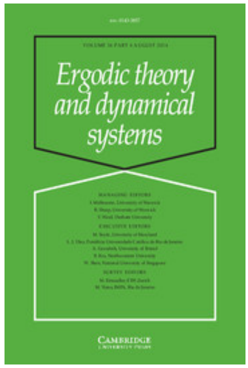

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Skinning measures in negative curvature and equidistribution of equidistant submanifolds

JOUNI PARKKONEN† and FRÉDÉRIC PAULIN‡

† *Department of Mathematics and Statistics, PO Box 35, 40014 University of Jyväskylä, Finland*
(e-mail: jouni.t.parkkonen@jyu.fi)

‡ *Département de mathématique, UMR 8628 CNRS, Bât. 425, Université Paris-Sud, 91405 Orsay Cedex, France*
(e-mail: frederic.paulin@math.u-psud.fr)

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Abstract. Let C be a locally convex closed subset of a negatively curved Riemannian manifold M. We define the skinning measure σ_C on the outer unit normal bundle to C in M by pulling back the Patterson–Sullivan measures at infinity, and give a finiteness result for σ_C, generalizing the work of Oh and Shah, with different methods. We prove that the skinning measures, when finite, of the equidistant hypersurfaces to C equidistribute to the Bowen–Margulis measure $m_{\rm BM}$ on T^1M, assuming only that $m_{\rm BM}$ is finite and mixing for the geodesic flow. Under additional assumptions on the rate of mixing, we give a control on the rate of equidistribution.

1. *Introduction*

Let M be a complete connected Riemannian manifold with sectional curvature at most -1. For any proper non-empty properly immersed locally convex closed subset C of M and $t > 0$, let Σ_t be the (Lipschitz) submanifold of T^1M that consists of images by the geodesic flow at time t of the outward-pointing unit normal vectors to the boundary of C (see §2 for precise definitions).

If M has constant curvature and finite volume and if C is an immersed totally geodesic submanifold of finite volume, we showed in [**PP1**, Theorem 2.2] that the Riemannian measure of Σ_t equidistributes to the Liouville measure of T^1M (which is the Riemannian measure of the Sasaki metric of T^1M). This result also follows from the equidistribution result of Eskin and McMullen [**EM**, Theorem 1.2] in affine symmetric spaces; see [**PP2**, §4] for details.

In this paper, we generalize the above result when C is no longer required to be totally geodesic and when M has variable curvature. Though the methods of locally homogeneous spaces as in [**EM**] are then completely inapplicable, the strategy of [**PP1**] remains helpful. Both the measures on T^1M and on Σ_t need to be adapted to variable curvature.

The measure on T^1M we will consider (when M is non-elementary and its fundamental group has finite critical exponent) is the well-known *Bowen–Margulis measure* m_{BM} (see [**Rob2**] for a nice presentation). It coincides with the Liouville measure (up to a multiplicative constant) when M is locally symmetric with finite volume (see, for instance, [**PP2**, §7] when M is real hyperbolic). It is, when finite and normalized, the unique probability measure of maximal entropy for the geodesic flow on T^1M (see [**Mar2, Bowe**] when M is compact, and [**OP**] under the sole assumption that m_{BM} is finite). The Bowen–Margulis measure is finite, for instance, when M is compact, or when M is geometrically finite and the critical exponent of its fundamental group is strictly bigger than the critical exponents of its parabolic subgroups (as it is the case when M is locally symmetric), by [**DOP**]. By [**Bab**, Theorem 1], the Bowen–Margulis measure, when finite, is mixing if the length spectrum of M is not contained in a discrete subgroup of $\mathbb{R}$. By [**Dal1, Dal2**], this condition holds, for instance, when M is compact or two-dimensional or locally symmetric, or if its fundamental group contains a parabolic element.

The measure on Σ_t we will consider is the *skinning measure* that we introduce in this generality in this paper (see §3), as an appropriate pushforward to Σ_t of the natural measures at infinity of the universal cover of M. It scales by $e^{\delta t}$, where δ is the critical exponent of M, under the geodesic flow map from Σ_s to Σ_{s+t}. When C is an immersed horoball, Σ_t is a leaf of the strong unstable foliation of the geodesic flow on T^1M, and the skinning measure on Σ_t is simply the conditional measure of the Bowen–Margulis measure on this leaf (see, for instance, [**Mar3, Rob2**]). When M is geometrically finite with constant curvature, and when C is an immersed ball, horoball or totally geodesic submanifold, the skinning measure on Σ_t has been introduced by Oh and Shah [**OS1, OS2**], who coined the term, with beautiful applications to circle packings, and coincides with the Riemannian measure up to a multiplicative constant (see [**PP2**, §7] for a computation of the constant) when, furthermore, M has finite volume. When the intersection of Σ_t with the non-wandering set of the geodesic flow of T^1M is compact, the skinning measure is finite.

When M is geometrically finite, generalizing (and giving an alternative proof of) Theorem 6.4 in [**OS2**] which assumes the curvature to be constant, we give in Theorem 10 a sharp criterion for the finiteness of the skinning measure, by studying its decay in the cusps of M. This decay is analogous to the decay of the Bowen–Margulis measure in the cusps, which was first studied by Sullivan [**Sul**], who called it the fluctuating density property (see also [**SV**] and [**HP2**, Theorem 4.1]). The criterion, as in the case of the Bowen–Margulis measure in [**DOP**], is a separation property of the critical exponents.

The following theorem is a simplified version of the main result of this paper. In the more general result, Theorem 19 in §5, we replace Σ_t by $g^t\Omega$, where Ω is an open set of outward-pointing unit normal vectors to ∂C with finite non-zero skinning measure.

THEOREM 1. *Let M be a non-elementary complete Riemannian manifold with pinched negative sectional curvature. Assume that the Bowen–Margulis measure on T^1M is finite and mixing for the geodesic flow. Let C be a proper non-empty properly immersed locally convex closed subset of M with finite non-zero skinning measure. Then as t tends to $+\infty$, the skinning measure on Σ_t equidistributes to the Bowen–Margulis measure on T^1M.*

When C is an immersed ball or horoball, this result is due to Margulis when M has finite volume (see, for example, [**Mar3**]), and to Babillot [**Bab**, Theorem 3] and Roblin [**Rob2**] under the weak assumptions of Theorem 1. Many ideas of our proof go back to [**Mar1**]. See also [**Sch, Mark, KO, Kim**] for other results on the equidistribution of horospheres and applications.

For instance, it follows from Theorem 1 that when M is a compact Riemannian manifold with negative sectional curvature, when C is the image in M of the convex hull of the limit set of an infinite index convex-cocompact subgroup of the covering group of a universal cover of M, the skinning measure on Σ_t equidistributes to the Bowen–Margulis measure on T^1M. But we make no compactness assumption on C in our theorem, only requiring the finiteness of the measures under consideration. The main tool is a general disintegration result of the Bowen–Margulis measure over any skinning measure (see Proposition 8).

We also give (see §6) estimates on the rate of equidistribution in the previous result, under assumptions on the rate of mixing of the geodesic flow. When M is locally symmetric and arithmetic, the rate of mixing of the geodesic flow for sufficiently smooth functions is exponential, by the work of Kleinbock and Margulis [**KM1**, Theorem 2.4.3] and Clozel [**Clo**, Theorem 3.1]. When the curvature is variable, the appropriate regularity is the Hölder one. The rate of mixing of the geodesic flow for Hölder-continuous functions is exponential if M is compact and has dimension two by the work of Dolgopyat [**Dol**] or if M is compact and locally symmetric (without the arithmetic assumption) by [**Sto**, Corollary 1.5] (see also [**Liv**] when M is compact; the result stated there for the Liouville measure should extend to the Bowen–Margulis measure, as has been done in [**GLP**, Corollary 2.7] when the sectional curvature of M is $\frac{1}{9}$-pinched).

THEOREM 2. *Under the hypotheses of Theorem 1, in any of the above cases when the geodesic flow of T^1M is mixing with exponential speed, the skinning measure σ_t of Σ_t equidistributes to the Bowen–Margulis measure with exponential speed.*

More precisely, in the Hölder case (see §6 for precise definitions), if M is compact and if the geodesic flow on T^1M is mixing with exponential speed for the Hölder regularity, then there exist $\alpha \in]0, 1[$ and $\tau > 0$ such that for every α-Hölder-continuous function $\psi : T^1M \to \mathbb{R}$ with α-Hölder norm $\|\psi\|_\alpha$, as t tends to $+\infty$,

$$\frac{1}{\|\sigma_t\|}\int_{\Sigma_t} \psi \, d\sigma_t = \frac{1}{\|m_{\mathrm{BM}}\|}\int_{T^1M} \psi \, dm_{\mathrm{BM}} + O(e^{-\tau\, t}\|\psi\|_\alpha).$$

In [**PP3**], we will use the tools introduced in this paper to study counting results of common perpendicular arcs between locally convex subsets in variable negative curvature.

2. *Geometry, dynamics and convexity in negative curvature*

In this section we briefly review the required background on negatively curved Riemannian manifolds, seen as locally CAT$(-\kappa)$ spaces, using, for instance, [**BH**] as a general reference, and their unit tangent bundles and geodesic flows. We introduce the geometric fibred neighbourhoods of the outer unit normal bundle of the boundary of a convex subset which will be used in what follows.

2.1. *Geometry and dynamics.* Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with sectional curvature bounded above by -1, and let $x_0 \in \widetilde{M}$. Let Γ be a discrete, non-elementary group of isometries of $\widetilde{M}$, and let us denote the quotient space of $\widetilde{M}$ under Γ by $M = \Gamma\backslash\widetilde{M}$. We denote by $\partial_\infty\widetilde{M}$ the boundary at infinity of $\widetilde{M}$ (with the Hölder structure defined by the visual distance recalled below), by $\Lambda\Gamma$ the limit set of Γ and by $\mathscr{C}\Lambda\Gamma$ the convex hull in $\widetilde{M}$ of $\Lambda\Gamma$. For every $\epsilon > 0$, we denote by $\mathscr{N}_\epsilon A$ the closed ϵ-neighbourhood of a subset A of $\widetilde{M}$, and by convention $\mathscr{N}_0 A = \overline{A}$.

For any point $\xi \in \partial_\infty\widetilde{M}$, let $\rho_\xi : [0, +\infty[\to \widetilde{M}$ be the geodesic ray with origin x_0 and point at infinity ξ. The *Busemann cocycle* of $\widetilde{M}$ is the map $\beta : \widetilde{M} \times \widetilde{M} \times \partial_\infty\widetilde{M} \to \mathbb{R}$ defined by

$$(x, y, \xi) \mapsto \beta_\xi(x, y) = \lim_{t\to+\infty} d(\rho_\xi(t), x) - d(\rho_\xi(t), y).$$

The above limit exists and is independent of x_0. If y is a point in the (image of the) geodesic ray from x to ξ, then $\beta_\xi(x, y) = d(x, y)$. The Busemann cocycle satisfies

$$\beta_{\gamma\xi}(\gamma x, \gamma y) = \beta_\xi(x, y) \quad \text{and} \quad \beta_\xi(x, y) + \beta_\xi(y, z) = \beta_\xi(x, z), \tag{1}$$

for all $\xi \in \partial_\infty\widetilde{M}$, all $x, y, z \in \widetilde{M}$ and every isometry γ of $\widetilde{M}$. The *visual distance* d_{x_0} (based at x_0) on $\partial_\infty\widetilde{M}$ is the distance defined by

$$d_{x_0}(\xi, \eta) = e^{-\frac{1}{2}(\beta_\xi(x_0, y) + \beta_\eta(x_0, y))} \tag{2}$$

for any y in the geodesic line between ξ and η if $\xi \neq \eta$, and $d_{x_0}(\xi, \eta) = 0$ if $\xi = \eta$.

The unit tangent bundle T^1N of a complete Riemannian manifold N can be identified with the set of locally geodesic lines $\ell : \mathbb{R} \to N$ in N, endowed with the compact-open topology. More precisely, we identify a locally geodesic line ℓ and its (unit) tangent vector $\dot{\ell}(0)$ at time $t = 0$ and, conversely, any $v \in T^1N$ is the tangent vector at time $t = 0$ of a unique locally geodesic line. We will use this identification without mention in this paper. We denote by $\pi : T^1N \to N$ the base point projection, which is given by $\pi(\ell) = \ell(0)$.

The *geodesic flow* on T^1N is the dynamical system $(g^t)_{t\in\mathbb{R}}$, where $g^t\ell\,(s) = \ell(s + t)$, for all $\ell \in T^1N$ and $s, t \in \mathbb{R}$. The isometry group of $\widetilde{M}$ acts on the space of geodesic lines in $\widetilde{M}$ by postcomposition, $(\gamma, \ell) \mapsto \gamma \circ \ell$, and this action commutes with the geodesic flow.

When Γ acts on $\widetilde{M}$ without fixed point, we have an identification $\Gamma\backslash T^1\widetilde{M} = T^1M$. Even in the general case with torsion, we denote by T^1M the quotient space $\Gamma\backslash T^1\widetilde{M}$. We use the notation $(g^t)_{t\in\mathbb{R}}$ also for the geodesic flow on T^1M (induced by the geodesic flow on $T^1\widetilde{M}$ by passing to the quotient).

We denote by $\iota : T^1\widetilde{M} \to T^1\widetilde{M}$ the *antipodal (flip) map* $v \mapsto -v$, and we again denote by $\iota : T^1M \to T^1M$ its quotient map. We have $\iota \circ g^t = g^{-t} \circ \iota$ for all $t \in \mathbb{R}$.

For every unit tangent vector $v \in T^1\widetilde{M}$, let $v_- = v(-\infty)$ and $v_+ = v(+\infty)$ be the two endpoints in the sphere at infinity of the geodesic line defined by v. Let $\partial^2_\infty\widetilde{M}$ be the subset of $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ which consists of pairs of distinct points at infinity. The *Hopf parametrization* of $T^1\widetilde{M}$ is the identification of $v \in T^1\widetilde{M}$ with the triple $(v_-, v_+, t) \in \partial^2_\infty\widetilde{M} \times \mathbb{R}$, where t is the signed (algebraic) distance of $\pi(v)$ from the closest point p_{v,x_0} to x_0 on the (oriented) geodesic line defined by v. This map is a homeomorphism, the geodesic flow acts by $g^s(v_-, v_+, t) = (v_-, v_+, t + s)$ and, for every isometry γ of $\widetilde{M}$, the image of γv is $(\gamma v_-, \gamma v_+, t + t_{\gamma,v_-,v_+})$, where t_{γ,v_-,v_+} is the signed distance from $\gamma p_{v,x_0}$ to $p_{\gamma v,x_0}$. Furthermore, in these coordinates, the antipodal map ι is $(v_-, v_+, t) \mapsto (v_+, v_-, -t)$.

The *strong stable manifold* of $v \in T^1\widetilde{M}$ is

$$W^{\rm ss}(v) = \{v' \in T^1\widetilde{M} : d(v(t), v'(t)) \to 0 \text{ as } t \to +\infty\},$$

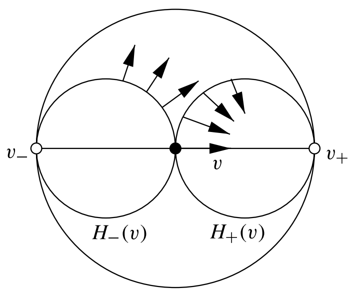
and its *strong unstable manifold* is

$$W^{\rm su}(v) = \{v' \in T^1\widetilde{M} : d(v(t), v'(t)) \to 0 \text{ as } t \to -\infty\}.$$

The union for $t \in \mathbb{R}$ of the images under g^t of the strong stable manifold of $v \in T^1\widetilde{M}$ is the *stable manifold* $W^{\rm s}(v) = \bigcup_{t\in\mathbb{R}} g^t W^{\rm ss}(v)$ of v, which consists of the elements $v' \in T^1\widetilde{M}$ with $v'_+ = v_+$. Similarly, the union of the images under the geodesic flow at all times of the strong unstable manifold of v is the *unstable manifold* $W^{\rm u}(v)$ of v, which consists of the elements $v' \in T^1\widetilde{M}$ with $v'_- = v_-$.

The strong stable manifolds, stable manifolds, strong unstable manifolds and unstable manifolds are the (smooth) leaves of (continuous) foliations, which are invariant under the geodesic flow and the isometry group of $\widetilde{M}$; they are denoted by $\mathscr{W}^{\rm ss}$, $\mathscr{W}^{\rm s}$, $\mathscr{W}^{\rm su}$ and $\mathscr{W}^{\rm u}$, respectively. These foliations are Hölder-continuous when $\widetilde{M}$ has pinched negative sectional curvature with bounded derivatives (see, for instance, [**Bri**], [**PPS**, §7.1]). The maps from $\mathbb{R} \times W^{\rm ss}(v)$ to $W^{\rm s}(v)$ defined by $(s, v') \mapsto g^s v'$ and from $\mathbb{R} \times W^{\rm su}(v)$ to $W^{\rm u}(v)$ defined by $(s, v') \mapsto g^s v'$ are smooth diffeomorphisms.

The images of the strong unstable and strong stable manifolds of $v \in T^1\widetilde{M}$ under the base point projection, denoted by $H_-(v) = \pi(W^{\rm su}(v))$ and $H_+(v) = \pi(W^{\rm ss}(v))$, are respectively called the *unstable and stable horospheres* of v, and are said to be *centred at* v_- and v_+, respectively. The unstable horosphere of v coincides with the zero set of the map $x \mapsto f_-(x) = \beta_{v_-}(x, \pi(v))$, and, similarly, the stable horosphere of v coincides with the zero set of $x \mapsto f_+(x) = \beta_{v_+}(x, \pi(v))$. The corresponding sublevel sets $HB_-(v) = f_-^{-1}(]-\infty, 0])$ and $HB_+(v) = f_+^{-1}(]-\infty, 0])$ are called the *horoballs* bounded by $H_-(v)$ and $H_+(v)$. Horoballs are (strictly) convex subsets of $\widetilde{M}$.



For every $w \in T^1\widetilde{M}$, let $d_{W^{\rm ss}(w)}$ be the *Hamenstädt distance* on the strong stable leaf of w, defined as follows (see [**Ham**], [**HP1**, Appendix], as well as [**HP3**, §2.2] for a generalization when the horosphere $H_+(w)$ is replaced by the boundary of any non-empty closed convex subset): for all $v, v' \in W^{\rm ss}(w)$,

$$d_{W^{\rm ss}(w)}(v, v') = \lim_{t\to+\infty} e^{\frac{1}{2}d(v(-t),v'(-t))-t}.$$

This limit exists, and the Hamenstädt distance is a distance inducing the original topology on $W^{ss}(w)$. For all $v, v' \in W^{ss}(w)$ and for every isometry γ of $\widetilde{M}$, we have $d_{W^{ss}(\gamma w)}(\gamma v, \gamma v') = d_{W^{ss}(w)}(v, v')$. By the triangle inequality, for all $v, v' \in W^{ss}(w)$,

$$d_{W^{ss}(w)}(v, v') \leq e^{\frac{1}{2}d(\pi(v), \pi(v'))}. \tag{3}$$

For all $w \in T^1\widetilde{M}$, $s \in \mathbb{R}$ and $v, v' \in W^{ss}(w)$,

$$d_{W^{ss}(g^s w)}(g^s v, g^s v') = e^{-s} d_{W^{ss}(w)}(v, v'). \tag{4}$$

The usual distance d on $T^1\widetilde{M}$ is defined, for all $v, v' \in T^1\widetilde{M}$, by

$$d(v, v') = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} d(v(t), v'(t)) e^{-t^2}\, dt.$$

This distance is invariant under the group of isometries of $\widetilde{M}$ and the antipodal map. Also note that, for all $s \in \mathbb{R}$ and $v \in T^1\widetilde{M}$,

$$d(g^s v, v) = |s|. \tag{5}$$

LEMMA 3. *There exists $c > 0$ such that, for all $w \in T^1\widetilde{M}$ and $v, v' \in W^{ss}(w)$,*

$$d(v, v') \leq c\, d_{W^{ss}(w)}(v, v').$$

Proof. We may assume that $v \neq v'$. By the convexity properties of the distance in $\widetilde{M}$, the map from $\mathbb{R}$ to $\mathbb{R}$ defined by $t \mapsto d(v(t), v'(t))$ is decreasing, with image $]0, +\infty[$. Let $S \in \mathbb{R}$ be such that $d(v(S), v'(S)) = 1$. For every $t \leq S$, let p and p' be the closest point projections of $v(S)$ and $v'(S)$ on the geodesic segment $[v(t), v'(t)]$. We have $d(p, v(S)), d(p', v'(S)) \leq 1$ by comparison. Hence, by convexity and the triangle inequality,

$$\begin{aligned} d(v(t), v'(t)) &\geq d(v(t), p) + d(p', v'(t)) \\ &\geq d(v(t), v(S)) - 1 + d(v'(t), v'(S)) - 1 = 2(S - t - 1). \end{aligned}$$

Thus, by the definition of the Hamenstädt distance $d_{W^{ss}(w)}$,

$$d_{W^{ss}(w)}(v, v') \geq e^{S-1}. \tag{6}$$

By the triangle inequality, if $t \leq S$, then

$$d(v(t), v'(t)) \leq d(v(t), v(S)) + d(v(S), v'(S)) + d(v'(S), v'(t)) = 2(S - t) + 1.$$

Since $\widetilde{M}$ is CAT(-1), if $t \geq S$, we have by comparison

$$d(v(t), v'(t)) \leq e^{S-t} d(v(S), v'(S)) = e^{S-t}.$$

Therefore, by the definition of the distance d on $T^1\widetilde{M}$,

$$d(v, v') \leq \int_{-\infty}^{S} (2(S-t) + 1) e^{-t^2}\, dt + \int_{S}^{+\infty} e^{S-t} e^{-t^2}\, dt = \mathrm{O}(e^S).$$

The result thus follows from equation (6). □

2.2. *Convexity.* Let C be a non-empty closed convex subset of $\widetilde{M}$. We denote by ∂C the boundary of C in $\widetilde{M}$ and by $\partial_\infty C$ its set of points at infinity (the set of endpoints of geodesic rays contained in C). Let $P_C : \widetilde{M} \cup (\partial_\infty \widetilde{M} - \partial_\infty C) \to C$ be the (continuous) closest point map: if $\xi \in \partial_\infty \widetilde{M} - \partial_\infty C$, then $P_C(\xi)$ is defined to be the unique point in C that minimizes the map $x \mapsto \beta_\xi(x, x_0)$ from C to $\mathbb{R}$. For every isometry γ of $\widetilde{M}$, we have $P_{\gamma C} \circ \gamma = \gamma \circ P_C$.

Let $\partial^1_+ C$ be the subset of $T^1\widetilde{M}$ consisting of the geodesic lines $v : \mathbb{R} \to \widetilde{M}$ with $v(0) \in \partial C$, $v_+ \notin \partial_\infty C$ and $P_C(v_+) = v(0)$. Note that $\pi(\partial^1_+ C) = \partial C$ and that for every isometry γ of $\widetilde{M}$, we have $\partial^1_+(\gamma C) = \gamma\, \partial^1_+ C$. In particular, $\partial^1_+ C$ is invariant under the isometries of $\widetilde{M}$ that preserve C. When $C = HB_-(v)$ is the unstable horoball of $v \in T^1\widetilde{M}$, then $\partial^1_+ C$ is the strong unstable manifold $W^{\rm su}(v)$ of v, and similarly, $W^{\rm ss}(v) = \iota\, \partial^1_+ HB_+(v)$.

The restriction of P_C to $\partial_\infty \widetilde{M} - \partial_\infty C$ (which is not necessarily injective) has a natural lift to a homeomorphism

$$\nu P_C : \partial_\infty \widetilde{M} - \partial_\infty C \to \partial^1_+ C$$

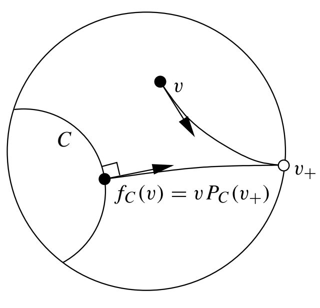
such that $\pi \circ \nu P_C = P_C$. The inverse of νP_C is the endpoint map $v \mapsto v_+$ from $\partial^1_+ C$ to $\partial_\infty \widetilde{M} - \partial_\infty C$. In particular, $\partial^1_+ C$ is a topological submanifold of $T^1\widetilde{M}$. For every $s \geq 0$, the geodesic flow induces a homeomorphism $g^s : \partial^1_+ C \to \partial^1_+ \mathscr{N}_s C$. For every isometry γ of $\widetilde{M}$, we have $\nu P_{\gamma C} \circ \gamma = \gamma \circ \nu P_C$. We refer for instance to [**Wal**] for the notion of $\mathrm{C}^{1,1}$ and Lipschitz manifolds. When C has non-empty interior and $\mathrm{C}^{1,1}$ boundary, then $\partial^1_+ C$ is the Lipschitz submanifold of $T^1\widetilde{M}$ consisting of the outward-pointing unit normal vectors to ∂C, and the map P_C itself is a homeomorphism (between $\partial_\infty \widetilde{M} - \partial_\infty C$ and ∂C). This holds, for instance, by [**Wal**], when C is the closed η-neighbourhood of any non-empty convex subset of $\widetilde{M}$ with $\eta > 0$.

We now define a canonical fundamental system of neighbourhoods, of dynamical origin, of these outer unit normal bundles of boundaries of convex sets. Let

$$U_C = \{v \in T^1\widetilde{M} : v_+ \notin \partial_\infty C\}. \tag{7}$$

Note that U_C is an open subset of $T^1\widetilde{M}$, invariant under the geodesic flow, which is empty if and only if $C = \widetilde{M}$, and is dense in $T^1\widetilde{M}$ if the interior of $\partial_\infty C$ in $\partial_\infty \widetilde{M}$ is empty. We have $U_{\gamma C} = \gamma U_C$ for every isometry γ of $\widetilde{M}$ and, in particular, U_C is invariant under the isometries of $\widetilde{M}$ preserving C.

Define a map $f_C : U_C \to \partial^1_+ C$, as the composition of the map from U_C onto $\partial_\infty \widetilde{M} - \partial_\infty C$ sending v to v_+ and the homeomorphism νP_C from $\partial_\infty \widetilde{M} - \partial_\infty C$ to $\partial^1_+ C$. The map f_C is a fibration as the composition of such a map with the homeomorphism νP_C. The fibre of $w \in \partial^1_+ C$ is precisely its stable leaf $W^{\rm s}(w) = \{v \in T^1\widetilde{M} : v_+ = w_+\}$. In particular, U_C is the disjoint union of the leaves $W^{\rm s}(w)$ for $w \in \partial^1_+ C$.



For every isometry γ of $\widetilde{M}$, we have $f_{\gamma C} \circ \gamma = \gamma \circ f_C$. We have $f_{\mathscr{N}_t C} = g^t \circ f_C$ for all $t \geq 0$, and $f_C \circ g^t = f_C$ for all $t \in \mathbb{R}$. In particular, the fibration f_C is invariant under the geodesic flow.

Let $\eta,\ R > 0$. For all $w \in T^1 M$, let

$$V_{w,R} = \{v' \in W^{\mathrm{ss}}(w) : d_{W^{\mathrm{ss}}(w)}(v',\, w) < R\} \tag{8}$$

be the open ball of radius R centred at w for the Hamenstädt distance in the strong stable leaf of w, and

$$\begin{aligned} V_{w,\eta,R} &= \{v \in W^{\mathrm{s}}(w) : \exists v' \in V_{w,R},\ \exists s \in\,]-\eta,\, \eta[,\ g^s v' = v\} \\ &= \bigcup_{s \in\,]-\eta,\eta[} g^s V_{w,R} = \bigcup_{s \in\,]-\eta,\eta[} V_{g^s w, e^{-s} R}. \end{aligned}$$

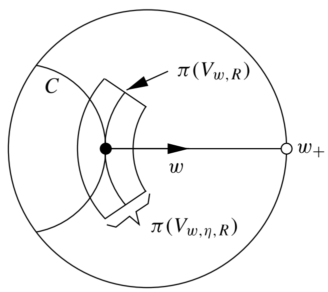
The latter equality follows from the fact that, by equation (4), $g^s V_{w,R} = V_{g^s w, e^{-s} R}$ for every $s \in \mathbb{R}$. For every isometry γ of $\widetilde{M}$, we have $\gamma V_{w,R} = V_{\gamma w,R}$ and $\gamma V_{w,\eta,R} = V_{\gamma w,\eta,R}$. The map from $]-\eta,\, \eta[\, \times V_{w,R}$ to $V_{w,\eta,R}$ defined by $(s,\, v') \mapsto g^s v'$ is a homeomorphism.

For every subset Ω of $\partial^1_+ C$, let

$$\mathscr{V}_{\eta,R}(\Omega) = \bigcup_{w \in \Omega} V_{w,\eta,R},$$

which is an open neighbourhood of Ω in $T^1 \widetilde{M}$ if Ω is open in $\partial^1_+ C$. For every isometry γ of $\widetilde{M}$ and every $t \geq 0$, we have $\gamma \mathscr{V}_{\eta,R}(\Omega) = \mathscr{V}_{\eta,R}(\gamma\Omega)$ and

$$g^t \mathscr{V}_{\eta,R}(\Omega) = \mathscr{V}_{\eta,e^{-t}R}(g^t \Omega).$$



These thickenings $\mathscr{V}_{\eta,R}(\Omega)$ are non-decreasing in η and in R and their intersection as $\eta,\ R$ range in $]0,\, +\infty[$ is Ω. Furthermore,

$$\bigcup_{\substack{\eta>0 \\ R>0}} \mathscr{V}_{\eta,R}(\partial^1_+ C) = U_C.$$

The restriction of f_C to $\mathscr{V}_{\eta,R}(\Omega)$ is a fibration over Ω, with fibre of $w \in \Omega$ the open subset $V_{w,\eta,R}$ of the stable leaf of w.

3. *Patterson, Bowen–Margulis and skinning measures*

Let $\widetilde{M}$, Γ, x_0, M and $T^1 M$ be as at the beginning of §2. In this section, we first review some background material on the measures associated with negatively curved manifolds (for which we refer to [**Rob2**]). We then define the skinning measure associated to any non-empty closed convex subset, generalizing the construction of [**OS1, OS2**], and we prove some basic properties of these measures, as well as a crucial disintegration result. Given a topological space X, we denote by $\mathscr{C}_c(X)$ the space of real-valued continuous functions on X with compact support.

Let $r > 0$. A family $(\mu_x)_{x\in\widetilde{M}}$ of non-zero finite measures on $\partial_\infty\widetilde{M}$ whose support is the limit set $\Lambda\Gamma$ is a *Patterson density of dimension r* for the group Γ if it is Γ-equivariant, that is, if it satisfies

$$\gamma_*\mu_x = \mu_{\gamma x} \tag{9}$$

for all $\gamma \in \Gamma$ and $x \in \widetilde{M}$, and if the pairwise Radon–Nikodym derivatives of the measures μ_x for $x \in \widetilde{M}$ exist and satisfy

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-r\beta_\xi(x,y)} \tag{10}$$

for all $x,\ y \in \widetilde{M}$ and $\xi \in \partial_\infty\widetilde{M}$.

The *critical exponent* of Γ is

$$\delta_\Gamma = \lim_{n\to+\infty}\frac{1}{n}\log \mathrm{Card}\{\gamma\in\Gamma : d(x_0, \gamma x_0) \le n\}.$$

The above limit exists and is positive (see [**Rob1**]), and the critical exponent is independent of the base point x_0 used in its definition. We assume that δ_Γ is finite, which is the case, in particular, if M has a finite lower bound on its sectional curvatures (see, for instance, [**Bowd**]). We say that the group Γ is *of divergence type* if its *Poincaré series* $P_\Gamma(s) = \sum_{\gamma\in\Gamma} e^{-sd(x_0,\gamma x_0)}$ diverges at $s = \delta_\Gamma$.

Let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension δ_Γ for Γ. The *Bowen–Margulis measure* $\widetilde{m}_{\mathrm{BM}}$ for Γ on $T^1\widetilde{M}$ is defined, using the Hopf parametrization, by

$$\begin{aligned} d\widetilde{m}_{\mathrm{BM}}(v) &= \frac{d\mu_{x_0}(v_-)d\mu_{x_0}(v_+)\,dt}{d_{x_0}(v_-,\,v_+)^{2\delta_\Gamma}} \\ &= e^{-\delta_\Gamma(\beta_{v_-}(\pi(v),x_0)+\beta_{v_+}(\pi(v),x_0))}d\mu_{x_0}(v_-)d\mu_{x_0}(v_+)\,dt. \end{aligned}$$

The Bowen–Margulis measure is independent of the base point x_0, and its support is (in the Hopf parametrization) $(\Lambda\Gamma\times\Lambda\Gamma - \Delta)\times\mathbb{R}$, where Δ is the diagonal in $\Lambda\Gamma\times\Lambda\Gamma$. It is invariant under the geodesic flow and the action of Γ, and thus it defines a measure m_{BM} on T^1M, invariant under the quotient geodesic flow. When the Bowen–Margulis measure m_{BM} is finite, there exists a unique (up to a multiplicative constant) Patterson density of dimension δ_Γ, and the set of points in $T^1\widetilde{M}$ fixed by a non-trivial element of Γ has measure 0 for $\widetilde{m}_{\mathrm{BM}}$; see, for instance, [**Rob2**, p. 19]. Denoting the total mass of a measure m by $\|m\|$, the probability measure $m_{\mathrm{BM}}/\|m_{\mathrm{BM}}\|$ is then uniquely defined. We will often make the assumption that m_{BM} is finite; see the introduction for examples.

Let C be a non-empty proper closed convex subset of $\widetilde{M}$. We define the *skinning measure* $\widetilde{\sigma}_C$ of Γ on ∂^1_+C, using the homeomorphism $w\mapsto w_+$ from ∂^1_+C to $\partial_\infty\widetilde{M}-\partial_\infty C$, by

$$\begin{aligned} d\widetilde{\sigma}_C(w) &= e^{-\delta_\Gamma\beta_{w(+\infty)}(\pi(w),x_0)}\,d(\nu P_C)_*(\mu_{x_0}|_{\partial_\infty\widetilde{M}-\partial_\infty C})(w) \\ &= e^{-\delta_\Gamma\beta_{w_+}(P_C(w_+),x_0)}\,d\mu_{x_0}(w_+)\,. \end{aligned} \tag{11}$$

We will also consider $\widetilde{\sigma}_C$ as a measure on $T^1\widetilde{M}$ with support contained in ∂^1_+C.

The skinning measure was first defined by Oh and Shah [**OS2**, §1.4] for the outer unit normal bundles of spheres, horospheres and totally geodesic subspaces in real hyperbolic spaces; see also [**HP3**, Lemma 4.3] for a closely related measure. The terminology comes from McMullen's proof of the contraction of the skinning map (capturing boundary information for surface subgroups of 3-manifold groups) introduced by Thurston to prove his hyperbolization theorem.

When C is a singleton $\{x\}$, we immediately have

$$d\widetilde{\sigma}_C(w) = d\mu_x(w_+).$$

Let $w \in T^1\widetilde{M}$. When $C = HB_-(w)$ is the unstable horoball of w, the measure

$$\mu_w^{\rm su} = \widetilde{\sigma}_{HB_-(w)}$$

is the well-known conditional measure of the Bowen–Margulis measure on the strong unstable leaf $W^{\rm su}(w)$ of w (see, for instance, [**Mar3, Rob2**]). Similarly, we denote by

$$\mu_w^{\rm ss} = \iota_*(\widetilde{\sigma}_{HB_+(w)}) = \iota_*(\mu_{\iota w}^{\rm su})$$

the conditional measure of the Bowen–Margulis measure on the strong stable leaf $W^{\rm ss}(w)$ of w. These two measures are independent of the element w of a given strong unstable leaf and given strong stable leaf, respectively. For future use, using the homeomorphism $v \mapsto v_-$ from $W^{\rm ss}(w)$ to $\partial_\infty\widetilde{M} - \{w_+\}$, we have

$$d\mu_w^{\rm ss}(v) = e^{-\delta_\Gamma \beta_{v_-}(P_{HB_+(w)}(v_-),x_0)}\, d\mu_{x_0}(v_-). \tag{12}$$

We also define the conditional measure of the Bowen–Margulis measure on the stable leaf $W^{\rm s}(w)$ of w, using the homeomorphism $(v', t) \mapsto v = g^t v'$ from $W^{\rm ss}(w) \times \mathbb{R}$ to $W^{\rm s}(w)$, by

$$d\mu_w^{\rm s}(v) = e^{-\delta_\Gamma t}\, d\mu_w^{\rm ss}(v')\, dt. \tag{13}$$

See, for instance, assertion (iii) of the next proposition for an explanation of the factor $e^{-\delta_\Gamma t}$. In this paper, we will not need the similarly defined measure $d\mu_w^{\rm u}(v) = e^{\delta_\Gamma t}\, d\mu_w^{\rm su}(v')\, dt$ on the unstable leaf $W^{\rm u}(w)$ of w.

The following propositions collect some basic properties of the skinning measures.

PROPOSITION 4. *Let C be a non-empty proper closed convex subset of $\widetilde{M}$, and let $\widetilde{\sigma}_C$ be the skinning measure of Γ on $\partial_+^1 C$.*

(i) *The skinning measure $\widetilde{\sigma}_C$ is independent of the base point x_0.*

(ii) *For all $\gamma \in \Gamma$, we have $\gamma_*\widetilde{\sigma}_C = \widetilde{\sigma}_{\gamma C}$. In particular, the measure $\widetilde{\sigma}_C$ is invariant under the stabilizer of C in Γ.*

(iii) *For all $s \geq 0$ and $w \in \partial_+^1 C$,*

$$(g^s)_*\,\widetilde{\sigma}_C = e^{-\delta_\Gamma s}\,\widetilde{\sigma}_{\mathscr{N}_s C}.$$

(iv) *The support of $\widetilde{\sigma}_C$ is*

$$\{w \in \partial_+^1 C : w_+ \in \Lambda\Gamma\} = \nu P_C(\Lambda\Gamma - \Lambda\Gamma \cap \partial_\infty C).$$

In particular, $\widetilde{\sigma}_C$ is the zero measure if and only if $\Lambda\Gamma$ is contained in $\partial_\infty C$.

It follows from (ii) that, for all $\gamma \in \Gamma$,

$$\gamma_*\mu^{\rm su}_w = \mu^{\rm su}_{\gamma w}, \quad \gamma_*\mu^{\rm ss}_w = \mu^{\rm ss}_{\gamma w}, \quad \gamma_*\mu^{\rm s}_w = \mu^{\rm s}_{\gamma w}. \tag{14}$$

It follows from (iii) and from the equality $\iota \circ g^t = g^{-t} \circ \iota$ that, for all $t \in \mathbb{R}$,

$$(g^t)_*\mu^{\rm su}_w = e^{-\delta_\Gamma t}\mu^{\rm su}_{g^t w}, \quad (g^{-t})_*\mu^{\rm ss}_w = e^{-\delta_\Gamma t}\mu^{\rm ss}_{g^{-t} w}, \quad (g^t)_*\mu^{\rm s}_w = e^{\delta_\Gamma t}\mu^{\rm s}_{w}. \tag{15}$$

Proof. Assertion (i) follows from equation (10) with $r = \delta_\Gamma$ and the second part of equation (1). Assertion (ii) follows from equation (9), the first part of equation (1), and assertion (i).

To prove assertion (iii), we note that, since $(g^s w)_+ = w_+$ and by the cocycle property (1),

$$\begin{aligned} d\widetilde{\sigma}_{\mathscr{N}_s C}(g^s w) &= e^{-\delta_\Gamma \beta_{w_+}(\pi(g^s w), x_0)}\, d\mu_{x_0}(w_+) = e^{-\delta_\Gamma \beta_{w_+}(\pi(g^s w), \pi(w))}\, d\widetilde{\sigma}_C(w) \\ &= e^{\delta_\Gamma s}\, d\widetilde{\sigma}_C(w). \end{aligned}$$

Assertion (iv) follows from the fact that the support of any Patterson measure is the limit set of Γ. □

Given two non-empty closed convex subsets C and C' of $\widetilde{M}$, let $\Omega_{C,C'} = \partial_\infty \widetilde{M} - (\partial_\infty C \cup \partial_\infty C')$ and let

$$h_{C,C'} : \nu P_C(\Omega_{C,C'}) \to \nu P_{C'}(\Omega_{C,C'})$$

be the restriction of $\nu P_{C'} \circ \nu P_C^{-1}$ to $\nu P_C(\Omega_{C,C'})$. It is a homeomorphism between open subsets of $\partial^1_+ C$ and $\partial^1_+ C'$, associating to the element w in the domain the unique element w' in the range with $w'_+ = w_+$.

PROPOSITION 5. *Let C and C' be non-empty proper closed convex subsets of $\widetilde{M}$ and let $h = h_{C,C'}$. The measures $h_*\widetilde{\sigma}_C$ and $\widetilde{\sigma}_{C'}$ on $\nu P_{C'}(\Omega_{C,C'})$ are absolutely continuous with respect to one another, with*

$$\frac{dh_*\widetilde{\sigma}_C}{d\widetilde{\sigma}_{C'}}(w') = e^{-\delta_\Gamma \beta_{w_+}(\pi(w), \pi(w'))},$$

for all $w \in \nu P_C(\Omega_{C,C'})$ and $w' = h(w)$.

Proof. Since $w'_+ = w_+$ and by the cocycle property (1), we have

$$d\widetilde{\sigma}_{C'}(w') = e^{-\delta_\Gamma \beta_{w'_+}(P_{C'}(w'_+), x_0)}\, d\mu_{x_0}(w'_+) = e^{-\delta_\Gamma \beta_{w_+}(P_{C'}(w'_+), P_C(w_+))}\, d\widetilde{\sigma}_C(w).$$

Since $w = \nu P_C(w_+)$ and $\pi \circ \nu P_C = P_C$, and similarly for w', the result follows from the anti-symmetry of the Busemann cocycle. □

We endow the set $\mathrm{Convex}(\widetilde{M})$ of non-empty closed proper convex subsets of $\widetilde{M}$ with the (metrizable, locally compact) topology of the Hausdorff convergence on compact subsets: a sequence $(C_i)_{i\in\mathbb{N}}$ of closed subsets of $\widetilde{M}$ converges to a closed subset C of $\widetilde{M}$ if and only if, for every compact subset K in $\widetilde{M}$, the Hausdorff distance between $(C_i \cap K) \cup {}^cK$ and $(C \cap K) \cup {}^cK$ tends to 0. Note that being convex is indeed a closed condition. We endow the set $\mathrm{Measure}(T^1\widetilde{M})$ of non-negative regular Borel measures on $T^1\widetilde{M}$ with the (metrizable, locally compact) topology of the weak-star convergence: a sequence $(\mu_i)_{i\in\mathbb{N}}$ of such measures on $\widetilde{M}$ converges to such a measure μ on $\widetilde{M}$ if and only if, for every compactly supported continuous function f on $\widetilde{M}$, the sequence $(\mu_i(f))_{i\in\mathbb{N}}$ converges to $\mu(f)$.

PROPOSITION 6. *The map from* $\mathrm{Convex}(\widetilde{M})$ *to* $\mathrm{Measure}(T^1\widetilde{M})$ *which associates to* C *its skinning measure* $\widetilde{\sigma}_C$ *is continuous.*

In particular, as the horoballs $HB_+(w)$ and $HB_-(w)$ depend continuously on $w \in T^1\widetilde{M}$, the measures μ_w^{su}, μ_w^{ss} and μ_w^{s} depend continuously on w.

Proof. Let $(C_i)_{i\in\mathbb{N}}$ be a sequence in $\mathrm{Convex}(\widetilde{M})$ which converges to $C \in \mathrm{Convex}(\widetilde{M})$ for the Hausdorff convergence on compact subsets of $\widetilde{M}$, and let us prove that $\widetilde{\sigma}_{C_i} \overset{*}{\rightharpoonup} \widetilde{\sigma}_C$.

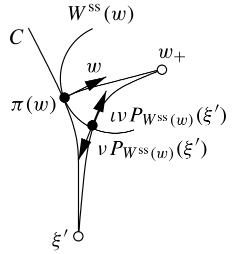
The sequence $(\partial_+^1 C_i)_{i\in\mathbb{N}}$ of closed subsets of $T^1\widetilde{M}$ converges to $\partial_+^1 C$ for the Hausdorff convergence on compact subsets of $T^1\widetilde{M}$. The sequence $(\partial_\infty\widetilde{M} - \partial_\infty C_i)_{i\in\mathbb{N}}$ of open subsets of $\partial_\infty\widetilde{M}$ converges to $\partial_\infty\widetilde{M} - \partial_\infty C$ for the Carathéodory convergence (that is, for the Hausdorff convergence of their complements). Hence, the sequences of maps $(P_{C_i})_{i\in\mathbb{N}}$ and $(\nu P_{C_i})_{i\in\mathbb{N}}$ converge to P_C and νP_C respectively for the uniform convergence of maps on compact subsets $\partial_\infty\widetilde{M} - \partial_\infty C$. Given two compact metric spaces X and Y and a finite Borel measure μ on X, the pushforward map $f \mapsto f_*\mu$ from the space of continuous maps from X to Y with the uniform topology to the space of finite Borel measures on Y with the weak-star topology is continuous. The claim follows from these observations, since the skinning measure on C is a multiple by a map depending continuously on C of the pushforward by a map depending continuously on C of the fixed measure μ_{x_0}. □

The following result will be useful in §5. Recall (see equation (8)) that $V_{w,R}$ is the open ball of radius R and centre w in the strong stable leaf $W^{\mathrm{ss}}(w)$ of $w \in T^1\widetilde{M}$ for the Hamenstädt distance.

LEMMA 7. *For every non-empty proper closed convex subset* C *in* $\widetilde{M}$, *there exists* $R_0 > 0$ *such that, for every* $R \geq R_0$ *and every* $w \in \partial_+^1 C$, *we have* $\mu_w^{\mathrm{ss}}(V_{w,R}) > 0$. *If* $\partial_\infty C \cap \Lambda\Gamma \neq \emptyset$, *we may take* $R_0 = 2$.

Proof. For all $w \in \partial_+^1 C$ and $x \in C \cup \partial_\infty C$, by a standard comparison and convexity argument in the CAT(-1)-space $\widetilde{M}$ applied to the geodesic triangle with vertices $\pi(w)$, w_+, x, the point $\pi(w)$ is at distance at most $2\log((1+\sqrt{5})/2)$ from the intersection between the stable horosphere $H_+(w)$ and the geodesic ray or line between x and w_+. Hence, by equation (3), for every $\xi' \in \partial_\infty C$,

$$d_{W^{\mathrm{ss}}(w)}(w, \iota\nu P_{W^{\mathrm{ss}}(w)}(\xi')) \leq \frac{1+\sqrt{5}}{2}.$$

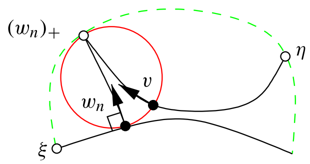


Thus, if $\partial_\infty C \cap \Lambda\Gamma \neq \emptyset$, then we may take $R_0 = 2 > (1+\sqrt{5})/2$, since by Proposition 4(iv), the support of $\mu^{\rm ss}_w$ is $\iota\, \nu P_{HB_+(w)}(\Lambda\Gamma - \Lambda\Gamma \cap \{w^+\})$.

Consider now the case $\partial_\infty C \cap \Lambda\Gamma = \emptyset$. For a contradiction, assume that, for all $n \in \mathbb{N}$, there exists $w_n \in \partial^1_+ C$ such that $\mu^{\rm ss}_{w_n}(V_{w_n,n}) = 0$. Assume first that $(w_n)_{n\in\mathbb{N}}$ has a convergent subsequence with limit $w \in \partial^1_+ C$. Since the measure $\mu^{\rm ss}_v$ depends continuously on v, for every compact subset K of $W^{\rm ss}(w)$, we have $\mu^{\rm ss}_w(K) = 0$. By Proposition 4(iv) and by equation (12), the support of the Patterson measure μ_{x_0}, which is the limit set of Γ, is contained in $\{w_+\}$. This is impossible, since Γ is non-elementary.

In the remaining case, the points $\pi(w_n)$ in C converge, up to extracting a subsequence, to a point ξ in $\partial_\infty C$. By definition of the map νP_C and of $\partial^1_+ C$, the points at infinity $(w_n)_+$ converge to ξ. For every η in $\partial_\infty \widetilde{M}$ different from ξ, the geodesic lines from η to $(w_n)_+$ converge to the geodesic line from η to ξ.

By convexity, if n is big enough, the geodesic line $]\eta, (w_n)_+[$ meets $\mathscr{N}_1 C$, and hence passes at distance at most 2 from $\pi(w_n)$. This implies (using equation (3) as above) that if n is big enough, then there exists $v \in V_{w_n,n}$ such that $\eta = v_-$.



Since we assumed that $\mu^{\rm ss}_{w_n}(V_{w_n,n}) = 0$ for all $n \in \mathbb{N}$, Proposition 4(iv) implies that we have $\eta \notin \Lambda\Gamma$. Hence $\Lambda\Gamma$ is contained in $\{\xi\}$, a contradiction since Γ is non-elementary. □

Let C be a non-empty closed convex subset of $\widetilde{M}$, and let U_C be the open subset of $T^1\widetilde{M}$ defined in equation (7). Note that U_C has full Bowen–Margulis measure in $T^1\widetilde{M}$ if the Patterson measure $\mu_x(\partial_\infty C)$ of $\partial_\infty C$ is equal to 0 (this being independent of $x \in \widetilde{M}$), by the quasi-product structure of $\widetilde{m}_{\rm BM}$.

The following disintegration result of the Bowen–Margulis measure over the skinning measure of C is the crucial tool for the equidistribution result in §5. When $\widetilde{M}$ has constant curvature and Γ is torsion-free and for special C, this result is implicit in [**OS2**].

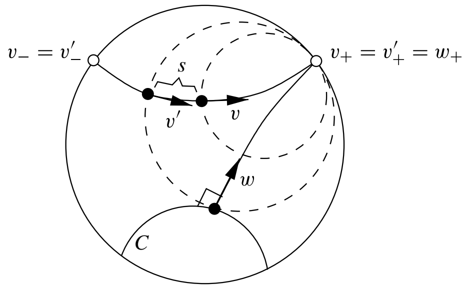
PROPOSITION 8. *Let C be a non-empty proper closed convex subset of $\widetilde{M}$. The restriction to U_C of the Bowen–Margulis measure $\widetilde{m}_{\rm BM}$ disintegrates by the fibration $f_C : U_C \to \partial^1_+ C$, over the skinning measure $\widetilde{\sigma}_C$ of C, with conditional measure $e^{\delta_\Gamma \beta_{w_+}(\pi(w),\pi(v))}\, d\mu^{\rm s}_w(v)$ on the fibre $f_C^{-1}(w) = W^s(w)$ of $w \in \partial^1_+ C$:*

$$d\widetilde{m}_{\rm BM}(v) = \int_{w\in\partial^1_+ C} e^{\delta_\Gamma \beta_{w_+}(\pi(w),\pi(v))}\, d\mu^{\rm s}_w(v)\, d\widetilde{\sigma}_C(w).$$

Proof. For every $\varphi \in \mathscr{C}_c(U_C)$, let $I_\varphi = \int_{v\in U_C} \varphi(v)\, d\widetilde{m}_{\rm BM}(v)$. By the definition of U_C and of the Bowen–Margulis measure in the Hopf parametrization, we have

$$I_\varphi = \int_{v_+\in\, \partial_\infty\widetilde{M} - \partial_\infty C} \int_{v_-\in\, \partial_\infty\widetilde{M} - \{v_+\}} \int_{t\in\mathbb{R}} \varphi(v)\, \frac{dt\, d\mu_{x_0}(v_-)\, d\mu_{x_0}(v_+)}{d_{x_0}(v_-, v_+)^{2\delta_\Gamma}}.$$

For every $v \in U_C$, let $w = f_C(v) = \nu P_C(v_+)$ and let $s \in \mathbb{R}$ be such that $v' = g^{-s} v$ belongs to the strong stable leaf $W^{\mathrm{ss}}(w)$ of w. Note that, with t the time parameter of v in the Hopf parametrization, the number $t - s$ depends only on v_+ and $v_- = v'_-$.



Since the map from $\partial^1_+ C$ to $\partial_\infty \widetilde{M} - \partial_\infty C$ defined by $w \mapsto w_+$ and the map from $W^{\mathrm{ss}}(w)$ to $\partial_\infty \widetilde{M} - \{w_+\}$ defined by $v' \mapsto v'_-$ are homeomorphisms, we have

$$I_\varphi = \int_{w \in \partial^1_+ C} \int_{v' \in W^{\mathrm{ss}}(w)} \int_{s \in \mathbb{R}} \varphi(g^s v') \, \frac{ds \, d\mu_{x_0}(v'_-) \, d\mu_{x_0}(w_+)}{d_{x_0}(v'_-, w_+)^{2\delta_\Gamma}}.$$

For every $w \in \partial^1_+ C$ and $v' \in W^{\mathrm{ss}}(w)$, we claim (explanations follow) that

$$\begin{aligned}
\frac{d\mu_{x_0}(v'_-) \, d\mu_{x_0}(w_+)}{d_{x_0}(v'_-, w_+)^{2\delta_\Gamma}} &= \frac{e^{\delta_\Gamma \beta_{v'_-}(\pi(v'), x_0))} e^{\delta_\Gamma \beta_{w_+}(\pi(w), x_0)}}{e^{-\delta_\Gamma(\beta_{v'_-}(x_0, \pi(v')) + \beta_{w_+}(x_0, \pi(v')))}} \, d\mu^{\mathrm{ss}}_w(v') \, d\widetilde{\sigma}_C(w) \\
&= e^{\delta_\Gamma \beta_{w_+}(\pi(w), \pi(v'))} \, d\mu^{\mathrm{ss}}_w(v') \, d\widetilde{\sigma}_C(w) \\
&= d\mu^{\mathrm{ss}}_w(v') \, d\widetilde{\sigma}_C(w).
\end{aligned}$$

The first equality holds by the definition of the measures μ^{ss}_w (see equation (12)) and $\widetilde{\sigma}_C$ (see equation (11)), by the definition of the visual distance d_{x_0} (see equation (2)), and since $\pi(v')$ belongs to the geodesic line between v'_- and $w_+ = v'_+$. The second equality follows from the cocycle property (1). The third one holds since $\pi(w)$ and $\pi(v')$ both belong to the stable horosphere of w.

Hence, since $\beta_{w_+}(\pi(w), \ \pi(v)) = s$ if $v = g^s v'$ and $v' \in W^{\mathrm{ss}}(w)$, and by the definition of the measure μ^{s}_w (see equation (13)), we have

$$\begin{aligned}
I_\varphi &= \int_{w \in \partial^1_+ C} \int_{v' \in W^{\mathrm{ss}}(w)} \int_{s \in \mathbb{R}} \varphi(g^s v') \, ds \, d\mu^{\mathrm{ss}}_w(v') \, d\widetilde{\sigma}_C(w) \\
&= \int_{w \in \partial^1_+ C} \int_{v \in W^{\mathrm{s}}(w)} \varphi(v) e^{\delta_\Gamma \beta_{w_+}(\pi(w), \pi(v))} \, d\mu^{\mathrm{s}}_w(v) \, d\widetilde{\sigma}_C(w), \qquad (16)
\end{aligned}$$

which proves the result. □

We conclude this section by defining the skinning measures of equivariant families of convex subsets.

Let I be an index set endowed with a left action $(\gamma, i) \mapsto \gamma i$ of Γ. A family $\mathscr{D} = (D_i)_{i \in I}$ of subsets of $\widetilde{M}$ or $T^1 \widetilde{M}$ indexed by I is Γ-*equivariant* if $\gamma D_i = D_{\gamma i}$ for all $\gamma \in \Gamma$ and all $i \in I$. We equip the index set I with the Γ-equivariant equivalence relation $\sim$ (or $\sim_{\mathscr{D}}$ when we want to stress the dependence on $\mathscr{D}$), defined by setting $i \sim j$ if and only if there exists $\gamma \in \mathrm{Stab}_\Gamma D_i$ such that $j = \gamma i$ (or equivalently if $D_j = D_i$ and $j = \gamma i$ for some $\gamma \in \Gamma$). Note that Γ acts on the left on the set of equivalence classes $I/\sim$.

An example of such a family is given by fixing a subset C of $\widetilde{M}$ or $T^1\widetilde{M}$, by setting $I = \Gamma$ with the left action by translations on the left $(\gamma, i) \mapsto \gamma i$, and by setting $D_i = iC$ for every $i \in \Gamma$. In this case, we have $i \sim j$ if and only if $i^{-1}j$ belongs to the stabilizer Γ_C of C in Γ, and $I/\sim\, = \Gamma/\Gamma_C$. More general examples include Γ-orbits of (usually finite) collections of subsets of $\widetilde{M}$ or $T^1\widetilde{M}$ with (usually finite) multiplicities.

A Γ-equivariant family $(D_i)_{i\in I}$ of closed subsets of $\widetilde{M}$ or $T^1\widetilde{M}$ is said to be *locally finite* if for every compact subset K in $\widetilde{M}$ or $T^1\widetilde{M}$, the quotient set $\{i \in I : D_i \cap K \neq \emptyset\}/\sim$ is finite. In particular, the union of the images of the sets D_i by the map $\widetilde{M} \to M$ or $T^1\widetilde{M} \to T^1M$ is closed. When $\Gamma\backslash I$ is finite, $(D_i)_{i\in I}$ is locally finite if and only if, for all $i \in I$, the canonical map from $\Gamma_{D_i}\backslash D_i$ to M or T^1M is proper, where Γ_{D_i} is the stabilizer of D_i in Γ.

Let $\mathscr{D} = (D_i)_{i\in I}$ be a locally finite Γ-equivariant family of non-empty proper closed convex subsets of $\widetilde{M}$. Then

$$\widetilde{\sigma}_{\mathscr{D}} = \sum_{i\in I/\sim} \widetilde{\sigma}_{D_i}$$

is a locally finite positive Borel measure on $T^1\widetilde{M}$ (independent on the choice of representatives in $I/\sim$), called the *skinning measure* of $\mathscr{D}$ on $T^1\widetilde{M}$. It is Γ-invariant by Proposition 4(ii), and its support is contained in $\bigcup_{i\in I/\sim} \partial^1_+ D_i$. Hence $\widetilde{\sigma}_{\mathscr{D}}$ induces a locally finite Borel positive measure $\sigma_{\mathscr{D}}$ on $T^1M = \Gamma\backslash T^1\widetilde{M}$, called the *skinning measure* of $\mathscr{D}$ on T^1M.

For every $t \in [0, +\infty[$, let $\mathscr{D}_t = (\mathscr{N}_t D_i)_{i\in I}$, which is also a Γ-equivariant locally finite family of non-empty proper closed convex subsets of $\widetilde{M}$. Note that by Proposition 4(iii),

$$(g^t)_*\sigma_{\mathscr{D}} = e^{-\delta_\Gamma t}\, \sigma_{\mathscr{D}_t}$$

and, in particular,

$$\|\sigma_{\mathscr{D}_t}\| = e^{\delta_\Gamma t}\|\sigma_{\mathscr{D}}\|.$$

Note that the measure $\sigma_{\mathscr{D}_t}$ is finite if and only if the measure $\sigma_{\mathscr{D}}$ is finite.

If the image in M of the support of $\sigma_{\mathscr{D}}$ is compact, then $\sigma_{\mathscr{D}}$ is finite. In particular, if Γ is geometrically finite, the skinning measure of a Margulis neighbourhood of a cusp in $\Gamma\backslash\widetilde{M}$ is finite, since for any parabolic fixed point p of Γ, the quotient of $\Lambda\Gamma - \{p\}$ by the stabilizer of p in Γ is compact.

Oh and Shah [**OS2**, Theorem 1.5] proved, in particular, that $\|\sigma_{\mathscr{D}}\|$ is finite if $\Gamma\backslash I$ is finite, Γ is torsion-free, M is geometrically finite with constant curvature -1, $\widetilde{\mathscr{D}}$ consists of codimension-one totally geodesic submanifolds, and $\delta_\Gamma > 1$. See [**OS2**, Theorem 6.4] for a statement without the codimension-one assumption, which we generalize in the following section.

The next result relates the finiteness of the skinning measure of $\mathscr{D}$ to that of a nested family $\mathscr{D}'$.

Remark 9. Let $\mathscr{D} = (D_i)_{i\in I}$ and $\mathscr{D}' = (D'_i)_{i\in I}$ be locally finite Γ-equivariant families of non-empty proper closed convex subsets of $\widetilde{M}$, with $D'_i \subset D_i$ for every $i \in I$. Assume that:

- $P_{D'_i}(\xi)$ is the closest point in D'_i to $P_{D_i}(\xi)$, for every $\xi \in \Lambda\Gamma - \partial_\infty D_i$;
- there exists $c > 0$ such that $d(P_{D_i}(\xi), D'_i) \leq c$, for every $i \in I$ and $\xi \in \Lambda\Gamma - \partial_\infty D_i$;
- for every $i \in I$, we have $\mu_{x_0}(\partial_\infty D_i - \partial_\infty D'_i) = 0$.

Then $\sigma_{\mathscr{D}}$ is finite if and only if $\sigma_{\mathscr{D}'}$ is finite.

It follows from this remark that for every $\epsilon \geq 0$, if $\mathscr{D}'' = (\mathscr{N}_\epsilon D_i)_{i \in I}$, then $\sigma_{\mathscr{D}''}$ is finite if and only if $\sigma_{\mathscr{D}}$ is finite.

The first assumption is also satisfied if $\widetilde{M}$ has constant curvature -1 and D_i is totally geodesic for all $i \in I$, since by homogeneity, for every ξ in $\widetilde{M}$ and $x \neq y$ in $\widetilde{M}$ such that $\angle_x(\xi, y) = \pi/2$, the value $\beta_\xi(y, x)$ is a strictly increasing function of only $d(x, y)$.

Proof. By the first assumption, the map $\theta : \nu P_{D_i}(\Lambda\Gamma - \partial_\infty D_i) \to \nu P_{D'_i}(\Lambda\Gamma - \partial_\infty D'_i)$ defined by $w \mapsto w'$, where $w'_+ = w_+$ and $\pi(w')$ is the closest point on D'_i to $\pi(w)$, is a homeomorphism onto its image such that

$$\nu P_{D'_i \,|\Lambda\Gamma - \partial_\infty D_i} = \theta \circ \nu P_{D_i \,|\Lambda\Gamma - \partial_\infty D_i}.$$

By the definition of the skinning measures, using this homeomorphism θ, we have, for all $w' \in \theta(\nu P_{D_i}(\Lambda\Gamma - \partial_\infty D_i))$,

$$d\widetilde{\sigma}_{D'_i}(w') = e^{-\delta_\Gamma \beta_{w'_+}(P_{D'_i}(w'_+), P_{D_i}(w'_+))}\, d\theta_* \widetilde{\sigma}_{D_i}(w').$$

The result then follows by the second and third assumptions. □

4. *Finiteness and fluctuation of the skinning measure*

We will say that a discrete group Γ' of isometries of $\widetilde{M}$ has *regular growth* if there exists $c > 0$ such that, for every $N \in \mathbb{N}$,

$$\frac{1}{c} e^{\delta_{\Gamma'} N} \leq \operatorname{Card}\{\gamma \in \Gamma' : d(x_0, \gamma x_0) \leq N\} \leq c e^{\delta_{\Gamma'} N}.$$

This does not depend on x_0, and the upper bound holds for all non-elementary groups Γ' (see, for instance, [**Rob2**, p. 11]). If the Bowen–Margulis measure m_{BM} on T^1M is finite, then Γ has regular growth (in fact, $\operatorname{Card}\{\gamma \in \Gamma : d(x_0, \gamma x_0) \leq N\} \sim c e^{\delta_\Gamma N}$ for an explicit $c > 0$; see, for instance, [**Rob2**]). If $\widetilde{M}$ is a symmetric space, then any discrete parabolic group of isometries of $\widetilde{M}$ has regular growth. In particular, if $\widetilde{M}$ is the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, then by a theorem of Bieberbach, any discrete parabolic group Γ' contains a finite index subgroup isomorphic to $\mathbb{Z}^k$ for some $k \in \{0, \ldots, n-1\}$ called the *rank* of the fixed point of Γ', and an easy and well-known computation in hyperbolic geometry proves that the critical exponent of Γ' is

$$\delta_{\Gamma'} = \frac{k}{2}, \tag{17}$$

and that Γ' has regular growth. Note that there exist complete simply connected Riemannian manifolds with pinched negative curvature having discrete parabolic groups of isometries which do not have regular growth; see, for instance, [**DOP**].

We will say that a convex subset C of $\widetilde{M}$ is *almost cone-like in cusps* for a discrete group Γ' of isometries of $\widetilde{M}$ if for any parabolic point p' of Γ' belonging to $\partial_\infty C$ and any horoball $\mathscr{H}'$ centred at p', there exist $r \geq 0$ and $x'_0 \in \partial\mathscr{H}'$ such that $C \cap \mathscr{H}' \cap$

$\mathscr{N}_{2\log(1+\sqrt{2})}(\mathscr{C}\Lambda\Gamma')$ is contained in the orbit of $\mathscr{N}_r([x_0', p'[)$ under the stabilizer in Γ' of p and C. It follows from the arguments of [**OS2**, §4] that if $\widetilde{M}$ has constant sectional curvature -1, if C is a totally geodesic submanifold and if Γ' is torsion-free and geometrically finite, then C is almost cone-like in cusps for Γ'.

THEOREM 10. *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with sectional curvature bounded above by -1. Let Γ be a geometrically finite discrete group of isometries of $\widetilde{M}$, of divergence type, with finite critical exponent. Let $\mathscr{D}=(D_i)_{i\in I}$ be a locally finite Γ-equivariant family of non-empty proper convex subsets of $\widetilde{M}$ which are almost cone-like in cusps for Γ, with $\Gamma\backslash I$ finite. Assume that for every parabolic point p of Γ and every $i\in I$ such that $p\in\partial_\infty D_i$, if Γ_p and Γ_{D_i} are the stabilizers in Γ of p and D_i, respectively, then Γ_p and $\Gamma_{D_i}\cap\Gamma_p$ have regular growth and satisfy*

$$\delta_\Gamma > 2(\delta_{\Gamma_p} - \delta_{\Gamma_{D_i}\cap\Gamma_p}). \tag{18}$$

Then the skinning measure $\sigma_{\mathscr{D}}$ of $\mathscr{D}$ on T^1M is finite.

We make some comments on this statement before giving its proof.

Remarks. (1) When $\widetilde{M}$ is a symmetric space (in particular, when $\widetilde{M}$ has constant sectional curvature -1), every geometrically finite group of isometries of $\widetilde{M}$ is of divergence type. This is not true in general, but holds true if $\delta_\Gamma > \delta_{\Gamma_p}$ for every parabolic point p of Γ; see [**DOP**]. As already said, δ_Γ is finite, for instance, if M has a finite lower bound on its sectional curvatures.

(2) Assume in this remark that the index of $\Gamma_{D_i}\cap\Gamma_p$ in Γ_p is finite for every parabolic point p of Γ and every $i\in I$ such that $p\in\partial_\infty D_i$. Then $\delta_{\Gamma_p}=\delta_{\Gamma_{D_i}\cap\Gamma_p}$, and this equality implies that condition (18) is satisfied. When $\widetilde{M}$ has constant sectional curvature -1, the subsets D_i are totally geodesic submanifolds, and Γ is torsion-free, the finiteness of $\sigma_{\mathscr{D}}$ follows from [**OS2**, Theorem 6.3].

(3) Assume in this remark that $\widetilde{M}$ has constant sectional curvature -1 and that the subsets D_i are totally geodesic submanifolds. Let us prove that for every parabolic point p of Γ belonging to $\partial_\infty D_i$, we have $\delta_{\Gamma_p}-\delta_{\Gamma_{D_i}\cap\Gamma_p}\le\frac{1}{2}\operatorname{codim}(D_i)$ (see also [**OS2**, Lemma 6.2] when Γ is torsion-free). This will imply that condition (18) is satisfied if $\delta_\Gamma>1$ and if the elements of $\mathscr{D}$ have codimension one.

Let k be the rank of Γ_p. In particular, $\delta_{\Gamma_p}=k/2$ by equation (17). Up to taking a finite index subgroup, and choosing appropriate coordinates, we may assume that p is the point at infinity in the upper halfspace model of $\widetilde{M}=\mathbb{H}^n_{\mathbb{R}}$, that Γ_p is the lattice $\mathbb{Z}^k$ of $\mathbb{R}^k$ acting by translations on the first factor (and trivially on the second one) on $\mathbb{R}^k\times\mathbb{R}^{n-k-1}=\mathbb{R}^{n-1}=\partial_\infty\mathbb{H}^n_{\mathbb{R}}-\{p\}$, and that $E=\partial_\infty D_i-\{p\}$ is a linear subspace of $\mathbb{R}^{n-1}$. Let $F=E\cap\mathbb{R}^k$, which is a linear subspace of $\mathbb{R}^k$. Since the family $\mathscr{D}$ is locally finite, the image of F in the torus $\mathbb{R}^k/\mathbb{Z}^k$ is closed, hence it is a subtorus. Since $0\in F$, the subgroup $\mathbb{Z}^k\cap F$ is thus a lattice in F. Therefore, by equation (17),

$$2(\delta_{\Gamma_p}-\delta_{\Gamma_{D_i}\cap\Gamma_p})=\operatorname{codim}_{\mathbb{R}^k}(F)\le\operatorname{codim}_{\mathbb{R}^{n-1}}(E)=\operatorname{codim}(D_i).$$

(4) Theorem 10 is optimal, since when $\widetilde{M}$ has constant sectional curvature -1, the subsets D_i are totally geodesic submanifolds and Γ is torsion-free, it is proved

in [**OS2**, Theorem 6.4] that the validity of equation (18) (translated using equation (17)), for all i, p as in the statement, is a necessary and sufficient condition for the skinning measure $\sigma_{\mathscr{D}}$ to be finite.

(5) The ideas of the proof of Theorem 10 are a blend of those of the finiteness of the Bowen–Margulis measure under a separation condition on the critical exponents in [**DOP**] (see also [**PPS**] for the case of Gibbs measures), and those of a generalization to variable curvature of Sullivan's fluctuating density property in [**HP2**, §4].

Proof. We may assume that $\Gamma\backslash I$ is a singleton. Let us fix $i \in I$. We may assume that $\partial_\infty D_i \cap \Lambda\Gamma$ is non-empty. Otherwise, indeed, since νP_{C_i} is a homeomorphism and $\Lambda\Gamma$ is closed, the support of $\widetilde{\sigma}_{D_i}$, which is the set of elements $v \in \partial^1_+ D_i$ such that $v_+ \in \Lambda\Gamma$ (see Proposition 4(iv)), is compact. Hence the support of $\sigma_{\mathscr{D}}$ is compact, therefore $\sigma_{\mathscr{D}}$ is finite. Let $\pi : T^1\widetilde{M} \to \widetilde{M}$ and again $\pi : T^1 M \to M$ be the base point projections. Note that the skinning measure $\sigma_{\mathscr{D}}$ is finite if and only if its pushforward measure $\pi_*\sigma_{\mathscr{D}}$ is finite.

In what follows, let $\varepsilon = \ln(1+\sqrt{2})$: note that for any geodesic triangle in $\mathbb{H}^2_{\mathbb{R}}$ with two ideal vertices and a right angle at the vertex $x \in \mathbb{H}^2_{\mathbb{R}}$, the distance from x to its opposite side is exactly ε.

LEMMA 11. *The support of the measure $\pi_*\widetilde{\sigma}_{D_i}$, which is $\{P_{D_i}(\xi) : \xi \in \Lambda\Gamma - \partial_\infty D_i\}$, is contained in the closed ε-neighbourhood of the convex hull $\mathscr{C}\Lambda\Gamma$.*

Proof. Let $\xi \in \Lambda\Gamma - \partial_\infty D_i$, let $\xi' \in \partial_\infty D_i \cap \Lambda\Gamma$, and let x be the closest point to ξ on D_i. Then the geodesic ray from x to ξ', which is contained in D_i by convexity, makes an angle at least $\pi/2$ at x with the geodesic ray from x to ξ. By a standard comparison result and the definition of ε, the point x is hence at distance at most ε from the geodesic line between ξ and ξ', which is contained in $\mathscr{C}\Lambda\Gamma$. □

Let Par_Γ be the set of parabolic fixed points of Γ. Since Γ is geometrically finite (see, for instance, [**Bowd**]):

- every $p \in \mathrm{Par}_\Gamma$ is *bounded*, that is, its stabilizer Γ_p in Γ acts properly with compact quotient on $\Lambda\Gamma - \{p\}$;
- the action of Γ on Par_Γ has only finitely many orbits;
- there exists a Γ-invariant family $(\mathscr{H}_p)_{p\in\mathrm{Par}_\Gamma}$ of pairwise disjoint closed horoballs, with $\mathscr{H}_p$ centred at p, such that the quotient

$$M_0 = \Gamma\Big\backslash\Big(\mathscr{C}\Lambda\Gamma - \bigcup_{p\in\mathrm{Par}_\Gamma} \mathscr{H}_p\Big)$$

  is compact. The inclusion $\mathscr{H}_p \subset \widetilde{M}$ induces an injection $\Gamma_p\backslash\mathscr{H}_p \to \Gamma\backslash\widetilde{M}$ and we will identify $\Gamma_p\backslash\mathscr{H}_p$ with its image in $\Gamma\backslash\widetilde{M}$. In particular, $\mathscr{H}_p$ is *precisely invariant* under Γ, that is, for all $\gamma \in \Gamma - \Gamma_p$, we have $\gamma\mathscr{H}_p \cap \mathscr{H}_p = \emptyset$.

By Lemma 11 (and since the ε-neighbourhood of M_0 is also compact), we thus only have to prove the finiteness of $\pi_*\sigma_{\mathscr{D}}(\Gamma_p\backslash\mathscr{H}_p)$ for all $p \in \mathrm{Par}_\Gamma$. By the local finiteness of $\mathscr{D}$ and the fact that parabolic fixed points are bounded, for all $p \in \mathrm{Par}_\Gamma$, if the orbit Γp does not meet $\partial_\infty D_i$, then $\pi_*\sigma_{\mathscr{D}}(\Gamma_p\backslash\mathscr{H}_p)$ is finite.

We assume, therefore, that there exist $p \in \mathrm{Par}_\Gamma \cap \partial_\infty D_i$, and we want to prove the finiteness of $\pi_*\sigma_{\mathscr{D}}(\Gamma_p\backslash\mathscr{H}_p)$. To simplify the notation, let $\Gamma_{p,i} = \Gamma_{D_i} \cap \Gamma_p$, $\delta_{p,i} = \delta_{\Gamma_{p,i}}$, $\delta_p = \delta_{\Gamma_p}$ and $\delta = \delta_\Gamma$. Let x_0 be a point in $D_i \cap \partial\mathscr{H}_p$ (which exists up to shrinking $\mathscr{H}_p$). Since p is the endpoint of a geodesic ray contained in D_i and of a geodesic ray contained in $\mathscr{C}\Lambda\Gamma$, and since geodesic rays with the same point at infinity become arbitrarily close, up to shrinking $\mathscr{H}_p$, we may assume that $x_0 \in \mathscr{N}_\varepsilon(\mathscr{C}\Lambda\Gamma)$.

Choose a set of representatives $\Gamma_p\backslash\backslash\Gamma$ of the right cosets in $\Gamma_p\backslash\Gamma$ such that, for all $\gamma' \in \Gamma_p\backslash\backslash\Gamma$,

$$d(x_0, \gamma' x_0) = \min_{\alpha\in\Gamma_p} d(x_0, \alpha\gamma' x_0).$$

Choose a set of representatives $\Gamma_{p,i}\backslash\backslash\Gamma_p$ of the right cosets in $\Gamma_{p,i}\backslash\Gamma_p$ such that, for all $\overline{\alpha} \in \Gamma_{p,i}\backslash\backslash\Gamma_p$,

$$d(x_0, \overline{\alpha} x_0) = \min_{\beta\in\Gamma_{p,i}} d(x_0, \beta\overline{\alpha} x_0).$$

Note that any $\gamma \in \Gamma$ may be uniquely written $\gamma = \beta\overline{\alpha}\gamma'$ with $\beta \in \Gamma_{p,i}$, $\overline{\alpha} \in \Gamma_{p,i}\backslash\backslash\Gamma_p$ and $\gamma' \in \Gamma_p\backslash\backslash\Gamma$.

LEMMA 12. *There exists $c_1 > 0$ such that the following assertions hold.*

(i) *For all $\gamma' \in \Gamma_p\backslash\backslash\Gamma$, the closest point on $\mathscr{H}_p$ to $\gamma' x_0$ is at distance at most c_1 from x_0. Furthermore, for all $\gamma' \in \Gamma_p\backslash\backslash\Gamma$ and $\alpha \in \Gamma_p$, for every y in the geodesic ray $[x_0, p[$,*

$$\begin{aligned} d(y, \alpha y) + d(y, x_0) + d(x_0, \gamma' x_0) - c_1 &\le d(y, \alpha\gamma' x_0) \\ &\le d(y, \alpha y) + d(y, x_0) + d(x_0, \gamma' x_0). \end{aligned}$$

(ii) *For all $\overline{\alpha} \in \Gamma_{p,i}\backslash\backslash\Gamma_p$, the closest point on D_i to $\overline{\alpha} x_0$ is at distance at most c_1 from the geodesic ray $[x_0, p[$. Furthermore, for all $\overline{\alpha} \in \Gamma_{p,i}\backslash\backslash\Gamma_p$ and $\beta \in \Gamma_{p,i}$,*

$$\begin{aligned} \max\{d(x_0, \overline{\alpha} x_0), d(x_0, \beta x_0)\} - c_1 &\le d(x_0, \beta\overline{\alpha} x_0) \\ &\le \max\{d(x_0, \overline{\alpha} x_0), d(x_0, \beta x_0)\} + c_1. \end{aligned}$$

Proof. (i) For all $\gamma \in \Gamma$, let p_γ be the closest point to γx_0 on $\mathscr{H}_p$, which lies on the geodesic ray $[\gamma x_0, p[$. Hence, by our choice of x_0 and by convexity, p_γ is at bounded distance from $\mathscr{C}\Lambda\Gamma$. Since $\mathscr{H}_p$ is precisely invariant and $x_0 \in \partial\mathscr{H}_p$, the point p_γ belongs to $\partial\mathscr{H}_p$. For all $\gamma \in \Gamma$ and $\alpha \in \Gamma_p$, if $p_\gamma \neq \alpha^{-1}x_0, \gamma x_0$, then the angle at p_γ between $[p_\gamma, \alpha^{-1}x_0]$ and $[p_\gamma, \gamma x_0]$ is at least $\pi/2$ by the convexity of $\mathscr{H}_p$. Hence, by a standard comparison argument and the definition of ε, the distance between p_γ and $[\alpha^{-1}x_0, \gamma x_0]$ is at most ε. By the triangle inequality,

$$d(\alpha^{-1}x_0, p_\gamma) + d(p_\gamma, \gamma x_0) - 2\varepsilon \le d(x_0, \alpha\gamma x_0) \le d(\alpha^{-1}x_0, p_\gamma) + d(p_\gamma, \gamma x_0).$$

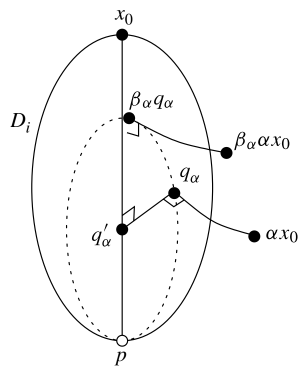
These inequalities are also true if p_γ is equal to $\alpha^{-1}x_0$ or to γx_0. Since p_γ is at bounded distance from $\mathscr{C}\Lambda\Gamma \cap \partial\mathscr{H}_p$ and since the action of Γ_p on $\mathscr{C}\Lambda\Gamma \cap \partial\mathscr{H}_p$ is cocompact, there exists $\alpha_\gamma \in \Gamma_p$ such that $d(p_\gamma, \alpha_\gamma x_0)$ is bounded, say by c_1'. Let $\gamma' \in \Gamma_p\backslash\backslash\Gamma$. Assume for a contradiction that $d(x_0, p_{\gamma'}) > 2\varepsilon + c_1'$. Then, using the above centred equation with $\alpha = 1$ and $\gamma = \gamma'$,

$$\begin{aligned} d(\alpha_{\gamma'}^{-1}\gamma' x_0, x_0) = d(\gamma' x_0, \alpha_{\gamma'} x_0) &\le d(\gamma' x_0, p_{\gamma'}) + d(p_{\gamma'}, \alpha_{\gamma'} x_0) \\ &\le d(\gamma' x_0, x_0) - d(x_0, p_{\gamma'}) + 2\varepsilon + c_1' < d(\gamma' x_0, x_0), \end{aligned}$$

which contradicts the minimality property of $d(\gamma' x_0, x_0)$. This proves the first claim of assertion (i) if $c_1 \geq 2\epsilon + c_1'$.

The first claim and the convexity of the horoball of centre p whose boundary contains y (which implies that if $y \neq x_0, \alpha^{-1} y$, then the angle at y between $[y, x_0]$ and $[y, \alpha^{-1} y]$ is at least $\pi/2$) imply that the length of the piecewise geodesic $[\gamma' x_0, x_0] \cup [x_0, y] \cup [y, \alpha^{-1} y]$ is almost additive, yielding the left-hand side of the second claim of assertion (i). Its right-hand side follows by the triangle inequality.

(ii) For all $\alpha \in \Gamma_p$, let q_α be the closest point to αx_0 on D_i. By the convexity of $\mathscr{H}_p$ and since $\alpha x_0 \in \partial \mathscr{H}_p$, we have $q_\alpha \in \mathscr{H}_p$. By the convexity of D_i and as in (i), the point q_α lies at distance at most ε of the geodesic ray $[\alpha x_0, p[$. Since $x_0 \in \mathscr{N}_\varepsilon(\mathscr{C}\Lambda\Gamma)$, the point q_α is at distance at most 2ε from a point in $\mathscr{C}\Lambda\Gamma$. Hence, $q_\alpha \in D_i \cap \mathscr{H}_p \cap \mathscr{N}_{2\varepsilon}(\mathscr{C}\Lambda\Gamma)$. Since D_i is almost cone-like in cusps for Γ, there exists $\beta_\alpha \in \Gamma_{p,i}$ such that the distance between $\beta_\alpha q_\alpha = q_{\beta_\alpha \alpha}$ and $[x_0, p[$ is less than a constant.



Let q'_α be the closest point to q_α on $[x_0, p[$. By quasi-geodesic arguments, there exists a constant $c > 0$ such that

$$|d(x_0, \beta_\alpha \alpha x_0) - 2d(x_0, \beta_\alpha q_\alpha)| \leq c,$$
$$|d(x_0, \alpha x_0) - 2d(x_0, \beta_\alpha q_\alpha) - 2d(q_\alpha, q'_\alpha)| \leq c.$$

Using a similar argument to that used in the proof of assertion (i), this proves that $q_{\overline{\alpha}}$ is at distance less than a constant from $[x_0, p[$ for every $\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p$.

For all $\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p$ and $\beta \in \Gamma_{p,i}$, since $\beta^{-1} x_0 \in D_i$ and $q_{\overline{\alpha}}$ is the closest point to $\overline{\alpha} x_0$ on D_i, we have

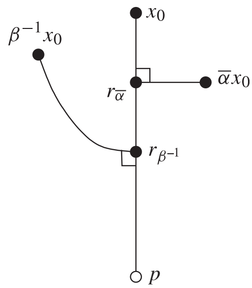
$$d(\beta^{-1} x_0, q_{\overline{\alpha}}) + d(q_{\overline{\alpha}}, \overline{\alpha} x_0) - 2\varepsilon \leq d(\beta^{-1} x_0, \overline{\alpha} x_0) \leq d(\beta^{-1} x_0, q_{\overline{\alpha}}) + d(q_{\overline{\alpha}}, \overline{\alpha} x_0).$$

For every $\alpha' \in \Gamma_p$, let $r_{\alpha'}$ be the closest point to $\alpha' x_0$ on $[x_0, p[$. Hence, by the above argument, there exists $c' > 0$ such that

$$|d(x_0, \beta \overline{\alpha} x_0) - d(\beta^{-1} x_0, r_{\overline{\alpha}}) - d(r_{\overline{\alpha}}, \overline{\alpha} x_0)| \leq c'.$$

For all $y \in [x_0, p[$,

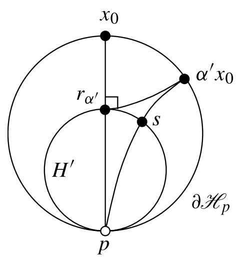
$$d(\alpha' x_0, r_{\alpha'}) + d(r_{\alpha'}, y) - 2\varepsilon \leq d(\alpha' x_0, y) \leq d(\alpha' x_0, r_{\alpha'}) + d(r_{\alpha'}, y).$$



Let H' be the horoball centred at p whose boundary contains $r_{\alpha'}$ and let s be the intersection point of $[\alpha' x_0, p[$ with $\partial H'$. Then

$$d(\alpha' x_0, r_{\alpha'}) \geq d(\alpha' x_0, s) = d(r_{\alpha'}, x_0),$$

since x_0 and $\alpha' x_0$ are on the same horosphere centred at p.



By an easy comparison argument in the geodesic triangle with vertices $r_{\alpha'}$, $\alpha' x_0$ and p, we have $d(s, r_{\alpha'}) \leq 1$. Hence

$$d(\alpha' x_0, r_{\alpha'}) \leq d(\alpha' x_0, s) + d(s, r_{\alpha'}) \leq d(x_0, r_{\alpha'}) + 1.$$

Applying this for $\alpha' = \beta^{-1}, \overline{\alpha}$ and $y = r_{\overline{\alpha}}, r_{\beta^{-1}}, x_0$, we have

$$|d(x_0, \beta\overline{\alpha}x_0) - d(\beta^{-1}x_0, x_0)| \leq c' + 2 + 2\varepsilon$$

if $r_{\overline{\alpha}}$ belongs to $[x_0, r_{\beta^{-1}}]$, and otherwise

$$|d(x_0, \beta\overline{\alpha}x_0) - d(x_0, \overline{\alpha}x_0)| \leq c' + 2 + 2\varepsilon.$$

This proves the result. □

The next lemma, which uses the regular growth property of Γ_p and $\Gamma_{p,i}$, implies, in particular, that the 'relative' critical exponent of Γ_p modulo $\Gamma_{p,i}$ is $\delta_p - \delta_{p,i}$ (see, for instance, [**Pau**] for background on relative Poincaré series).

LEMMA 13. *There exists $c_2 > 0$ such that, for every $t \in [0, +\infty[$,*

$$\frac{1}{c_2} e^{(\delta_p - \delta_{p,i})t} \leq \mathrm{Card}\{\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p : d(x_0, \overline{\alpha}x_0) \leq t\} \leq c_2 e^{(\delta_p - \delta_{p,i})t}.$$

Proof. For all $t \in [0, +\infty[$, define

$$f(t) = \mathrm{Card}\{\alpha \in \Gamma_p : d(x_0, \alpha x_0) \leq t\} \quad \text{and} \quad g(t) = \mathrm{Card}\{\beta \in \Gamma_{p,i} : d(x_0, \beta x_0) \leq t\}.$$

Since Γ_p and $\Gamma_{p,i}$ have regular growth, there exists a constant $c > 0$ such that, for all $t \in [0, +\infty[$,

$$\frac{1}{c}e^{\delta_p t} \leq f(t) \leq ce^{\delta_p t} \quad \text{and} \quad \frac{1}{c}e^{\delta_{p,i} t} \leq g(t) \leq ce^{\delta_{p,i} t}.$$

Also define $E = \Gamma_{p,i} \times (\Gamma_{p,i} \backslash\backslash \Gamma_p)$ and $h(t) = \mathrm{Card}\{\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p : d(x_0, \overline{\alpha} x_0) \leq t\}$.

For all $t \geq c_1$, using Lemma 12(ii) to get the inequality, we have

$$\begin{aligned} f(t - c_1) &= \mathrm{Card}\{(\beta, \overline{\alpha}) \in E : d(x_0, \beta\overline{\alpha} x_0) \leq t - c_1\} \\ &= \mathrm{Card}\{(\beta, \overline{\alpha}) \in E : d(x_0, \beta\overline{\alpha} x_0) \leq t - c_1,\, d(x_0, \beta x_0) \leq d(x_0, \overline{\alpha} x_0)\} \\ &\quad + \mathrm{Card}\{(\beta, \overline{\alpha}) \in E : d(x_0, \beta\overline{\alpha} x_0) \leq t - c_1,\, d(x_0, \beta x_0) > d(x_0, \overline{\alpha} x_0)\} \\ &\leq \mathrm{Card}\{(\beta, \overline{\alpha}) \in E : d(x_0, \overline{\alpha} x_0) \leq t,\, d(x_0, \beta x_0) \leq t\} \\ &\quad + \mathrm{Card}\{(\beta, \overline{\alpha}) \in E : d(x_0, \beta x_0) \leq t,\, t \geq d(x_0, \overline{\alpha} x_0)\} \\ &= 2g(t)h(t). \end{aligned}$$

This gives the lower bound in Lemma 13.

Similarly, for all $t \geq c_1$,

$$\begin{aligned} f(t + c_1 + 1) &\geq \mathrm{Card}\{(\beta, \overline{\alpha}) \in E : t - c_1 < d(x_0, \beta\overline{\alpha} x_0) \leq t + c_1 + 1, \\ &\qquad d(x_0, \beta x_0) \leq d(x_0, \overline{\alpha} x_0)\} \\ &\geq \mathrm{Card}\{(\beta, \overline{\alpha}) \in E : t < d(x_0, \overline{\alpha} x_0) \leq t + 1,\, d(x_0, \beta x_0) \leq t + 1\} \\ &= g(t + 1)(h(t + 1) - h(t)). \end{aligned}$$

A geometric series summation argument gives the upper bound in Lemma 13. □

Now let $\mathscr{F}^+_{p,i}$ be the set of accumulation points in $\partial_\infty \widetilde{M}$ of the orbit points $\overline{\alpha}\gamma' x_0$, where $\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p$ and $\gamma' \in \Gamma_p \backslash\backslash \Gamma$.

LEMMA 14. *We have* $\Lambda\Gamma = \{p\} \cup \bigcup_{\beta \in \Gamma_{p,i}} \beta \mathscr{F}^+_{p,i}$.

Note that in general this union is not a disjoint union.

Proof. Every element ξ in $\Lambda\Gamma$ is the limit of a sequence $(\beta_i \overline{\alpha}_i \gamma'_i x_0)_{i \in \mathbb{N}}$ where $(\beta_i)_{i \in \mathbb{N}}$, $(\overline{\alpha}_i)_{i \in \mathbb{N}}$, $(\gamma'_i)_{i \in \mathbb{N}}$ are sequences in respectively $\Gamma_{p,i}$, $\Gamma_{p,i} \backslash\backslash \Gamma_p$ and $\Gamma_p \backslash\backslash \Gamma$. Up to extraction, if $\xi \neq p$, since the limit set of Γ_p is reduced to $\{p\}$, we may assume that $\lim_{i \to +\infty} \gamma'_i x_0 = \xi' \in \partial_\infty \widetilde{M} - \{p\}$. Since any compact neighbourhood of ξ' not containing p is mapped into any given neighbourhood of p by all except finitely many elements of Γ_p, if the sequence $(\beta_i \overline{\alpha}_i)_{i \in \mathbb{N}}$ in Γ_p takes infinitely many values, then $\xi = p$. Hence, up to extraction, if $\xi \neq p$, the sequence $(\beta_i \overline{\alpha}_i)_{i \in \mathbb{N}}$ is constant, and so is $(\beta_i)_{i \in \mathbb{N}}$: therefore, $\xi \in \beta_0 \mathscr{F}^+_{p,i}$. This proves the result. □

Let $\mathscr{F}_{p,i} = \nu P_{D_i}(\mathscr{F}^+_{p,i} - \partial_\infty D_i) \cap \pi^{-1}(\mathscr{H}_p)$. The images of $\mathscr{F}_{p,i}$ under the elements of $\Gamma_{p,i}$ cover $\pi^{-1}(\mathscr{H}_p) \cap \mathrm{Supp}\, \widetilde{\sigma}_{D_i}$. It follows from Lemma 12 (ii) that there exists $c_3 > 0$ such that $P_{D_i}(\mathscr{F}^+_{p,i} - \partial_\infty D_i) \cap \mathscr{H}_p = \pi(\mathscr{F}_{p,i})$ is contained in the c_3-neighbourhood of the geodesic ray $[x_0, p[$.

In order to prove the finiteness of $\pi_* \sigma_{\mathscr{D}}(\Gamma_p \backslash \mathscr{H}_p)$, we thus only have to prove the finiteness of $\widetilde{\sigma}_{D_i}(\mathscr{F}_{p,i})$.

The next lemma, which uses the assumption that Γ is of divergence type, gives a control on the Patterson measure μ_y of $\mathscr{F}^+_{p,i}$ as y converges radially to p.

LEMMA 15. *There exists $c_4 > 0$ such that, for every y on the geodesic ray* $[x_0, p[$,

$$\mu_y(\mathscr{F}^+_{p,i}) \leq c_4 e^{(2(\delta_p - \delta_{p,i}) - \delta) d(x_0, y)}.$$

Proof. For all $s \geq 0$ and $y \in \widetilde{M}$, for every subgroup Γ' of Γ, let

$$P_{\Gamma', y}(s) = \sum_{\gamma \in \Gamma'} e^{-sd(y, \gamma x_0)} \in [0, +\infty],$$

and let $\mathscr{D}_y$ be the unit Dirac mass at the point y. Since Γ is of divergence type, the Patterson measure μ_y is the weak-star limit as $s \to \delta^+$ of the measures

$$\mu_{y,s} = \frac{1}{P_{\Gamma, x_0}(s)} \sum_{\gamma \in \Gamma} e^{-sd(y, \gamma x_0)} \mathscr{D}_{\gamma x_0}$$

(see, for instance, [**Rob2**]). By discreteness and Lemma 14, there exists a finite subset F of $\Gamma_{p,i}$ such that $\bigcup_{\beta \in F} \beta \mathscr{F}^+_{p,i} - \{p\}$ is a neighbourhood of $\mathscr{F}^+_{p,i} - \{p\}$ in $\Lambda\Gamma - \{p\}$. Since Γ is of divergence type, the measure μ_y has no atom at p (see, for instance, [**Rob2**, Corollary 1.8]). Hence there exists $c > 0$ such that, for every $y \in [x_0, p[$,

$$\mu_y(\mathscr{F}^+_{p,i}) \leq c \lim_{s \to \delta^+} \frac{1}{P_{\Gamma, x_0}(s)} \sum_{\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p, \gamma' \in \Gamma_p \backslash\backslash \Gamma} e^{-sd(y, \overline{\alpha} \gamma' x_0)}.$$

Let

$$Q_y(s) = \sum_{\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p} e^{-sd(y, \overline{\alpha} y)} \quad \text{and} \quad R(s) = \sum_{\gamma' \in \Gamma_p \backslash\backslash \Gamma} e^{-sd(x_0, \gamma' x_0)}.$$

By the lower bound in Lemma 12(i),

$$\sum_{\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p, \gamma' \in \Gamma_p \backslash\backslash \Gamma} e^{-sd(y, \overline{\alpha} \gamma' x_0)} \leq e^{sc_1} e^{-sd(y, x_0)} Q_y(s) R(s).$$

Similarly, by the upper bound in Lemma 12(i), we have $P_{\Gamma, x_0}(s) \geq P_{\Gamma_p, x_0}(s) R(s)$. We will prove below that the series $Q_y(\delta)$ converges. Thus, even if $P_{\Gamma_p, x_0}(\delta) = +\infty$,

$$\mu_y(\mathscr{F}^+_{p,i}) \leq \frac{c e^{\delta c_1}}{P_{\Gamma_p, x_0}(\delta)} e^{-\delta d(y, x_0)} Q_y(\delta). \tag{19}$$

By the convexity of the horoball of centre p whose boundary contains y and by standard quasi-geodesic arguments, there exist two constants $c', c'' > 0$ such that, for every $\overline{\alpha} \in \Gamma_{p,i} \backslash\backslash \Gamma_p$, if $d(y, \overline{\alpha} y) > c'$, then

$$d(y, \overline{\alpha} y) + 2d(y, x_0) - c'' \leq d(x_0, \overline{\alpha} x_0) \leq d(y, \overline{\alpha} y) + 2d(y, x_0).$$

If $d(y, \overline{\alpha} y) \le c'$ then $d(x_0, \overline{\alpha} x_0) \le 2d(y, x_0) + c'$, by the triangle inequality. We thus have, using the notation $t \mapsto h(t)$ introduced in the proof of Lemma 13,

$$\begin{aligned} Q_y(\delta) &= \sum_{\substack{\overline{\alpha}\in\Gamma_{p,i}\backslash\backslash\Gamma_p \\ d(y,\overline{\alpha}y)\le c'}} e^{-\delta d(y,\overline{\alpha}y)} + \sum_{\substack{\overline{\alpha}\in\Gamma_{p,i}\backslash\backslash\Gamma_p \\ d(y,\overline{\alpha}y)> c'}} e^{-\delta d(y,\overline{\alpha}y)} \\ &\le \sum_{\substack{\overline{\alpha}\in\Gamma_{p,i}\backslash\backslash\Gamma_p \\ d(x_0,\overline{\alpha}x_0)\le 2d(y,x_0)+c'}} 1 + \sum_{\substack{\overline{\alpha}\in\Gamma_{p,i}\backslash\backslash\Gamma_p \\ d(x_0,\overline{\alpha}x_0)\ge 2d(y,x_0)+c'-c''}} e^{-\delta d(x_0,\overline{\alpha}x_0)+2\delta d(y,x_0)} \\ &\le h(2\,d(y,x_0)+c') + e^{2\delta d(y,x_0)} \sum_{n\ge 2d(y,x_0)+c'-c''-1} h(n+1)e^{-\delta n}. \end{aligned}$$

Since $h(t) \le c_2 e^{(\delta_p - \delta_{p,i})t}$ by Lemma 13, and by a geometric series summation argument since $\delta - \delta_p + \delta_{p,i} > 0$ by assumption (18), we therefore have

$$\begin{aligned} Q_y(\delta) &\le c_2 e^{(\delta_p-\delta_{p,i})c'} e^{2(\delta_p-\delta_{p,i})d(y,x_0)} \\ &\quad + c_2 e^{\delta_p-\delta_{p,i}+2\delta d(y,x_0)} \sum_{n\ge 2d(y,x_0)+c'-c''-1} e^{(\delta_p-\delta_{p,i}-\delta)n} \\ &\le c''' e^{2(\delta_p-\delta_{p,i})d(y,x_0)}, \end{aligned}$$

for some $c''' > 0$. Using equation (19), this proves Lemma 15. □

Let $\rho : [0, +\infty[\to \widetilde{M}$ be the geodesic ray with origin x_0 and point at infinity p. For every $n \in \mathbb{N}$, let A_n be the set of points $\xi \in \partial_\infty \widetilde{M} - \{p\}$ such that the closest point to ξ on the geodesic ray ρ belongs to $\rho([n, n+1])$. Note that $\bigcup_{n\in\mathbb{N}} A_n = \partial_\infty \widetilde{M} - \{p\}$.

LEMMA 16. *There exists $c_5 > 0$ such that, for every $n \in \mathbb{N}$,*

$$\mu_{x_0}(\mathscr{F}^+_{p,i} \cap A_n) \le c_5 e^{2(\delta_p-\delta_{p,i}-\delta)n}.$$

Proof. For every $\xi \in A_n$, since the angle at $\rho(n)$ between $[\rho(n), x_0]$ and $[\rho(n), \xi[$ is at least $\pi/2$ if $n \ne 0$, we have $\beta_\xi(x_0, \rho(n)) \ge d(x_0, \rho(n)) - 2\varepsilon = n - 2\varepsilon$. Hence, by equation (10) and Lemma 15,

$$\begin{aligned} \mu_{x_0}(\mathscr{F}^+_{p,i} \cap A_n) &= \int_{\xi\in\mathscr{F}^+_{p,i}\cap A_n} e^{-\delta\beta_\xi(x_0,\rho(n))} d\mu_{\rho(n)}(\xi) \le e^{-\delta n + 2\delta\varepsilon} \mu_{\rho(n)}(\mathscr{F}^+_{p,i}) \\ &\le c_4 e^{2\delta\varepsilon} e^{2(\delta_p-\delta_{p,i}-\delta)n}. \end{aligned}$$

□

After this series of lemmas, let us prove the finiteness of $\widetilde{\sigma}_{D_i}(\mathscr{F}_{p,i})$, which concludes the proof of Theorem 10.

With $a_n = \widetilde{\sigma}_{D_i}(\mathscr{F}_{p,i} \cap \nu P_{D_i}(A_n))$, we only have to prove that the series $\sum_{n\in\mathbb{N}} a_n$ converges. For every $\xi \in (\mathscr{F}^+_{p,i} - \partial_\infty D_i) \cap A_n \cap P^{-1}_{D_i}(\mathscr{H}_p)$, by the definition of c_3, the point $P_{D_i}(\xi)$ lies at distance less than a constant from $\rho([n, n+1])$. Hence there exists a constant $c > 0$ such that $\beta_\xi(P_{D_i}(\xi), x_0) \ge -n - c$. By the definition of the skinning measures in equation (11) and by Lemma 16, therefore,

$$\begin{aligned} \widetilde{\sigma}_{D_i}(\mathscr{F}_{p,i} \cap \nu P_{D_i}(A_n)) &= \int_{\xi\in(\mathscr{F}^+_{p,i}-\partial_\infty D_i)\cap A_n\cap P^{-1}_{D_i}(\mathscr{H}_p)} e^{-\delta\beta_\xi(P_{D_i}(\xi),x_0)} d\mu_{x_0}(\xi) \\ &\le e^{\delta n+\delta c} \mu_{x_0}(\mathscr{F}^+_{p,i} \cap A_n) \\ &\le c_5 e^{\delta c} e^{(2(\delta_p-\delta_{p,i})-\delta)n}. \end{aligned}$$

By the assumption $\delta > 2(\delta_p - \delta_{p,i})$ in equation (18), a geometric series summation argument proves that $\sum_{n\in\mathbb{N}} a_n$ converges. This completes the proof of Theorem 10. $\Box$

5. *Equidistribution of equidistant submanifolds*

Let $\widetilde{M}$, Γ, x_0, M and T^1M be as in §2. Assume that the critical exponent δ_Γ of Γ is finite. Let $(\mu_x)_{x\in\widetilde{M}}$ be a Patterson density of dimension δ_Γ, with Bowen–Margulis measures $\widetilde{m}_{\rm BM}$ and $m_{\rm BM}$ on $T^1\widetilde{M}$ and T^1M, respectively. Let $\mathscr{C} = (C_i)_{i\in I}$ be a Γ-equivariant family of proper non-empty closed convex subsets of $\widetilde{M}$. Let $\mathscr{C}_t = (\mathscr{N}_t C_i)_{i\in I}$ (in particular, $\mathscr{C}_0 = \mathscr{C}$), and let $\widetilde{\sigma}_{\mathscr{C}_t}$ and $\sigma_{\mathscr{C}_t}$ be the skinning measures of $\mathscr{C}_t$ on $T^1\widetilde{M}$ and T^1M, respectively. Let $\Omega = (\Omega_i)_{i\in I}$ be a locally finite Γ-equivariant family of subsets of $T^1\widetilde{M}$, where Ω_i is a measurable subset of $\partial^1_+ C_i$ with $\widetilde{\sigma}_{C_i}(\partial\Omega_i) = 0$ for every $i \in I$. Let $\sim\, = \,\sim_\Omega$ be the equivalence relation on I defined at the end of §3. As we have already defined when $\Omega = \mathscr{C}$, let

$$\widetilde{\sigma}_\Omega = \sum_{i\in I/\sim} \widetilde{\sigma}_{C_i}|_{\Omega_i},$$

which is a Γ-invariant locally finite positive Borel measure on $T^1\widetilde{M}$ (independent of the choice of representatives in $I/\sim$). Hence, $\widetilde{\sigma}_\Omega$ induces a locally finite positive Borel measure σ_Ω on T^1M. Note that $g^t\Omega_i \subset \partial^1_+\mathscr{N}_t C_i$ and, as in the end of §3, for every $t > 0$,

$$\|\sigma_{g^t\Omega}\| = e^{\delta_\Gamma t}\|\sigma_\Omega\|. \tag{20}$$

The aim of this section is to prove, under some finiteness assumptions, that the measures $\sigma_{g^t\Omega}$ on T^1M equidistribute to the Bowen–Margulis measure on T^1M as $t \to +\infty$. We start by introducing the test functions approximating the support of the measures $\sigma_{g^t\Omega}$.

Assume that the number of orbits of Γ on the set of elements $i \in I$, such that the intersection $\partial_\infty C_i \cap \Lambda\Gamma$ is empty, is finite (this condition is stronger than the requirement on $\mathscr{C}$ to be locally finite). Under this assumption, by Lemma 7, there exists $R > 0$ such that for every $i \in I$, for every $w \in \partial^1_+ C_i$, we have $\mu^{\rm ss}_w(V_{w,R}) > 0$, where $V_{w,R}$ is the open ball of radius R and centre w for the Hamenstädt distance in the strong stable leaf $W^{\rm ss}(w)$. We fix such an R.

For every $\eta > 0$, let $h_{\eta,R} : T^1\widetilde{M} \to [0, +\infty]$ be the measurable map defined by

$$h_{\eta,R}(w) = \frac{1}{2\eta\,\mu^{\rm ss}_w(V_{w,R})}.$$

Note that $h_{\eta,R}$ is Γ-invariant by equation (14) and that $h_{\eta,R}\circ g^{-t} = e^{-\delta_\Gamma t}\, h_{\eta,e^{-t}R}$ for every $t \in \mathbb{R}$: indeed, for every $w \in T^1\widetilde{M}$, by equation (15),

$$h_{\eta,R}(g^{-t}w) = \frac{1}{2\eta\,\mu^{\rm ss}_{g^{-t}w}(V_{g^{-t}w,R})} = \frac{1}{2\eta e^{\delta_\Gamma t}\mu^{\rm ss}_w(g^t V_{g^{-t}w,R})} = \frac{e^{-\delta_\Gamma t}}{2\eta\mu^{\rm ss}_w(V_{w,e^{-t}R})}.$$

For every $i \in I$, let $\mathscr{V}_{\eta,R,i} = \mathscr{V}_{\eta,R}(\Omega_i)$ be the dynamical thickening of Ω_i defined at the end of §2. Note that $\gamma\mathscr{V}_{\eta,R,i} = \mathscr{V}_{\eta,R,\gamma i}$ for every $\gamma \in \Gamma$ and every $i \in I$.

We denote by χ_A the characteristic function of a subset A. We will use the test function $\widetilde{\phi}_\eta = \widetilde{\phi}_{\eta,R,\Omega} : T^1\widetilde{M} \to [0, +\infty[$ defined by (using the convention $\infty \times 0 = 0$)

$$\widetilde{\phi}_\eta(v) = \sum_{i\in I/\sim} h_{\eta,R}\circ f_{C_i}(v)\; \chi_{\mathscr{V}_{\eta,R,i}}(v),$$

where $f_{C_i} : U_{C_i} \to \partial^1_+ C_i$ is the fibration defined in §2.2. Note that $\mathscr{V}_{\eta,R,i}$ is contained in U_{C_i}, and we define $h_{\eta,R} \circ f_{C_i}(v)\, \chi_{\mathscr{V}_{\eta,R,i}}(v) = 0$ if $v \notin \mathscr{V}_{\eta,R,i}$.

LEMMA 17. *The function $\widetilde{\phi}_\eta$ is well defined, measurable and Γ-invariant. Furthermore, for every $t \in [0, +\infty[$,*

$$\widetilde{\phi}_{\eta,R,\Omega} \circ g^{-t} = e^{-\delta_\Gamma t} \widetilde{\phi}_{\eta,e^{-t}R,g^t\Omega}.$$

Proof. The function $\widetilde{\phi}_\eta$ is well defined, since $\Omega_i = \Omega_j$ and $\mathscr{V}_{\eta,R,i} = \mathscr{V}_{\eta,R,j}$ if $i \sim j$, since $h_{\eta,R} \circ f_{C_i}(v)$ is finite if $v \in \mathscr{V}_{\eta,R,i}$ (by the definition of R), and since the sum defining $\widetilde{\phi}_\eta(v)$ has only finitely many non-zero terms, by the local finiteness of the family Ω (given v, the summation over $I/\!\sim$ may be replaced by a summation over the finite set $\{i \in I : v \in \mathscr{V}_{\eta,R,i}\}/\!\sim$).

The function $\widetilde{\phi}_\eta$ is Γ-invariant since

$$\chi_{\mathscr{V}_{\eta,R,i}} \circ \gamma = \chi_{\gamma^{-1}\mathscr{V}_{\eta,R,i}} = \chi_{\mathscr{V}_{\eta,R,\gamma^{-1}i}}$$

and

$$h_{\eta,R} \circ f_{C_i} \circ \gamma = h_{\eta,R} \circ \gamma \circ f_{\gamma^{-1}C_i} = h_{\eta,R} \circ f_{C_{\gamma^{-1}i}},$$

and by a change of index in the above sum.

Let $t \geq 0$. The last claim follows by noting that

$$\chi_{\mathscr{V}_{\eta,R}(\Omega_i)} \circ g^{-t} = \chi_{g^t\mathscr{V}_{\eta,R}(\Omega_i)} = \chi_{\mathscr{V}_{\eta,e^{-t}R}(g^t\Omega_i)}$$

and

$$h_{\eta,R} \circ f_{C_i} \circ g^{-t} = h_{\eta,R} \circ f_{C_i} = h_{\eta,R} \circ g^{-t} \circ g^t \circ f_{C_i} = e^{-\delta_\Gamma t} h_{\eta,e^{-t}R} \circ f_{\mathscr{N}_t C_i}. \qquad \square$$

Hence the test function $\widetilde{\phi}_\eta$ defines, by passing to the quotient, a measurable function $\phi_\eta = \phi_{\eta,R,\Omega} : T^1M \to [0, +\infty[$, such that, for every $t \in [0, +\infty[$,

$$\phi_{\eta,R,\Omega} \circ g^{-t} = e^{-\delta_\Gamma t} \phi_{\eta,e^{-t}R,g^t\Omega}. \tag{21}$$

PROPOSITION 18. *Assume that the Bowen–Margulis measure of T^1M is finite. For every $\eta > 0$, we have $\int \phi_\eta \, dm_{\mathrm{BM}} = \|\sigma_\Omega\|$. In particular, the function ϕ_η is integrable for the Bowen–Margulis measure if and only if the measure σ_Ω is finite.*

Proof. Let $i \in I$ and let K_i be a measurable subset of Ω_i. By the disintegration result of Proposition 8 (more precisely by equation (16)), and by the definitions of the function $h_{\eta,R}$ and of the set $\mathscr{V}_{\eta,R,i} = \bigcup_{w\in\Omega_i} \bigcup_{s\in\,]-\eta,\eta[} g^s V_{w,R}$, we have

$$\begin{aligned}\int_{\mathscr{V}_{\eta,R,i}\cap f_{C_i}^{-1}(K_i)} h_{\eta,R} \circ f_{C_i}\, d\widetilde{m}_{\mathrm{BM}} &= \int_{w\in K_i} h_{\eta,R}(w) \int_{v'\in V_{w,R}} \int_{-\eta}^{\eta} ds\, d\mu^{\mathrm{ss}}_w(v')\, d\widetilde{\sigma}_{C_i}(w)\\ &= \widetilde{\sigma}_{C_i}(K_i).\end{aligned}$$

Let Δ_Γ be a *fundamental domain* for the action of Γ on $T^1\widetilde{M}$, that is, Δ_Γ is the closure of its interior, its boundary has measure zero for the Bowen–Margulis measure, the images of Δ_Γ by the elements of Γ have pairwise disjoint interiors and cover $T^1\widetilde{M}$, and any compact subset of $T^1\widetilde{M}$ meets only finitely many images of Δ_Γ by elements of Γ. Such a fundamental domain exists since the Bowen–Margulis measure of T^1M is finite (see, for instance, [**Rob2**, p. 13]). By the definition of the test function $\widetilde{\phi}_\eta$,

$$\int_{T^1M} \phi_\eta \, dm_{\mathrm{BM}} = \int_{\Delta_\Gamma} \widetilde{\phi}_\eta \, d\widetilde{m}_{\mathrm{BM}} = \sum_{i \in I/\sim} \int_{\mathscr{V}_{\eta,R,i} \cap \Delta_\Gamma} h_{\eta,R} \circ f_{C_i} \, d\widetilde{m}_{\mathrm{BM}}.$$

By the definition of the measure σ_Ω,

$$\|\sigma_\Omega\| = \widetilde{\sigma}_\Omega(\Delta_\Gamma) = \sum_{i \in I/\sim} \widetilde{\sigma}_{C_i}(\Delta_\Gamma \cap \Omega_i).$$

By an easy multiplicity argument, the result follows. □

We can now state and prove the main result of this paper.

THEOREM 19. *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with sectional curvature bounded above by -1. Let Γ be a discrete, non-elementary group of isometries of $\widetilde{M}$, with finite critical exponent. Assume that the Bowen–Margulis measure m_{BM} of Γ on T^1M is finite and mixing for the geodesic flow. Let $\mathscr{C} = (C_i)_{i \in I}$ be a Γ-equivariant family of non-empty proper closed convex subsets of $\widetilde{M}$. Let $\Omega = (\Omega_i)_{i \in I}$ be a locally finite Γ-equivariant family of measurable subsets $\Omega_i \subset \partial^1_+ C_i$ with $\widetilde{\sigma}_{C_i}(\partial \Omega_i) = 0$. Assume that σ_Ω is finite and non-zero. Then, as $t \to +\infty$,*

$$\frac{1}{\|\sigma_{g^t\Omega}\|} \, \sigma_{g^t\Omega} \overset{*}{\rightharpoonup} \frac{1}{\|m_{\mathrm{BM}}\|} m_{\mathrm{BM}}.$$

In particular, if $\mathscr{C} = (C_i)_{i \in I}$ is a locally finite Γ-equivariant family of non-empty proper closed convex subsets of $\widetilde{M}$ with finite non-zero skinning measure, then the skinning measure $\sigma_{\mathscr{C}_t}$ on T^1M of $\mathscr{C}_t = (\mathscr{N}_t C_i)_{i \in I}$ equidistributes to the Bowen–Margulis measure as $t \to +\infty$.

Proof. Given three numbers a, b, c (depending on some parameters), we write $a = b \pm c$ if $|a - b| \leq c$.

We may assume that $\Gamma \backslash I$ is finite. Indeed, if $\overline{J}$ is a big enough finite subset of $\Gamma \backslash I$, if J is the preimage of $\overline{J}$ by the canonical map $I \to \Gamma \backslash I$, since the measure σ_Ω is finite, the contribution of the family $(g^t \Omega_i)_{i \in I - J}$ is negligible compared to that of $(g^t \Omega_i)_{i \in J}$ (they grow at equal rate as t tends to $+\infty$, by equation (20)).

Hence we may consider $R > 0$ as was fixed in the beginning of §5 and, for every $\eta > 0$, the test function $\phi_\eta = \phi_{\eta,R,\Omega}$ as defined above.

Fix $\psi \in \mathscr{C}_c(T^1M)$. Let us prove that

$$\lim_{t \to +\infty} \frac{1}{\|\sigma_{g^t\Omega}\|} \int_{T^1M} \psi \, d\sigma_{g^t\Omega} = \frac{1}{\|m_{\mathrm{BM}}\|} \int_{T^1M} \psi \, dm_{\mathrm{BM}}.$$

Given a fundamental domain Δ_Γ for the action of Γ on $T^1\widetilde{M}$ as above, by a standard argument of finite partition of unity, we may assume that there exists a map $\widetilde{\psi} : T^1\widetilde{M} \to \mathbb{R}$ whose support is contained in Δ_Γ such that $\widetilde{\psi} = \psi \circ p$, where $p : T^1\widetilde{M} \to \Gamma \backslash T^1\widetilde{M}$ is the canonical projection (which is 1-Lipschitz). Fix $\epsilon > 0$. Since $\widetilde{\psi}$ is uniformly continuous, for every $\eta > 0$ small enough, and for every $t \geq 0$ big enough, for every $w \in T^1\widetilde{M}$ and $v \in V_{w,\eta,e^{-t}R}$, we have

$$\widetilde{\psi}(v) = \widetilde{\psi}(w) \pm \frac{\epsilon}{2}. \tag{22}$$

We have, using respectively:

- equation (21) and the definition of $\widetilde{\psi}$ for the first and second equality;
- the definition of the test function $\widetilde{\phi}_\eta$ for the third equality;
- equation (22) and the fact that the support of $\widetilde{\psi}$ is contained in Δ_Γ for the fourth equality;
- the invariance of the Bowen–Margulis measure under the geodesic flow, and equation (16) as in the proof of Proposition 18 for the fifth equality;
- the definition of $h_{\eta,e^{-t}R}$ and Proposition 18 for the sixth equality:

$$\begin{aligned}
&\int_{T^1M} \phi_\eta \circ g^{-t}\psi \, dm_{\mathrm{BM}} \\
&= e^{-\delta_\Gamma t} \int_{T^1M} \phi_{\eta,e^{-t}R,g^t\Omega}\psi \, dm_{\mathrm{BM}} = e^{-\delta_\Gamma t} \int_{T^1\widetilde{M}} \widetilde{\phi}_{\eta,e^{-t}R,g^t\Omega}\widetilde{\psi} d\widetilde{m}_{\mathrm{BM}} \\
&= e^{-\delta_\Gamma t} \sum_{i\in I/\sim} \int_{\mathscr{V}_{\eta,e^{-t}R}(g^t\Omega_i)} h_{\eta,e^{-t}R} \circ f_{\mathscr{N}_t C_i} \widetilde{\psi} \, d\widetilde{m}_{\mathrm{BM}} \\
&= e^{-\delta_\Gamma t} \sum_{i\in I/\sim} \int_{\mathscr{V}_{\eta,e^{-t}R}(g^t\Omega_i)} (h_{\eta,e^{-t}R}\widetilde{\psi}) \circ f_{\mathscr{N}_t C_i} \, d\widetilde{m}_{\mathrm{BM}} \pm \frac{\epsilon}{2} \int_{\Delta_\Gamma} \widetilde{\phi}_\eta \circ g^t d\widetilde{m}_{\mathrm{BM}} \\
&= e^{-\delta_\Gamma t} \sum_{i\in I/\sim} \int_{w\in g^t\Omega_i} h_{\eta,e^{-t}R}(w)\widetilde{\psi}(w)(2\eta)\mu^{\mathrm{ss}}_w(V_{w,e^{-t}R}) \, d\widetilde{\sigma}_{\mathscr{N}_t C_i} \pm \frac{\epsilon}{2} \int_{T^1M} \phi_\eta \, dm_{\mathrm{BM}} \\
&= e^{-\delta_\Gamma t} \sum_{i\in I/\sim} \int_{w\in g^t\Omega_i} \widetilde{\psi}(w) \, d\widetilde{\sigma}_{\mathscr{N}_t C_i} \pm \frac{\epsilon}{2} \, \|\sigma_\Omega\| \\
&= e^{-\delta_\Gamma t} \int \psi \, d\sigma_{g^t\Omega} \pm \frac{\epsilon}{2} \, \|\sigma_\Omega\|.
\end{aligned}$$

Hence, using equation (20) for the first equality, the previous computation for the second equality, the invariance of the Bowen–Margulis measure under the geodesic flow for the third equality, and Proposition 18 for the last one, we have, for $\eta > 0$ small enough and $t \geq 0$ big enough,

$$\begin{aligned}
\frac{\int \psi \, d\sigma_{g^t\Omega}}{\|\sigma_{g^t\Omega}\|} = \frac{\int \psi \, d\sigma_{g^t\Omega}}{e^{\delta_\Gamma t}\|\sigma_\Omega\|} &= \frac{\int_{T^1M} \phi_\eta \circ g^{-t}\psi \, dm_{\mathrm{BM}}}{\|\sigma_\Omega\|} \pm \frac{\epsilon}{2} \\
&= \frac{\int_{T^1M} \phi_\eta\psi \circ g^t \, dm_{\mathrm{BM}}}{\|\sigma_\Omega\|} \pm \frac{\epsilon}{2} \\
&= \frac{\int_{T^1M} \phi_\eta\psi \circ g^t \, dm_{\mathrm{BM}}}{\int_{T^1M} \phi_\eta \, dm_{\mathrm{BM}}} \pm \frac{\epsilon}{2}. \qquad (23)
\end{aligned}$$

By the mixing property of the geodesic flow on T^1M, for $t \geq 0$ big enough (while η is fixed),

$$\frac{\int_{T^1M} \phi_\eta\psi \circ g^t \, dm_{\mathrm{BM}}}{\int_{T^1M} \phi_\eta \, dm_{\mathrm{BM}}} = \frac{\int_{T^1M} \psi \, dm_{\mathrm{BM}}}{\|m_{\mathrm{BM}}\|} \pm \frac{\epsilon}{2}.$$

This proves the result. □

We conclude this section by proving Theorem 1 in the introduction. The definition of a properly immersed locally convex subset is recalled at the beginning of the proof.

Proof of Theorem 1. Let M, C be as in the statement of Theorem 1, that is, they satisfy the following property. Let $\widetilde{M} \to M$ be a universal covering of M, with covering group Γ. Let $\widetilde{C} \to C$ be a covering map which is a universal covering over each component of C. The immersion from C to M lifts to an immersion $f : \widetilde{C} \to \widetilde{M}$, which is, on each connected component of $\widetilde{C}$, an embedding whose image is a closed convex subset of $\widetilde{M}$.

Let $I = \Gamma \times \pi_0\widetilde{C}$ with the action of Γ defined by $\gamma \cdot (\alpha, c) = (\gamma\alpha, c)$ for all $\gamma, \alpha \in \Gamma$ and every component c of $\widetilde{C}$. Consider the family $\mathscr{C} = (C_i)_{i\in I}$, where $C_i = \alpha f(c)$ if $i = (\alpha, c)$. Then $\mathscr{C}$ is a Γ-equivariant family of non-empty closed convex subsets of $\widetilde{M}$, which is locally finite since C is properly immersed in M. The result then follows from Theorem 19. □

6. *Exponential rate of equidistribution*

Let $\widetilde{M}$, Γ, M, T^1M, m_{BM}, $\mathscr{C}$, $\mathscr{C}_t$ and $\sigma_{\mathscr{C}_t}$ be as at the beginning of §5. When the Bowen–Margulis measure m_{BM} is finite, we denote by $\overline{m}_{\mathrm{BM}}$ its normalization to a probability measure.

In this section we show, under the finiteness assumptions of Theorem 19, that in the known cases when the geodesic flow is exponentially mixing, the skinning measure equidistributes to the Bowen–Margulis measure with exponential speed. To begin with, we recall the two types of exponential mixing results that are available. In order to prove our estimates for the rate of equidistribution using these results, we will smoothen (in accordance with the two regularities) our test function ϕ_η defined in the previous section.

Firstly, when M is locally symmetric with finite volume, then the boundary at infinity of $\widetilde{M}$, the strong unstable, unstable, stable, and strong stable foliations of $T^1\widetilde{M}$ are smooth. Hence, for all $\ell \in \mathbb{N}$, talking about $\mathscr{C}^\ell$-smooth leafwise defined functions on T^1M makes sense. We will denote by $\mathscr{C}^\ell_c(T^1M)$ the vector space of $\mathscr{C}^\ell$-smooth functions on T^1M with compact support and by $\|\psi\|_\ell$ the Sobolev $W^{\ell,2}$-norm of any $\psi \in \mathscr{C}^\ell_c(T^1M)$. Note that now the Bowen–Margulis measure m_{BM} of T^1M is the unique (up to a multiplicative constant) locally homogeneous smooth measure on T^1M (hence it coincides, up to a multiplicative constant, with the Liouville measure).

Given $\ell \in \mathbb{N}$, we will say that the geodesic flow on T^1M is *exponentially mixing for the Sobolev regularity* ℓ (or that it has *exponential decay of ℓ-Sobolev correlations*) if there exist $c, \kappa > 0$ such that, for all $\phi, \psi \in \mathscr{C}^\ell_c(T^1M)$ and all $t \in \mathbb{R}$,

$$\left| \int_{T^1M} \phi \circ g^{-t}\, \psi\, d\overline{m}_{\mathrm{BM}} - \int_{T^1M} \phi\, d\overline{m}_{\mathrm{BM}} \int_{T^1M} \psi\, d\overline{m}_{\mathrm{BM}} \right| \le c e^{-\kappa|t|} \|\psi\|_\ell \|\phi\|_\ell.$$

When Γ is an arithmetic lattice in the isometry group of $\widetilde{M}$, this property, for some $\ell \in \mathbb{N}$, follows from [**KM1**, Theorem 2.4.5], with the help of [**Clo**, Theorem 3.1] to check its spectral gap property, and of [**KM2**, Lemma 3.1] to deal with finite cover problems.

Secondly, when $\widetilde{M}$ has pinched negative sectional curvature with bounded derivatives, then the boundary at infinity of $\widetilde{M}$, the strong unstable, unstable, stable, and strong stable foliations of $T^1\widetilde{M}$ are only Hölder-smooth (see, for instance, [**Bri**] when $\widetilde{M}$ has a compact quotient, and [**PPS**, Theorem 7.3]). Hence the appropriate regularity on functions on $T^1\widetilde{M}$ is the Hölder one. For every $\alpha \in \,]0, 1[$, we denote by $\mathrm{C}^\alpha_{\mathrm{c}}(X)$ the space of α-Hölder continuous real-valued functions with compact support on a metric space (X, d), endowed with the Hölder norm

$$\|f\|_\alpha = \|f\|_\infty + \sup_{x,y\in X, x\neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}.$$

Assuming the Bowen–Margulis measure m_{BM} on T^1M to be finite, given $\alpha \in]0, 1[$, we will say that the geodesic flow on T^1M is *exponentially mixing for the Hölder regularity* α (or that it has *exponential decay of* α*-Hölder correlations*) if there exist $\kappa, c > 0$ such that, for all $\phi, \psi \in \mathrm{C}_{\mathrm{c}}^\alpha(T^1M)$ and all $t \in \mathbb{R}$,

$$\left| \int_{T^1M} \phi \circ g^{-t} \psi \, d\overline{m}_{\mathrm{BM}} - \int_{T^1M} \phi d\overline{m}_{\mathrm{BM}} \int_{T^1M} \psi \, d\overline{m}_{\mathrm{BM}} \right| \leq c e^{-\kappa|t|} \|\phi\|_\alpha \|\psi\|_\alpha.$$

This holds if M is compact and has dimension two by the work of Dolgopyat [**Dol**] or if M is compact and locally symmetric by [**Sto**, Corollary 1.5] (see also [**Liv**] when M is compact; the result stated there for the Liouville measure should extend to the Bowen–Margulis measure).

The following result gives exponentially small error terms in the equidistribution of the skinning measures to the Bowen–Margulis measure, in the known situations when the geodesic flow is exponentially mixing. Here we state the result for skinning measures but, clearly, it remains valid if $\sigma_{\mathscr{C}_t}$ is replaced by $\sigma_{g^t\Omega}$ as in Theorem 19.

THEOREM 20. *Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with negative sectional curvature. Let Γ be a discrete, non-elementary group of isometries of $\widetilde{M}$. Let $\mathscr{C} = (C_i)_{i\in I}$ be a locally finite Γ-equivariant family of proper non-empty closed convex subsets of $\widetilde{M}$, with finite non-zero skinning measure $\sigma_{\mathscr{C}}$. Let $M = \Gamma\backslash\widetilde{M}$.*

(i) *If M is compact and if the geodesic flow on T^1M is mixing with exponential speed for the Hölder regularity, then there exist $\alpha \in]0, 1[$ and $\kappa'' > 0$ such that, for all $\psi \in \mathrm{C}_{\mathrm{c}}^\alpha(T^1M)$, as $t \to +\infty$,*

$$\frac{1}{\|\sigma_{\mathscr{C}_t}\|} \int \psi \, d\sigma_{\mathscr{C}_t} = \frac{1}{\|m_{\mathrm{BM}}\|} \int \psi \, dm_{\mathrm{BM}} + O(e^{-\kappa'' t} \|\psi\|_\alpha).$$

(ii) *If $\widetilde{M}$ is a symmetric space, if M has finite volume and if the geodesic flow on T^1M is mixing with exponential speed for the Sobolev regularity, then there exists $\ell \in \mathbb{N}$ and $\kappa'' > 0$ such that, for all $\psi \in \mathscr{C}_c^\ell(T^1M)$, as $t \to +\infty$,*

$$\frac{1}{\|\sigma_{\mathscr{C}_t}\|} \int \psi \, d\sigma_{\mathscr{C}_t} = \frac{1}{\|m_{\mathrm{BM}}\|} \int \psi \, dm_{\mathrm{BM}} + O(e^{-\kappa'' t} \|\psi\|_\ell).$$

Proof. Up to rescaling, we may assume that the sectional curvature is bounded from above by -1. The critical exponent and the Bowen–Margulis measure are finite in all cases considered.

Let us consider claim (i). Under these assumptions, there is some $\alpha \in]0, 1[$ such that the geodesic flow on T^1M is exponentially mixing for the Hölder regularity α and such that the strong stable foliation of $T^1\widetilde{M}$ is α-Hölder (see, for instance, [**PPS**, §7.1]).

Fix $R > 0$ and, for every $\eta > 0$, let us consider the test function $\phi_\eta = \phi_{\eta,R,\mathscr{C}}$ as in §5. Up to replacing C_i by $\mathscr{N}_1 C_i$, we may assume that the boundary of C_i is $C^{1,1}$-smooth, for every $i \in I$ (see §2.2).

Fix $\psi \in \mathrm{C}^\alpha_{\mathrm{c}}(T^1 M)$. We may assume as above that there exists a lift $\widetilde{\psi} : T^1\widetilde{M} \to \mathbb{R}$ of ψ whose support is contained in a given fundamental domain Δ_Γ for the action of Γ on $T^1\widetilde{M}$. First assume that $\Gamma\backslash I$ is finite. There exist $\eta_0 > 0$ and $t_0 \geq 0$ such that, for all $\eta \in\,]0, \eta_0]$, $t \in [t_0, +\infty[$, $w \in T^1\widetilde{M}$ and $v \in V_{w,\eta,e^{-t}R}$,

$$\widetilde{\psi}(v) = \widetilde{\psi}(w) + \mathrm{O}((\eta + e^{-t})^\alpha \|\psi\|_\alpha), \tag{24}$$

since $d(v, w) = \mathrm{O}(\eta + e^{-t})$ by equation (5) and Lemma 3.

As in the proof of Theorem 19 using equation (24) instead of equation (22) (see equation (23)), we have

$$\frac{\int \psi\, d\sigma_{\mathscr{C}_t}}{\|\sigma_{\mathscr{C}_t}\|} = \frac{\int_{T^1M} \phi_\eta \psi\circ g^t\, d\overline{m}_{\mathrm{BM}}}{\int_{T^1M}\phi_\eta\, d\overline{m}_{\mathrm{BM}}} + \mathrm{O}((\eta + e^{-t})^\alpha\|\psi\|_\alpha).$$

As M is compact, the Patterson densities and the Bowen–Margulis measure are doubling measures and, using discrete convolution approximation (see, for instance, [**Sem**, pp. 290–292] or [**KKST**]), there exist $\kappa' > 0$ and, for every $\eta > 0$, a non-negative function $\Phi_\eta \in \mathrm{C}^\alpha_{\mathrm{c}}(T^1M)$ such that:

- $\int_{T^1M}\Phi_\eta\, d\overline{m}_{\mathrm{BM}} = \int_{T^1M}\phi_\eta\, d\overline{m}_{\mathrm{BM}}$;
- $\int_{T^1M}|\Phi_\eta - \phi_\eta|\, d\overline{m}_{\mathrm{BM}} = \mathrm{O}(\eta \int_{T^1M}\phi_\eta\, d\overline{m}_{\mathrm{BM}})$;
- $\|\Phi_\eta\|_\alpha = \mathrm{O}(\eta^{-\kappa'}\int_{T^1M}\phi_\eta\, d\overline{m}_{\mathrm{BM}})$.

Hence, applying the exponential mixing of the geodesic flow, with $\kappa > 0$ as in its definition, since $\int_{T^1M}\phi_\eta\, dm_{\mathrm{BM}}$, which is equal to $\|\sigma_{\mathscr{C}}\|$ by Proposition 18, is independent of η, we have, for $\eta \in\,]0, \eta_0]$ and $t \in [t_0, +\infty[$,

$$\begin{aligned}
\frac{\int \psi\, d\sigma_{\mathscr{C}_t}}{\|\sigma_{\mathscr{C}_t}\|} &= \frac{\int_{T^1M}\Phi_\eta\psi\circ g^t\, d\overline{m}_{\mathrm{BM}}}{\int_{T^1M}\phi_\eta\, d\overline{m}_{\mathrm{BM}}} + \mathrm{O}(\eta\|\psi\|_\infty + (\eta + e^{-t})^\alpha\|\psi\|_\alpha)\\
&= \frac{\int_{T^1M}\Phi_\eta\, d\overline{m}_{\mathrm{BM}}}{\int_{T^1M}\phi_\eta\, d\overline{m}_{\mathrm{BM}}}\int_{T^1M}\psi\, d\overline{m}_{\mathrm{BM}}\\
&\quad + \mathrm{O}(e^{-\kappa t}\|\Phi_\eta\|_\alpha\|\psi\|_\alpha + \eta\|\psi\|_\infty + (\eta + e^{-t})^\alpha\|\psi\|_\alpha)\\
&= \int_{T^1M}\psi\, d\overline{m}_{\mathrm{BM}} + \mathrm{O}((e^{-\kappa t}\eta^{-\kappa'} + \eta + (\eta + e^{-t})^\alpha)\|\psi\|_\alpha).
\end{aligned}$$

Taking $\eta = e^{-t\lambda}$ for λ small enough (for instance, $\lambda = \kappa/(2\kappa')$), the result follows (for instance, with $\kappa'' = \min\{\kappa/2, \kappa/(2\kappa'), \alpha\min\{1, \kappa/(2\kappa')\}\}$), when $\Gamma\backslash I$ is finite. As the implied constants do not depend on the family $\mathscr{C}$, the result holds in general.

The proof of claim (ii) is similar. In this case, the strong stable foliation is smooth and the Bowen–Margulis measure coincides, up to a scalar multiple, with the Liouville measure. Thus, we can use the usual convolution approximation (see, for instance, [**Zie**, §1.6]) to approximate the test function by smooth functions.

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