduction, we concentrate on the special case of the field of rational fractions and its valuation at infinity. We refer to Section 3 for the definitions and results in the general case, allowing congruence considerations.

Let \mathbb{F}_q be a finite field of order a positive power q of a positive prime. Let $R = \mathbb{F}_q[Y]$, $K = \mathbb{F}_q(Y)$ and $\widehat{K} = \mathbb{F}_q((Y^{-1}))$ be respectively the ring of polynomials in one variable Y over \mathbb{F}_q , the field of rational functions in Y over \mathbb{F}_q and the field of formal Laurent series in Y^{-1} over \mathbb{F}_q . Then \widehat{K} is a nonarchimedean local field, the completion of K with respect to its place at infinity, that is, the absolute value $|\frac{P}{Q}| = q^{\deg P - \deg Q}$ for all $P, Q \in R - \{0\}$. Let

$$K^{(2)} = \{f \in \widehat{K} : [K(f) : K] = 2\}$$

be the set of quadratic irrationals over K in \widehat{K} . Given $f \in \widehat{K} - K$, it is well known that $f \in K^{(2)}$ if and only if the continued fraction expansion² of f is eventually periodic. The projective action of $\Gamma = \mathrm{PGL}_2(R)$ on $\mathbb{P}_1(\widehat{K}) = \widehat{K} \cup \{\infty\}$ preserves $K^{(2)}$, keeping the periodic part of the continued fraction expansions unchanged (up to cyclic permutation

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²See Section 4 for a definition.

and invertible elements). We refer for instance to [Las, Sch, Pau] for background on the above notions.

Now let us fix $\alpha \in K^{(2)}$. We denote by $\alpha^\sigma \in K^{(2)}$ the Galois conjugate of α over K . The *complexity* $h(\alpha) = \frac{1}{|\alpha - \alpha^\sigma|}$ of α was introduced in [HP] and developed in [BPP, §17.2]. It plays the role of the (naive) height of a rational number in Diophantine approximation by rationals, and is an appropriate complexity when studying the approximation by elements in the orbit under the modular group of a given quadratic irrational.³ We refer to the above references for motivations and results, in particular to [HP, Thm. 1.6] for a Khintchine type result and to [BPP, §17.2] for an equidistribution result of the orbit of α under $\mathrm{PGL}_2(R)$.

Let

$$\Theta_\alpha = \mathrm{PGL}_2(R) \cdot \{\alpha, \alpha^\sigma\}$$

be the union of the orbits of α and α^σ under the projective action of $\mathrm{PGL}_2(R)$. Given $x \in \widehat{K} - (K \cup \Theta_\alpha)$, we define the *quadratic approximation constant* of x by

$$c_\alpha(x) = \liminf_{\beta \in \Theta_\alpha, |\beta - \beta^\sigma| \rightarrow 0} \frac{|x - \beta|}{|\beta - \beta^\sigma|}.$$

We define the *quadratic Lagrange spectrum* of α as

$$\mathrm{Sp}(\alpha) = \{c_\alpha(x) : x \in \widehat{K} - (K \cup \Theta_\alpha)\}.$$

Note that $\mathrm{Sp}(\alpha) \subset q^{\mathbb{Z}} \cup \{0, +\infty\}$. It follows from [HP, Thm. 1.6] that if $m_{\widehat{K}}$ is a Haar measure on the locally compact additive group of \widehat{K} , then for $m_{\widehat{K}}$ -almost every $x \in \widehat{K}$, we have $c_\alpha(x) = 0$. Hence in particular, $0 \in \mathrm{Sp}(\alpha)$ and the quadratic Lagrange spectrum is therefore closed. In Section 3, we prove that it is bounded, and we can thus define the (*quadratic*) *Hurwitz constant* of α as $\max \mathrm{Sp}(\alpha) \in q^{\mathbb{Z}}$.

The following theorems, giving nonarchimedean analogs of the results of Lin, Bugeaud and Pejkovic [Lin, Bug1, Pej], say that the quadratic Lagrange spectrum of α is a closed bounded subset of $q^{\mathbb{Z}} \cup \{0\}$ which contains an initial interval, and computes various Hurwitz constants.

Theorem 1.1. *Let α be a quadratic irrational over K in \widehat{K} .*

- (1) (*Upper bound*) *Its quadratic Hurwitz constant satisfies $\max \mathrm{Sp}(\alpha) \leq q^{-2}$.*
- (2) (*Hall ray*) *There exists $m_\alpha \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq m_\alpha$, we have $q^{-n} \in \mathrm{Sp}(\alpha)$.*

In Section 3, we even prove that Assertion (2) of this theorem is valid when K is any function field over \mathbb{F}_q , \widehat{K} is the completion of K at any place of K , and R is the corresponding affine function ring.

Theorem 1.2. *The Hurwitz constant of any quadratic irrational over K in \widehat{K} , whose continued fraction expansion is eventually k -periodic with $k \leq q - 1$, is equal to q^{-2} .*

There are examples of quadratic irrationals for which the quadratic Lagrange spectrum coincides with the maximal Hall ray. The following theorem gives a special case, see Theorem 4.11 for a more general result.

³see [PaP4] for a comparison between the naive height of the algebraic number α and the above complexity of α in the Archimedean case.

Theorem 1.3. *If*

$$\varphi = Y + \frac{1}{Y + \frac{1}{Y + \dots}},$$

then $\text{Sp}(\varphi) = \{0\} \cup \{q^{-n-2} : n \in \mathbb{N}\}$.

In Proposition 4.12, we give a class of quadratic irrationals whose quadratic Lagrange spectrum does not coincide with its maximal Hall ray, in other words, who have *gaps* in their spectrum.

After the first version of this paper was posted on ArXiv, Yann Bugeaud [Bug2] has given a completely different proof of the above results (except the generalisation to function fields), and proved several new theorems giving a more precise description of these spectra. In particular, he proved that all approximation constants for a given quadratic irrational are attained on the other quadratic irrationals, that for every $m \geq 2$ there exists $\beta \in K^{(2)}$ such that $\max \text{Sp}(\beta) = q^{-m}$, and that for all $\ell \in \mathbb{N}$, there exists $\beta \in K^{(2)}$ such that $\text{Sp}(\beta)$ contains exactly ℓ gaps.

In order to explain the origin of our results, recall that for $x \in \mathbb{R} - \mathbb{Q}$, the *approximation constant* of x by rational numbers is

$$c(x) = \liminf_{p,q \in \mathbb{Z}, q \rightarrow +\infty} q^2 \left| x - \frac{p}{q} \right|,$$

and that the *Lagrange spectrum* is $\text{Sp}_{\mathbb{Q}} = \{c(x) : x \in \mathbb{R} - \mathbb{Q}\}$. Numerous properties of the Lagrange spectrum are known, see for instance [CF]. In particular, $\text{Sp}_{\mathbb{Q}}$ is bounded and closed, has maximum $\frac{1}{\sqrt{5}}$, and contains a maximal interval $[0, \mu]$ with $0 < \mu < \frac{1}{\sqrt{5}}$ called a *Hall ray*. Khinchin [Khi] proved that almost every real number is badly approximable by rational numbers, so that the approximation constant vanishes almost surely. Many of these results have been generalised to the Diophantine approximation of complex numbers, Hamiltonian quaternions and for the Heisenberg group, see for example [Poi, Sch1, Sch2, PaP1, PaP2, PaP3].

Let α_0 be a fixed real quadratic irrational number over \mathbb{Q} . For every such number α , let α^σ be its Galois conjugate. Let $\mathcal{E}_{\alpha_0} = \text{PSL}_2(\mathbb{Z}) \cdot \{\alpha_0, \alpha_0^\sigma\}$ be its (countable, dense in \mathbb{R}) orbit for the action by homographies and anti-homographies of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{R} \cup \{\infty\}$. For every $x \in \mathbb{R} - (\mathbb{Q} \cup \mathcal{E}_{\alpha_0})$, the *approximation constant* of x by elements of \mathcal{E}_{α_0} was defined in [PaP4] by

$$c_{\alpha_0}(x) = \liminf_{\alpha \in \mathcal{E}_{\alpha_0} : |\alpha - \alpha^\sigma| \rightarrow 0} 2 \frac{|x - \alpha|}{|\alpha - \alpha^\sigma|},$$

the *quadratic Lagrange spectrum* (or *approximation spectrum*) of α_0 by

$$\text{Sp}(\alpha_0) = \{c_{\alpha_0}(x) : x \in \mathbb{R} - (\mathbb{Q} \cup \mathcal{E}_{\alpha_0})\},$$

and the *Hurwitz constant* of α_0 by $\sup \text{Sp}(\alpha_0)$. We proved that the quadratic Lagrange spectrum of α_0 is bounded and closed, and that an analog of Khinchin's theorem holds. We generalised the definitions and the above results to the approximation of complex numbers and elements of the Heisenberg group. In the latter cases, we also proved the existence of a Hall ray in the spectrum.

In the real case, the existence of a Hall ray in $\mathrm{Sp}(\alpha_0)$ is due to Lin [Lin]. Bugeaud [Bug1] proved that the Hurwitz constant of the Golden Ratio ϕ is equal to $\frac{3}{\sqrt{5}} - 1$, and his conjecture that the Hurwitz constant of any real quadratic irrational is at most $\frac{3}{\sqrt{5}} - 1$ was confirmed by Pejkovic [Pej]. The Hurwitz constant is known explicitly in many 2-periodic continued fraction expansion cases, see [Pej, Lin].

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2 Background on function fields and Bruhat-Tits trees

In this section, we recall the basic notations and properties of function fields K over \mathbb{F}_q and their valuations v , the associated Bruhat-Tits trees \mathbb{T}_v and modular groups Γ_v acting on \mathbb{T}_v . We refer to [Gos, Ros, Ser] for definitions, proofs and further information, see also [BPP, Ch. 14 and 15].

Let \mathbb{F}_q be a finite field of order q with q a positive power of a positive prime.

2.1 Function fields

Let K be a function field over \mathbb{F}_q and let $v : K^\times \rightarrow \mathbb{Z}$ be a (normalised discrete) valuation of K . Let R_v be the affine function ring associated with (K, v) . Let $|\cdot|_v$ be the absolute value on K corresponding to v and let K_v be the completion of K with respect to $|\cdot|_v$. We again denote by v and $|\cdot|_v$ the extensions of v and $|\cdot|_v$ to K_v . Let

$$\mathcal{O}_v = \{x \in K_v : v(x) \geq 0\}$$

be the *valuation ring* of K_v . Its unique maximal ideal is

$$\mathfrak{m}_v = \{x \in K_v : v(x) > 0\}.$$

We denote the cardinality of the *residual field* $k_v = \mathcal{O}_v/\mathfrak{m}_v$ by q_v , so that $|f|_v = q_v^{-v(f)}$ for all $f \in K_v^\times$.

Example 2.1. Let $K = \mathbb{F}_q(Y)$ be the field of rational functions in one variable Y with coefficients in \mathbb{F}_q , let $\mathbb{F}_q[Y]$ be the ring of polynomials in one variable Y with coefficients in \mathbb{F}_q , and let $v_\infty : K^\times \rightarrow \mathbb{Z}$ be the *valuation at infinity* of K , defined on every $P/Q \in K$ with $P, Q \in \mathbb{F}_q[Y] - \{0\}$ by

$$v_\infty(P/Q) = \deg Q - \deg P.$$

Then $R_{v_\infty} = \mathbb{F}_q[Y]$ and the absolute value $|\cdot|_{v_\infty}$ on K associated with v_∞ is the absolute value defined in the introduction. The completion K_{v_∞} of K for $|\cdot|_{v_\infty}$ is the field $\mathbb{F}_q((Y^{-1}))$ of formal Laurent series in one variable Y^{-1} with coefficients in \mathbb{F}_q , denoted by \widehat{K} in the introduction. The elements x in $\mathbb{F}_q((Y^{-1}))$ are of the form

$$x = \sum_{i \in \mathbb{Z}} x_i Y^{-i}$$

where $x_i \in \mathbb{F}_q$ for all $i \in \mathbb{Z}$ and $x_i = 0$ for i small enough. The valuation at infinity of $\mathbb{F}_q((Y^{-1}))$ extending the valuation at infinity of $\mathbb{F}_q(Y)$ is

$$\forall x \in \mathbb{F}_q((Y^{-1}))^\times, \quad v_\infty(x) = \sup\{i \in \mathbb{Z} : \forall j < i, \quad x_j = 0\}.$$

The valuation ring \mathcal{O}_{v_∞} of v_∞ is the ring $\mathbb{F}_q[[Y^{-1}]]$ of formal power series in one variable Y^{-1} with coefficients in \mathbb{F}_q . The residual field $k_{v_\infty} = \mathcal{O}_{v_\infty}/\mathfrak{m}_{v_\infty}$ of v_∞ is \mathbb{F}_q , which has order $q_{v_\infty} = q$.

We identify the projective line $\mathbb{P}_1(K_v)$ with $K_v \cup \{\infty\}$ using the map $K_v(x, y) \mapsto \frac{x}{y}$, so that

$$\infty = [1 : 0].$$

The projective action of $\mathrm{PGL}_2(K_v)$ on $\mathbb{P}^1(K_v)$ is the action by homographies on $K_v \cup \{\infty\}$, given by $(g, z) \mapsto g \cdot z = \frac{az+b}{cz+d}$ if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(K_v)$. As usual, we define $g \cdot \infty = \frac{a}{c}$ and $g \cdot (-\frac{d}{c}) = \infty$.

2.2 Bruhat-Tits trees

An \mathcal{O}_v -lattice Λ in the K_v -vector space $K_v \times K_v$ is a rank 2 free \mathcal{O}_v -submodule of $K_v \times K_v$, generating $K_v \times K_v$ as a vector space. The Bruhat-Tits tree \mathbb{T}_v of (PGL_2, K_v) is the graph whose set of vertices $V\mathbb{T}_v$ is the set of homothety classes (under $(K_v)^\times$) $[\Lambda]$ of \mathcal{O}_v -lattices Λ in $K_v \times K_v$, and whose set of edges $E\mathbb{T}_v$ is the set of pairs (x, x') of vertices such that there exist representatives Λ of x and Λ' of x' for which $\Lambda \subset \Lambda'$ and Λ'/Λ is isomorphic to $\mathcal{O}_v/\mathfrak{m}_v$. The graph \mathbb{T}_v is a regular tree of degree $|\mathbb{P}_1(k_v)| = q_v + 1$. The *standard base point* $*_v$ of \mathbb{T}_v is the homothety class $[\mathcal{O}_v \times \mathcal{O}_v]$ of the \mathcal{O}_v -lattice $\mathcal{O}_v \times \mathcal{O}_v$ generated by the canonical basis of $K_v \times K_v$. The *link*

$$\mathrm{lk}(*_v) = \{y \in V\mathbb{T}_v : d(y, *_v) = 1\}$$

of $*_v$ in \mathbb{T}_v identifies with the projective line $\mathbb{P}_1(k_v)$.

The left linear action of $\mathrm{GL}_2(K_v)$ on $K_v \times K_v$ induces a faithful, vertex-transitive left action of $\mathrm{PGL}_2(K_v)$ by automorphisms on \mathbb{T}_v . The stabiliser of $*_v$ in $\mathrm{PGL}_2(K_v)$ is $\mathrm{PGL}_2(\mathcal{O}_v)$, which acts projectively on $\mathrm{lk}(*_v) = \mathbb{P}_1(k_v)$ by reduction modulo v , and in particular $\mathrm{PGL}_2(k_v)$ acts simply transitively on triples of pairwise distinct points on $\mathrm{lk}(*_v)$. We identify the boundary at infinity $\partial_\infty \mathbb{T}_v$ of \mathbb{T}_v with $\mathbb{P}_1(K_v)$ by the unique homeomorphism from $\partial_\infty \mathbb{T}_v$ to $\mathbb{P}_1(K_v)$ such that the (continuous) extension to $\partial_\infty \mathbb{T}_v$ of the isometric action of $\mathrm{PGL}_2(K_v)$ on \mathbb{T}_v corresponds to the projective action of $\mathrm{PGL}_2(K_v)$ on $\mathbb{P}_1(K_v)$.

Let $\Gamma_v = \mathrm{PGL}_2(R_v)$. The group Γ_v is a lattice in the locally compact group $\mathrm{PGL}_2(K_v)$, called the *modular group* at v of K . The quotient graph $\Gamma \backslash \mathbb{T}_v$ is called the *modular graph* of K , and the quotient graph of groups $\Gamma \backslash \backslash \mathbb{T}_v$ is called the *modular graph of groups* at v of K . We refer to [Ser] for background information on these objects, and for instance to [Pau] for a geometric treatment when $K = \mathbb{F}_q(Y)$ and $v = v_\infty$.

Recall that the *open horoballs* centred at $\xi \in \partial_\infty \mathbb{T}_v$ are the subsets of the geometric realisation $|\mathbb{T}_v|$ of \mathbb{T}_v defined by

$$\mathcal{H}(\rho_\xi) = \{y \in |\mathbb{T}_v| : \lim_{t \rightarrow +\infty} (t - d(\rho_\xi(t), y)) > 0\}$$

where ρ_ξ is a geodesic ray converging to ξ . The boundary of $\mathcal{H}(\rho_\xi)$ is the *horosphere*

$$\partial\mathcal{H}(\rho_\xi) = \{y \in |\mathbb{T}_v| : \lim_{t \rightarrow +\infty} (t - d(\rho_\xi(t), y)) = 0\}.$$

We refer to [BH] for background on these notions. The *height* in $\mathcal{H}(\rho_\xi)$ of a point $x \in |\mathbb{T}_v|$ is $\lim_{t \rightarrow +\infty} (t - d(\rho_\xi(t), x))$. It is positive if and only if x belongs to $\mathcal{H}(\rho_\xi)$. We denote by \mathcal{H}_∞ the unique horoball centred at $\infty \in \partial_\infty \mathbb{T}_v$ whose associated horosphere passes through $*_v$.

Let Γ be a finite index subgroup of Γ_v . By for instance [Ser, Pau], there exists a Γ -equivariant family of pairwise disjoint open horoballs $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}_1(K)}$ in \mathbb{T}_v with \mathcal{H}_ξ centered at ξ and the stabiliser Γ_ξ of ξ in Γ acting transitively on the boundary of \mathcal{H}_ξ for every $\xi \in \mathbb{P}_1(K) \subset \partial_\infty \mathbb{T}_v$, so that the quotient by Γ of

$$\tilde{E}_\Gamma = \mathbb{T}_v - \bigcup_{\xi \in \mathbb{P}_1(K)} \mathcal{H}_\xi$$

is a finite connected graph, denoted by E_Γ . The *set of cusps* $\Gamma \backslash \mathbb{P}_1(K)$ is finite. For every representative ξ of a cusp in $\Gamma \backslash \mathbb{P}_1(K)$, the injective image by the canonical projection $\mathbb{T}_v \rightarrow \Gamma \backslash \mathbb{T}_v$ of any geodesic ray starting from a point on the boundary of \mathcal{H}_ξ with point at infinity ξ is called a *cuspidal ray*. The quotient graph $\Gamma \backslash \mathbb{T}_\infty$ is the union of the finite subgraph E_Γ and the finite collection of cuspidal rays that are glued to E_Γ at their origin.

Example 2.2. (See for instance [BaL].) Let $K = \mathbb{F}_q(Y)$ and $v = v_\infty$. Then $\Gamma_{v_\infty} = \text{PGL}_2(\mathbb{F}_q[Y])$ and the quotient graph of groups $\Gamma_{v_\infty} \backslash \mathbb{T}_{v_\infty}$ is the following *modular ray*

$$\begin{array}{cccccccc} \Gamma_{-1} & & \Gamma_0 & & \Gamma_1 & & \Gamma_2 & & \dots \\ \blacksquare & & \bullet & & \bullet & & \bullet & & \bullet \text{---} \\ & \Gamma'_0 & & \Gamma_0 & & \Gamma_1 & & \Gamma_2 & \end{array}$$

where $\Gamma_{-1} = \text{PGL}_2(\mathbb{F}_q)$, $\Gamma'_0 = \Gamma_0 \cap \Gamma_{-1}$ and, for every $n \in \mathbb{N}$,

$$\Gamma_n = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \text{PGL}_2(\mathbb{F}_q[Y]) : a, d \in \mathbb{F}_q^\times, b \in \mathbb{F}_q[Y], \deg b \leq n + 1 \right\}.$$

The canonical Γ_{v_∞} -equivariant family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}_1(K)}$ of pairwise disjoint maximal open horoballs in \mathbb{T}_{v_∞} , with \mathcal{H}_ξ centered at ξ for every $\xi \in \mathbb{P}_1(K)$, consists of the connected components of $|\mathbb{T}_{v_\infty}| - \Gamma_{v_\infty} *_v$. The graph \tilde{E}_Γ is reduced to the orbit of the base point $*_{v_\infty}$, and E_Γ is reduced to one vertex, the origin of the modular ray (see the above picture). In particular, the diameter of E_Γ is zero.

All geodesic lines in \mathbb{T}_{v_∞} pass through the Γ_{v_∞} -orbit of $*_{v_\infty}$. Indeed, no geodesic is completely contained in a horoball and since $\bigcup_{\xi \in \mathbb{P}_1(K)} \partial\mathcal{H}_\xi = \Gamma_{v_\infty} *_v$, the only way a geodesic line exits a horoball of the canonical family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}_1(K)}$ is through the orbit of $*_{v_\infty}$. The intersection of a geodesic line with the orbit $\Gamma_{v_\infty} *_v$ is hence finite if and only if its endpoints are both in $\mathbb{P}_1(K)$.

We end this section with the following lemma, which is an effective version of a special case of [HP, Prop. 2.6]. It controls the intersection length of a translation axis of an element of a discrete group of automorphisms of a tree with its images under this group. We will use it in Section 3 in order to prove Theorem 3.4.

Recall that an automorphism γ of a simplicial tree \mathbb{T} with geometric realisation $|\mathbb{T}|$ is *loxodromic* if it fixes no point of $|\mathbb{T}|$, that its *translation length* $\ell(\gamma) = \min_{x \in V\mathbb{T}} d(x, \gamma x)$ is then positive and that its *translation axis* $\text{Ax}_\gamma = \{x \in |\mathbb{T}| : d(x, \gamma x) = \ell(\gamma)\}$ is then a geodesic line in $|\mathbb{T}|$.

Lemma 2.3. *Let Γ be a discrete group of automorphisms of a locally finite tree \mathbb{T} . Let $\gamma_0 \in \Gamma$ be a loxodromic element on \mathbb{T} . Let $k_0 = \min_{x \in \text{Ax}_{\gamma_0}} |\Gamma_x|$ be the minimal order of the stabiliser in Γ of a vertex of Ax_{γ_0} and let Γ_0 be the stabiliser of Ax_{γ_0} in Γ . Then for every $\gamma \in \Gamma - \Gamma_0$, the length of the geodesic segment $\gamma \text{Ax}_{\gamma_0} \cap \text{Ax}_{\gamma_0}$ is less than $(k_0 + 1)\ell(\gamma_0) - 1$.*

Proof. Assume for a contradiction that the length $L \in \mathbb{N}$ of $\gamma \text{Ax}_{\gamma_0} \cap \text{Ax}_{\gamma_0}$ is at least $(k_0 + 1)\ell(\gamma_0) - 1$. Denote by $[x, y]$ the geodesic segment $\gamma \text{Ax}_{\gamma_0} \cap \text{Ax}_{\gamma_0}$, such that $\gamma_0 x$ and y are on same side of x on Ax_{γ_0} . Let $\epsilon = 1$ if $\gamma \gamma_0 \gamma^{-1} x$ and y are on same side of x on $\gamma \text{Ax}_{\gamma_0}$, and $\epsilon = -1$ otherwise.

Since γ_0 acts by a translation of length $\ell(\gamma_0)$ on Ax_{γ_0} , there exists a point $x' \in [x, y]$ at distance at most $\ell(\gamma_0) - 1$ from x such that $|\Gamma_{x'}| = k_0$. Note that $\gamma \gamma_0^\epsilon \gamma^{-1}$ acts by a translation of length $\ell(\gamma_0)$ on $\gamma \text{Ax}_{\gamma_0}$ and that the translation directions of γ_0 and $\gamma \gamma_0^\epsilon \gamma^{-1}$ coincide on $[x, y]$. Hence for every $k \in \{0, 1, \dots, k_0\}$, the point $\gamma \gamma_0^{\epsilon k} \gamma^{-1} x'$ belongs to $[x, y]$ by the assumption on L , and $\gamma_0^{-k} \gamma \gamma_0^{\epsilon k} \gamma^{-1} x' = x'$. Since the stabiliser of x' has order less than $k_0 + 1$, there are hence distinct $k, k' \in \{0, 1, \dots, k_0\}$ such that

$$\gamma_0^{-k} \gamma \gamma_0^{\epsilon k} \gamma^{-1} = \gamma_0^{-k'} \gamma \gamma_0^{\epsilon k'} \gamma^{-1},$$

that is, $\gamma_0^{k'-k} \gamma = \gamma \gamma_0^{\epsilon(k'-k)}$. Since $\text{Ax}_{\gamma_0^m} = \text{Ax}_{\gamma_0}$ and $\gamma \text{Ax}_{\gamma_0^{m'}} = \text{Ax}_{\gamma \gamma_0^{m'} \gamma^{-1}}$ for all $m, m' \in \mathbb{Z} - \{0\}$, this implies that γ preserves Ax_{γ_0} , a contradiction since $\gamma \notin \Gamma_0$. \square

3 Quadratic Diophantine approximation in completions of function fields

Let K be a function field over \mathbb{F}_q , let v be a (normalised discrete) valuation of K , let R_v be the affine function ring associated with v , and let Γ be a finite index subgroup of $\Gamma_v = \text{PGL}_2(R_v)$ (for instance a congruence subgroup).

We denote by

$$K_v^{(2)} = \{x \in K_v : [K(x) : K] = 2\}$$

the set of quadratic irrationals in K_v over K , and we fix $\alpha \in K_v^{(2)}$. We denote by $\alpha^\sigma \in K_v^{(2)}$ the Galois conjugate of α over K , and by

$$\Theta_{\alpha, \Gamma} = \Gamma \cdot \{\alpha, \alpha^\sigma\}$$

the union of the orbits of α and α^σ under the projective action of Γ , with $\Theta_\alpha = \Theta_{\alpha, \Gamma_v}$. Note that $\alpha^\sigma \neq \alpha$, since an irreducible quadratic polynomial over K which is inseparable does not split over K_v (see for instance [BPP, Lem. 17.2]), and that there exists a loxodromic element $\gamma_\alpha \in \Gamma_v$ such that $]\alpha, \alpha^\sigma[= \text{Ax}_{\gamma_\alpha}$ (see for instance [BPP, Prop. 17.3]). For all $x \in K_v$ and $\beta \in K_v^{(2)}$ with $x \neq \beta$, let

$$c(x, \beta) = \frac{|x - \beta|_v}{|\beta - \beta^\sigma|_v} \in \mathbb{Q}_v^\times.$$

The following result gives a geometric interpretation to this quantity.

Lemma 3.1. Let $x \in K_v$ and $\beta \in K_v^{(2)}$ with $x \neq \beta, \beta^\sigma$.

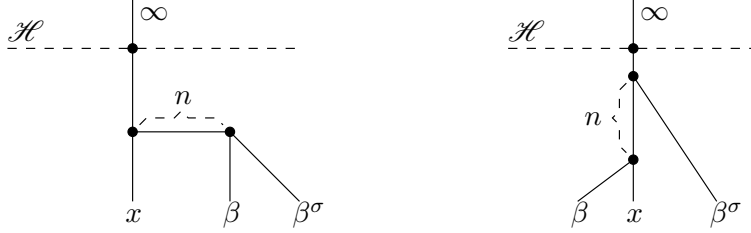
(1) If the geodesic lines $] \infty, x[$ and $] \beta, \beta^\sigma[$ in \mathbb{T}_v are disjoint or meet at only one vertex, then, with n the distance between them,

$$c(x, \beta) = c(x, \beta^\sigma) = q_v^n \geq 1.$$

(2) If the geodesic lines $] \infty, x[$ and $] \beta, \beta^\sigma[$ in \mathbb{T}_v meet along a geodesic segment of length $n > 0$, with the closest point to β on $] \infty, x[$ closer to x than the closest point to β^σ on $] \infty, x[$, then

$$c(x, \beta) = q_v^{-n} < 1 = c(x, \beta^\sigma).$$

In particular, $\min\{c(x, \beta), c(x, \beta^\sigma)\} < 1$.



Proof. For all distinct $y, z \in K_v$ and for every horoball \mathcal{H} in \mathbb{T}_v centered at ∞ , let

$$d_{\mathcal{H}}(y, z) = \lim_{t \rightarrow +\infty} e^{\frac{1}{2}d(y_t, z_t) - t},$$

where $t \mapsto y_t$, $t \mapsto z_t$ are the geodesic lines starting from ∞ , through $\partial\mathcal{H}$ at time $t = 0$, ending at the points at infinity y, z respectively. By for instance [BPP, Eq. (15.2)], we have

$$|y - z|_v = d_{\mathcal{H}\infty}(y, z)^{\ln q_v}.$$

Since the ratio $\frac{d_{\mathcal{H}}(y, z)}{d_{\mathcal{H}}(y', z')}$ does not depend on the horoball \mathcal{H} centered at ∞ , if \mathcal{H} is small enough, we hence have

$$c(x, \beta) = \frac{|x - \beta|_v}{|\beta - \beta^\sigma|_v} = q_v^{-d(\mathcal{H},]x, \beta]) + d(\mathcal{H},]\beta, \beta^\sigma])}.$$

The result follows. \square

For every $x \in K_v - (K \cup \Theta_{\alpha, \Gamma})$, we define the *approximation constant* of x by the (extended) Γ -orbit of α as

$$c_{\alpha, \Gamma}(x) = \liminf_{\beta \in \Theta_{\alpha, \Gamma}, |\beta - \beta^\sigma|_v \rightarrow 0} c(x, \beta).$$

When x is itself a quadratic irrational, the following result gives a geometric computation of the approximation constant $c_{\alpha, \Gamma}(x)$.

Remark 3.2. For all $\alpha, \beta \in K_v^{(2)}$ such that $\beta \notin \Theta_{\alpha, \Gamma}$, we have

$$c_{\beta, \Gamma}(\alpha) = c_{\alpha, \Gamma}(\beta) = q^{-n(\alpha, \beta)}$$

where $n(\alpha, \beta) = \max_{\gamma \in \Gamma} \text{length}(] \alpha^\sigma, \alpha[\cap] \beta^\sigma, \beta[)$ if some image of $] \beta^\sigma, \beta[$ by an element of Γ meets $] \alpha^\sigma, \alpha[$, and $n(\alpha, \beta) = -\min_{\gamma \in \Gamma} d(] \alpha^\sigma, \alpha[,] \gamma] \beta^\sigma, \beta[)$ otherwise.

Proof. Note that since the elements of Γ preserve the lengths of segments, and by a change of variable $\gamma \mapsto \gamma^{-1}$, we have $n(\beta, \alpha) = n(\alpha, \beta)$, so that the first equality follows from the second one.

By a proof similar to the one of Proposition 2.3, for all $\alpha, \beta \in K_v^{(2)}$ such that $\beta \notin \Theta_{\alpha, \Gamma}$, there exists a constant $\ell_{\alpha, \beta}$ (depending only on the translation lengths of primitive elements in Γ preserving the geodesic lines $] \alpha^\sigma, \alpha[$ and $] \beta^\sigma, \beta[$, as well as on the cardinalities of the stabilisers in Γ of the vertices of these geodesic lines) such that for every $\gamma \in \Gamma$, the length of the intersection $] \alpha^\sigma, \alpha[\cap \gamma] \beta^\sigma, \beta[$ is at most $\ell_{\alpha, \beta}$.

First assume that some image of $] \beta^\sigma, \beta[$ by an element of Γ meets $] \alpha^\sigma, \alpha[$. Using Lemma 3.1 for the second equality, the fact that $] \infty, \beta[$ and $] \beta^\sigma, \beta[$ share a subray with endpoint β for the third equality, and Lemma 3.1 (2) for the fourth one, we have

$$\begin{aligned} c_{\alpha, \Gamma}(\beta) &= \liminf_{\alpha' \in \Theta_{\alpha, \Gamma}, |\alpha'^\sigma - \alpha'|_v \rightarrow 0} c(\beta, \alpha') = \liminf_{\alpha' \in \Theta_{\alpha, \Gamma}, |\alpha'^\sigma - \alpha'|_v \rightarrow 0,] \alpha'^\sigma, \alpha'[\cap] \infty, \beta[\neq \emptyset} c(\beta, \alpha') \\ &= \liminf_{\alpha' \in \Theta_{\alpha, \Gamma}, |\alpha'^\sigma - \alpha'|_v \rightarrow 0,] \alpha'^\sigma, \alpha'[\cap] \beta^\sigma, \beta[\neq \emptyset} c(\beta, \alpha') \\ &= \liminf_{\alpha' \in \Theta_{\alpha, \Gamma}, |\alpha'^\sigma - \alpha'|_v \rightarrow 0,] \alpha'^\sigma, \alpha'[\cap] \beta^\sigma, \beta[\neq \emptyset} q^{-\text{length}(] \alpha'^\sigma, \alpha'[\cap] \beta^\sigma, \beta[)} = q^{-n(\beta, \alpha)}. \end{aligned}$$

Otherwise, the result follows by using Lemma 3.1 (1). \square

We define the *quadratic Lagrange spectrum* of α relative to Γ as

$$\text{Sp}(\alpha, \Gamma) = \{c_{\alpha, \Gamma}(x) : x \in K_v - (K \cup \Theta_{\alpha, \Gamma})\},$$

and $\text{Sp}(\alpha) = \text{Sp}(\alpha, \Gamma_v)$. Note that $\text{Sp}(\alpha, \Gamma)$ is contained in $q_v^{\mathbb{Z}} \cup \{0, +\infty\}$ and that for every $\beta \in \Theta_{\alpha, \Gamma}$, the functions $c_{\alpha, \Gamma}$ and $c_{\beta, \Gamma}$ on $K_v - (K \cup \Theta_{\alpha, \Gamma}) = K_v - (K \cup \Theta_{\beta, \Gamma})$ coincide, so that $\text{Sp}(\alpha, \Gamma) = \text{Sp}(\beta, \Gamma)$.

Proposition 3.3. *The quadratic Lagrange spectrum $\text{Sp}(\alpha, \Gamma)$ is closed and bounded in \mathbb{R} .*

The above result allows us to define the *Hurwitz constant* of α relative to Γ as

$$\max \text{Sp}(\alpha, \Gamma) \in q_v^{\mathbb{Z}},$$

and the *Hurwitz constant* of α as $\max \text{Sp}(\Gamma_v, \alpha)$. The proof of Proposition 3.3 actually gives an upper bound on $\text{Sp}(\alpha, \Gamma)$ which does not depend on the quadratic irrational α , see Equation (1) below. In the special case when $(K, v, \Gamma) = (\mathbb{F}_q(Y), v_\infty, \Gamma_{v_\infty})$, we will prove more precisely in Section 4 that $\max_{\alpha \in K_v^{(2)}} \max \text{Sp}(\alpha) = \frac{1}{q^2}$.

Proof. It follows from [HP, Thm. 1.6]⁴ that if m_{K_v} is a Haar measure on the locally compact additive group of K_v , then $c_{\alpha, \Gamma}(x) = 0$ for m_{K_v} -almost every $x \in K_v$. Therefore $0 \in \text{Sp}(\alpha, \Gamma)$, and the quadratic Lagrange spectrum of α relative to Γ is closed.

Let us fix $x \in K_v - K$ and let us prove that $c_{\alpha, \Gamma}(x) \leq q_v^{\text{diam } E_\Gamma}$, where E_Γ is as defined in Section 2. This proves Proposition 3.3 with a uniform bound on the Hurwitz constants

$$\forall \alpha \in K_v^{(2)}, \quad \max \text{Sp}(\alpha, \Gamma) \leq q_v^{\text{diam } E_\Gamma}. \quad (1)$$

⁴Actually, Thm. 1.6 of [HP] is stated only for $K = \mathbb{F}_q(Y)$, $v = v_\infty$ and $\Gamma = \Gamma_v$, but it has an analogous version for general (K, v, Γ) by using [HP, Prop. 1.5].

Since x is irrational and since any geodesic ray entering into a horoball and not converging towards its point at infinity has to exit the horoball, the geodesic line $] \infty, x[$ from ∞ to x cannot stay after a given time in a given horoball of the family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}_1(K)}$ defined in Section 2. Hence there exists a sequence $(p_n)_{n \in \mathbb{N}}$ of points of \tilde{E}_Γ converging to x along the geodesic line $] \infty, x[$. Since $E_\Gamma = \Gamma \setminus \tilde{E}_\Gamma$ is finite and since no geodesic line is contained in a horoball, there exists a sequence $(\gamma'_n)_{n \in \mathbb{N}}$ in Γ such that $d(p_n, \gamma'_n] \alpha, \alpha^\sigma] \leq \text{diam } E_\Gamma$ for all $n \in \mathbb{N}$.

By Lemma 3.1, there exists $\beta_n \in \{\gamma'_n \alpha, \gamma'_n \alpha^\sigma\} \subset \Theta_{\alpha, \Gamma}$ such that $c(x, \beta_n) < 1$ if $] \beta_n, \beta_n^\sigma [= \gamma'_n] \alpha, \alpha^\sigma [$ meets $] \infty, x[$ in at least an edge, and $c(x, \beta_n) \leq q_v^{\text{diam } E_\Gamma}$ otherwise. Hence

$$\liminf_{n \rightarrow +\infty} c(x, \beta_n) \leq q_v^{\text{diam } E_\Gamma} .$$

Let $\gamma_\alpha \in \Gamma_v$ be a loxodromic element such that $] \alpha, \alpha^\sigma [= \text{Ax}_{\gamma_\alpha}$. Since Γ has finite index in Γ_v , up to replacing γ_α by a positive power, we may assume that γ_α belongs to Γ . Since the length of the intersection of two distinct translates of $\text{Ax}_{\gamma_\alpha}$ by elements of Γ is uniformly bounded by Lemma 2.3, we have $\lim_{n \rightarrow +\infty} |\beta_n - \beta_n^\sigma|_v = 0$. Hence by the definition of the approximation constants, we have as wanted $c_{\alpha, \Gamma}(x) \leq q_v^{\text{diam } E_\Gamma}$. \square

The following result, which implies Theorem 1.1 (2) in the introduction, says that the nonarchimedean quadratic Lagrange spectra contain Hall rays. Note that its proof gives an explicit upper bound on the constant whose existence is claimed.

Theorem 3.4. *There exists $m_\alpha \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m \geq m_\alpha$, we have $q_v^{-m} \in \text{Sp}(\alpha, \Gamma)$.*

Proof. Let

$$k_\alpha = \min_{x \in] \alpha^\sigma, \alpha [\cap V\mathbb{T}_v} |\Gamma_x|$$

be the minimal order of the stabiliser in Γ of a vertex of the geodesic line $] \alpha^\sigma, \alpha [$. Let $\gamma_\alpha \in \Gamma$ be a loxodromic element (with minimal translation length) such that $\text{Ax}_{\gamma_\alpha} =] \alpha^\sigma, \alpha [$ and let $\kappa_\alpha = (k_\alpha + 1)\ell(\gamma_\alpha) - 2$. By Lemma 2.3, for all $\beta, \beta' \in \Theta_{\alpha, \Gamma}$, if $\beta' \notin \{\beta, \beta^\sigma\}$, then the intersection $] \beta^\sigma, \beta [\cap] \beta'^\sigma, \beta' [$ is a (possibly empty) segment of length at most κ_α . Take

$$m_\alpha = 2\kappa_\alpha + 1 .$$

Let us fix $m \in \mathbb{N}$ with $m \geq m_\alpha$, and let us prove that q_v^{-m} belongs to $\text{Sp}(\alpha, \Gamma)$, which gives Theorem 3.4. For this, let us construct $\xi \in K_v - (K \cup \Theta_{\alpha, \Gamma})$ and a sequence $(\beta_n)_{n \in \mathbb{N}}$ in $\Theta_{\alpha, \Gamma}$ such that

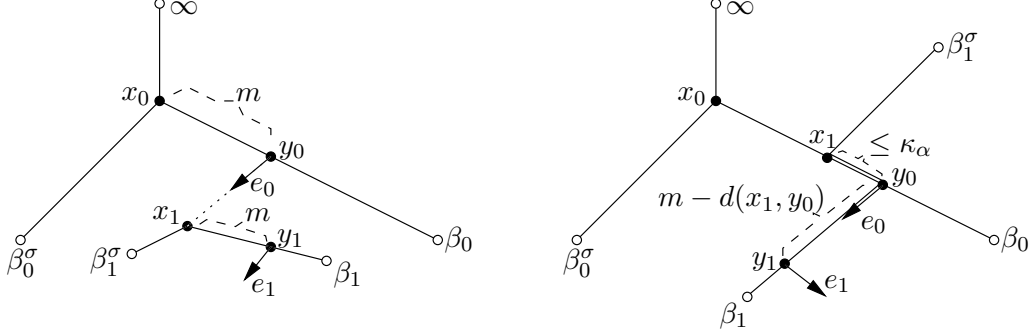
- $|\beta_n - \beta_n^\sigma|_v \rightarrow 0$ as $n \rightarrow +\infty$,
- the length of the intersection $] \beta_n^\sigma, \beta_n [\cap] \infty, \xi [$ is exactly m ,
- the closest point to β_n on $] \infty, \xi [$ is closer to ξ than the closest point to β_n^σ on $] \infty, \xi [$,
- for every $\beta \in \Theta_{\alpha, \Gamma}$, either $] \beta^\sigma, \beta [$ and $] \infty, \xi [$ are disjoint or their nonempty intersection has length at most m .

By Lemma 3.1, this proves that

$$c_{\alpha, \Gamma}(\xi) = \liminf_{\beta \in \Theta_{\alpha, \Gamma}, |\beta - \beta^\sigma|_v \rightarrow 0} c(\xi, \beta) = \liminf_{n \rightarrow +\infty} c(\xi, \beta_n) = q_v^{-m} ,$$

so that q_v^{-m} does belong to $\text{Sp}(\alpha, \Gamma)$.

Since the image of $]\alpha^\sigma, \alpha[$ in $\Gamma \backslash \mathbb{T}_v$ is compact, and since $\Gamma \cdot \infty$ is a cusp, there exists $\beta_0 \in \Theta_{\alpha, \Gamma}$ such that if x_0 is the closest point to ∞ on $]\beta_0^\sigma, \beta_0[$, then the open geodesic ray $]x_0, \infty[$ does not meet any $]\beta^\sigma, \beta[$ for $\beta \in \Theta_{\alpha, \Gamma}$. Let y_0 be the vertex on $[x_0, \beta_0[$ at distance exactly m from x_0 , and let e_0 be an edge with origin y_0 and not contained in $]\beta_0^\sigma, \beta_0[$.



Assume first (see the above picture on the left) that there exists no $\beta \in \Theta_{\alpha, \Gamma}$ such that e_0 is contained in $]\beta^\sigma, \beta[$. Since Γ is a lattice, the set $\Theta_{\alpha, \Gamma}$, which contains the orbit $\Gamma \cdot \alpha$, is dense in $\partial_\infty \mathbb{T}_v$. Hence there exists $\beta_1 \in \Theta_{\alpha, \Gamma}$ such that if x_1 is the closest point to y_0 on $]\beta_1^\sigma, \beta_1[$, then the open segment $]y_0, x_1[$ contains e_0 in its closure and meets no $]\beta^\sigma, \beta[$ for $\beta \in \Theta_{\alpha, \Gamma}$. Let y_1 be the vertex on the geodesic ray $[x_1, \beta_1[$ at distance equal to m from x_1 , and let e_1 be an edge with origin y_1 and not contained in $]\beta_1^\sigma, \beta_1[$. Note that $d(y_0, y_1) \geq d(x_1, y_1) = m \geq m_\alpha \geq \kappa_\alpha + 1$. By Lemma 2.3, for every $\beta \in \Theta_{\alpha, \Gamma}$, the length of the (possibly empty) intersection $[y_1, \infty[\cap]\beta^\sigma, \beta[$ is at most m (exactly m if $\beta \in \{\beta_0, \beta_0^\sigma, \beta_1, \beta_1^\sigma\}$ and at most κ_α otherwise).

Assume now (see the above picture on the right) that there exists $\beta_1 \in \Theta_{\alpha, \Gamma}$ such that e_0 is contained in $]\beta_1^\sigma, \beta_1[$. Up to exchanging β_1 and β_1^σ , we may assume that e_0 is contained in $[y_0, \beta_1[$.

Let $x_1 \in V\mathbb{T}_v$ be such that $[y_0, \infty[\cap [y_0, \beta_1^\sigma[= [y_0, x_1]$. Note that by Lemma 2.3, we have $d(x_1, y_0) \leq \kappa_\alpha$ and $x_1 \in [y_0, x_0[$ since $d(x_0, y_0) = m \geq m_\alpha > \kappa_\alpha$. Let y_1 be the point at distance equal to $m - d(x_1, y_0)$ from y_0 on $[y_0, \beta_1[$. Note that $d(y_0, y_1) \geq \kappa_\alpha + 1$ since $m \geq m_\alpha = 2\kappa_\alpha + 1$, and in particular by Lemma 2.3, there is no $\beta \in \Theta_{\alpha, \Gamma}$ such that $]\beta^\sigma, \beta[$ contains both e_1 and y_0 . By Lemma 2.3, for every $\beta \in \Theta_{\alpha, \Gamma}$, the length of $[y_1, \infty[\cap]\beta^\sigma, \beta[$ is at most m (exactly m if $\beta \in \{\beta_0, \beta_0^\sigma, \beta_1, \beta_1^\sigma\}$ and at most $2\kappa_\alpha$ otherwise).

By iterating this construction, we obtain sequences $(\beta_n)_{n \in \mathbb{N}}$ in $\Theta_{\alpha, \Gamma}$, $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ in $V\mathbb{T}_v$ and $(e_n)_{n \in \mathbb{N}}$ in $E\mathbb{T}_v$ such that, for all $n \in \mathbb{N}$,

- $y_n \in [y_{n+1}, \infty[$ and $d(y_n, y_{n+1}) \geq \kappa_\alpha + 1$,
- for every $\beta \in \Theta_{\alpha, \Gamma}$, the length of $[y_n, \infty[\cap]\beta^\sigma, \beta[$ is at most m , and the length of $[y_n, \infty[\cap]\beta_n^\sigma, \beta_n[= [y_n, x_n]$ is exactly m ,
- $y_n \in [x_n, \beta_n[$,
- e_n exits $]\beta_n^\sigma, \beta_n[$ at y_n and is contained in $[y_n, y_{n+1}]$,
- x_{n+1} and y_n belong to $[x_n, y_{n+1}]$, and either y_n belongs to $[x_n, x_{n+1}[$ in which case $]y_n, x_{n+1}[$ meets no $]\beta^\sigma, \beta[$ for $\beta \in \Theta_{\alpha, \Gamma}$, or y_n belongs to $[x_{n+1}, y_{n+1}[$.

By the first point, the sequence $(y_n)_{n \in \mathbb{N}}$ converges to $\xi \in \partial_\infty \mathbb{T}_v$, such that y_0, y_1, y_2, \dots are in this order on the geodesic line $]\infty, \xi[$ oriented from ∞ to ξ . The point at infinity ξ belongs neither to K (since y_n belongs to $]\beta_n^\sigma, \beta_n[$ and the geodesic lines $]\beta^\sigma, \beta[$ for $\beta \in \Theta_{\alpha, \Gamma}$

do not enter a small enough horoball centered at any point of K) nor to $\Theta_{\alpha, \Gamma}$ (otherwise $]\infty, \xi[\cap]\xi^\sigma, \xi[$ would be infinitely long, contradicting the second point for n large enough).

The length of $]\infty, \xi[\cap]\beta_n^\sigma, \beta_n[$, which is equal to $[y_n, x_n]$ since e_n exits $]\beta_n^\sigma, \beta_n[$ at y_n , is exactly m , and in particular is bounded in n . Since $(y_n)_{n \in \mathbb{N}}$ converges to the point at infinity ξ , we have $|\beta_n - \beta_n^\sigma|_v \rightarrow 0$ as $n \rightarrow +\infty$. By construction, there is no $\beta \in \Theta_{\alpha, \Gamma}$ such that the length of $]\infty, \xi[\cap]\beta^\sigma, \beta[$ is larger than m .

Therefore ξ satisfies the properties required at the beginning of the proof, and Theorem 3.4 follows. \square

In the next section, we will give several computations, using the continued fraction expansions, in the special case when $K = \mathbb{F}_q(Y)$, $v = v_\infty$ is the valuation at infinity, and $\Gamma = \Gamma_{v_\infty}$ is the full Nagao lattice $\mathrm{PGL}_2(\mathbb{F}_q[Y])$.

4 Computations of approximation constants, Hurwitz constants and quadratic Lagrange spectra for fields of formal Laurent series

In this section, we use the notation \mathbb{F}_q , $R = \mathbb{F}_q[Y]$, $K = \mathbb{F}_q(Y)$, $\widehat{K} = \mathbb{F}_q((Y^{-1}))$, $|\cdot|$ given in the introduction. Let

$$\mathcal{O} = \{f \in \widehat{K} : |f| \leq 1\} = \mathbb{F}_q[[Y^{-1}]]$$

be the ring of formal power series in one variable Y^{-1} over \mathbb{F}_q . Its unique maximal ideal is $\mathfrak{m} = Y^{-1}\mathcal{O}$. We denote by \mathbb{T} the Bruhat-Tits tree of $(\mathrm{PGL}_2, \widehat{K})$, with standard base point $* = [\mathcal{O} \times \mathcal{O}]$, and $\Gamma = \mathrm{PGL}_2(R)$, see Section 2.

Any element $f \in \widehat{K}$ may be uniquely written as a sum $f = [f] + \{f\}$ of its *integral part* $[f] \in R$ and its *fractional part* $\{f\} \in \mathfrak{m}$. The *Artin map* $\Psi : \mathfrak{m} - \{0\} \rightarrow \mathfrak{m}$ is defined by $f \mapsto \{\frac{1}{f}\}$. Any $f \in \widehat{K} - K$ has a unique continued fraction expansion

$$f = [a_0, a_1, a_2, a_3, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}}$$

with $a_0 = a_0(f) = [f] \in R$, and $a_n = a_n(f) = \left[\frac{1}{\Psi^{n-1}(f - a_0)} \right] \in R$ a nonconstant polynomial for $n \geq 1$. The polynomials $a_n(f)$ are called the *coefficients* of the continued fraction expansion of f . For every $n \in \mathbb{N}$, the rational element

$$\frac{P_n}{Q_n} = [a_0, a_1, a_2, \dots, a_{n-1}, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}}$$

is the n -th *convergent* of f . We refer to [Las, Sch, Pau] for details and further information on continued fraction expansions of formal Laurent series and their geometric interpretation in terms of the Bruhat-Tits tree \mathbb{T} .

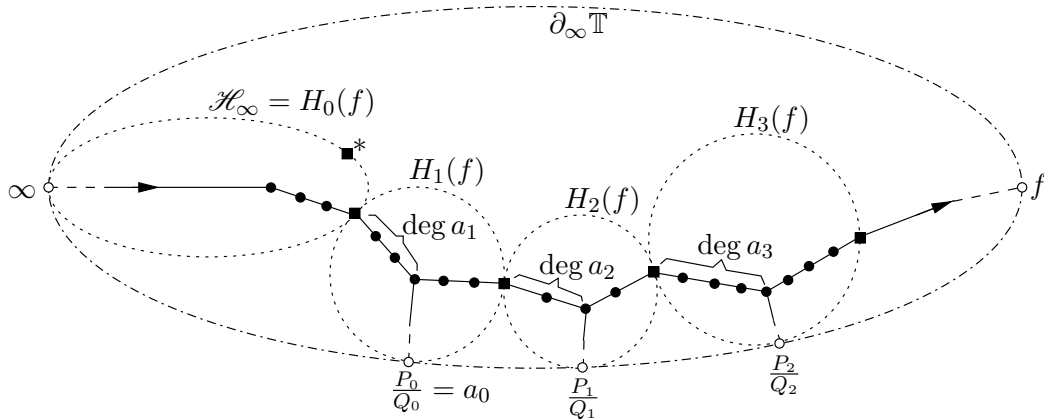
For every $f \in \widehat{K} - K$, let

$$\begin{aligned} M(f) &= \limsup_{k \rightarrow +\infty} \deg a_k \geq 1, \\ M_2(f) &= \limsup_{k \rightarrow +\infty} (\deg a_k + \deg a_{k+1}) \geq 2, \\ m(f) &= \liminf_{k \rightarrow +\infty} \deg a_k \geq 1. \end{aligned}$$

As recalled in the introduction, an irrational element $\alpha \in \widehat{K} - K$ is quadratic over K if and only if its continued fraction expansion is eventually periodic: For every $p \in \mathbb{N}$ large enough, the sequence of coefficients $(a_{k+p}(\alpha))_{k \in \mathbb{N}}$ is periodic with period $m \in \mathbb{N} - \{0\}$ and, as usual, we then write the continued fraction expansion of α as $\alpha = [a_0, a_1, a_2, \dots, a_{p-1}, \overline{a_p, a_{p+1}, \dots, a_{p+m-1}}]$. We then have

$$M(\alpha) = \max_{0 \leq k \leq m-1} \deg a_{p+k}(\alpha) \quad \text{and} \quad m(\alpha) = \min_{0 \leq k \leq m-1} \deg a_{p+k}(\alpha).$$

Let us recall from [Pau, §6.3, Rem. 2] the penetration properties of the geodesic lines of \mathbb{T} inside the canonical equivariant family of pairwise disjoint open horoballs whose closures cover \mathbb{T} , see the picture below. For every $f \in \widehat{K} - K$, the geodesic line $] \infty, f[$ (oriented from ∞ to f) starts in $H_0(f) = \mathcal{H}_\infty$ by an initial negative subray; after \mathcal{H}_∞ , it successively passes through an infinite sequence of open horoballs in the family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}^1(K)}$, denoted by $(H_n(f))_{n \in \mathbb{N} - \{0\}}$ and for every $n \in \mathbb{N}$, we have $H_{n+1}(f) = \mathcal{H}_{P_n/Q_n}$, where P_n/Q_n is the n -th convergent of f . For every $n \in \mathbb{N}$, the maximum height of a point of $] \infty, f[$ inside $H_{n+1}(f)$ is equal to $\deg a_{n+1}$, or, equivalently, the intersection of $H_{n+1}(f)$ with $] \infty, f[$ is an open segment of length $2 \deg a_{n+1}$.



We will need the following lemmas, which also follow from the geometric interpretation of the continued fraction expansion given in [Pau], in order to estimate the quadratic approximation constants of elements $f \in \widehat{K} - K$.

Lemma 4.1. *Let $f, f' \in \widehat{K} - K$ with $a_0(f) = a_0(f')$. If the geodesic lines $] \infty, f[$ and $] \infty, f'[$ coincide inside the horoball $\mathcal{H}_{a_0(f)}$, then the polynomial $a_1(f) - a_1(f')$ is constant. \square*

Lemma 4.2. *Let $f \in \widehat{K} - K$ and $\alpha \in K^{(2)}$ with purely periodic continued fraction expansion. If there exist $p, q \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $a_{p+1}(f) = a_{q+1}(\alpha), \dots, a_{p+m}(f) = a_{q+m}(\alpha)$, then there exists $\gamma \in \Gamma$ such that the geodesic lines $] \infty, f[$ and $\gamma] \alpha^\sigma, \alpha [$ coincide at least in $H_{p+1}(f) \cup \dots \cup H_{p+m}(f)$, so that their intersection has length at least*

$$2 \sum_{i=1}^m \deg a_{p+i}(f). \quad \square$$

The proof of Proposition 3.3, and in particular Equation (1) since $\text{diam } E_\Gamma = 0$ for the Nagao lattice Γ as seen in Example 2.2, shows that the quadratic Lagrange spectrum with respect to the valuation at infinity of any quadratic irrational is contained in $\{0\} \cup \{q^{-n} : n \in \mathbb{N}\}$. The following result, which implies Theorem 1.1 (1) in the introduction, improves the upper bound of the spectrum. In Corollary 4.7 and Proposition 4.8, we will show that this upper bound is realised for certain quadratic irrationals.

Proposition 4.3. *For every quadratic irrational α , the spectrum $\text{Sp}(\alpha)$ is contained in $\{0\} \cup \{q^{-n-2} : n \in \mathbb{N}\}$.*

Proof. Up to replacing α by an element of Θ_α , we may assume that $\alpha \in \mathfrak{m}$ and $\alpha^\sigma \in \widehat{K} - \mathcal{O}$, so that the base point $*$ belongs to the geodesic line $] \alpha, \alpha^\sigma [$.

Let $f \in \widehat{K} - (K \cup \Theta_\alpha)$. Since f is irrational, no positive subray of the geodesic line $] \infty, f[$ (oriented from ∞ to f) is contained in a single horoball of the canonical family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}^1(K)}$. Hence there exists a sequence $(x_n = \gamma_n *)_{n \in \mathbb{N}}$, with $\gamma_n \in \Gamma$, of vertices in the Γ -orbit of the base point $*$, converging to f on the geodesic line $] \infty, f[$. As $*$ belongs to $] \alpha, \alpha^\sigma [$, we thus have that $] \infty, f[$ and $] \gamma_n \cdot \alpha, \gamma_n \cdot \alpha^\sigma [$ meet at least at the vertex $\gamma_n *$.

The stabiliser $\text{PGL}_2(\mathbb{F}_q)$ of $*$ in Γ acts transitively on the set of pairs of distinct elements of the link of $*$. Thus, up to multiplying γ_n on the right by an element of $\text{PGL}_2(\mathbb{F}_q)$, the geodesic line $] \infty, f[$ meets $] \gamma_n \cdot \alpha, \gamma_n \cdot \alpha^\sigma [$ in a segment of length at least 2 for all $n \in \mathbb{N}$. Thus, by Lemma 3.1 (2), we have

$$\min\{c(x, \gamma_n \cdot \alpha), c(x, \gamma_n \cdot \alpha^\sigma)\} \leq q^{-2}.$$

Since $|\gamma_n \cdot \alpha - \gamma_n \cdot \alpha^\sigma|$ tends to 0 by Lemma 2.3, we have

$$c_\alpha(f) = \liminf_{\beta \in \Theta_\alpha, |\beta^\sigma - \beta| \rightarrow 0} c(x, \beta) \leq q^{-2}.$$

Hence the result follows. □

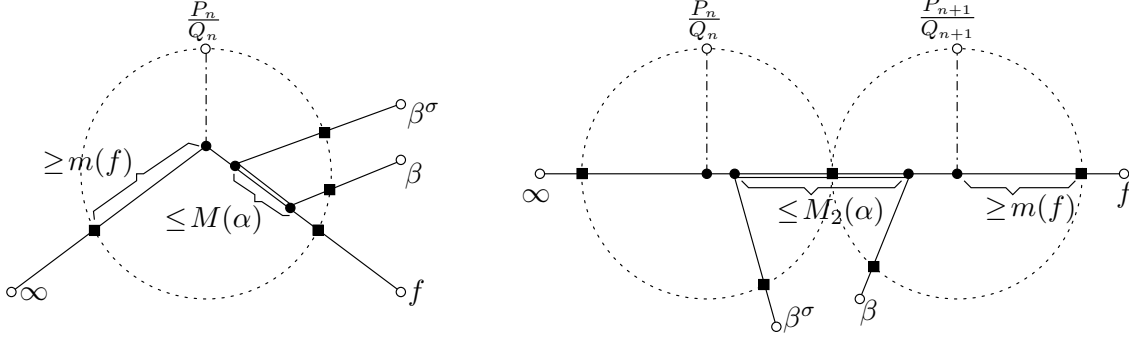
We are now going to give a series of computations of quadratic approximation constants. We start by two preliminary results.

Lemma 4.4. *Let $\alpha \in K^{(2)}$ and let $f \in \widehat{K} - (K \cup \Theta_\alpha)$.*

- (1) *If $m(f) > M(\alpha)$, then $c_\alpha(f) \geq q^{-M_2(\alpha)}$.*
- (2) *If $M(f) < m(\alpha)$, then $c_\alpha(f) \geq q^{-M_2(f)}$.*

Proof. (1) By the penetration properties of geodesic lines in the horoballs of the canonical family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}^1(K)}$, for every $\beta \in \Theta_\alpha$, the maximum height the geodesic line $] \beta^\sigma, \beta [$ enters in one of these horoballs is $M(\alpha)$. Similarly, the minimum height the geodesic line $] \infty, f [$ enters one of these horoballs except finitely many of them is $m(f)$, which is strictly bigger

than $M(\alpha)$. Hence for all $\beta \in \Theta_\alpha$, the geodesic lines $] \infty, f[$ and $] \beta^\sigma, \beta[$ can meet at most in two consecutive horoballs $H_n(f)$ for $n \in \mathbb{N}$ large enough, and their intersection has length at most $M_2(\alpha)$ (and even at most $M(\alpha) \leq M_2(\alpha)$ if $] \beta^\sigma, \beta[$ meets at most one of the horoballs $H_n(f)$ for $n \in \mathbb{N}$ large enough), see the picture below.

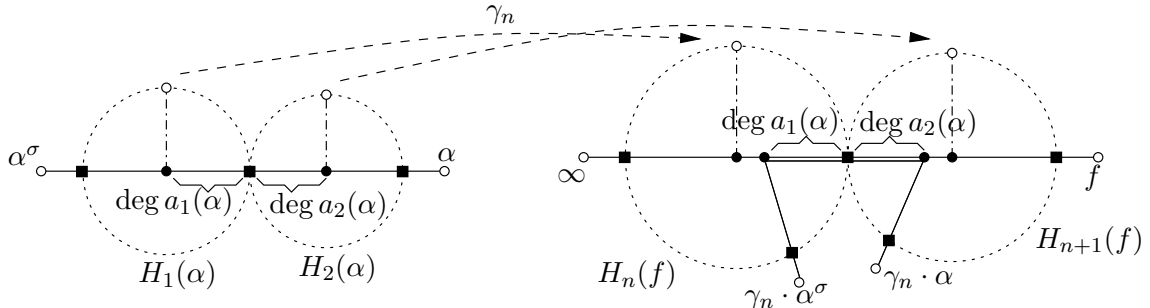


By Lemma 3.1, we have $c(f, \beta) \geq q^{-M_2(\alpha)}$ for all $\beta \in \Theta_\alpha$, which proves the result.

The proof of Assertion (2) is similar. \square

Lemma 4.5. *For all $\alpha \in K^{(2)}$ and $f \in \widehat{K} - (K \cup \Theta_\alpha)$ such that $m(f) > M(\alpha)$, there exists a sequence $(\beta_n)_{n \in \mathbb{N}}$ in Θ_α such that $c(f, \beta_n) = q^{-M_2(\alpha)}$ and $|\beta_n - \beta_n^\sigma| \rightarrow 0$ as $n \rightarrow +\infty$.*

Proof. Replacing α by an element in its Γ -orbit if necessary, we can assume that the continued fraction expansion of α is periodic, that $\alpha \in \mathfrak{m}$ and $\alpha^\sigma \in \widehat{K} - \mathcal{O}$, and that $M_2(\alpha) = \deg a_1(\alpha) + \deg a_2(\alpha)$. The unipotent upper triangular subgroup $\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in R \right\}$ of Γ fixes $\infty \in \partial_\infty \mathbb{T}$, and acts transitively on the subset R of $\partial_\infty \mathbb{T}$. Since the horoballs in the canonical family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}^1(K)}$ whose closure meets the closure of \mathcal{H}_∞ are (besides \mathcal{H}_∞ itself) the ones centred at an element of R , the group Γ acts transitively on the ordered pairs of horoballs in this family whose closures meet at one point. In particular, for all $n \in \mathbb{N}$ large enough, there exists $\gamma_n \in \Gamma$ sending $H_1(\alpha)$ to $H_n(f)$ and $H_2(\alpha)$ to $H_{n+1}(f)$.



For all $x \in \mathbb{T}$ and $\gamma \in \text{Aut}(\mathbb{T})$, and for every horoball \mathcal{H} in \mathbb{T} , the height of γx with respect to $\gamma \mathcal{H}$ is equal to the height of x with respect to \mathcal{H} . Hence for n large enough, since $m(f) > M(\alpha)$, the geodesic lines $] \infty, f[$ and $] \gamma_n \alpha^\sigma, \gamma_n \alpha[=] \gamma_n \cdot \alpha^\sigma, \gamma_n \cdot \alpha[$ intersect along a segment of length equal to $\deg a_1(\alpha) + \deg a_2(\alpha) = M_2(\alpha)$. Let $\beta_n = \gamma_n \cdot \alpha$, we then have $c(f, \beta_n) = q^{-M_2(\alpha)}$ by Lemma 3.1. The fact that $|\beta_n - \beta_n^\sigma| \rightarrow 0$ as $n \rightarrow +\infty$ follows from Lemma 2.3. \square

Corollary 4.6. *Let $\alpha \in K^{(2)}$. If $f \in \widehat{K} - (K \cup \Theta_\alpha)$ satisfies $m(f) > M(\alpha)$, then $c_\alpha(f) = q^{-M_2(\alpha)} \in \text{Sp}(\alpha)$.*

Proof. This follows immediately from Lemmas 4.4 (1) and 4.5, by the definition of the quadratic approximation constants. \square

Corollary 4.7. *Let $\alpha \in K^{(2)}$. Then*

(1) $\max \text{Sp}(\alpha) \geq q^{-M_2(\alpha)}$.

(2) If $M(\alpha) = 1$ or $m(\alpha) \geq 2$, then $\max \text{Sp}(\alpha) = \max_{P \in \mathbb{F}_q[X], \deg P=1} c_\alpha([\overline{P}]) = q^{-2}$.

Proof. (1) This follows from Corollary 4.6 since for instance

$$f = [\overline{a_{p+1}(\alpha) + Y^{M(\alpha)+1}, a_p(\alpha) + Y^{M(\alpha)+1}}]$$

satisfies the assumption of Corollary 4.6 if $p \in \mathbb{N}$ is such that $M_2(\alpha) = \deg a_p(\alpha) + \deg a_{p+1}(\alpha)$.

(2) If $M(\alpha) = 1$, then $M_2(\alpha) = 2$ and this follows from Proposition 4.3 and Corollary 4.6. If $m(\alpha) \geq 2$, with $f = [\overline{Y}]$, we have $M(f) = 1 < m(\alpha)$ and $M_2(f) = 2$, thus $c_\alpha(f) \geq q^{-2}$ by Lemma 4.4 (2), and the result follows from Proposition 4.3. \square

The above corollary shows that the maximum Hurwitz constant is attained for many quadratic irrationals α . In fact, the same holds for all quadratic irrationals with small enough period length.

Proposition 4.8. *If α is a quadratic irrational over K in \widehat{K} whose period of its continued fraction expansion contains at most $q - 2$ coefficients of degree 1, then $\max \text{Sp}(\alpha) = \max_{P \in \mathbb{F}_q[X], \deg P=1} c_\alpha([\overline{P}]) = q^{-2}$.*

The first equality is a (strengthened) nonarchimedean version of the 2-periodic case of Bugeaud's conjecture solved by Lin [Lin, Rem. 1.3]. In particular, if $\alpha \in K^{(2)}$ is eventually k -periodic with $k \leq q - 1$ (as for instance with $k = 2$ since $q \geq 3$), then $\max \text{Sp}(\alpha) = \max_{P \in \mathbb{F}_q[X], \deg P=1} c_\alpha([\overline{P}]) = q^{-2}$. Indeed, either all coefficients of the period of α have degrees 1, in which case $M(\alpha) = 1$ and Corollary 4.7 (2) applies, or α satisfies the assumption of Proposition 4.8. This proves Theorem 1.2 in the introduction.

Proof. Since $\text{Card}(\mathbb{F}_q - \{0\}) = q - 1$ and by the assumption, there exists a polynomial $P \in R$ of degree 1 such that for every degree 1 coefficient $a_i(\alpha)$ of the period of α , the polynomial $P - a_i(\alpha)$ is nonconstant. Let $f = [\overline{P}]$. For all $\beta \in \Theta_\alpha$ and $n \in \mathbb{N}$ large enough, we claim that $] \beta^\sigma, \beta[$ agrees with $] \infty, f[$ on a segment with length at most 1 inside any horoball $H_n(f)$ for $n \in \mathbb{N}$. By an argument as in the proof of Proposition 4.3, this implies that $c_\alpha(f) = q^{-2}$. This in turn implies that $\max \text{Sp}(\alpha) \geq q^{-2}$, and the result follows since q^{-2} is an upper bound on $\max \text{Sp}(\alpha)$ (see Proposition 4.3).

Assume for a contradiction that the geodesic segment $] \beta^\sigma, \beta[$ agrees with $] \infty, f[$ on a segment of length at least 2 inside $H_n(f)$. Since $\deg a_n(f) = \deg P = 1$, this implies that $] \beta^\sigma, \beta[$ and $] \infty, f[$ actually coincide inside of $H_n(f)$. Assume that the orientations of the geodesic lines $] \beta^\sigma, \beta[$ and $] \infty, f[$ respectively from β^σ to β and from ∞ to f agree. By Lemma 4.1, this implies that if $a_i(\beta)$ is the coefficient in the period of β such that $H_n(f) = H_i(\beta)$, then the polynomial $P - a_i(\beta)$ is constant. This implies that $\deg a_i(\beta) = 1$ and this contradicts the definition of P , since α and β have the same period (up to a cyclic permutation). \square

If the period of a quadratic irrational α is longer than $q - 1$, then its Hurwitz constant $\max \text{Sp}(\alpha)$ may be arbitrarily small, as the following result shows.

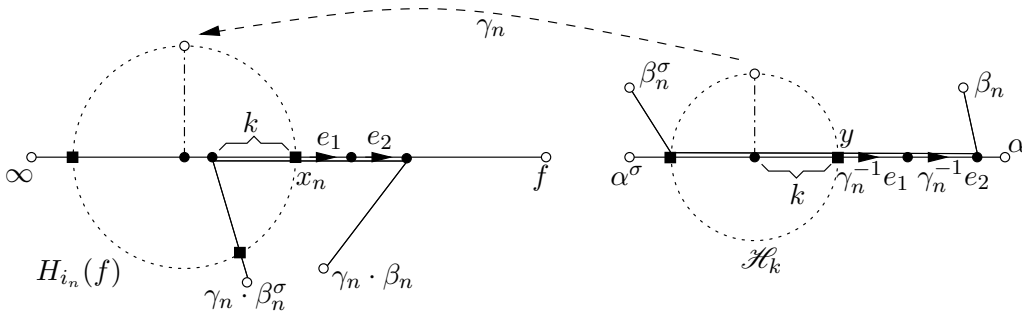
Proposition 4.9. *For all $m, k \in \mathbb{N} - \{0, 1\}$, let us denote by $\{b_1, \dots, b_N\}$ the set⁵ of the finite sequences of length m of polynomials of degree at least 1 and at most k , and let*

$$\mathcal{E}_{m,k} = \bigcup_{\sigma \in \mathcal{S}_N} \Gamma \cdot [\overline{b_{\sigma(1)}, \dots, b_{\sigma(N)}}].$$

For every $\alpha \in \mathcal{E}_{m,k}$, we have $\max \text{Sp}(\alpha) \leq q^{-\min(2m, k+2)}$.

Proof. Let us fix $\alpha \in \mathcal{E}_{m,k}$ and $f \in \widehat{K} - (K \cup \Theta_\alpha)$, and let us prove that $c_\alpha(f) \leq q^{-\min(2m, k+2)}$, which gives the result.

Assume first that there exists a subsequence of coefficients $(a_{i_n}(f))_{n \in \mathbb{N}}$ of f such that $i_0 \geq 1$ and $\deg a_{i_n}(f) > k$ for all $n \in \mathbb{N}$. Let x_n be the point of $\partial H_{i_n}(f)$ at which the geodesic line $]\infty, f[$ exits the horoball $H_{i_n}(f)$. Since any polynomial $P \in R$ of degree k occurs as a coefficient of α (in its periodic part), there exists a horoball \mathcal{H}_k in the canonical family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}^1(K)}$ which intersects the geodesic $]\alpha^\sigma, \alpha[$ in a segment of length exactly $2k$. Let y be the point of $\partial \mathcal{H}_k$ at which the geodesic $]\alpha^\sigma, \alpha[$ exits the horoball \mathcal{H}_k .



By the transitivity of the action of Γ on the pairs (\mathcal{H}, x) where \mathcal{H} belongs to the canonical family $(\mathcal{H}_\xi)_{\xi \in \mathbb{P}^1(K)}$ and x belongs to $\partial \mathcal{H}$, there exists $\gamma_n \in \Gamma$ which sends (\mathcal{H}_k, y) to $(H_{i_n}(f), x_n)$. Let e_1, e_2 be the consecutive edges along $]\infty, f[$ such that the origin of e_1 is x_n . Note that the endpoint of e_2 (which is at distance 2 from x_n) might belong to the orbit $\Gamma*$ of the base point of \mathbb{T} or not. Since any pair (P, P') with $P, P' \in R$, $\deg P = k$ and $\deg P' \in \{1, 2\}$ occurs as a pair of consecutive coefficients in the continued fraction expansion of α , there exists $\beta_n \in \Theta_\alpha$ such that the geodesic line $]\beta_n^\sigma, \beta_n[$ contains $\gamma_n^{-1}e_1$ and $\gamma_n^{-1}e_2$, and coincides with $]\alpha^\sigma, \alpha[$ inside \mathcal{H}_k . Hence $\gamma_n]\beta_n^\sigma, \beta_n[$ coincides with $]\infty, f[$ on e_1, e_2 as well as on a segment of length exactly k inside $H_{i_n}(f)$ (since $\deg a_{i_n}(f) > k$). Thus

$$\min\{c(f, \gamma_n \cdot \beta_n^\sigma), c(f, \gamma_n \cdot \beta_n)\} \leq q^{-(k+2)}$$

by Lemma 3.1. Since $|\gamma_n \cdot \beta_n - \gamma_n \cdot \beta_n^\sigma| \rightarrow 0$ as $n \rightarrow +\infty$ by Lemma 2.3, we have $c_\alpha(f) \leq q^{-(k+2)}$.

Otherwise, there exists $i_0 \in \mathbb{N}$ such that $\deg a_i(f) \leq k$ for all $i \geq i_0$. For every $n \in \mathbb{N}$, consider the string of m consecutive horoballs $H_{i_0+nm}(f), \dots, H_{i_0+(n+1)m-1}(f)$ crossed by the geodesic line $]\infty, f[$. The quadratic irrational α has been constructed in such a way that there exists a string of consecutive coefficients in the period of α that agrees with those

⁵with an arbitrary ordering and $N = (q^{k+1} - q)^m$

of f inside the above horoballs. Using Lemma 4.2, this gives the estimate $c_\alpha(f) \leq q^{-2m}$. Together with the previous estimate, this completes the proof. \square

Example 4.10. If α belongs to the set $\mathcal{E}_{m,k}$ constructed in Proposition 4.9 with $k = m = 2$, then

$$\max \text{Sp}(\alpha) = \max_{P \in R} c_\alpha([\overline{P}]) = \limsup_{P \in R, \deg P \rightarrow +\infty} c_\alpha([\overline{P}]) = q^{-4}.$$

Indeed, Proposition 4.9 gives $\max \text{Sp}(\alpha) \leq q^{-4}$. For any $P \in R$ with $\deg P \geq 3$ and $f = [\overline{P}]$, we have $c_\alpha(f) = q^{-4}$ by Corollary 4.6. The result follows.

Let

$$\varphi = [\overline{Y}].$$

As a special case of Corollary 4.7, we get $\max \text{Sp}(\varphi) = q^{-2}$. The following result gives in particular a complete description of the quadratic Lagrange spectrum of φ , proving Theorem 1.3 in the introduction.

Theorem 4.11. *For every $P \in R$ with $\deg P = 1$, we have*

$$\text{Sp}([\overline{P}]) = \{0\} \cup \{q^{-(2+n)} : n \in \mathbb{N}\}.$$

Proof. Let $\alpha = [\overline{P}]$. For every $m \in \mathbb{N}$, let $f_m = [0, \overline{Y^2, P, P, \dots, P}]$ and $f'_m = [0, \overline{Y^2, P, P, \dots, P, P+1}]$, which are elements of $K^{(2)} - \Theta_\alpha$ with period lengths of the periodic part of their continued fraction equal to $m+1$ and $m+2$ respectively. Let us prove that $c_\alpha(f_m) = q^{-2m-2}$ and $c_\alpha(f'_m) = q^{-2m-3}$, which implies that q^{-n-2} belongs to $\text{Sp}(\alpha)$ for all $n \in \mathbb{N}$. This gives the result by Proposition 4.3.

We know that $c_\alpha(f_0) = q^{-2}$ by Corollary 4.6. Assume that $m \geq 1$. Note that for every $n \in \mathbb{N}$, the maximum height of the geodesic line $] \infty, f_m[$ inside the horoball $H_{n(m+1)+i+1}(f_m)$ is equal to 2 if $i = 0, m+1$ and to 1 if $1 \leq i \leq m$. Since the geodesic lines $]f_m^\sigma, f_m[$ and $] \alpha^\sigma, \alpha[$ both contain points in $\Gamma*$, some image of $] \alpha^\sigma, \alpha[$ by an element of Γ meets $]f_m^\sigma, f_m[$. Hence by Remark 3.2, we have $c_\alpha(f_m) = q^{-n(f_m, \alpha)}$ where

$$n(f_m, \alpha) = \max_{\gamma \in \Gamma} \text{length} (]f_m^\sigma, f_m[\cap \gamma] \alpha^\sigma, \alpha[).$$

By Lemma 4.2, for every $n \in \mathbb{N}$, there exists $\beta_n \in \Theta_\alpha$ such that the geodesic line $] \beta_n^\sigma, \beta_n[$ coincides with the geodesic line $] \infty, f_m[$ inside the horoballs $H_{n(m+1)+i+1}(f_m)$ for $1 \leq i \leq m$. Therefore $n(f_m, \alpha) \geq 2m$. For every such β_n , the height of the geodesic line $] \beta_n^\sigma, \beta_n[$ inside $H_{n(m+1)+i+1}(f_m)$ for $i = 0, m+1$ is at most 1. Hence for n large enough, the geodesic lines $] \infty, f_m[$ and $] \beta_n^\sigma, \beta_n[$ coincide at most in one edge inside $H_{n(m+1)+i+1}(f_m)$ for $i = 0, m+1$. Thus $n(f_m, \alpha) \leq 2m+2$. Since the stabiliser in Γ of both $*$ and the edge e_∞ of $\text{lk}(*)$ pointing towards ∞ acts transitively on the edges of $\text{lk}(*)$ different from e_∞ , it is possible to adjust β_n so that $] \infty, f_m[$ and $] \beta_n^\sigma, \beta_n[$ do coincide in exactly an edge inside $H_{n(m+1)+i+1}(f_m)$ for both $i = 0, m+1$. Thus $n(f_m, \alpha) = 2m+2$, as wanted.

Similarly, for every $n \in \mathbb{N}$, consider the elements $\beta'_n \in \Theta_\alpha$ such that the geodesic line $] \beta'_n{}^\sigma, \beta'_n[$ coincides with the geodesic line $] \infty, f'_m[$ inside the horoballs $H_{n(m+2)+i+1}(f'_m)$ for $1 \leq i \leq m$. Using the fact that the upper unipotent action of $\mathbb{F}_q[X]$ is simply transitive on the edges going out of the horoball \mathcal{H}_∞ , and that two polynomials have the same action on $\partial \mathcal{H}_\infty$ if and only if they differ by a constant, it is possible to adjust β'_n so that for n large enough $] \infty, f'_m[$ and $] \beta'_n{}^\sigma, \beta'_n[$ do coincide in exactly one edge inside $H_{n(m+2)+1}(f'_m)$ and two

edges inside $H_{n(m+2)+m+1}(f'_m)$, but do not coincide in an edge inside $H_{n(m+2)+m+2}(f'_m)$. Thus $n(f'_m, \alpha) = 2m + 3$, as wanted. \square

The next result shows that there are examples of quadratic Lagrange spectra which contain a gap, that is, are not always of the form $\{0\} \cup \{q^{-n} : n \in \mathbb{N}, n \geq N\}$ for some $N \in \mathbb{N}$

Proposition 4.12. *Let $k \in \mathbb{N} - \{0, 1\}$, let $\{b_1, \dots, b_{N'}\}$ be the set⁶ of the polynomials of degree at least k and at most $2k$, and let $\mathcal{F}_k = \bigcup_{\sigma \in \mathcal{S}_{N'}} \Gamma \cdot [b_{\sigma(1)}, \dots, b_{\sigma(N')}]$. For every $\alpha \in \widehat{\mathcal{F}}_k$, the number q^{-2k+1} does not belong to $\text{Sp}(\alpha)$.*

Proof. Let $f \in \widehat{K} - (K \cup \Theta_\alpha)$. Let us prove that $c_\alpha(f) \neq q^{-2k+1}$, which gives the result. There are three cases to consider.

Assume first that there exists $i_0 \in \mathbb{N}$ such that $\deg a_i(f) < k$ for all $i \geq i_0$. By Lemma 4.4 (2), we have $c_\alpha(f) \geq q^{-M_2(f)} \geq q^{-2(k-1)} = q^{-2k+2}$, and in particular $c_\alpha(f) \neq q^{-2k+1}$.

Assume then that there exists a subsequence of coefficients $a_{i_n}(f)$ for $n \in \mathbb{N}$ such that $i_0 \geq 1$ and $k \leq \deg a_{i_n}(f) \leq 2k$. Then $a_{i_n}(f) \in \{b_1, \dots, b_{N'}\}$, and again there exists an element $\beta_{i_n} \in \Theta_\alpha$ for which the intersection $] \beta_{i_n}^\sigma, \beta_{i_n} [\cap] \infty, f [$ has length at least $2 \deg a_{i_n}(f) \geq 2k$. Hence $c_\alpha(f) \leq q^{-2k}$, and in particular $c_\alpha(f) \neq q^{-2k+1}$.

If neither of the previous two cases occurs, we have $\deg a_i(f) > 2k$ for i large enough. By Lemma 4.5, we have $c_\alpha(f) \leq q^{-4k}$, and in particular $c_\alpha(f) \neq q^{-2k+1}$. \square

Corollary 4.13. *Let α be as in Proposition 4.12. Then $\text{Sp}(\alpha)$ contains a gap.*

Proof. By Corollary 4.7, we have $\max \text{Sp}(\alpha) = q^{-2}$, and $\text{Sp}(\alpha)$ contains q^{-n} for all n large enough by Theorem 3.4. Thus, the spectrum has a gap that contains q^{-2k+1} by Proposition 4.12. \square

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⁶with an arbitrary order and $N' = q^{2k+1} - q^k$

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