Joint effective equidistribution of partial lattices in positive characteristic

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Abstract

Let ν be a place of a global function field K over a finite field, with associated affine function ring R_ν and completion K_ν, and let $1 \leqslant \mathfrak{m} < \mathbf{d}$. The aim of this paper is to prove an effective triple joint equidistribution result for primitive partial R_ν-lattices Λ of rank $\mathfrak{m}$ in ${K_\nu}^{\mathbf{d}}$ as their covolume tends to infinity: of their K_ν-linear span V_Λ in the rank-$\mathfrak{m}$ Grassmannian space of ${K_\nu}^{\mathbf{d}}$; of their shape in the modular quotient by $\mathrm{PGL}_{\mathfrak{m}}(R_\nu)$ of the Bruhat-Tits buildings of $\mathrm{PGL}_{\mathfrak{m}}(K_\nu)$; and of the shape of $\Lambda^\perp$ in the similar quotient for $\mathrm{PGL}_{\mathbf{d}-\mathfrak{m}}(K_\nu)$, where $\Lambda^\perp$ is the orthogonal partial R_ν-lattice of rank $\mathbf{d}-\mathfrak{m}$ in the dual space of ${K_\nu}^{\mathbf{d}}$. The main tools are a new refined LU decomposition by blocks of elements of $\mathrm{SL}_{\mathbf{d}}(K_\nu)$, techniques of Gorodnik and Nevo for counting integral points in well-rounded families of subsets of algebraic groups, and computations of volumes of various homogeneous spaces associated with partial R_ν-lattices. [1]

1 Introduction

We fix throughout the paper three positive integers $\mathfrak{m}$, $\mathfrak{n}$, $\mathbf{d}$ such that $\mathbf{d} = \mathfrak{m} + \mathfrak{n}$. A *primitive integral vector* in $\mathbb{R}^{\mathbf{d}}$ is an element of $\mathbb{Z}^{\mathbf{d}}$ with coprimes componants, so that the cyclic group it generates is a free abelian factor of rank 1 of $\mathbb{Z}^{\mathbf{d}}$. More generally, a *primitive* $\mathfrak{m}$*-lattice* in $\mathbb{R}^{\mathbf{d}}$ is a free abelian factor of rank $\mathfrak{m}$ in $\mathbb{Z}^{\mathbf{d}}$. The distribution problems of primitive integral vectors and of primitive $\mathfrak{m}$-lattices have first been studied by Linnik and Maass (see for instance [Lin] and [Maa]), and have given rise to a huge amount of works using various tools, see for instance [Sch1, Sch2, Duk1, Duk2, Mar, ElMV, EiMSS, HK1].

Let us define the *covolume* $\mathrm{Covol}(\Lambda)$ of a primitive $\mathfrak{m}$-lattice Λ in $\mathbb{R}^{\mathbf{d}}$ as the Lebesgue volume of the parallelepiped generated by any $\mathbb{Z}$-basis of Λ, and its *shape* $\mathrm{sh}(\Lambda)$ as its equivalence class modulo rotations and homotheties (or its "similarity class" with the terminology of [Sch2]), which belongs to the double coset space $\mathrm{PSh}_{\mathfrak{m}} = \mathrm{PSO}(\mathfrak{m})\backslash \mathrm{PGL}_{\mathfrak{m}}(\mathbb{R}) / \mathrm{PGL}_{\mathfrak{m}}(\mathbb{Z})$. Let us denote by V_Λ the $\mathbb{R}$-linear subspace of $\mathbb{R}^{\mathbf{d}}$ generated by Λ, which belongs to the Grassmannian space $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ of $\mathfrak{m}$-dimensional $\mathbb{R}$-linear subspaces of $\mathbb{R}^{\mathbf{d}}$. Schmidt in [Sch3] proved that the pairs $(V_\Lambda, \mathrm{sh}(\Lambda))$ equidistribute in $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}} \times \mathrm{PSh}_{\mathfrak{m}}$ (towards the natural product measure) as the covolumes of the primitive $\mathfrak{m}$-lattices Λ tend to infinity in average. Without average, stronger equidistribution results have been obtained when fixing the covolume of the primitive $\mathfrak{m}$-lattices and letting it go to ∞ (possibly requiring some congruence properties), see for instance [AES1, AES2, EiRW] when $\mathfrak{m} = 1$, [AEW] when

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$\mathfrak{m} = \mathfrak{n} = 2$, and [Ber]. Schmidt's result in average was strengthen (with an effective version) in [HK2], by adding in the third factor $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}} \times \mathrm{PSh}_{\mathfrak{m}} \times \mathrm{PSh}_{\mathfrak{n}}$ the equidistribution of the shapes of the orthogonal $\mathfrak{n}$-lattices $\Lambda^\perp = V_\Lambda^\perp \cap \mathbb{Z}^D$ of the primitive $\mathfrak{m}$-lattices Λ. Removing the average aspect and under appropriate congruence conditions, this triple equidistribution result has been extended in [AMW].

In this paper, replacing $\mathbb{Q}$ by any global function field over a finite field $\mathbb{F}_q$ (not only the field $\mathbb{F}_q(Y)$ of rational fractions, though already new and interesting), and $\mathbb{Z}$ by the corresponding affine ring for any fixed choice of place at infinity (not only the polynomial ring $\mathbb{F}_q[Y]$), we give the first complete treatment of the triple equidistribution of primitive partial lattices in positive characteristic. Our results are effective and allow versions with congruences. Since this affine ring is no longer principal in general, we will need to adapt our tools. Our result is not a result in average, we will also fix the covolume of the primitive $\mathfrak{m}$-lattices and let it go to ∞ (also requiring an appropriate congruence properties). This is a major extension of the case $\mathbf{d} = 2$ considered in [HP1], since only a double equidistribution makes sense in this dimension, and many moduli spaces constructions and volumes computations where not available. We refer to [HP2] for the consequences of this work purely in terms of equidistribution of rational points in the Grassmannian spaces (and in particular to Frobenius numbers). Class number issues prevent the correspondence between primitive partial lattices and rational points, that was satisfied in the real field case, to take place. An analogous (though different) study of the distribution of the rational points just in the Grassmannian spaces over some function fields had been considered in [Thu].

More precisely, referring to [Gos, Ros] and Subsection 2.1 for definitions and complements, we fix a global function field K of genus $\mathfrak{g}$ over a finite field $\mathbb{F}_q$ of order q, a (discrete normalized) valuation ν of K and a uniformizer π_ν of ν. We denote by ζ_K the Dedekind zeta function of K, by K_ν the associated completion of K, by $\mathscr{O}_\nu$ its valuation ring, by q_ν the order of its residual field, by $|\cdot| = {q_\nu}^{-\nu(\,\cdot\,)}$ its (normalized) absolute value, and by R_ν the affine function ring associated with ν (for instance, $K = \mathbb{F}_q(Y)$, $\mathfrak{g} = 0$, $\nu(P/Q) = \deg Q - \deg P$ for all $P, Q \in \mathbb{F}_q[Y]$, $\mathscr{O}_\nu = \mathbb{F}_q[[Y^{-1}]]$, $q_\nu = q$ and $R_\nu = \mathbb{F}_q[Y]$).

We endow ${K_\nu}^{\mathbf{d}}$ with the supremum norm, and any K_ν-linear subspace of ${K_\nu}^{\mathbf{d}}$ with its induced norm and with its associated normalized Haar measure (giving mass one to its closed unit ball), see Subsection 2.2. A *partial R_ν-lattice* Λ *of rank* $\mathfrak{m}$ (or $\mathfrak{m}$-lattice for short) in ${K_\nu}^{\mathbf{d}}$ is a discrete free R_ν-submodule of rank $\mathfrak{m}$ generating a $\mathfrak{m}$-dimensional K_ν-vector subspace V_Λ of ${K_\nu}^{\mathbf{d}}$. We denote by $\mathrm{Covol}(\Lambda)$ the covolume of Λ in V_Λ. We say that Λ is *primitive* if it is a free direct factor of ${R_\nu}^{\mathbf{d}}$. Among all the definitions of primitiveness that were equivalent for partial $\mathbb{Z}$-lattices in $\mathbb{R}^n$ and non longer are, this turns out to be the appropriate one. We denote by $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}$ the set of primitive $\mathfrak{m}$-lattices in ${K_\nu}^{\mathbf{d}}$.

Our first result is a joint equidistribution result in modular quotients of Bruhat-Tits buildings. For $k = \mathfrak{m}, \mathfrak{n}$, let $\mathscr{I}_{\nu,k}$ be the Bruhat-Tits building of the simple algebraic group PGL_k over the local field K_ν (see for instance [BrT]). Its $\mathrm{PGL}_k(K_\nu)$-homogeneous set of vertices $V\mathscr{I}_{\nu,k}$ is the (discrete) set of ${K_\nu}^\times$-homothety classes $[L]$ of $\mathscr{O}_\nu$-lattices L of ${K_\nu}^k$. The unimodular group $\mathrm{GL}_k^1(K_\nu) = \{g \in \mathrm{GL}_k(K_\nu) : |\det g| = 1\}$ acts projectively on $V\mathscr{I}_{\nu,k}$ with finitely many orbits. We denote by $V_0\mathscr{I}_{\nu,k}$ the orbit by $\mathrm{GL}_k^1(K_\nu)$ of the vertex $[{\mathscr{O}_\nu}^k] \in V\mathscr{I}_{\nu,k}$, identified with $\mathrm{GL}_k^1(K_\nu)/\,\mathrm{GL}_k(\mathscr{O}_\nu)$. The action of the modular group $\widetilde{\Gamma}_k = \mathrm{GL}_k(R_\nu)$ on $V\mathscr{I}_{\nu,k}$ is proper (with finite stabilisers $\widetilde{\Gamma}_{k,x}$ of every vertex $x \in V\mathscr{I}_{\nu,k}$) with (discrete) infinite quotient $\widetilde{\Gamma}_k\backslash V\mathscr{I}_{\nu,k}$. See for instance [Ser2] [BrPP, §15.2] when $k = 2$

and [Sou] when $\mathfrak{g} = 0$ for the structure of the quotient complex of groups (in the sense of [BrH]) $\widetilde{\Gamma}_k \backslash\backslash \mathscr{I}_{\nu,k}$. The measure on $\widetilde{\Gamma}_k \backslash V_0 \mathscr{I}_{\nu,k}$ induced by the counting measure on $V_0 \mathscr{I}_{\nu,k}$ is the finite measure

$$\mu_{\widetilde{\Gamma}_k \backslash V_0 \mathscr{I}_{\nu,k}} = \sum_{[x] \in \widetilde{\Gamma}_k \backslash V_0 \mathscr{I}_{\nu,k}} \frac{1}{\operatorname{Card} \widetilde{\Gamma}_{k,x}} \Delta_{[x]}$$

(see [BaL, §1.5] when $n = 2$). We identify the quotient space $\widetilde{\Gamma}_k \backslash V_0 \mathscr{I}_{\nu,k}$ with the double coset space $\mathrm{Sh}^1_k = \mathrm{GL}_k(\mathscr{O}_\nu) \backslash \mathrm{GL}^1_k(K_\nu) / \widetilde{\Gamma}_k$ by the map induced by $g \mapsto g^{-1}$ on double cosets, that we also denote by $D \mapsto D^{-1}$.

Let $\Lambda \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}$ be such that there exists $i \in \mathbb{Z}$ with $\frac{\operatorname{Covol}(\Lambda)}{\operatorname{Covol}}(R_\nu{}^{\mathfrak{m}}) = q_\nu{}^{\operatorname{lcm}(\mathfrak{m},\mathfrak{n})\, i}$ and $g \in \mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$ sending V_Λ to $K_\nu{}^{\mathfrak{m}} \times \{0\}$ (so that $g\Lambda$ becomes a full R_ν-lattice of $K_\nu{}^{\mathfrak{m}}$). We define (see Equation (27)) the *shape* of Λ as the class of the $\mathfrak{m}$-lattice Λ modulo scaling and the maximal compact subgroup action, that is,

$$\mathrm{sh}(\Lambda) = \mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)\, \pi_\nu{}^{-\frac{\operatorname{lcm}(\mathfrak{m},\mathfrak{n})}{\mathfrak{m}}\, i}\, g\, \Lambda \in \mathrm{Sh}^1_{\mathfrak{m}}\ .$$

Contrarily to the real field case, the rescaling process is much harder when the absolute value is discrete, hence the above restriction on the covolumes. We also define the *orthogonal R_ν-lattice* $\Lambda^\perp$ of the primitive $\mathfrak{m}$-lattice Λ (which is a primitive $\mathfrak{n}$-lattice in the dual space $(K_\nu{}^{\mathbf{d}})^*$ of $K_\nu{}^{\mathbf{d}}$, see Subsection 2.3) by $\Lambda^\perp = V_\Lambda{}^\perp \cap R_\nu{}^{\mathbf{d},*}$, where $V_\Lambda{}^\perp$ is the subspace of $(K_\nu{}^{\mathbf{d}})^*$ consisting in the K_ν-linear forms on $K_\nu{}^{\mathbf{d}}$ vanishing on V_Λ and $R_\nu{}^{\mathbf{d},*}$ is the standard R_ν-lattice in $(K_\nu{}^{\mathbf{d}})^*$ generated by the dual basis of the canonical basis of $K_\nu{}^{\mathbf{d}}$. The following joint equidistribution result of the pairs of shapes of primitive $\mathfrak{m}$-lattices and their orthogonal $\mathfrak{n}$-lattices as their covolume tends to infinity, in the product of the quotients of the Bruhat-Tits buildings $\mathscr{I}_{\nu,\mathfrak{m}}$ and $\mathscr{I}_{\nu,\mathfrak{n}}$ by their modular groups $\widetilde{\Gamma}_{\mathfrak{m}}$ and $\widetilde{\Gamma}_{\mathfrak{n}}$, is a corollary of Theorem 1.2, see the end of Subsection 4.3 for its proof.

Corollary 1.1 *With* $c' = \frac{q^{(\mathfrak{g}-1)(\mathbf{d}^2-\mathbf{d}+1-\mathfrak{m}\,\mathfrak{n})}\,(q_\nu{}^{\mathbf{d}}-1)^2 \prod_{i=2}^{\mathbf{d}}((q_\nu{}^{i-1}-1)\,\zeta_K(i))}{(q-1)\,q_\nu{}^{2\,\mathfrak{m}\,\mathfrak{n}-2}\,(q_\nu-1)^2 \prod_{i=1}^{\mathfrak{m}}(q_\nu{}^i-1)^2 \prod_{i=1}^{\mathfrak{n}}(q_\nu{}^i-1)^2}$, *for the weak-star convergence on the (discrete) locally compact space* $\widetilde{\Gamma}_{\mathfrak{m}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{m}} \times \widetilde{\Gamma}_{\mathfrak{n}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{n}}$, *we have*

$$\lim_{i \to +\infty} \frac{c'}{q_\nu{}^{\operatorname{lcm}(\mathfrak{m},\mathfrak{n})\,\mathbf{d}\,i}} \sum_{\Lambda \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}\,:\, \frac{\operatorname{Covol}(\Lambda)}{\operatorname{Covol}(R_\nu{}^{\mathfrak{m}})} = q_\nu{}^{\operatorname{lcm}(\mathfrak{m},\mathfrak{n})\, i}} \Delta_{\mathrm{sh}(\Lambda)^{-1}} \otimes \Delta_{\mathrm{sh}(\Lambda^\perp)^{-1}} = \mu_{\widetilde{\Gamma}_{\mathfrak{m}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{m}}} \otimes \mu_{\widetilde{\Gamma}_{\mathfrak{n}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{n}}}\ .$$

We refer to Equation (78) for error terms and for versions with congruences of this corollary. The main result of this paper is the following triple joint equidistribution theorem. We endow the unimodular group $\mathrm{GL}^1_{\mathfrak{m}}(K_\nu)$ with its Haar measure giving mass 1 to its maximal compact subgroup $\mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)$ and the (discrete, infinite) double quotient $\mathrm{Sh}^1_{\mathfrak{m}}$ with its induced measure $\mu_{\mathrm{Sh}^1_{\mathfrak{m}}}$, which is finite (see Subsection 3.4). We denote by $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ the compact Grassmannian space of $\mathfrak{m}$-dimensional K_ν-linear subspaces of $K_\nu{}^{\mathbf{d}}$, and by $\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}$ its $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$-invariant probability measure.

Theorem 1.2 *With* $c'' = \frac{q^{\,(\mathfrak{g}-1)(\mathbf{d}^2-\mathbf{d}+1-\mathfrak{m}\,\mathfrak{n})}\,({q_\nu}^{\mathbf{d}}-1)^2\prod_{i=2}^{\mathbf{d}}(({q_\nu}^{i-1}-1)\,\zeta_K(i))}{(q-1)\,{q_\nu}^{2\,\mathfrak{m}\,\mathfrak{n}}\prod_{i=1}^{\mathfrak{m}}({q_\nu}^{i}-1)^2\prod_{i=1}^{\mathfrak{n}}({q_\nu}^{i}-1)^2}$*, for the weak-star convergence of Borel measures on the locally compact space* $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}\times\mathrm{Sh}^1_{\mathfrak{m}}\times\mathrm{Sh}^1_{\mathfrak{n}}$*, we have*

$$\lim_{i\to+\infty}\ \frac{c''}{{q_\nu}^{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}\,\mathbf{d}\,i}}\sum_{\Lambda\in\mathscr{PL}_{\mathfrak{m},\mathbf{d}}\,:\,\frac{\mathrm{Covol}(\Lambda)}{\mathrm{Covol}({R_\nu}^{\mathfrak{m}})}={q_\nu}^{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}\,i}}\Delta_{V_\Lambda}\otimes\Delta_{\mathrm{sh}(\Lambda)}\otimes\Delta_{\mathrm{sh}(\Lambda^\perp)}$$
$$=\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}\otimes\mu_{\mathrm{Sh}^1_{\mathfrak{m}}}\otimes\mu_{\mathrm{Sh}^1_{\mathfrak{n}}}\ . \qquad (1)$$

We refer to Corollary 4.8 for error terms and for versions with congruences of this theorem. We will actually prove in Theorem 4.7 a much stronger (albeit more technical) equidistribution result. We will define in Subsection 3.3 a (non discrete) moduli space $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ of pairs $(L,L')\in\mathrm{GL}^1_{\mathfrak{m}}(K_\nu)/\mathrm{GL}_{\mathfrak{m}}(R_\nu)\times\mathrm{GL}^1_{\mathfrak{n}}(K_\nu)/\mathrm{GL}_{\mathfrak{n}}(R_\nu)$ of unimodular $\mathfrak{m}$- and $\mathfrak{n}$-lattices with an appropriate correlation on the determinant of any of their R_ν-basis. Using a refined LU decomposition of elements of $\mathrm{SL}_{\mathbf{d}}(K_\nu)$ introduced in Subsection 2.5, we will associate such a pair $[\![\Lambda]\!]\in\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ to any primitive $\mathfrak{m}$-lattice Λ, under a restriction that the linear subspace V_Λ of the primitive $\mathfrak{m}$-lattices Λ belongs to the unit ball $\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$ of the lower maximal Bruhat cell of the Grassmannian space $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ (see Subsection 3.1 for more details). We will then prove in Theorem 4.7 the equidistribution of the pairs $(V_\Lambda,[\![\Lambda]\!])$ in $\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}\times\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ for the primitive $\mathfrak{m}$-lattices Λ whose covolume is fixed (satisfying some congruence property) and tends to $+\infty$. Theorem 1.2 will follow by a tricky consideration of compound matrices.

We refer to Subsection 2.4 for the description of the appropriate congruence subgroup of $\mathrm{SL}_{\mathbf{d}}(K_\nu)$ that we will use for the version with congruences of our theorems. A major part of the paper consists of a fine study of the homogeneous measures on the various homogeneous spaces $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ (see Subsection 3.1), $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ (see Subsections 3.2 and 3.3), and on the double coset spaces Sh^1_k for $k=\mathfrak{m},\mathfrak{n}$ (see Subsection 3.4), besides the precise disintegration of the Haar measure of $\mathrm{SL}_{\mathbf{d}}(K_\nu)$ by the refined LU decomposition in Subsection 2.5. A key tool of this paper is the counting result in well-rounded sets of integral points of algebraic groups over K by Gorodnik and Nevo [GN]. A long study is necessary in order to introduce the appropriate well-rounded sets, to prove that they are indeed well-rounded, and to compute their measures: see Subsection 4.1 which gives a precise relationship between primitive $\mathfrak{m}$-lattices in ${K_\nu}^{\mathbf{d}}$ and integral matrices in $\mathrm{SL}_{\mathbf{d}}(R_\nu)$, and Subsection 4.2.

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2 Background definitions and notation

2.1 On global function fields

We refer for instance to [Gos, Ros] and [BrPP, Chap. 14] for the content of this Section.

Let $\mathbb{F}_q$ be a finite field of order q, where q is a positive power of a positive prime. Let K be a (global) *function field* over $\mathbb{F}_q$, that is, the function field of a geometrically connected smooth projective curve $\mathbf{C}$ over $\mathbb{F}_q$, or equivalently an extension of $\mathbb{F}_q$ of transcendence degree 1, in which $\mathbb{F}_q$ is algebraically closed. We denote by $\mathbf{g}$ the genus of the curve $\mathbf{C}$.

There is a bijection between the set of closed points of $\mathbf{C}$ and the set of (normalized discrete) valuations ν of its function field K, where the valuation of a given element $f\in K$

is the order of the zero or the opposite of the order of the pole of f at the given closed point. We fix such a valuation ν from now on.

We denote by K_ν the completion of K for the valuation ν, and by

$$\mathscr{O}_\nu = \{x \in K_\nu \ : \ \nu(x) \geqslant 0\}$$

the valuation ring of (the unique extension to K_ν) of ν. Let us fix a uniformiser $\pi_\nu \in K$ of ν, that is, an element in K with $\nu(\pi_\nu) = 1$. We denote by q_ν the order of the residual field $\mathscr{O}_\nu/\pi_\nu\mathscr{O}_\nu$ of ν, which is a (possibly proper) power of q. We normalize the absolute value associated with ν as usual: for every $x \in K_\nu$, we have the equality

$$|\,x\,| = q_\nu^{-\nu(x)} \ .$$

Finally, let R_ν denote the affine algebra of the affine curve $\mathbf{C} - \{\nu\}$, consisting of the elements of K whose only poles (if any) are at the closed point ν of $\mathbf{C}$. It is a Dedeking ring and its field of fractions is equal to K. Note that (see for instance [BrPP, Eq. (14.2) and (14.3)]

$$R_\nu \cap \mathscr{O}_\nu = \mathbb{F}_q \quad \text{and} \quad R_\nu^{\,\times} = \mathbb{F}_q^{\,\times} \subset \mathscr{O}_\nu^{\,\times} \ . \tag{2}$$

The *Dedekind zeta function* of K is (see for instance [Ros, §5]) defined if Re $s > 1$ by

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s} \ ,$$

where the summation is over the nonzero ideals I of R_ν, with norm $N(I) = [R_\nu : I]$. By [Ros, Theo. 5.9]), it is a rational function of q^{-s} and has an analytic continuation on $\mathbb{C}\smallsetminus\{0, 1\}$ with simple poles at $s = 0$, $s = 1$. Furthermore, it has positive values at $s = -i$ for all $i \in \mathbb{N}\smallsetminus\{0\}$, since by the functional equation of ζ_K (see loc. cit.), we have

$$\zeta_K(-i) = q^{(\mathfrak{g}-1)(1+2\,i)}\zeta_K(1 + i) > 0 \ . \tag{3}$$

The simplest example corresponds to $\mathbf{C} = \mathbb{P}^1$ (so that $\mathbf{g} = 0$) and $\nu = \nu_\infty$ the valuation associated with the point at infinity $[1 : 0]$. Then

- $K = \mathbb{F}_q(Y)$ is the field of rational functions in one variable Y over $\mathbb{F}_q$,
- ν_∞ is the valuation defined, for all $P, Q \in \mathbb{F}_q[Y]$, by

$$\nu_\infty(P/Q) = \deg Q - \deg P \ .$$

- $R_{\nu_\infty} = \mathbb{F}_q[Y]$ is the (principal) ring of polynomials in one variable Y over $\mathbb{F}_q$,
- $K_{\nu_\infty} = \mathbb{F}_q((Y^{-1}))$ is the field of formal Laurent series in one variable Y^{-1} over $\mathbb{F}_q$,
- $\mathscr{O}_{\nu_\infty} = \mathbb{F}_q[[Y^{-1}]]$ is the ring of formal power series in one variable Y^{-1} over $\mathbb{F}_q$, $\pi_{\nu_\infty} = Y^{-1}$ is the usual choice of a uniformizer, and $q_{\nu_\infty} = q$.

2.2 Partial lattices

Let V be a K_ν-vector space with finite dimension $D \geqslant 1$ endowed with an ultrametric norm $\|\ \|$, and let $k \in [\![1, D]\!]$. We denote by $B_V(0, 1)$ the closed unit ball of V. We endow V with the unique Haar measure μ_V of the abelian locally compact topological group $(V, +)$ such that $\mu_V(B_V(0, 1)) = 1$. This measure scales as follows under linear maps: for all $x \in V$ and $g \in \mathrm{GL}(V)$, we have

$$d\mu_V(gx) = |\det g|\ d\mu_V(x) \ . \tag{4}$$

When $V = K_\nu^{\mathbf{d}}$ with its canonical basis $(e_1, \ldots, e_{\mathbf{d}})$, we will take the supremum norm $\|\lambda_1 e_1 + \cdots + \lambda_{\mathbf{d}} e_{\mathbf{d}}\| = \max_{1 \leqslant i \leqslant \mathbf{d}} |\lambda_i|$. The Haar measure of $K_\nu^{\mathbf{d}}$ is then normalized so that $\mu_{K_\nu^{\mathbf{d}}}(\mathscr{O}_\nu^{\mathbf{d}}) = 1$. On the dual K_ν-vector space V^*, we will consider the dual norm. When $V = K_\nu^{\mathbf{d}}$, the dual norm on V^* is then the supremum norm with respect to the dual basis $(e_1^*, \ldots, e_{\mathbf{d}}^*)$ of $(e_1, \ldots, e_{\mathbf{d}})$.

Recall that for every $g \in \mathrm{GL}(V)$, its (left) action $\check{g} : \ell \mapsto \ell \circ g^{-1}$ on the dual space V^* satisfies, for every K_ν-basis $\mathscr{B}$ of V with dual K_ν-basis $\mathscr{B}^*$ of V^*, that

$$\mathrm{Mat}_{\mathscr{B}^*}(\check{g}) = {}^t\,\mathrm{Mat}_{\mathscr{B}}(g)^{-1} . \tag{5}$$

For every K_ν-vector subspace W of V, its *orthogonal space* is the K_ν-vector subspace $W^\perp$ of the dual K_ν-vector space V^* defined by

$$W^\perp = \{\ell \in V^* : \forall\, x \in W,\ \ell(x) = 0\} .$$

It is well-known that $\dim(W^\perp) = D - \dim(W)$, that $(W^\perp)^\perp = W$ and that for every $g \in \mathrm{GL}(V)$, we have $(gW)^\perp = \check{g}\,(W^\perp)$.

A *partial R_ν-lattice* Λ *of rank* k in V, or *k-lattice* for short, is a discrete free R_ν-submodule of rank k generating a k-dimensional K_ν-vector subspace V_Λ of V. When $k = D$, we say that Λ is *full R_ν-lattice*. We endow V_Λ with the restriction of the norm of V, hence with its unique Haar measure μ_{V_Λ} such that $\mu_{V_\Lambda}(B_V(0,1) \cap V_\Lambda) = 1$. We define the *covolume* $\mathrm{Covol}(\Lambda)$ of Λ as the total mass of the induced measure (again denoted by μ_{V_Λ}) on the quotient space V_Λ/Λ, that is,

$$\mathrm{Covol}(\Lambda) = \mu_{V_\Lambda}(V_\Lambda/\Lambda) . \tag{6}$$

The set $\mathrm{Lat}_k(V)$ of k-lattices in V is invariant under the linear action of the linear group $\mathrm{GL}(V)$. This action of $\mathrm{GL}(V)$ on $\mathrm{Lat}_k(V)$ is transitive, by taking an R_ν-basis in two k-lattices, by completing them to two K_ν-basis $\mathscr{B}$ and $\mathscr{B}'$ of V, and by taking the K_ν-linear map sending $\mathscr{B}$ to $\mathscr{B}'$. For all $g \in \mathrm{GL}(V)$ and $\Lambda \in \mathrm{Lat}_k(V)$, we have

$$V_{g\Lambda} = gV_\Lambda \quad \text{and} \quad \mathrm{Covol}(g\Lambda) = \frac{dg_*\mu_{V_\Lambda}}{d\mu_{V_{g\Lambda}}}\ \mathrm{Covol}(\Lambda) . \tag{7}$$

In particular, for every $\lambda \in K_\nu$, we have $\mathrm{Covol}(\lambda\,\Lambda) = |\lambda|^k\,\mathrm{Covol}(\Lambda)$ and if $k = D$, then

$$\mathrm{Covol}(g\Lambda) = |\det g|\ \mathrm{Covol}(\Lambda) . \tag{8}$$

An *integral structure* (or *R_ν-structure*) on V is the choice of a full R_ν-lattice in V. Alternatively, it is the choice of an equivalence class of K_ν-basis of V, where two K_ν-bases are equivalent if their transition matrix belongs to $\mathrm{GL}_D(R_\nu)$. These two definitions agree by identifying the equivalence class of a K_ν-basis $(b_1, \ldots, b_D)$ with the R_ν-lattice $R_\nu\, b_1 + \cdots + R_\nu\, b_D$ it generates. An *integral K_ν-space* is a finite dimensional K_ν-vector space W endowed with an integral structure, denoted by W_{R_ν}. We denote by $\mathrm{GL}(W_{R_\nu})$ the subgroup of $\mathrm{GL}(W)$ preserving the integral structure W_{R_ν} of W. The dual K_ν-vector space W^* will be endowed with the *dual* integral structure (see the appendix A for developments), denoted by $W^*_{R_\nu}$ and defined by

$$W^*_{R_\nu} = \{\ell \in W^* : \forall\, x \in W_{R_\nu},\ \ \ell(x) \in R_\nu\} .$$

Equivalently, $W^*_{R_\nu}$ is the integral structure on W^* whose equivalence class of R_ν-bases is the set of the dual K_ν-bases of the elements in the equivalence class of the R_ν-bases for W_{R_ν}. This is well defined by Equation (5) since $\mathrm{GL}_{\mathbf{d}}(R_\nu)$ is stable by inversion and transposition. Note that $W^{**}_{R_\nu} = W_{R_\nu}$.

For instance, we will endow the product K_ν-vector space $K_\nu^{\mathbf{d}}$ with its integral structure $R_\nu^{\mathbf{d}}$ (or equivalently with the equivalence class of its canonical basis $(e_1, \dots, e_{\mathbf{d}})$). By for instance [BrPP, Lem. 14.4)], we have

$$\mathrm{Covol}(R_\nu^{\mathbf{d}}) = (\mathrm{Covol}(R_\nu))^{\mathbf{d}} = q^{(\mathfrak{g}-1)\mathbf{d}} . \tag{9}$$

We will endow the dual K_ν-vector space $(K_\nu^{\mathbf{d}})^*$ with the equivalence class of the dual basis $(e_1^*, \dots, e_{\mathbf{d}}^*)$ of $(e_1, \dots, e_{\mathbf{d}})$ (or with the full R_ν-lattice $R_\nu\, e_1^* + \dots + R_\nu\, e_{\mathbf{d}}^*$). For every k-lattice Λ in V, the pair (V_Λ, Λ) is an integral K_ν-space with $(V_\Lambda)_{R_\nu} = \Lambda$.

Since the standard R_ν-lattice $R_\nu^{\mathbf{d}}$ in $K_\nu^{\mathbf{d}}$ does not have covolume 1 (contrarily to the case of the real field), we define the *normalized covolume* of an R_ν-lattice Λ in V by

$$\overline{\mathrm{Covol}}(\Lambda) = \frac{\mathrm{Covol}(\Lambda)}{\mathrm{Covol}(R_\nu^{\mathbf{d}})} .$$

Let V be an integral K_ν-space with finite dimension D, and $k \in [\![1, D]\!]$. A k-lattice in V is

- *unimodular* if its normalized covolume $\overline{\mathrm{Covol}}(\Lambda)$ is equal to 1;
- *rational* if it is contained in the K-vector space $V_K = V_{R_\nu} \otimes K$ generated by the integral structure V_{R_ν} of V ;
- *integral* if it is contained in V_{R_ν};
- *primitive* if it is integral and satisfies one of the following equivalent properties:

(1) the R_ν-module Λ is a free direct factor of V_{R_ν} (or equivalently, there exists an R_ν-basis $(b_1 \dots, b_D)$ of V_{R_ν} such that $(b_1 \dots, b_k)$ is an R_ν-basis of Λ),

(2) the R_ν-module V_{R_ν}/Λ is a free R_ν-module of rank $D - k$.

Note that this definition is appropriate in the setting where R_ν is not necessarily principal, and that definitions that were equivalent in the case of $(\mathbb{R}, \mathbb{Q}, \mathbb{Z})$ instead of (K_ν, K, R_ν) no longer are. For instance, if Λ is a primitive k-lattice, then V_Λ determines Λ, with

$$\Lambda = V_\Lambda \cap V_{R_\nu} .$$

But this equality is no longer sufficient for an integral k-lattice Λ to be primitive.

Note that an integral k-lattice is a rational k-lattice. By taking an R_ν-basis in two rational k-lattices, by completing them to two K-bases $\mathscr{B}$ and $\mathscr{B}'$ of V_K, and by taking the K-linear map sending $\mathscr{B}$ to $\mathscr{B}'$, we see that the linear group $\mathrm{GL}(V_K)$ acts transitively on the set of rational k-lattices in V.

Let $\mathrm{Lat}^1(V)$ be the space of unimodular full R_ν-lattices in V. The closed unimodular subgroup

$$\mathrm{GL}^1(V) = \{g \in \mathrm{GL}(V) : |\det g\,| = 1\} \tag{10}$$

acts transitively on the set of (partial) k-lattices if $k < D$. It also acts transitively on $\mathrm{Lat}^1(V)$ (the determinant of every element $g \in \mathrm{GL}(V)$ mapping a unimodular full R_ν-lattice to another one has absolute value 1 by Equation (8)). Note that the discrete group $\mathrm{GL}(V_{R_\nu})$ is contained in $\mathrm{GL}^1(V)$, and is exactly the stabilizer in $\mathrm{GL}^1(V)$

of the full R_ν-lattice V_{R_ν}. We hence identify from now on the set $\mathrm{GL}^1(V)/\mathrm{GL}(V_{R_\nu})$ with $\mathrm{Lat}^1(V)$ by the map $g\,\mathrm{GL}(V_{R_\nu}) \mapsto gV_{R_\nu}$. In particular, we identify $\mathrm{Lat}^1_D = \mathrm{Lat}^1({K_\nu}^D)$ with $\mathrm{GL}^1_D(K_\nu)/\mathrm{GL}_D(R_\nu)$ (by taking the matrix of a linear automorphism of ${K_\nu}^D$ in the canonical basis of ${K_\nu}^D$).

Let $\mathscr{PL}_k(V)$ be the set of primitive k-lattices in V, and $\mathscr{PL}_{\mathfrak{m},\mathbf{d}} = \mathscr{PL}_{\mathfrak{m}}({K_\nu}^{\mathbf{d}})$. This set can be described as a (discrete) homogeneous space as follows. For every commutative ring A and for all $k, k' \in \mathbb{N} \smallsetminus \{0\}$, we denote by $\mathscr{M}_{k,k'}(A)$ (and by $\mathscr{M}_k(A)$ when $k = k'$) the A-module of $k \times k'$ matrices with coefficients in A. Let

$$P^+(R_\nu) = \Big\{ \big(\begin{smallmatrix} \alpha & \gamma \\ 0 & \delta \end{smallmatrix}\big) : \alpha \in \mathrm{GL}_{\mathfrak{m}}(R_\nu),\ \delta \in \mathrm{GL}_{\mathfrak{n}}(R_\nu),\ \gamma \in \mathscr{M}_{\mathfrak{m},\mathfrak{n}}(R_\nu),\ \det(\alpha)\det(\delta) = 1 \Big\} . \tag{11}$$

Lemma 2.1 *The group $\Gamma = \mathrm{SL}_{\mathbf{d}}(R_\nu)$ acts transitively on $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}$.*

The validity of this lemma is one of the main reasons for our definition of primitive $\mathfrak{m}$-lattices, and could be no longer true with other definitions. Since the stabilizer in $\mathrm{SL}_{\mathbf{d}}(R_\nu)$ of the first coordinates primitive $\mathfrak{m}$-lattice $R_\nu^{\mathfrak{m}} \times \{0\}$ is equal to $P^+(R_\nu)$, we will from now on, as we may, identify the quotient $\Gamma/P^+(R_\nu)$ and $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}$ by the map

$$gP^+(R_\nu) \mapsto \Lambda_g = g(R_\nu^{\mathfrak{m}} \times \{0\}) .$$

Proof. Let $\Lambda \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}$. By the definition of a primitive $\mathfrak{m}$-lattice in ${K_\nu}^{\mathbf{d}}$, there exists an R_ν-basis $(b_1, \dots, b_{\mathbf{d}})$ of $R_\nu^{\mathbf{d}}$ such that $(b_1, \dots, b_{\mathfrak{m}})$ is an R_ν-basis of Λ. Let g be the transition matrix from the canonical basis of $R_\nu^{\mathbf{d}}$ to $(b_1, \dots, b_{\mathbf{d}})$. Note that a priori $g \in \mathrm{GL}_{\mathbf{d}}(R_\nu)$, but since $\det g \in {R_\nu}^\times$, up to replacing $b_{\mathbf{d}}$ by $\lambda\, b_{\mathbf{d}}$ for some $\lambda \in {R_\nu}^\times$, which does not change Λ_g since $\mathfrak{m} < \mathbf{d}$, we may assume that $g \in \mathrm{SL}_{\mathbf{d}}(R_\nu)$. Then $\Lambda_g = \Lambda$, and the map $g \mapsto \Lambda_g$ is indeed onto. □

2.3 Orthogonal primitive partial lattices

Let V be an integral K_ν-space with finite dimension D, and let $k \in [\![1, D-1]\!]$. Let Λ be a primitive k-lattice in V.

The *orthogonal $(D-k)$-lattice* of Λ is the R_ν-submodule of the dual integral K_ν-vector space V^* defined by

$$\Lambda^\perp = {V_\Lambda}^\perp \cap V^*_{R_\nu} .$$

For instance, if V is ${K_\nu}^{\mathbf{d}}$ with its canonical basis $(e_1, \dots, e_{\mathbf{d}})$ (defining its integral structure) and its dual basis $(e_1^*, \dots, e_{\mathbf{d}}^*)$, if $\Lambda = \oplus_{1 \leqslant i \leqslant \mathfrak{m}} R_\nu e_i$, that we have already denoted by $R_\nu^{\mathfrak{m}} \times \{0\}$, then $\Lambda^\perp = \oplus_{\mathfrak{m}+1 \leqslant i \leqslant \mathbf{d}} R_\nu e_i^*$, that we will also denote by $\{0\} \times R_\nu^{\mathfrak{n}}$.

Proposition 2.2 *The R_ν-submodule $\Lambda^\perp$ of V^* is a primitive $(D-k)$-lattice in V^*. For every $g \in \mathrm{GL}(V_{R_\nu})$, we have*

$$V_{\Lambda^\perp} = (V_\Lambda)^\perp, \qquad \Lambda = (\Lambda^\perp)^\perp \qquad \textit{and} \qquad (g\Lambda)^\perp = \check{g}\, \Lambda^\perp . \tag{12}$$

Furthermore, if we endow V with the supremum norm associated with any R_ν-basis of V_{R_ν} and V^* with its dual norm, then Λ and $\Lambda^\perp$ have the same normalized covolume:

$$\overline{\mathrm{Covol}}(\Lambda^\perp) = \frac{\mathrm{Covol}(\Lambda^\perp)}{\mathrm{Covol}({R_\nu}^{D-k})} = \frac{\mathrm{Covol}(\Lambda)}{\mathrm{Covol}({R_\nu}^{k})} = \overline{\mathrm{Covol}}(\Lambda) . \tag{13}$$

Proof. Since the k-lattice Λ is primitive, there exists an R_ν-basis $(b_1, \ldots, b_D)$ of V_{R_ν} such that $(b_1, \ldots, b_k)$ is an R_ν-basis of Λ. The dual K_ν-basis $(b_1^*, \ldots, b_D^*)$ of $(b_1, \ldots, b_D)$ is also an R_ν-basis of the integral structure $V^*_{R_\nu}$ of V^*. We have $V_\Lambda = \oplus_{1 \leqslant i \leqslant k} K_\nu b_i$, hence ${V_\Lambda}^\perp = \oplus_{k+1 \leqslant i \leqslant D} K_\nu b_i^*$. Therefore $\Lambda^\perp = {V_\Lambda}^\perp \cap V^*_{R_\nu} = \oplus_{k+1 \leqslant i \leqslant D} R_\nu b_i^*$ is an integral and primitive $(D-k)$-lattice in V^*. Moreover, we have $V_{\Lambda^\perp} = \oplus_{k+1 \leqslant i \leqslant D} K_\nu b_i^* = {V_\Lambda}^\perp$, and $(\Lambda^\perp)^\perp = \oplus_{1 \leqslant i \leqslant k} R_\nu b_i = \Lambda$.

For every $g \in \mathrm{GL}(V)$, we have

$$(g\Lambda)^\perp = (V_{g\Lambda})^\perp \cap V^*_{R_\nu} = (gV_\Lambda)^\perp \cap V^*_{R_\nu} = \check{g}\,(V_\Lambda)^\perp \cap V^*_{R_\nu} \;.$$

In particular, if $g \in \mathrm{GL}(V_{R_\nu})$, then $\check{g} \in \mathrm{GL}(V^*_{R_\nu})$ and we do have $(g\Lambda)^\perp = \check{g}\,\Lambda^\perp$.

For a proof of Equation (13), we refer to the end of the appendix A. □

2.4 Congruence properties on primitive partial lattices

In this subsection, we fix a nonzero ideal I of the Dedekind ring R_ν, and we define a class of primitive partial lattices in $K_\nu^{\mathbf{d}}$ that have specific congruence properties modulo the ideal I.

A primitive partial lattice Λ in $K_\nu^{\mathbf{d}}$ is said to be *horizontal modulo* I if $\Lambda \subset R_\nu^{\mathfrak{m}} \times I^{\mathfrak{n}}$, as for instance $R_\nu^{\mathfrak{m}} \times \{0\}$. We will denote by $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)$ the set of primitive partial lattices in $K_\nu^{\mathbf{d}}$ that are horizontal modulo I. If $I = R_\nu$, then $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I) = \mathscr{PL}_{\mathfrak{m},\mathbf{d}}$.

Let $\Gamma = \mathrm{SL}_{\mathbf{d}}(R_\nu)$. We consider the following *Hecke congruence subgroup by blocks* :

$$\Gamma_I = \{\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in \Gamma : \;\; \beta \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(I)\} \;.$$

Note that $\Gamma_{R_\nu} = \Gamma$ and $P^+(R_\nu) \subset \Gamma_I$ where $P^+(R_\nu)$ is defined in Equation (11).

The first assertion of the following lemma is a congruence version of Lemma 2.1, and implies that the map $gP^+(R_\nu) \mapsto \Lambda_g = g(R_\nu^{\mathfrak{m}} \times \{0\})$ for $g \in \Gamma_I$ identifies $\Gamma_I / P^+(R_\nu)$ with $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)$. The second one is exactly [BrPP, Lem. 16.5] when $\mathbf{d} = 2$ and $\mathfrak{m} = 1$.

Lemma 2.3 *(1) The group Γ_I acts transitively on $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)$. Furthermore, for every $g \in \Gamma$, we have $\Lambda_g = g(R_\nu^{\mathfrak{m}} \times \{0\}) \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)$ if and only if $g \in \Gamma_I$.*

(2) We have

$$[\Gamma : \Gamma_I] = N(I)^{\mathfrak{m}\,\mathfrak{n}} \prod_{\mathfrak{p}\,|\,I} \prod_{i=1}^{\mathfrak{m}} \frac{N(\mathfrak{p})^i - N(\mathfrak{p})^{-\mathfrak{n}}}{N(\mathfrak{p})^i - 1} \;,$$

where the first product ranges over the prime factors $\mathfrak{p}$ of the ideal I.

Proof. (1) Let $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in \Gamma$ with β an $\mathfrak{n} \times \mathfrak{m}$ matrix, so that

$$\Lambda_g = g(R_\nu^{\mathfrak{m}} \times \{0\}) = \{(\alpha x, \beta x) : x \in R_\nu^{\mathfrak{m}}\} \;.$$

Then $\Lambda_g \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)$ if and only if $\beta x \in I^{\mathfrak{n}}$ for every $x \in R_\nu^{\mathfrak{m}}$, which occurs if and only if $\beta \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(I)$, that is, when $g \in \Gamma_I$.

(2) We denote by $|E|$ the cardinality of a finite set E. For every commutative ring A with finite group of invertible elements $A^\times$ and for every $\ell \in \mathbb{N} \smallsetminus \{0\}$, it is well known that $[\mathrm{GL}_\ell(A) : \mathrm{SL}_\ell(A)] = |A^\times|$ and that if A is a finite field, then

$$|\,\mathrm{GL}_\ell(A)| = |A|^{\frac{\ell(\ell-1)}{2}} \prod_{i=1}^{\ell} (|A|^i - 1) \;.$$

The group morphism of reduction modulo I from $\mathrm{SL}_{\mathfrak{m}}(R_\nu)$ to $\mathrm{SL}_{\mathfrak{m}}(R_\nu/I)$ is onto, by an argument of further reduction to the various prime power factors of I and of lifting elementary matrices. The order of the upper triangular subgroup by blocks

$$T_{\mathfrak{m},\mathbf{d}}(I) = \{\left(\begin{smallmatrix}\alpha & \gamma\\ 0 & \delta\end{smallmatrix}\right) \in \mathrm{SL}_{\mathbf{d}}(R_\nu/I) : \quad \alpha \in \mathscr{M}_{\mathfrak{m}}(R_\nu/I)\}$$

of $\mathrm{SL}_{\mathbf{d}}(R_\nu/I)$ is $|(R_\nu/I)^\times|^{-1}|\,\mathrm{GL}_{\mathfrak{m}}(R_\nu/I)|\,|\,\mathrm{GL}_{\mathfrak{n}}(R_\nu/I)|\,|R_\nu/I|^{\mathfrak{m}\,\mathfrak{n}}$. Hence

$$[\Gamma : \Gamma_I] = \frac{|\,\mathrm{SL}_{\mathbf{d}}(R_\nu/I)|}{|T_{\mathfrak{m},\mathbf{d}}(I)|} = \frac{|\,\mathrm{GL}_{\mathbf{d}}(R_\nu/I)|}{|\,\mathrm{GL}_{\mathfrak{m}}(R_\nu/I)|\,|\,\mathrm{GL}_{\mathfrak{n}}(R_\nu/I)|\,|R_\nu/I|^{\mathfrak{m}\,\mathfrak{n}}}\,. \tag{14}$$

By the multiplicativity of the norm and by the Chinese remainder theorem, the result reduces to the case when $I = \mathfrak{p}^k$ is the k-th power of a fixed prime ideal $\mathfrak{p}$, where $k \in \mathbb{N}$. Let $N = N(\mathfrak{p})$ so that $N(I) = |R_\nu/I| = N^k$, and note that $R_\nu/\mathfrak{p}$ is a field of order N. For every $\ell \in \mathbb{N} \smallsetminus \{0\}$, the kernel of the morphism of reduction modulo $\mathfrak{p}$ from $\mathrm{GL}_\ell(R_\nu/\mathfrak{p}^k)$ to $\mathrm{GL}_\ell(R_\nu/\mathfrak{p})$ has order $N^{\ell^2(k-1)}$. Hence

$$|\,\mathrm{GL}_\ell(R_\nu/I)| = N^{\ell^2(k-1)+\frac{\ell(\ell-1)}{2}} \prod_{i=1}^{\ell}(N^i - 1)\,.$$

Therefore, by Equation (14), we have after simplifications

$$[\Gamma : \Gamma_I] = N^{\mathfrak{m}\,\mathfrak{n}(k-1)} \prod_{i=1}^{\mathfrak{m}} \frac{N^{i+\mathfrak{n}} - 1}{N^i - 1} = N(I)^{\mathfrak{m}\,\mathfrak{n}} \prod_{i=1}^{\mathfrak{m}} \frac{N(\mathfrak{p})^i - N(\mathfrak{p})^{-\mathfrak{n}}}{N(\mathfrak{p})^i - 1}\,.$$

This proves the result. □

2.5 Refined LU decomposition by blocks

Let $G = \mathrm{SL}_{\mathbf{d}}(K_\nu)$, which is a unimodular totally disconnected locally compact topological group. In this subsection, we define some closed subgroups of G and we study their Haar measures. We will denote an element $g \in G$ by blocks as $g = \left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$ with α an $\mathfrak{m} \times \mathfrak{m}$ matrix. For every $k \in \mathbb{N} \smallsetminus \{0\}$, let I_k be the identity $k \times k$ matrix.

We will consider throughout this paper the following subgroups of G. Let

$$U^- = \left\{\left(\begin{smallmatrix}I_{\mathfrak{m}} & 0\\ \beta & I_{\mathfrak{n}}\end{smallmatrix}\right) : \beta \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu)\right\} \quad \text{and} \quad U^+ = \left\{\left(\begin{smallmatrix}I_{\mathfrak{m}} & \gamma\\ 0 & I_{\mathfrak{n}}\end{smallmatrix}\right) : \gamma \in \mathscr{M}_{\mathfrak{m},\mathfrak{n}}(K_\nu)\right\}$$

be the lower and upper unipotent triangular subgroups by blocks of the matrix group G. For every $k \in \mathbb{N} \smallsetminus \{0\}$, we define $\mathrm{GL}^1_k(K_\nu) = \{g \in \mathrm{GL}_k(K_\nu) : |\det g| = 1\}$, which is a split extension of its normal closed subgroup $\mathrm{SL}_k(K_\nu)$ by the compact group $\{\left(\begin{smallmatrix}a & 0\\ 0 & I_{k-1}\end{smallmatrix}\right) : a \in \mathscr{O}_\nu{}^\times\}$. Let

$$G'' = \left\{\left(\begin{smallmatrix}\alpha & 0\\ 0 & \delta\end{smallmatrix}\right) : \alpha \in \mathrm{GL}^1_{\mathfrak{m}}(K_\nu),\ \delta \in \mathrm{GL}^1_{\mathfrak{n}}(K_\nu),\ \ \det\alpha\ \det\delta = 1\right\}$$

be the intersection with G of the product group $\mathrm{GL}^1_{\mathfrak{m}}(K_\nu) \times \mathrm{GL}^1_{\mathfrak{n}}(K_\nu)$ diagonally embedded by blocks in $\mathrm{GL}_{\mathbf{d}}(K_\nu)$. Note that U^-, G'', U^+ are closed unimodular subgroups of G, and that G'' normalizes U^- and U^+. Let

$$Z = \left\{\left(\begin{smallmatrix}\pi_\nu{}^r I_{\mathfrak{m}} & 0\\ 0 & \pi_\nu{}^s I_{\mathfrak{n}}\end{smallmatrix}\right) : r, s \in \mathbb{Z},\ \mathfrak{m}r + \mathfrak{n}s = 0\right\}\,,$$

which is a discrete abelian subgroup of G that centralises G'' (actually G'' is the centralizer of Z in G), and normalizes U^- and U^+.

Let $Z' = \left\{ \begin{pmatrix} {\pi_\nu}^i & 0 & 0 \\ 0 & I_{\mathbf{d}-2} & 0 \\ 0 & 0 & {\pi_\nu}^{-i} \end{pmatrix} : 0 \leqslant i \leqslant \operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\} - 1 \right\}$ which is a finite subset of order $\operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\}$ of G (not a subgroup). Let

$$\mathscr{U}_G = \left\{ \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in G : \nu(\det(\alpha)) \in \operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\}\, \mathbb{Z} \right\} \tag{15}$$

and $\mathscr{U}_G^0 = \left\{ \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in G : \det(\alpha) = 0 \right\}$, which are disjoint closed subsets of G (not subgroups), with $\mathscr{U}_G$ open in G, such that

$$G = \mathscr{U}_G^0 \sqcup \bigsqcup_{z' \in Z'} z'\, \mathscr{U}_G = \mathscr{U}_G^0 \sqcup \bigsqcup_{z' \in Z'} \mathscr{U}_G\, z'$$

is a finite disjoint union of $\mathscr{U}_G^0$ and finitely many left (or right) translates of $\mathscr{U}_G$. Let $\mathscr{S}_{\mathbf{d}}^{\pm}$ be the subgroup of G consisting in the elements of G that act by a permutation and a possible change of sign on the elements of the canonical basis of ${K_\nu}^{\mathbf{d}}$ (in order for their determinant to be 1). Multiplying on the left an element $g \in G$ by an element in $\mathscr{S}_{\mathbf{d}}^{\pm}$ amounts to permuting the rows of g by the inverse of the associated permutation and possibly changing their sign. Since the rank of the submatrix of any invertible matrix consisting of the first $\mathfrak{m}$ columns is $\mathfrak{m}$, we have $\mathscr{U}_G^0 \subset \mathscr{S}_{\mathbf{d}}^{\pm}(G \smallsetminus \mathscr{U}_G^0)$.

For every closed subgroup H of G, we denote by $H(\mathscr{O}_\nu)$ the compact-open subgroup $H \cap \mathrm{GL}_m(\mathscr{O}_\nu)$ of H, and by μ_H the left Haar measure of H normalized so that

$$\mu_H(H(\mathscr{O}_\nu)) = 1\ . \tag{16}$$

In particular, $G(\mathscr{O}_\nu) = \mathrm{SL}_{\mathbf{d}}(\mathscr{O}_\nu)$ and $\mu_G(G(\mathscr{O}_\nu)) = 1$. Note that $\mathrm{GL}^1_{\mathbf{d}}(\mathscr{O}_\nu) = \mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$ by Lemma 2.7 and similarly $\mathrm{GL}^1_{\mathbf{d}}(R_\nu) = \mathrm{GL}_{\mathbf{d}}(R_\nu)$. For all $k, k' \in \mathbb{N} \smallsetminus \{0\}$, we endow the locally compact additive group $\mathscr{M}_{k,k'}(K_\nu)$ with its Haar measure $\mathrm{Haar}_{k,k'}$ normalized so that

$$\mathrm{Haar}_{k,k'}(\mathscr{M}_{k,k'}(\mathscr{O}_\nu)) = 1\ .$$

The group $\mathrm{GL}_k(K_\nu) \times \mathrm{GL}_{k'}(K_\nu)$ acts linearly on the K_ν-vector space $\mathscr{M}_{k,k'}(K_\nu)$ by the action $\phi(g,h) : x \mapsto g x h^{-1}$ for all $x \in \mathscr{M}_{k,k'}(K_\nu)$ and $(g,h) \in \mathrm{GL}_k(K_\nu) \times \mathrm{GL}_{k'}(K_\nu)$. The following claim is well-known.

Lemma 2.4 *This action scales the Haar measure* $\mathrm{Haar}_{k,k'}$ *as follows:*

$$\forall\ (g,h) \in \mathrm{GL}_k(K_\nu) \times \mathrm{GL}_{k'}(K_\nu), \quad \phi(g,h)_*\, \mathrm{Haar}_{k,k'} = |\det g\,|^{k'}\ |\det h\,|^{-k}\, \mathrm{Haar}_{k,k'}\ . \tag{17}$$

Proof. For every $(g,h) \in \mathrm{GL}_k(K_\nu) \times \mathrm{GL}_{k'}(K_\nu)$, we have $\phi(g,h) = \phi(g,\mathrm{id}) \circ \phi(\mathrm{id},h)$, and $\phi(g,\mathrm{id})$ acts on $x \in \mathscr{M}_{k,k'}(K_\nu)$ by the diagonal linear action of g on the k' columns of x, and $\phi(\mathrm{id},h)$ acts on $x \in \mathscr{M}_{k,k'}(K_\nu)$ by the transpose of the diagonal linear action by the transpose-inverse of h on the k columns of ${}^t x$, since ${}^t(x\, h^{-1}) = {}^t h^{-1}\, {}^t x$. The result hence follows from Equation (4) and a diagonal by block computation of determinants. □

The maps $\mathfrak{u}^- : \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu) \to U^-$ and $\mathfrak{u}^+ : \mathscr{M}_{\mathfrak{m},\mathfrak{n}}(K_\nu) \to U^+$ defined respectively by $\beta \mapsto \left(\begin{smallmatrix} I_{\mathfrak{m}} & 0 \\ \beta & I_{\mathfrak{n}} \end{smallmatrix}\right)$ and $\gamma \mapsto \left(\begin{smallmatrix} I_{\mathfrak{m}} & \gamma \\ 0 & I_{\mathfrak{n}} \end{smallmatrix}\right)$ are topological group isomorphisms, satisfying

$$\mathfrak{u}^-{}_*\, \mathrm{Haar}_{\mathfrak{n},\mathfrak{m}} = \mu_{U_-} \quad \text{and} \quad \mathfrak{u}^+{}_*\, \mathrm{Haar}_{\mathfrak{m},\mathfrak{n}} = \mu_{U_+}\ . \tag{18}$$

We denote by $\chi_{\mathfrak{m}}$ (respectively $\chi_{\mathfrak{n}}$) the characters from Z to $K_\nu^\times$ sending $\left(\begin{smallmatrix} \lambda I_{\mathfrak{m}} & 0 \\ 0 & \mu I_{\mathfrak{n}} \end{smallmatrix}\right)$ to λ (respectively μ). Note that ${\chi_{\mathfrak{m}}}^{\mathfrak{m}}\,{\chi_{\mathfrak{n}}}^{\mathfrak{n}}$ is the trivial character. For all $z \in Z$, $\beta \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu)$ and $\gamma \in \mathscr{M}_{\mathfrak{m},\mathfrak{n}}(K_\nu)$, we have

$$z\,\mathfrak{u}^-(\beta)\,z^{-1} = \mathfrak{u}^-(\,(\chi_{\mathfrak{n}}(z)I_{\mathfrak{n}})\,\beta\,(\chi_{\mathfrak{m}}(z)I_{\mathfrak{m}})^{-1})\;. \tag{19}$$

The Haar measure μ_Z on Z is exactly the counting measure, since $Z(\mathscr{O}_\nu) = \{I_{\mathfrak{m}}\}$:

$$\mu_Z = \sum_{z\in Z} \Delta_z\;. \tag{20}$$

The next result gives a refined LU decomposition by blocks of G and the corresponding decomposition of its Haar measure.

Proposition 2.5 *The product map $(u^-, g'', z, u^+) \mapsto u^-\,g''\,z\,u^+$ from $U^- \times G'' \times Z \times U^+$ to $\mathscr{U}_G$ is a homeomorphism and if*

$$c_1 = \frac{q_\nu^{\mathfrak{m}\,\mathfrak{n}}\;\prod_{i=1}^{\mathfrak{m}}(q_\nu^{\,i}-1)\prod_{i=1}^{\mathfrak{n}}(q_\nu^{\,i}-1)}{\prod_{i=1}^{\mathbf{d}}(q_\nu^{\,i}-1)} \leqslant 1\;,$$

then

$$d\mu_G(u^-\,g''\,z\,u^+) = c_1\,|\chi_{\mathfrak{m}}(z)|^{\mathbf{d}\,\mathfrak{m}}\,d\mu_{U^-}(u^-)\;d\mu_{G''}(g'')\;d\mu_Z(z)\;d\mu_{U^+}(u^+)\;.$$

Proof. For every $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in G$ such that $\det\alpha \neq 0$ (which is the case if $g \in \mathscr{U}_G$), we have

$$g = \begin{pmatrix} I_{\mathfrak{m}} & 0 \\ \beta\alpha^{-1} & I_{\mathfrak{n}} \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ 0 & \delta-\beta\alpha^{-1}\gamma \end{pmatrix} = \begin{pmatrix} I_{\mathfrak{m}} & 0 \\ \beta\alpha^{-1} & I_{\mathfrak{n}} \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \delta-\beta\alpha^{-1}\gamma \end{pmatrix}\begin{pmatrix} I_{\mathfrak{m}} & \alpha^{-1}\gamma \\ 0 & I_{\mathfrak{n}} \end{pmatrix}\;. \tag{21}$$

In particular, the matrix $\delta - \beta\alpha^{-1}\gamma$ is invertible, and $\det(\alpha)\det(\delta-\beta\alpha^{-1}\gamma) = 1$, so that $\nu(\det(\delta-\beta\alpha^{-1}\gamma)) = -\nu(\det(\alpha))$. Thus if $g \in \mathscr{U}_G$, then $\nu(\det(\alpha))$ is divisible by $\mathfrak{m}$ and $\nu(\det((\delta-\beta\alpha^{-1}\gamma))$ is divisible by $\mathfrak{n}$. Furthermore,

$$|\det(\pi_\nu^{-\frac{\nu(\det\alpha)}{\mathfrak{m}}}\alpha)| = |\pi_\nu^{-\nu(\det\alpha)}|\,|\det\alpha| = q_\nu^{\,\nu(\det\alpha)}\,|\det\alpha| = |\det\alpha|^{-1}\,|\det\alpha| = 1\;.$$

Consider the map Ξ from $\mathscr{U}_G$ to $U^- \times G'' \times Z \times U^+$ which to $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in \mathscr{U}_G$ associates

$$\left(u^- = \begin{pmatrix} I_{\mathfrak{m}} & 0 \\ \beta\alpha^{-1} & I_{\mathfrak{n}} \end{pmatrix},\; g'' = \begin{pmatrix} \pi_\nu^{-\frac{\nu(\det\alpha)}{\mathfrak{m}}}\alpha & 0 \\ 0 & \pi_\nu^{-\frac{\nu(\det(\delta-\beta\alpha^{-1}\gamma))}{\mathfrak{n}}}(\delta-\beta\alpha^{-1}\gamma) \end{pmatrix}\right.$$

$$\left. z = \begin{pmatrix} \pi_\nu^{\frac{\nu(\det\alpha)}{\mathfrak{m}}} I_{\mathfrak{m}} & 0 \\ 0 & \pi_\nu^{\frac{\nu(\det(\delta-\beta\alpha^{-1}\gamma))}{\mathfrak{n}}} I_{\mathfrak{n}} \end{pmatrix},\; u^+ = \begin{pmatrix} I_{\mathfrak{m}} & \alpha^{-1}\gamma \\ 0 & I_{\mathfrak{n}} \end{pmatrix}\right)\;, \tag{22}$$

which is well defined as we just checked. Let us prove that Ξ is onto. Let

$$u^- = \begin{pmatrix} I_{\mathfrak{m}} & 0 \\ \beta' & I_{\mathfrak{n}} \end{pmatrix} \in U^-,\quad g'' = \begin{pmatrix} \alpha' & 0 \\ 0 & \delta' \end{pmatrix} \in G'',\quad z = \begin{pmatrix} \pi_\nu^{\,r} I_{\mathfrak{m}} & 0 \\ 0 & \pi_\nu^{\,s} \end{pmatrix} \in Z,\quad u^+ = \begin{pmatrix} I_{\mathfrak{m}} & \gamma' \\ 0 & I_{\mathfrak{n}} \end{pmatrix} \in U^+\;.$$

Let $\alpha = \pi_\nu^{\,r}\alpha'$, $\beta = \beta'\alpha$, $\gamma = \gamma'\alpha$ and $\delta = \pi_\nu^{\,s}\delta' + \beta\alpha^{-1}\gamma$. The equality $\mathfrak{m}\,r + \mathfrak{n}\,s = 0$ implies by Gauss Lemma that $\frac{\mathfrak{n}}{\gcd\{\mathfrak{m},\mathfrak{n}\}}$ divides r, hence that $\operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\} = \frac{\mathfrak{m}\,\mathfrak{n}}{\gcd\{\mathfrak{m},\mathfrak{n}\}}$ divides $\mathfrak{m}\,r$. Since $\alpha' \in \mathrm{GL}^1_{\mathfrak{m}}(K_\nu)$, we have

$$\nu(\det\alpha) = \nu((\pi_\nu^{\,r})^{\mathfrak{m}}\det\alpha') = \mathfrak{m}\,r + \nu(\det\alpha') = \mathfrak{m}\,r \in \operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\}\,\mathbb{Z}\;. \tag{23}$$

Thus $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$ belongs to $\mathscr{U}_G$ and by construction $\Xi(g) = (u^-, g'', z, u^+)$. It is immediate to see that Ξ is continuous on $\mathscr{U}_G$ and is the inverse of the continuous multiplication map $(u^-, g'', z, u^+) \mapsto u^- \, g'' \, z \, u^+$.

Let $P^- = U^- G'' Z$, which is a closed subgroup of G, since Z centralises G'' and $G''Z$ normalizes U^-, so that $\mathscr{U}_G = P^- U^+$. By [LanS, §III.1], since G and U^+ are unimodular, there exists a constant $c_2 > 0$ such that the restriction to the open set $\mathscr{U}_G$ of the Haar measure of G satisfies $d\mu_G(p^- u^+) = c_2 \, d\mu_{P^-}(p^-) \, d\mu_{U^+}(u^+)$ for (almost) all $p^- \in P^-$ and $u^+ \in U^+$.

For all $g'' = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right) \in G''$ and $\beta \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu)$, we have $g'' \mathfrak{u}^-(\beta) g''^{-1} = \mathfrak{u}^-(\delta \beta \alpha^{-1})$. Therefore, with $\iota_{g''}$ the conjugation map $x \mapsto g'' \, x \, g''^{-1}$ by g'' on U^-, by Equation (18), by Equation (17) with $k = \mathfrak{n}$ and $k' = \mathfrak{m}$, and since $|\det \alpha| = |\det \delta| = 1$, we have

$$\begin{aligned} (\iota_{g''})_* \mu_{U^-} &= (\iota_{g''} \circ \mathfrak{u}^-)_* \operatorname{Haar}_{\mathfrak{n},\mathfrak{m}} = (\mathfrak{u}^- \circ \phi(\delta, \alpha))_* \operatorname{Haar}_{\mathfrak{n},\mathfrak{m}} \\ &= (\mathfrak{u}^-)_* \operatorname{Haar}_{\mathfrak{n},\mathfrak{m}} = \mu_{U^-} \; . \end{aligned} \tag{24}$$

Let $z \in Z$. By the relation between the two characters $\chi_{\mathfrak{m}}$ et $\chi_{\mathfrak{n}}$ of Z, we have

$$\begin{aligned} &|\det(\chi_{\mathfrak{n}}(z) I_{\mathfrak{n}})|^{\mathfrak{m}} \, |\det(\chi_{\mathfrak{m}}(z) I_{\mathfrak{m}})|^{-\mathfrak{n}} = |\chi_{\mathfrak{n}}(z)^{\mathfrak{n}}|^{\mathfrak{m}} \, |\chi_{\mathfrak{m}}(z)^{\mathfrak{m}}|^{-\mathfrak{n}} \\ &= |\chi_{\mathfrak{m}}(z)^{-\mathfrak{m}}|^{\mathfrak{m}} \, |\chi_{\mathfrak{m}}(z)|^{-\mathfrak{m}\,\mathfrak{n}} = |\chi_{\mathfrak{m}}(z)|^{-\mathbf{d}\,\mathfrak{m}} \; . \end{aligned}$$

Therefore, with ι_z the conjugation map by z on U^-, by Equation (18), by Equation (17) with $k = \mathfrak{n}$ and $k' = \mathfrak{m}$, we have

$$\begin{aligned} (\iota_z)_* \mu_{U^-} &= (\iota_z \circ \mathfrak{u}^-)_* \operatorname{Haar}_{\mathfrak{n},\mathfrak{m}} = (\mathfrak{u}^- \circ \phi(\chi_{\mathfrak{n}}(z) I_{\mathfrak{n}}, \chi_{\mathfrak{m}}(z) I_{\mathfrak{m}}))_* \operatorname{Haar}_{\mathfrak{n},\mathfrak{m}} \\ &= |\chi_{\mathfrak{m}}(z)|^{-\mathfrak{m}\mathbf{d}} \mu_{U^-} \; . \end{aligned} \tag{25}$$

The product in the group $P^- = U^- G'' Z$ may be written as follows: for all (u^-, g'', z) and $(\widehat{u}^-, \widehat{g}'', \widehat{z})$ in $U^- \times G'' \times Z$, we have

$$(u^- \, g'' \, z) \, (\widehat{u}^- \, \widehat{g}'' \, \widehat{z}) = \big(u^- \, (\, g'' \, (z \, \widehat{u}^- \, z^{-1}) \, g''^{-1})\big) \, \big(g'' \, \widehat{g}''\big) \big(z \; \widehat{z}\big) \; .$$

By Equations (25) and (24), the image of the measure

$$|\chi_{\mathfrak{m}}(z)|^{\mathfrak{m}\mathbf{d}} \; d\mu_{U^-}(u^-) \; d\mu_{G''}(g'') \; d\mu_Z(z)$$

on $U^- \times G'' \times Z$ by the product map $(u^-, g'', z) \mapsto u^- \, g'' \, z$ is hence a left Haar measure on P^-. Therefore there exists a constant $c_3 > 0$ such that, for (almost) all $u^- \in U^-$, $g'' \in G''$ and $z \in Z$, we have

$$|\chi_{\mathfrak{m}}(z)|^{\,\mathfrak{m}\mathbf{d}} \; d\mu_{U^-}(u^-) \; d\mu_{G''}(g'') \; d\mu_Z(z) = c_3 \; d\mu_{P^-}(u^- g'' z) \; .$$

Therefore, with $c_1 = c_2 c_3^{-1}$, we have

$$d\mu_G(u^- \, g'' \, z \, u^+) = c_1 \; |\chi_{\mathfrak{m}}(z)|^{\mathbf{d}\,\mathfrak{m}} \, d\mu_{U^-}(u^-) \; d\mu_{G''}(g'') \; d\mu_Z(z) \; d\mu_{U^+}(u^+) \; . \tag{26}$$

In order to compute the constant c_1, we evaluate the measures on both sides of Equation (26) on the compact-open subgroup

$$H = \left\{ \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in G(\mathscr{O}_\nu) : \begin{array}{l} \alpha \in I_{\mathfrak{m}} + \pi_\nu \mathscr{M}_{\mathfrak{m}}(\mathscr{O}_\nu), \; \delta \in I_{\mathfrak{n}} + \pi_\nu \mathscr{M}_{\mathfrak{n}}(\mathscr{O}_\nu), \\ \beta \in \pi_\nu \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu), \; \gamma \in \pi_\nu \mathscr{M}_{\mathfrak{m},\mathfrak{n}}(\mathscr{O}_\nu) \end{array} \right\} .$$

We are going to need the following well-known result.

Lemma 2.6 *For every $N \in \mathbb{N}$, let H_N be the kernel of the morphism from $G(\mathscr{O}_\nu)$ to $\mathrm{SL}_{\mathbf{d}}(\mathscr{O}_\nu/\pi_\nu^N\mathscr{O}_\nu)$ of reduction modulo $\pi_\nu^N\mathscr{O}_\nu$. Then*

$$[G(\mathscr{O}_\nu) : H_N] = q_\nu^{\,N(\mathbf{d}^2-1)-\frac{\mathbf{d}(\mathbf{d}+1)}{2}+1}\prod_{i=2}^{\mathbf{d}}(q_\nu^{\,i}-1)\;.$$

Proof. The reduction morphism $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu) \to \mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu/\pi_\nu^N\mathscr{O}_\nu)$ is onto, and the reduction morphism $\mathrm{SL}_{\mathbf{d}}(\mathscr{O}_\nu) \to \mathrm{SL}_{\mathbf{d}}(\mathscr{O}_\nu/\pi_\nu^N\mathscr{O}_\nu)$ is hence also onto. By for instance [Han, Theo. 2.7] (applied with $k = N$ to the finite local commutative ring $R = \mathscr{O}_\nu/\pi_\nu^{\,k}\mathscr{O}_\nu$ with maximal ideal $P = \pi_\nu\mathscr{O}_\nu/\pi_\nu^{\,k}\mathscr{O}_\nu$), we have

$$|\,\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu/\pi_\nu^N\mathscr{O}_\nu)| = |\pi_\nu\mathscr{O}_\nu/\pi_\nu^N\mathscr{O}_\nu|^{\mathbf{d}^2}\;|\,\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu/\pi_\nu\mathscr{O}_\nu)| = q_\nu^{\,(N-1)\mathbf{d}^2}\prod_{i=0}^{\mathbf{d}-1}(q_\nu^{\,\mathbf{d}}-q_\nu^{\,i})$$
$$= q_\nu^{\,N\mathbf{d}^2-\frac{\mathbf{d}(\mathbf{d}+1)}{2}}\prod_{i=1}^{\mathbf{d}}(q_\nu^{\,i}-1)\;.$$

The index of $\mathrm{SL}_{\mathbf{d}}(\mathscr{O}_\nu/\pi_\nu^{\,N}\mathscr{O}_\nu)$ in $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu/\pi_\nu^{\,N}\mathscr{O}_\nu)$ is equal to $|(\mathscr{O}_\nu/\pi_\nu^N\mathscr{O}_\nu)^\times| = q_\nu^{\,N-1}(q_\nu-1)$. The result follows. □

Since $\mu_G(G(\mathscr{O}_\nu)) = 1$ and by Lemma 2.6 with $N = 1$, the group H has Haar measure

$$\mu_G(H) = \frac{\mu_G(G(\mathscr{O}_\nu))}{[G(\mathscr{O}_\nu) : H]} = \frac{1}{q_\nu^{\frac{\mathbf{d}(\mathbf{d}-1)}{2}}\prod_{i=2}^{\mathbf{d}}(q_\nu^{\,i}-1)}\;.$$

The group $H \cap U^- = \big\{\big(\begin{smallmatrix} I_{\mathfrak{m}} & 0\\ \beta & I_{\mathfrak{n}}\end{smallmatrix}\big) : \beta \in \pi_\nu\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)\big\}$ has index $q_\nu^{\,\mathfrak{m}\,\mathfrak{n}}$ in $U^-(\mathscr{O}_\nu)$, and so does $H \cap U^+ = \big\{\big(\begin{smallmatrix} I_{\mathfrak{m}} & \gamma\\ 0 & I_{\mathfrak{n}}\end{smallmatrix}\big) : \gamma \in \pi_\nu\mathscr{M}_{\mathfrak{m},\mathfrak{n}}(\mathscr{O}_\nu)\big\}$ in $U^+(\mathscr{O}_\nu)$. Hence

$$\mu_{U^-}(H \cap U^-) = \mu_{U^+}(H \cap U^+) = \frac{1}{q_\nu^{\,\mathfrak{m}\,\mathfrak{n}}}\;.$$

We have $H \cap Z = \{I_{\mathbf{d}}\}$, hence $\int_{H\cap Z}|\chi_{\mathfrak{m}}(z)|^{\mathbf{d}\,\mathfrak{m}}\,d\mu_Z(z) = 1$ by Equation (20).

The index of the subgroup $H \cap G'' = \left\{\big(\begin{smallmatrix} \alpha & 0\\ 0 & \delta\end{smallmatrix}\big) : \begin{array}{l}\alpha \in I_{\mathfrak{m}} + \pi_\nu\mathscr{M}_{\mathfrak{m}}(\mathscr{O}_\nu),\\ \delta \in I_{\mathfrak{n}} + \pi_\nu\mathscr{M}_{\mathfrak{n}}(\mathscr{O}_\nu),\end{array}\ \det\alpha\ \det\delta = 1\right\}$ in the group $G''(\mathscr{O}_\nu) = \big\{\big(\begin{smallmatrix} \alpha & 0\\ 0 & \delta\end{smallmatrix}\big) : \alpha \in \mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu),\ \delta \in \mathrm{GL}_{\mathfrak{n}}(\mathscr{O}_\nu),\ \det\alpha\ \det\delta = 1\big\}$ is equal to $\frac{1}{|\mathbb{F}_{q_\nu}^\times|}(|\,\mathrm{GL}_{\mathfrak{m}}(\mathbb{F}_{q_\nu})| \times |\,\mathrm{GL}_{\mathfrak{n}}(\mathbb{F}_{q_\nu})|)$. Since $\mu_{G''}(G''(\mathscr{O}_\nu)) = 1$, we hence have

$$\mu_{G''}(H \cap G'') = \frac{(q_\nu-1)}{q_\nu^{\frac{\mathfrak{m}(\mathfrak{m}-1)}{2}}\prod_{i=1}^{\mathfrak{m}}(q_\nu^{\,i}-1)\;q_\nu^{\frac{\mathfrak{n}(\mathfrak{n}-1)}{2}}\prod_{i=1}^{\mathfrak{n}}(q_\nu^{\,i}-1)}\;.$$

Note that H is contained in $\mathscr{U}_G$ since for every $\big(\begin{smallmatrix} \alpha & \gamma\\ \beta & \delta\end{smallmatrix}\big) \in H$, we have $\nu(\det(\alpha)) = 0$ as $\det\alpha \equiv \det I_{\mathfrak{m}} \equiv 1 \mod \pi_\nu$. We then also have $\nu(\det(\delta - \beta\alpha^{-1}\alpha)) = 0$. It follows from Equation (22) that the product map $(u^-, g'', z, u^+) \mapsto u^-\,g''\,z\,u^+$ from $U^- \times G'' \times Z \times U^+$ to $\mathscr{U}_G$ induces a homeomorphism from $(H \cap U^-) \times (H \cap G'') \times (H \cap Z) \times (H \cap U^+)$ to H. By Equation (26) and the above computations, we thus have

$$c_1 = \frac{\mu_G(H)}{\mu_{U^-}(H \cap U^-)\;\mu_{G''}(H \cap G'')\;\mu_{U^+}(H \cap U^+)}$$
$$= \frac{q_\nu^{\,\mathfrak{m}\,\mathfrak{n}}\;\prod_{i=1}^{\mathfrak{m}}(q_\nu^{\,i}-1)\prod_{i=1}^{\mathfrak{n}}(q_\nu^{\,\mathfrak{n}}-1)}{\prod_{i=1}^{\mathbf{d}}(q_\nu^{\,i}-1)}\;.$$

Note that $c_1 = \prod_{i=1}^{\mathfrak{m}}\frac{q_\nu^{\,i+\mathfrak{n}}-q_\nu^{\,\mathfrak{n}}}{q_\nu^{\,i+\mathfrak{n}}-1} \leqslant 1$. This concludes the proof of Proposition 2.5. □

2.6 Refined LU decomposition by blocks and partial lattices

Let V be a K_ν-vector space with finite dimension $D \geqslant 1$, and let $k \in [\![1, D]\!]$. We denote by $\mathrm{Gr}_k(V)$ the Grassmannian space of k-dimensional K_ν-linear subspaces of V (endowed with the compact metrisable Chabauty topology, see Subsection 3.1 for its measure theoretic and metric aspects). We define $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}} = \mathrm{Gr}_{\mathfrak{m}}({K_\nu}^{\mathbf{d}})$.

An $\mathscr{O}_\nu$*-structure* on V is the choice of a finitely generated $\mathscr{O}_\nu$-submodule $V_{\mathscr{O}_\nu}$ generating V as a K_ν-vector space, or equivalently an equivalence class of K_ν-basis of V, where two K_ν-bases are equivalent if their transition matrix belongs to $\mathrm{GL}_D(\mathscr{O}_\nu)$. For instance, we endow ${K_\nu}^D$ and $({K_\nu}^D)^*$ with the $\mathscr{O}_\nu$-structure defined by their canonical basis and its dual basis, respectively. We denote by $\mathrm{GL}(V_{\mathscr{O}_\nu})$ the subgroup of $\mathrm{GL}(V)$ preserving $V_{\mathscr{O}_\nu}$, and we define $\mathrm{SL}(V_{\mathscr{O}_\nu}) = \mathrm{GL}(V_{\mathscr{O}_\nu}) \cap \mathrm{SL}(V)$. The following claim is well-known.

Lemma 2.7 *The group* $\mathrm{GL}(V_{\mathscr{O}_\nu})$ *is contained in* $\mathrm{GL}^1(V)$ *and acts transitively on the Grassmannian space* $\mathrm{Gr}_k(V)$*. If* $k < D$*, then* $\mathrm{SL}(V_{\mathscr{O}_\nu})$ *also acts transitively on* $\mathrm{Gr}_k(V)$*.*

Proof. The first claim follows from the fact that the determinant of a matrix in $\mathrm{GL}(V_{\mathscr{O}_\nu})$, being an element of ${\mathscr{O}_\nu}^\times$, has absolute value 1. By for instance [Wei4, Theo. 1], every complete flag $(V_1, \dots, V_D)$ of V admits a K_ν-basis $(x_1, \dots, x_D)$ which is both adapted to this flag (that is, $V_i = K_\nu x_1 + \dots + K_\nu x_i$ for every $i \in [\![1, D]\!]$) and is an $\mathscr{O}_\nu$-basis of $V_{\mathscr{O}_\nu}$. Hence every k-dimensional K_ν-linear subspace of V admits a K_ν-basis that can be completed to an $\mathscr{O}_\nu$-basis of $V_{\mathscr{O}_\nu}$. Since $\mathrm{GL}(V_{\mathscr{O}_\nu})$ acts transitively on the set of $\mathscr{O}_\nu$-bases of $V_{\mathscr{O}_\nu}$, the second claim follows. The last claim when $k < D$ follows by multiplying the last element of the above $\mathscr{O}_\nu$-basis by an appropriate element of ${\mathscr{O}_\nu}^\times$. □

Assume that V is endowed with a K_ν-basis $(f_1, \dots, f_D)$ defining both an R_ν-structure $V_{R_\nu} = R_\nu f_1 + \dots + R_\nu f_D$, an $\mathscr{O}_\nu$-structure $V_{\mathscr{O}_\nu} = \mathscr{O}_\nu f_1 + \dots + \mathscr{O}_\nu f_D$ and an ultrametric norm $\|x_1 f_1 + \dots + x_D f_D\| = \max_{1 \leqslant i \leqslant D} |x_i|$ whose (closed) unit ball is $V_{\mathscr{O}_\nu}$. For instance, V could be ${K_\nu}^D$ with its canonical basis, or its dual space $({K_\nu}^D)^*$ with its dual basis, or K_ν-linear subspaces of them generated by basis elements. We denote the *space of shapes* of unimodular full R_ν-lattices of V by

$$\mathrm{Sh}^1(V) = \mathrm{GL}(V_{\mathscr{O}_\nu}) \backslash \mathrm{Lat}^1(V) = \mathrm{GL}(V_{\mathscr{O}_\nu}) \backslash \mathrm{GL}^1(V) / \mathrm{GL}(V_{R_\nu}) \ ,$$

endowed with the quotient topology (see Subsection 3.4 for its measure theoretic aspects). For simplicity, we denote $\mathrm{Sh}^1_D = \mathrm{Sh}^1({K_\nu}^D)$.

The *shape map* of k-lattices of V is the map

$$\mathrm{sh} : \big\{\Lambda \in \mathrm{Lat}_k(V) : \overline{\mathrm{Covol}}(\Lambda) \in {q_\nu}^{k\mathbb{Z}}\big\} \to \mathrm{Sh}^1_k$$

defined as follows. Let $\Lambda \in \mathrm{Lat}_k(V)$. By Lemma 2.7, choose an element $g \in \mathrm{GL}(V_{\mathscr{O}_\nu})$ such that we have $gV_\Lambda = K_\nu f_1 + \dots + K_\nu f_k$. Note that g preserves the covolume of partial lattices, since it maps the unit ball $V_\Lambda \cap V_{\mathscr{O}_\nu}$ of V_Λ to the unit ball $\mathscr{O}_\nu f_1 + \dots + \mathscr{O}_\nu f_k$ of $K_\nu f_1 + \dots + K_\nu f_k$. Let $\Theta : K_\nu f_1 + \dots + K_\nu f_k \to {K_\nu}^k$ be the isometric (hence Haar measure preserving) K_ν-linear isomorphism mapping $(f_1, \dots, f_k)$ to the canonical basis of ${K_\nu}^k$, that preserves the covolume of full R_ν-lattices. We define

$$\mathrm{sh}(\Lambda) = \mathrm{GL}_k(\mathscr{O}_\nu) {\pi_\nu}^{\frac{1}{k} \log_{q_\nu} \overline{\mathrm{Covol}}(\Lambda)} \Theta(g\Lambda) \ . \qquad (27)$$

Note that $\pi_\nu^{\frac{1}{k}\log_{q_\nu}\overline{\mathrm{Covol}}(\Lambda)}\Lambda$ is a unimodular k-lattice in V, and that homotheties and linear maps commute. Furthermore, the shape $\mathrm{sh}(\Lambda)$ of Λ does not depend on the choice of g as above, since given two choices g_1 and g_2, the linear maps $\Theta \circ g_1$ and $\Theta \circ g_2$ differ by multiplication on the left by an element of $\mathrm{GL}_k(\mathscr{O}_\nu)$. Note that when Λ is a unimodular full R_ν-latice in the product space $V = K_\nu{}^k$, then Equation (27) greatly simplifies to

$$\mathrm{sh}(\Lambda) = \mathrm{GL}_k(\mathscr{O}_\nu)\Lambda\ ,$$

since we can take $(f_1, \ldots, f_k)$ to be the canonical basis of V and $g = \Theta$ to be the identity map of V.

The next result gives the relationship between the refined LU decomposition by blocks of elements of $\mathscr{U}_G$ and the partial lattices generated by their first $\mathfrak{m}$ columns. Let us first give the notation that will be used. For every $D \in \mathbb{N} \smallsetminus \{0\}$, we identify $\mathrm{GL}(K_\nu{}^D)$ and $\mathrm{GL}_D(K_\nu)$ (respectively $\mathrm{GL}((K_\nu{}^D)^*)$ and $\mathrm{GL}_D(K_\nu)$) by taking matrices of linear automorphisms in the canonical basis $(e_1, \ldots, e_D)$ of $K_\nu{}^D$ (respectively its dual basis $(e_1^*, \ldots, e_D^*)$ of $(K_\nu{}^D)^*$). Recall that the map $h \mapsto \check{h} = {}^t h^{-1}$ is a group isomorphism from $\mathrm{GL}(K_\nu{}^{\mathbf{d}})$ to $\mathrm{GL}((K_\nu{}^{\mathbf{d}})^*)$. For practical reasons, we denote by $R_\nu^{\mathfrak{m}} \times \{0\}$ the $\mathfrak{m}$-lattice $R_\nu e_1 + \cdots + R_\nu e_{\mathfrak{m}}$ of $K_\nu{}^{\mathbf{d}}$ and by $\{0\} \times R_\nu^{\mathfrak{n}}$ the $\mathfrak{n}$-lattice $R_\nu e^*_{\mathfrak{m}+1} + \cdots + R_\nu e^*_{\mathbf{d}}$ of $(K_\nu{}^{\mathbf{d}})^*$. For every $g \in \mathrm{GL}(K_\nu{}^{\mathbf{d}})$ (respectively $g' \in \mathrm{GL}((K_\nu{}^{\mathbf{d}})^*)$), we define

$$\Lambda_g = g(R_\nu^{\mathfrak{m}} \times \{0\}) \quad (\text{respectively} \quad \Lambda_{g'} = g'(\{0\} \times R_\nu^{\mathfrak{n}})\ )\ ,$$

which is the $\mathfrak{m}$-lattice generated by the first $\mathfrak{m}$ columns of g (respectively the $\mathfrak{n}$-lattice generated by the last $\mathfrak{n}$ columns of g'). For every element $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in \mathscr{U}_G$, we denote by $(u^-, g'' = \left(\begin{smallmatrix} \overline{g} & 0 \\ 0 & \underline{g} \end{smallmatrix}\right), z, u^+)$ the decomposition of g given by Proposition 2.5, and by

$$t = -\frac{\nu(\det \alpha)}{\mathrm{lcm}\{\mathfrak{m}, \mathfrak{n}\}} = \frac{\nu(\det(\delta - \beta\alpha^{-1}\gamma))}{\mathrm{lcm}\{\mathfrak{m}, \mathfrak{n}\}} \in \mathbb{Z}$$

(which is indeed an integer since $g \in \mathscr{U}_G$ by Equation (15)). By Equation (22), we then have

$$z = \begin{pmatrix} \pi_\nu^{-\frac{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}}{\mathfrak{m}}t} I_{\mathfrak{m}} & 0 \\ 0 & \pi_\nu^{\frac{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}}{\mathfrak{n}}t} I_{\mathfrak{n}} \end{pmatrix} . \tag{28}$$

To conclude this list of notation for Proposition 2.8, let us define

$$G^\sharp = \{g \in \mathscr{U}_G : u^- \in G(\mathscr{O}_\nu)\}\ .$$

Proposition 2.8 *For every $g \in \mathscr{U}_G$, we have*

(i) $V_{\Lambda_g} = V_{\Lambda_{u^-}}$ *(i)*$^\perp$ $V_{(\Lambda_g)^\perp} = V_{\Lambda_{\check{u}^-}}$

(ii) $\overline{\mathrm{Covol}}(\Lambda_g) = q_\nu^{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}\,t}$ *(ii)*$^\perp$ $\overline{\mathrm{Covol}}((\Lambda_g)^\perp) = q_\nu^{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}\,t}$

and if furthermore $g \in G^\sharp$, then

(iii) $\mathrm{sh}(\Lambda_g) = \mathrm{sh}(\overline{g}\, R_\nu^{\mathfrak{m}})$ *(iii)*$^\perp$ $\mathrm{sh}((\Lambda_g)^\perp) = \mathrm{sh}(\check{\underline{g}}\, R_\nu^{\mathfrak{n}})$.

Proof. Let $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in G \smallsetminus \mathscr{U}_G^0$ (that is, let g be an element of G whose upper-left $\mathfrak{m} \times \mathfrak{m}$ submatrix is invertible), let $u_- = \left(\begin{smallmatrix} I_{\mathfrak{m}} & 0 \\ \beta\alpha^{-1} & I_{\mathfrak{n}} \end{smallmatrix}\right)$ and let $t = -\frac{\nu(\det \alpha)}{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}} \in \mathbb{Q}$, so that these two notations coincide with the above ones when $g \in \mathscr{U}_G \subset G \smallsetminus \mathscr{U}_G^0$. Let us prove that Assertions (i), (i)$^\perp$, (ii), (ii)$^\perp$ are actually satisfied under this greater generality on g. Since we have $\big(\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}\big)\big(\begin{smallmatrix}x \\ 0\end{smallmatrix}\big) = \big(\begin{smallmatrix}I_{\mathfrak{m}} & 0 \\ \beta\alpha^{-1} & I_{\mathfrak{n}}\end{smallmatrix}\big)\big(\begin{smallmatrix}\alpha x \\ 0\end{smallmatrix}\big)$ for every $x \in \mathscr{M}_{\mathfrak{m},1}(K_\nu)$, we have

$$\Lambda_g = g(R_\nu^{\ \mathfrak{m}} \times \{0\}) = u^-((\alpha R_\nu^{\ \mathfrak{m}}) \times \{0\}) \ . \tag{29}$$

Since α is invertible, we hence have $V_{\Lambda_g} = u^-(K_\nu^{\ \mathfrak{m}} \times \{0\})$, thus proving (i).

Let $\delta' = \delta - \beta\alpha^{-1}\gamma$. Since $g = u^-\big(\begin{smallmatrix}\alpha & 0 \\ 0 & \delta'\end{smallmatrix}\big)u^+$ by Equation (21), we have $\check{g} = \widecheck{u^-}\big(\begin{smallmatrix}\check{\alpha} & 0 \\ 0 & \check{\delta}'\end{smallmatrix}\big)\widecheck{u^+}$. Note that $\widecheck{u^+}$, being lower unipotent by blocks, preserves $\{0\} \times R_\nu^{\ \mathfrak{n}}$. By the last equality in Equation (12), we hence have

$$(\Lambda_g)^\perp = \big(g(R_\nu^{\ \mathfrak{m}} \times \{0\})\big)^\perp = \check{g}\,(R_\nu^{\ \mathfrak{m}} \times \{0\})^\perp = \check{g}\,(\{0\} \times R_\nu^{\ \mathfrak{n}}) = \widecheck{u^-}(\{0\} \times (\check{\delta}' R_\nu^{\ \mathfrak{n}})) \ . \tag{30}$$

Since $\check{\delta}'$ is invertible, we thus have $V_{(\Lambda_g)^\perp} = \widecheck{u^-}(\{0\} \times K_\nu^{\ \mathfrak{n}}) = \widecheck{u^-}\, V_{\{0\}\times R_\nu^{\ \mathfrak{n}}} = V_{\widecheck{u^-}(\{0\}\times R_\nu^{\ \mathfrak{n}})}$, thereby proving Assertion (i)$^\perp$.

By Equations (29), (7) (its left-hand side) and (8), since we have $\det(u^-) = 1$ and by the definition of $t = -\frac{\nu(\det\alpha)}{\operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\}} = \frac{\log_{q_\nu}|\det\alpha|}{\operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\}}$, we have

$$\operatorname{Covol}(\Lambda_g) = \operatorname{Covol}(\alpha R_\nu^{\ \mathfrak{m}}) = |\det\alpha|\operatorname{Covol}(R_\nu^{\ \mathfrak{m}}) = q_\nu^{\ \operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\}\,t}\operatorname{Covol}(R_\nu^{\ \mathfrak{m}}) \ ,$$

thus proving (ii). Assertion (ii)$^\perp$ follows from Equation (13).

Assume from now on in this proof that $g \in G^\sharp$. Note that by Assertion (ii), the $\mathfrak{m}$-lattice Λ_g has normalized covolume which is an integral power of $q_\nu^{\ \mathfrak{m}}$, hence $\operatorname{sh}(\Lambda_g)$ is well defined by Equation (27) with $k = \mathfrak{m}$. Recall that the shape of a partial lattice of $K_\nu^{\ \mathbf{d}}$ is invariant by every homothety and by taking the image by any element in $\operatorname{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$. Again by Equation (29), since $u^- \in G(\mathscr{O}_\nu)$ when $g \in G^\sharp$, and since $\overline{g} = \pi_\nu^{\ -\frac{\nu(\det\alpha)}{\mathfrak{m}}}\alpha = \pi_\nu^{\ \frac{\ell(\mathfrak{m},\mathfrak{n})}{\mathfrak{m}}\,t}\alpha$ by Equation (22), we have

$$\operatorname{sh}(\Lambda_g) = \operatorname{sh}((\alpha R_\nu^{\ \mathfrak{m}}) \times \{0\}) = \operatorname{sh}(\alpha R_\nu^{\ \mathfrak{m}}) = \operatorname{sh}(\,\overline{g}\,R_\nu^{\ \mathfrak{m}}) \ ,$$

thus proving (iii). Note that $\overline{g}\,' \in \operatorname{GL}^1_{\mathfrak{m}}(K_\nu)$, so that $\overline{g}\,R_\nu^{\ \mathfrak{m}}$ is a unimodular full R_ν-lattice in $K_\nu^{\ \mathfrak{m}}$.

By Assertion (ii)$^\perp$, the $\mathfrak{n}$-lattice $(\Lambda_g)^\perp$ has normalized covolume which is an integral power of $q_\nu^{\ \mathfrak{n}}$, hence $\operatorname{sh}((\Lambda_g)^\perp)$ is well defined by Equation (27) with $k = \mathfrak{n}$. As previously, since $\widecheck{u^-}$ (which is now upper unipotent by blocks) still belongs to $G(\mathscr{O}_\nu)$, and since $\underline{g}$ is a scalar multiple of $\delta' = \delta - \beta\alpha^{-1}\gamma$, we have by Equation (30) that

$$\operatorname{sh}((\Lambda_g)^\perp) = \operatorname{sh}(\{0\} \times (\,\check{\delta}' R_\nu^{\ \mathfrak{n}})) = \operatorname{sh}(\,\check{\delta}' R_\nu^{\ \mathfrak{n}}) = \operatorname{sh}(\,\check{\underline{g}}\,R_\nu^{\ \mathfrak{n}}) \ ,$$

thus proving (iii)$^\perp$. Note that $\check{\underline{g}}\,R_\nu^{\ \mathfrak{n}}$ is a unimodular full R_ν-lattice in $K_\nu^{\ \mathfrak{n}}$. □

3 Metric measured moduli spaces of partial lattices

In this section, we define the natural measures and distances on the moduli spaces $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}$ (see Subsection 3.1), $\operatorname{Sh}^1_{\mathfrak{m}}$ and $\operatorname{Sh}^1_{\mathfrak{n}}$ (see Subsection 3.4), that were introduced in Subsection 2.6 and on whose products the equidistribution results of the Introduction will take place. We introduce (and similarly analyse) in Subsection 3.3 an avatar $\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ of the product $\operatorname{Lat}^1_{\mathfrak{m}} \times \operatorname{Lat}^1_{\mathfrak{n}}$ of the spaces (described in Subsection 3.2) of $\mathfrak{m}$- and $\mathfrak{n}$-lattices, on which our stronger equidistribution result (Theorem 4.7) will take place in the subsequent Section 4.

3.1 The Grassmannian spaces

As defined in Subsection 2.6, we denote by $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}} = \operatorname{Gr}_{\mathfrak{m}}(K_\nu{}^{\mathbf{d}})$ the Grassmannian space of $\mathfrak{m}$-dimensional K_ν-linear subspaces of $K_\nu{}^{\mathbf{d}}$. Recalling that $G = \operatorname{SL}_{\mathbf{d}}(K_\nu)$, we define

$$Q^+ = \left\{ \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in G : \beta = 0 \right\} ,$$

which is a nonunimodular closed subgroup of G. It contains the closed and open subgroup $P^+ = G''ZU^+ = \left\{ \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in Q^+ : \nu(\det(\alpha)) \in \operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\}\,\mathbb{Z} \right\}$ with finite index. The compact metrisable group $G(\mathscr{O}_\nu)$ acts continuously and transitively by Lemma 2.7 on the compact metrisable space $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}$. The stabilizer of the K_ν-linear subspace $K_\nu{}^{\mathfrak{m}} \times \{0\}$ of $K_\nu{}^{\mathbf{d}}$ corresponding to the first $\mathfrak{m}$ coordinates is exactly $Q^+(\mathscr{O}_\nu)$. Hence the continuous onto orbital map $g \mapsto g(K_\nu{}^{\mathfrak{m}} \times \{0\})$ from $G(\mathscr{O}_\nu)$ to $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}$ induces a continuous bijection

$$G(\mathscr{O}_\nu)/Q^+(\mathscr{O}_\nu) \to \operatorname{Gr}_{\mathfrak{m},\mathbf{d}} ,$$

which is hence a homeomorphism by compactness arguments. We identify from now on $G(\mathscr{O}_\nu)/Q^+(\mathscr{O}_\nu)$ and $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}$ by this map.

By the normalisation convention of the Haar measure of the closed subgroups of G (see Equation (16)), the Haar measures $\mu_{G(\mathscr{O}_\nu)}$ and $\mu_{Q^+(\mathscr{O}_\nu)}$ are normalized to be probability measures. We denote by $\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}$ the unique $G(\mathscr{O}_\nu)$-invariant probability measure on the Grassmannian space $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}} = G(\mathscr{O}_\nu)/Q^+(\mathscr{O}_\nu)$. This is in accordance with Weil's normalization process of measures on homogeneous spaces (see [Wei3, §9]). Indeed, the probability measure $\mu_{G(\mathscr{O}_\nu)}$ disintegrates with respect to the canonical projection $G(\mathscr{O}_\nu) \to G(\mathscr{O}_\nu)/Q^+(\mathscr{O}_\nu)$ over the measure $\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}$, with conditionnal measures on the fibers $g\,Q^+(\mathscr{O}_\nu)$ the probability pushforward measures $g_*\mu_{Q^+(\mathscr{O}_\nu)}$: for every $f \in C^0(G(\mathscr{O}_\nu))$, we have

$$\int_{g\in G(\mathscr{O}_\nu)} f(g)\, d\mu_{G(\mathscr{O}_\nu)} = \int_{g\,Q^+(\mathscr{O}_\nu)\in \operatorname{Gr}_{\mathfrak{m},\mathbf{d}}} \int_{h\in Q^+(\mathscr{O}_\nu)} f(gh)\, d\mu_{Q^+(\mathscr{O}_\nu)}(h)\, d\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}(g\,Q^+(\mathscr{O}_\nu)) . \tag{31}$$

In particular, we have

$$\|\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}\| = 1 . \tag{32}$$

We denote by $\operatorname{orb}_{\mathfrak{m}} : \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu) \to \operatorname{Gr}_{\mathfrak{m},\mathbf{d}}$ the (continuous injective) map defined by

$$\operatorname{orb}_{\mathfrak{m}} : \beta \mapsto \left(\begin{smallmatrix} I_{\mathfrak{m}} & 0 \\ \beta & I_{\mathfrak{n}} \end{smallmatrix}\right)(K_\nu{}^{\mathfrak{m}} \times \{0\}) ,$$

and by $\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}} = \operatorname{orb}_{\mathfrak{m}}(\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu))$ its image. Every element $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in G(\mathscr{O}_\nu)$ such that $\det(\alpha) \neq 0$ and $\beta\alpha^{-1} \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$ belongs to $\left(\begin{smallmatrix} I_{\mathfrak{m}} & 0 \\ \beta\alpha^{-1} & I_{\mathfrak{n}} \end{smallmatrix}\right) Q^+(\mathscr{O}_\nu)$ by Equation (21). Conversely, if an element $g = \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$ belongs to $U^-(\mathscr{O}_\nu)\,Q^+(\mathscr{O}_\nu)$, then we have $\det\alpha \neq 0$ and $\beta\alpha^{-1} \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$, so that

$$\begin{aligned} \operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}} &= U^-(\mathscr{O}_\nu)\,Q^+(\mathscr{O}_\nu) = U^-(\mathscr{O}_\nu)(K_\nu{}^{\mathfrak{m}} \times \{0\}) \\ &= \left\{ \left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right) \in G(\mathscr{O}_\nu) : {}^{\det\alpha\neq 0}_{\beta\alpha^{-1}\in\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)} \right\}(K_\nu{}^{\mathfrak{m}} \times \{0\}) . \end{aligned}$$

Hence $\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$ is a compact (as the image by the continuous map $\operatorname{orb}_{\mathfrak{m}}$ of the compact space $\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$) and open subset of the open Bruhat cell U^-Q^+ of the Grassmannian space $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}$ (corresponding to the longest element in the Weyl group of $\operatorname{SL}_{\mathbf{d}}(K_\nu)$). By Equation

(31) applied with f the characteristic function of the compact subset $U^-(\mathscr{O}_\nu)\,Q^+(\mathscr{O}_\nu)$ of $G(\mathscr{O}_\nu)$ for the first equality, by Proposition 2.5 for the third equality and by the normalisation in Equation (16) of the Haar measures for the last equality, we have

$$\begin{aligned}\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}(\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}) &= \mu_{G(\mathscr{O}_\nu)}(U^-(\mathscr{O}_\nu)\,Q^+(\mathscr{O}_\nu)) = \mu_G(U^-(\mathscr{O}_\nu)\,G''(\mathscr{O}_\nu)\,U^+(\mathscr{O}_\nu))\\ &= c_1\;\mu_{U^-}(U^-(\mathscr{O}_\nu))\,\mu_{G''}(G''(\mathscr{O}_\nu))\,\mu_{U^+}(U^+(\mathscr{O}_\nu)) = c_1\;.\end{aligned}$$

By the normalization of the Haar measure $\mathrm{Haar}_{\mathfrak{n},\mathfrak{m}}$ of $\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu)$ so that its restriction $\mu_{\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)}$ to $\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$ is a probability measure, we hence have

$$(\mathrm{orb}_{\mathfrak{m}})_*(\mu_{\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)}) = c_1\;(\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}})_{|\,\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}}\;. \tag{33}$$

In order to be able to define locally constant functions on the Grassmannian space $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ for error term estimates, one way is to define an appropriate distance on this space. The standard construction is the following one. We endow the K_ν-vector space $V = K_\nu^{\,\mathbf{d}}$ with the usual norm, the maximum of the absolute values of the coordinates in its canonical K_ν-basis $(e_1,\dots,e_{\mathbf{d}})$, and for every $k\in\mathbb{N}\smallsetminus\{0\}$, its k-th exterior power $\wedge^k V$ with the corresponding norm, the maximum of the absolute values of the coordinates in its corresponding K_ν-basis $(e_{i_1}\wedge\dots\wedge e_{i_k})_{1\leqslant i_1<\dots<i_k\leqslant\mathbf{d}}$. We now endow the projective space $\mathbb{P}(V)$ with its usual distance d defined by $d(K_\nu x, K_\nu y) = \frac{\|x\wedge y\|}{\|x\|\,\|y\|}$ for all $x,y\in V\smallsetminus\{0\}$, and $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ with its induced distance d by the Plücker embedding $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}\to\mathbb{P}(\wedge^{\mathfrak{m}}V)$ defined by $W\mapsto K_\nu(b_1\wedge\dots\wedge b_{\mathfrak{m}})$ if $(b_1,\dots,b_{\mathfrak{m}})$ is any K_ν-basis of W. Since the linear action of $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$ on V preserves the norm, the exterior action of $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$ on $\wedge^{\mathfrak{m}}V$ preserves the norm, hence the projective action of $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$ on $\mathbb{P}(\wedge^{\mathfrak{m}}V)$ preserves the distance. Since the Plücker embedding is equivariant with respect to the actions of $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$, the action of $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$ on $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ preserves its distance d.

For all $D,D',D''\in\mathbb{N}\smallsetminus\{0\}$, we endow the K_ν-vector space $\mathscr{M}_{D,D'}(K_\nu)$ with its supremum norm $\|\cdot\|$ defined, for every element $X=(X_{i,j})_{1\leqslant i\leqslant D,1\leqslant j\leqslant D'}\in\mathscr{M}_{D,D'}(K_\nu)$, by

$$\|X\| = \max\{|X_{i,j}| : 1\leqslant i\leqslant D, 1\leqslant j\leqslant D'\}\in q_\nu^{\,\mathbb{Z}}\cup\{0\}\;.$$

This norm is an ultrametric norm and satisfies the following properties:

- The transposition map $A\mapsto {}^tA$ from $\mathscr{M}_{D,D'}(K_\nu)$ to $\mathscr{M}_{D',D}(K_\nu)$ is a linear isometry for the norms $\|\;\|$.
- By the ultrametric property of the absolute value, the norm $\|\;\|$ is a submultiplicative norm: For all $A\in\mathscr{M}_{D,D'}(K_\nu)$ and $B\in\mathscr{M}_{D',D''}(K_\nu)$, we have $\|AB\|\leqslant\|A\|\,\|B\|$.
- For every $A\in\mathscr{M}_{D,D'}(K_\nu)$, we have $\|A\|\leqslant 1$ if and only if $A\in\mathscr{M}_{D,D'}(\mathscr{O}_\nu)$. Hence the unit ball of $\|\cdot\|$ is $\mathscr{M}_{D,D'}(\mathscr{O}_\nu)$ and the right and left multiplications by elements of $\mathscr{M}_D(\mathscr{O}_\nu)$ and $\mathscr{M}_{D'}(\mathscr{O}_\nu)$ are 1-Lipschitz maps on $\mathscr{M}_{D,D'}(\mathscr{O}_\nu)$: For all $A\in\mathscr{M}_D(K_\nu)$ and $B\in\mathscr{M}_{D,D'}(\mathscr{O}_\nu)$ and $C\in\mathscr{M}_{D'}(\mathscr{O}_\nu)$, we have $\|ABC\|\leqslant\|B\|$. In particular, $\|ABC\|=\|B\|$ if $A\in\mathrm{GL}_D(\mathscr{O}_\nu)$ and $C\in\mathrm{GL}_{D'}(\mathscr{O}_\nu)$.

Lemma 3.1 *For all $\beta,\beta'\in\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$, we have*

$$d(\mathrm{orb}_{\mathfrak{m}}(\beta),\mathrm{orb}_{\mathfrak{m}}(\beta')) = \|\beta-\beta'\|\;.$$

Proof. Every matrix $\beta\in\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$ will be seen as a linear map from $K_\nu^{\,\mathfrak{m}}\times\{0\}$ to $\{0\}\times K_\nu^{\,\mathfrak{n}}$, so that $\beta e_j=\sum_{1\leqslant i\leqslant\mathfrak{n}}\beta_{i,j}\,e_{i+\mathfrak{m}}$ for every $j\in[\![1,\mathfrak{m}]\!]$. A K_ν-basis of the K_ν-linear subspace $\mathrm{orb}_{\mathfrak{m}}(\beta)=\begin{pmatrix}I_{\mathfrak{m}}&0\\ \beta&I_{\mathfrak{n}}\end{pmatrix}(K_\nu^{\,\mathfrak{m}}\times\{0\})$ of $K_\nu^{\,\mathbf{d}}$ is hence $(e_1+\beta e_1,\dots,e_{\mathfrak{m}}+\beta e_{\mathfrak{m}})$.

Let $x_\beta = (e_1 + \beta e_1) \wedge \cdots \wedge (e_{\mathfrak{m}} + \beta e_{\mathfrak{m}})$. We have $\|x_\beta\| = 1$ since $\|e_1 \wedge \cdots \wedge e_{\mathfrak{m}}\| = 1$ and since the entries of β have absolute value at most 1. For every $\beta, \beta' \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$, we have $x_\beta \wedge x_{\beta'} = (x_\beta - x_{\beta'}) \wedge x_{\beta'}$ and $x_\beta - x_{\beta'} = v_1 + v_2 + \cdots + v_{\mathfrak{m}}$ where, separating the terms according to the number of occurrences of β's in them,

$$v_1 = \sum_{1 \leqslant i \leqslant \mathfrak{m}} e_1 \wedge \cdots \wedge e_{i-1} \wedge (\beta - \beta')e_i \wedge e_{i+1} \wedge \cdots \wedge e_{\mathfrak{m}} ,$$

$$v_2 = \sum_{1 \leqslant i < j \leqslant \mathfrak{m}} \big(e_1 \wedge \cdots \wedge e_{i-1} \wedge \beta e_i \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge \beta e_j \wedge e_{j+1} \wedge \cdots \wedge e_{\mathfrak{m}}$$
$$- e_1 \wedge \cdots \wedge e_{i-1} \wedge \beta' e_i \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge \beta' e_j \wedge e_{j+1} \wedge \cdots \wedge e_{\mathfrak{m}}\big)$$

and so on, and $v_{\mathfrak{m}} = \beta e_1 \wedge \cdots \wedge \beta e_{\mathfrak{m}} - \beta' e_1 \wedge \cdots \wedge \beta' e_{\mathfrak{m}}$. By the ultrametric properties of the norms, we have $\|v_k\| \leqslant \|\beta - \beta'\|$ for all $k \geqslant 2$ and $\|v_1\| = \|\beta - \beta'\|$. By considering the elements of the K_ν-basis of $\wedge^{\mathfrak{m}} V$ involved in the formulation of $v_1, v_2, \ldots, v_{\mathfrak{m}}$, and again by the ultrametric properties of the norms, we hence have $\|x_\beta - x_{\beta'}\| = \|\beta - \beta'\|$. Furthermore, the coordinate of $x_\beta - x_{\beta'}$ corresponding to the basis vector $e_1 \wedge \cdots \wedge e_{\mathfrak{m}}$ is 0 while the one of $x_{\beta'}$ is 1. Thus $\|x_\beta \wedge x_{\beta'}\| = \|x_\beta - x_{\beta'}\|$ and the result follows. □

3.2 The spaces of unimodular full lattices

For every unimodular locally compact group H endowed with a Haar measure μ_H, and for every discrete subgroup Γ' of H, we again denote by μ_H the unique left H-invariant measure on H/Γ' such that the covering map $H \to H/\Gamma'$ locally preserves the measure.

Let $k \in \mathbb{N} \smallsetminus \{0\}$. Note that $\mathrm{SL}_k(K_\nu)$ is a closed unimodular subgroup of the unimodular locally compact group $\mathrm{GL}^1_k(K_\nu)$, whose Haar measure $\mu_{\mathrm{SL}_k(K_\nu)}$ is normalized so that $\mu_{\mathrm{SL}_k(K_\nu)}(\mathrm{SL}_k(\mathscr{O}_\nu)) = 1$ (as we did for $k = \mathbf{d}$ in Subsection 2.5). The restriction to $\mathscr{O}_\nu{}^\times$ of the Haar measure of the additive group $(K_\nu, +)$ is a Haar measure $\mu_{\mathscr{O}_\nu{}^\times}$ of the multiplicative group $(\mathscr{O}_\nu{}^\times, \times)$ by Equation (4). By the normalization of the Haar measure of $(K_\nu, +)$, we have

$$\|\mu_{\mathscr{O}_\nu{}^\times}\| = \mu_{K_\nu}(\mathscr{O}_\nu \smallsetminus \pi_\nu \mathscr{O}_\nu) = 1 - q_\nu^{-1} . \tag{34}$$

We have a split short exact sequence of locally compact groups

$$1 \longrightarrow \mathrm{SL}_k(K_\nu) \longrightarrow \mathrm{GL}^1_k(K_\nu) \longrightarrow \mathscr{O}_\nu{}^\times ,$$

with section $s_k : \mathscr{O}_\nu{}^\times \to \mathrm{GL}^1_k(K_\nu)$ defined for instance by $\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & I_{k-1} \end{pmatrix}$. We define the Haar measure $\mu_{\mathrm{GL}^1_k(K_\nu)}$ of $\mathrm{GL}^1_k(K_\nu)$, for all $g \in \mathrm{SL}_k(K_\nu)$ and $\lambda \in \mathscr{O}_\nu{}^\times$, by

$$d\mu_{\mathrm{GL}^1_k(K_\nu)}(s_k(\lambda)g) = d\mu_{\mathscr{O}_\nu{}^\times}(\lambda)\, d\mu_{\mathrm{SL}_k(K_\nu)}(g) . \tag{35}$$

In particular, we have

$$\mu_{\mathrm{GL}^1_k(K_\nu)}(\mathrm{GL}_k(\mathscr{O}_\nu)) = 1 - q_\nu^{-1} . \tag{36}$$

After Equation (10), we identified the space $\mathrm{Lat}^1_k = \mathrm{Lat}^1(K_\nu{}^k)$ of unimodular full R_ν-lattices in $K_\nu{}^k$ with the homogeneous space $\mathrm{GL}^1_k(K_\nu)/\mathrm{GL}_k(R_\nu)$. Since $\mathrm{GL}_k(R_\nu)$ is a discrete subgroup of the unimodular group $\mathrm{GL}^1_k(K_\nu)$, we endow Lat^1_k with the unique $\mathrm{GL}^1_k(K_\nu)$-invariant measure $\mu_{\mathrm{Lat}^1_k}$ such that the orbital map $\mathrm{GL}^1_k(K_\nu) \to \mathrm{Lat}^1_k$ defined by $g \mapsto g\, R_\nu{}^k$ locally preserves the measure.

Since the index of $\mathrm{SL}_k(R_\nu)$ in $\mathrm{GL}_k(R_\nu)$ is equal to $\mathrm{Card}(R_\nu^\times) = \mathrm{Card}(\mathbb{F}_q^\times) = q-1$ (see the formula on the right in Equation (2)), and by Equations (35) and (34), we have

$$\begin{aligned}\|\mu_{\mathrm{Lat}^1_k}\| &= \|\mu_{\mathrm{GL}^1_k(K_\nu)/\mathrm{GL}_k(R_\nu)}\| = \frac{1}{q-1}\,\|\mu_{\mathrm{GL}^1_k(K_\nu)/\mathrm{SL}_k(R_\nu)}\| \\ &= \frac{1-q_\nu^{-1}}{q-1}\,\|\mu_{\mathrm{SL}_k(K_\nu)/\mathrm{SL}_k(R_\nu)}\|\;. \end{aligned} \tag{37}$$

Let us apply [Ser1, §3] in order to compute the total mass of $\mu_{\mathrm{Lat}^1_k}$, using boldface letters in order to denote the notation of this reference, thus facilitating the reference process. Let $\boldsymbol{L} = \mathrm{SL}_k$, which is a simple simply connected split algebraic group defined over the global field $\boldsymbol{k} = K$, with relative rank $\boldsymbol{\ell} = k-1$, and with exponents of its Weyl group $\boldsymbol{m}_i = i$ for $i \in [\![1, \boldsymbol{\ell}]\!]$ (see [Bou, page 251]). Let $\boldsymbol{S} = \{\boldsymbol{v} = \nu\}$, which is a finite nonempty set of places of $\boldsymbol{k}$, and note that there are no archimedean places since K is a function field. The function ring $\boldsymbol{O_S}$ defined in [Ser1, page 123] is then exactly our function ring R_ν, and $\boldsymbol{L_0} = \mathrm{SL}_k$ is a split, simple, simply connected group scheme over $\boldsymbol{O_S} = R_\nu$ (as required in [Ser1, page 157]) such that $\boldsymbol{L} = \boldsymbol{L_0} \otimes_{\boldsymbol{O_S}} \boldsymbol{k}$. The zeta function $\boldsymbol{\zeta_{k,S}}$ of $\boldsymbol{k}$ related to $\boldsymbol{S}$ defined in [Ser1, page 156] is exactly our zeta function ζ_K, and the $\boldsymbol{S}$-arithmetic group $\boldsymbol{\Gamma_S}$ defined in [Ser1, page 157] is exactly our arithmetic group $\mathrm{SL}_k(R_\nu)$.

Let $\boldsymbol{G} = \boldsymbol{L}(\boldsymbol{k_v}) = \mathrm{SL}_k(K_\nu)$. Motivated by the relationship with the Euler characteristic, Serre defines a canonical signed measure (with constant sign by homogeneity) $\boldsymbol{\mu_G}$ on $\boldsymbol{G}$, whose associated positive measure $|\boldsymbol{\mu_G}|$ is a Haar measure on $\boldsymbol{G}$. We don't need to recall its definition, only to understand its normalisation. By the second claim of Theorem 7 (see top of page 151) of [Ser1], using when $k=1$ the standard convention that an empty product is equal to 1, and since the order of the residual field of $\boldsymbol{k_v} = K_\nu$ is $\boldsymbol{q} = q_\nu$, we have

$$|\boldsymbol{\mu_G}|(\mathrm{SL}_k(\mathscr{O}_\nu)) = \prod_{i=1}^{\boldsymbol{\ell}} (\boldsymbol{q}^{\boldsymbol{m}_i} - 1) = \prod_{i=1}^{k-1} (q_\nu^{\,i} - 1)\;.$$

Since our Haar measure of $\mathrm{SL}_k(K_\nu)$ is normalized so that $\mu_{\mathrm{SL}_k(K_\nu)}(\mathrm{SL}_k(\mathscr{O}_\nu)) = 1$, we hence have

$$\mu_{\mathrm{SL}_k(K_\nu)} = \frac{1}{|\boldsymbol{\mu_G}|(\mathrm{SL}_k(\mathscr{O}_\nu))}|\boldsymbol{\mu_G}| = \prod_{i=1}^{k-1} (q_\nu^{\,i} - 1)^{-1} |\boldsymbol{\mu_G}|\;. \tag{38}$$

By, for instance, [LanR] (see also [Wei2, Theo 3.3.1] and [Wei1, p. 257]), the Tamagawa number $\boldsymbol{\tau}$ of $\boldsymbol{L}$ is 1. By the footnote 10 on page 158 of [Ser1], we hence have

$$|\boldsymbol{\mu_G}|(\boldsymbol{G}/\boldsymbol{\Gamma_S}) = \Big|\boldsymbol{\tau}\prod_{i=1}^{\boldsymbol{\ell}} \boldsymbol{\zeta_{k,S}}(-\boldsymbol{m}_i)\Big| = \prod_{i=1}^{k-1} \zeta_K(-i)\;. \tag{39}$$

Thus, by Equations (38), (39) and (3), we have

$$\|\mu_{\mathrm{SL}_k(K_\nu)/\mathrm{SL}_k(R_\nu)}\| = \prod_{i=1}^{k-1} \frac{\zeta_K(-i)}{q_\nu^{\,i} - 1} = q^{\,(\mathfrak{g}-1)(k^2-k+1)} \prod_{i=1}^{k-1} \frac{\zeta_K(1+i)}{q_\nu^{\,i} - 1}\;. \tag{40}$$

Therefore, by Equation (37), we have

$$\|\mu_{\mathrm{Lat}^1_k}\| = \frac{q_\nu - 1}{q_\nu(q-1)} \prod_{i=1}^{k-1} \frac{\zeta_K(-i)}{q_\nu^{\,i} - 1}\;. \tag{41}$$

3.3 The space of correlated pairs of R_ν-lattices

We define in this subsection a measured space $\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ of pairs of full R_ν-lattices in dimensions $\mathfrak{m}$ and $\mathfrak{n}$ with correlated normalized covolume, in which a version of the equidistribution results stronger than the ones stated in the introduction will take place (see Theorem 4.7).

For every $k \in \mathbb{N} \smallsetminus \{0\}$, using the canonical K_ν-basis $(e_1, \dots, e_k)$ of $V = K_\nu{}^k$, we identify the maximal exterior power $\wedge^k V$ with K_ν by the map $\lambda(e_1 \wedge \cdots \wedge e_k) \mapsto \lambda$. Using this identification, given $\Lambda \in \operatorname{Lat}^1_k$ and an R_ν-basis $(b_1, \dots, b_k)$ of Λ, which is also a K_ν-basis of V and therefore satisfies $b_1 \wedge \cdots \wedge b_k \neq 0$, we define

$$\det \Lambda = [b_1 \wedge \cdots \wedge b_k] \in K_\nu{}^\times / \mathbb{F}_q{}^\times ,$$

where the class $[b_1 \wedge \cdots \wedge b_k]$ of $b_1 \wedge \cdots \wedge b_k$ modulo multiplication by an element of $\mathbb{F}_q^\times$ does not depend on the choice of the R_ν-basis of Λ, since the change of R_ν-basis matrix belongs to $\operatorname{GL}_k(R_\nu)$, hence has determinant in $R_\nu^\times = \mathbb{F}_q^\times$. Note that the ratio $\frac{a}{a'}$ of two elements $a = \lambda \mathbb{F}_q{}^\times$ and $a' = \lambda' \mathbb{F}_q{}^\times$ of $K_\nu{}^\times / \mathbb{F}_q{}^\times$ is a well-defined element $\frac{a}{a'} = \frac{\lambda}{\lambda'} \mathbb{F}_q{}^\times$ of $K_\nu{}^\times / \mathbb{F}_q{}^\times$, and we will also denote by $\mathbb{F}_q{}^\times$ the class of 1 in $K_\nu{}^\times / \mathbb{F}_q{}^\times$. For instance, $\det R_\nu{}^k = [e_1 \wedge \cdots \wedge e_k] = \mathbb{F}_q{}^\times$.

Let us define

$$\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}} = \Big\{ (\Lambda, \Lambda') \in \operatorname{Lat}^1_{\mathfrak{m}} \times \operatorname{Lat}^1_{\mathfrak{n}} : \frac{\det \Lambda}{\det \Lambda'} = \mathbb{F}_q^\times \Big\} .$$

For instance, $(R_\nu^{\mathfrak{m}}, R_\nu^{\mathfrak{n}})$ belongs to $\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$. We endow the product space $\operatorname{Lat}^1_{\mathfrak{m}} \times \operatorname{Lat}^1_{\mathfrak{n}}$ with the product measure $\mu_{\operatorname{Lat}^1_{\mathfrak{m}}} \otimes \mu_{\operatorname{Lat}^1_{\mathfrak{n}}}$ of the measures $\mu_{\operatorname{Lat}^1_{\mathfrak{m}}}$ and $\mu_{\operatorname{Lat}^1_{\mathfrak{n}}}$ defined in Section 3.2. For every $\lambda \in \mathscr{O}_\nu{}^\times$, let

$$s_{\mathbf{d}}(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & I_{\mathbf{d}-1} \end{pmatrix} \in \operatorname{GL}_{\mathbf{d}}(\mathscr{O}_\nu) .$$

We endow $\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ with the measure $\mu_{\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}}$ defined, for every Borel subset B of $\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$, by

$$\mu_{\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(B) = (\mu_{\operatorname{Lat}^1_{\mathfrak{m}}} \otimes \mu_{\operatorname{Lat}^1_{\mathfrak{n}}})\big(s_{\mathbf{d}}(\mathscr{O}_\nu{}^\times) B\big) , \tag{42}$$

which satisfies

$$\| \mu_{\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}} \| = \| \mu_{\operatorname{Lat}^1_{\mathfrak{m}}} \| \, \| \mu_{\operatorname{Lat}^1_{\mathfrak{n}}} \| \tag{43}$$

since $s_{\mathbf{d}}(\mathscr{O}_\nu{}^\times) \operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}} = \operatorname{Lat}^1_{\mathfrak{m}} \times \operatorname{Lat}^1_{\mathfrak{n}}$.

Recall that $g \mapsto \check{g} = {}^t g^{-1}$ for every $g \in \operatorname{GL}^1_{\mathfrak{n}}(K_\nu)$ is the standard Cartan involution of $\operatorname{GL}^1_{\mathfrak{n}}(K_\nu)$. The group $G'' = \big\{ \big(\begin{smallmatrix} \overline{g} & 0 \\ 0 & \underline{g} \end{smallmatrix}\big) : \overline{g} \in \operatorname{GL}^1_{\mathfrak{m}}(K_\nu),\ \underline{g} \in \operatorname{GL}^1_{\mathfrak{n}}(K_\nu),\ \det \overline{g} \ \det \underline{g} = 1 \big\}$ acts continuously on the product space $\operatorname{Lat}^1_{\mathfrak{m}} \times \operatorname{Lat}^1_{\mathfrak{n}}$ by

$$\Big(\big(\begin{smallmatrix} \overline{g} & 0 \\ 0 & \underline{g} \end{smallmatrix}\big), (\Lambda, \Lambda') \Big) \mapsto \big(\overline{g}\, \Lambda, \check{\underline{g}}\, \Lambda' \big) ,$$

since $\operatorname{GL}^1_k(K_\nu)$ preserves the set of unimodular lattices Lat^1_k for every $k \in \mathbb{N} \smallsetminus \{0\}$.

Lemma 3.2 *The orbit of $(R_\nu^{\mathfrak{m}}, R_\nu^{\mathfrak{n}})$ by this action of G'' is exactly $\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$, and the restriction to $\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ of the action of G'' preserves the measure $\mu_{\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}}$.*

We denote by $\pi_{\mathfrak{m},\mathfrak{n}} : G'' \to \mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ the twisted canonical projection defined by

$$\pi_{\mathfrak{m},\mathfrak{n}} : \left(\begin{smallmatrix}\alpha & 0\\ 0 & \delta\end{smallmatrix}\right) \mapsto (\alpha R_\nu^{\mathfrak{m}}, \check{\delta} R_\nu^{\mathfrak{n}}) \;. \tag{44}$$

Since the stabilizer of $(R_\nu^{\mathfrak{m}}, R_\nu^{\mathfrak{n}})$ by the above action of G'' is exactly the discrete subgroup $G''(R_\nu)$, and since $\pi_{\mathfrak{m},\mathfrak{n}}$ is the orbital map of the above action at $(R_\nu^{\mathfrak{m}}, R_\nu^{\mathfrak{n}})$, we will identify from now on the quotient space $G''/G''(R_\nu)$ and the subspace $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ by the homeomorphism $g\,G''(R_\nu) \mapsto \pi_{\mathfrak{m},\mathfrak{n}}(g) = g(R_\nu^{\mathfrak{m}}, R_\nu^{\mathfrak{n}})$.

Proof. For all $k \in \mathbb{N} \smallsetminus \{0\}$, $g \in \mathrm{GL}^1_k(K_\nu)$ and $\Lambda \in \mathrm{Lat}^1_k$, we have $\det(g\Lambda) = \det(g)\det(\Lambda)$. Since $\det(\,\check{\underline{g}}\,) = (\det \underline{g}\,)^{-1}$, the action of G'' preserves $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$.

Conversely, let $(\Lambda, \Lambda') \in \mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$. Since the action of $\mathrm{GL}^1_k(K_\nu)$ on Lat^1_k is transitive for every $k \in \mathbb{N} \smallsetminus \{0\}$ and since the map $g \mapsto \check{g}$ is an automorphism of $\mathrm{GL}^1_k(K_\nu)$, there exist $\overline{g} \in \mathrm{GL}^1_{\mathfrak{m}}(K_\nu)$ and $\underline{g} \in \mathrm{GL}^1_{\mathfrak{n}}(K_\nu)$ such that $\Lambda = \overline{g}\, R_\nu^{\mathfrak{m}}$ and $\Lambda' = \check{\underline{g}}\, R_\nu^{\mathfrak{n}}$. Since $(\Lambda, \Lambda') \in \mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$, we have $\lambda = \det \overline{g}\, \det \underline{g} \in \mathbb{F}_q^{\,\times} = R_\nu^{\,\times}$. Then the $\mathfrak{m} \times \mathfrak{m}$ diagonal matrix $s_{\mathfrak{m}}(\lambda)$ with diagonal entries $\lambda, 1, \dots, 1$ belongs to $\mathrm{GL}_{\mathfrak{m}}(R_\nu)$, and in particular $s_{\mathfrak{m}}(\lambda)^{-1} R_\nu^{\mathfrak{m}} = R_\nu^{\mathfrak{m}}$. Then the matrix $\left(\begin{smallmatrix}\overline{g}\, s_{\mathfrak{m}}(\lambda)^{-1} & 0\\ 0 & \underline{g}\end{smallmatrix}\right)$ belongs to G'' and maps $(R_\nu^{\mathfrak{m}}, R_\nu^{\mathfrak{n}})$ to (Λ, Λ'). This proves the first claim of the lemma.

The second claim follows by the invariance of the product measure $\mu_{\mathrm{Lat}^1_{\mathfrak{m}}} \otimes \mu_{\mathrm{Lat}^1_{\mathfrak{n}}}$ under the product group $\mathrm{GL}^1_{\mathfrak{m}}(K_\nu) \times \mathrm{GL}^1_{\mathfrak{n}}(K_\nu)$. □

As for the Grassmannian space $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$, in order to be able to define locally constant functions on $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ for our error term estimates, we now define a natural distance on the space $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$.

Let $k \in \mathbb{N} \smallsetminus \{0\}$. Since the supremum norm $\|\ \|$ on $\mathscr{M}_k(K_\nu)$ is a submultiplicative norm (see above Lemma 3.1), the map

$$d : (g, h) \mapsto \log_{q_\nu}(1 + \max\{\|\, gh^{-1} - I_k \,\|,\ \|\, hg^{-1} - I_k \,\|\}) \tag{45}$$

is well-known to be a distance on the locally compact group $\mathrm{GL}_k(K_\nu)$ (inducing its topology). By construction, this distance is invariant by translations on the right by all elements of $\mathrm{GL}_k(K_\nu)$. It is also invariant by translations on the left by the elements of $\mathrm{GL}_k(\mathscr{O}_\nu)$, since the supremum norm $\|\ \|$ on $\mathscr{M}_k(K_\nu)$ is invariant under conjugation by any element of $\mathrm{GL}_k(\mathscr{O}_\nu)$. Since the transposition map preserves the supremum norm $\|\ \|$ on $\mathscr{M}_k(K_\nu)$ and by the symmetry of the distance d on $\mathrm{GL}_k(K_\nu)$, the map $g \mapsto \check{g}$ is an isometry of d. In particular, for all $g \in \mathrm{GL}_k(K_\nu)$ and $\rho > 0$, we have

$$\widetilde{B(g,\rho)} = B(\,\check{g}, \rho) \;.$$

The following lemma will be needed in Subsection 4.2.

Lemma 3.3 *For all $h, h_0 \in \mathrm{GL}_k(K_\nu)$, we have*

$$\|h\| \leqslant \|h_0\|\; q_\nu^{\,d(h,\,h_0)} \quad \textit{and} \quad \|h^{-1}\| \leqslant \|h_0^{-1}\|\; q_\nu^{\,d(h,\,h_0)} \;.$$

Proof. Let $t = d(h,\, h_0)$. We have $\log_{q_\nu}(1 + \|\, hh_0^{-1} - I_k \,\|) \leqslant t$, hence $\|\, hh_0^{-1} - I_k \,\| \leqslant q_\nu^{\,t} - 1$, thus

$$\|h\| - \|h_0\| \leqslant \|h - h_0\| = \|(hh_0^{-1} - I_k)h_0\| \leqslant \|hh_0^{-1} - I_k\|\; \|h_0\| \leqslant (q_\nu^{\,t} - 1)\|h_0\| \;.$$

The first result follows. The second result follows similarly (or since the map $g \mapsto \check{g}$ is an isometry and the transposition preserves the supremum norm of matrices). $\square$

We endow every closed subgroup H of $\mathrm{GL}_k(K_\nu)$ with the induced distance, and, with $H(R_\nu) = H \cap \mathrm{GL}_k(R_\nu)$ which is a discrete subgroup of H, we endow the quotient space $H/H(R_\nu)$ with the quotient distance

$$\forall\, g, h \in H, \quad d(g\,H(R_\nu), h\,H(R_\nu)) = \min_{\gamma \in H(R_\nu)} d(g, h\gamma)\ .$$

It is easy to check by Equation (45) that for all $g = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \delta\end{smallmatrix}\right), g' = \left(\begin{smallmatrix}\alpha' & 0\\ 0 & \delta'\end{smallmatrix}\right) \in G''$, we have

$$d_{\mathrm{GL}_{\mathfrak{d}}(K_\nu)}(g, g') = \max\{d_{\mathrm{GL}_{\mathfrak{m}}(K_\nu)}(\alpha, \alpha'), d_{\mathrm{GL}_{\mathfrak{n}}(K_\nu)}(\delta, \delta')\}\ . \tag{46}$$

The canonical projection $H \to H/H(R_\nu)$ is 1-lipschitz and is a local isometry since the discrete subgroup $H(R_\nu)$ of H acts isometrically by right-translations on H. The action of $H(\mathscr{O}_\nu) = H \cap \mathrm{GL}_k(\mathscr{O}_\nu)$ by translations on the left on $H/H(R_\nu)$ is isometric. This process provides the homogeneous spaces $\mathrm{Lat}^1_{\mathfrak{m}} = \mathrm{GL}^1_{\mathfrak{m}}(K_\nu)/\mathrm{GL}_{\mathfrak{m}}(R_\nu)$, $\mathrm{Lat}^1_{\mathfrak{n}} = \mathrm{GL}^1_{\mathfrak{n}}(K_\nu)/\mathrm{GL}_{\mathfrak{n}}(R_\nu)$ and $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}} = G''/G''(R_\nu)$ with distances invariant under $\mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)$, $\mathrm{GL}_{\mathfrak{n}}(\mathscr{O}_\nu)$ and $G''(\mathscr{O}_\nu)$ respectively, that from now on we will consider on these spaces. In particular, the map $\pi_{\mathfrak{m},\mathfrak{n}}$ from G'' to $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ defined in Equation (44) is a local isometry.

3.4 The spaces of shapes of unimodular full lattices

Let $k \in \mathbb{N} \smallsetminus \{0\}$. As defined in Subsection 2.6, the space of shapes of full unimodular R_ν-lattices of $K_\nu{}^k$ is the locally compact metrisable separable (actually discrete and countably infinite) quotient space

$$\mathrm{Sh}^1_k = \mathrm{GL}_k(\mathscr{O}_\nu)\backslash \mathrm{Lat}^1_k = \mathrm{GL}_k(\mathscr{O}_\nu)\backslash \mathrm{GL}^1_k(K_\nu)\,/\,\mathrm{GL}_k(R_\nu)\ .$$

We endow Sh^1_k with the unique finite measure $\mu_{\mathrm{Sh}^1_k}$ such that the left invariant finite measure $\mu_{\mathrm{Lat}^1_k}$ on the right homogeneous space $\mathrm{Lat}^1_k = \mathrm{GL}^1_k(K_\nu)/\mathrm{GL}_k(R_\nu)$ disintegrates with respect to the proper canonical projection $\mathrm{sh} : \mathrm{Lat}^1_k \to \mathrm{Sh}^1_k = \mathrm{GL}_k(\mathscr{O}_\nu)\backslash \mathrm{Lat}^1_k$ over $\mu_{\mathrm{Sh}^1_k}$ with conditional measures on the fibers $\mathrm{GL}_k(\mathscr{O}_\nu)\Lambda$ the pushforward measures $\mu_{\mathrm{GL}_k(\mathscr{O}_\nu)\Lambda}$ of the finite Haar measure $\mu_{\mathrm{GL}_k(\mathscr{O}_k)} = \mu_{\mathrm{GL}^1_k(K_\nu)}{}_{|\mathrm{GL}_k(\mathscr{O}_\nu))}$ of the compact (hence unimodular) group $\mathrm{GL}_k(\mathscr{O}_\nu)$ by the orbital maps $g \mapsto g\Lambda$: for every $f \in C^0(\mathrm{Lat}^1_k)$, we have

$$\begin{aligned}&\int_{\Lambda \in \mathrm{Lat}^1_k} f(\Lambda)\; d\mu_{\mathrm{Lat}^1_k}(\Lambda)\\ =\ &\int_{\mathrm{GL}_k(\mathscr{O}_\nu)\Lambda \in \mathrm{Sh}^1_k} \int_{g \in \mathrm{GL}_k(\mathscr{O}_\nu)} f(g\Lambda)\; d\mu_{\mathrm{GL}_k(\mathscr{O}_\nu)}(g)\; d\mu_{\mathrm{Sh}^1_k}(\mathrm{GL}_k(\mathscr{O}_\nu)\Lambda)\ .\end{aligned}$$

In particular, using Equation (36), we have

$$\mathrm{sh}_*\, \mu_{\mathrm{Lat}^1_k} = \|\mu_{\mathrm{GL}_k(\mathscr{O}_\nu)}\|\; \mu_{\mathrm{Sh}^1_k} = (1 - q_\nu{}^{-1})\; \mu_{\mathrm{Sh}^1_k}\ , \tag{47}$$

and by Equation (41), we have

$$\|\mu_{\mathrm{Sh}^1_k}\| = \frac{q_\nu\, \|\mu_{\mathrm{Lat}^1_k}\|}{q_\nu - 1} = \frac{1}{q-1} \prod_{i=1}^{k-1} \frac{\zeta_K(-i)}{q_\nu^{\,i} - 1}\ .$$

With the notation of Equation (27) (that greatly simplifies as indicated below it), we define a map

$$\begin{array}{rcl}\varphi_{\mathfrak{m},\mathfrak{n}}: \ \mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}} & \rightarrow & \mathrm{Sh}^1_{\mathfrak{m}}\times\mathrm{Sh}^1_{\mathfrak{n}} = \big(\,\mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)\backslash\,\mathrm{Lat}^1_{\mathfrak{m}}\,\big)\times\big(\,\mathrm{GL}_{\mathfrak{n}}(\mathscr{O}_\nu)\backslash\,\mathrm{Lat}^1_{\mathfrak{n}}\,\big)\\ (\Lambda,\Lambda') & \mapsto & (\mathrm{sh}(\Lambda),\mathrm{sh}(\Lambda')) = (\mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)\Lambda,\ \mathrm{GL}_{\mathfrak{n}}(\mathscr{O}_\nu)\Lambda')\ .\end{array} \tag{48}$$

We summarize its properties in the following lemma, after giving some notation.

Since $G''(\mathscr{O}_\nu)$ is compact and open in G'', there exists a maximal $\rho_0 > 0$ such that $G''(\mathscr{O}_\nu)$ contains the (closed) ball $B_{G''}(I_{\mathbf{d}},\rho_0)$ of center $I_{\mathbf{d}}$ and radius ρ_0 for the distance on G'' defined at the end of the previous subsection 3.3. Thus every map $f:\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}\rightarrow\mathbb{C}$ which is constant on every left $G''(\mathscr{O}_\nu)$-orbit in $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ is ρ_0-locally constant, that is constant on every ball of radius ρ_0 in $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$.

For every $g\in\mathscr{U}_G$, if $g = u^-\,g''\,z\,u^+$ with $g'' = \big(\begin{smallmatrix}\overline{g} & 0\\ 0 & \underline{g}\end{smallmatrix}\big)$ is its unique writing given by Proposition 2.5, we define the *correlated pair of lattices* $[\![\Lambda_g]\!]$ *associated with* g by

$$[\![\Lambda_g]\!] = \pi_{\mathfrak{m},\mathfrak{n}}(g'') = (\,\overline{g}\,R_\nu^{\ \mathfrak{m}},\,\check{\underline{g}}\,R_\nu^{\ \mathfrak{n}}\,)\in\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}\ . \tag{49}$$

Lemma 3.4 *The map $\varphi_{\mathfrak{m},\mathfrak{n}}$ is proper, surjective, and satisfies the following properties.*

(1) *We have* $(\varphi_{\mathfrak{m},\mathfrak{n}})_*\mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}} = (1-q_\nu{}^{-1})^2\,\mu_{\mathrm{Sh}^1_{\mathfrak{m}}}\otimes\mu_{\mathrm{Sh}^1_{\mathfrak{n}}}$.

(2) *For every $g\in G^\sharp$ (as defined just above Proposition 2.8), we have*

$$\varphi_{\mathfrak{m},\mathfrak{n}}([\![\Lambda_g]\!]) = (\mathrm{sh}(\Lambda_g),\mathrm{sh}((\Lambda_g)^\perp))\ .$$

(3) *For all functions $f_1:\mathrm{Sh}^1_{\mathfrak{m}}\rightarrow\mathbb{R}$ and $f_2:\mathrm{Sh}^1_{\mathfrak{n}}\rightarrow\mathbb{R}$ with finite support, denoting by $f_1\times f_2:\mathrm{Sh}^1_{\mathfrak{m}}\times\mathrm{Sh}^1_{\mathfrak{n}}\rightarrow\mathbb{R}$ their product map $(x,y)\mapsto f_1(x)f_2(y)$, the composition function $(f_1\times f_2)\circ\varphi_{\mathfrak{m},\mathfrak{n}}:\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}\rightarrow\mathbb{R}$ is compactly supported and ρ_0-locally constant with $\|(f_1\times f_2)\circ\varphi_{\mathfrak{m},\mathfrak{n}}\|_{\rho_0}\leqslant\frac{1}{\rho_0}\,\|f_1\|_\infty\,\|f_2\|_\infty$.*

Proof. Since the groups $\mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)$ and $\mathrm{GL}_{\mathfrak{n}}(\mathscr{O}_\nu)$ are compact, the map $\varphi_{\mathfrak{m},\mathfrak{n}}$ is proper.

(1) By Equations (42) and (47) for $k=\mathfrak{m}$ and $k=\mathfrak{n}$, the measures $(\varphi_{\mathfrak{m},\mathfrak{n}})_*\mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}$ and $\mu_{\mathrm{Sh}^1_{\mathfrak{m}}}\otimes\mu_{\mathrm{Sh}^1_{\mathfrak{n}}}$ are proportional. The proportionality constant is given by Equations (43) and (47).

(2) Let $g\in G^\sharp$. Let $u^-\in U^-(\mathscr{O}_\nu)$, $g'' = \big(\begin{smallmatrix}\overline{g} & 0\\ 0 & \underline{g}\end{smallmatrix}\big)\in G''$, $z\in Z$ and $u^+\in U^+$ be such that $g = u^-\,g''\,z\,u^+$, as in Proposition 2.5. Then by Equation (49), by the definition of $\varphi_{\mathfrak{m},\mathfrak{n}}$ and by Proposition 2.8 (iii) and (iii)$^\perp$, we have

$$\varphi_{\mathfrak{m},\mathfrak{n}}([\![\Lambda_g]\!]) = \varphi_{\mathfrak{m},\mathfrak{n}}(\,\overline{g}\,R_\nu^{\ \mathfrak{m}},\,\check{\underline{g}}\,R_\nu^{\ \mathfrak{n}}\,) = (\mathrm{sh}(\,\overline{g}\,R_\nu^{\ \mathfrak{m}}),\,\mathrm{sh}(\,\check{\underline{g}}\,R_\nu^{\ \mathfrak{n}}\,)) = (\mathrm{sh}(\Lambda_g),\mathrm{sh}((\Lambda_g)^\perp))\ .$$

(3) Since $\varphi_{\mathfrak{m},\mathfrak{n}}$ is proper and $f_{1,2} = f_1\times f_2$ compactly supported, the function $f_{1,2}\circ\varphi_{\mathfrak{m},\mathfrak{n}}$ is compactly supported. Let us prove that for all points $x,x_0\in\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ at distance at most ρ_0, we have $f_{1,2}\circ\varphi_{\mathfrak{m},\mathfrak{n}}(x) = f_{1,2}\circ\varphi_{\mathfrak{m},\mathfrak{n}}(x_0)$. Since $\|f_{1,2}\circ\varphi_{\mathfrak{m},\mathfrak{n}}\|_\infty = \|f_{1,2}\|_\infty\leqslant\|f_1\|_\infty\,\|f_2\|_\infty$, this will prove Assertion (3).

Recall that by the end of Subsection 3.3, the distance on $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ is the quotient distance of the distance $d_{G''}$ on G'' by the onto map $\pi_{\mathfrak{m},\mathfrak{n}}$. Let $\widetilde{x}_0\in G''$ be such that $\pi_{\mathfrak{m},\mathfrak{n}}(\widetilde{x}_0) = x_0$. Since the action by right translations of G'' on itself is isometric and since the preimages of $\pi_{\mathfrak{m},\mathfrak{n}}$ are the right orbits of $G''(R_\nu)$ in G'', there exists $\widetilde{x}\in G''$ such that $\pi_{\mathfrak{m},\mathfrak{n}}(\widetilde{x}) = x$ and $d_{G''}(\widetilde{x},\widetilde{x}_0)\leqslant\rho_0$. Again since the action by right translations of G'' on itself is isometric and by the definition of ρ_0, we have $g = \widetilde{x}_0\,\widetilde{x}^{-1}\in B_{G''}(\mathrm{id},\rho_0)\subset G''(\mathscr{O}_\nu)$. Since $\pi_{\mathfrak{m},\mathfrak{n}}$ is G''-equivariant, we have $g\,x = \pi_{\mathfrak{m},\mathfrak{n}}(g\,\widetilde{x}) = \pi_{\mathfrak{m},\mathfrak{n}}(\widetilde{x_0}) = x_0$. Since $\varphi_{\mathfrak{m},\mathfrak{n}}$ is constant on the left orbits of $G''(\mathscr{O}_\nu)$, we have $\varphi_{\mathfrak{m},\mathfrak{n}}(x) = \varphi_{\mathfrak{m},\mathfrak{n}}(x_0)$, therefore $f_{1,2}\circ\varphi_{\mathfrak{m},\mathfrak{n}}(x) = f_{1,2}\circ\varphi_{\mathfrak{m},\mathfrak{n}}(x_0)$. $\square$

4 Joint equidistribution of primitive partial lattices

4.1 The correspondence between primitive partial lattices and integral group elements

The aim of this subsection is to naturally and injectively associate elements in the modular group $\Gamma = \mathrm{SL}_{\mathbf{d}}(R_\nu)$ of integral points of $G = \mathrm{SL}_{\mathbf{d}}(K_\nu)$ to primitive $\mathfrak{m}$-lattices in $R_\nu^{\mathbf{d}}$. We start by introducing the subsets of the group G and of the moduli spaces $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}$, $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$, $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ which will be technically useful.

We fix from now on a compact-open strict fundamental domain $\mathscr{D}$ for the action by translations of R_ν on K_ν (for instance $\mathbf{d} = \pi_\nu\mathscr{O}_\nu$ when $K = \mathbb{F}_q(Y)$ and $\nu = \nu_\infty$), such that for every $x \in \mathscr{D}$, the (closed) ball $B(x, q_\nu^{-1}) = x + \pi_\nu\mathscr{O}_\nu$ in K_ν is contained in $\mathscr{D}$. This is possible since $R_\nu \cap \pi_\nu\mathscr{O}_\nu = \{0\}$ by Equation (2). We thus have a compact-open strict fundamental domain

$$\square = \big\{(x_{i,j})_{1\leqslant i\leqslant \mathfrak{m}, 1\leqslant j\leqslant \mathfrak{n}} \in \mathscr{M}_{\mathfrak{m},\mathfrak{n}}(K_\nu) : \forall\ i \in [\![1,\mathfrak{m}]\!],\ \forall\ j \in [\![1,\mathfrak{n}]\!],\ \ x_{i,j} \in \mathscr{D}\big\}$$

for the action by translations of the R_ν-lattice $\mathscr{M}_{\mathfrak{m},\mathfrak{n}}(R_\nu)$ on the K_ν-vector space $\mathscr{M}_{\mathfrak{m},\mathfrak{n}}(K_\nu)$, for instance $\square = \mathscr{M}_{\mathfrak{m},\mathfrak{n}}(\pi_\nu\mathscr{O}_\nu)$ if $K = \mathbb{F}_q(Y)$ and $\nu = \nu_\infty$. We also fix a closed-open strict fundamental domain $G''_\lozenge$ for the action by translations on the right of the discrete subgroup $G''(R_\nu)$ on G'', so that we have $G'' = G''_\lozenge\, G''(R_\nu)$ with unique writing.

For every $r \in \mathbb{Z}$ and for all measurable subsets Ψ of $\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu)$ and $\mathscr{F}$ of G'', we define

$$U^-_\Psi = \Big\{\big(\begin{smallmatrix} I_\mathfrak{m} & 0 \\ \beta & I_\mathfrak{n}\end{smallmatrix}\big) \in U^- : \beta \in \Psi\Big\}, \quad G''_{\mathscr{F}} = G''_\lozenge \cap \mathscr{F}\ , \tag{50}$$

$$Z_r = \Big\{\big(\begin{smallmatrix} \lambda I_\mathfrak{m} & 0 \\ 0 & \lambda' I_\mathfrak{n}\end{smallmatrix}\big) \in Z : \nu(\lambda) = \frac{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}}{\mathfrak{m}}\, r\Big\} \quad\text{and}\quad U^+_\square = \Big\{\big(\begin{smallmatrix} I_\mathfrak{m} & \gamma \\ 0 & I_\mathfrak{n}\end{smallmatrix}\big) \in U^+ : \gamma \in \square\Big\}\ .$$

Note that $Z = \bigsqcup_{r\in\mathbb{Z}} Z_r$ by Equation (23). As defined just before Proposition 2.8, let

$$G^\sharp = \{\big(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta\end{smallmatrix}\big) \in G : \nu(\det\alpha) \in \mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}\,\mathbb{Z},\ \ \beta\alpha^{-1} \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)\}\ ,$$

so that the product map $(u^-, g'', z, u^+) \mapsto u^-\, g''\, z\, u^+$ from $U^-(\mathscr{O}_\nu) \times G'' \times Z \times U^+$ to $G^\sharp$ is a homeomorphism, by Proposition 2.5. For every closed subgroup H of G, let

$$H^\sharp = H \cap G^\sharp\ . \tag{51}$$

We also define the corresponding subset of the set $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}$ of primitive $\mathfrak{m}$-lattices in $K_\nu^{\mathbf{d}}$ by

$$\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}} = \Gamma^\sharp(R_\nu^{\mathfrak{m}} \times \{0\})\ .$$

For every measurable subset $\mathscr{E}$ of $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$, with the map $\pi_{\mathfrak{m},\mathfrak{n}} : G'' \to \mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ introduced in Equation (44), we define

$$\widetilde{\mathscr{E}} = \pi_{\mathfrak{m},\mathfrak{n}}^{-1}(\mathscr{E}) \subset G''\ , \tag{52}$$

which is a measurable subset of G'' invariant by the translations on the right by $G''(R_\nu)$.

For every measurable subset Φ of $\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$, with $\mathrm{orb}_\mathfrak{m} : \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu) \to \mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$ the isometric map defined in Subsection 3.1, we define

$$\widetilde{\Phi} = \mathrm{orb}_\mathfrak{m}^{-1}(\Phi) \subset \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)\ , \tag{53}$$

which is a measurable subset of $\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$.

For every $r \in \mathbb{Z}$ and for all measurable subsets Φ of $\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$ and $\mathscr{E}$ of $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$, using the notation of Equation (50) with $\Psi = \widetilde{\Phi}$ and $\mathscr{F} = \widetilde{\mathscr{E}}$, we finally define

$$\Omega = U^-(\mathscr{O}_\nu)\, G''_\lozenge\, Z\, U^+_\square \subset G^\sharp \quad \text{and} \quad \Omega_{\Phi,\mathscr{E},r} = U^-_{\widetilde{\Phi}}\, G''_{\widetilde{\mathscr{E}}}\, Z_r\, U^+_\square \subset \Omega\,. \tag{54}$$

Lemma 4.1 *With* $c_1' = (q-1)\, q^{(\mathfrak{g}-1)\,\mathfrak{m}\,\mathfrak{n}}\, q_\nu^{\,2\,\mathfrak{m}\,\mathfrak{n}+2}\, \frac{\prod_{i=1}^{\mathfrak{m}}(q_\nu^{\,i}-1)^2 \prod_{i=1}^{\mathfrak{n}}(q_\nu^{\,i}-1)^2}{(q_\nu-1)^2\, \prod_{i=1}^{\mathbf{d}}(q_\nu^{\,i}-1)^2}$, *we have*

$$\mu_G(\Omega_{\Phi,\mathscr{E},r}) = c_1'\, q_\nu^{\,\mathbf{d}\,\mathrm{lcm}(\mathfrak{m},\mathfrak{n})\,r}\, \mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}(\Phi)\, \mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E})\,.$$

Proof. By Proposition 2.5, we have

$$\mu_G(\Omega_{\Phi,\mathscr{E},r}) = c_1\, \mu_{U^-}(U^-_{\widetilde{\Phi}})\, \mu_{G''}(G''_{\widetilde{\mathscr{E}}})\, \mu_{U^+}(U^+_\square) \int_{z\in Z_r} |\chi_{\mathfrak{m}}(z)|^{\mathbf{d}\,\mathfrak{m}}\, d\mu_Z(z)\,. \tag{55}$$

By Equations (18), (53) and (33), and since $\Phi \subset \mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$, we have

$$\mu_{U^-}(U^-_{\widetilde{\Phi}}) = \mathrm{Haar}_{\mathfrak{n},\mathfrak{m}}(\widetilde{\Phi}) = \mathrm{Haar}_{\mathfrak{n},\mathfrak{m}}(\mathrm{orb}_{\mathfrak{m}}^{-1}(\Phi)) = c_1\, \mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}(\Phi)\,. \tag{56}$$

By Equations (18) and (9), we have

$$\mu_{U^+}(U^+_\square) = \mathrm{Haar}_{\mathfrak{m},\mathfrak{n}}(\square) = \mathrm{Covol}(R_\nu^{\,\mathfrak{m}\,\mathfrak{n}}) = q^{(\mathfrak{g}-1)\,\mathfrak{m}\,\mathfrak{n}}\,. \tag{57}$$

Note that $z = \left(\begin{smallmatrix}\lambda I_{\mathfrak{m}} & 0\\ 0 & \mu I_{\mathfrak{n}}\end{smallmatrix}\right)$ belongs to Z_r if and only if $\chi_{\mathfrak{m}}(z) = \lambda = \pi_\nu^{\;-\frac{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}}{\mathfrak{m}}\,r}$. Since the Haar measure μ_Z is normalized so that $\mu_Z(Z(\mathscr{O}_\nu)) = 1$, we have

$$\int_{z\in Z_r} |\chi_{\mathfrak{m}}(z)|^{\mathbf{d}\,\mathfrak{m}} d\mu_Z(z) = |\pi_\nu^{\;-\frac{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\}}{\mathfrak{m}}\,r}|^{\mathbf{d}\,\mathfrak{m}} = q_\nu^{\,\mathbf{d}\,\mathrm{lcm}(\mathfrak{m},\mathfrak{n})\,r}\,. \tag{58}$$

Note that $\mathrm{GL}^1_{\mathfrak{n}}(A)$ for $A = R_\nu, \mathscr{O}_\nu, K_\nu$ is stable by the Cartan involution $\underline{g} \mapsto \underline{\breve{g}} = {}^t\underline{g}^{-1}$, and that this map preserves the Haar measure $\mu_{\mathrm{GL}^1_{\mathfrak{n}}(K_\nu)}$ (defined in Subsection 3.2) of the selfadjoint unimodular group $\mathrm{GL}^1_{\mathfrak{n}}(K_\nu)$. For every $\lambda \in \mathscr{O}_\nu^{\;\times}$, let

$$s'_{\mathfrak{m}}(\lambda) = \left(\left(\begin{smallmatrix}\lambda & 0\\ 0 & I_{\mathfrak{m}-1}\end{smallmatrix}\right), I_{\mathfrak{n}}\right) \in \mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)\times \mathrm{GL}_{\mathfrak{n}}(\mathscr{O}_\nu)\,.$$

Let $\iota : G'' \to \mathrm{GL}^1_{\mathfrak{m}}(K_\nu)\times \mathrm{GL}^1_{\mathfrak{n}}(K_\nu)$ be the group morphism defined by $\left(\begin{smallmatrix}\overline{g} & 0\\ 0 & \underline{g}\end{smallmatrix}\right) \mapsto (\overline{g}, \underline{\breve{g}})$. Any element of $\mathrm{GL}^1_{\mathfrak{m}}(K_\nu)\times \mathrm{GL}^1_{\mathfrak{n}}(K_\nu)$ may be written as $s'_{\mathfrak{m}}(\lambda)\,\iota(g'')$ for unique elements $\lambda \in \mathscr{O}_\nu^{\;\times}$ and $g'' \in G''$. Hence we have a split exact sequence

$$1 \longrightarrow G'' \overset{\iota}{\longrightarrow} \mathrm{GL}^1_{\mathfrak{m}}(K_\nu)\times \mathrm{GL}^1_{\mathfrak{n}}(K_\nu) \longrightarrow \mathscr{O}_\nu^{\;\times} \longrightarrow 1\,,$$

which induces two split exact sequences

$$1 \longrightarrow G''(\mathscr{O}_\nu) \overset{\iota}{\longrightarrow} \mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)\times \mathrm{GL}_{\mathfrak{n}}(\mathscr{O}_\nu) \longrightarrow \mathscr{O}_\nu^{\;\times} \longrightarrow 1\,,$$

$$1 \longrightarrow G''(R_\nu) \overset{\iota}{\longrightarrow} \mathrm{GL}_{\mathfrak{m}}(R_\nu)\times \mathrm{GL}_{\mathfrak{n}}(R_\nu) \longrightarrow R_\nu^{\;\times} \longrightarrow 1\,.$$

By the normalisation of the measures (see Equations (36), (16) and (34)), the Haar measure $\mu_{\mathrm{GL}^1_{\mathfrak{m}}(K_\nu)} \otimes \mu_{\mathrm{GL}^1_{\mathfrak{n}}(K_\nu)}$ of $\mathrm{GL}^1_{\mathfrak{m}}(K_\nu)\times \mathrm{GL}^1_{\mathfrak{n}}(K_\nu)$ satisfies

$$d(\mu_{\mathrm{GL}^1_{\mathfrak{m}}(K_\nu)} \otimes \mu_{\mathrm{GL}^1_{\mathfrak{n}}(K_\nu)})(s'_{\mathfrak{m}}(\lambda)\iota(g'')) = (1-q_\nu^{\;-1})\, d\mu_{G''}(g'')\, d\mu_{\mathscr{O}_\nu^{\;\times}}(\lambda)\,. \tag{59}$$

We endow from now on the product space $(\mathrm{GL}^1_{\mathfrak{m}}(K_\nu)/\mathrm{GL}_{\mathfrak{m}}(R_\nu)) \times (\mathrm{GL}^1_{\mathfrak{n}}(K_\nu)/\mathrm{GL}_{\mathfrak{n}}(R_\nu))$ with the product measure $\mu_{\mathrm{Lat}^1_{\mathfrak{m}}} \otimes \mu_{\mathrm{Lat}^1_{\mathfrak{n}}}$ (see Subsection 3.2 for the definition of the measures $\mu_{\mathrm{Lat}^1_k}$ for $k \in \mathbb{N} \smallsetminus \{0\}$) and the quotient space $G''/G''(R_\nu)$ with the measure $\mu_{G''/G''(R_\nu)}$ so that the two canonical projections (that are covering maps)

$$\mathrm{GL}^1_{\mathfrak{m}}(K_\nu) \times \mathrm{GL}^1_{\mathfrak{n}}(K_\nu) \to (\mathrm{GL}^1_{\mathfrak{m}}(K_\nu)/\mathrm{GL}_{\mathfrak{m}}(R_\nu)) \times (\mathrm{GL}^1_{\mathfrak{n}}(K_\nu)/\mathrm{GL}_{\mathfrak{n}}(R_\nu))$$

and $G'' \to G''/G''(R_\nu)$ locally preserve the measures. Using Equation (43) for the first equality below, and Equation (59) for the second one, since we have $\mathrm{Card}\, R_\nu^\times = q - 1$ by Equation (2) and again by Equation (34), we hence have

$$\|\, \mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}} \,\| = \|\, \mu_{\mathrm{Lat}^1_{\mathfrak{m}}} \otimes \mu_{\mathrm{Lat}^1_{\mathfrak{n}}} \,\| = \frac{(1 - {q_\nu}^{-1})^2}{q - 1} \,\|\, \mu_{G''/G''(R_\nu)} \,\| \,.$$

Since the measures $\mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}$ and $\mu_{G''/G''(R_\nu)}$ on $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}} = G''/G''(R_\nu)$ are both G''-invariant (see Equation (42) for the first one), we thus have $\mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}} = \frac{(q_\nu - 1)^2}{{q_\nu}^2 (q-1)} \; \mu_{G''/G''(R_\nu)}$. In particular, we have

$$\mu_{G''}(G''_{\widetilde{\mathscr{E}}}) = \mu_{G''}(G''_{\lozenge} \cap \widetilde{\mathscr{E}}) = \mu_{G''/G''(R_\nu)}(\mathscr{E}) = \frac{(q-1)\, {q_\nu}^2}{(q_\nu - 1)^2} \; \mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E}) \,. \tag{60}$$

Lemma 4.1 follows from Equation (55) by plugging in the computations of Equations (56), (60), (57) and (58), by defining the constant $c_1' = {c_1}^2 \; \frac{(q-1)\, {q_\nu}^2}{(q_\nu - 1)^2} \; q^{(\mathfrak{g}-1)\, \mathfrak{m}\, \mathfrak{n}}$ and by expliciting c_1 using Proposition 2.5. □

The following result gives a precise 1-to-1 correspondence between partial lattices in $\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}} = \Gamma^\sharp(R_\nu^{\mathfrak{m}} \times \{0\})$ and appropriate matrices in the discrete group $\Gamma = \mathrm{SL}_{\mathbf{d}}(R_\nu)$.

Proposition 4.2 *The map $g \mapsto \Lambda_g = g(R_\nu^{\mathfrak{m}} \times \{0\})$ from $\Gamma \cap \Omega$ to $\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}$ is a bijection such that for every nonzero ideal I of R_ν, for every $r \in \mathbb{Z}$ and for all measurable subsets Φ of $\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$ and $\mathscr{E}$ of $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$, the following two assertions are equivalent*

1. *the integral matrix $g \in \Gamma \cap \Omega$ lies in $\Omega_{\Phi,\mathscr{E},r} \cap \Gamma_I$,*
2. *the primitive $\mathfrak{m}$-lattice $\Lambda_g \in \mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}$ satisfies that $\Lambda_g \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)$ (as defined in the beginning of Subsection 2.4), $V_{\Lambda_g} \in \Phi$ (as defined in Subsection 2.2), $[\![\Lambda_g]\!] \in \mathscr{E}$ (as defined in Equation (49)) and $\overline{\mathrm{Covol}}(\Lambda_g) = \overline{\mathrm{Covol}}((\Lambda_g)^\perp) = {q_\nu}^{\mathrm{lcm}\{\mathfrak{m},\mathfrak{n}\} r}$.*

Proof. Since $\Omega \subset G^\sharp$, if $g \in \Gamma \cap \Omega$, then $g \in \Gamma^\sharp$, thus $\Lambda_g = g(R_\nu^{\mathfrak{m}} \times \{0\}) \in \mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}$. Hence the above map is well defined.

For all $g = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in G^\sharp$ and $p^+ = \begin{pmatrix} \alpha' & \gamma' \\ 0 & \delta' \end{pmatrix} \in P^+ = G'' Z U^+ \subset G^\sharp$, since the first column of $g\,p^+$ is $\begin{pmatrix} \alpha\alpha' \\ \beta\alpha' \end{pmatrix}$, since $(\beta\alpha')(\alpha\alpha')^{-1} = \beta\alpha^{-1}$ and since $\nu(\det(\alpha\alpha')) = \nu(\det\alpha) + \nu(\det\alpha')$, we have $g\,p^+ \in G^\sharp$. In particular, the action by right translations on $\Gamma = \mathrm{SL}_{\mathbf{d}}(R_\nu)$ of its subgroup $P^+(R_\nu)$ defined in Equation (11), which satisfies $P^+(R_\nu) = G''(R_\nu)\, U^+(R_\nu)$, preserves $\Gamma^\sharp = \Gamma \cap G^\sharp$. By the identification given just after the statement of Lemma 2.1, we thus have

$$\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}} = \Gamma^\sharp / P^+(R_\nu) \,. \tag{61}$$

Let us prove that

$$G^\sharp = \coprod_{\gamma \in P^+(R_\nu)} \Omega\, \gamma \,. \tag{62}$$

Since $P^+(R_\nu) \subset \Gamma$, this will imply that $\Gamma^\sharp = \Gamma \cap G^\sharp = \coprod_{\gamma \in P^+(R_\nu)} (\Gamma \cap \Omega)\gamma$. By Equation (61), this will imply that the map $g \mapsto \Lambda_g = g(R_\nu^{\mathfrak m} \times \{0\})$ from $\Gamma \cap \Omega$ to $\mathscr{PL}^\sharp_{\mathfrak m,\mathbf d}$ is a bijection.

In order to prove Equation (62), let us fix $g \in G^\sharp$. By Proposition 2.5, there exist unique elements $u^- \in U^-(\mathscr O_\nu)$, $g'' = \left(\begin{smallmatrix} \overline{g} & 0 \\ 0 & \underline{g} \end{smallmatrix}\right) \in G''$, $z \in Z$ and $u^+ \in U^+$ such that $g = u^- \, g'' \, z \, u^+$. By the definition of the fundamental domain $G''_\diamondsuit$, there exist a unique $f'' = \left(\begin{smallmatrix} \overline{f} & 0 \\ 0 & \underline{f} \end{smallmatrix}\right) \in G''_\diamondsuit$ and a unique $\gamma'' = \left(\begin{smallmatrix} \overline{\gamma} & 0 \\ 0 & \underline{\gamma} \end{smallmatrix}\right) \in G''(R_\nu)$ such that

$$\overline{f}\,\overline{\gamma} = \overline{g}, \quad \underline{f}\,\underline{\gamma} = \underline{g} \quad \text{and} \quad g'' = f''\,\gamma''\,. \tag{63}$$

Since Z centralizes G'', we have $g = u^- \, f'' \, z \, \gamma'' \, u^+$. Since G'' normalizes U^+, and by the definition of $\square$, there exist unique elements $u_0^+ \in U_\square^+$ and $\gamma_0 \in U^+(R_\nu)$ such that $\gamma'' \, u^+ (\gamma'')^{-1} = u_0^+ \, \gamma_0$. Defining $\gamma = \gamma_0 \gamma'' \in P^+(R_\nu)$, we have

$$g = u^- \, f'' \, z \, (\gamma'' \, u^+ (\gamma'')^{-1}) \, \gamma'' = u^- \, f'' \, z \, u_0^+ \gamma \tag{64}$$

and $u^- \, f'' \, z \, u_0^+ \in \Omega$ by the definition of Ω in Equation (54). Since the writing $h = u_0^+ \gamma$ of an element $h \in P^+(R_\nu) = U^+(R_\nu) \, G''(R_\nu)$ with $u_0^+ \in U^+(R_\nu)$ and $\gamma \in G''(R_\nu)$ is unique, this proves Equation (62).

Let us now assume that $g \in \Gamma \cap \Omega$. By the uniqueness of the writing in Equation (64), we may uniquely write $g = u^- \, f'' \, z \, u_0^+$ with $u^- \in U^-(\mathscr O_\nu)$, $f'' \in G''_\diamondsuit$, $z \in Z$ and $u_0^+ \in U_\square^+$. By the definition of $\Omega_{\Phi,\mathscr E,r}$ in Equation (54), we have $g \in \Omega_{\Phi,\mathscr E,r}$ if and only if $u^- \in U^-_{\widetilde\Phi}$, $f'' \in G''_{\widetilde{\mathscr E}}$ and $z \in Z_r$.

We have $u^- \in U^-_{\widetilde\Phi}$ if and only if there exists $\beta \in \widetilde\Phi$ with $u^- = \left(\begin{smallmatrix} I_{\mathfrak m} & 0 \\ \beta & I_{\mathfrak n} \end{smallmatrix}\right)$, hence if and only if there exists $\beta \in \widetilde\Phi = {\operatorname{orb}_{\mathfrak m}}^{-1}(\Phi)$ with $\operatorname{orb}_{\mathfrak m}(\beta) = u^-(K_\nu^{\mathfrak m} \times \{0\}) = V_{\Lambda_{u^-}}$ by the definition of $\operatorname{orb}_{\mathfrak m}$ in Subsection 3.1, therefore if and only if $V_{\Lambda_g} \in \Phi$ by Proposition 2.8 (i).

By Equation (52) and by the definition of $\pi_{\mathfrak m,\mathfrak n}$ in Equation (44), we have

$$G''_{\widetilde{\mathscr E}} = G''_\diamondsuit \cap \pi_{\mathfrak m,\mathfrak n}^{-1}(\mathscr E) = \left\{ \left(\begin{smallmatrix} \overline{g} & 0 \\ 0 & \underline{g} \end{smallmatrix}\right) \in G''_\diamondsuit : (\, \overline{g}\, R_\nu^{\mathfrak m}, \, \underline{\breve g}\, R_\nu^{\mathfrak n}\,) \in \mathscr E \right\}.$$

Hence by Equation (49) and since $f'' \in G''_\diamondsuit$, we have $f'' \in G''_{\widetilde{\mathscr E}}$ if and only if $[\![\Lambda_g]\!] \in \mathscr E$.

We have $z \in Z_r$ if and only if $z = \begin{pmatrix} {\pi_\nu}^{-\frac{\operatorname{lcm}\{\mathfrak m,\mathfrak n\}}{\mathfrak m} r} I_{\mathfrak m} & 0 \\ 0 & {\pi_\nu}^{\frac{\operatorname{lcm}\{\mathfrak m,\mathfrak n\}}{\mathfrak n} r} I_{\mathfrak n} \end{pmatrix}$, hence if and only if $\overline{\operatorname{Covol}}(\Lambda_g) = {q_\nu}^{\operatorname{lcm}\{\mathfrak m,\mathfrak n\}\, r}$ by Proposition 2.8 (ii) and by Equation (28). Note that we have $\overline{\operatorname{Covol}}((\Lambda_g)^\perp) = \overline{\operatorname{Covol}}(\Lambda_g)$ by Proposition 2.8 (ii)$^\perp$.

The fact that $g \in \Gamma_I$ if and only if $\Lambda_g \in \mathscr{PL}_{\mathfrak m,\mathbf d}(I)$ has been shown in Lemma 2.3 (1). This concludes the proof of Proposition 4.2. □

4.2 Counting in well-rounded families

A crucial tool of this paper is a counting result of lattice points by Gorodnik and Nevo [GN]. In this subsection, after the necessary definitions, we recall from [HP1] an adaptation of the Gorodnik-Nevo result, and we proceed to the construction of the well-rounded family of subsets to which we will apply it.

Let $\mathbf{G}'$ be an absolutely connected and simply connected semi-simple algebraic group over K_ν, which is almost K_ν-simple. Let $G' = \mathbf{G}'(K_\nu)$ be the locally compact group of K_ν-points of $\mathbf{G}'$. Let Γ' be a nonuniform[2] lattice in G', and let $\mu_{G'}$ be any (left) Haar measure of G'. Note that $G' = G$ and $\Gamma' = \Gamma_I$ (defined in Subsection 2.4) satisfy these assumptions for every nonzero ideal I of R_ν.

Let $\rho > 0$. Let $(\mathscr{V}'_\epsilon)_{\epsilon>0}$ be a fundamental system of neighborhoods of the identity in G', which

- is symmetric (that is, $x \in \mathscr{V}'_\epsilon$ if and only if $x^{-1} \in \mathscr{V}'_\epsilon$),
- is nondecreasing with ϵ (that is, $\mathscr{V}'_\epsilon \subset \mathscr{V}'_{\epsilon'}$ if $\epsilon \leqslant \epsilon'$), and
- has *upper local dimension* ρ, that is, there exist $m_1, \epsilon_1 > 0$ such that $\mu_{G'}(\mathscr{V}'_\epsilon) \geqslant m_1\, \epsilon^\rho$ for every $\epsilon \in \,]0, \epsilon_1[\,$.

Let $C \geqslant 0$. Let $(\mathscr{Z}_n)_{n\in\mathbb{N}}$ be a family of measurable subsets of G'. We define

$$(\mathscr{Z}_n)^{+\epsilon} = \mathscr{V}'_\epsilon \mathscr{Z}_n \mathscr{V}'_\epsilon = \bigcup_{g,h\in\mathscr{V}'_\epsilon} g\mathscr{Z}_n h \quad \text{and} \quad (\mathscr{Z}_n)^{-\epsilon} = \bigcap_{g,h\in\mathscr{V}'_\epsilon} g\mathscr{Z}_n h\ .$$

The family $(\mathscr{Z}_n)_{n\in\mathbb{N}}$ is *C-Lipschitz well-rounded* with respect to $(\mathscr{V}'_\epsilon)_{\epsilon>0}$ if there exists $\epsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $\epsilon \in \,]0, \epsilon_0[$ and $n \geqslant n_0$, we have

$$\mu_{G'}((\mathscr{Z}_n)^{+\epsilon}) \leqslant (1 + C\,\epsilon)\ \mu_{G'}((\mathscr{Z}_n)^{-\epsilon})\ .$$

We refer to [HP1, Theo. 4.1] for a proof of the following adaptation of results of Gorodnik-Nevo [GN].

Theorem 4.3 *For every $\rho > 0$, there exists $\tau(\Gamma') \in \,]0, \frac{1}{2(1+\rho)}]$ such that for every symmetric nondecreasing fundamental system $(\mathscr{V}'_\epsilon)_{\epsilon>0}$ of neighborhoods of the identity in G' with upper local dimension ρ, for every $C \geqslant 0$, for every family $(\mathscr{Z}_n)_{n\in\mathbb{N}}$ of measurable subsets of G' that is C-Lipschitz well-rounded with respect to $(\mathscr{V}'_\epsilon)_{\epsilon>0}$, and for every $\delta > 0$, we have that, as $n \to +\infty$,*

$$\Big|\mathrm{Card}(\mathscr{Z}_n \cap \Gamma') - \frac{1}{\|\mu_{G'/\Gamma'}\|}\ \mu_{G'}(\mathscr{Z}_n)\Big| = \mathrm{O}\left(\mu_{G'}(\mathscr{Z}_n)^{1-\tau(\Gamma')+\delta}\right)\ ,$$

where the function $\mathrm{O}(\cdot)$ depends only on $G', \Gamma', \delta, C, (\mathscr{V}'_\epsilon)_{\epsilon>0}, \rho$. □

We will use, as a fundamental system of neighborhoods of the identity element in G, a family of compact-open subgroups of $G(\mathscr{O}_\nu)$ given by the kernels of the morphisms of reduction modulo $\pi_\nu^N \mathscr{O}_\nu$ for appropriate $N \in \mathbb{N}$. For every $\epsilon > 0$, let $N_\epsilon = \big\lfloor -\log_{q_\nu} \epsilon \big\rfloor$ so that $N_\epsilon \geqslant 1$ if and only if $\epsilon \leqslant \frac{1}{q_\nu}$. Let $\mathscr{V}_\epsilon = G(\mathscr{O}_\nu)$ if $\epsilon > \frac{1}{q_\nu}$ and otherwise let

$$\begin{aligned}\mathscr{V}_\epsilon &= \ker(G(\mathscr{O}_\nu) \to \mathrm{SL}_{\mathbf{d}}(\mathscr{O}_\nu/\pi_\nu^{N_\epsilon}\mathscr{O}_\nu))\\ &= \big\{I_{\mathbf{d}} + \pi_\nu^{N_\epsilon} X : X \in \mathscr{M}_{\mathbf{d}}(\mathscr{O}_\nu)\big\} \cap G\ .\end{aligned} \tag{65}$$

The family $(\mathscr{V}_\epsilon)_{\epsilon>0}$ is indeed nondecreasing and we have $\bigcap_{\epsilon>0} \mathscr{V}_\epsilon = \{\mathrm{id}\}$. Note that for all $\epsilon_1, \dots, \epsilon_k > 0$, we have

$$\begin{aligned}\min\{N_{\epsilon_1}, \cdots, N_{\epsilon_k}\} &\geqslant \min\{-\log_{q_\nu} \epsilon_1, \cdots, -\log_{q_\nu} \epsilon_k\} - 1\\ &\geqslant -\log_{q_\nu}(\epsilon_1 + \cdots + \epsilon_k) - 1 \geqslant N_{q_\nu(\epsilon_1+\cdots+\epsilon_k)}\ ,\end{aligned}$$

[2]This implies that $\mathbf{G}'$ is isotropic over K_ν, as part of the assumptions of [GN].

hence

$$\mathscr{V}_{\epsilon_1}\mathscr{V}_{\epsilon_2}\cdots\mathscr{V}_{\epsilon_k} \;\subset\; \mathscr{V}_{q_\nu(\epsilon_1+\cdots+\epsilon_k)}\,. \tag{66}$$

For every subgroup H of G, let $\mathscr{V}_\epsilon^H = \mathscr{V}_\epsilon \cap H$. The index of $\mathscr{V}_\epsilon$ in $G(\mathscr{O}_\nu)$ is given by Lemma 2.6 with $N = N_\epsilon$.

We denote the operator norm of a linear operator ℓ of the normed K_ν-algebra $\mathscr{M}_{\mathbf{d}}(K_\nu)$ (for the supremum norm defined before Lemma 3.1) by

$$\|\ell\| = \max\Big\{\frac{\|\ell(X)\|}{\|X\|} : X \in \mathscr{M}_{\mathbf{d}}(K_\nu) - \{0\}\Big\} \in q_\nu^{\mathbb{Z}} \cup \{0\}\,,$$

so that $\ell(\mathscr{M}_{\mathbf{d}}(\mathscr{O}_\nu)) \subset \mathscr{M}_{\mathbf{d}}(\pi_\nu^{-\log_{q_\nu}\|\ell\|}\mathscr{O}_\nu)$ if ℓ is invertible. For every $g \in G$, recall that $\operatorname{Ad} g$ is the linear automorphism $x \mapsto gxg^{-1}$ of $\mathscr{M}_{\mathbf{d}}(K_\nu)$. Also recall that $P^- = U^- G'' Z$.

Lemma 4.4 *For all $\epsilon \in \,]0, \frac{1}{q_\nu}]$ and $g \in G$, we have*

$$g\,\mathscr{V}_\epsilon\, g^{-1} \;\subset\; \mathscr{V}_{\|\operatorname{Ad} g\|\,\epsilon}\,, \quad \mathscr{V}_\epsilon = \mathscr{V}_\epsilon^{P^-}\;\mathscr{V}_\epsilon^{U^+} \quad\text{and}\quad \mathscr{V}_\epsilon^{P^-} = \mathscr{V}_\epsilon^{U^-}\;\mathscr{V}_\epsilon^{G''}\,.$$

Furthermore, the number $\rho = \mathbf{d}^2 - 1$ is an upper local dimension of the family $(\mathscr{V}_\epsilon)_{\epsilon>0}$.

Proof. The first claim follows from the fact that

$$\begin{aligned} g\,\mathscr{V}_\epsilon\, g^{-1} &= \big(I_{\mathbf{d}} + \pi_\nu^{N_\epsilon} g\mathscr{M}_{\mathbf{d}}(\mathscr{O}_\nu)g^{-1}\big) \cap G \\ &\subset \big(I_{\mathbf{d}} + \pi_\nu^{N_\epsilon - \log_{q_\nu}\|\operatorname{Ad} g\|}\mathscr{M}_{\mathbf{d}}(\mathscr{O}_\nu)\big) \cap G = \mathscr{V}_{\epsilon\,\|\operatorname{Ad} g\|}\,. \end{aligned}$$

Note that $\mathscr{V}_\epsilon$ is contained in $\mathscr{U}_G$ (defined in Equation (15)). Indeed, if $g = \begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix} \in \mathscr{V}_\epsilon$ then $\alpha \in I_{\mathfrak{m}} + \pi_\nu^{N_\epsilon}\mathscr{M}_{\mathfrak{m}}(\mathscr{O}_\nu)$, hence $\det(\alpha) \in 1 + \pi_\nu^{N_\epsilon}\mathscr{O}_\nu$, so that $\nu(\det\alpha) = 0 \in \operatorname{lcm}\{\mathfrak{m},\mathfrak{n}\}\mathbb{Z}$ since $N_\epsilon \geqslant 1$ and therefore $g \in \mathscr{U}_G$.

By Proposition 2.5, we may hence uniquely write any $g = \begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix} \in \mathscr{V}_\epsilon$ as $g = u^-\, g''\, z\, u^+$ with $u^- \in U^-$, $g'' \in G''$, $z \in Z$ and $u^+ \in U^+$. By Equation (22) and since $\nu(\det\alpha) = 0$, we have $\alpha \in \mathrm{GL}_{\mathfrak{m}}(\mathscr{O}_\nu)$, $u^\pm \in U^\pm(\mathscr{O}_\nu)$, $g'' \in G''(\mathscr{O}_\nu)$ and $z = I_{\mathbf{d}}$. Furthermore, since $g \in \mathscr{V}_\epsilon$, we have $\alpha = I_{\mathfrak{m}} \mod \pi_\nu^{N_\epsilon}$, $\gamma = 0 \mod \pi_\nu^{N_\epsilon}$, $\beta = 0 \mod \pi_\nu^{N_\epsilon}$ and $\delta = I_{\mathfrak{n}} \mod \pi_\nu^{N_\epsilon}$. Therefore again by Equation (22), we have $u^\pm \in \mathscr{V}_\epsilon^{U^\pm}$ and $g'' \in \mathscr{V}_\epsilon^{G''}$. This proves the second and third claims.

In order to prove the last claim, let us apply Lemma 2.6 with $N = N_\epsilon = \lfloor -\log_{q_\nu}\epsilon\rfloor$, so that $N_\epsilon - 1 \leqslant -\log_{q_\nu}\epsilon$. Since $\mu_G(G(\mathscr{O}_\nu)) = 1$, we hence have

$$\mu_G(\mathscr{V}_\epsilon) = \frac{\mu_G(G(\mathscr{O}_\nu))}{[G(\mathscr{O}_\nu):\mathscr{V}_\epsilon]} = \frac{q_\nu - 1}{q_\nu^{(N_\epsilon-1)(\mathbf{d}^2-1)}}\prod_{i=0}^{\mathbf{d}-1}(q_\nu^{\mathbf{d}} - q_\nu^{\,i})^{-1} \geqslant \epsilon^{\mathbf{d}^2-1}(q_\nu - 1)\prod_{i=0}^{\mathbf{d}-1}(q_\nu^{\mathbf{d}} - q_\nu^{\,i})^{-1}\,.$$

This proves the result. □

We will need the following effective version of the refined LU decomposition by blocks given in Proposition 2.5. We denote by $c : h \mapsto c_h$ the continuous function from G to $[0, +\infty[$ defined by $c_h = \|\operatorname{Ad} h\|$ for every $h \in G$.

Lemma 4.5 *For all $\epsilon \in \,]0, \frac{1}{q_\nu}]$, $u^- \in U^-$, $g'' \in G''$, $z \in Z$ and $u^+ \in U^+$, if $|\chi_{\mathfrak{m}}(z)| \geqslant 1$ and $g = u^-\, g''\, z\, u^+$, then*

$$\mathscr{V}_\epsilon\; g\; \mathscr{V}_\epsilon \;\subset\; u^-\;\mathscr{V}^{U^-}_{c_{g''}q_\nu(c_{u^-g''}+c_{u^+})\,\epsilon}\; g''\;\mathscr{V}^{G''}_{q_\nu(c_{u^-g''}+c_{u^+})\,\epsilon}\; z\;\mathscr{V}^{U^+}_{q_\nu(c_{u^-g''}+2c_{u^+})\,\epsilon}\; u^+\,.$$

Proof. In order to simplify the notation, let $p = u^- g''$ and $u = u^+$, so that $g = pzu$. If $z = \begin{pmatrix} \lambda I_{\mathfrak{m}} & 0 \\ 0 & \mu I_{\mathfrak{n}} \end{pmatrix}$, as seen in Equation (19) for the first equality, we have

$$z \begin{pmatrix} I_{\mathfrak{m}} & 0 \\ \beta & I_{\mathfrak{n}} \end{pmatrix} z^{-1} = \begin{pmatrix} I_{\mathfrak{m}} & 0 \\ \mu\,\lambda^{-1}\beta & I_{\mathfrak{n}} \end{pmatrix} \quad \text{and} \quad z^{-1} \begin{pmatrix} I_{\mathfrak{m}} & \gamma \\ 0 & I_{\mathfrak{n}} \end{pmatrix} z = \begin{pmatrix} I_{\mathfrak{m}} & \lambda^{-1}\mu\,\gamma \\ 0 & I_{\mathfrak{n}} \end{pmatrix}.$$

Since $|\lambda| = |\chi_{\mathfrak{m}}(z)| \geqslant 1$ and $\det z = 1$ so that $|\mu| = |\lambda|^{-\frac{\mathfrak{m}}{\mathfrak{n}}} \leqslant 1$, for every $\epsilon' > 0$, we have

$$z\,\mathscr{V}_{\epsilon'}^{U^-} z^{-1} \subset \mathscr{V}_{\epsilon'}^{U^-} \quad \text{and} \quad z^{-1}\,\mathscr{V}_{\epsilon'}^{U^+} z \subset \mathscr{V}_{\epsilon'}^{U^+}\ . \tag{67}$$

Using for the following sequence of equalities and inclusions respectively

- the first claim of Lemma 4.4 for the first inclusion,
- the second and third claims of Lemma 4.4 for the second equality,
- the claim on the left in Equation (67) and the fact that Z centralises G'' for the second inclusion,
- the fact that $\mathscr{V}_{c_u\epsilon}$ is a normal subgroup of $G(\mathscr{O}_\nu)$ that contains $\mathscr{V}_{c_u\epsilon}^{U^-}\,\mathscr{V}_{c_u\epsilon}^{G''}$ for the third inclusion,
- twice the claim on the right in Equation (67) for the fourth inclusion,
- twice Equation (66) with $k = 2$ and $k = 3$, defining the constants $c_1'' = q_\nu(c_p + c_u)$ and $c_2'' = q_\nu(c_p + 2c_u)$ for the fifth inclusion,
- again the third claim of Lemma 4.4 for the sixth equality,
- the fact that G'' normalizes U^- and again the first claim of Lemma 4.4 for the last inclusion,

we have

$$\begin{aligned}
\mathscr{V}_\epsilon\, g\, \mathscr{V}_\epsilon &= p\, p^{-1} \mathscr{V}_\epsilon\, p\, z\, u\, \mathscr{V}_\epsilon\, u^{-1}\, u \ \subset\ p\, \mathscr{V}_{c_p\epsilon}\, z\, \mathscr{V}_{c_u\epsilon}\, u = p\, \mathscr{V}_{c_p\epsilon}^{P^-}\, \mathscr{V}_{c_p\epsilon}^{U^+}\, z\, \mathscr{V}_{c_u\epsilon}^{U^-}\, \mathscr{V}_{c_u\epsilon}^{G''}\, \mathscr{V}_{c_u\epsilon}^{U^+}\, u \\
&= p\, \mathscr{V}_{c_p\epsilon}^{P^-}\, \mathscr{V}_{c_p\epsilon}^{U^+}\, z\, \mathscr{V}_{c_u\epsilon}^{U^-}\, z^{-1}\, z\, \mathscr{V}_{c_u\epsilon}^{G''}\, \mathscr{V}_{c_u\epsilon}^{U^+}\, u \subset p\, \mathscr{V}_{c_p\epsilon}^{P^-}\, \mathscr{V}_{c_p\epsilon}^{U^+}\, \mathscr{V}_{c_u\epsilon}^{U^-}\, \mathscr{V}_{c_u\epsilon}^{G''}\, z\, \mathscr{V}_{c_u\epsilon}^{U^+}\, u \\
&\subset p\, \mathscr{V}_{c_p\epsilon}^{P^-}\, \mathscr{V}_{c_u\epsilon}\, \mathscr{V}_{c_p\epsilon}^{U^+}\ z\, \mathscr{V}_{c_u\epsilon}^{U^+}\, u = p\, \mathscr{V}_{c_p\epsilon}^{P^-}\, \mathscr{V}_{c_u\epsilon}^{P^-}\, \mathscr{V}_{c_u\epsilon}^{U^+}\, \mathscr{V}_{c_p\epsilon}^{U^+}\ z\, \mathscr{V}_{c_u\epsilon}^{U^+}\, u \\
&= p\, \mathscr{V}_{c_p\epsilon}^{P^-}\, \mathscr{V}_{c_u\epsilon}^{P^-}\, z\, z^{-1}\, \mathscr{V}_{c_u\epsilon}^{U^+}\, z\, z^{-1}\, \mathscr{V}_{c_p\epsilon}^{U^+}\ z\, \mathscr{V}_{c_u\epsilon}^{U^+}\, u \\
&\subset p\, \mathscr{V}_{c_p\epsilon}^{P^-}\, \mathscr{V}_{c_u\epsilon}^{P^-}\, z\, \mathscr{V}_{c_u\epsilon}^{U^+}\, \mathscr{V}_{c_p\epsilon}^{U^+}\, \mathscr{V}_{c_u\epsilon}^{U^+}\, u \subset p\, \mathscr{V}_{c_1''\epsilon}^{P^-}\, z\, \mathscr{V}_{c_2''\epsilon}^{U^+}\, u \\
&= u^-\, g''\, \mathscr{V}_{c_1''\epsilon}^{U^-}\, \mathscr{V}_{c_1''\epsilon}^{G''}\, z\, \mathscr{V}_{c_2''\epsilon}^{U^+}\, u = u^-\, g''\, \mathscr{V}_{c_1''\epsilon}^{U^-}\, g''^{-1}\, g''\, \mathscr{V}_{c_1''\epsilon}^{G''}\, z\, \mathscr{V}_{c_2''\epsilon}^{U^+}\, u \\
&\subset u^-\, \mathscr{V}_{c_{g''}c_1''\epsilon}^{U^-}\, g''\, \mathscr{V}_{c_1''\epsilon}^{G''}\, z\, \mathscr{V}_{c_2''\epsilon}^{U^+}\, u\ ,
\end{aligned}$$

as wanted. □

Let Φ be a closed ball of radius less than 1 in the metric space $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$, contained in $\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$. By Lemma 3.1 and with the notation of Equation (53), the set $\widetilde{\Phi} = \mathrm{orb}_{\mathfrak{m}}^{-1}(\Phi)$ is a closed ball of same radius in $\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$. Let $\mathscr{E}$ be a closed ball in $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$, small enough so that there exists a closed ball in the clopen fundameental domain $G''_\Diamond$ which maps isometrically to $\mathscr{E}$ by the locally isometric map $\pi_{\mathfrak{m},\mathfrak{n}} : G'' \to \mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ defined in Equation (44). Let $\widetilde{\mathscr{E}} = \pi_{\mathfrak{m},\mathfrak{n}}^{-1}(\mathscr{E})$ and let $r \in \mathbb{N}$. Using the notation $\Omega_{\Phi,\mathscr{E},r}$ defined in Equation (54), the family of Lipschitz well-rounded subsets with respect to $(\mathscr{V}_\epsilon)_{\epsilon>0}$ that we will use in order to apply Theorem 4.3 is given by the following result.

Proposition 4.6 *With Φ and $\mathscr{E}$ as above, the family $\big(\Omega_{\Phi,\mathscr{E},r}\big)_{r\in\mathbb{N}}$ is 0-Lipschitz well-rounded with respect to $(\mathscr{V}_\epsilon)_{\epsilon>0}$.*

Proof. Recall that $\Omega_{\Phi,\mathscr{E},r} = U^-_{\widetilde{\Phi}}\ G''_{\widetilde{\mathscr{E}}}\ Z_r\ U^+_{\square}$ with the notation at the beginning of Subsection 4.1. We will actually prove (as allowed by the ultrametric situation) the stronger statement that given Φ and $\mathscr{E}$ as above, if ϵ is small enough, then for every $r \in \mathbb{N}$, we have

$$\big(U^-_{\widetilde{\Phi}}\, G''_{\widetilde{\mathscr{E}}}\, Z_r\, U^+_{\square}\big)^{-\epsilon} = U^-_{\widetilde{\Phi}}\, G''_{\widetilde{\mathscr{E}}}\, Z_r\, U^+_{\square} = \big(U^-_{\widetilde{\Phi}}\, G''_{\widetilde{\mathscr{E}}}\, Z_r\, U^+_{\square}\big)^{+\epsilon} .$$

Let

$$c = \max\big\{q_\nu \max\{c_{g''}(c_{u^-g''} + c_{u^+}), c_{u^-g''} + 2\,c_{u^+}\} : u^- \in U^-_{\widetilde{\Phi}},\ g'' \in G''_{\widetilde{\mathscr{E}}},\ u^+ \in U^+_{\square}\big\}, \quad (68)$$

which is finite since $U^-_{\widetilde{\Phi}}$, $G''_{\widetilde{\mathscr{E}}}$ and $U^+_{\square}$ are compact subsets of G. Since $\widetilde{\Phi}$ is a ball of radius less than 1 in $\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu)$, let $v_0 \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu)$ and $k \in \mathbb{N}\smallsetminus\{0\}$ be such that

$$\widetilde{\Phi} = v_0 + {\pi_\nu}^k \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu) .$$

Let $r_{\mathscr{E}}$ be the radius of the ball $\mathscr{E}$ (satisfying the assumptions of Proposition 4.6). By Equation (46) and since the map $\underline{g} \mapsto \check{\underline{g}}$ is an isometry of $\mathrm{GL}_{\mathfrak{n}}(K_\nu)$, there exists $\big(\begin{smallmatrix}\overline{g}_0 & 0\\ 0 & \underline{g}_0\end{smallmatrix}\big) \in G''$ such that

$$G''_{\widetilde{\mathscr{E}}} = \big\{\big(\begin{smallmatrix}\overline{g} & 0\\ 0 & \underline{g}\end{smallmatrix}\big) \in G'' : \max\{d(\overline{g},\overline{g}_0),\ d(\underline{g},\underline{g}_0)\} \leqslant r_{\mathscr{E}}\big\} . \quad (69)$$

Let us now consider $\epsilon_0 = \frac{1}{c}\, q_\nu^{\,-\max\{k,\, r_{\mathscr{E}}+\log_{q_\nu}(\frac{1}{r_{\mathscr{E}}}\|\overline{g}_0\|\,\|\overline{g}_0{}^{-1}\|),\, r_{\mathscr{E}}+\log_{q_\nu}(\frac{1}{r_{\mathscr{E}}}\|\underline{g}_0\|\,\|\underline{g}_0{}^{-1}\|)\}-2} > 0$, so that for every $\epsilon \in \,]0,\epsilon_0[\,$, we have

$$N_{c\epsilon} > 1 + \max\big\{k, r_{\mathscr{E}} + \log_{q_\nu}\big(\frac{1}{r_{\mathscr{E}}}\|\overline{g}_0\|\,\|\overline{g}_0{}^{-1}\|\big), r_{\mathscr{E}} + \log_{q_\nu}\big(\frac{1}{r_{\mathscr{E}}}\|\underline{g}_0\|\,\|\underline{g}_0{}^{-1}\|\big)\big\} \geqslant 1 . \quad (70)$$

Let us prove that for every $\epsilon \in \,]0,\epsilon_0[\,$, we have

$$U^-_{\widetilde{\Phi}}\ \mathscr{V}^{U^-}_{c\epsilon} = U^-_{\widetilde{\Phi}}, \quad G''_{\widetilde{\mathscr{E}}}\,\mathscr{V}^{G''}_{c\epsilon} = G''_{\widetilde{\mathscr{E}}} \quad\text{and}\quad \mathscr{V}^{U^+}_{c\epsilon}\ U^+_{\square} = U^+_{\square} . \quad (71)$$

For all $u \in U^-_{\widetilde{\Phi}}$ and $u' \in \mathscr{V}^{U^-}_{c\epsilon}$, let $\beta, \beta' \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$ be such that $u = \big(\begin{smallmatrix}I_{\mathfrak{m}} & 0\\ v_0+{\pi_\nu}^k\beta & I_{\mathfrak{n}}\end{smallmatrix}\big)$ and $u' = \big(\begin{smallmatrix}I_{\mathfrak{m}} & 0\\ {\pi_\nu}^{N_{c\epsilon}}\beta' & I_{\mathfrak{n}}\end{smallmatrix}\big)$. Then since $N_{c\epsilon} > k$, we have $uu' = \big(\begin{smallmatrix}I_{\mathfrak{m}} & 0\\ v_0+{\pi_\nu}^k\beta+{\pi_\nu}^{N_{c\epsilon}}\beta' & I_{\mathfrak{n}}\end{smallmatrix}\big) \in U^-_{\widetilde{\Phi}}$. Therefore we have $U^-_{\widetilde{\Phi}}\mathscr{V}^{U^-}_{c\epsilon} \subset U^-_{\widetilde{\Phi}}$ and the opposite inclusion is clear. This proves the equality on the left-hand side of Formula (71).

The proof of the equality on the right-hand side is similar. For all $u \in U^+_{\square}$ and $u' \in \mathscr{V}^{U^+}_{c\epsilon}$, let $\gamma \in \square$ and $\gamma' \in \mathscr{M}_{\mathfrak{m},\mathfrak{n}}(\mathscr{O}_\nu)$ be such that $u = \big(\begin{smallmatrix}I_{\mathfrak{m}} & \gamma\\ 0 & I_{\mathfrak{n}}\end{smallmatrix}\big)$ and $u' = \big(\begin{smallmatrix}I_{\mathfrak{m}} & {\pi_\nu}^{N_{c\epsilon}}\gamma'\\ 0 & I_{\mathfrak{n}}\end{smallmatrix}\big)$. Then since $N_{c\epsilon} > 1$ and since $\square + \pi_\nu\mathscr{M}_{\mathfrak{m},\mathfrak{n}}(\mathscr{O}_\nu) = \square$ by the construction of the fundamental domain $\mathscr{D}$ at the beginning of Subsection 4.1, we have $u'u = \big(\begin{smallmatrix}I_{\mathfrak{m}} & \gamma+{\pi_\nu}^{N_{c\epsilon}}\gamma'\\ 0 & I_{\mathfrak{n}}\end{smallmatrix}\big) \in U^+_{\square}$. Therefore we have $\mathscr{V}^{U^+}_{c\epsilon}U^+_{\square} \subset U^+_{\square}$ and the opposite inclusion is clear.

Let $g = \begin{pmatrix}\overline{g} & 0\\ 0 & \underline{g}\end{pmatrix} \in G''_{\widetilde{\mathscr{E}}}$. For every $g' = \begin{pmatrix}\overline{g'} & 0\\ 0 & \underline{g'}\end{pmatrix} \in \mathscr{V}^{G''}_{c\epsilon}$, there exist $\alpha \in \mathscr{M}_{\mathfrak{m}}(\mathscr{O}_\nu)$ and $\delta \in \mathscr{M}_{\mathfrak{n}}(\mathscr{O}_\nu)$ such that

$$\overline{g'} = I_{\mathfrak{m}} + {\pi_\nu}^{N_{c\epsilon}}\alpha \quad\text{and}\quad \underline{g'} = I_{\mathfrak{n}} + {\pi_\nu}^{N_{c\epsilon}}\delta .$$

We have $gg' = \begin{pmatrix}\overline{g}\,\overline{g'} & 0\\ 0 & \underline{g}\,\underline{g'}\end{pmatrix} \in G''$. Let us prove that $d(\overline{g}\,\overline{g'},\, \overline{g}_0) \leqslant r_{\mathscr{E}}$. A similar proof gives that $d(\underline{g}\,\underline{g'},\, \underline{g}_0) \leqslant r_{\mathscr{E}}$, thus proving that $G''_{\widetilde{\mathscr{E}}}\ \mathscr{V}^{G''}_{c\epsilon} \subset G''_{\widetilde{\mathscr{E}}}$ by Equation (69). The opposite inclusion being clear, this proves the middle equality of Formula (71).

By the submultiplicativity of the supremum norm, since $\alpha \in \mathscr{M}_{\mathfrak{m}}(\mathscr{O}_\nu)$ so that $\|\alpha\| \leqslant 1$, by Lemma 3.3 since $\overline{g} \in B(\overline{g}_0, r_{\mathscr{E}})$ and by Equation (70), we have

$$\|\pi_\nu^{N_{c\epsilon}}\,\overline{g}\,\alpha\,\overline{g}_0^{\;-1}\| \leqslant q_\nu^{\;-N_{c\epsilon}}\|\overline{g}\|\,\|\alpha\|\,\|\overline{g}_0^{\;-1}\| \leqslant q_\nu^{\;-N_{c\epsilon}+r_{\mathscr{E}}}\|\overline{g}_0\|\,\|\overline{g}_0^{\;-1}\| \leqslant \frac{r_{\mathscr{E}}}{q_\nu}\;.$$

We also have, by the ultrametric triangle inequality,

$$\begin{aligned}\|\,\overline{g}\,\overline{g'}\,\overline{g}_0^{\;-1} - I_{\mathfrak{m}}\| &= \|\,\overline{g}(I_{\mathfrak{m}} + \pi_\nu^{N_{c\epsilon}}\alpha)\,\overline{g}_0^{\;-1} - I_{\mathfrak{m}}\| = \|\,\overline{g}\,\overline{g}_0^{\;-1} - I_{\mathfrak{m}} + \pi_\nu^{N_{c\epsilon}}\,\overline{g}\,\alpha\,\overline{g}_0^{\;-1}\|\\ &\leqslant \max\{\|\,\overline{g}\,\overline{g}_0^{\;-1} - I_{\mathfrak{m}}\|,\ \|\pi_\nu^{N_{c\epsilon}}\,\overline{g}\,\alpha\,\overline{g}_0^{\;-1}\|\}\;.\end{aligned}$$

Thus since $\overline{g} \in B(\overline{g}_0, r_{\mathscr{E}})$ and $\ln(1+t) \leqslant t$ for every $t \geqslant 0$, we have

$$\log_{q_\nu}(1 + \|\,\overline{g}\,\overline{g'}\,\overline{g}_0^{\;-1} - I_{\mathfrak{m}}\|) \leqslant \max\{r_{\mathscr{E}}, \log_{q_\nu}(1+\frac{r_{\mathscr{E}}}{q_\nu})\} \leqslant r_{\mathscr{E}}\;.$$

Since $N_{c\epsilon} \geqslant 1$, the standard formula for the inverse of $I_{\mathfrak{m}}+X$ when $\|X\| < 1$ gives that there exists $\alpha' \in \mathscr{M}_{\mathfrak{n}}(\mathscr{O}_\nu)$ such that $\overline{g'}^{\;-1} = I_{\mathfrak{m}} + \pi_\nu^{N_{c\epsilon}}\alpha'$. A proof similar to the one above thus gives that $\log_{q_\nu}(1+\|\,\overline{g}_0(\,\overline{g}\,\overline{g'}\,)^{-1} - I_{\mathfrak{m}}\|) \leqslant r_{\mathscr{E}}$, which proves as wanted that $d(\overline{g}\,\overline{g'},\,\overline{g}_0) \leqslant r_{\mathscr{E}}$. This concludes the proof of Formula (71).

Now, for every $r \in \mathbb{N}$, we have by Lemma 4.5 and by Equations (68) and (71) that

$$\begin{aligned}\big(U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}\big)^{+\epsilon} &= \mathscr{V}_\epsilon\,U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}\,\mathscr{V}_\epsilon\\ \subset\ U^-_{\widetilde{\Phi}}\,\mathscr{V}^{U^-}_{c\epsilon}\,G''_{\widetilde{\mathscr{E}}}\,\mathscr{V}^{G''}_{c\epsilon}\,Z_r\,\mathscr{V}^{U^+}_{c\epsilon}\,U^+_{\square} &= U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}\;.\end{aligned}$$

Since the converse inclusion is immediate, we have $\big(U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}\big)^{+\epsilon} = U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}$.

Since $\mathscr{V}_\epsilon$, being a subgroup, is stable by $g \mapsto g^{-1}$, this implies that $g\,U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}\,h$ contains $U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}$ for all $g, h \in \mathscr{V}_\epsilon$, so that $\big(U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}\big)^{-\epsilon} \supset U^-_{\widetilde{\Phi}}\,G''_{\widetilde{\mathscr{E}}}\,Z_r\,U^+_{\square}$. Since the converse inclusion is immediate, this concludes the proof of Proposition 4.6. □

4.3 The main statement and its proof

Error terms in equidistribution results usually require smoothness properties on test functions. The appropriate smoothness regularity of functions defined on ultrametric spaces as $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ and $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ is the locally constant one. The locally constant regularity on such homogeneous spaces of totally discontinuous groups could be defined (as for instance in [AtGP], [KPS, §4.3]) by using the familly of small compact-open subgroups $(\mathscr{V}_\epsilon)_{\epsilon\in\,]0,1]}$ of G defined in Subsection 4.2, and by defining an ϵ-locally constant map on $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ to be a map which is constant on every orbit of $\mathscr{V}_\epsilon \cap G''$ on $\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}$. But it turns out to be more convenient in this paper to use a general purely metric definition. For every ultrametric space E and $\epsilon \in\,]0,1]$, a bounded map $f : E \to \mathbb{R}$ is *ϵ-locally constant* if it is constant on every closed ball of radius ϵ in E. With $\|f\|_\infty = \sup_{x\in E}|f(x)|$ the supremum norm of f, the *ϵ-locally constant norm* of f is $\|f\|_\epsilon = \frac{\|f\|_\infty}{\epsilon}$.

The key result of this paper is the following one. Let $\ell = \mathrm{lcm}(\mathfrak{m},\mathfrak{n})$. For every nonzero ideal I of R_ν, let $\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}(I) = \Gamma^\sharp_I(R_\nu^{\;\mathfrak{m}} \times \{0\})$ (see Subsection 2.4 for the definition of Γ_I and Equation (51) for the one of $\Gamma^\sharp_I$) and

$$c_I = \frac{q^{\,(\mathfrak{g}-1)(\mathbf{d}^2-\mathbf{d}+1-\mathfrak{m}\,\mathfrak{n})}\,(q_\nu^{\;\mathbf{d}}-1)^2\,\prod_{i=2}^{\mathbf{d}}((q_\nu^{\;i-1}-1)\,\zeta_K(i))\,N(I)^{\mathfrak{m}\,\mathfrak{n}}\prod_{\mathfrak{p}\,|\,I}\prod_{i=1}^{\mathfrak{m}}\frac{N(\mathfrak{p})^i-N(\mathfrak{p})^{-\mathfrak{n}}}{N(\mathfrak{p})^i-1}}{(q-1)\,q_\nu^{\;2\,\mathfrak{m}\,\mathfrak{n}+2}\,\prod_{i=1}^{\mathfrak{m}}(q_\nu^{\;i}-1)^2\prod_{i=2}^{\mathfrak{n}}(q_\nu^{\;i}-1)^2}\;. \tag{72}$$

Theorem 4.7 *For every nonzero ideal I of R_ν, for the weak-star convergence of Borel measures on the locally compact space* $\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}} \times \operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$, *we have*

$$\lim_{i\to+\infty} \frac{c_I}{q_\nu^{\,\ell\,\mathbf{d}\,i}} \sum_{\Lambda\in\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}(I)\,:\,\overline{\operatorname{Covol}}\,\Lambda = q_\nu^{\,\ell\, i}} \Delta_{V_\Lambda}\otimes\Delta_{[\![\Lambda]\!]} = \mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}|\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}} \otimes \mu_{\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}}\;. \tag{73}$$

Furthermore, there exists $\tau\in\,]0,\frac{1}{2\,\mathbf{d}^2}]$ such that for all $\delta\in\,]0,\tau[$ and $\epsilon\in\,]0,1]$, there is an additive error term of the form $\operatorname{O}_{\nu,\delta,I}\big(q_\nu^{\,\ell\,\mathbf{d}\,i(-\tau+\delta)}\,\|f\|_\epsilon\,\|g\|_\epsilon\big)$ in the above equidistribution claim when evaluated on pairs (f,g) for all compactly supported ϵ-locally constant maps $f:\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}\to\mathbb{R}$ and $g:\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}\to\mathbb{R}$: as $i\to+\infty$, we have

$$\begin{aligned}&\frac{c_I}{q_\nu^{\,\ell\,\mathbf{d}\,i}} \sum_{\Lambda\in\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}(I)\,:\,\overline{\operatorname{Covol}}\,\Lambda = q_\nu^{\,\ell\, i}} f(V_\Lambda)\, g([\![\Lambda]\!])\\ &= \Big(\int_{\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}} f\,d\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}\Big)\Big(\int_{\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}} g\,d\mu_{\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}}\Big) + \operatorname{O}_{\nu,\delta,I}\big(q_\nu^{\,\ell\,\mathbf{d}\,i(-\tau+\delta)}\,\|f\|_\epsilon\,\|g\|_\epsilon\big)\;.\end{aligned}$$

Proof. Let $\Lambda\in\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}$. Then there exists $g\in\Gamma^\sharp$ (thus $g\in\mathscr{U}_G$) such that we have $\Lambda=\Lambda_g=g(R_\nu^{\,\mathfrak{m}}\times\{0\})$, so that if $g=u^-\,g''\,z\,u^+$ is the decomposition given by Proposition 2.5, then $u^-\in U^-(\mathscr{O}_\nu)$. Hence by Proposition 2.8 (i), we have

$$V_\Lambda = V_{\Lambda_g} = V_{\Lambda_{u^-}} \in \operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}} = U^-(\mathscr{O}_\nu)V_{R_\nu^{\,\mathfrak{m}}\times\{0\}}\;.$$

Furthermore, we have $[\![\Lambda]\!]\in\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ by Equation (49), so that the statement of Theorem 4.7 is well defined.

Let I be a nonzero ideal of R_ν. Let $\tau=\tau(\Gamma_I)\in\,]0,\frac{1}{2\,\mathbf{d}^2}]$ be as in Theorem 4.3 applied with $G'=G$, with $\Gamma'=\Gamma_I$ and with the family $(\mathscr{V}'_\epsilon)_{\epsilon>0}=(\mathscr{V}_\epsilon)_{\epsilon>0}$ given by Equation (65), which has an upper local dimension $\rho=\mathbf{d}^2-1$ according to the final claim of Lemma 4.4. Let $\delta\in\,]0,\tau[$.

Let Φ be a closed ball in $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}$ of radius $r_\Phi\in\,]0,c_1^{\frac{1}{\mathfrak{m}\,\mathfrak{n}}}]$, where $c_1\leqslant 1$ is given by Proposition 2.5. Besides, we assume that Φ is contained in $\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$. By Lemma 3.1 and with the notation of Equation (53), it follows that $\widetilde{\Phi}=\operatorname{orb}_{\mathfrak{m}}^{-1}(\Phi)$ is a closed ball of radius r_Φ in $\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$. Let χ_Φ be the characteristic function of Φ, which is r_Φ-locally constant with $\|\chi_\Phi\|_{r_\Phi}=\frac{1}{r_\Phi}\geqslant 1$. By Equation (56), we have

$$\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}(\Phi)=\frac{1}{c_1}\operatorname{Haar}_{\mathfrak{n},\mathfrak{m}}(\widetilde{\Phi})=\frac{1}{c_1}\,r_\Phi^{\,\mathfrak{m}\,\mathfrak{n}}\leqslant 1\;,$$

so that, since $\tau\leqslant\frac{1}{2\,\mathbf{d}^2}\leqslant\frac{1}{\mathfrak{m}\,\mathfrak{n}}$, we have

$$\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}(\Phi)^{-\tau+\delta}\leqslant\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}(\Phi)^{-\tau}=\operatorname{O}(\|\,\chi_\Phi\|_{r_\Phi})\;. \tag{74}$$

Let $\mathscr{E}$ be a closed ball in $\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$ of radius $r_\mathscr{E}\in\,]0,1]$ small enough so that $\mu_{\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E})\leqslant 1$ and there exists a closed ball $\widetilde{\mathscr{E}}_0$ in $G''_\diamondsuit$ mapping isometrically to $\mathscr{E}$ by $\pi_{\mathfrak{m},\mathfrak{n}}:G''\to\operatorname{Lat}^1_{\mathfrak{m},\mathfrak{n}}$. Let $\widetilde{\mathscr{E}}=\pi_{\mathfrak{m},\mathfrak{n}}^{-1}(\mathscr{E})=\bigsqcup_{\gamma\in G''(R_\nu)}\widetilde{\mathscr{E}}_0\,\gamma$. Let $\chi_\mathscr{E}$ be the characteristic function of $\mathscr{E}$, which is $r_\mathscr{E}$-locally constant with $\|\chi_\mathscr{E}\|_{r_\mathscr{E}}=\frac{1}{r_\mathscr{E}}\geqslant 1$. By Equation (60) and by the Alhfors regularity of the homogeneous measure $\mu_{G''}$ of the group G'' with dimension $\dim G''\leqslant\mathfrak{m}^2+\mathfrak{n}^2\leqslant 2\mathbf{d}^2$

for the distance d defined in Section 3.3 (see in particular Equation (45)), there exists a constant $c > 0$ such that

$$\mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E}) = \frac{(q_\nu - 1)^2}{(q-1)\, q_\nu^{\,2}}\, \mu_{G''}(G''_{\widetilde{\mathscr{E}}}) = \frac{(q_\nu - 1)^2}{(q-1)\, q_\nu^{\,2}}\, \mu_{G''}(\widetilde{\mathscr{E}}_0) \geqslant c\, r_{\mathscr{E}}^{\dim G''} \geqslant c\, r_{\mathscr{E}}^{2\mathbf{d}^2}\ ,$$

so that, since $\tau \leqslant \frac{1}{2\,\mathbf{d}^2}$ and $\mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E}) \leqslant 1$, we have

$$\mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E})^{-\tau+\delta} = \mathrm{O}(\|\,\chi_{\mathscr{E}}\|_{r_{\mathscr{E}}})\ . \tag{75}$$

For every $r \in \mathbb{N}$, let us define

$$\mathscr{P}\!\mathscr{L}^{\sharp}_{\mathfrak{m},\mathbf{d}}(I, \Phi, \mathscr{E}, r) = \{\Lambda \in \mathscr{P}\!\mathscr{L}^{\sharp}_{\mathfrak{m},\mathbf{d}}(I) : V_\Lambda \in \Phi,\ [\![\Lambda]\!] \in \mathscr{E},\ \overline{\mathrm{Covol}}\,\Lambda\ =\ q_\nu^{\,\ell\, r}\}$$

Using respectively

- Proposition 4.2 for the first equality,
- Theorem 4.3 applied to the family $(\mathscr{Z}_r = \Omega_{\Phi,\mathscr{E},r})_{r\in\mathbb{N}}$ (where the set $\Omega_{\Phi,\mathscr{E},r}$ is defined in Equation (54)), which is 0-Lipschitz well-rounded with respect to $(\mathscr{V}_\epsilon)_{\epsilon>0}$ by Proposition 4.6 for the second equality,
- Lemma 4.1 for the third equality,
- Equations (74) and (75) for the last equality (and the fact that $\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}$ and $\mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}$ are finite measures),

we have

$$\begin{aligned}
&\mathrm{Card}\, \mathscr{P}\!\mathscr{L}^{\sharp}_{\mathfrak{m},\mathbf{d}}(I, \Phi, \mathscr{E}, r) = \mathrm{Card}(\Omega_{\Phi,\mathscr{E},r} \cap \Gamma_I)\\
&= \frac{1}{\|\mu_{G/\Gamma_I}\|}\, \mu_G(\Omega_{\Phi,\mathscr{E},r})\big(1 + \mathrm{O}\left(\mu_G(\Omega_{\Phi,\mathscr{E},r})^{-\tau+\delta}\right)\big)\\
&= \frac{c'_1\, q_\nu^{\,\mathbf{d}\,\ell\, r}}{\|\mu_{G/\Gamma_I}\|}\, \mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}(\Phi)\, \mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E})\big(1 + \mathrm{O}\left(q_\nu^{\,\mathbf{d}\,\ell\, r(-\tau+\delta)} \mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}(\Phi)^{-\tau+\delta} \mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E})^{-\tau+\delta}\right)\big)\\
&= \frac{c'_1\, q_\nu^{\,\mathbf{d}\,\ell\, r}}{\|\mu_{G/\Gamma_I}\|}\, \big(\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}(\Phi)\, \mu_{\mathrm{Lat}^1_{\mathfrak{m},\mathfrak{n}}}(\mathscr{E}) + \mathrm{O}\left(q_\nu^{\,\mathbf{d}\,\ell\, r(-\tau+\delta)}\ \|\,\chi_\Phi\|_{r_\Phi}\ \|\,\chi_{\mathscr{E}}\|_{r_{\mathscr{E}}}\right)\big)\ .
\end{aligned} \tag{76}$$

Let $c_I = \frac{\|\mu_{G/\Gamma_I}\|}{c'_1} = \frac{[\Gamma:\Gamma_I]\,\|\mu_{G/\Gamma}\|}{c'_1}$. With the value of $[\Gamma : \Gamma_I]$ given by Lemma 2.3 (2), the value of $\|\mu_{G/\Gamma}\|$ given by Equation (40) with $k = \mathbf{d}$ and the value of c'_1 given in Lemma 4.1, we have

$$c_I = \frac{\mathscr{N}(I)^{\mathfrak{m}\,\mathfrak{n}} \prod_{\mathfrak{p}\,|\,I} \prod_{i=1}^{\mathfrak{m}} \frac{N(\mathfrak{p})^i - N(\mathfrak{p})^{-\mathfrak{n}}}{N(\mathfrak{p})^i - 1}\ q^{\,(\mathfrak{g}-1)(\mathbf{d}^2 - \mathbf{d}+1)} \prod_{i=1}^{\mathbf{d}-1} \frac{\zeta_K(1+i)}{q_\nu^{\,i}-1}}{(q-1)\, q^{(\mathfrak{g}-1)\,\mathfrak{m}\,\mathfrak{n}}\ q_\nu^{\,2\,\mathfrak{m}\,\mathfrak{n}+2}\ \frac{\prod_{i=1}^{\mathfrak{m}} (q_\nu^{\,i}-1)^2 \prod_{i=1}^{\mathfrak{n}} (q_\nu^{\,i}-1)^2}{(q_\nu - 1)^2\ \prod_{i=1}^{\mathbf{d}} (q_\nu^{\,i}-1)^2}}\ ,$$

as wanted in Equation (72). Note that every compactly supported ϵ-locally constant map on an ultrametric space is a finite linear combination of characteristic functions of balls of radius ϵ. By a finite bilinearity argument, Theorem 4.7 follows from Equation (76). □

Corollary 4.8 *For every nonzero ideal I of R_ν, for the weak-star convergence of Borel measures on the locally compact space* $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}} \times \mathrm{Sh}^1_{\mathfrak{m}} \times \mathrm{Sh}^1_{\mathfrak{n}}$*, we have*

$$\lim_{i\to+\infty} \frac{c_I\,(1 - q_\nu^{\,-1})^{-2}}{q_\nu^{\,\ell\,\mathbf{d}\, i}} \sum_{\Lambda \in \mathscr{P}\!\mathscr{L}_{\mathfrak{m},\mathbf{d}}(I)\,:\,\overline{\mathrm{Covol}}\,\Lambda\, =\, q_\nu^{\,\ell\, i}} \Delta_{V_\Lambda} \otimes \Delta_{\mathrm{sh}(\Lambda)} \otimes \Delta_{\mathrm{sh}(\Lambda^\perp)}$$
$$= \mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}} \otimes \mu_{\mathrm{Sh}^1_{\mathfrak{m}}} \otimes \mu_{\mathrm{Sh}^1_{\mathfrak{n}}}\ . \tag{77}$$

Furthermore, there exists $\tau \in \,]0, \frac{1}{2\,\mathbf{d}^2}]$ such that for all $\delta \in \,]0, \tau[$ and $\epsilon \in \,]0,1]$, there is an additive error term of the form $\operatorname{O}_{\nu,\delta,I}\big(q_\nu^{\;\ell\,\mathbf{d}\,i(-\tau+\delta)}\,\|f\|_\epsilon\,\|f_1\|_\infty\|f_2\|_\infty\big)$ in the above equidistribution claim when evaluated on (f, f_1, f_2) for every compactly supported ϵ-locally constant map $f : \operatorname{Gr}_{\mathfrak{m},\mathbf{d}} \to \mathbb{R}$ and for all finitely supported maps $f_1 : \operatorname{Sh}^1_{\mathfrak{m}} \to \mathbb{R}$ and $f_2 : \operatorname{Sh}^1_{\mathfrak{n}} \to \mathbb{R}$: as $i \to +\infty$, we have

$$\frac{c_I\,(1-q_\nu^{\;-1})^{-2}}{q_\nu^{\;\ell\,\mathbf{d}\,i}} \sum_{\Lambda \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)\,:\,\overline{\operatorname{Covol}}\,\Lambda = q_\nu^{\;\ell\,i}} f(V_\Lambda)\; f_1(\operatorname{sh}(\Lambda))\; f_2(\operatorname{sh}(\Lambda^\perp))$$
$$= \Big(\int_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}} f\, d\mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}}\Big)\Big(\int_{\operatorname{Sh}^1_{\mathfrak{m}}} f_1\, d\mu_{\operatorname{Sh}^1_{\mathfrak{m}}}\Big)\Big(\int_{\operatorname{Sh}^1_{\mathfrak{n}}} f_2\, d\mu_{\operatorname{Sh}^1_{\mathfrak{n}}}\Big)$$
$$+ \operatorname{O}_{\nu,\delta,I}\big(q_\nu^{\;\ell\,\mathbf{d}\,i(-\tau+\delta)}\,\|f\|_\epsilon\,\|f_1\|_\infty\|f_2\|_\infty\big)\;.$$

Theorem 1.2 in the Introduction follows from the first claim of this corollary by taking $I = R_\nu$.

Proof. Step 1. We first prove the result with $\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$ instead of $\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}$ and $\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}(I)$ instead of $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)$.

Since the map $\varphi_{\mathfrak{m},\mathfrak{n}}$ (defined in Equation (48)) is proper by Lemma 3.4, the pushforward map $(\varphi_{\mathfrak{m},\mathfrak{n}})_*$ of Borel measures by $\varphi_{\mathfrak{m},\mathfrak{n}}$ is linear and weak-star continuous. Hence applying the map $(\operatorname{id}\times\varphi_{\mathfrak{m},\mathfrak{n}})_*$ to Equation (73), using Lemma 3.4 (2) on the left hand side of Equation (73), and Lemma 3.4 (1) on the right hand side of Equation (73), we have

$$\lim_{i\to+\infty} \frac{c_I}{q_\nu^{\;\ell\,\mathbf{d}\,i}} \sum_{\Lambda \in \mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}(I)\,:\,\overline{\operatorname{Covol}}\,\Lambda = q_\nu^{\;\ell\,i}} \Delta_{V_\Lambda} \otimes \Delta_{\operatorname{sh}(\Lambda)} \otimes \Delta_{\operatorname{sh}(\Lambda^\perp)}$$
$$= (1-q_\nu^{\;-1})^2\, \mu_{\operatorname{Gr}_{\mathfrak{m},\mathbf{d}}\,|\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}} \otimes \mu_{\operatorname{Sh}^1_{\mathfrak{m}}} \otimes \mu_{\operatorname{Sh}^1_{\mathfrak{n}}}\;.$$

It follows from Lemma 3.4 (3) and from the error term in Theorem 4.7 applied with the compactly supported ρ_0-locally constant function $g = (f_1 \times f_2)\circ\varphi_{\mathfrak{m},\mathfrak{n}}$ that we have an additive error term of the form $\operatorname{O}_{\nu,\delta,I}\big(q_\nu^{\;\ell\,\mathbf{d}\,i(-\tau+\delta)}\,\|f\|_\epsilon\,\|f_1\|_\infty\|f_2\|_\infty\big)$ in this equidistribution claim when evaluated on (f, f_1, f_2) for every compactly supported ϵ-locally constant map $f : \operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}} \to \mathbb{R}$ and for all finitely supported maps $f_1 : \operatorname{Sh}^1_{\mathfrak{m}} \to \mathbb{R}$ and $f_2 : \operatorname{Sh}^1_{\mathfrak{n}} \to \mathbb{R}$.

Step 2. We now explain how to deduce Corollary 4.8 from the equidistribution of $(V_\Lambda, \operatorname{sh}(\Lambda), \operatorname{sh}(\Lambda^\perp))$ in $\operatorname{Gr}^\sharp_{\mathfrak{m},\mathbf{d}} \times \operatorname{Sh}^1_{\mathfrak{m}} \times \operatorname{Sh}^1_{\mathfrak{n}}$ when Λ varies in $\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}(I)$ with the appropriate covolume. The key technical lemma is the following one.

Let us denote by $W_{\mathbf{d}}$ the Weyl subgroup of $\operatorname{GL}_{\mathbf{d}}(K_\nu)$ consisting in the permutation matrices of the canonical basis of $K_\nu^{\;\mathbf{d}}$. Note that $W_{\mathbf{d}}$ is contained in $\operatorname{GL}_{\mathbf{d}}(R_\nu) \cap \operatorname{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$.

Lemma 4.9 *For every $g \in \operatorname{GL}_{\mathbf{d}}(K_\nu)$, there exists $\sigma \in W_{\mathbf{d}}$ such that if $\sigma g = \left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$ with $\alpha \in \mathscr{M}_{\mathfrak{m}}(K_\nu)$, then $\alpha \in \operatorname{GL}_{\mathfrak{m}}(K_\nu)$ and $\beta\,\alpha^{-1} \in \mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$.*

Proof. For every $g \in \mathscr{M}_{\mathbf{d}}(K_\nu)$, we denote by $g_{\mathfrak{m}|}$ the submatrix of g consisting of its first $\mathfrak{m}$ columns. For all $\alpha \in \mathscr{M}_{\mathfrak{m}}(K_\nu)$ and $j,k \in [\![1,\mathfrak{m}]\!]$, we denote by $\alpha_{\widehat{j},\widehat{k}}$ the submatrix of α where the j-th row and k-th column have been removed. Recall that the (j,k) coefficient of the comatrix $\operatorname{Comm}(\alpha)$ of α is $\operatorname{Comm}(\alpha)_{j,k} = (-1)^{j+k}\det\alpha_{\widehat{j},\widehat{k}}$.

Note that the statement of Lemma 4.9 is invariant by multiplication of the left of g by an element of $W_{\mathbf{d}}$. Since multiplying on the left g by an element of $W_{\mathbf{d}}$ amounts to permuting the rows of g, up to such a multiplication, we may assume that the absolute value of the upper-left $\mathfrak{m}\times\mathfrak{m}$ minor of $g_{\mathfrak{m}|}$ (hence of g) is maximal over the absolute values of all $\mathfrak{m}\times\mathfrak{m}$ minors of $g_{\mathfrak{m}|}$. Let us then prove that if $g=\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}$ with $\alpha\in\mathscr{M}_{\mathfrak{m}}(K_\nu)$, then $\alpha\in\mathrm{GL}_{\mathfrak{m}}(K_\nu)$ and $\beta\,\alpha^{-1}\in\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(\mathscr{O}_\nu)$, which proves Lemma 4.9 by taking $\sigma=\mathrm{id}$.

Since g is invertible, the rank of its submatrix $g_{\mathfrak{m}|}$ is $\mathfrak{m}$. Hence $g_{\mathfrak{m}|}$ has at least one nonzero $\mathfrak{m}\times\mathfrak{m}$ minor, so that $|\det\alpha|\neq 0$ by the above maximality property.

For all $i\in[\![1,\mathfrak{n}]\!]$ and $j\in[\![1,\mathfrak{m}]\!]$, let us prove that the (i,j)-coefficient $(\beta\alpha^{-1})_{i,j}$ of the matrix $\beta\alpha^{-1}\in\mathscr{M}_{\mathfrak{n},\mathfrak{m}}(K_\nu)$ has absolute value at most 1, which proves Lemma 4.9. We denote by $A(i,j)\in\mathscr{M}_{\mathfrak{m}}(K_\nu)$ the matrix α where its j-th row has been replaced by the $(i+\mathfrak{m})$-th row of $g_{\mathfrak{m}|}$. By the above maximality property and since $\det A(i,j)$ is an $\mathfrak{m}\times\mathfrak{m}$ minor of $g_{\mathfrak{m}|}$, we have

$$|\det A(i,j)|\leqslant|\det\alpha|\ .$$

Since the i-th row of β is the $(i+\mathfrak{m})$-th row of $g_{\mathfrak{m}|}$, since $\alpha^{-1}=\frac{1}{\det\alpha}\,{}^t\mathrm{Comm}(\alpha)$, and by the Laplace expansion formula for the determinant of $A(i,j)$ with respect to its j-th row, we have

$$\begin{aligned}(\beta\alpha^{-1})_{ij}&=\sum_{k=1}^{\mathfrak{m}}\beta_{i,k}(\alpha^{-1})_{k,j}=\frac{1}{\det\alpha}\sum_{k=1}^{\mathfrak{m}}g_{i+\mathfrak{m},k}\,\mathrm{Comm}(\alpha)_{j,k}\\&=\frac{1}{\det\alpha}\sum_{k=1}^{\mathfrak{m}}(-1)^{j+k}g_{i+\mathfrak{m},k}\det\alpha_{\widehat{j},\widehat{k}}=\frac{\det A(i,j)}{\det\alpha}\ .\end{aligned}$$

Therefore $|(\beta\alpha^{-1})_{ij}|\leqslant 1$, as wanted. □

The linear action of an element σ of the Weyl group $W_{\mathbf{d}}$ on an element $\Lambda\in\mathscr{PL}_{\mathfrak{m},\mathbf{d}}$ satisfies the following properties.

- Since $\sigma\in\mathrm{GL}_{\mathbf{d}}(R_\nu)$, the $\mathfrak{m}$-lattice $\sigma\Lambda$ is primitive, and by Equation (8), we have

$$\overline{\mathrm{Covol}}(\sigma\Lambda)=\overline{\mathrm{Covol}}(\Lambda)\ .$$

- We have $V_{\sigma\Lambda}=\sigma V_\Lambda$ by the left hand side of Equation (7).
- Since $\sigma\in\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$, and by the construction of the shape map sh in and above Equation (27), we have $\mathrm{sh}(\sigma\Lambda)=\mathrm{sh}(\Lambda)$.
- Let ${R_\nu}^{\mathbf{d},*}$ be the standard full R_ν-lattice of the dual space of ${K_\nu}^{\mathbf{d}}$, which is invariant under the dual action of σ since $\check{\sigma}={}^t\sigma^{-1}\in\mathrm{GL}_{\mathbf{d}}(R_\nu)$. As seen in Equation (12), we have $(\sigma\Lambda)^\perp=\check{\sigma}(\Lambda^\perp)$. Hence $\mathrm{sh}((\sigma\Lambda)^\perp)=\mathrm{sh}(\Lambda^\perp)$ since $\check{\sigma}\in\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$.
- By the $\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$-invariance of the probability measure $\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}$ (see Subsection 3.1), and since $\sigma\in\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$, we have $\sigma_*\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}=\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}$.
- Since $\sigma\in\mathrm{GL}_{\mathbf{d}}(\mathscr{O}_\nu)$, the left action of σ on $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ is isometric for the distance d on $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ constructed in Subsection 3.1.

By Lemma 4.9, for every $\Lambda\in\mathscr{PL}_{\mathfrak{m},\mathbf{d}}\smallsetminus\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}$ with $\overline{\mathrm{Covol}}(\Lambda)\in{q_\nu}^{\ell\mathbb{Z}}$, there exists $\sigma\in W_{\mathbf{d}}$ such that $\sigma\Lambda\in\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}$ and $V_{\sigma\Lambda}=\sigma V_\Lambda\in\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$. Furthermore, σ maps a small ball centered at V_Λ contained in $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}\smallsetminus\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$ to a ball contained in $\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}$ centered at $V_{\sigma\Lambda}$ of the same radius, by the last point above. Hence the equidistribution with error term as $i\to+\infty$ of $(V_\Lambda,\mathrm{sh}(\Lambda),\mathrm{sh}(\Lambda^\perp))$ in $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}\times\mathrm{Sh}^1_{\mathfrak{m}}\times\mathrm{Sh}^1_{\mathfrak{n}}$ when Λ varies in $\mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)$ with $\overline{\mathrm{Covol}}(\Lambda)={q_\nu}^{\ell i}$ follows from the equidistribution with error term as $i\to+\infty$ of $(V_\Lambda,\mathrm{sh}(\Lambda),\mathrm{sh}(\Lambda^\perp))$ in $\mathrm{Gr}^\sharp_{\mathfrak{m},\mathbf{d}}\times\mathrm{Sh}^1_{\mathfrak{m}}\times\mathrm{Sh}^1_{\mathfrak{n}}$ when Λ varies in $\mathscr{PL}^\sharp_{\mathfrak{m},\mathbf{d}}(I)$ with $\overline{\mathrm{Covol}}(\Lambda)={q_\nu}^{\ell i}$.
□

We conclude this paper with a proof of Corollary 1.1 in the introduction.

Proof that Theorem 1.2 implies Corollary 1.1. For $k = \mathfrak{m}, \mathfrak{n}$, with $\widetilde{\Gamma}_k = \mathrm{GL}_k(R_\nu)$, let us consider the map $\iota : D \mapsto D^{-1}$ defined in the Introduction from the discrete set $\mathrm{Sh}^1_k = \mathrm{GL}_k(\mathscr{O}_\nu) \backslash \mathrm{GL}^1_k(K_\nu) / \widetilde{\Gamma}_k$ to the discrete set $\widetilde{\Gamma}_k \backslash V_0 \mathscr{I}_{\nu,k} = \widetilde{\Gamma}_k \backslash \mathrm{GL}^1_k(K_\nu) / \mathrm{GL}_k(\mathscr{O}_\nu)$. By Equation (36) , this map satisfies

$$\iota_* \mu_{\mathrm{Sh}^1_k} = \mu_{\mathrm{GL}^1_k(K_\nu)}(\mathrm{GL}_k(\mathscr{O}_\nu))\, \mu_{\widetilde{\Gamma}_k \backslash V_0 \mathscr{I}_{\nu,k}} = (1 - q_\nu^{-1})\, \mu_{\widetilde{\Gamma}_k \backslash V_0 \mathscr{I}_{\nu,k}} \;.$$

Let $\varphi' : \mathrm{Gr}_{\mathfrak{m},\mathbf{d}} \times \mathrm{Sh}^1_{\mathfrak{m}} \times \mathrm{Sh}^1_{\mathfrak{n}} \to \widetilde{\Gamma}_{\mathfrak{m}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{m}} \times \widetilde{\Gamma}_{\mathfrak{n}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{n}}$ be the continuous map defined by $(x, y, z) \mapsto (y^{-1}, z^{-1})$. Since $\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}$ is compact, this map φ' is proper, and the pushforward map φ'_* of Borel measures by φ' is linear and weak-star continuous. With $\ell = \mathrm{lcm}(\mathfrak{m}, \mathfrak{n})$, the image by φ'_* of the left hand side of Equation (1) is hence

$$\lim_{i \to +\infty} \frac{c''}{q_\nu^{\,\ell\,\mathbf{d}\, i}} \sum_{\Lambda \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}\,:\, \overline{\mathrm{Covol}}(\Lambda) = q_\nu^{\,\ell\, i}} \Delta_{\mathrm{sh}(\Lambda)^{-1}} \otimes \Delta_{\mathrm{sh}(\Lambda^\perp)^{-1}} \;.$$

Since $\|\mu_{\mathrm{Gr}_{\mathfrak{m},\mathbf{d}}}\| = 1$ by Equation (32), the image by φ'_* of the right hand side of Equation (1) is $(1 - q_\nu^{-1})^2\, \mu_{\widetilde{\Gamma}_{\mathfrak{m}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{m}}} \otimes \mu_{\widetilde{\Gamma}_{\mathfrak{n}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{n}}}$. Hence with $c' = \frac{c''}{(1 - q_\nu^{-1})^2}$ as given in Corollary 1.1, Theorem 1.2 does imply Corollary 1.1. □

Remark. Proceeding as in the proof of Corollary 4.8, for every nonzero ideal I of R_ν, we have the following version with error term and congruences of Corollary 1.1 : There exists $\tau \in\,]0, \frac{1}{2\,\mathbf{d}^2}]$ such that for all finitely supported maps $f_1 : \widetilde{\Gamma}_{\mathfrak{m}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{m}} \to \mathbb{R}$ and $f_2 : \widetilde{\Gamma}_{\mathfrak{n}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{n}} \to \mathbb{R}$ and for every $\delta \in\,]0, \tau[$, we have

$$\frac{c'}{q_\nu^{\,\ell\,\mathbf{d}\, i}} \sum_{\Lambda \in \mathscr{PL}_{\mathfrak{m},\mathbf{d}}(I)\,:\, \overline{\mathrm{Covol}}\,\Lambda = q_\nu^{\,\ell\, i}} f_1(\mathrm{sh}(\Lambda)^{-1})\; f_2(\mathrm{sh}(\Lambda^\perp)^{-1})$$
$$= \Big(\int f_1\, d\mu_{\widetilde{\Gamma}_{\mathfrak{m}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{m}}}\Big)\Big(\int f_2\, d\mu_{\widetilde{\Gamma}_{\mathfrak{n}} \backslash V_0 \mathscr{I}_{\nu,\mathfrak{n}}}\Big) + \mathrm{O}_{\nu,\delta,I}\big(q_\nu^{\,\ell\,\mathbf{d}\, i(-\tau+\delta)}\, \|f_1\|_\infty \|f_2\|_\infty\big) \;. \tag{78}$$

A Dual and factor partial lattices

Let V be a K_ν-vector space with finite dimension $D \geqslant 2$ endowed with an ultrametric norm $\|\;\|$. Let $k \in [\![1, D-1]\!]$ and let W be a k-dimensional K_ν-vector subspace of V, endowed with the restriction norm. We endow the quotient $(D-k)$-dimensional K_ν-vector space V/W with the quotient norm. We denote by $\pi : V \to V/W$ the canonical projection. Then the Haar measures μ_V, μ_W and $\mu_{V/W}$ on respectively V, W, V/W, normalized by these choices of norms as explained in Subsection 2.2, satisfy the following Weil's normalization process (see [Wei3, §9]). For all $\overline{x} \in V/W$ and $x \in V$ such that $\pi(x) = \overline{x}$, let $\mu_{\pi^{-1}(\overline{x})}$ be the measure on the K_ν-affine subspace $\pi^{-1}(\overline{x}) = x + W$ such that the translation $\tau_x : y \mapsto x + y$, which is a homeomorphism from W to $\pi^{-1}(\overline{x})$, satisfies $(\tau_x)_* \mu_W = \mu_{\pi^{-1}(\overline{x})}$ (this does not depend on the choice of x in $\pi^{-1}(\overline{x})$). Then *Weil's normalisation* is asking that we have the following disintegration property of the measure μ_V over the measure $\mu_{V/W}$ by π:

$$d\mu_V = \int_{\overline{x} \in V/W} d\mu_{\pi^{-1}(\overline{x})}\; d\mu_{V/W}(\overline{x}) \;. \tag{79}$$

This formula implies that the normalizations of μ_V and μ_W uniquely determine the normalization of $\mu_{V/W}$. In order to check that this normalization coincides with the one coming from the quotient norm on V/W, we apply the above formula on $B_V(0,1)$, noting that $\pi(B_V(0,1)) = B_{V/W}(0,1)$ and that, by the ultrametric property, for every $x \in B_V(0,1)$, we have $-x + B_V(0,1) \cap (x+W) = B_W(0,1)$.

Let Λ be a k-lattice in V. Its *dual lattice* is the R_ν-submodule of the dual K_ν-vector space V_Λ^* defined by

$$\Lambda^* = \{\ell \in V_\Lambda^* : \forall\, x \in \Lambda,\ \ \ell(x) \in R_\nu\}\ .$$

Lemma A.1 *The dual lattice Λ^* is a full R_ν-lattice in V_Λ^*, we have $(\Lambda^*)^* = \Lambda$ and*

$$\operatorname{Covol}(\Lambda^*)\operatorname{Covol}(\Lambda) = q^{2k(\mathfrak{g}-1)}\ .$$

Proof. Let $(b_1, \dots, b_k)$ be an R_ν-basis of Λ. Then $(b_1, \dots, b_k)$ is a K_ν-basis in V_Λ, and we denote by $(b_1^*, \dots, b_k^*)$ its dual basis in V_Λ^*. A linear form $\ell = \sum_{i=1}^k \lambda_i b_1^* \in V_\Lambda^*$ takes integral values on all elements of Λ if and only if it takes integral values on $b_1, \dots, b_k$, that is, if and only if its coordinates $\lambda_1, \dots, \lambda_k$ are integral. Thus $\Lambda^* = \oplus_{i=1}^k R_\nu b_i^*$, which is a k-lattice in the k-dimensional vector space V_Λ^*. Since the dual basis in $(V_\Lambda^*)^* = V_\Lambda$ of the K_ν-basis $(b_1^*, \dots, b_k^*)$ of V_Λ^* is the K_ν-basis $(b_1, \dots, b_k)$ of V_Λ, we have $(\Lambda^*)^* = \Lambda$.

The duality pairing $V_\Lambda^* \times V_\Lambda \to K_\nu$ defined by $(\ell, x) \mapsto \ell(x)$, which sends $\Lambda^* \times \Lambda$ to R_ν induces a K_ν-linear isomorphism $V_\Lambda^* \times V_\Lambda \to {K_\nu}^{2k}$ that sends the full R_ν-lattice $\Lambda^* \times \Lambda$ to ${R_\nu}^{2k}$ and the supremum product norm of the norm on V_Λ and of its dual norm on V_Λ^* to the standard supremum norm on ${K_\nu}^{2k}$. Hence, using Equation (9) for the last equality, we have

$$\operatorname{Covol}(\Lambda^*)\operatorname{Covol}(\Lambda) = \operatorname{Covol}(\Lambda^* \times \Lambda) = \operatorname{Covol}({R_\nu}^{2k}) = q^{2k(\mathfrak{g}-1)}\ . \quad \Box$$

Assume that V is also endowed with an integral structure V_{R_ν}. Let Λ be a primitive k-lattice in the integral K_ν-space V. The *factor lattice* of Λ is the R_ν-submodule Λ^π of the quotient K_ν-vector space V/V_Λ which is the image of V_{R_ν} by the canonical projection $\pi : V \to V/V_\Lambda$.

Lemma A.2 *The factor lattice Λ^π is a full R_ν-lattice in V/V_Λ. The canonical K_ν-linear isomorphism $V/V_\Lambda \to (V_\Lambda^\perp)^*$ maps Λ^π to $(\Lambda^\perp)^*$. We have*

$$\operatorname{Covol}(\Lambda^\pi) = \operatorname{Covol}((\Lambda^\perp)^*) \quad \text{and} \quad \operatorname{Covol}(\Lambda^\pi)\operatorname{Covol}(\Lambda) = \operatorname{Covol}(V_{R_\nu})$$

Proof. Since the k-lattice Λ is primitive, there exists an R_ν-basis $(b_1, \dots, b_D)$ of V_{R_ν} such that $(b_1, \dots, b_k)$ is an R_ν-basis of Λ, hence a K_ν-basis of V_Λ. Then $(\pi(b_k+1), \dots, \pi(b_D))$ is a K_ν-basis of V/V_Λ, and an R_ν-basis of Λ^π by definition. Hence Λ^π is a $(D-k)$-lattice in the $(D-k)$-dimensional vector space V/V_Λ.

Identifying a K_ν-vector space W with its bidual $(W^*)^*$ by the map $x \mapsto (\ell \mapsto \ell(x))$ as usual, the map $\widetilde{\Theta''} : V \to (V_\Lambda^\perp)^*$ defined by $x \mapsto x_{|V_\Lambda^\perp}$ induces a K_ν-linear isomorphism $\Theta'' : V/V_\Lambda \to (V_\Lambda^\perp)^*$. With $(b_1, \dots, b_D)$ as above, we have seen in the proof of Proposition 2.2 that $(b_{k+1}^*, \dots, b_D^*)$ is an R_ν-basis of $\Lambda^\perp$, hence a K_ν-basis of $V_{\Lambda^\perp} = V_\Lambda^\perp$. As seen in the proof of Lemma A.1, the dual K_ν-basis $({b_{k+1}^*}^*, \dots, {b_D^*}^*)$ of $(b_{k+1}^*, \dots, b_D^*)$ is an R_ν-basis of $(\Lambda^\perp)^*$. But $({b_{k+1}^*}^*, \dots, {b_D^*}^*)$ is exactly $\big(\Theta''(\pi(b_{k+1})), \dots, \Theta''(\pi(b_{k+1}))\big)$. Hence $\Theta''(\Lambda^\pi) = (\Lambda^\perp)^*$.

When V/V_Λ is endowed with the quotient norm, and $(V_\Lambda^\perp)^*$ with the dual norm of the restriction to $V_\Lambda^\perp$ of the dual norm on V^*, the above map Θ'' is an isometry. Hence $\operatorname{Covol}(\Lambda^\pi) = \operatorname{Covol}((\Lambda^\perp)^*)$.

Let F be a clopen strict fundamental domain for the action of R_ν on K_ν. The formula $\operatorname{Covol}(V_{R_\nu}) = \operatorname{Covol}(\Lambda) \operatorname{Covol}(\Lambda^\pi)$ follows by integrating Equation (79) with $W = V_\Lambda$ on the strict fundamental domain $F\,b_1 + \cdots + F\,b_D$ of V_{R_ν}. □

Proof of Equation (13)**.** Under its assumptions, we may assume that $V = {K_\nu}^D$ and $V_{R_\nu} = {R_\nu}^D$ and that the norm of V is the standard supremum norm of ${K_\nu}^D$. We then have $\operatorname{Covol}(V_{R_\nu}) = q^{D(\mathfrak{g}-1)}$ by Equation (9). Hence respectively by Lemma A.1 (recalling that $\Lambda^\perp$ is a $(D-k)$-lattice), by the first equality in Lemma A.2, and by the second equality in Lemma A.2, we have

$$\begin{aligned}\operatorname{Covol}(\Lambda^\perp) &= q^{2(D-k)(\mathfrak{g}-1)} \operatorname{Covol}((\Lambda^\perp)^*)^{-1} = q^{2(D-k)(\mathfrak{g}-1)} \operatorname{Covol}(\Lambda^\pi)^{-1}\\ &= q^{2(D-k)(\mathfrak{g}-1)} \operatorname{Covol}(\Lambda) q^{-D(\mathfrak{g}-1)} = \operatorname{Covol}(\Lambda)\, q^{(D-2k)(\mathfrak{g}-1)}\,,\end{aligned}$$

as wanted. □

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