Counting orbits of integral points in families of affine homogeneous varieties and diagonal flows

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Abstract

In this paper, we study the distribution of integral points on parametric families of affine homogeneous varieties. By the work of Borel and Harish-Chandra, the set of integral points on each such variety consists of finitely many orbits of arithmetic groups, and we establish an asymptotic formula (on average) for the number of the orbits indexed by their Siegel weights. In particular, we deduce asymptotic formulas for the number of inequivalent integral representations by decomposable forms and by norm forms in division algebras, and for the weighted number of equivalence classes of integral points on sections of quadratic surfaces. Our arguments use the exponential mixing property of diagonal flows on homogeneous spaces. [1]

1 Introduction

Let $\mathbf{L}$ be a reductive linear algebraic group defined and anisotropic over $\mathbb{Q}$, let $\pi : \mathbf{L} \to \mathrm{GL}(\mathbf{V})$ be a rational linear representation of $\mathbf{L}$ defined over $\mathbb{Q}$ and let Λ be a $\mathbb{Z}$-lattice in $\mathbf{V}(\mathbb{Q})$ invariant under $\mathbf{L}(\mathbb{Z})$. For every $v \in \mathbf{V}(\mathbb{Q})$ whose orbit $\mathbf{X}_v$ under $\mathbf{L}$ is Zariski-closed in $\mathbf{V}$, Borel-Harish-Chandra's finiteness theorem [BHC, Theo. 6.9] says that the number of orbits of $\mathbf{L}(\mathbb{Z})$ in $\mathbf{X}_v \cap \Lambda$ is finite. The aim of this paper is to study the asymptotic behaviour of this number as v tends to ∞ in $\mathbf{V}(\mathbb{Q})$, in appropriate averages (for instance in order to take into account the fact that $\mathbf{X}_v \cap \Lambda$ could be empty). We will count the orbits using appropriate weights. For every $u \in \mathbf{X}_v(\mathbb{Q})$ with stabiliser $\mathbf{L}_u$ in $\mathbf{L}$, define the *Siegel weight* of u as

$$w_{\mathbf{L},\pi}(u) = \frac{\mathrm{vol}\left(\mathbf{L}_u(\mathbb{Z})\backslash\mathbf{L}_u(\mathbb{R})\right)}{\mathrm{vol}\left(\mathbf{L}(\mathbb{Z})\backslash\mathbf{L}(\mathbb{R})\right)} \; ,$$

using Weil's convention for the normalisation of the measures on $\mathbf{L}_u(\mathbb{R})$ (depending on the choice of a left Haar measure on $\mathbf{L}(\mathbb{R})$ and of a $\mathbf{L}(\mathbb{R})$-invariant measure on $\mathbf{X}_v(\mathbb{R})$, see Section 2). These weights generalise the ones occuring in Siegel's weight formula when $\mathbf{L}$ is an orthogonal group (see for instance [Sie, ERS], and [Vos, Chap. 5] for general $\mathbf{L}$) and are used in many works (see for instance [BR, Oh1]; contrarily to the last two references, we will also need the reductive non semisimple case for our applications). We do not assume $\mathbf{X}_v(\mathbb{R})$ to be an affine symmetric space, contrarily to [DRS] and many other references.

In order to motivate the statement of our main result, let us give two applications using the above setting.

The first one is an asymptotic estimate on the number of inequivalent representations of integers by decomposable forms. Recall that a *decomposable form* $F(x_1, \ldots, x_n)$ is a polynomial in n variables with coefficients in $\mathbb{Q}$ which is the product of d linear forms with coefficients in $\overline{\mathbb{Q}}$. In particular, a *norm form* is a decomposable form $N_{K/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_n x_n)$ where $\alpha_1, \ldots, \alpha_n$ are fixed elements in a number field K of degree d and $x_1, \ldots, x_n$ are rational variables. We will work only with $d = n$. The aim is not to study the existence of integral solutions to one equation $F(x) = m$ (see for instance [CTX], building on work of J. Sansuc, J.-L. Colliot-Thélène, D. Harari, R. Heath-Brown, A. Skorobogatov, M. Borovoi, C. Demarche and others for the existence of rational solutions, see for instance [Pey]), but to consider the inequality $|F(x)| \leq m$ which goes around the existence problem. There are many works on integral solutions to decomposable form inequalities, and in particular for norm forms, by W. M. Schmidt, K. Győry, J.-H. Evertse, H. P. Schlickewei, J. Thunder, Z. Chen, M. Ru, see for instance [Sch, Győ, Thu, Koc] and their references. But, besides the frequent assumption that $d > n$, most of these references work under a hypothesis (nondegenerate as in [Sch, Győ] or of finite type as in [Thu]) which is not satisfied in our situation, since the number of our solutions might be infinite. As explained in [Sch], a natural approach is to count the solutions by families of them, in our case by orbits of naturally acting arithmetic groups. Another approach, formulated by Linnik and Sarnak, and especially developed in [EO, GO1, Oh1], is to use dilations of relatively compact subsets. For the finiteness of this number of orbits, see for instance [Koc, Theo. 2.14.1, page 63].

Theorem 1 *Let $n \geq 2$, let $F \in \mathbb{Q}[x_1, \ldots, x_n]$ be a rational polynomial in n variables, which is irreducible over $\mathbb{Q}$, splits as a product of n linearly independant over $\mathbb{C}$ linear forms with coefficients in $\overline{\mathbb{Q}}$, and satisfies $F^{-1}(]0, +\infty[) \neq \emptyset$. Let $\Gamma_F = \{g \in \mathrm{SL}_n(\mathbb{Z}) : F \circ g = F\}$, and for every $k \in \mathbb{Q}$, let Σ_k be the set of $x \in \mathbb{Z}^n$ such that $F(x) = k$. Then there exist $c = c(F) > 0$ and $\delta = \delta(n) \in]0, 1[$ such that, as $r \to +\infty$,*

$$\sum_{k \in [1, r]} \mathrm{Card}\big(\Gamma_F \backslash \Sigma_k\big) = c\, r + \mathrm{O}\left(r^{\delta}\right).$$

With $\mathbf{L}$ the stabiliser of F in $\mathrm{SL}_n(\mathbb{C})$, $\mathbf{V} = \mathbb{C}^n$, $\Lambda = \mathbb{Z}^n$ and π the inclusion of $\mathbf{L}$ in $\mathrm{GL}(\mathbf{V})$, this result fits into the above program, since $\Gamma_F = \mathbf{L}(\mathbb{Z})$, the algebraic torus $\mathbf{L}$ is anisotropic over $\mathbb{Q}$ (see Lemma 19) and acts simply transitively on the affine subvariety of $\mathbf{V}$ with equation $F(x) = k$ if $k \neq 0$, noting that the Siegel weights $w_{\mathbf{L}, \pi}(u) = 1/\mathrm{vol}\big(\mathbf{L}(\mathbb{Z})\backslash\mathbf{L}(\mathbb{R})\big)$ are then constant. We will explicit c in Section 3.1.

When K is a number field of degree n with ring of integers $\mathscr{O}_K$, taking an integral basis $(\alpha_1, \ldots, \alpha_n)$ of K, and $F(x_1, \ldots, x_n)$ the particular norm form $N_{K/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_n x_n)$, we recover the well-known counting result of the number of nonzero integral ideals of $\mathscr{O}_K$ with trivial ideal class and norm at most s (see for instance [Lan, Theorem 3, page 132]), giving

$$\{\mathfrak{a} \text{ ideal in } \mathscr{O}_K : N_{K/\mathbb{Q}}(\mathfrak{a}) \leq s\} = \frac{2^{r_1}(2\pi)^{r_2} R_K h_K}{\omega_K \sqrt{|D_K|}}\, s + \mathrm{O}(s^{1-\epsilon}), \tag{1}$$

where r_1 and r_2 are the numbers of real and complex conjugate embeddings of K, R_K is the regulator of K, h_K is the ideal class number of K, ω_K is the number of roots of unity of $\mathscr{O}_K$, D_K is the discriminant of K and $\epsilon = 1/n$.

The second application is an asymptotic estimate on the (weighted) number of inequivalent integral points on hyperplane sections of affine quadratic surfaces. See for instance [Sie, DRS, ERS, BR, EMS, GO1, Oh1, EO, GO2], as well as the surveys [Bab, Oh2], for counting results of integral or rational points in affine homogeneous varieties. Our result is quite different, since we are counting whole orbits, weighted by the Siegel weights, of integral points. Another approach to properties of sums of Siegel weights is to express them as product of local densities, using the Siegel weight formula as in [Sie, ERS, Oh1]. But we believe that our results do not follow from this formula in any obvious way, and offer a really new approach to the asymptotic of Siegel weights.

Theorem 2 *Let $n \geq 3$, let $q : \mathbb{C}^n \to \mathbb{C}$ be a nondegenerate rational quadratic form, which is isotropic over $\mathbb{Q}$, let $\ell : \mathbb{C}^n \to \mathbb{C}$ be a nonzero rational linear form, and let $\mathbf{L} = \{g \in \mathrm{SL}_n(\mathbb{C}) : q \circ g = q, \ell \circ g = \ell\}$. For every $k \in \mathbb{Q}$, let Σ_k be the set of primitive $x \in \mathbb{Z}^n$ such that $q(x) = 0$ and $\ell(x) = k$. Assume that the restriction of q to the kernel of ℓ is nondegenerate and anisotropic over $\mathbb{Q}$. Then there exist $c = c(q, \ell) > 0$ and $\delta = \delta(q) > 0$ such that, as $r \to +\infty$,*

$$\sum_{k \in [1,r]} \sum_{[u] \in (\mathbf{L}(\mathbb{Z}) \cap \mathbf{L}(\mathbb{R})_0) \backslash \Sigma_k} \mathrm{vol}\left((\mathbf{L}_u(\mathbb{Z}) \cap \mathbf{L}(\mathbb{R})_0) \backslash (\mathbf{L}_u \cap \mathbf{L}(\mathbb{R})_0)\right) = c\, r^{n-2} + \mathrm{O}\left(r^{n-2-\delta}\right).$$

This result also fits into the above program (up to a slight modification of the Siegel weights, see Section 2), by taking $\mathbf{V} = \mathbb{C}^n$, $\Lambda = \mathbb{Z}^n$, and $\pi : \mathbf{L} \to \mathrm{GL}(\mathbf{V})$ the inclusion map, noting that $\mathbf{L}$ is semisimple, and defined and anisotropic over $\mathbb{Q}$ as a consequence of the assumptions (see Section 3.2 for details, where we also explicit c).

We will prove the above two results in Section 3. As another application of our main result, we also give there an asymptotic formula for the number of orbits under the group of integral units of the integral points of given norm in a division algebra over $\mathbb{Q}$.

A particular case of the main result of this paper is the following one.

Theorem 3 *Let $\mathbf{G}$ be a simply connected reductive linear algebraic group defined over $\mathbb{Q}$, without nontrivial $\mathbb{Q}$-characters. Let $\mathbf{P}$ be a maximal parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$, and let $\mathbf{P} = \mathbf{A}\mathbf{M}\mathbf{U}$ be a relative Langlands decomposition of $\mathbf{P}$, such that $\mathbf{A}(\mathbb{R})_0$ is a one-parameter subgroup $(a_s)_{s \in \mathbb{R}}$, with $\lambda = \log \det (\mathrm{Ad}\, a_1)_{|\mathfrak{U}} > 0$, where $\mathfrak{U}$ is the Lie algebra of $\mathbf{U}(\mathbb{R})$. Let $\rho : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ be a rational representation of $\mathbf{G}$ defined over $\mathbb{Q}$ such that there exists $v_0 \in \mathbf{V}(\mathbb{Q})$ whose stabiliser in $\mathbf{G}$ is $\mathbf{M}\mathbf{U}$. Let $\mathbf{L}$ be a reductive algebraic subgroup of $\mathbf{G}$ defined and anisotropic over $\mathbb{Q}$. Assume that $\mathbf{L}\mathbf{P}$ is Zariski-open in $\mathbf{G}$ and that for every $s \in \mathbb{R}$, the orbit $\mathbf{X}_s = \rho(\mathbf{L}a_s)v_0$ is Zariski-closed in $\mathbf{V}$.*

Let Λ be a $\mathbb{Z}$-lattice in $\mathbf{V}(\mathbb{Q})$ invariant under $\mathbf{G}(\mathbb{Z})$, and let Λ^{prim} be the subset of indivisible elements of Λ. Assume ρ to be irreducible over $\mathbb{C}$. Then there exist $c, \delta > 0$ such that, as t tends to $+\infty$,

$$\sum_{0 \leq s \leq t} \sum_{[x] \in \mathbf{L}(\mathbb{Z}) \backslash (\mathbf{X}_s \cap \Lambda^{\mathrm{prim}})} w_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) = c\, e^{\lambda t} + \mathrm{O}(e^{(\lambda - \delta)t}).$$

More precisely, let $\mathbf{G}, \mathbf{P}, \mathbf{A}, \mathbf{M}, \mathbf{U}, \mathbf{V}, \mathbf{L}, \rho, v_0, (a_s)_{s \in \mathbb{R}}$ be as above (ρ not necessarily irreducible). Endow $\mathbf{G}(\mathbb{R})$ with a left-invariant Riemannian metric, for which the Lie algebras of $\mathbf{M}\mathbf{U}(\mathbb{R})$ and $\mathbf{A}(\mathbb{R})$ are orthogonal, and the orthogonal of the Lie algebra of $\mathbf{P}(\mathbb{R})$ is contained in the Lie algebra of $\mathbf{L}(\mathbb{R})$.

Theorem 4 *There exists $\delta > 0$ such that, as t tends to $+\infty$,*

$$\sum_{0 \le s \le t} \sum_{[x] \in \mathbf{L}(\mathbb{Z}) \backslash (\rho(\mathbf{L}(\mathbb{R}) a_s) v_0 \cap \rho(\mathbf{G}(\mathbb{Z})) v_0)} w_{\mathbf{L}, \rho_{|\mathbf{L}}}(x)$$

$$= \frac{\mathrm{vol}\left(\mathbf{MU}(\mathbb{Z}) \backslash \mathbf{MU}(\mathbb{R})\right) \mathrm{vol}(a_1^{\mathbb{Z}} \backslash \mathbf{A}(\mathbb{R})_0)}{\lambda \, \mathrm{vol}(\mathbf{G}(\mathbb{Z}) \backslash \mathbf{G}(\mathbb{R}))} \, e^{\lambda t} + \mathrm{O}(e^{(\lambda - \delta)t}) \ .$$

We will prove a more general version of this result in Section 2 without the maximality condition on $\mathbf{P}$, involving the more elaborate root data of $\mathbf{P}$, and without the simple connectedness assumption on $\mathbf{G}$ (up to a slight modification of the Siegel weights), see Theorem 6 and Theorem 16. We are using the proof of the main result of [PP1] as a guideline.

Another main difference with the counting results of [EMS, Oh1, EO, GO2] is that these papers are using the dynamics of unipotent flows, as instead we are using here the mixing property with exponential decay of correlations of diagonalisable flows, in the spirit of [KM1] (see also [EM, BO]).

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2 Counting Siegel weights

Here are a few notational conventions. By linear algebraic group $\mathbf{G}'$ defined over a subfield k of $\mathbb{C}$, we mean a subgroup of $\mathrm{GL}_N(\mathbb{C})$ for some $N \in \mathbb{N}$ which is a closed algebraic subset of $\mathscr{M}_N(\mathbb{C})$ defined over k, and we define $\mathbf{G}'(\mathbb{Z}) = \mathbf{G}' \cap \mathrm{GL}_N(\mathbb{Z})$. For every linear algebraic group $\mathbf{G}'$ defined over $\mathbb{R}$, we denote by $\mathbf{G}'(\mathbb{R})_0$ the identity component of the Lie group of real points of $\mathbf{G}'$. We denote by $\log$ the natural logarithm.

Let us first recall *Weil's normalisation* of measures on homogeneous spaces. Let G' be a unimodular real Lie group, endowed with a transitive smooth left action of G' on a smooth manifold X', with unimodular stabilisers. A triple $(\nu_{G'}, \nu_{X'}, (\nu_{G'_x})_{x \in X'})$ of a left Haar measure $\nu_{G'}$ on G', a left-invariant (Borel, positive, regular) measure $\nu_{X'}$ on X' and of a left Haar measure $\nu_{G'_x}$ on the stabiliser G'_x of every $x \in X'$, is *compatible* if, for every $x \in X'$, for every $f : G' \to \mathbb{R}$ continuous with compact support, with $f_x : X' \to \mathbb{R}$ the map (well) defined by $gx \mapsto \int_{h \in G'_x} f(gh) \, d\nu_{G'_x}(h)$ for every $g \in G$, we have

$$\int_{G'} f \, d\nu_{G'} = \int_{X'} f_x \, d\nu_{X'} \ .$$

Weil proved (see for instance [Wei, §9]) that, for every left-invariant measure $\nu_{X'}$ on X', then
• for every left Haar measure $\nu_{G'}$ on G', there exists a unique compatible triple $(\nu_{G'}, \nu_{X'}, (\nu_{G'_x})_{x \in X'})$.
• for every $x_0 \in X'$, for every left Haar measure ν_0 on G'_{x_0}, there exists a unique compatible triple $(\nu_{G'}, \nu_{X'}, (\nu_{G'_x})_{x \in X'})$ with $\nu_{G'_{x_0}} = \nu_0$.

The following remark should be well-known, though we did not found a precise reference.

Lemma 5 *If $(\nu_{G'}, \nu_{X'}, (\nu_{G'_x})_{x \in X'})$ is a compatible triple, then for every $\ell \in G'$ and $x \in X'$, with $i_\ell : h \mapsto \ell h \ell^{-1}$ the conjugation by ℓ, we have*

$$\nu_{G'_{\ell x}} = (i_\ell)_* \nu_{G'_x} .$$

Proof. Let $x \in X'$, $\ell \in G'$, $H' = G'_x$ and $H'' = G'_{\ell x} = \ell H' \ell^{-1}$. Using the left invariance of $\nu_{X'}$ for the first inequality and the bi-invariance of the Haar measure on G' for the last one, we have, for every $f : G' \to \mathbb{R}$ continuous with compact support,

$$\int_{g'\ell x \in X'} \int_{h'' \in H''} f(g'h'') \, d(i_\ell)_* \nu_{G'_x}(h'') \, d\nu_{X'}(g'\ell x)$$

$$= \int_{g'\ell x \in X'} \int_{h'' \in H''} f(\ell g'h'') \, d(i_\ell)_* \nu_{G'_x}(h'') \, d\nu_{X'}(g'\ell x)$$

$$= \int_{g'\ell x \in X'} \int_{h' \in H'} f(\ell g'\ell h'\ell^{-1}) \, d\nu_{G'_x}(h') \, d\nu_{X'}(g'\ell x)$$

$$= \int_{gx \in X'} \int_{h' \in H'} f \circ i_\ell(gh') \, d\nu_{G'_x}(h') \, d\nu_{X'}(gx)$$

$$= \int_{G'} f \circ i_\ell \, d\nu_{G'} = \int_{G'} f \, d\nu_{G'} .$$

The result then follows by uniqueness. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In order to deal with non simply connected groups, we introduce a modified version of the Siegel weights.

Let $\mathbf{L}'$ be a reductive linear algebraic group defined and anisotropic over $\mathbb{Q}$, let $\pi : \mathbf{L}' \to \mathrm{GL}(\mathbf{V}')$ be a rational linear representation of $\mathbf{L}'$ defined over $\mathbb{Q}$, let $v \in \mathbf{V}'(\mathbb{Q})$ be such that its orbit $\mathbf{X}'_v$ under $\mathbf{L}'$ is Zariski-closed in $\mathbf{V}'$, let $u \in \mathbf{X}'_v(\mathbb{Q})$ and let $\mathbf{L}'_u$ be the stabiliser of u in $\mathbf{L}'$. We defined the *modified Siegel weight* of u as

$$w'_{\mathbf{L}',\pi}(u) = \frac{\mathrm{vol}\left((\mathbf{L}'_u(\mathbb{Z}) \cap \mathbf{L}'(\mathbb{R})_0) \backslash (\mathbf{L}'_u \cap \mathbf{L}'(\mathbb{R})_0)\right)}{\mathrm{vol}\left((\mathbf{L}'(\mathbb{Z}) \cap \mathbf{L}'(\mathbb{R})_0) \backslash \mathbf{L}'(\mathbb{R})_0\right)} , \qquad (2)$$

using Weil's convention for the normalisation of the measures on $\mathbf{L}'_u(\mathbb{R})$ (depending on the choice of a left Haar measure on $\mathbf{L}'(\mathbb{R})$ and of a $\mathbf{L}'(\mathbb{R})$-invariant measure on $\mathbf{X}'_v(\mathbb{R})$). Note that the denominator of the standard Siegel weight $w_{\mathbf{L}',\pi}(u)$ is an integral multiple (depending only on $\mathbf{L}'$) of the denominator of the modified one, since $(\mathbf{L}'(\mathbb{Z}) \cap \mathbf{L}'(\mathbb{R})_0) \backslash \mathbf{L}'(\mathbb{R})_0$ is a connected component of $\mathbf{L}'(\mathbb{Z}) \backslash \mathbf{L}'(\mathbb{R})$. But the ratio of the numerator of the Siegel weight by the numerator of the modified one may depend on u.

Let us now describe the framework of our main result. Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$. Let $\mathbf{P}$ be a (proper) parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ (see for instance [BJ, §III.1], [Spr, §5.2])). Recall that a linear algebraic group defined over $\mathbb{Q}$ is $\mathbb{Q}$-*anisotropic* if it contains no nontrivial $\mathbb{Q}$-split torus.

Recall that there exist a (nontrivial) maximal $\mathbb{Q}$-split torus $\mathbf{S}$ in $\mathbf{G}$ (contained in $\mathbf{P}$ and unique modulo conjugation by an element of $\mathbf{P}(\mathbb{Q})$), such that if $\Phi^{\mathbb{C}} = \Phi(\mathbf{G}, \mathbf{S})$ is the root system of $\mathbf{G}$ relative to $\mathbf{S}$ (seen contained in the set of characters of $\mathbf{S}$), if $\mathfrak{g}^{\mathbb{C}}_\beta$ is the root space of $\beta \in \Phi^{\mathbb{C}}$, then there exist a unique set of simple roots $\Delta = \Delta_{\mathbf{P}}$ in $\Phi^{\mathbb{C}}$ and a unique proper subset $I = I_{\mathbf{P}}$ of Δ, such that, with $\Phi^{\mathbb{C}}_+$ the set of positive roots of $\Phi^{\mathbb{C}}$ defined by

Δ and $\Phi_I^{\mathbb{C}}$ the set of roots of Φ that are linear combinations of elements of I, if $\mathbf{A}$ is the identity component of

$$\bigcap_{\alpha \in I} \ker \alpha \; ,$$

which is a $\mathbb{Q}$-split subtorus of $\mathbf{S}$, if $\mathbf{U}$ is the connected algebraic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ whose Lie algebra is

$$\mathfrak{u}^{\mathbb{C}} = \bigoplus_{\beta \in \Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}} \mathfrak{g}_\beta^{\mathbb{C}} \; ,$$

then $\mathbf{P}$ is the semi-direct product of its unipotent radical $\mathbf{U}$ and of the centraliser of $\mathbf{A}$ in $\mathbf{G}$. Note that $\mathbf{A}$ is one-dimensional if $\mathbf{P}$ is a maximal (proper) parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ (that is, if $\Delta - I$ is a singleton).

Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}(\mathbb{R})$. Using the multiplicative notation on the group of characters of $\mathbf{S}$, for every $\alpha \in \Delta$, we define $m_\alpha = m_{\alpha, \mathbf{P}} \in \mathbb{N}$ by

$$\prod_{\beta \in \Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}} \beta^{\dim_{\mathbb{R}}(\mathfrak{g}_\beta^{\mathbb{C}} \cap \mathfrak{g})} = \prod_{\alpha \in \Delta} \alpha^{m_\alpha} \; .$$

Let $(\alpha^\vee)_{\alpha \in \Delta - I}$ in $\mathbf{A}(\mathbb{R})_0^{\Delta - I}$ be such that $\log \beta(\alpha^\vee)$ is equal to 1 if $\alpha = \beta$ and to 0 otherwise. Let $\Lambda^\vee$ be the lattice in $\mathbf{A}(\mathbb{R})_0$ generated by $\{\alpha^\vee \; : \; \alpha \in \Delta - I\}$. For every element $T = (t_\alpha)_{\alpha \in \Delta - I}$ of $[0, +\infty[^{\Delta - I}$, let

$$A_T = \{a \in \mathbf{A}(\mathbb{R})_0 \; : \; \forall \, \alpha \in \Delta - I, \; 0 \leq \log(\alpha(a)) \leq t_\alpha\} \; .$$

Recall that by the definition of *a relative Langlands decomposition* of the parabolic subgroup $\mathbf{P}$ defined over $\mathbb{Q}$, there exists a connected reductive algebraic subgroup $\mathbf{M}$ of $\mathbf{P}$ defined over $\mathbb{Q}$ without nontrivial $\mathbb{Q}$-characters such that $\mathbf{AM}$ is the centraliser of $\mathbf{A}$ in $\mathbf{G}$. In particular, $\mathbf{AM}$ is a Levi subgroup of $\mathbf{P}$ defined over $\mathbb{Q}$, $\mathbf{A}$ centralises $\mathbf{M}$ and is the largest $\mathbb{Q}$-split subtorus of the centre of $\mathbf{AM}$, $\mathbf{AM}$ normalises $\mathbf{U}$, and

$$\mathbf{P} = \mathbf{AMU} \; .$$

For every Lie group G' endowed with a left Haar measure, for every discrete subgroup Γ' of G', we endow $\Gamma' \backslash G'$ with the unique measure such that the canonical covering map $G' \to \Gamma' \backslash G'$ locally preserves the measures.

In what follows, we will need a normalisation of the Haar measures, which behaves appropriately when passing to some subgroups. We will start with a Riemannian metric on $\mathbf{G}(\mathbb{R})$, take the induced Riemannian volumes on the real points of the various algebraic subgroups of $\mathbf{G}$ defined over $\mathbb{Q}$ that will appear, which will give us the choices necessary for using Weil's normalisation to define the Siegel weights.

The main result of this paper is the following one.

Theorem 6 *Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$, without nontrivial $\mathbb{Q}$-characters. Let $G = \mathbf{G}(\mathbb{R})_0$ and $\Gamma = \mathbf{G}(\mathbb{Z}) \cap G$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$, and let $\mathbf{P} = \mathbf{AMU}$ be a relative Langlands decomposition of $\mathbf{P}$. Let $\rho : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ be a rational representation of $\mathbf{G}$ defined over $\mathbb{Q}$ such that there exists $v_0 \in \mathbf{V}(\mathbb{Q})$ whose stabiliser in $\mathbf{G}$ is $\mathbf{H} = \mathbf{MU}$. Let $\mathbf{L}$ be a reductive algebraic subgroup of $\mathbf{G}$ defined and anisotropic over $\mathbb{Q}$.*

Assume that $\mathbf{LP}$ *is Zariski-open in* $\mathbf{G}$ *and that for every* $a \in \mathbf{A}$, *the orbit* $\rho(\mathbf{L}a)v_0$ *is Zariski-closed in* $\mathbf{V}$. *Endow* $\mathbf{G}(\mathbb{R})$ *with a left-invariant Riemannian metric, for which the Lie algebras of* $\mathbf{H}(\mathbb{R})$ *and* $\mathbf{A}(\mathbb{R})$ *are orthogonal, and the orthogonal of the Lie algebra of* $\mathbf{P}(\mathbb{R})$ *is contained in the Lie algebra of* $\mathbf{L}(\mathbb{R})$.

Then there exists $\delta > 0$ *such that, as* $T = (t_\alpha)_{\alpha \in \Delta-I} \in [0, +\infty[^{\Delta-I}$ *and* $\min_{\alpha \in \Delta-I} t_\alpha$ *tends to* $+\infty$,

$$\sum_{a \in A_T} \quad \sum_{[x] \in (\mathbf{L}(\mathbb{R})_0 \cap \Gamma) \backslash (\rho(\mathbf{L}(\mathbb{R})_0 a)v_0 \cap \rho(\Gamma)v_0)} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) =$$

$$\frac{\mathrm{vol}\left((\mathbf{H} \cap \Gamma) \backslash (\mathbf{H} \cap G)\right) \mathrm{vol}(\Lambda^\vee \backslash \mathbf{A}(\mathbb{R})_0)}{\mathrm{vol}(\Gamma \backslash G)} \left(\prod_{\alpha \in \Delta-I} \frac{e^{m_\alpha t_\alpha}}{m_\alpha} \right) \left(1 + \mathrm{O}(e^{-\delta \min_{\alpha \in \Delta-I} t_\alpha})\right) .$$

Proof. Let us start by fixing the notation that will be used throughout the proof of Theorem 6, and by making more explicit the above-mentionned conventions about the various volumes that occur in the asymptotic formula.

Consider the connected real Lie group $G = \mathbf{G}(\mathbb{R})_0$, its (closed) Lie subgroups

$$A = \mathbf{A}(\mathbb{R})_0, \; H = \mathbf{H} \cap G, \; L = \mathbf{L}(\mathbb{R})_0, \; M = \mathbf{M} \cap G, \; P = \mathbf{P} \cap G, \; U = \mathbf{U}(\mathbb{R}) .$$

We have $H = MU$ and $P = AMU = MUA$, since A and U are connected. Note that L is also connected, but H and M are not necessarily connected. We denote by

$$\mathfrak{a}, \quad \mathfrak{g}, \quad \mathfrak{h}, \quad \mathfrak{l}, \quad \mathfrak{m}, \quad \mathfrak{p}, \quad \mathfrak{u}$$

the Lie algebras of the real Lie groups A, G, H, L, M, P, U respectively, endowed with the restriction of the scalar product on $\mathfrak{g}$ defined by the Riemannian metric of G. Since $\mathbf{L}$ is $\mathbb{Q}$-anisotropic, so is $\mathbf{L} \cap \mathbf{P}$. Since the map $\mathbf{L} \cap \mathbf{P} \to \mathbf{P}/\mathbf{H} \simeq \mathbf{A}$ is defined over $\mathbb{Q}$ and $\mathbf{A}$ is a $\mathbb{Q}$-split torus, this implies that the identity component of $\mathbf{L} \cap \mathbf{P}$ is contained in $\mathbf{H}$. In particular

$$\mathfrak{l} \cap \mathfrak{h} = \mathfrak{l} \cap \mathfrak{p} . \tag{3}$$

Note that $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$ since $\mathbf{LP}$ is Zariski-open in $\mathbf{G}$. We have assumed that $\mathfrak{a}$ is orthogonal to $\mathfrak{h}$ and that the orthogonal $\mathfrak{p}^\perp$ of $\mathfrak{p}$ is contained in $\mathfrak{l}$. In particular, with $\mathfrak{q}$ the orthogonal of $\mathfrak{l} \cap \mathfrak{h}$ in $\mathfrak{h}$, we have the following orthogonal decompositions

$$\mathfrak{g} = \mathfrak{p}^\perp \overset{\perp}{\oplus} (\mathfrak{l} \cap \mathfrak{h}) \overset{\perp}{\oplus} \mathfrak{q} \overset{\perp}{\oplus} \mathfrak{a}, \quad \mathfrak{h} = (\mathfrak{l} \cap \mathfrak{h}) \overset{\perp}{\oplus} \mathfrak{q}, \quad \mathfrak{l} = \mathfrak{p}^\perp \overset{\perp}{\oplus} (\mathfrak{l} \cap \mathfrak{h}), \quad \mathfrak{p} = \mathfrak{h} \overset{\perp}{\oplus} \mathfrak{a} . \tag{4}$$

The left-invariant Riemannian metric on G induces a left Haar measure ω_G on G, and a left-invariant Riemannian metric on every Lie subgroup G' of G, hence a left Haar measure

$$\omega_{G'}$$

on G' (which is the counting measure if G' is discrete). Note that $A, G, H, L, M, U, L \cap H$ are unimodular: indeed A, G, L, M are reductive and U is unipotent; furthermore, $\mathbf{L} \cap \mathbf{H}$ is the stabiliser of v_0 in $\mathbf{L}$, the orbit of v_0 under $\mathbf{L}$ is affine and hence $\mathbf{L} \cap \mathbf{H}$ is reductive by [BHC, Theo. 3.5]. But P is not unimodular.

The map $A \times M \times U \to P$ defined by $(a, m, u) \mapsto amu$ is a smooth diffeomorphism (see for instance [BJ, page 273]). We will denote by $d\omega_A d\omega_H$ the measure on P which is

the push-forward of the product measure by the diffeomorphism $(a, h) \mapsto ah$. Since A normalises H, the measure $d\omega_A d\omega_H$ is left-invariant by P, so that $d\omega_P(ah)$ and $d\omega_A(a)d\omega_H(h)$ are proportional. Since these measures are induced by Riemannian metrics, and since $\mathfrak{a}$ and $\mathfrak{h}$ are orthogonal, we hence have

$$d\omega_P(ah) = d\omega_A(a)d\omega_H(h) \ .$$

Since A normalises U, the group A acts on the Lie algebra $\mathfrak{U}$ of U by the adjoint representation. The roots of this linear representation of A are exactly the restrictions to A of the elements β in $\Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}$, with root spaces $\mathfrak{g}_\beta^{\mathbb{C}} \cap \mathfrak{g}$ and a set of simple roots is the set of restrictions of the elements of $\Delta - I$ to A (see for instance [BJ, Rem. III.1.14]). Since A is connected, these roots have value in $]0, +\infty[$. The map $A \to \mathbb{R}^{\Delta-I}$ defined by $a \mapsto (\log(\alpha(a)))_{\alpha \in \Delta-I}$ is hence a smooth diffeomorphism. We will denote by $\prod_{\alpha \in \Delta-I} dt_\alpha$ the measure on A which is the push-forward of the product Lebesgue measure by the inverse of this diffeomorphism. By invariance, there exists a constant $c_A > 0$ such that

$$d\omega_A = c_A \prod_{\alpha \in \Delta-I} dt_\alpha \ .$$

By the definition of $\Lambda^\vee$, we have $c_A = \text{Vol}(\Lambda^\vee \backslash A)$.

Let $\Gamma = \mathbf{G}(\mathbb{Z}) \cap G$, which is a discrete subgroup of G acting isometrically for the Riemannian metric of G by left translations. Let $Y_G = \Gamma \backslash G$ and let $\pi : G \to Y_G = \Gamma \backslash G$ be the canonical projection, which is equivariant under the right actions of G. Then Y_G is a connected Riemannian manifold (for the unique Riemannian metric such that π is a local isometry) endowed with the transitive right action of G by translations on the right. To simplify the notation, for every Lie subgroup G' of G, define

$$Y_{G'} = \pi(G') \ ,$$

which is a injectively immersed submanifold in Y_G, endowed with the Riemannian metric induced by Y_G, and identified with $(G' \cap \Gamma) \backslash G'$ by the map induced by the inclusion of G' in G. Note that Y_L and Y_U are connected, but Y_H and Y_M are not necessarily connected. For every Lie subgroup G' of G, let

$$\mu_{G'}$$

be the Riemannian measure on $Y_{G'}$, which locally is the push-forward of the left Haar measure $\omega_{G'}$.

Since $\mathbf{G}$ and the identity component of $\mathbf{MU}$ have no nontrivial $\mathbb{Q}$-character, the Riemannian manifolds Y_G and Y_H have finite volume (see [BHC, Theo. 9.4]) and Y_H is closed in Y_G (see for instance [Rag, Theo. 1.13]). Since $\mathbf{L}$ is reductive and $\mathbb{Q}$-anisotropic, the submanifold Y_L is compact (see [BHC, Theo. 11.6]). Since $\mathbf{U}$ is unipotent, the submanifold Y_U is compact (see for instance [BHC, § 6.10]).

For every Lie subgroup G' of G such that $Y_{G'}$ has finite measure (that is, such that $\Gamma \cap G'$ is a lattice in G'), we denote by

$$\overline{\mu}_{G'} = \frac{\mu_{G'}}{\|\mu_{G'}\|}$$

the finite measure $\mu_{G'}$ normalised to be a probability measure. In particular, $\overline{\mu}_G, \overline{\mu}_H, \overline{\mu}_L, \overline{\mu}_U$ are well defined.

For every $T = (t_\alpha)_{\alpha \in \Delta - I}$ and $T' = (t'_\alpha)_{\alpha \in \Delta - I}$ in $[0, +\infty[^{\Delta - I}$, let

$$A_{[T,T']} = \{ a \in A \ : \ \forall \, \alpha \in \Delta - I, \ \ t_\alpha \le \log(\alpha(a)) \le t'_\alpha \} \ .$$

and $P_{[T,T']} = UMA_{[T,T']}{}^{-1} = HA_{[T,T']}{}^{-1}$. Define $Y_{P_{[T,T']}} = \pi(P_{[T,T']})$, which is a submanifold with boundary of Y_G, invariant under the right action of H, since A normalises H. To shorten the notation, we define

$$A_T = A_{[0,T]} \ , \quad P_T = P_{[0,T]} = HA_T{}^{-1} \quad \text{and} \quad Y_{P_T} = Y_{P_{[0,T]}} = \pi(P_T) = Y_H A_T{}^{-1} \ ,$$

as well as $\min T = \min_{\alpha \in \Delta - I} t_\alpha \ge 0$, which measures the complexity of T and will converge to $+\infty$. We will need to estimate the volume of $\pi(P_T)$ for μ_P.

Lemma 7 *For every $T = (t_\alpha)_{\alpha \in \Delta - I}$ in $[0, +\infty[^{\Delta - I}$, we have*

$$\mu_P(Y_{P_T}) = \mathrm{vol}(\Lambda^\vee \backslash \mathbf{A}(\mathbb{R})_0) \ \|\mu_H\| \prod_{\alpha \in \Delta - I} \frac{e^{m_\alpha t_\alpha}}{m_\alpha} \ .$$

Proof. Denote by du_β the Lebesgue measure on the Euclidean space $\mathfrak{g}_\beta = \mathfrak{g}_\beta^{\mathbb{C}} \cap \mathfrak{g}$. For any order on $\Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}$, the map from $\prod_{\beta \in \Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}} \mathfrak{g}_\beta$ to U defined by $(u_\beta)_{\beta \in \Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}} \mapsto \prod_{\beta \in \Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}} \exp u_\beta$ is a smooth diffeomorphism, and there exists $c_U > 0$ such that $d\omega_U$ is the push-forward by this diffeomorphism of the measure $c_U \prod_{\beta \in \Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}} du_\beta$.

For every $a \in A$, if $i_a : g \mapsto aga^{-1}$ is the conjugation by a, then for every $u_\beta \in \mathfrak{g}_\beta$, we have $i_a(\exp u_\beta) = \exp((\mathrm{Ad}\, a)(u_\beta)) = \exp(\beta(a)u_\beta)$. Hence

$$(i_a^{-1})_*(\omega_U) = \prod_{\beta \in \Phi_+^{\mathbb{C}} - \Phi_I^{\mathbb{C}}} \beta(a)^{\dim \mathfrak{g}_\beta} \omega_U = \prod_{\alpha \in \Delta - I} \alpha(a)^{m_\alpha} \omega_U$$

by the definition of $(m_\alpha)_{\alpha \in \Delta}$ and since the elements of I are trivial on A. Since A commutes with M, we hence have $(i_a^{-1})_*(\omega_H) = \prod_{\alpha \in \Delta - I} \alpha(a)^{m_\alpha} \omega_H$.

We have, since A is unimodular,

$$d\omega_P(ha^{-1}) = d\omega_P(a^{-1}aha^{-1}) = d\omega_A(a^{-1})d\omega_H(aha^{-1}) = d\omega_A(a)d((i_a^{-1})_*\omega_H)(h) \ .$$

Since $\Gamma \cap P = \Gamma \cap H$ (see for instance the lines following Proposition III.2.21 in [BJ, page 285]) and $A \cap H = \{e\}$, we have $\pi(Ha) \ne \pi(Ha')$ if $a \ne a'$. Hence

$$\mu_P(Y_{P_T}) = \int_{y \in Y_H} \int_{a \in A_T} d\mu_P(ya^{-1}) = \int_{A_T} \prod_{\alpha \in \Delta - I} \alpha(a)^{m_\alpha} d\omega_A(a) \int_{Y_H} d\mu_H \qquad (5)$$

$$= \|\mu_H\| \, c_A \prod_{\alpha \in \Delta - I} \int_0^{t_\alpha} e^{m_\alpha s} \, ds \ .$$

Since $m_\alpha > 0$, the result follows. $\qquad\qquad\square$

To simplify the notation, we write $\rho(g)x = gx$ for every $g \in \mathbf{G}$ and $x \in \mathbf{V}$, we define $v_a = av_0$ for every $a \in A$, and we denote by $L_x = \mathbf{G}_x \cap L$ the stabiliser of x in L for every $x \in \mathbf{V}(\mathbb{R})$.

Since we have a left Haar measure ω_L on L and $\omega_{L \cap H}$ on $L \cap H$, Weil's normalisation gives a L-invariant measure on the homogeneous space $L/(L \cap H)$, and hence a left Haar

measure on the stabilisers $\ell(L \cap H)\ell^{-1}$ for every ℓ in L, as explained above. As announced, the modified Siegel weights $w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(\cdot)$ are defined using this Weil's normalisation, as follows.

For every $\ell \in L$ and $a \in A$, if $x = \ell a v_0$, since $H = MU$ is normalised by A and is the stabiliser of v_0 in G, we have

$$L_x = L \cap \operatorname{Stab}_G x = L \cap (\ell H \ell^{-1}) = \ell(L \cap H)\ell^{-1} \; . \tag{6}$$

Note that

$$\mathbf{L}(\mathbb{R})_0 \cap \mathbf{L}(\mathbb{Z}) = \mathbf{L}(\mathbb{R})_0 \cap \mathbf{G}(\mathbb{Z}) = \mathbf{L}(\mathbb{R})_0 \cap \mathbf{G}(\mathbb{R})_0 \cap \mathbf{G}(\mathbb{Z}) = L \cap \Gamma \; , \tag{7}$$

and similarly $\mathbf{L}(\mathbb{R})_0 \cap \mathbf{L}_x(\mathbb{Z}) = L_x \cap \Gamma$ for every $x \in Lv_a \cap \Gamma v_0$. Hence the denominator of the modified Siegel weight $w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x)$ is equal to $\operatorname{vol}\big((L \cap \Gamma)\backslash L\big) = \operatorname{vol}(Y_L)$, using the measure μ_L on Y_L induced by the Haar measure ω_L on L. Its numerator $\operatorname{vol}\big((L_x \cap \Gamma)\backslash L_x\big)$ is defined by using the measure on $(L_x \cap \Gamma)\backslash L_x$ induced by the left Haar measure on $L_x = \ell(L \cap H)\ell^{-1}$ given by Weil's normalisation.

We now proceed to the proof of Theorem 6.

Step 1. The first step of the proof is the following group theoretic lemma, which relates the counting function of modified Siegel weights to the counting function of volumes of orbits of $L \cap H$. We denote with square brackets the (left or right) appropriate orbit of an element.

Lemma 8 *For every $a \in A$, there exists a bijection between finite subsets*

$$\Theta_a \; : \; (L \cap \Gamma)\backslash(Lv_a \cap \Gamma v_0) \; \longrightarrow \; (Y_L \cap Y_H a^{-1})/(L \cap H)$$

such that for every $x \in Lv_a \cap \Gamma v_0$, if $[y] = \Theta_a([x])$, then

$$\operatorname{vol}\big((L_x \cap \Gamma)\backslash L_x\big) = \operatorname{vol}\big(y(L \cap H)\big) \; . \tag{8}$$

In particular, for every $T \in [0, +\infty[^{\Delta - I}$, we have

$$\sum_{a \in A_T} \sum_{[x] \in (L \cap \Gamma)\backslash(Lv_a \cap \Gamma v_0)} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) = \sum_{a \in A_T} \sum_{[y] \in (Y_L \cap Y_H a^{-1})/(L \cap H)} \frac{\operatorname{vol}\big(y(L \cap H)\big)}{\operatorname{vol}(Y_L)} \; .$$

Proof. Fix $a \in A$. First note that the groups $L \cap \Gamma$ and $L \cap H$ do preserve the subsets $Lv_a \cap \Gamma v_0$ of $V(\mathbb{R})$ and $Y_L \cap Y_H a^{-1}$ of Y respectively for their left and right action, since A normalises H. The finiteness of the set $(L \cap \Gamma)\backslash(Lv_a \cap \Gamma v_0)$ follows from Borel-Harish-Chandra's finiteness theorem as in the introduction. Also recall that $H = MU$ is the stabiliser of v_0 in G.

Define

$$\Theta_a : [\ell v_a] \mapsto [\pi(\ell)] \; .$$

Let us prove that this map is well defined and bijective. Let $\ell, \ell' \in L$.

We have $\ell v_a \in \Gamma v_0$ if and only if there exists $\gamma \in \Gamma$ such that $\ell a v_0 = \gamma v_0$, that is, if and only if there exist $\gamma \in \Gamma$ and $h \in H$ such that $\ell = \gamma h a^{-1}$, that is, if and only if $\pi(\ell) \in Y_L \cap Y_H a^{-1}$. This proves that Θ_a has values in $(Y_L \cap Y_H a^{-1})/(L \cap H)$ and is surjective.

Let us prove that Θ_a does not depend on the choice of representatives and is injective. We have $[\ell' v_a] = [\ell v_a]$ if and only if there exists $\gamma \in L \cap \Gamma$ such that $\ell' a v_0 = \gamma \ell a v_0$, hence if and only if there exist $\gamma \in L \cap \Gamma$ and $h \in H$ such that $\ell' a = \gamma \ell a h$. Note that this equation implies that $a h a^{-1} \in L$ if and only if $\gamma \in L$. Since A normalises H, we hence have $[\ell' v_a] = [\ell v_a]$ if and only if $\ell' \in \Gamma \ell (L \cap H)$, that is if and only if $[\pi(\ell)] = [\pi(\ell')]$.

To prove the second assertion, let $\ell \in L$ be such that $x = \ell v_a \in \Gamma v_0$ and let $y = \pi(\ell)$, so that $\Theta_a([x]) = [y]$. The orbit of y under $L \cap H$ in Y_G is the image by the locally isometric map π of the Riemannian submanifold $\ell(L \cap H)$ of G. The left translation by ℓ^{-1} is an isometry (hence is volume preserving) from $\ell(L \cap H)$ to $(L \cap H)$. By Lemma 5, the map $g \mapsto \ell g \ell^{-1}$ from $L \cap H$ to $\ell(L \cap H)\ell^{-1}$, which is equal to L_x by Equation (6), is measure preserving. Therefore the map $\varphi : [z] \mapsto [z\ell]$ from $(L_x \cap \Gamma)\backslash L_x$ to $y(L \cap H)$ is a measure preserving bijection. This proves the volume equality of Equation (8).

The last claim follows from the other ones, since the numerator of the modified Siegel weight $w'_{\mathbf{L}, \rho|\mathbf{L}}(x)$ is $\mathrm{vol}\big((L_x \cap \Gamma)\backslash L_x\big)$. $\qquad\square$

Step 2. The second step of the proof is an equidistribution result, in the spirit of [KM1], saying that the piece Y_{P_T} of orbit of P equidistributes in Y_G as $\min T \to +\infty$.

For every smooth Riemannian manifold Z and $q \in \mathbb{N}$, we denote by $\mathscr{C}_c^q(Z)$ the normed vector space of C^q maps with compact support on Z, with norm $\|\cdot\|_q$.

Proposition 9 *There exist $q \in \mathbb{N}$ and $\kappa > 0$ such that for every $f \in \mathscr{C}_c^q(Y_G)$ and $T = (t_\alpha)_{\alpha \in \Delta - I} \in [0, +\infty[^{\Delta - I}$, we have, as $\min T$ tends to $+\infty$,*

$$\frac{1}{\mu_P(Y_{P_T})} \int_{Y_{P_T}} f \, d\mu_P = \int_{Y_G} f \, d\overline{\mu}_G + \mathrm{O}\left(e^{-\kappa \min T} \|f\|_q\right) .$$

To prove the proposition, we will use the disintegration formula already seen in the proof of Lemma 7

$$\int_{Y_{P_T}} f \, d\mu_P = \int_{A_T} \left(\int_{Y_H} f(ya^{-1}) \, d\mu_H(y) \right) \left(\prod_{\alpha \in \Delta - I} \alpha(a)^{m_\alpha} \right) d\omega_A(a) .$$

This formula indicates that the proposition would follow from (an averaging of) the equidistribution of the translates $Y_H a^{-1}$, which is established in Proposition 10 below. To state this proposition, we need to introduce additional notation.

The linear algebraic group $\mathbf{G}$ decomposes as an almost direct product

$$\mathbf{G} = \mathbf{Z}(\mathbf{G})\mathbf{G}_1 \cdots \mathbf{G}_s$$

where $\mathbf{Z}(\mathbf{G})$ is the centre of $\mathbf{G}$, and $\mathbf{G}_1, \ldots, \mathbf{G}_s$ are $\mathbb{Q}$-simple connected algebraic subgroups of $\mathbf{G}$. The maximal $\mathbb{Q}$-split torus $\mathbf{S}$ decomposes as an almost direct product

$$\mathbf{S} = \mathbf{S}_1 \cdots \mathbf{S}_s$$

where $\mathbf{S}_i$ is a maximal $\mathbb{Q}$-split torus in $\mathbf{G}_i$. We also get an almost direct product decomposition

$$G = Z(G)G_1 \cdots G_s, \tag{9}$$

where $Z(G)$ is the centre of G (which is equal to $\mathbf{Z}(\mathbf{G})_0$ since $\mathbf{G}$ is connected and G is Zariski-dense in $\mathbf{G}$) and $G_i = \mathbf{G}_i(\mathbb{R})_0$ for $1 \leq i \leq s$. Since $\mathbf{G}$ has no nontrivial $\mathbb{Q}$-character, and since $\mathbf{M}$ is the centraliser of $\mathbf{A}$, this gives corresponding almost direct product decompositions of the Lie groups $A = A_1 \dots A_s$ (this one being a direct product), $U = U_1 \dots U_s$, $M = Z(G)M_1 \dots M_s$, $H = Z(G)H_1 \dots H_s$. The set of simple roots Δ decomposes as a disjoint union

$$\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_s$$

where Δ_i is a set of simple roots of $\mathbf{G}_i$ relatively to $\mathbf{S}_i$, and the positive (closed) Weyl chamber A^+ in A associated to Δ decomposes as

$$A^+ = A_1^+ \cdots A_s^+,$$

where A_i^+ is the positive (closed) Weyl chamber in $A \cap G_i$ associated to Δ_i.

For $1 \leq i \leq s$ and $a \in A_i$, we define

$$E_i(a) = \exp\left(-\left(\max_{\alpha \in \Delta_i - I} \log \alpha(a)\right)\right) > 0 \tag{10}$$

if $\Delta_i - I \neq \emptyset$, and $E_i(a) = 0$, otherwise. For every $\kappa > 0$, we also define

$$E^\kappa(a) = \sum_{i=1}^s E_i(a_i)^\kappa$$

for every $a \in A^+$ with $a_1 \in A_1^+, \cdots, a_s \in A_s^+$ and $a = a_1 \cdots a_s$.

Proposition 10 *There exist $q \in \mathbb{N}$ and $\kappa > 0$ such that for all $f \in \mathscr{C}_c^q(Y_G)$ and $a \in A^+$,*

$$\int_{Y_H} f(ya^{-1}) \, d\overline{\mu}_H(y) = \int_{Y_G} f \, d\overline{\mu}_G + \mathrm{O}\left(E^\kappa(a) \, \|f\|_q\right).$$

Given a Lie subgroup D of G such that $\Gamma \cap D$ is a lattice in D, we denote by $\overline{\nu}_D$ the normalised right invariant measure on $(\Gamma \cap D) \backslash D$. Recall that $Y_D = \pi(D)$ is a closed submanifold of Y_G, and that μ_D is the invariant measure on Y_D induced by the Riemannian metric, with normalised measure $\overline{\mu}_D$.

We identify $(\Gamma \cap H) \backslash H$ with Y_H using the (well defined) map $h \mapsto \Gamma h$ (denoting again by $h \in H$ a representative of a coset $h \in (\Gamma \cap H) \backslash H$). Since the groups $Z(G), H_1, \cdots, H_s$ commute, we also have the map

$$(\Gamma \cap Z(G)) \backslash Z(G) \times (\Gamma \cap H_1) \backslash H_1 \cdots \times (\Gamma \cap H_s) \backslash H_s \to Y_H$$

well defined by $(h_0, h_1, \ldots, h_s) \mapsto \Gamma h_0 h_1 \cdots h_s$ (using conventions similar to the above one for coset representatives). Then the normalised invariant measures $\overline{\mu}_H, \overline{\mu}_{H_1}, \ldots, \overline{\mu}_{H_s}$ satisfy, for all $f \in \mathscr{C}_c(Y_G)$,

$$\int_{Y_H} f(y) \, d\overline{\mu}_H(y) = \int_{(\Gamma \cap H) \backslash H} f(\Gamma h) d\overline{\nu}_H(h)$$

$$= \int_{(\Gamma \cap Z(G)) \backslash Z(G) \times \cdots \times (\Gamma \cap H_s) \backslash H_s} f(\Gamma h_0 h_1 \cdots h_s) d\overline{\nu}_{Z(G)}(h_0) \cdots d\overline{\nu}_{H_s}(h_s). \tag{11}$$

We will prove Proposition 10 by using an inductive argument on the number of factors. We start by analysing the distribution of $Y_{U_i}a^{-1}$ in Lemma 11 and then the distribution of $Y_{H_i}a^{-1}$ in Lemma 12.

Let D be a product of almost direct factors of G in the decomposition (9). For every $f \in \mathscr{C}_c^0(Y_G)$, we define a map $\mathscr{P}_D f : Y_G \to \mathbb{C}$ by

$$(\mathscr{P}_D f)(\Gamma g) = \int_{(\Gamma \cap D) \backslash D} f(\Gamma dg) \, d\overline{\nu}_D(d)$$

which does not depend on the choice of the representative of Γg, by the right invariance of $\overline{\nu}_D$ under D. Note that $\mathscr{P}_D f$ is continuous and invariant under the right action of D.

Lemma 11 *There exist $q \in \mathbb{N}$ and $\kappa_1 > 0$ such that for every $i \in \{1, \dots, s\}$ with $A_i \neq \{1\}$, for every $f \in \mathscr{C}_c^q(Y_G)$ and $a \in A_i^+$,*

$$\int_{Y_{U_i}} f(ya^{-1}) \, d\overline{\mu}_{U_i}(y) = (\mathscr{P}_{G_i} f)(\Gamma e) + \mathrm{O}\left(E_i(a)^{\kappa_1} \|f|_{Y_{G_i}}\|_q\right).$$

Proof. For $1 \leq i \leq s$, we consider the unitary representation of the group G_i on the orthogonal complement of the space of G_i-invariant (hence constant on Y_{G_i}) functions in the Hilbert space $\mathbb{L}^2(Y_{G_i}, \overline{\mu}_{G_i})$, whose scalar product we denote by $\langle \cdot, \cdot \rangle_{Y_{G_i}}$ (using the normalised measure $\overline{\mu}_{G_i}$). We note that for every $f \in \mathscr{C}_c(Y_G)$, the function $f|_{Y_{G_i}} - (\mathscr{P}_{G_i} f)(\Gamma e)$ belongs to this space.

We say that a unitary representation of a connected real semisimple Lie group G' has the *strong spectral gap property* if the restriction to every noncompact simple factor of G' is isolated from the trivial representation for the Fell topology (see for instance [Cow], [BHV, Appendix], [KM2, Appendix] for equivalent definitions and examples, and compare for instance with [Nev, KS] for variations on the terminology). We claim that the above unitary representation of G_i has the strong spectral gap property. Indeed, if $\mathbf{G}_i$ is simply connected and Γ_i is a congruence subgroup in G_i, then the strong spectral gap property on $\Gamma_i \backslash G_i$ is a direct consequence of the property τ proved in [Clo], see Theorem 3.1 therein. By [KM2, Lemma 3.1], this also implies, when $\mathbf{G}_i$ is simply connected, the strong spectral property on $\Gamma_i \backslash G_i$ for subgroups Γ_i that are commensurable with congruence subgroups, and, in particular, for arithmetic subgroups of G_i. Now let $p_i : \widetilde{\mathbf{G}}_i \to \mathbf{G}_i$ be a simply connected cover of $\mathbf{G}_i$, and let $\widetilde{G}_i = \widetilde{\mathbf{G}}_i(\mathbb{R})$. Then $Y_{G_i} \simeq p_i^{-1}(\Gamma \cap G_i) \backslash \widetilde{G}_i$, and the strong spectral gap property for $\mathbb{L}^2(Y_{G_i}, \overline{\mu}_{G_i})$ follows from the above arguments.

Applying [KM1, Theorem 2.4.3], we deduce that there exist $q \in \mathbb{N}$ and $\kappa_1', C > 0$ such that for every $i \in \{1, \dots, s\}$ such that $A_i \neq \{e\}$, for every $\phi \in \mathscr{C}_c^q(Y_{G_i})$ and $a \in A_i^+$,

$$\left\langle (f|_{Y_{G_i}} - (\mathscr{P}_{G_i} f)(\Gamma e)) \circ a^{-1}, \phi \right\rangle_{Y_{G_i}} \leq C \, E_i(a)^{\kappa_1'} \|f|_{Y_{G_i}}\|_q \|\phi\|_q , \tag{12}$$

where $E_i(a)$ is defined in Equation (10).

Let P_i^- denote the parabolic subgroup in G_i opposite to U_i. The product map $U_i \times P_i^- \to G_i$ is a diffeomorphism between neighbourhoods of the identities. Since $Y_{U_i} = \pi(U_i)$ is compact, if $\epsilon > 0$ is small enough, there exists an open ϵ-neighbourhood Ω_ϵ of the identity in P_i^- such that the product map $Y_{U_i} \times \Omega_\epsilon \to Y_{G_i}$ is a diffeomorphism onto its image $Y_{U_i}\Omega_\epsilon$. We have (see also Lemma 14)

$$\forall \, y \in Y_{U_i}, \, \forall \, p \in \Omega_\epsilon, \qquad d\overline{\mu}_{G_i}(yp) = d\overline{\mu}_{U_i}(y) d\omega(p),$$

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for a suitably normalised smooth measure ω on Ω_ϵ. There exists $\sigma > 0$ (depending on q) such that for every $\epsilon > 0$ small enough, there exists a nonnegative function $\psi_\epsilon \in \mathscr{C}_c^q(\Omega_\epsilon)$ satisfying

$$\int_{\Omega_\epsilon} \psi_\epsilon \, d\omega = 1 \quad \text{and} \quad \|\psi_\epsilon\|_q = \mathrm{O}(\epsilon^{-\sigma}) \,.$$

Define a C^q function $\phi_\epsilon : Y_{G_i} \to [0, +\infty[$ supported on $Y_{U_i}\Omega_\epsilon$ by

$$\forall \, y \in Y_{U_i}, \; \forall \, p \in \Omega_\epsilon, \qquad \phi_\epsilon(yp) = \psi_\epsilon(p) \,.$$

Then

$$\int_{Y_{G_i}} \phi_\varepsilon \, d\overline{\mu}_{G_i} = 1 \quad \text{and} \quad \|\phi_\epsilon\|_q = \mathrm{O}(\epsilon^{-\sigma}) \,.$$

Since for all $a \in A_i^+$ and $p \in \Omega_\epsilon$,

$$d(apa^{-1}, e) = \mathrm{O}(\epsilon),$$

we obtain

$$
\begin{aligned}
\langle f|_{Y_{G_i}} \circ a^{-1}, \phi_\epsilon \rangle_{Y_{G_i}} &= \int_{Y_{U_i} \times \Omega_\epsilon} f(ypa^{-1}) \psi_\epsilon(p) \, d\overline{\mu}_{U_i}(y) d\omega(p) \\
&= \int_{Y_{U_i}} f(ya^{-1}) \, d\overline{\mu}_{U_i}(y) + \mathrm{O}\left(\epsilon \|f|_{Y_{G_i}}\|_1\right) \,.
\end{aligned}
$$

Since $\mathscr{P}_{G_i} f$ is G_i-invariant,

$$\langle \mathscr{P}_{G_i} f, \phi_\epsilon \rangle_{Y_{G_i}} = (\mathscr{P}_{G_i} f)(\Gamma e)\left(\int_{Y_{G_i}} \phi_\varepsilon \, d\overline{\mu}_{G_i} \right) = (\mathscr{P}_{G_i} f)(\Gamma e).$$

Combining these estimates with (12) (we may assume that $q \geq 1$), we conclude that

$$\int_{Y_{U_i}} f(ya^{-1}) \, d\overline{\mu}_{U_i}(y) = (\mathscr{P}_{G_i} f)(\Gamma e) + \mathrm{O}\left(\left(\epsilon + E_i(a)^{\kappa_1'}\epsilon^{-\sigma}\right) \|f|_{Y_{G_i}}\|_q\right).$$

Finally, taking $\epsilon = E_i(a)^{\kappa_1'/(1+\sigma)}$ which is small if a lies outside a compact subset of A_i^+, we deduce that

$$\int_{Y_{U_i}} f(ya^{-1}) \, d\overline{\mu}_{U_i}(y) = (\mathscr{P}_{G_i} f)(\Gamma e) + \mathrm{O}\left(E_i(a)^{\kappa_1'/(1+\sigma)} \|f|_{Y_{G_i}}\|_q\right),$$

as required. $\qquad\square$

Lemma 12 *There exist $q \in \mathbb{N}$ and $\kappa_2 > 0$ such that for every $f \in \mathscr{C}_c^q(Y_G)$ and $a \in A_i^+$, for every $i \in \{1, \ldots, s\}$, we have*

$$\int_{Y_{H_i}} f(ya^{-1}) \, d\overline{\mu}_{H_i}(y) = (\mathscr{P}_{G_i} f)(\Gamma e) + \mathrm{O}\left(E_i(a)^{\kappa_2} \|f|_{Y_{G_i}}\|_q\right).$$

Proof. We first observe that if $A_i = \{e\}$, then $H_i = G_i$, and the claim of the lemma is obvious. Now we assume that $A_i \neq \{e\}$ in which case Lemma 11 applies.

Let $N_i = (\Gamma \cap M_i)\backslash M_i$. The space $Y_{H_i} = \pi(U_i M_i)$ is a bundle over N_i with fibres isomorphic to Y_{U_i}, and the invariant measure $\overline{\mu}_{H_i}$ on Y_{H_i} decomposes with respect to this structure. Explicitly, for every $m \in M_i$, the integrals $\int_{Y_{U_i}} f(ym) \, d\overline{\mu}_{U_i}(y)$ for all $f \in \mathscr{C}_c(Y_G)$ define a U_i-invariant probability measure on $Y_{U_i} m$, which depends only on the coset $n = [m]$ of m in $N_i = (\Gamma \cap M_i)\backslash M_i$, and the H_i-invariant probability measure on Y_{H_i} is given by $\int_{N_i} \left(\int_{Y_{U_i}} f(ym) \, d\overline{\mu}_{U_i}(y) \right) d\overline{\nu}_{M_i}([m])$ for all $f \in \mathscr{C}_c(Y_G)$. Hence, denoting again by n any representative of a coset n in N_i, since A centralises M,

$$\int_{Y_{H_i}} f(ya^{-1}) \, d\overline{\mu}_{H_i}(y) = \int_{N_i} \int_{Y_{U_i}} f(yna^{-1}) \, d\overline{\mu}_{U_i}(y) \, d\overline{\nu}_{M_i}(n)$$
$$= \int_{N_i} \int_{Y_{U_i}} f(ya^{-1}n) \, d\overline{\mu}_{U_i}(y) \, d\overline{\nu}_{M_i}(n) \, .$$

For $m \in M_i$ and $f \in \mathscr{C}_c(Y_G)$, we consider the function $f_m : Y_G \to \mathbb{C}$ defined by $y \mapsto f(ym)$. We note that there exist $c_1, C' > 0$ such that for every $f \in \mathscr{C}_c^q(Y_G)$, we have

$$\|f_m|_{Y_{G_i}}\|_q \leq C' e^{c_1 d(e,m)} \|f|_{Y_{G_i}}\|_q \, .$$

Hence, by Lemma 11, for every $f \in \mathscr{C}_c^q(Y_G)$ and $m \in M_i$, since $\mathscr{P}_{G_i} f_m = \mathscr{P}_{G_i} f$ by invariance under G_i,

$$\int_{Y_{U_i}} f(ya^{-1}m) \, d\overline{\mu}_{U_i}(y) = (\mathscr{P}_{G_i} f)(\Gamma e) + \mathrm{O}\left(E_i(a)^{\kappa_1} e^{c_1 d(e,m)} \|f|_{Y_{G_i}}\|_q \right) \, . \tag{13}$$

We fix $n_0 \in N_i$ and for $R > 0$, we set

$$(N_i)_R = \{n \in N_i : d(n_0, n) \leq R\} \, ,$$

where $d(\cdot, \cdot)$ denotes the distance on N_i with respect to the induced Riemannian metric. We shall use the following estimate on the volumes of the "cusp": there exists $c_2 > 0$ such that for every $R > 0$,

$$\overline{\nu}_{M_i}(N_i - (N_i)_R) = \mathrm{O}(e^{-c_2 R}). \tag{14}$$

To prove this estimate, we may pass to an equivalent Riemannian metric and to a finite index subgroup of $\Gamma \cap M_i$. This way, we reduce the proof to the case when M_i is semisimple, $\Gamma \cap M_i$ is an nonuniform lattice in M_i, and the Riemannian metric on M_i is bi-invariant under a maximal compact subgroup in M_i. Then Equation (14) follows from [KM2, §5.1] (which notes that the irreducible assumption on $\Gamma \cap M_i$ is not necessary).

Equation (14) implies that

$$\int_{N_i - (N_i)_R} \int_{Y_{U_i}} f(ya^{-1}n) \, d\overline{\mu}_{U_i}(y) \, d\overline{\nu}_{M_i}(n) = \mathrm{O}\left(e^{-c_2 R} \|f|_{Y_{G_i}}\|_0 \right) \, . \tag{15}$$

Given $m \in M_i$ such that $(\Gamma \cap M_i)m \in (N_i)_R$, there exists $m' \in M_i$ such that $(\Gamma \cap M_i)m = (\Gamma \cap M_i)m'$ and $d(e, m') \leq R$. Therefore, it follows from Equation (13) that

$$\int_{(N_i)_R} \int_{Y_{U_i}} f(ya^{-1}n) \, d\overline{\mu}_{U_i}(y) \, d\overline{\nu}_{M_i}(n)$$
$$= \overline{\nu}_{M_i}((N_i)_R)(\mathscr{P}_{G_i} f)(\Gamma e) + \mathrm{O}\left(E_i(a)^{\kappa_1} e^{c_1 R} \|f|_{Y_{G_i}}\|_q \right)$$
$$= (\mathscr{P}_{G_i} f)(\Gamma e) + \mathrm{O}\left((e^{-c_2 R} + E_i(a)^{\kappa_1} e^{c_1 R}) \|f|_{Y_{G_i}}\|_q \right) \, . \tag{16}$$

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Finally, combining (15) and (16), we obtain that

$$\int_{N_i} \int_{Y_{U_i}} f(ya^{-1}n) \, d\overline{\mu}_{U_i}(y) \, d\overline{\nu}_{M_i}(n)$$
$$= (\mathscr{P}_{G_i} f)(\Gamma e) + \mathrm{O}\left((e^{-c_2 R} + E_i(a)^{\kappa_1} e^{c_1 R}) \, \|f|_{Y_{G_i}}\|_q\right).$$

Taking $R = \log E_i(a)^{-\frac{\kappa_1}{c_1+c_2}}$, we deduce the claim of Lemma 12 with $\kappa_2 = \frac{\kappa_1 c_2}{c_1+c_2}$. $\qquad\Box$

Proof of Proposition 10. For a subsemigroup D which decomposes as a product $D = D_p \cdots D_q$ and $p \le i \le q$, we write

$$D_{\le i} = D_p \cdots D_i \quad \text{and} \quad D_{>i} = D_{i+1} \cdots D_q.$$

We show inductively on $i \in \{0, \dots, s\}$ that for every $a = a_1 \dots a_i \in A_{\le i}^+$ (by convention $a = e$ if $i = 0$) and $g \in G_{>i}$, we have

$$\int_{Y_{H_{\le i}}} f(ya^{-1}g) \, d\overline{\mu}_{H_{\le i}}(y) = (\mathscr{P}_{G_{\le i}} f)(\Gamma g) + \sum_{j=1}^{i} \mathrm{O}\left(E_j(a_j)^{\kappa} \|f\|_q\right) \qquad (17)$$

with $\kappa = \kappa_2$ and q as in Lemma 12. Since $H_{\le 0} = G_{\le 0} = Z(G)$, this is obvious for $i = 0$. To get this estimate for $i = 1$, we apply Lemma 12 to the function $f_g(y) = (\mathscr{P}_{G_{\le 0}} f)(yg)$ with $g \in G_{>1}$. Since G_1 commutes with $G_{\le 0}$ and $G_{>1}$, we have

$$\|f_g|_{Y_{G_1}}\|_q \le \|f\|_q \quad \text{and} \quad (\mathscr{P}_{G_1} \mathscr{P}_{G_{\le 0}} f)(\Gamma g) = (\mathscr{P}_{G_{\le 1}} f)(\Gamma g).$$

This proves Equation (17) with $i = 1$.

Now suppose that Equation (17) is proved at rank i. As in Equation (11), for $f \in \mathscr{C}_c(Y_G)$,

$$\int_{Y_{H_{\le i+1}}} f(y) \, d\overline{\mu}_{H_{\le i+1}}(y) = \int_{(\Gamma \cap H_i)\backslash H_i} \int_{Y_{H_{\le i}}} f(yh) \, d\overline{\mu}_{H_{\le i}}(y) \, d\overline{\nu}_{H_i}(h).$$

Hence, for every $a' = a_1 \dots a_i \in A_{\le i}^+$, $a_{i+1} \in A_{i+1}^+$ and $g \in G_{>i+1}$, with $a = a' a_{i+1}$, by the right invariance of $\overline{\nu}_{H_{i+1}}$ under H_{i+1} and by Equation (17), we have

$$\int_{Y_{H_{\le i+1}}} f(ya^{-1}g) \, d\overline{\mu}_{H_{\le i+1}}(y)$$
$$= \int_{(\Gamma \cap H_{i+1})\backslash H_{i+1}} \int_{Y_{H_{\le i}}} f(y(a')^{-1} h a_{i+1}^{-1} g) \, d\overline{\mu}_{H_{\le i}}(y) \, d\overline{\nu}_{H_{i+1}}(h)$$
$$= \int_{(\Gamma \cap H_{i+1})\backslash H_{i+1}} (\mathscr{P}_{G_{\le i}} f)(\Gamma h a_{i+1}^{-1} g) \, d\overline{\nu}_{H_{i+1}}(h) + \sum_{j=1}^{i} \mathrm{O}\left(E_j(a_j)^{\kappa} \|f\|_q\right).$$

Applying Lemma 12 to the functions $\overline{f}_g : y \mapsto (\mathscr{P}_{G_{\le i}} f)(yg)$ on Y_G, we obtain

$$\int_{(\Gamma \cap H_{i+1})\backslash H_{i+1}} (\mathscr{P}_{G_{\le i}} f)(\Gamma h a_{i+1}^{-1} g) \, d\overline{\nu}_{H_{i+1}}(h)$$
$$= (\mathscr{P}_{G_{i+1}} \overline{f}_g)(\Gamma e) + \mathrm{O}\left(E_{i+1}(a_{i+1})^{\kappa} \|\overline{f}_g|_{Y_{G_{i+1}}}\|_q\right)$$
$$= (\mathscr{P}_{G_{\le i+1}} f)(\Gamma g) + \mathrm{O}\left(E_{i+1}(a_{i+1})^{\kappa} \|f\|_q\right).$$

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This completes the proof of Equation (17). Since

$$(\mathscr{P}_{G_{\leq s}} f)(\Gamma e) = \int_{Y_G} f \, d\overline{\mu}_G,$$

the proposition follows. □

Proof of Proposition 9. Since

$$\int_{Y_{P_T}} f \, d\mu_P = \int_{A_T} \Big(\int_{Y_H} f(ya^{-1}) \, d\mu_H(y) \Big) \Big(\prod_{\alpha \in \Delta - I} \alpha(a)^{m_\alpha} \Big) \, d\omega_A(a) \, ,$$

it follows from Proposition 10 and from Equation (5) that

$$\int_{Y_{P_T}} f \, d\mu_P = \mu_P(Y_{P_T}) \int_{Y_G} f \, d\overline{\mu}_G + \mathrm{O}\Big(\|f\|_q \int_{A_T} E^\kappa(a) \Big(\prod_{\alpha \in \Delta - I} \alpha(a)^{m_\alpha} \Big) \, d\omega_A(a) \Big).$$

For every $i \in \{1, \ldots, s\}$ such that $\Delta_i - I \neq \emptyset$, let $\beta \in \Delta_i - I$. For every $b_i \in A_i$, we have

$$E_i(b_i) \leq e^{-\log \beta(b_i)}.$$

Hence, by Lemma 7, we have, assuming that $\kappa < \min_{\alpha \in \Delta - I} m_\alpha$ (which is possible),

$$\int_{A_T} E_i(a_i)^\kappa \Big(\prod_{\alpha \in \Delta - I} \alpha(a)^{m_\alpha} \Big) \, d\omega_A(a) \leq c_A \Big(\prod_{\alpha \in \Delta - I - \{\beta\}} \int_0^{t_\alpha} e^{m_\alpha s} \, ds \Big) \int_0^{t_\beta} e^{(m_\beta - \kappa)s} \, ds$$
$$= \mathrm{O}\big(\mu_P(Y_{P_T}) \, e^{-\kappa t_\beta} \big) \, .$$

Therefore, since $E^\kappa(a) = \sum_{1 \leq i \leq s \, : \, \Delta_i - I \neq \emptyset} E_i(a_i)^\kappa$, we have

$$\frac{1}{\mu_P(Y_{P_T})} \int_{Y_{P_T}} f \, d\mu_P = \int_{Y_G} f \, d\overline{\mu}_G + \mathrm{O}(e^{-\kappa \min T} \|f\|_q),$$

as required. □

Step 3. In this last step of the proof of Theorem 6, we will diffuse the orbits of $L \cap H$ we want to count using bump functions, and apply the equidistribution result given by Proposition 9 in Step 2 to infer our main theorem.

Before starting this program, we rewrite the sum whose asymptotic we want to study in a more concise way. Let $T, T' \in [0, +\infty[^{\Delta - I}$. By transversality (see for instance [Hir, p. 22, Theo. 3.3]), the intersection

$$Z_{[T,T']} = Y_L \cap Y_{P_{[T,T']}}$$

is a compact Riemannian submanifold of Y_G, invariant under the right action of $L \cap H$, and for every $x \in Z_{[T,T']}$, we have $T_x Z_{[T,T']} = (T_x Y_L) \cap (T_x Y_{P_{[T,T']}})$. Since $\mathfrak{l} \cap \mathfrak{p} = \mathfrak{l} \cap \mathfrak{h}$ by Equation (3), the Lie group $L \cap H$ has open orbits in $Z_{[T,T']}$. Hence the compact subset $Z_{[T,T']}$ is a finite union of orbits of $L \cap H$ (see the picture below when A is 1-dimensional).
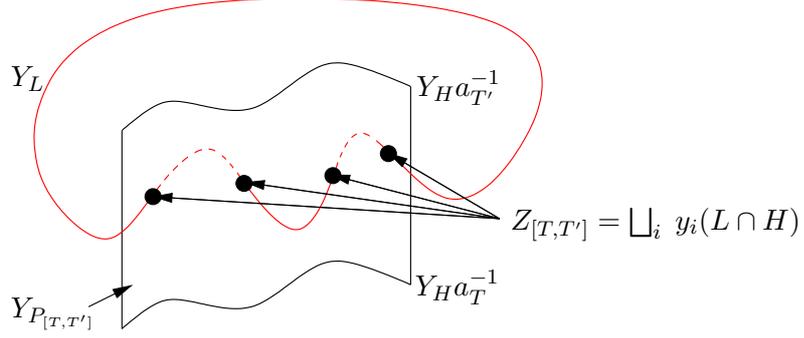
We will denote by $\mu_{Z_{[T,T']}}$ the Riemannian measure on $Z_{[T,T']}$. Using Riemannian volumes, we hence have

$$\mu_{Z_{[T,T']}}(Z_{[T,T']}) = \sum_{[y]\in (Y_L \cap Y_{P_{[T,T']}})/(L\cap H)} \mathrm{vol}\left(y(L\cap H)\right)$$

$$= \sum_{a\in A_{[T,T']}} \sum_{[y]\in (Y_L \cap Y_H a^{-1})/(L\cap H)} \mathrm{vol}\left(y(L\cap H)\right).$$

By Lemma 8 in Step 1, the quantity $\mu_{Z_{[0,T]}}(Z_{[0,T]})$, when divided by $\mathrm{vol}(Y_L)$, is the sum whose asymptotic we want to study.

We first start by studying the supports of the bump functions we will define: they will be appropriate neighbourhoods of Y_L and $Z_{[T,T']}$. Fix $\epsilon > 0$, which will be appropriately choosen small enough later on. Consider the open ball $B(0,\epsilon)$ of center 0 and radius ϵ in the orthogonal complement $\mathfrak{q}\oplus\mathfrak{a}$ of $\mathfrak{l}\cap\mathfrak{p}$ in $\mathfrak{p}$, and let $\mathscr{O}_\epsilon = \exp B(0,\epsilon)$, which is contained in P.

Since L is compact, if ϵ is small enough, the right action of G on Y_G induces a map $Y_L \times \mathscr{O}_\epsilon \to Y_G$, with $(y,g)\mapsto yg$, which is a smooth diffeomorphism onto an open neighbourhood $Y_L\mathscr{O}_\epsilon$ of the submanifold Y_L in Y_G. Similarly, if ϵ is small enough, then for every $T,T' \in [0,+\infty[^{\Delta-I}$, the map $Z_{[T,T']}\times \mathscr{O}_\epsilon \to Y_P$ defined by $(y,g)\mapsto yg$ is a smooth diffeomorphism onto an open neighbourhood $Z_{[T,T']}\mathscr{O}_\epsilon$ of the submanifold $Z_{[T,T']}$ in Y_P. If $\eta \in \mathbb{R}$ and $T'' = (t''_\alpha)_{\alpha\in\Delta-I} \in [0,+\infty[^{\Delta-I}$, we denote $T''+\eta = (t''_\alpha + \eta)_{\alpha\in\Delta-I}$.

Lemma 13 *There exists $c > 0$ such that if $\epsilon > 0$ is small enough, for every $T, T' \in [0,+\infty[^{\Delta-I}$, then*

$$Z_{[T+c\epsilon,T'-c\epsilon]}\mathscr{O}_\epsilon \subset Y_L\mathscr{O}_\epsilon \cap Y_{P_{[T,T']}} \subset Z_{[T-c\epsilon,T'+c\epsilon]}\mathscr{O}_\epsilon \,.$$

Proof. We first claim that there exists $c > 0$ such that

$$P_{[T+c\epsilon,T'-c\epsilon]} \subset P_{[T,T']}\mathscr{O}_\epsilon \subset P_{[T-c\epsilon,T'+c\epsilon]} \,.$$

Since the product map $(h,a) \mapsto ha$ is a diffeomorphism from $H \times A$ to P, since $\mathscr{O}_\epsilon$ is contained in P, and since the distances are Riemannian ones, there exists $c_1 > 0$ such that if $\epsilon > 0$ is small enough, then for every $g \in \mathscr{O}_\epsilon$, there exist $h \in H$ and $a \in A$ with $g = ha$ and $d(a,e) \leq c_1\epsilon$. Since the Riemannian distance on A is equivalent to the image by $\exp$ of the distance on $\mathfrak{a}$ defined by the norm $\|x\| = \max_{\alpha\in\Delta-I}|\log(\alpha(\exp x))|$, there exists $c_2 > 0$ such that $|\log\alpha(a)| \leq c_2 d(a,e)$ for every $a \in A$.

Let $g \in \mathscr{O}_\epsilon$, $h \in H$ and $a \in A$ be such that $g = ha$ and $d(a, e) \leq c_1 \epsilon$. Since A normalises H, we have $H A_{[T,T']}^{-1} g = H A_{[T,T']}^{-1} ha = H A_{[T,T']}^{-1} a$. Hence $H A_{[T,T']}^{-1} g$ is contained in $H A_{[T-c_1 c_2 \epsilon, T'+c_1 c_2 \epsilon]}^{-1}$ and contains $H A_{[T+c_1 c_2 \epsilon, T'-c_1 c_2 \epsilon]}^{-1}$. This proves the first claim.

Now, let $y \in Y_L$, $g \in \mathscr{O}_\epsilon$ and $p \in P_{[T,T']}$ be such that $yg = \pi(p)$. Then $y = \pi(pg^{-1})$. Since $\mathscr{O}_\epsilon$ is invariant by taking inverses, pg^{-1} belongs to $P_{[T,T']} \mathscr{O}_\epsilon$, hence by the first claim, $yg \in Z_{[T-c\epsilon, T'+c\epsilon]} \mathscr{O}_\epsilon$. The left inclusion is proven similarly. $\qquad\square$

We now study the properties of the Riemannian measures on the neighbourhoods $Y_L \mathscr{O}_\epsilon$ and $Z_{[T,T']} \mathscr{O}_\epsilon$.

Lemma 14 *For every $\epsilon > 0$ small enough, there exist smooth measures ν and $\widetilde{\nu}$ on $\mathscr{O}_\epsilon$ such that the product maps $Y_L \times \mathscr{O}_\epsilon \to Y_G$ and $Z_{[T,T']} \times \mathscr{O}_\epsilon \to Y_P$ send the product measures $\mu_L \otimes \nu$ and $\mu_{Z_{[T,T']}} \otimes \widetilde{\nu}$ to the restricted measures $\mu_{G|Y_L \mathscr{O}_\epsilon}$ and $\mu_{P|Z_{[T,T']} \mathscr{O}_\epsilon}$, respectively. Furthermore, $\frac{d\widetilde{\nu}}{d\nu}(e) = 1$.*

Proof. Since the measure $\mu_{G|Y_L \mathscr{O}_\epsilon}$ (respectively $\mu_{P|Z_{[T,T']} \mathscr{O}_\epsilon}$) is Riemannian, it disintegrates with respect to the trivialisable fibration $Y_L \mathscr{O}_\epsilon \to Y_L$ (respectively $Z_{[T,T']} \mathscr{O}_\epsilon \to Z_{[T,T']}$ with measure on the basis μ_L (respectively $\mu_{Z_{[T,T']}}$), and conditional measures ν_y (respectively $\widetilde{\nu}_y$ on the fibers $y \mathscr{O}_\epsilon$ for all $y \in Y_L$ (respectively $y \in Z_{[T,T']}$). By left invariance of the measures ω_L and $\omega_{L \cap H}$, there exist smooth measures ν (respectively $\widetilde{\nu}$) on $\mathscr{O}_\epsilon$ such that the maps $\mathscr{O}_\epsilon \to y \mathscr{O}_\epsilon$ defined by $g \mapsto yg$ send ν (respectively $\widetilde{\nu}$) to ν_y (respectively $\widetilde{\nu}_y$) for all $y \in Y_L$ (respectively $y \in Z_{[T,T']}$). This proves the first claim.

Since $\mathfrak{q} + \mathfrak{a}$ is orthogonal to $\mathfrak{l}$ (respectively $\mathfrak{l} \cap \mathfrak{h}$) by Equation (4), the manifold $y \mathscr{O}_\epsilon$ is orthogonal to Y_L (respectively $Z_{[T,T']}$) at every $y \in Y_L$ (respectively $y \in Z_{[T,T']}$). Hence any orthonormal frame F of $T_y(y \mathscr{O}_\epsilon)$ at a given $y \in Z_{[T,T']}$ may be completed to an orthonormal frame whose last vectors form a basis of $T_y Y_L$, whose first vectors form a basis of $T_y Z_{[T,T']}$. By desintegration, the orthogonal frame F has the same infinitesimal volume for ν and $\widetilde{\nu}$. The last assertion follows. $\qquad\square$

Let us now define our bump functions. By the standard construction of bump functions on manifolds, for every $q \in \mathbb{N}$, there exists $\kappa' > 0$ such that for every $\epsilon > 0$ small enough, there exists a C^q map ψ_ϵ from $\mathscr{O}_\epsilon$ to $[0, +\infty[$, with compact support, such that $\int \psi_\epsilon \, d\nu = 1$ and $\|\psi_\epsilon\|_q = O(\epsilon^{-\kappa'})$. Since $\frac{d\widetilde{\nu}}{d\nu} = 1 + O(\epsilon)$ on $\mathscr{O}_\epsilon$ by Lemma 14, we have

$$\int_{\mathscr{O}_\epsilon} \psi_\epsilon \, d\widetilde{\nu} = 1 + O(\epsilon) \ .$$

For every $\epsilon > 0$ small enough, define $f_\epsilon : Y_G \to [0, +\infty[$ by $f_\epsilon(y) = 0$ if $y \notin Y_L \mathscr{O}_\epsilon$ and $f_\epsilon(yg) = \psi_\epsilon(g)$ for every $y \in Y_L$ and $g \in \mathscr{O}_\epsilon$. Note that f_ϵ is C^q with compact support, since Y_L is compact. We have

$$\int_{Y_G} f_\epsilon \, d\overline{\mu} = \frac{\int_{Y_L \mathscr{O}_\epsilon} f_\epsilon \, d\mu_G}{\mathrm{vol}(Y_G)} = \frac{\int_{g \in \mathscr{O}_\epsilon} \int_{y \in Y_L} \psi_\epsilon(g) \, d\mu_L(y) d\nu(g)}{\mathrm{vol}(Y_G)} = \frac{\mathrm{vol}(Y_L)}{\mathrm{vol}(Y_G)} \ ,$$

and $\|f_\epsilon\|_q = O(\epsilon^{-\kappa'})$.

Since the support of f_ϵ is contained in $Y_L \mathscr{O}_\epsilon$, by Lemma 14, and by the right inclusion in Lemma 13, we have, for every $T \in [0, +\infty[^{\Delta-I}$,

$$
\begin{aligned}
\int_{Y_{P_T}} f_\epsilon \, d\mu_P &\leq \int_{Z_{[-c\epsilon, T+c\epsilon]}\mathscr{O}_\epsilon} f_\epsilon \, d\mu_P \\
&= \int_{g \in \mathscr{O}_\epsilon} \int_{y \in Z_{[-c\epsilon, T+c\epsilon]}} \psi_\epsilon(g) \, d\mu_{Z_{[-c\epsilon, T+c\epsilon]}}(y) d\widetilde{\nu}(g) \\
&= \mathrm{vol}(Z_{[-c\epsilon, T+c\epsilon]})\big(1 + \mathrm{O}(\epsilon)\big) \,. \tag{18}
\end{aligned}
$$

Similarly, since $f_\epsilon \geq 0$ and by the left inclusion in Lemma 13, we have, for every $T \in [0, +\infty[^{\Delta-I}$,

$$
\int_{Y_{P_T}} f_\epsilon \, d\mu_P \geq \int_{Z_{[c\epsilon, T-c\epsilon]}\mathscr{O}_\epsilon} f_\epsilon \, d\mu_P = \mathrm{vol}(Z_{[c\epsilon, T-c\epsilon]})\big(1 + \mathrm{O}(\epsilon)\big) \,. \tag{19}
$$

Finally, we apply Step 2 to our bump functions. By Proposition 9, we have the equality $\frac{1}{\mu_P(Y_{P_T})} \int_{Y_{P_T}} f_\epsilon \, d\mu_P = \int_{Y_G} f_\epsilon \, d\overline{\mu}_G + \mathrm{O}\left(e^{-\kappa \min T} \|f_\epsilon\|_q\right)$. Hence, by the properties of f_ϵ,

$$
\int_{Y_{P_T}} f_\epsilon \, d\mu_P = \frac{\mathrm{vol}(Y_L)\mu_P(Y_{P_T})}{\mathrm{vol}(Y_G)}\big(1 + \mathrm{O}(\epsilon^{-\kappa'} e^{-\kappa \min T})\big) \,. \tag{20}
$$

Let $\delta = \frac{\kappa}{\kappa'+1} > 0$ and $\epsilon = e^{-\delta \min T}$ (which tends to 0 as $\min T$ tends to $+\infty$). Then $\epsilon^{-\kappa'} e^{-\kappa \min T} = e^{(\kappa'\delta - \kappa)\min T} = e^{-\delta \min T}$. By the equations (19) and (20), and by Lemma 7, we have, as $\min T$ tends to $+\infty$,

$$
\begin{aligned}
\mathrm{vol}(Z_{[c\epsilon, T-c\epsilon]}) &\leq \left(\int_{Y_{P_T}} f_\epsilon \, d\mu_P\right)\big(1 + \mathrm{O}(e^{-\delta \min T})\big) \\
&= \frac{\mathrm{vol}(Y_L)\mu_P(Y_{P_T})}{\mathrm{vol}(Y_G)}\big(1 + \mathrm{O}(e^{-\delta \min T})\big) \\
&= \frac{\mathrm{Vol}(\Lambda^\vee \backslash A)\,\mathrm{vol}(Y_L)\,\mathrm{vol}(Y_H)}{\mathrm{vol}(Y_G)}\Big(\prod_{\alpha \in \Delta-I} \frac{e^{m_\alpha t_\alpha}}{m_\alpha}\Big)\big(1 + \mathrm{O}(e^{-\delta \min T})\big) \,.
\end{aligned}
$$

Since $e^x = 1 + \mathrm{O}(x)$ as x tends to 0, we have $e^{c\, e^{-\delta \min T} \sum_{\alpha \in \Delta-I} m_\alpha} = 1 + \mathrm{O}(e^{-\delta \min T})$ as $\min T$ tends to $+\infty$. Since $Z_{[0, c\epsilon]}$ is bounded, we hence have, as $\min T$ tends to $+\infty$,

$$
\mathrm{vol}(Z_{[0, T]}) \leq \frac{\mathrm{Vol}(\Lambda^\vee \backslash A)\,\mathrm{vol}(Y_L)\,\mathrm{vol}(Y_H)}{\mathrm{vol}(Y_G)}\Big(\prod_{\alpha \in \Delta-I} \frac{e^{m_\alpha t_\alpha}}{m_\alpha}\Big)\big(1 + \mathrm{O}(e^{-\delta \min T})\big) \,.
$$

The converse inequality is proven similarly, using Equation (18) instead of Equation (19)

Since $\sum_{a \in A_T} \sum_{[x] \in (L \cap \Gamma) \backslash (Lv_a \cap \Gamma v_0)} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) = \frac{\mathrm{vol}(Z_{[0, T]})}{\mathrm{vol}(Y_L)}$ as said in the beginning of Step 3, this ends the proof of Theorem 6. $\qquad \square$

Remark 15 Let $\mathbf{G}, \mathbf{P}, \mathbf{A}, \mathbf{M}, \mathbf{U}, \mathbf{L}, \mathbf{V}, \rho, v_0$ be as in the statement of Theorem 6, and assume furthermore that $\mathbf{G}$ is simply connected. Then we have the following counting results using the standard Siegel weights.

There exists $\delta > 0$ such that, as $T = (t_\alpha)_{\alpha \in \Delta - I} \in [0, +\infty[^{\Delta - I}$ and $\min_{\alpha \in \Delta - I} t_\alpha$ tends to $+\infty$,

$$\sum_{a \in A_T} \sum_{[x] \in \mathbf{L}(\mathbb{Z}) \backslash (\rho(\mathbf{L}(\mathbb{R})a)v_0 \cap \rho(\Gamma)v_0)} w_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) =$$

$$\frac{\operatorname{vol}\left(\mathbf{MU}(\mathbb{Z}) \backslash \mathbf{MU}(\mathbb{Z})\right) \operatorname{vol}(\Lambda^\vee \backslash \mathbf{A}(\mathbb{R})_0)}{\operatorname{vol}(\mathbf{G}(\mathbb{Z}) \backslash \mathbf{G}(\mathbb{R}))} \Big(\prod_{\alpha \in \Delta - I} \frac{e^{m_\alpha t_\alpha}}{m_\alpha} \Big) \big(1 + \mathrm{O}(e^{-\delta \min_{\alpha \in \Delta - I} t_\alpha})\big) .$$

The proof is the same as the one of Theorem 6, with the following modifications. Since $\mathbf{G}$ is simply connected, $\mathbf{G}(\mathbb{R})$ is connected (see for instance [PR, §7.2]). Hence with the previous notation, we have $G = \mathbf{G}(\mathbb{R})$ and $\Gamma = \Gamma(\mathbb{Z})$ (and the connectedness of G was useful). Now take $L = \mathbf{L}(\mathbb{R})$ instead of $L = \mathbf{L}(\mathbb{R})_0$ (which is still contained in G, but would not have been if G was only taken to be $\mathbf{G}(\mathbb{R})_0$ while $\mathbf{G}(\mathbb{R})$ is not connected). Though L and Y_L may be no longer connected, the proof stays valid.

To end this section, we give two slightly different versions of Theorem 6 when $\mathbf{P}$ is maximal.

Theorem 16 *Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{Q}$, without nontrivial $\mathbb{Q}$-characters. Let $\mathbf{P}$ be a maximal (proper) parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$, and let $\mathbf{P} = \mathbf{AMU}$ be a relative Langlands decomposition of $\mathbf{P}$, such that $\mathbf{A}(\mathbb{R})_0$ is a one-parameter subgroup $(a_s)_{s \in \mathbb{R}}$, with $\lambda = \log \det (\operatorname{Ad} a_1)_{|\mathfrak{U}} > 0$, where $\mathfrak{U}$ is the Lie algebra of $\mathbf{U}(\mathbb{R})$. Let $\rho : \mathbf{G} \to \operatorname{GL}(\mathbf{V})$ be a rational representation of $\mathbf{G}$ defined over $\mathbb{Q}$ such that there exists $v_0 \in \mathbf{V}(\mathbb{Q})$ whose stabiliser in $\mathbf{G}$ is $\mathbf{MU}$. Let $\mathbf{L}$ be a reductive algebraic subgroup of $\mathbf{G}$ defined and anisotropic over $\mathbb{Q}$. Assume that $\mathbf{LP}$ is Zariski-open in $\mathbf{G}$ and that for every $s \in \mathbb{R}$, the orbit $\mathbf{X}_s = \rho(\mathbf{L}a_s)v_0$ is Zariski-closed in $\mathbf{V}$.*

(1) Endow $\mathbf{G}(\mathbb{R})$ with a left-invariant Riemannian metric, for which the Lie algebras of $\mathbf{MU}(\mathbb{R})$ and $\mathbf{A}(\mathbb{R})$ are orthogonal, and the orthogonal of the Lie algebra of $\mathbf{P}(\mathbb{R})$ is contained in the Lie algebra of $\mathbf{L}(\mathbb{R})$. Let $G = \mathbf{G}(\mathbb{R})_0$ and $\Gamma = \mathbf{G}(\mathbb{Z}) \cap G$. There exists $\delta > 0$ such that, as $t \geq 0$ tends to $+\infty$,

$$\sum_{0 \leq s \leq t} \sum_{[x] \in (\mathbf{L}(\mathbb{R})_0 \cap \Gamma) \backslash (\rho(\mathbf{L}(\mathbb{R})_0 a_s)v_0 \cap \rho(\Gamma)v_0)} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x)$$
$$= \frac{\operatorname{vol}\left((\mathbf{MU} \cap \Gamma) \backslash (\mathbf{MU} \cap G)\right) \operatorname{vol}(a_1^{\mathbb{Z}} \backslash \mathbf{A}(\mathbb{R})_0)}{\lambda \operatorname{vol}(\Gamma \backslash G)} e^{\lambda t} + \mathrm{O}(e^{(\lambda - \delta)t}) .$$

(2) Let Λ be a $\mathbb{Z}$-lattice in $\mathbf{V}(\mathbb{Q})$ invariant under $\mathbf{G}(\mathbb{Z})$, and let Λ^{prim} be the subset of indivisible elements of Λ. Assume ρ to be irreducible over $\mathbb{C}$. Then there exist $c, \delta > 0$ such that, as $t \geq 0$ tends to $+\infty$,

$$\sum_{0 \leq s \leq t} \sum_{[x] \in (\mathbf{L}(\mathbb{Z}) \cap \mathbf{L}(\mathbb{R})_0) \backslash (\mathbf{X}_s \cap \Lambda^{\mathrm{prim}})} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) = c \, e^{\lambda t} + \mathrm{O}(e^{(\lambda - \delta)t}) .$$

Proof. (1) In this case, $\Delta - I$ consists of one simple root α_0. Changing the parametrisation of the one-parameter subgroup $(a_s)_{s \in \mathbb{R}}$ appearing in Theorem 16 by multiplying s by a positive constant does not change the asymptotic formula in the statement of Theorem 16 (1). Hence we may assume that $a_1 = (\alpha_0)^\vee$, hence that the group $a_1^{\mathbb{Z}}$ generated by a_1 is equal to the lattice $\Lambda^\vee$. The constant λ defined in Theorem 16 is then equal to m_{α_0}. The first part of Theorem 16 hence follows from Theorem 6.

(2) We start by proving two lemmas.

Lemma 17 *If ρ is irreducible, then the stabiliser of $\mathbb{C}v_0$ in $\mathbf{G}$ is $\mathbf{P}$ and there exists $\chi \in \mathbb{R}$ such that $a_s v_0 = e^{\chi s} v_0$ for every $s \in \mathbb{R}$.*

Proof. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ containing $\mathbf{S}$, and let $\Delta_\mathbf{T}$ be a set of primitive roots of $\mathbf{G}$ relative to $\mathbf{T}$, whose set of nonzero restrictions to $\mathbf{S}$ is Δ (see for instance [Bor3, §21.8]. Then the unipotent subgroup $\mathbf{U}_\mathbf{T}^+$, whose Lie algebra is the sum of the positive root spaces of $\mathbf{G}$ relative to $\mathbf{T}$, is contained in $\mathbf{MU}$. By the properties of the highest weights, if ρ is irreducible, the space $\{v \in \mathbf{V} \ : \ \mathbf{U}_\mathbf{T}^+ v = v\}$ is one-dimensional, hence equal to $\mathbb{C}v_0$. Since $\mathbf{A}$ normalises $\mathbf{MU}$, hence $\mathbf{U}_\mathbf{T}^+$, it preserves $\mathbb{C}v_0$, and the result follows, by the connectedness of A. $\qquad\square$

Lemma 18 *There exist $v_1, \ldots, v_k$ in Λ^{prim} such that $\Lambda^{\mathrm{prim}} \cap \mathbf{G}v_0 = \bigsqcup_{i=1}^{k} \Gamma v_i$.*

Proof. By [Bor3, Prop. 20.5], the natural map $\mathbf{G}(\mathbb{Q}) \to (\mathbf{G}/\mathbf{P})(\mathbb{Q})$ is onto. Since $\mathbf{G}v_0 \simeq \mathbf{G}/\mathbf{MU}$, this implies that every $x \in (\mathbf{G}v_0)(\mathbb{Q})$ may be written as $x = gpv_0$ for some $g \in \mathbf{G}(\mathbb{Q})$ and $p \in \mathbf{P}$. Hence by Lemma 17,

$$(\mathbf{G}v_0)(\mathbb{Q}) \subset \mathbb{C}^\times \mathbf{G}(\mathbb{Q})v_0.$$

By [Bor2, Prop. 15.6], there exists a finite subset F of $\mathbf{G}(\mathbb{Q})$ such that $\mathbf{G}(\mathbb{Q}) = \Gamma F \mathbf{P}(\mathbb{Q})$. Hence,

$$(\mathbf{G}v_0)(\mathbb{Q}) \subset \mathbb{C}^\times \Gamma F v_0.$$

In particular, we conclude that there exist $v_1, \ldots v_k$ in Λ^{prim} such that

$$\Lambda^{prim} \cap \mathbf{G}v_0 \subset \bigsqcup_{i=1}^{k} \mathbb{C}^\times \Gamma v_i.$$

Since for every $v \in \Lambda^{prim}$,

$$\mathbb{C}^\times v \cap \Lambda^{prim} = \{\pm v\},$$

this implies the lemma. $\qquad\square$

Now, since the identity component L of $\mathbf{L}(\mathbb{R})$ has finite index in $\mathbf{L}(\mathbb{R})$, there exist $\ell_1 \ldots, \ell_{k'}$ in $\mathbf{L}(\mathbb{R})$ such that $\mathbf{L}(\mathbb{R}) = \bigsqcup_{j=1}^{k'} L\,\ell_j$. Hence, since v_0 belongs to $\mathbf{V}(\mathbb{R})$ and $\mathbf{X}_s \subset \mathbf{G}v_0$, by Lemma 17 and Lemma 18, we have

$$\mathbf{X}_s \cap \Lambda^{\mathrm{prim}} = (\mathbf{L}(\mathbb{R})e^{\chi s}v_0) \cap (\Lambda^{\mathrm{prim}} \cap \mathbf{G}v_0) = \bigsqcup_{1 \le i \le k\,,\,1 \le j \le k'} e^{\chi s} L\,\ell_j v_0 \cap \Gamma v_i\,. \qquad (21)$$

If $L\ell_j v_0 \cap \Gamma v_i$ is nonempty, fix $v_{i,j} \in L\ell_j v_0 \cap \Gamma v_i$. In particular, there exist $\gamma \in \Gamma$ and $\ell \in L$ such that $v_{i,j} = \ell\,\ell_j v_0 = \gamma v_i$. Since $v_i \in \mathbf{V}(\mathbb{Q})$, we have $v_{i,j} \in \mathbf{V}(\mathbb{Q})$. Hence the stabiliser $\mathbf{P}_{i,j}$ of $v_{i,j}$ in $\mathbf{G}$ is an algebraic subgroup defined over $\mathbb{Q}$. Since $v_{i,j}$ is in the $\mathbf{G}$-orbit of v_0, the stabilisers of v_0 and of $v_{i,j}$ are conjugate, hence $\mathbf{P}_{i,j}$ is a parabolic subgroup of $\mathbf{G}$. Since two parabolic subgroups of $\mathbf{G}$, which are defined over $\mathbb{Q}$ and conjugate in $\mathbf{G}$, are conjugated by an element of $\mathbf{G}(\mathbb{Q})$ (see for instance [Bor4, Theo. 20.9 (iii)]), there exists $\alpha_{i,j} \in \mathbf{G}(\mathbb{Q})$ such that $\mathbf{P}_{i,j} = \alpha_{i,j}\mathbf{P}\alpha_{i,j}^{-1}$. Furthermore, using Lemma 17, we have $\mathbb{C}v_{i,j} = \mathbb{C}\alpha_{i,j}v_0$. A relative Langlands decomposition of $\mathbf{P}_{i,j}$ is $\mathbf{P}_{i,j} = \mathbf{A}_{i,j}\mathbf{M}_{i,j}\mathbf{U}_{i,j}$ where

$$\mathbf{A}_{i,j} = \alpha_{i,j}\mathbf{P}\alpha_{i,j}^{-1}, \quad \mathbf{M}_{i,j} = \alpha_{i,j}\mathbf{M}\alpha_{i,j}^{-1}, \quad \mathbf{U}_{i,j} = \alpha_{i,j}\mathbf{U}\alpha_{i,j}^{-1}\,.$$

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We have $\mathbf{A}_{i,j}(\mathbb{R})_0 = \left(a_s^{i,j} = \alpha_{i,j} a_s \alpha_{i,j}^{-1}\right)_{s \in \mathbb{R}}$ and the Lie algebra of $\mathbf{U}_{i,j}(\mathbb{R})$ is $\mathfrak{U}_{i,j} = \operatorname{Ad}\alpha_{i,j}(\mathfrak{U})$. Hence $a_s^{i,j} v_{i,j} = e^{\chi s} v_{i,j}$ for every $s \in \mathbb{R}$ and

$$\log \det(\operatorname{Ad} a_1^{i,j})_{|\mathfrak{U}_{i,j}} = \lambda \;,$$

for every i, j with $L\,\ell_j v_0 \cap \Gamma v_i \neq \emptyset$.

By Assertion (1) of Theorem 16 applied to the (maximal) parabolic subgroup $\mathbf{P}_{i,j}$ defined over $\mathbb{Q}$, there exist $c_{i,j}, \delta_{i,j} > 0$ (with $c_{i,j}$ explicit) such that, as $t \geq 0$ tends to $+\infty$,

$$\sum_{0 \leq s \leq t} \sum_{[x] \in (L \cap \Gamma) \backslash (La_s^{i,j} v_{i,j} \cap \Gamma v_{i,j})} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) = c_{i,j}\, e^{\lambda t} + \mathrm{O}(e^{(\lambda - \delta_{i,j})t}) \;.$$

Hence, using the equations (7) and (21), with $\delta = \min_{i,j} \delta_{i,j}$ and $c = \sum_{i,j} c_{i,j}$, we have, as $t \geq 0$ tends to $+\infty$,

$$\begin{aligned}
&\sum_{0 \leq s \leq t} \sum_{[x] \in (\mathbf{L}(\mathbb{Z}) \cap \mathbf{L}(\mathbb{R})_0) \backslash (\mathbf{X}_s \cap \Lambda^{\mathrm{Prim}})} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) \\
&= \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k' \\ L\ell_j v_0 \,\cap\, \Gamma v_i \,\neq\, \emptyset}} \sum_{0 \leq s \leq t} \sum_{[x] \in (L \cap \Gamma) \backslash (La_s^{i,j} v_{i,j} \cap \Gamma v_{i,j})} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(x) \\
&= c\, e^{\lambda t} + \mathrm{O}(e^{(\lambda - \delta)t}) \;.
\end{aligned}$$

This ends the proof of Assertion (2) of Theorem 16. $\qquad\square$

Remark. Using Remark 15 instead of Theorem 6 in the above proof gives Theorem 4 and Theorem 3 in the introduction.

3 Applications

3.1 Counting inequivalent representations of integers by norm forms

In this subsection, we fix $n \geq 2$, an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$, and $F \in \mathbb{Q}[x_1, \dots, x_n]$ a rational polynomial in n variables, which is irreducible over $\mathbb{Q}$ and splits as a product of n linearly independant linear forms with coefficients in $\overline{\mathbb{Q}}$. We assume that $F^{-1}(]0, +\infty[)$ is nonempty.

Remarks. (1) With the notation of Theorem 1, for every $k \in \mathbb{R}$, define $N(k) = \operatorname{Card}\left(\Gamma_F \backslash \Sigma_k\right)$. If Σ_k is not empty and if e is the least common multiple of the denominators of the coefficients of F, then $k \in \frac{1}{e}\mathbb{Z}$. In particular, there are only finitely many k in any compact interval of $\mathbb{R}$ such that $N(k) \neq 0$ (and moreover these numbers k are rational, and even integral if F has integral coefficients). In particular, the sum in the left hand side of the asymptotic formula in Theorem 1 is a finite sum.

(2) If $n = 2$, Theorem 1 is well known. It is easy to see that

$$F(x_1, x_2) = a(x_1 + \alpha x_2)(x_1 + \overline{\alpha} x_2)$$

where $a \in \mathbb{Q}$ and α is a quadratic irrational with Galois conjugate $\overline{\alpha}$. Theorem 1 follows from Equation (1) when α is an algebraic integer, where $K = \mathbb{Q}(\alpha)$. When the binary

quadratic form F is indefinite, we refer to [Coh, page 164] for an algebraic proof and to [PP1, Coro. 1.3] for a geometric proof of the main term (and [PP2] for the error term), and these last two papers for geometric extensions to higher dimensional hyperbolic manifolds.

(3) By for instance [Koc, Theo. 2.3.3, page 38], any polynomial F as in the beginning of this subsection is a rational multiple of a norm form. Let us give a quick proof for completeness.

The absolute Galois group $\mathrm{Gal} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ naturally acts on the $\overline{\mathbb{Q}}$-vector space $\overline{\mathbb{Q}}[x_1, \ldots, x_n]$. Let $L_1, \ldots, L_n \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ be linear forms such that $F = \prod_{i=1}^{n} L_i$. By the uniqueness property of irreducible decompositions, the group Gal preserves the set of lines $\{\overline{\mathbb{Q}}L_1, \ldots, \overline{\mathbb{Q}}L_n\}$. If this action is not transitive, and $\{\overline{\mathbb{Q}}L_{k_1}, \ldots, \overline{\mathbb{Q}}L_{k_m}\}$ is an orbit, then the nonzero polynomial $\prod_{i=1}^{m} L_{k_i}$ has its coefficients that are invariant by Gal up to multiplication by an element of $\overline{\mathbb{Q}}$. Dividing $\prod_{i=1}^{m} L_{k_i}$ by one of its nonzero coefficients, we hence get an element of $\mathbb{Q}[x_1, \ldots, x_n]$ (with degree different from 0 and n) which divides F. This contradicts the irreducibility of F over $\mathbb{Q}$.

We may assume that one of the coefficients of L_1 is 1 (up to dividing L_1 by one of its nonzero coefficients, and multiplying L_2 by it). Hence the stabiliser of $\overline{\mathbb{Q}}L_1$ in Gal is equal to the stabiliser Gal_1 of L_1, which is equal to the Galois group $\mathrm{Gal}_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ where K is the number field generated by the coefficients of L_1. Hence there exists $a \in \overline{\mathbb{Q}}$ such that

$$F = a \prod_{\sigma \in \mathrm{Gal}\,/\,\mathrm{Gal}_1} \sigma L_1 = a \prod_{\sigma \in \mathrm{Gal}\,/\,\mathrm{Gal}_K} \sigma L_1 = a\, N_{K/\mathbb{Q}} \circ L_1 \,.$$

Since $N_{K/\mathbb{Q}}$ takes rational values on K, this proves that F is a rational multiple of a norm form.

(4) The assumption that the polynomial F is irreducible over $\mathbb{Q}$ is essential for Theorem 1. For instance, consider $F(x) = x_1 \cdots x_n$. Then the cardinality of $F^{-1}(k) \cap \mathbb{Z}^n$ is nonzero if and only if $k \in \mathbb{Z}$, and, for every $\epsilon > 0$, there exists $\kappa > 0$ such that for every $k \in \mathbb{Z}$,

$$\mathrm{Card}(F^{-1}(k) \cap \mathbb{Z}^n) \leq d(k)^n \leq \kappa\, k^\epsilon \,,$$

where $d(k)$ denotes the number of divisors of k (see for instance [Apo, page 296]).

(5) Let $\mathscr{O}$ be an order in the ring of integers $\mathscr{O}_K$ of a number field K of degree n. Generalizing the case of $\mathscr{O} = \mathscr{O}_K$ (see Equation (1)), with $\alpha_1, \ldots, \alpha_n$ a $\mathbb{Z}$-basis of $\mathscr{O}$, applying Theorem 1 to the norm form $F(x) = N_{K/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_n x_n)$, we prove in a dynamical way that there are constants $c, \delta > 0$ such that $\mathrm{Card}\big(\mathscr{O}^\times \backslash \{x \in \mathscr{O} \,:\, 1 \leq |N_{K/\mathbb{Q}}(x)| \leq r\}\big) = c\, r + \mathrm{O}(r^{1-\delta})$ as $r \to \infty$.

Proof of Theorem 1. In order to apply Theorem 3, let us first define the objects appearing in its statement.

Let $\mathbf{G} = \mathrm{SL}_n(\mathbb{C})$ which is a ($\mathbb{Q}$-split) quasi-simple simply connected linear algebraic group without nontrivial $\mathbb{Q}$-characters. Let $\mathbf{V} = \mathbb{C}^n$, $\Lambda = \mathbb{Z}^n$ (which is a $\mathbb{Z}$-lattice in $\mathbf{V}(\mathbb{Q})$ invariant under $\mathbf{G}(\mathbb{Z})$), $(e_1, \ldots, e_n)$ the canonical basis of $\mathbf{V}$ and $\rho : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ the monomorphism mapping a matrix x to the linear automorphism of $\mathbf{V}$ whose matrix in the canonical basis is x, which is an irreducible rational representation over $\mathbb{C}$. To simplify the notation, we denote $\rho(g)v = gv$ for every $g \in \mathbf{G}$ and $v \in \mathbf{V}$. Let $\mathbf{P}$ be the stabiliser in $\mathbf{G}$ of the line generated by e_1, which is a maximal (proper) parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$. With I_k the identity $k \times k$ matrix and $s \in \mathbb{R}$, let $\mathbf{U} = \left\{ \begin{pmatrix} 1 & u \\ 0 & I_{n-1} \end{pmatrix} : u \in \right.$

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$\mathscr{M}_{1,n-1}(\mathbb{C})\Big\}$, $a_s = \begin{pmatrix} e^{\frac{s}{n}} & 0 \\ 0 & e^{-\frac{s}{n(n-1)}}I_{n-1} \end{pmatrix}$, and $\mathbf{M} = \Big\{ \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \ : \ m \in \mathrm{SL}_{n-1}(\mathbb{C}) \Big\}$. With $\mathbf{A}$ the centraliser of $\mathbf{M}$ in $\mathbf{G}$, we have that $\mathbf{P} = \mathbf{AMU}$ is a relative Langlands decomposition of $\mathbf{P}$ over $\mathbb{Q}$, and the identity component of $\mathbf{A}(\mathbb{R})$ is the one-parameter subgroup $(a_s)_{s\in\mathbb{R}}$. With $\mathfrak{U}$ the Lie algebra of $\mathbf{U}(\mathbb{R})$, an immediate computation gives

$$\lambda = \log \det(\mathrm{Ad}\, a_1)_{|\mathfrak{U}} = 1 > 0 \ . \tag{22}$$

Since F is homogeneous, as F takes a positive value (and equivalently), there exists $v_0 \in \mathbb{Z}^n$ such that $F(v_0) > 0$. We may assume that v_0 is primitive up to rescaling it, and after an integral linear change of variable (which does not change the set of integral representations of a real number by F), we may assume that $v_0 = e_1$. Note that the stabiliser of v_0 in $\mathbf{G}$ is then precisely $\mathbf{MU}$.

We denote by $\mathbf{L}$ the stabiliser of F in $\mathbf{G}$ and by $\pi : \mathbf{L} \to \mathrm{GL}(\mathbf{V})$ the restriction of ρ to $\mathbf{L}$. By the linear independence over $\mathbb{C}$ assumption, $\mathbf{L}$ is a maximal algebraic torus defined over $\mathbb{Q}$ in $\mathbf{G}$ (hence $\mathbf{L}$ is reductive, but not semisimple). For every $z \in \mathbb{C} - \{0\}$, the group $\mathbf{L}$ acts simply transitively on the affine hypersurface $F^{-1}(z)$. Hence, with $v_s = a_s v_0 = e^{\frac{s}{n}} v_0$, the orbit

$$\mathbf{X}_s = \mathbf{L} v_s = F^{-1}(F(v_s)) = F^{-1}(e^s F(v_0)) \tag{23}$$

(since F is homogeneous of degree n) is Zariski-closed in $\mathbf{V}$.

Let us now check in two lemmas that the hypotheses of Theorem 3 are satisfied by these objects.

Lemma 19 *The algebraic torus $\mathbf{L}$ is anisotropic over $\mathbb{Q}$.*

Proof. As seen in Remark (3) above, there exist $a \in \mathbb{Q} - \{0\}$ and linearly independant linear forms $\ell_1, \dots, \ell_n$ on $\mathbb{C}^n$ with coefficients in $\overline{\mathbb{Q}}$ such that $F = a \prod_{i=1}^n \ell_i$ and the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on the set $\{\ell_1, \dots, \ell_n\}$. Let $\mathscr{B}$ be the basis of $\mathbb{C}^n$ whose dual basis is $(\ell_1, \dots, \ell_n)$. The algebraic torus $\mathbf{L}$ is the subgroup of the elements of $\mathbf{G}$ whose matrix in the basis $\mathscr{B}$ is diagonal. For $1 \leq i \leq n$, let χ_i be the character (defined over $\overline{\mathbb{Q}}$) of $\mathbf{L}$ which associates to an element of $\mathbf{L}$ the i-th diagonal element of its matrix in $\mathscr{B}$. Note that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on the set $\{\chi_1, \dots, \chi_n\}$. Any character of $\mathbf{L}$ may be uniquely written $\prod_{i=1}^n \chi_i^{k_i}$ with $k_1, \dots, k_n \in \mathbb{Z}$. Any $\mathbb{Q}$-character $\prod_{i=1}^n \chi_i^{k_i}$ of $\mathbf{L}$, being invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, should have $k_1 = \cdots = k_n$ by transitivity, hence is trivial. The result follows, since an algebraic torus defined over $\mathbb{Q}$ without nontrivial $\mathbb{Q}$-characters is anisotropic over $\mathbb{Q}$, that is, it contains no nontrivial $\mathbb{Q}$-split torus (see for instance [Bor3, page 121], though this reference uses a different meaning of anisotropic). $\qquad\square$

Lemma 20 *The intersection $\mathbf{L} \cap \mathbf{P}$ is finite and $\mathbf{LP}$ is Zariski-open in $\mathbf{G}$.*

Proof. Let us prove that the algebraic group $\mathbf{L} \cap \mathbf{P}$ is finite. Since an algebraic group has only finitely many components, we only have to prove that its identity component $\mathbf{S} = (\mathbf{L} \cap \mathbf{P})_0$ is trivial. The algebraic torus $\mathbf{S}$ is defined over $\mathbb{Q}$, hence is contained in a maximal torus of $\mathbf{P}$ defined over $\mathbb{Q}$. By [Bor3, Theo. 19.2], two maximal tori of $\mathbf{P}$ defined over $\mathbb{Q}$ are conjugated over $\mathbb{Q}$. Since $\mathbf{G}$ splits over $\mathbb{Q}$, this implies that $\mathbf{L} \cap \mathbf{P}$ splits over $\mathbb{Q}$. Since $\mathbf{L}$ is anisotropic over $\mathbb{Q}$ by Lemma 19, this implies that $\mathbf{S}$ is trivial, and proves the first claim.

Now, the homogeneous space $\mathbf{G}/\mathbf{P}$ is identified with the complex projective space $\mathbb{P}(\mathbb{C}^n)$ by the map $g \mapsto \mathbb{C}ge_1$. We write $e_1 = \sum_{i=1}^n c_i w_i$ where $\mathscr{B} = (w_i)_{1 \leq i \leq n}$ is a diagonalisation basis of $\mathbf{V}$ for the action of the algebraic torus $\mathbf{L}$, as in the proof of Lemma 19. Since the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on $\{w_1, \ldots w_n\}$ and fixes e_1, it follows that the coefficients c_i are all different from 0. Hence $\mathbf{L}(\mathbb{C}e_1) = \{\mathbb{C}\sum_{i=1}^n b_i w_i : b_i \neq 0\}$, which implies the second claim. $\qquad\square$

To conclude the proof of Theorem 1, we relate the two counting functions in the statements of Theorem 1 and Theorem 3.

For every $s > 0$ and $p \in \mathbb{N} - \{0\}$, let $A_s^{(p)}$ be the set of integral points of $\mathbf{X}_s$ whose coefficients have their greatest common divisor equal to p. Note that $A_s^{(1)} = \mathbf{X}_s \cap \Lambda^{\mathrm{prim}}$ is the set of primitive integral points of $\mathbf{X}_s$. With $N_s^{(p)} = \mathrm{Card}(\mathbf{L}(\mathbb{Z})\backslash A_s^{(p)})$, we have $\mathrm{Card}(\mathbf{L}(\mathbb{Z})\backslash\mathbf{X}_s(\mathbb{Z})) = \sum_{p=1}^{+\infty} N_s^{(p)}$, and $N_s^{(p)} = N_{s-\ln(p^n)}^{(1)}$, since $\mathbf{X}_{s-\log(p^n)} = \frac{1}{p}\mathbf{X}_s$ by the homogeneity of F and Equation (23).

Since $\mathbf{L}$ acts simply transitively on each $\mathbf{X}_s$, the stabiliser $\mathbf{L}_x$ of every $x \in \mathbf{X}_s$ is trivial, hence the Siegel weight $w_{\mathbf{L},\pi}(x)$ is constant, equal to $\frac{1}{\mathrm{vol}(\mathbf{L}(\mathbb{Z})\backslash\mathbf{L}(\mathbb{R}))}$. By Theorem 3 and Equation (22), there exist $\delta > 0$, that we may assume to be in $]0, 1 - \frac{1}{n}[$, and $c > 0$ such that, as $t \geq 0$ and $t \to +\infty$,

$$\sum_{s \in [0,t]} N_s^{(1)} = c\, e^t + \mathrm{O}(e^{t(1-\delta)})\,.$$

For every $r \geq F(v_0)+1$, by setting $t = \log \frac{r}{F(v_0)} \geq 0$ and by using the change of variables $k = e^s F(v_0)$ (see Equation (23)), we have, with $\Sigma_k = F^{-1}(k) \cap \mathbb{Z}^n$ and ζ Riemann's zeta function,

$$\sum_{k \in [F(v_0),r]} \mathrm{Card}(\mathbf{L}(\mathbb{Z})\backslash\Sigma_k) = \sum_{s \in [0,t]} \mathrm{Card}(\mathbf{L}(\mathbb{Z})\backslash\mathbf{X}_s(\mathbb{Z})) = \sum_{s \in [0,t]} \sum_{p=1}^{+\infty} N_s^{(p)}$$

$$= \sum_{p=1}^{+\infty} \sum_{s \in [0,t]} N_{s-\ln(p^n)}^{(1)} = \sum_{p=1}^{+\infty} c\, p^{-n}\, e^t + \mathrm{O}(p^{n(\delta-1)}e^{t(1-\delta)})$$

$$= c\, \zeta(n)\, e^t + \mathrm{O}\left(e^{t(1-\delta)}\right) = \frac{c\, \zeta(n)}{F(v_0)}\, r + \mathrm{O}\left(r^{1-\delta}\right)\,.$$

Note that $\sum_{k \in [\min\{1,F(v_0)\},\max\{1,F(v_0)\}]} \mathrm{Card}(\mathbf{L}(\mathbb{Z})\backslash\Sigma_k)$ is finite. The result follows. $\qquad\square$

3.2 Counting inequivalent integral points on hyperplane sections of affine quadratic surfaces

Let $n \geq 3$, let $q : \mathbb{C}^n \to \mathbb{C}$ with $q(x) = \sum_{i=1}^n q_{ij}x_i x_j$ for every $x = (x_1, \ldots, x_n)$ be a nondegenerate quadratic form in n variables with coefficients q_{ij} in $\mathbb{Q}$, and let $\ell : \mathbb{C}^n \to \mathbb{C}$ with $\ell(x) = \sum_{i=1}^n \ell_i x_i$ for every $x = (x_1, \ldots, x_n)$ be a nonzero linear form in n variables with coefficients ℓ_i in $\mathbb{Q}$.

The aim of this section is to count the number of orbits of integral points on the sections, by the hyperplanes parallel to the kernel of ℓ, of the isotropic cone $q^{-1}(0)$ of q.
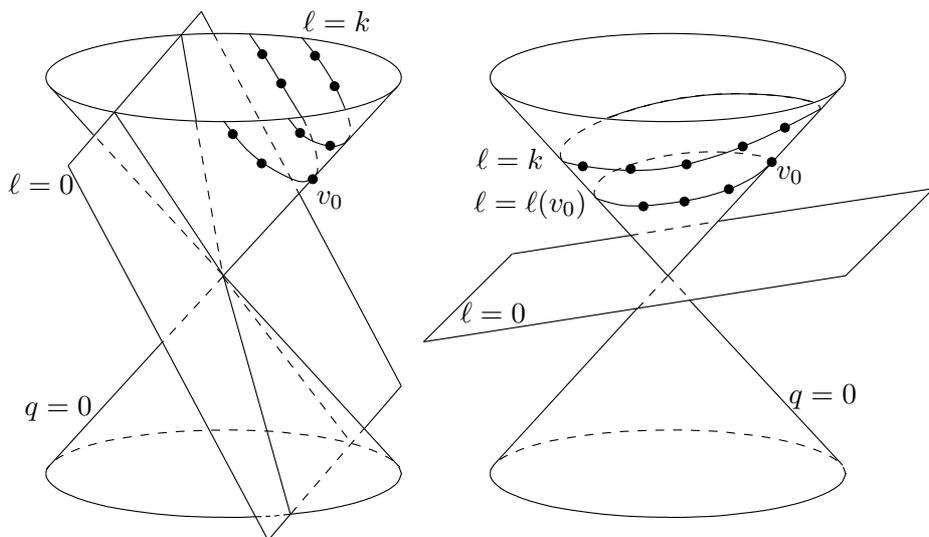
For $\mathbb{K} = \mathbb{R}$ or $\mathbb{Q}$, recall that q is *isotropic (or indefinite when* $\mathbb{K} = \mathbb{R}$*) over* $\mathbb{K}$ or *represents* 0 *over* $\mathbb{K}$ if there exists $x \in \mathbb{K}^n - \{0\}$ such that $q(x) = 0$, and that q is *anisotropic over* $\mathbb{K}$ otherwise. For instance, $x^2 + 2y^2 - 7z^2$ is anisotropic over $\mathbb{Q}$, but indefinite over $\mathbb{R}$. By A. Meyer's 1884 result (see for instance [Ser, page 77]), if $n \geq 5$, then q is isotropic over $\mathbb{Q}$ if and only if q is indefinite over $\mathbb{R}$.

Proof of Theorem 2. In order to apply Theorem 16 (2), let us first define the objects appearing in its statement.

Let $\mathbf{G} = O_q$ be the orthogonal group of the nondegenerate rational quadratic form q, which is a connected semisimple linear algebraic group defined over $\mathbb{Q}$, hence is reductive without nontrivial $\mathbb{Q}$-characters. Let $\mathbf{V} = \mathbb{C}^n$ and let $\rho : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ be the monomorphism mapping a matrix x to the linear automorphism of $\mathbf{V}$ whose matrix in the canonical basis is x, which is an irreducible rational representation over $\mathbb{C}$. Let $\Lambda = \mathbb{Z}^n$, which is a $\mathbb{Z}$-lattice in $\mathbf{V}(\mathbb{Q})$ invariant under $\mathbf{G}(\mathbb{Z})$. To simplify the notation, we denote $\rho(g)v = gv$ for every $g \in \mathbf{G}$ and $v \in \mathbf{V}$.

Since q is assumed to be isotropic over $\mathbb{Q}$, there exists v_0 in $\Lambda - \{0\}$ such that $q(v_0) = 0$ and we assume that $\ell(v_0) \geq 0$ up to replacing v_0 by $-v_0$. Since the restriction of q to the kernel of ℓ is assumed to be anisotropic over $\mathbb{Q}$, we have $\ell(v_0) > 0$. Let $\mathbf{P}$ be the stabiliser in $\mathbf{G}$ of the line generated by v_0, which is a maximal (proper) parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ since this line is isotropic. Let $\mathscr{B} = (e_1, \dots, e_n)$ be a basis of $\mathbf{V}$ over $\mathbb{Q}$ such that $e_1 = v_0$, (e_1, e_2) is a standard basis of a hyperbolic plane over $\mathbb{Q}$ for q, which is orthogonal for q to the vector subspace $\mathbf{V}'$ generated by $\mathscr{B}' = (e_3, \dots, e_n)$. In particular, the matrix of q in the basis $\mathscr{B}$ is $Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & Q' \end{pmatrix}$ with Q' the (rational symmetric) matrix in the basis $\mathscr{B}'$ of the restriction q' of q to $\mathbf{V}'$. Denoting in the same way a vector v (resp. u) of $\mathbf{V}$ (resp. $\mathbf{V}'$) and the column vector of its coordinates in $\mathscr{B}$ (resp. $\mathscr{B}'$), we have $q(v) = {}^t v Q v$ (resp. $q'(u) = {}^t u Q' u$). With I_k the identity $k \times k$ matrix and $s \in \mathbb{R}$, define

$$a_s = \begin{pmatrix} e^s & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} , \quad \mathbf{A} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} : a \in \mathbb{C}^* \right\} ,$$

27

$$\mathbf{M} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m \end{pmatrix} \ : \ m \in O_{q'} \right\} \quad \text{and} \quad \mathbf{U} = \left\{ \begin{pmatrix} 1 & -q'(u)/2 & -{}^t u Q' \\ 0 & 1 & 0 \\ 0 & u & I_{n-2} \end{pmatrix} \ : \ u \in \mathbf{V}' \right\} .$$

It is easy to check that $\mathbf{P} = \mathbf{AMU}$ is a relative Langlands decomposition of $\mathbf{P}$, that the identity component of $\mathbf{A}(\mathbb{R})$ is the one-parameter subgroup $(a_s)_{s \in \mathbb{R}}$, and that the stabiliser of $v_0 = e_1$ in $\mathbf{G}$ is exactly $\mathbf{MU}$. With $\mathfrak{U}$ the Lie algebra of $\mathbf{U}(\mathbb{R})$, an immediate computation gives (since $n \geq 3$)

$$\lambda = \log \det(\mathrm{Ad}\, a_1)_{|\mathfrak{U}} = n - 2 > 0 . \tag{24}$$

We denote by $\mathbf{L} = \{g \in \mathbf{G} \ : \ \ell \circ g = \ell\}$ the stabiliser of ℓ in $\mathbf{G}$, which is a linear algebraic group defined over $\mathbb{Q}$. Let $\mathbf{W}$ be the kernel of ℓ and $\mathbf{W}^\perp$ be its orthogonal for q. Since $q_{|\mathbf{W}}$ is assumed to be nondegenerate, $\mathbf{W}^\perp$ is a line, $\mathbf{V} = \mathbf{W}^\perp \oplus \mathbf{W}$, and the bloc matrix of q in this decomposition is diagonal.

Let us now check in the next lemma that the hypotheses of Theorem 16 are satisfied by these objects.

Lemma 21 *(1) The linear algebraic group $\mathbf{L}$ is reductive and anisotropic over $\mathbb{Q}$.*
(2) For every $s \in \mathbb{R}$, if $k = e^s \ell(v_0)$ and $\mathbf{X}_s = \mathbf{L} a_s v_0$, then $\mathbf{X}_s = \{v \in \mathbf{V} \ : \ q(v) = 0, \ell(v) = k\}$. In particular, $\mathbf{X}_s$ is Zariski-closed in $\mathbf{V}$.
(3) The subset $\mathbf{LP}$ is Zariski-open in $\mathbf{G}$.

Proof. (1) For every $g \in \mathrm{GL}(\mathbf{V})$, if $\ell \circ g = \ell$, then g preserves $\mathbf{W}$. If furthermore $g \in \mathbf{G} = O_q$, then g preserves $\mathbf{W}^\perp$. Since $\mathbf{W}^\perp$ is a line, there exists $\lambda \in \mathbb{C}$ such that g acts by $x \mapsto \lambda x$ on $\mathbf{W}^\perp$. As $\ell_{|\mathbf{W}^\perp}$ is nonzero and g preserves ℓ, we have $\lambda = 1$. Hence the elements of $\mathbf{L}$ are exactly the elements of $\mathrm{GL}(\mathbf{V})$ whose bloc matrix in the decomposition $\mathbf{V} = \mathbf{W}^\perp \oplus \mathbf{W}$ has the form $\begin{pmatrix} 1 & 0 \\ 0 & g' \end{pmatrix}$ with $g' \in O_{q_{|\mathbf{W}}}$. In particular, the linear algebraic group $\mathbf{L}$, isomorphic over $\mathbb{Q}$ to the orthogonal group of the nondegenerate rational quadratic form $q_{|\mathbf{W}}$, is semisimple hence reductive.

It is well-known (see for instance [Bor1][BJ, page 270]) that the $\mathbb{Q}$-rank of the orthogonal group $O_{q''}$ of a nondegenerate rational quadratic form q'' is zero (or equivalently that $O_{q''}$ is anisotropic over $\mathbb{Q}$) if and only if q'' does not represents 0 over $\mathbb{Q}$. For instance, this follows from the fact that the spherical Tits building over $\mathbb{Q}$ of $O_{q''}$ is the building of isotropic flags over $\mathbb{Q}$. Hence by assumption, $\mathbf{L}$ is anisotropic over $\mathbb{Q}$.

(2) Note that by the definition of a_s, we have $a_s v_0 = e^s v_0$, hence by the linearity of ℓ, we may assume that $s = 0$. Recall that $q(v_0) = 0$ and $\ell(v_0) > 0$. By the definition of $\mathbf{L}$, the orbit $\mathbf{X}_0 = \mathbf{L} v_0$ is contained in $\{v \in \mathbf{V} \ : \ q(v) = 0, \ \ell(v) = \ell(v_0)\}$. To prove the opposite inclusion, write $v = v' + v''$ the decomposition of any $v \in \mathbf{V}$ in the direct sum $\mathbf{V} = \mathbf{W}^\perp \oplus \mathbf{W}$. If $\ell(v) = \ell(v_0)$ and $q(v) = 0$, then $v' = v_0'$ and $q(v'') = -q(v') = -q(v_0')$, and in particular $q(v'') = q(v_0'')$. By Witt's theorem, there exists $g' \in O_{q_{|\mathbf{W}}}$ such that $v'' = g' v_0''$. Hence the linear transformation of $\mathbf{V}$ which is the identity on $\mathbf{W}^\perp$ and is equal to g' on $\mathbf{W}$, is an element of $\mathbf{L}$ sending $v = v' + v''$ to $v_0 = v_0' + v_0''$. The second assertion follows.

(3) The algebraic group $\mathbf{G} = O_q$ acts transitively on the projective variety of isotropic lines in $\mathbf{V}$, the stabiliser of the line generated by v_0 being $\mathbf{P}$ by definition. As we have seen in (2), the orbit under $\mathbf{L}$ of the line generated by v_0 is hence the Zariski-open subset

of $\mathbf{G}/\mathbf{P}$ consisting of the isotropic lines not contained in $\mathbf{W}$. The last claim of Lemma 21 follows. $\qquad\square$

To conclude the proof of Theorem 2, we relate the two counting functions in the statements of Theorem 2 and Theorem 16 (2). Let $L = \mathbf{L}(\mathbb{R})_0$ and $\Gamma = \mathbf{G}(\mathbb{R})_0 \cap \mathbf{G}(\mathbb{Z})$.

We have $\ell(v_0) > 0$ by the definition of v_0. For every $r \geq \ell(v_0) + 1$, let $t = \ln \frac{r}{\ell(v_0)} > 0$. With Σ_k as in the statement of Theorem 2, using the change of variables $k = e^s \ell(v_0)$ and Lemma 21 (2), by the definition of the modified Siegel weights in Equation (2), we have

$$
\sum_{k \in [\ell(v_0), r]} \sum_{[u] \in (\mathbf{L}(\mathbb{Z}) \cap L) \setminus \Sigma_k} \operatorname{vol} \left((\mathbf{L}_u(\mathbb{Z}) \cap L) \setminus (\mathbf{L}_u \cap L) \right) =
$$

$$
\operatorname{vol} \left((L \cap \Gamma) \setminus L \right) \sum_{s \in [0, t]} \sum_{[u] \in (\mathbf{L}(\mathbb{Z}) \cap L) \setminus \mathbf{X_s} \cap \Lambda^{\mathrm{prim}}} w'_{\mathbf{L}, \rho_{|\mathbf{L}}}(u) . \tag{25}
$$

By Theorem 16 (2) and Equation (24), there exist $c, \delta > 0$ such that as $t \to +\infty$, the quantity (25) is equal to

$$
c \, e^{(n-2)t} + \mathrm{O}\left(e^{(n-2-\delta)t} \right) = \frac{c}{\ell(v_0)^{n-2}} \, r^{n-2} + \mathrm{O}\left(r^{n-2-\delta} \right) .
$$

Note that $\sum_{k \in [\min\{1, \ell(v_0)\}, \max\{1, \ell(v_0)\}]} \sum_{[u] \in (\mathbf{L}(\mathbb{Z}) \cap L) \setminus \Sigma_k} \operatorname{vol} \left((\mathbf{L}_u(\mathbb{Z}) \cap L) \setminus (\mathbf{L}_u \cap L) \right)$ is finite. This concludes the proof of Theorem 2. $\qquad\square$

Remarks (1) If $n \geq 6$, since q is isotropic over $\mathbb{Q}$ and the restriction of q to the kernel of ℓ is anisotropic over $\mathbb{Q}$, then the signature of q over $\mathbb{R}$ is $(1, n-1)$ or $(n-1, 1)$, and $\mathbf{L}(\mathbb{R})$ is compact (see the above picture on the right); hence $\mathbf{L}(\mathbb{Z})$ is finite, and our result allows to count integral points on the quadratic hypersurface $q^{-1}(0)$ (see the references given in the introduction for related works).

(2) If $n \geq 4$, then we have a result similar to Theorem 2 where we consider all the integral points and not only the primitive ones: under the other assumptions of Theorem 2 and with c as above, we have, for every $r \geq 1$ with $r \to +\infty$,

$$
\sum_{k \in [1, r]} \sum_{[u] \in (\mathbf{L}(\mathbb{Z}) \cap L) \setminus (q^{-1}(0) \cap \ell^{-1}(k) \cap \mathbb{Z}^n)} \operatorname{vol} \left((\mathbf{L}_u(\mathbb{Z}) \cap L) \setminus (\mathbf{L}_u \cap L) \right)
$$

$$
= \frac{c \, \zeta(n-2)}{\ell(v_0)^{n-2}} \, r^{n-2} + \mathrm{O}\left(r^{n-2-\delta} \right) .
$$

The proof is similar to the one at the end of Section 3.1. For every $s \in \mathbb{R}$ and $p \in \mathbb{N} - \{0\}$, we denote by $A_s^{(p)}$ the set of integral points of $\mathbf{X}_s$ whose greatest common divisor of their coefficients is p. We note that by Lemma 21 (2), the map from $A_s^{(p)}$ to $A_{s-\ln p}^{(1)}$ defined by $x \mapsto \frac{x}{p}$ is a bijection such that $\mathbf{L}_{\frac{x}{p}} = \mathbf{L}_x$ for every $x \in A_s^{(p)}$. Hence with

$$
N_s^{(p)} = \sum_{[u] \in (\mathbf{L}(\mathbb{Z}) \cap L) \setminus A_s^{(p)}} \operatorname{vol} \left((\mathbf{L}_u(\mathbb{Z}) \cap L) \setminus (\mathbf{L}_u(\mathbb{R}) \cap L) \right) ,
$$

we have $N_s^{(p)} = N_{s-\ln p}^{(1)}$ and

$$
\sum_{[u] \in (\mathbf{L}(\mathbb{Z}) \cap L) \setminus (q^{-1}(0) \cap \ell^{-1}(k) \cap \mathbb{Z}^n)} \operatorname{vol} \left((\mathbf{L}_u(\mathbb{Z}) \cap L) \setminus (\mathbf{L}_u \cap L) \right) = \sum_{p=1}^{\infty} N_s^{(p)} ,
$$

and one concludes as in the end of Section 3.1.

When $n = 3$, the same argument gives

$$\sum_{k \in [1,r]} \quad \sum_{[u] \in (\mathbf{L}(\mathbb{Z}) \cap L) \backslash (q^{-1}(0) \cap \ell^{-1}(k) \cap \mathbb{Z}^n)} \mathrm{vol}\left((\mathbf{L}_u(\mathbb{Z}) \cap L) \backslash (\mathbf{L}_u \cap L)\right) = \frac{c}{\ell(v_0)} \ r \log r + \mathrm{O}\left(r\right) .$$

3.3 Counting inequivalent integral points of given norm in central division algebras

Let $n \geq 2$, let D be a central simple algebra over $\mathbb{Q}$ of dimension n^2, let $N : D \to \mathbb{Q}$ be its reduced norm, and let $\mathscr{O}$ be an *order* in D (that is, a finitely generated $\mathbb{Z}$-submodule of D, generating D as a $\mathbb{Q}$-vector space, which is a unitary subring). We refer for instance to [Rei] and [PR, Chap. I, §1.4]) for generalities. The aim of this section is to use our main result to deduce asymptotic counting results of elements of $\mathscr{O}$ (modulo units) of given norm.

Theorem 22 *If D is a division algebra over $\mathbb{Q}$, then there exist $c = c(D, \mathscr{O}) > 0$ and $\delta = \delta(D) > 0$ such that, for every $r \geq 1$ with $r \to +\infty$,*

$$\mathrm{Card}_{\mathscr{O}^\times \backslash} \{x \in \mathscr{O} \ : \ 1 \leq |N(x)| \leq r\} = c \ r^n \left(1 + \mathrm{O}\left(r^{-\delta}\right)\right) .$$

Proof. In order to apply Theorem 3, let us first define the objects appearing in its statement.

Let $\mathbf{V}$ be the vector space over $\mathbb{Q}$ such that $\mathbf{V}(K) = D \otimes_{\mathbb{Q}} K$ for every characteristic zero field, with the integral structure such that $\Lambda = \mathbf{V}(\mathbb{Z}) = \mathscr{O}$, which is (for the extended multiplication) a central simple algebra over $\mathbb{C}$. Let $\mathbf{D}^1$ be the group of elements of (reduced) norm ± 1 in $\mathbf{V}$.

We take $\mathbf{G} = \mathrm{SL}(\mathbf{V})$ (which is connected, simply connected, semisimple, defined over $\mathbb{Q}$, hence reductive without nontrivial $\mathbb{Q}$-characters) and ρ the inclusion of $\mathbf{G}$ in $\mathrm{GL}(\mathbf{V})$ (which is an irreducible rational representation). To simplify the notation, we denote $\rho(g)v = gv$ for every $g \in \mathbf{G}$ and $v \in \mathbf{V}$.

Let $\mathbf{L}$ be the algebraic subgroup of $\mathbf{G}$ which is the image of $\mathbf{D}^1$ into $\mathbf{G}$ by the (left) regular representation $d \mapsto \{v \mapsto dv\}$. Note that the linear algebraic groups $\mathbf{L}$ and $\mathbf{D}^1$ are defined over $\mathbb{Q}$ and are isomorphic by this representation. We have

$$\mathbf{L}(\mathbb{Z}) = \mathbf{D}^1 \cap \mathscr{O} = \mathscr{O}^\times . \tag{26}$$

We take $v_0 \in \mathbf{V}$ to be the identity element in D. The stabiliser of the line $\mathbb{C}v_0$ in $\mathbf{G}$ is a (maximal) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ defined over $\mathbb{Q}$. We note that $\dim(\mathbf{P}) = \dim(D)^2 - \dim(D) - 1$ and $\dim(\mathbf{L}) = \dim(D) - 1$. We have a relative Langlands decomposition $\mathbf{P} = \mathbf{AMU}$ with $\mathbf{MU}$ the stabiliser of v_0 in $\mathbf{G}$, and we may write $\mathbf{A}(\mathbb{R})_0 = (a_s)_{s \in \mathbb{R}}$ such that $a_s v_0 = e^{\frac{s}{n}} v_0$. An easy computation gives

$$\lambda = \log \det(\mathrm{Ad}\, a_1)_{|\mathfrak{u}} = n > 0 . \tag{27}$$

Let us now check that the hypotheses of Theorem 3 are satisfied by these objects.

We claim that the group $\mathbf{L} \cap \mathbf{P}$ is finite. The action of this group on v_0 defines a $\mathbb{Q}$-character of $\mathbf{L} \cap \mathbf{P}$. Since $\mathbf{L} \simeq \mathbf{D}^1$ is anisotropic over $\mathbb{Q}$ (see for instance [PR, Chap. II, §2.3]), this character must be trivial on $(\mathbf{L} \cap \mathbf{P})_0$, and $(\mathbf{L} \cap \mathbf{P})_0 v_0 = v_0$. Since $\mathrm{Stab}_{\mathbf{L}}(v_0) =$

$\{e\}$, it follows that $(\mathbf{L} \cap \mathbf{P})_0 = \{e\}$, which proves the claim. Comparing dimensions, we deduce that $\mathbf{LP}$ is Zariski-open in $\mathbf{G}$.

For every $s \in \mathbb{R}$, we have

$$\mathbf{X}_s = \mathbf{L}a_s v_0 = e^{\frac{s}{n}}\mathbf{L}v_0 = e^{\frac{s}{n}}\mathbf{D}^1 \ . \tag{28}$$

Hence $\mathbf{X}_s$ is Zariski-closed in $\mathbf{V}$.

To conclude the proof of Theorem 22, we relate the two counting functions in the statements of Theorem 22 and Theorem 3.

Since $\mathbf{L}$ acts simply transitively on the orbit of v_0, the Siegel weights are constant, equal to $\frac{1}{\mathrm{vol}(\mathbf{L}(\mathbb{Z})\backslash\mathbf{L}(\mathbb{R}))}$. For every $k \in \mathbb{N} - \{0\}$, denote by $\mathscr{O}^{(k)}$ the subset of nonzero elements of $\mathscr{O}$ whose greatest common divisor of their coefficients in a $\mathbb{Z}$-basis of $\mathscr{O}$ is k. In particular, since the norm is a homogeneous polynomial of degree n and by Equation (28), we have

$$\mathbf{X}_s \cap \Lambda^{\mathrm{prim}} = \{x \in \mathscr{O}^{(1)} \ : \ N(x) = e^s\} \ .$$

Note that the map $x \mapsto \frac{x}{k}$ is a bijection from $\mathscr{O}^{(k)}$ to $\mathscr{O}^{(1)}$. Hence, using Equation (26) and Theorem 3, there exist $\delta > 0$, that we may assume to be in $]0,1[$, and $c > 0$ such that, as $r \geq 1$ and $r \to +\infty$,

$$\begin{aligned}
\mathrm{Card}\,_{\mathscr{O}^\times\backslash}\{x \in \mathscr{O} \ : \ 1 \leq |N(x)| \leq r\} &= \sum_{k=1}^{+\infty} \mathrm{Card}\,_{\mathscr{O}^\times\backslash}\{x \in \mathscr{O}^{(k)} \ : \ 1 \leq |N(x)| \leq r\} \\
&= \sum_{k=1}^{+\infty} \mathrm{Card}\,_{\mathscr{O}^\times\backslash}\{x \in \mathscr{O}^{(1)} \ : \ 1 \leq |N(x)| \leq \frac{r}{k^n}\} \\
&= \sum_{k=1}^{+\infty} \sum_{0 \leq s \leq \log \frac{r}{k^n}} \mathrm{Card}\,\big(\mathbf{L}(\mathbb{Z})\backslash(\mathbf{X}_s \cap \Lambda^{\mathrm{prim}})\big) \\
&= \sum_{k=1}^{+\infty} c\ \big(\frac{r}{k^n}\big)^n\Big(1 + \mathrm{O}\,\big((\frac{r}{k^n})^{-\delta}\big)\Big) \\
&= c\,\zeta(n^2)\,r^n\big(1 + \mathrm{O}(r^{-\delta})\big) \ .
\end{aligned}$$

This ends the proof of Theorem 22. $\qquad\square$

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