Spiraling spectra of geodesic lines in negatively curved manifolds

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Abstract

Given a negatively curved geodesic metric space M, we study the asymptotic penetration behaviour of geodesic lines of M in small neighbourhoods of closed geodesics and of other compact convex subsets of M. We define a *spiraling spectrum* which gives precise information on the asymptotic spiraling lengths of geodesic lines around these objects. We prove analogs of the theorems of Dirichlet, Hall and Cusick in this context. As a consequence, we obtain Diophantine approximation results of elements of $\mathbb{R}, \mathbb{C}$ or the Heisenberg group by quadratic irrational ones. [1]

1 Introduction

Let M be a finite volume connected complete Riemannian manifold with dimension n at least 2 and sectional curvature at most -1. Let e be an end of M, and let C be a closed geodesic in M. One of the aims of this paper is to study the asymptotic spiraling behaviour of the (locally) geodesic lines in M starting from e around the closed geodesic C.

Just for the sake of normalization, fix a Margulis neighbourhood N of the cusp e in M (see for instance [BK]). Let $\mathrm{Lk}_N(M)$ be the set of geodesic lines starting from e that first meet ∂N at time 0, and do not converge to a cusp of M. Let d_N be the Hamenstädt distance on $\mathrm{Lk}_N(M)$ (see [HP2]), which is a natural distance inducing the compact-open topology on $\mathrm{Lk}_N(M)$, and which coincides with the induced Riemannian distance on the first intersection points with ∂N if N has constant curvature.

Let $\mathrm{Lk}_{N,C}(M)$ be the (countable, dense) set of elements ρ in $\mathrm{Lk}_N(M)$ that spiral indefinitely around C, that is such that $\lim_{t\to+\infty}\ d(\rho(t), C) = 0$. For every r in $\mathrm{Lk}_{N,C}(M)$, let $D(r)$ be the shortest length of a path between ∂N and C which is homotopic (while its endpoints stay in ∂N and C respectively), for any t big enough, to the path obtained by following r from $r(0)$ to $r(t)$, and then a shortest geodesic between $r(t)$ and its closest point on C. This number $D(r)$ naturally measures the wandering of r in M before r seriously starts to spiral indefinitely around C. See the end of Section 3 for explicit computations when M is locally symmetric.

We define the *spiraling constant* around C of $\xi \in \mathrm{Lk}_N(M)$ by

$$c(\xi) = \liminf_{r\in \mathrm{Lk}_{N,C}(M)\ ,\ D(r)\to+\infty} \quad e^{D(r)} d_N(\xi, r)\ ,$$

which measures how well ξ is approximated by geodesic lines spiraling indefinitely around C, and, when small, says that, asymptotically, ξ has long periods of time during which it

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spirals around C. We define the *spiraling spectrum* around C in M by

$$\mathrm{Sp}_{N,C}(M) = \big\{c(\xi) \;:\; \xi \in \mathrm{Lk}_N(M) - \mathrm{Lk}_{N,C}(M)\big\} \;.$$

Here is a sample of our results.

Theorem 1.1 (Dirichlet-type theorem) *The spiraling spectrum* $\mathrm{Sp}_{N,C}(M)$ *is a bounded subset of* $[0, +\infty[$.

Theorem 1.2 (Cusick-type and Hall-type theorem) *If M has constant curvature, then the spiraling spectrum* $\mathrm{Sp}_{N,C}(M)$ *is closed. If, in addition, the dimension of M is at least* 3*, then the spectrum contains an interval* $[0, c]$ *for some* $c > 0$.

When C is replaced by a cusp (and spiraling a long time around C is replaced by having a long excursion in a fixed cusp neighbourhood), the analogous results are motivated by Diophantine approximation results: see for instance [For, Coh, Pat, Sul, Ser, Haa, CF, Vul, Dal], as well as below in this introduction, and Remark 3.3. In that context, the boundedness of the spectrum was proved in [HP2, Theorem 1.1], the closedness of the spectrum was shown in [Mau], and the existence of a Hall ray was proved in [PP3, Theorem 1.6].

Although our arithmetic applications are going to be in the setting defined at the beginning of this introduction, our results are true in much more general situations (see the beginning of Section 3). In particular, M does not need to have a cusp (and for instance could be compact): we may replace e by a point x_0 in M, and then consider the geodesic rays starting from x_0. Or M could be allowed to have a compact totally geodesic boundary, and we may replace e by a connected component $\partial_0 M$ of ∂M, considering the geodesic rays starting from a point of $\partial_0 M$ perpendicularly to $\partial_0 M$. Furthermore, C can be replaced by a connected embedded totally geodesic submanifold of positive nonmaximal dimension, or by the convex core of a precisely invariant quasifuchsian subgroup (see for instance [MT] for definitions). The theorems 1.1 and 1.2 remain valid under certain more general hypotheses on M (see the theorems 4.4, 4.8 and Corollary 5.5 for statements). Section 5, where we prove the existence of Hall rays in spiraling spectra, relies on [PP3]. In Section 4.4, we also give upper bounds on the spiraling spectra in several classical examples.

To conclude this introduction, we give Diophantine approximation results which follow from the above theorems in Riemannian geometry. Recall that for $x \in \mathbb{R} - \mathbb{Q}$, the *approximation constant* of x by rational numbers is

$$c(x) = \liminf_{p,q \in \mathbb{Z},\; q \to +\infty} \; q^2 \Big| x - \frac{p}{q} \Big| ,$$

and that the *Lagrange spectrum* is $\mathrm{Sp}_{\mathbb{Q}} = \{c(\xi) \;:\; \xi \in \mathbb{R} - \mathbb{Q}\}$. Numerous properties of the Lagrange spectrum are known (see for instance [CF]). In particular, $\mathrm{Sp}_{\mathbb{Q}}$ is bounded (Dirichlet 1842), has maximum $\frac{1}{\sqrt{5}}$ (Korkine-Zolotareff 1873, Hurwitz 1891), is closed (Cusick 1975), contains a *Hall ray*, that is a maximal non trivial interval $[0, \mu]$ (Hall 1947), with $\mu = 491993569/(2221564096+283748\sqrt{468})$ (Freiman 1975). Also, recall Khintchine's result [Khi] saying that almost every real number is badly approximable by rational numbers. The following result, which is a quite particular case of the results of Section 6, gives analogous Diophantine approximation results of real numbers by (families of) quadratic irrational elements.

For every real quadratic irrational number α over $\mathbb{Q}$, let α^σ be its Galois conjugate. Let α_0 be a fixed real quadratic irrational number over $\mathbb{Q}$. Let $\mathscr{E}_{\alpha_0} = \mathrm{PSL}_2(\mathbb{Z}) \cdot \{\alpha_0, \alpha_0^\sigma\}$ be its (countable, dense in $\mathbb{R}$) orbit for the action by homographies and anti-homographies of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{R} \cup \{\infty\}$. For instance, if ϕ is the Golden Ratio $\frac{1+\sqrt{5}}{2}$, then $\mathscr{E}_\phi$ is the set of real numbers whose continued fraction expansion ends with an infinite string of 1's.

For every $x \in \mathbb{R} - (\mathbb{Q} \cup \mathscr{E}_{\alpha_0})$, define the *approximation constant* of x by elements of $\mathscr{E}_{\alpha_0}$, as

$$c_{\alpha_0}(x) = \liminf_{\alpha \in \mathscr{E}_{\alpha_0} \,:\, |\alpha - \alpha^\sigma| \to 0} \; 2\,\frac{|x-\alpha|}{|\alpha - \alpha^\sigma|}\;,$$

and the corresponding *approximation spectrum*, by

$$\mathrm{Sp}_{\alpha_0} = \{c_{\alpha_0}(x) \;:\; x \in \mathbb{R} - (\mathbb{Q} \cup \mathscr{E}_{\alpha_0})\}\;.$$

Theorem 1.3 *Let α_0 be a real quadratic irrational number over $\mathbb{Q}$. Then Sp_{α_0} is a closed bounded subset of $[0, +\infty[$.*

Furthermore, let $\psi : \,]0, +\infty[\, \to \,]0, +\infty[$ be a map such that $t \mapsto \log(\psi(e^{-t}))$ is Lipschitz. If $\int_0^1 \psi(t)/t^2\, dt$ diverges (resp. converges), then for Lebesgue almost all $x \in \mathbb{R}$,

$$\liminf_{\alpha \in \mathscr{E}_{\alpha_0} \,:\, |\alpha - \alpha^\sigma| \to 0} \;\frac{|x-\alpha|}{\psi(|\alpha - \alpha^\sigma|)} = 0 \;\;(\text{resp.} = +\infty)\;.$$

In this particular case, the last statement can be derived from [BV] or [DMPV]. Except for the following result, we do not know the exact value of the maximum K_{α_0} of Sp_{α_0} (an analog of Hurwitz's constant). We prove an upper bound $K_{\alpha_0} \le (1+\sqrt{2})\sqrt{3} \approx 4.19$ for any α_0, see Section 4.4.

Proposition 1.4 *For the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$, we have $K_\phi = 1 - 1/\sqrt{5} \approx 0.55$, and K_ϕ is not isolated in Sp_ϕ.*

There are many papers on the Diophantine approximation of real numbers by algebraic numbers. After pioneering work by Mahler, Koksma, Roth and Wirsing, the following Dirichlet type theorem has been proved by Davenport and Schmidt. Let $\mathbb{Q}_{\rm quad}$ be the set of real quadratic irrational numbers over $\mathbb{Q}$, and denote by $H(\alpha)$ the naive height of an algebraic number α (the maximal absolute value of the coefficients of its minimal polynomial over $\mathbb{Z}$). For every nonquadratic irrational real number x, Davenport and Schmidt [DS] proved that

$$\liminf_{\alpha \in \mathbb{Q} \cup \mathbb{Q}_{\rm quad} \,:\, H(\alpha) \to +\infty} \; H(\alpha)^3\,|x-\alpha| \;<\; +\infty\;.$$

Sprindžuk [Spr] proved that this result is generically optimal: For every $\epsilon > 0$, for Lebesgue almost every x in $\mathbb{R}$,

$$\liminf_{\alpha \in \mathbb{Q} \cup \mathbb{Q}_{\rm quad} \,:\, H(\alpha) \to +\infty} \; H(\alpha)^{3+\epsilon}\,|x-\alpha| \;= +\infty\;.$$

We refer to [Bug] and its impressive bibliography for further references. But note that none of the works that we know of is approximating by elements in the orbit under integral homographies of a given algebraic number; almost all of them are approximating using (a simple function of) the naive height as a complexity, but none using our complexity $h(\alpha) = 2/|\alpha - \alpha^\sigma|$. This complexity (see [PP4, Lem. 5.2] for an algebraic interpretation) behaves very differently from the naive height $H(\alpha)$, even in such an orbit, see Section 6.1.

In Section 6, expanding Theorem 1.3, we will give arithmetic applications analogous to the results of Dirichlet, Cusick, and Khintchine for the Diophantine approximation of points of $\mathbb{R}$ (resp. $\mathbb{C}$, the Heisenberg group $\mathrm{Heis}_{2n-1}(\mathbb{R})$) by classes of quadratic irrational elements over $\mathbb{Q}$ (resp. quadratic irrational elements over imaginary quadratic extensions of $\mathbb{Q}$, elements whose coefficients are rational or quadratic over an imaginary quadratic extension of $\mathbb{Q}$), and to Hall's result in $\mathbb{C}$ and $\mathrm{Heis}_{2n-1}(\mathbb{R})$.

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2 Preliminaries

Throughout the paper, (X, d) will be a proper CAT(-1) geodesic metric space, and $\partial_\infty X$ its boundary at infinity. We use [BH] as a general reference for this section. Unless otherwise stated, balls and horoballs are closed. If $\epsilon > 0$ and A is a subset of X, we denote by $\mathscr{N}_\epsilon A$ the (closed) ϵ-neighbourhood of A in X.

Let Γ be a discrete group of isometries of X. The limit set of Γ is denoted by $\Lambda\Gamma$. The conical limit set of Γ is denoted by $\Lambda_c\Gamma$. When $\Lambda\Gamma$ contains at least two points, the convex hull of $\Lambda\Gamma$ is denoted by $\mathscr{C}\Gamma$. The group Γ is *convex-cocompact* if $\Lambda\Gamma$ contains at least two points, and if the action of Γ on $\mathscr{C}\Gamma$ has compact quotient.

We will say that a subgroup H of a group G is *almost malnormal* if, for every g in $G - H$, the subgroup $gHg^{-1} \cap H$ is finite. We refer for instance to [HP5, Prop. 2.6] for a proof of the following well known result.

Proposition 2.1 *Let Γ_0 be a convex-cocompact subgroup of Γ. The following assertions are equivalent.*

(1) Γ_0 is almost malnormal in Γ;

(2) the limit set of Γ_0 is precisely invariant under Γ_0, that is for every $\gamma \in \Gamma - \Gamma_0$, the set $\Lambda\Gamma_0 \cap \gamma\Lambda\Gamma_0$ is empty;

(3) $\mathscr{C}\Gamma_0 \cap \gamma\mathscr{C}\Gamma_0$ is compact for every $\gamma \in \Gamma - \Gamma_0$;

(4) for every $\epsilon > 0$, there exists $\kappa = \kappa(\epsilon) > 0$ such that $\mathrm{diam}\big(\mathscr{N}_\epsilon\mathscr{C}\Gamma_0 \cap \gamma\mathscr{N}_\epsilon\mathscr{C}\Gamma_0\big) \leq \kappa$ for every $\gamma \in \Gamma - \Gamma_0$. □

For every ξ in $\partial_\infty X$, the *Busemann function at ξ* is the map β_ξ from $X \times X$ to $\mathbb{R}$ defined by

$$\beta_\xi(x, y) = \lim_{t\to+\infty} d(x, \xi_t) - d(y, \xi_t) ,$$

for any geodesic ray $t \mapsto \xi_t$ ending at ξ.

Let C be a nonempty closed convex subset of X. We denote by $\partial_\infty C$ its set of points at infinity, and by ∂C its boundary in X. The *closest point map* of C is the map $\pi_C : (X \cup \partial_\infty X) \to (C \cup \partial_\infty C)$ which associates to a point $x \in X$ its closest point in C in the usual sense, which fixes all points of $\partial_\infty C$, and which associates to a point $\xi \in \partial_\infty X - \partial_\infty C$ the point of C which minimizes the map $x \mapsto \beta_\xi(x, x_0)$ for any x_0 in X. This map is continuous.

As in [HP5], we define the *distance-like map* $d_C : \big(\partial_\infty X - \partial_\infty C\big)^2 \to [0, +\infty[$ associated to C as follows: For $\xi, \xi' \in \partial_\infty X - \partial_\infty C$, let $x = \pi_C(\xi), x' = \pi_C(\xi')$ be their closest points in C. Let $\xi_t, \xi'_t : [0, +\infty[\to X$ be the geodesic rays starting at x, x' and converging to ξ, ξ' as $t \to \infty$. Let

$$d_C(\xi, \xi') = \lim_{t \to +\infty} e^{\frac{1}{2} d(\xi_t, \xi'_t) - t}. \qquad (\text{-}\,1\,\text{-})$$

The distance-like map is invariant under the diagonal action of the isometries of X preserving C, and generalizes the visual and Hamenstädt distances: If C consists of a single point x, then d_C is the *visual distance* on $\partial_\infty X$ based at x (see for instance [Bou]), and we denote it by d_x. If C is a ball, then the distance-like map d_C is a positive constant multiple of the visual distance based at the center of C. If C is a horoball with point at infinity ξ_0, then d_C is the *Hamenstädt distance* on $\partial_\infty X - \{\xi_0\}$, and we also denote it by $d_{\xi_0, \partial C}$ to put the emphasis on ξ_0.

Although d_C is not always an actual distance on $\partial_\infty X - \partial_\infty C$, it follows from [HP5] that for every $\epsilon > 0$, there exists $\eta > 0$ such that for every ξ, ξ', ξ'' in $\partial_\infty X - \partial_\infty C$, if $d_C(\xi, \xi') < \eta$ and $d_C(\xi', \xi'') < \eta$, then $d_C(\xi, \xi'') < \epsilon$. Indeed, if $d_C(\xi, \xi')$ and $d_C(\xi', \xi'')$ are small, then by [HP5, Lemma 2.3 (4)], the geodesic lines $]\xi, \xi'[$ and $]\xi', \xi''[$ are far from C. By hyberbolicity of X, the geodesic line $]\xi, \xi'''[$ is also far from C. Hence $\pi_C(\xi)$ and $\pi_C(\xi'')$ are close. Therefore, by [HP5, Lemma 2.3 (3)], the value of $d_C(\xi, \xi'')$ is small. In particular, the family of subsets

$$\Big\{ W_n = \big\{ (\xi, \xi') \in \big(\partial_\infty X - \partial_\infty C\big)^2 \;:\; d_C(\xi, \xi') < \frac{1}{n+1} \big\} \Big\}_{n \in \mathbb{N}}$$

is a countable separating system of entourages of a metrisable uniform structure on $\partial_\infty X - \partial_\infty C$ (see [Bou, TG II.1]), whose induced topology is the usual one, by [HP5, Lemma 2.3 (1)], and which is invariant by the diagonal action of the isometries of X preserving C.

The *crossratio* of four pairwise distinct points $a, b, c, d \in \partial_\infty X$ is

$$[a, b, c, d] = \frac{1}{2} \lim_{t \to +\infty} d(a_t, c_t) - d(c_t, b_t) + d(b_t, d_t) - d(d_t, a_t), \qquad (\text{-}\,2\,\text{-})$$

where a_t, b_t, c_t, d_t are any geodesic rays converging to a, b, c, d, respectively. For the existence of the limit and the properties of the crossratios, see [Ota] where the order convention is different, and [Bou] whose crossratio is the exponential of ours; we will be using the same expression as in [HP1, PP3]. If $x_0 \in X$, then

$$[a, b, c, d] = \log \frac{d_{x_0}(c, a)}{d_{x_0}(c, b)} \frac{d_{x_0}(d, b)}{d_{x_0}(d, a)} .$$

If H is a horosphere with center $\xi \in \partial_\infty X$, then for $a, b, c, d \in \partial_\infty X - \{\xi\}$,

$$[a, b, c, d] = \log \frac{d_H(c, a)}{d_H(c, b)} \frac{d_H(d, b)}{d_H(d, a)} .$$

If ξ and a coincide, the above expression simplifies as follows, see [PP3, Section 3.1]:

$$[\xi, b, c, d] = \log \frac{d_H(d, b)}{d_H(c, b)} . \tag{-3-}$$

Let $\xi \in X \cup \partial_\infty X$. We say that a geodesic line $\rho :]-\infty, +\infty[\to X$ (resp. geodesic ray $\rho : [\iota_0, +\infty[\to X$) *starts* from ξ if $\xi = \rho(-\infty)$ (resp. $\xi = \rho(\iota_0)$). We denote by $T^1_\xi X$ the space of geodesic rays or lines starting from ξ, endowed with the compact-open topology.

In [PP3], the penetration of geodesic rays and lines in neighbourhoods of convex subsets of X was studied by means of penetration maps. We now recall the definitions of three classes of such maps $\ell_C, \mathfrak{crp}_L, \mathfrak{ftp}_L : T^1_\xi X \to [0, +\infty]$, where C is the (closed) ϵ-neighbourhood of a closed convex subset in X for some $\epsilon > 0$, and L is a geodesic line in X, with endpoints L_1, L_2.

(1) The *penetration length map* ℓ_C associates to every ρ in $T^1_\xi X$ the length of the intersection of C and of the image of ρ. (This intersection is connected by convexity; there was the assumption in [PP3] that $\xi \notin C \cup \partial_\infty C$, which is not necessary here.)

(2) The *fellow-traveller penetration map* $\mathfrak{ftp}_L$ is defined by

$$\mathfrak{ftp}_L : \rho \mapsto d\big(\pi_L(\xi), \pi_L(\rho(+\infty))\big) ,$$

with the convention that this distance is $+\infty$ if $\pi_L(\xi)$ or $\pi_L(\rho(+\infty))$ is in $\partial_\infty X$ (there was the assumption in [PP3] that $\xi \notin \mathscr{N}_\epsilon L \cup \partial_\infty L$ where $\epsilon > 0$ was arbitrary but fixed, which is not necessary here).

(3) When $\xi \in \partial_\infty X$, the *crossratio penetration map* $\mathfrak{crp}_L$ is defined by

$$\mathfrak{crp}_L : \rho \mapsto \max\big\{0, [\xi, L_1, \rho(+\infty), L_2], [\xi, L_2, \rho(+\infty), L_1]\big\} ,$$

if $\xi, \rho(+\infty) \notin \{L_1, L_2\}$, and $\mathfrak{crp}_L(\rho) = +\infty$ otherwise (there was the assumption in [PP3] that $\xi \notin \partial_\infty L$, which is not necessary here).

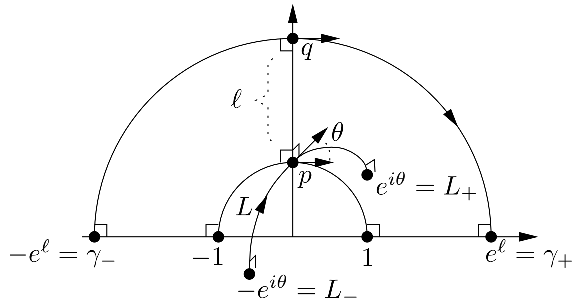
It is shown in Section 3.1 of [PP3] (and it is easy to see that the result is still true if ξ belongs respectively to $C \cup \partial_\infty C$, $\mathscr{N}_\epsilon L \cup \partial_\infty L$, $\partial_\infty L$) that the above maps are continuous and that, for every $\epsilon > 0$, we have, with the convention that $x - y = 0$ if $x = y = +\infty$, the following inequalities

$$||\mathfrak{ftp}_L - \ell_{\mathscr{N}_\epsilon L}||_\infty = \sup_{\rho \in T^1_\xi X} |\mathfrak{ftp}_L(\rho) - \ell_{\mathscr{N}_\epsilon L}(\rho)| \le 2\, c'_1(\epsilon) + 2\epsilon , \tag{-4-}$$

where $c'_1(\epsilon) = 2 \operatorname{argsinh}(\coth \epsilon)$, and when $\xi \in \partial_\infty X$,

$$||\mathfrak{crp}_L - \mathfrak{ftp}_L||_\infty = \sup_{\rho \in T^1_\xi X} |\mathfrak{crp}_L(\rho) - \mathfrak{ftp}_L(\rho)| \le 4 \log(1 + \sqrt{2}) . \tag{-5-}$$

In constant curvature, the crossratio penetration map has the following geometric interpretation (see for instance [Bea, §7.23–7.24], and [Fen, §V.3], for related formulas). Recall that the *complex distance* $\ell + i\theta$ between two oriented geodesic lines γ and L (in this order) in $\mathbb{H}^n_\mathbb{R}$ is defined as follows. It is $0 + i0$ if they are simultaneously asymptotic at $+\infty$ or at $-\infty$, and $0 + i\pi$ if the terminal point at infinity of one is the original point at infinity of the other. Otherwise, if $[p, q]$ is the common perpendicular arc (with $p = q$ the common intersection point of γ and L if they intersect), where $p \in L$, then $\ell = d(p, q)$ and θ is the angle at p between the parallel transport of γ along $[p, q]$ and L.



Lemma 2.2 *Let γ and L be oriented geodesic lines in $\mathbb{H}^n_{\mathbb{R}}$ with pairwise distinct endpoints γ_-, γ_+ and L_-, L_+, respectively, and with complex distance $\ell + i\theta$. Then*

$$[\gamma_-, L_-, \gamma_+, L_+] = -\log\ \frac{\cosh \ell + \cos\theta}{2}\ .$$

In particular,

$$\mathfrak{crp}_L(\gamma) = \max\big\{0, -\log\ \frac{\cosh\ell \pm \cos\theta}{2}\big\}\ .$$

Proof. Using isometries, we may assume that γ and L are both contained in the upper halfspace model of $\mathbb{H}^3_{\mathbb{R}}$, that the common perpendicular segment $[p, q]$ is on the vertical axis, with p at Euclidean height one and q above p, and that γ_+ is a positive real number. By an easy computation, we then have

$$[\gamma_-, L_-, \gamma_+, L_+] = \log\ \frac{|\gamma_+ - \gamma_-|\ |L_+ - L_-|}{|\gamma_+ - L_-|\ |L_+ - \gamma_-|} = \log\ \frac{4e^{\ell}}{|e^{\ell} + e^{i\theta}|^2} = -\log\ \frac{\cosh\ell + \cos\theta}{2}\ .$$

The result follows. □

Note that in $\mathbb{H}^2_{\mathbb{R}}$, we have $\ell > 0$ if and only if $\theta = 0$ or $\theta = \pi$.

3 The approximation and spiraling spectra

In this section, we set up the general framework for our approximation results. We begin by the definition of the quadruples of data that we study.

The definition of $\mathscr{D}$. *Let Γ be a discrete group of isometries of a proper* CAT(-1) *geodesic metric space X. Let Γ_0 be an almost malnormal convex-cocompact subgroup of infinite index in Γ and let $C_0 = \mathscr{C}\Gamma_0$ be the convex hull of Γ_0. Let C_∞ be a nonempty closed convex subset of X, with stabilizer Γ_∞ in Γ. Assume that C_∞ does not contain $\mathscr{C}\Gamma$, that $\Gamma_\infty \backslash \partial C_\infty$ is compact, and that the intersection of C_∞ and γC_0 is nonempty for only finitely many classes $[\gamma]$ in $\Gamma_\infty\backslash\Gamma/\Gamma_0$. We will denote the quadruple of data $(X, \Gamma, \Gamma_0, C_\infty)$ by $\mathscr{D}$.*

Since $\Lambda\Gamma_0$ has at least two points and since Γ_0 has infinite index in Γ, note that Γ is nonelementary. By Proposition 2.1 (3) and since C_0 is noncompact, the subgroup Γ_0 is the stabilizer of C_0 in Γ. By the discreteness of Γ and the cocompactness of Γ_0 on C_0, a compact subset of X intersects only finitely many γC_0 for $\gamma \in \Gamma/\Gamma_0$.

Recall that the *distance* between two subsets A, B of X is $d(A, B) = \inf_{a\in A, b\in B} d(a, b)$. For every $r = [\gamma]$ in $\Gamma_\infty\backslash\Gamma/\Gamma_0$, define

$$D(r) = d(C_\infty, \gamma C_0)\ ,$$

which does not depend on the choice of the representative γ of r. By the cocompactness of the action of Γ_∞ on ∂C_∞ and the fact that only finitely many translates of C_0 meet a given compact subset, the intersection $C_\infty \cap \gamma C_0$ is empty if and only if $D(r) > 0$. By convexity, this condition implies that $\partial_\infty C_\infty \cap \gamma \partial_\infty C_0$ is also empty. For the same reasons, the following result holds, see [HP5, Lemma 4.1] for a proof in the case $C_\infty = C_0$.

Lemma 3.1 *For every $T \geq 0$, there are only finitely many elements r in $\Gamma_\infty \backslash \Gamma / \Gamma_0$ such that $D(r) \leq T$.* □

Lemma 3.2 *The set of double cosets $\Gamma_\infty \backslash \Gamma / \Gamma_0$ is infinite.*

Proof. We first claim that there exists a hyperbolic element γ in Γ whose attractive fixed point γ_+ does not belong to $\partial_\infty C_\infty$. The limit set $\Lambda\Gamma$ is the closure of the set of attractive fixed points of elements of Γ, since Γ is nonelementary. Thus, if no such γ_+ exists, $\Lambda\Gamma$ is contained in $\partial_\infty C_\infty$, which contradicts the hypotheses on $\mathscr{D}$ by the convexity of C_∞.

By Lemma 2.1 (2), and since $\Gamma \neq \Gamma_0$, we have $\bigcap_{\gamma' \in \Gamma} \gamma' \Lambda \Gamma_0 = \emptyset$, hence there exists $\gamma' \in \Gamma$ such that the repulsive fixed point γ_- of γ does not belong to $\gamma' \Lambda \Gamma_0$. Hence the sequence of closed subsets $(\gamma^n \gamma' C_0)_{n \in \mathbb{N}}$ of the compact space $X \cup \partial_\infty X$ converges to the singleton $\{\gamma_+\}$ as n goes to $+\infty$. This implies that $D([\gamma^n \gamma']) = d(C_\infty, \gamma^n \gamma' C_0)$ converges to $+\infty$ as n goes to $+\infty$. In particular, the set $\{D(r) : r \in \Gamma_\infty \backslash \Gamma / \Gamma_0\}$ is infinite, and the result follows. □

The *link* of $\mathscr{D}$ (which depends only on X, Γ and C_∞) is

$$\mathrm{Lk}_\infty = \Gamma_\infty \backslash (\Lambda_c \Gamma - \partial_\infty C_\infty).$$

The quotient space $\Gamma_\infty \backslash (\Lambda\Gamma - \partial_\infty C_\infty)$, which contains Lk_∞, is compact, since the closest point map from $\partial_\infty X - \partial_\infty C_\infty$ to ∂C_∞ is continuous and Γ_∞-equivariant. Furthermore, Lk_∞ is dense in $\Gamma_\infty \backslash (\Lambda\Gamma - \partial_\infty C_\infty)$. For every $r = [\gamma]$ in $\Gamma_\infty \backslash \Gamma / \Gamma_0$ such that $D(r) > 0$, let

$$\Lambda_r = \pi_\infty(\gamma \partial_\infty C_0)$$

be the image by the canonical projection $\pi_\infty : \Lambda_c \Gamma - \partial_\infty C_\infty \to \mathrm{Lk}_\infty$ of $\gamma \partial_\infty C_0 = \gamma \Lambda \Gamma_0$. Note that $\gamma \partial_\infty C_0$ is indeed contained in $\Lambda_c \Gamma$ since Γ_0 is convex-cocompact, and that $\partial_\infty C_\infty$ is disjoint from $\gamma \partial_\infty C_0$ if $D(r) > 0$, as seen before. Furthermore, the sets Λ_r are compact subsets of Lk_∞, that are pairwise disjoint by Lemma 2.1 (2), and the union

$$\mathrm{Lk}_{\infty,0} = \bigsqcup_{r \in \Gamma_\infty \backslash \Gamma / \Gamma_0 \,,\, D(r) > 0} \Lambda_r \qquad (\text{-}6\text{-})$$

is dense in Lk_∞. In this paper, we study how dense $\mathrm{Lk}_{\infty,0}$ is in Lk_∞.

Let $\widetilde{d}_\infty : (\partial_\infty X - \partial_\infty C_\infty)^2 \to [0, +\infty[$ be the distance-like map associated to C_∞, and let d_∞ be its quotient map on Lk_∞, which defines, as in Section 2, a metrisable uniform structure on $\Gamma_\infty \backslash (\partial_\infty X - \partial_\infty C_\infty)$, inducing the quotient topology. We endow the double coset space $\Gamma_\infty \backslash \Gamma / \Gamma_0$ with the Fréchet filter of the complements of the finite subsets, and denote by $\liminf_r f(r)$ the lower limit of a real valued map f along this filter. The *approximation constant* of $\xi \in \mathrm{Lk}_\infty - \mathrm{Lk}_{\infty,0}$ is

$$c(\xi) = \liminf_r \; e^{D(r)} d_\infty(\xi, \Lambda_r) \,, \qquad (\text{-}7\text{-})$$

and the subset of $[0,+\infty]$ defined by

$$\mathrm{Sp}(\mathscr{D}) \;=\big\{c(\xi)\;:\;\xi\in \mathrm{Lk}_\infty-\mathrm{Lk}_{\infty,0}\big\}$$

is called the *approximation spectrum* of points of Lk_∞ by points of $\mathrm{Lk}_{\infty,0}$. We define the *Hurwitz constant* of $\mathscr{D}$ as

$$K_{\mathscr{D}}=\sup\,\mathrm{Sp}(\mathscr{D})\;\in[0,+\infty]\;.$$

Remark 3.3 Let us give some background and motivations for the terminology introduced in this paper: In the definition of the quadruple of data $\mathscr{D}$, let us specialise to the situation when X is a Riemannian manifold with pinched negative curvature and Γ is geometrically finite. If we change the assumptions on Γ_0 and Γ_∞ such that $\Gamma_0=\Gamma_\infty$ is the stabilizer of a parabolic fixed point ξ_∞ of Γ and C_∞ is the maximal precisely invariant horoball centered at ξ_∞, then we recover the framework of Diophantine approximation in negatively curved manifolds that was developped in [HP2, HP3, HP4, PP3, PP1]. In this situation, Γ_0 is not convex-cocompact and $C_0=C_\infty$, and the new quadruple does not have the properties we require of the quadruples of data in this paper. However, if we take $\mathrm{Lk}_\infty=\Gamma_\infty\backslash\Lambda_c\Gamma$, $\Lambda_{[\gamma]}=\Gamma_\infty\gamma(+\infty)$, $\mathrm{Lk}_{\infty,0}=\pi_\infty(\Gamma\cdot\infty)$, all the constructions in Section 3 are still valid.

In particular, let X be the upper halfplane model of the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$, let $\Gamma=\mathrm{PSL}_2(\mathbb{Z})$, let C_∞ be the horoball in X of points having Euclidean height at least 1, let $\Gamma_0=\Gamma_\infty$ be the cyclic group generated by $z\mapsto z+1$, and let $\mathscr{D}=(X,\Gamma,\Gamma_0,\Gamma_\infty)$. Then (see [HP2, section 2.3], [PP1]) $\mathrm{Lk}_\infty=(\mathbb{R}-\mathbb{Q})/\mathbb{Z}$; for every $r=[\gamma]\in\Gamma_\infty\backslash(\Gamma-\Gamma_\infty)/\Gamma_\infty$, we have $D(r)=2\log q$ if $\gamma\infty=p/q$ with $p\in\mathbb{Z}$ and $q\in\mathbb{N}-\{0\}$ relatively prime; for every $\xi\in\mathbb{R}-\mathbb{Q}$, the approximation constant $c(\xi \mod \mathbb{Z})$ is the classical approximation constant of the irrational number ξ by rational numbers; the approximation spectrum $\mathrm{Sp}(\mathscr{D})$ is the classical Lagrange spectrum, and the Hurwitz constant $K_{\mathscr{D}}$ is the classical Hurwitz constant $\frac{1}{\sqrt{5}}$ (see the introduction for the definition of these objects).

The end of this section is devoted to the study of geometric examples.

Let M be a nonelementary complete connected Riemannian manifold with sectional curvature at most -1, and dimension at least 2. Let A_0 be a closed geodesic in M, not necessarily simple (for more general A_0's, as for instance in the introduction, we refer to the general setup). Let A_∞ be a closed codimension 0 submanifold of M with smooth connected compact locally convex boundary, disjoint from A_0.

Recall that a locally geodesic ray ρ in M is *recurrent* if, as a map from $[0,+\infty[$ to M, it is not proper, that is if there exist a compact subset K of M and a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,+\infty[$ converging to $+\infty$ such that $\rho(t_n)\in K$ for every n. We say that a locally geodesic ray ρ in M *spirals around* A_0 if $d(\rho(t),A_0)$ converges to 0 as t goes to $+\infty$.

Let $\mathrm{Lk}_{A_\infty}(M)$ be the set of recurrent locally geodesic rays starting perpendicularly from ∂A_∞ and exiting A_∞, and let $\mathrm{Lk}_{A_\infty,A_0}(M)$ be the subset of elements of $\mathrm{Lk}_{A_\infty}(M)$ that spiral around A_0. Recall that a geodesic line in a complete simply connected manifold that crosses a horosphere perpendicularly starts from (up to time reversal) the point at infinity of this horosphere. Hence when A_∞ is a small Margulis neighbourhood N_∞ of a cusp e_∞ with compact boundary, it is equivalent to require that a geodesic ray exits perpendicularly from A_∞ or that the negative subray of the geodesic line containing it is a minimizing geodesic ray starting from the boundary of N_∞ and converging to e_∞.

For every ρ,ρ' in $\mathrm{Lk}_{A_\infty}(M)$, and every $t\in[0+\infty[$, let ℓ_t be the shortest length of a path homotopic (relative to the endpoints) to the path obtained by following (the inverse of) ρ from $\rho(t)$ to $\rho(0)$, then following a shortest path contained in A_∞ between $\rho(0)$ and $\rho'(0)$, then following ρ' from $\rho'(0)$ to $\rho'(t)$; define

$$\delta_{A_\infty}(\rho,\rho') = \lim_{t\to+\infty} e^{\frac{1}{2}\ell_t - t}\ . \tag{-8-}$$

(We will show below that the limit does exist). For every $\bar r$ in $\mathrm{Lk}_{A_\infty,A_0}(M)$, let $D(\bar r)$ be the shortest length of a geodesic segment $[a,b]$ with a in ∂A_∞, b in A_0 such that there exists a (locally) geodesic ray ρ starting from b, contained in A_0, such that the path obtained by following $[a,b]$ from a to b and then ρ is properly homotopic to $\bar r$ while its origin remains in ∂A_∞. The *spiraling constant* around A_0 of an element $\bar\xi$ of $\mathrm{Lk}_{A_\infty}(M)$ is

$$c(\bar\xi) = \liminf_{\bar r\in \mathrm{Lk}_{A_\infty,A_0}(M)\ ,\ D(\bar r)\to+\infty} \ e^{D(\bar r)}\delta_{A_\infty}(\bar\xi,\bar r)\ ,$$

and the subset of $[0,+\infty]$ defined by

$$\mathrm{Sp}_{A_\infty,A_0}(M) = \big\{c(\bar\xi)\ :\ \bar\xi\in \mathrm{Lk}_{A_\infty}(M) - \mathrm{Lk}_{A_\infty,A_0}(M)\big\}$$

is called the *spiraling spectrum* of geodesic rays in $\mathrm{Lk}_{A_\infty}(M)$ around A_0. These notions coincide with the similarly named ones in the introduction if M has finite volume and A_∞ is the chosen Margulis neighbourhood of the cusp e.

To see the connection with the framework outlined at the beginning of this section, we may define a quadruple of data

$$\mathscr{D}_{M,A_0,A_\infty} = (X,\Gamma,\Gamma_0,C_\infty)$$

as follows. If $\widetilde M\to M$ is a universal Riemannian covering of M with covering group Γ, let $X=\mathscr{C}\Gamma$ be the convex hull of Γ, let Γ_0 be the stabilizer in Γ of a fixed lift C_0 of A_0 to $\widetilde M$, and let C_∞ be the intersection with X of a fixed connected component $\widetilde A_\infty$ of the preimage of A_∞ in $\widetilde M$.

Note that the image in M of a geodesic ray ρ in $\widetilde M$ is recurrent if and only if the endpoint at infinity of ρ is a conical limit point of Γ. Consider the map $\widetilde\Phi$ from $\Lambda_c\Gamma - \partial_\infty C_\infty$ to $\mathrm{Lk}_{A_\infty}(M)$, which associates to an element ξ of $\Lambda_c\Gamma - \partial_\infty C_\infty$ the image in M of the geodesic ray in X starting from the closest point on $\widetilde A_\infty$ to ξ and converging to ξ. By taking the quotient by Γ_∞, this map induces a homeomorphism $\Phi:\mathrm{Lk}_\infty\to \mathrm{Lk}_{A_\infty}(M)$, which maps $\mathrm{Lk}_{\infty,0}$ to $\mathrm{Lk}_{A_\infty,A_0}(M)$. By construction, the map Φ preserves the maps d_∞ and δ_{A_∞} (which proves along the way that the limit in (-8-) exists).

For every $\bar r$ in $\mathrm{Lk}_{A_\infty,A_0}(M)$, by definition of $\mathrm{Lk}_{\infty,0}$ (see Equation (-6-)), there exists a unique r in $\Gamma_\infty\backslash\Gamma/\Gamma_0$ such that $\Phi^{-1}(\bar r)$ belongs to Λ_r. The map $\bar r\mapsto r$ from $\mathrm{Lk}_{A_\infty,A_0}(M)$ to $\Gamma_\infty\backslash\Gamma/\Gamma_0$ satisfies $D(r)=D(\bar r)$ if $D(r)>0$; the complementary subset in $\Gamma_\infty\backslash\Gamma/\Gamma_0$ of its image is finite, since there are only finitely many $r\in\Gamma_\infty\backslash\Gamma/\Gamma_0$ such that $D(r)\le 0$; every point in its image has at most two preimages, since Λ_r has at most two points.

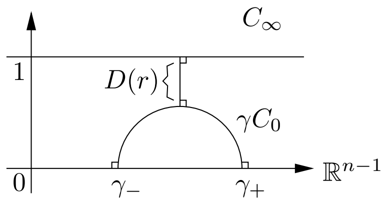
Hence, for every ξ in Lk_∞, we have by construction $c(\xi)=c(\Phi(\xi))$. Therefore, as Φ is surjective,

$$\mathrm{Sp}(\mathscr{D}_{M,A_0,A_\infty}) = \mathrm{Sp}_{A_\infty,A_0}(M)\ , \tag{-9-}$$

and we conclude that to obtain results on the spiraling spectrum, it is sufficient to prove results on the approximation spectrum.

Example 1: Spiraling around a closed geodesic in a real hyperbolic manifold

We will use the upper halfspace model of the real hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$, with constant sectional curvature -1, so that $\partial_\infty \mathbb{H}^n_{\mathbb{R}} = \mathbb{R}^{n-1} \cup \{\infty\}$. Let Γ be a nonelementary discrete subgroup of isometries of $\mathbb{H}^n_{\mathbb{R}}$. Assume that ∞ is a parabolic fixed point of Γ, with stabilizer Γ_∞, and that the interior of the horoball $\mathscr{H}_1$ of points of Euclidean height at least one is *precisely invariant* under Γ_∞ (that is for all $\gamma \in \Gamma - \Gamma_\infty$, the horoballs $\mathscr{H}_1$ and $\gamma\mathscr{H}_1$ have disjoint interior). When Γ is torsion-free and has finite covolume, $\mathscr{H}_1$ covers a Margulis neighbourhood of a cusp of $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$. Define $C_\infty = \mathscr{H}_1 \cap \mathscr{C}\Gamma$, and assume that $\Gamma_\infty\backslash\partial C_\infty$ is compact (for instance if Γ is geometrically finite, and in particular if Γ has finite covolume). Let γ_0 be a hyperbolic element of Γ, with translation axis C_0. Let Γ_0 be the stabilizer of C_0 in Γ, which contains the cyclic group generated by γ_0 as a subgroup of finite index. The quadruple $\mathscr{D} = (\mathscr{C}\Gamma, \Gamma, \Gamma_0, C_\infty)$ satisfies the hypotheses of the beginning of the section.



For every γ in Γ, let $\gamma_\pm$ be the fixed points of the hyperbolic element $\gamma\gamma_0\gamma^{-1}$. If r is the double coset of γ in $\Gamma_\infty\backslash\Gamma/\Gamma_0$, and if $D(r) = d(C_\infty, \gamma C_0) > 0$, then by an easy computation in hyperbolic geometry

$$D(r) = -\log \frac{1}{2}\|\gamma_+ - \gamma_-\| \,, \qquad (\text{-}\,10\,\text{-})$$

where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^{n-1}$.

Let $\mathscr{R}_{\Gamma_0}$ be the set of fixed points of the conjugates of γ_0, endowed with its Fréchet filter. For every $\alpha \in \mathscr{R}_{\Gamma_0}$, let α^* be the other endpoint of the translation axis of a conjugate of γ_0 containing α at infinity. The distance-like map d_{C_∞} coincides with the Hamenstädt distance $d_{\infty,\partial\mathscr{H}_1}$ on the limit set $\Lambda\Gamma$. In $\mathbb{R}^{n-1} = \partial_\infty\mathbb{H}^n_{\mathbb{R}} - \{\infty\}$, the Hamenstädt distance $d_{\infty,\partial\mathscr{H}_1}$ coincides with the Euclidean metric (see for instance [HP2]). For every ξ in $\Lambda_c\Gamma - \mathscr{R}_{\Gamma_0}$ and $\alpha \in \mathscr{R}_{\Gamma_0}$, define $\ell_\alpha + i\theta_\alpha$ to be the complex distance between the oriented geodesic lines from ∞ to ξ and from α^* to α. Note that $\ell_{\alpha^*} = \ell_\alpha$ and $\theta_{\alpha^*} = \theta_\alpha + \pi$. Then, we have by Equation (-10-), by Equation (-3-) and by Lemma 2.2, respectively,

$$c(\Gamma_\infty\xi) = \liminf_{\alpha\in\mathscr{R}_{\Gamma_0}} \; 2\,\frac{\|\xi-\alpha\|}{\|\alpha-\alpha^*\|} = \liminf_{\alpha\in\mathscr{R}_{\Gamma_0}} \; 2\,e^{-[\infty,\alpha,\xi,\alpha^*]} = \liminf_{\alpha\in\mathscr{R}_{\Gamma_0}} \; \big(\cosh\ell_\alpha - \cos\theta_\alpha\big)\,. \quad (\text{-}\,11\,\text{-})$$

Furthermore, by definition, $\mathrm{Sp}(\mathscr{D}) = \big\{\, c(\xi) \;:\; \xi \in \Gamma_\infty\backslash(\Lambda_c\Gamma - \mathscr{R}_{\Gamma_0})\big\}$.

Example 2: Spiraling around a closed geodesic in a complex hyperbolic manifold

Let $n \geq 2$. The elements of $\mathbb{C}^{n-1}$ are identified with their coordinate column vectors and for every w, w' in $\mathbb{C}^{n-1}$, we denote by w^*w' their standard Hermitian product, where w^* is the conjugate transpose of w, and $|w|^2 = w^*w$.

Let $\mathbb{H}^n_{\mathbb{C}}$ be the Siegel domain model of the complex hyperbolic n-space. Its underlying manifold is

$$\mathbb{H}^n_{\mathbb{C}} = \big\{(w_0, w) \in \mathbb{C}\times\mathbb{C}^{n-1} \;:\; 2\,\mathrm{Re}\; w_0 - |w|^2 > 0\big\}\,.$$

The complex hyperbolic distance $d_{\mathbb{H}^n_{\mathbb{C}}}$ is defined by the Riemannian metric

$$ds^2 = \frac{4}{(2\,\mathrm{Re}\; w_0 - |w|^2)^2}\big((dw_0 - dw^*\,w)(\overline{dw_0} - w^*\,dw) + (2\,\mathrm{Re}\;w_0 - |w|^2)\,dw^*\,dw\big)$$

(see for instance [Gol, Sect. 4.1]). The complex hyperbolic space has constant holomorphic sectional curvature -1, hence its real sectional curvatures are bounded between -1 and $-\frac{1}{4}$.

When we want to consider $\mathbb{H}^n_{\mathbb{C}}$ as a CAT(-1) space, we will use the distance $d'_{\mathbb{H}^n_{\mathbb{C}}} = \frac{1}{2} d_{\mathbb{H}^n_{\mathbb{C}}}$. The boundary at infinity of $\mathbb{H}^n_{\mathbb{C}}$ is

$$\partial_\infty \mathbb{H}^n_{\mathbb{C}} = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \;:\; 2\,\mathrm{Re}\, w_0 - |w|^2 = 0\} \cup \{\infty\}\;.$$

The horoballs centered at ∞ in $\mathbb{H}^n_{\mathbb{C}}$ are the subspaces

$$\mathscr{H}_s = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \;:\; 2\,\mathrm{Re}\, w_0 - |w|^2 \geq s\}\;,$$

for $s > 0$. The submanifold $\{(w_0, w) \in \mathbb{H}^n_{\mathbb{C}} \;:\; w = 0\}$, with the induced Riemannian metric, is the right halfplane model of the real hyperbolic plane with constant curvature -1, and it is totally geodesic in $\mathbb{H}^n_{\mathbb{C}}$. Hence the map $c_0 : \mathbb{R} \to \mathbb{H}^n_{\mathbb{C}}$ defined by $c_0 : t \mapsto (e^{-t}, 0)$ is a unit speed geodesic line for $d_{\mathbb{H}^n_{\mathbb{C}}}$, starting from ∞, ending at $(0,0) \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ and meeting the horosphere $\partial \mathscr{H}_2$ at time $t = 0$. In particular, the distance between two horospheres centered at ∞ is

$$d_{\mathbb{H}^n_{\mathbb{C}}}(\partial \mathscr{H}_s, \partial \mathscr{H}_{s'}) = |\log(s'/s)|. \qquad (\text{-}\,12\,\text{-})$$

Let $G_{\mathscr{H}_2}$ be the group of isometries of $\mathbb{H}^n_{\mathbb{C}}$ preserving (globally) $\mathscr{H}_2$. The *Cygan distance* $d_{\rm Cyg}$ (see for instance [Gol, page 160]) is the unique distance on $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ invariant under $G_{\mathscr{H}_2}$ such that

$$d_{\rm Cyg}((w_0, w), (0,0)) = \sqrt{2|w_0|}.$$

Similarly, we introduced in [PP3, Lem. 6.1] the *modified Cygan distance* $d'_{\rm Cyg}$, as the unique distance on $\partial_\infty \mathbb{H}^n_{\mathbb{C}} - \{\infty\}$ invariant under $G_{\mathscr{H}_2}$ such that

$$d'_{\rm Cyg}((w_0, w), (0,0)) = \sqrt{2|w_0| + |w|^2}.$$

Let Γ be a discrete subgroup of isometries of $\mathbb{H}^n_{\mathbb{C}}$ with finite covolume. Assume that ∞ is a parabolic fixed point, whose stabilizer in Γ we denote by Γ_∞, such that the horoball $\mathscr{H}_2$ is precisely invariant under Γ_∞. Let γ_0 be a hyperbolic element of Γ, with translation axis C_0. Let Γ_0 be the stabilizer of C_0 in Γ. The quadruple $\mathscr{D} = (\mathbb{H}^n_{\mathbb{C}}, \Gamma, \Gamma_0, \mathscr{H}_2)$ satisfies the hypotheses of the beginning of the section.

In the following result, we compute the associated map $D : \Gamma_\infty \backslash \Gamma / \Gamma_0 \to \mathbb{R}$ where $D([\gamma]) = d'_{\mathbb{H}^n_{\mathbb{C}}}(\mathscr{H}_2, \gamma C_0)$.

Lemma 3.4 *If $[\gamma] \in \Gamma_\infty \backslash \Gamma / \Gamma_0$ and $D([\gamma]) > 0$ then, with $\gamma_\pm$ the fixed points of the hyperbolic element $\gamma\gamma_0\gamma^{-1}$, we have*

$$D([\gamma]) = -\log \frac{1}{2} \frac{d_{\rm Cyg}(\gamma_-, \gamma_+)^2}{d'_{\rm Cyg}(\gamma_-, \gamma_+)}\;.$$

Proof. Identify $\mathbb{H}^n_{\mathbb{C}} \cup \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ with its image in the projective space $\mathbb{P}_n(\mathbb{C})$ by $(w_0, w) \mapsto [w_0 : w : 1]$ and $\infty \mapsto [1 : 0 : 0]$. Note that γC_0 is the geodesic line between γ_- and γ_+. By invariance under $G_{\mathscr{H}_2}$, we may assume that $\gamma_- = (0,0)$ and $\gamma_+ = (w_0, w) \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ such that $w_0 \neq 0$. The projective action of $g = \begin{pmatrix} 1 & 0 & 0 \\ \frac{w}{w_0} & \mathrm{Id} & 0 \\ \frac{1}{w_0} & \frac{w^*}{\overline{w_0}} & 1 \end{pmatrix}$ is an isometry of (the image of) the Siegel domain, fixing the point $(0,0)$ of $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$, and mapping ∞ to (w_0, w), hence sending the geodesic line between $(0,0)$ and ∞ to the one between $(0,0)$ and (w_0, w). Therefore the map $\gamma_{w_0,w} : \mathbb{R} \to \mathbb{H}^n_{\mathbb{C}}$ defined by

$$\gamma_{w_0,w}(t) = \Big(\frac{w_0}{1+w_0 e^t}, \frac{w}{1+w_0 e^t}\Big)$$

is a geodesic line with endpoints (w_0, w) and $(0,0)$ in $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$. The point $\gamma_{w_0,w}(t)$ belongs to the horosphere $\partial\mathscr{H}_{s(t)}$, where

$$s(t) = 2\mathrm{Re}\big(\frac{w_0}{1+w_0 e^t}\big) - \big|\frac{w}{1+w_0 e^t}\big|^2 = \frac{2\mathrm{Re}(w_0(1+\overline{w}_0 e^t)) - |w|^2}{|1+e^t w_0|^2} = \frac{2e^t |w_0|^2}{|1+e^t w_0|^2}.$$

If $w_0 = re^{i\phi}$ (in polar coordinates) and if $T = e^t$, then

$$s(t) = \frac{2Tr^2}{T^2 r^2 + 2Tr\cos\phi + 1}\;.$$

The map $t \mapsto s(t)$ reaches its maximum at $T = 1/r$, that is at $t = -\log|w_0|$, and its maximum value is

$$s = \frac{|w_0|}{1 + \frac{\mathrm{Re}\, w_0}{|w_0|}} = \frac{|w_0|^2}{|w_0| + |w|^2/2} = \frac{1}{2}\frac{d_{\mathrm{Cyg}}((0,0),(w_0,w))^4}{d'_{\mathrm{Cyg}}((0,0),(w_0,w))^2}\;.$$

The result then follows from Equation (-12-), since $\mathscr{H}_2$ and γC_0 are disjoint if and only if $s < 2$. □

In $\mathbb{H}^n_{\mathbb{C}}$ with its CAT(-1) distance $d'_{\mathbb{H}^n_{\mathbb{C}}}$, the distance-like map $d_{\mathscr{H}_2}$ coincides (as seen in Section 2) with the Hamenstädt distance $d_{\infty,\partial\mathscr{H}_2}$. Recall (see [HP3, Prop. 3.12]) that

$$d_{\infty,\partial\mathscr{H}_2} = \frac{1}{\sqrt{2}}\, d_{Cyg}\;. \tag{-13-}$$

Let $\mathscr{R}_{\Gamma_0}$ be the set of fixed points of the conjugates of γ_0, endowed with its Fréchet filter. For every $\alpha \in \mathscr{R}_{\Gamma_0}$, let α^* be the other endpoint of the translation axis of a conjugate of γ_0 containing α at infinity. We therefore have

$$\mathrm{Sp}(\mathscr{D}) = \big\{c(\Gamma_\infty\xi) = \liminf_{\alpha\in\mathscr{R}_{\Gamma_0}} \sqrt{2}\,\frac{d'_{\mathrm{Cyg}}(\alpha,\alpha^*)\; d_{\mathrm{Cyg}}(\xi,\alpha)}{d_{\mathrm{Cyg}}(\alpha,\alpha^*)^2} \;:\; \xi \in \Lambda_c\Gamma - \mathscr{R}_{\Gamma_0}\big\}\;. \tag{-14-}$$

4 The basic properties of the approximation spectra

Let $\mathscr{D} = (X, \Gamma, \Gamma_0, C_\infty)$ be a quadruple of data as defined in Section 3. In this section, we study the upper bound of the approximation spectrum $\mathrm{Sp}(\mathscr{D}) \subset \mathbb{R}$ of $\mathscr{D}$, and we give a closedness result for $\mathrm{Sp}(\mathscr{D})$.

4.1 The nontriviality of the approximation spectra

A map $f : [0,+\infty[\; \to\;]0,+\infty[$ is called *slowly varying* if it is measurable and if there exist constants $B > 0$ and $A \geq 1$ such that for every x, y in $\mathbb{R}_+$, if $|x-y| \leq B$, then $f(y) \leq A\, f(x)$. Recall that this implies that f is locally bounded, hence it is locally integrable; also, if $\log f$ is Lipschitz, then f is slowly varying.

Let ϵ be a positive real number, and let $f, g : [0, +\infty[\to]0, +\infty[$. A geodesic ray or line ρ in X will be called *(ϵ, g)-Liouville* (with respect to $\mathscr{D}$) if there exist a sequence $(t_n)_{n\in\mathbb{N}}$ of positive times converging to $+\infty$ and a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ such that $\rho(t)$ belongs to $\mathscr{N}_\epsilon(\gamma_n C_0)$ for every t in $[t_n, t_n + g(t_n)]$. A geodesic ray or line ρ in X such that $\rho(+\infty) \notin \partial_\infty C_\infty$ will be called *f-well approximated* (with respect to $\mathscr{D}$) if there exist infinitely many γ in Γ/Γ_0 such that

$$\widetilde{d}_\infty\big(\rho(+\infty), \gamma\Lambda\Gamma_0\big) \leq f\big(D([\gamma])\big)\ e^{-D([\gamma])}\ .$$

The following result is proved in [HP5, Lemma 5.2] (when $C_\infty = C_0$, but the proof is the same).

Lemma 4.1 *Let $f : [0, +\infty[\to]0, 1[$ be slowly varying, and let $g : t \mapsto -\log f(t)$. Let $\epsilon > 0$. There exists $c = c(\epsilon, f) > 0$ such that for every geodesic ray or line ρ in X such that $\rho(+\infty) \notin \partial_\infty C_\infty \cup \bigcup_{\gamma\in\Gamma} \gamma\, \partial_\infty C_0$, if ρ is (ϵ, g)-Liouville, then ρ is (cf)-well approximated, and conversely, if ρ is $(\frac{1}{c} f)$-well approximated, then ρ is (ϵ, g)-Liouville.* □

Our first result says in particular that $\{0\} \subsetneq \mathrm{Sp}(\mathscr{D})$. We refer to Section 5 for much stronger results for particular cases of $\mathscr{D}$.

Proposition 4.2 *The approximation spectrum of $\mathscr{D}$ contains 0 as a nonisolated point, and hence the Hurwitz constant of $\mathscr{D}$ is positive.*

The following consequence, amongst other similar ones, follows from Equation (-9-).

Corollary 4.3 *Let M be a nonelementary complete connected Riemannian manifold with sectional curvature at most -1 and dimension at least 2. Let A_0 be a closed geodesic in M, and let A_∞ be a closed codimension 0 submanifold of M with smooth connected compact locally convex boundary, disjoint from A_0. Then the spiraling spectrum $\mathrm{Sp}_{A_\infty, A_0}(M)$ around A_0 contains 0 as a nonisolated point.* □

Proof of Proposition 4.2. Let us first prove that there exists an element γ in $\Gamma - \Gamma_0$ such that $d(C_0, \gamma C_0)$ and $d(C_\infty, \gamma C_0)$ are both as big as we need.

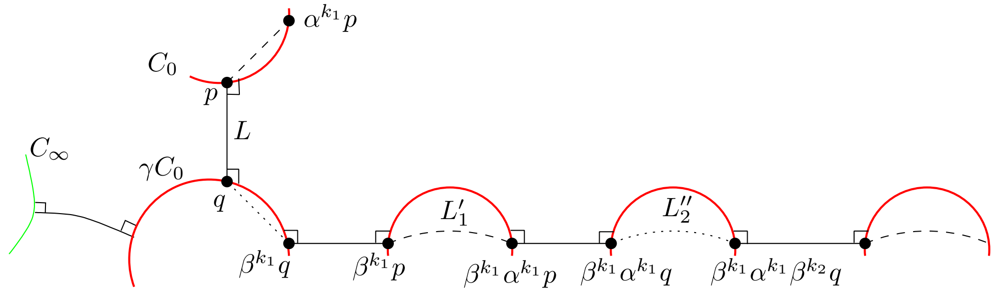
By the lemmae 3.1 and 3.2, there exists a nontrivial double class $[\gamma_0] \in \Gamma_\infty\backslash\Gamma/\Gamma_0$ such that $d(C_\infty, \gamma_0 C_0)$ is big. Since Γ_0 contains a hyperbolic element, there exists a hyperbolic element γ_1 in Γ whose attractive fixed point $(\gamma_1)_+$ belongs to $\gamma_0\partial_\infty C_0$, and in particular is not in $\partial_\infty C_\infty$. As $\gamma_0 \notin \Gamma_0$ and Γ_0 is almost malnormal, we have $(\gamma_1)_+ \notin \partial_\infty C_0$ by Lemma 2.1 (2). Since $(\gamma_1)_+ \notin \partial_\infty C_0 \cup \partial_\infty C_\infty$, if n is big enough, then $\gamma = \gamma_1^n$ is an element in $\Gamma - \Gamma_0$ such that $d(C_0, \gamma C_0)$ and $d(C_\infty, \gamma C_0)$ are both big enough.

Now, let γ be as above. Let $[p, q]$ be the shortest segment between C_0 and γC_0, with $p \in C_0$. Let α be a hyperbolic element in Γ_0 with big translation length, and $\beta = \gamma\alpha\gamma^{-1}$. Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers. In particular, $L = d(p, q)$, $L'_n = d(p, \alpha^{k_n} p)$ and $L''_n = d(q, \beta^{k_n} q)$ are big (independently of $(k_n)_{n\in\mathbb{N}}$).

For every n in $\mathbb{N}$, define $\gamma_n = \beta^{k_1}\alpha^{k_1}\beta^{k_2}\alpha^{k_2}\dots\beta^{k_n}\alpha^{k_n}$, so that $\gamma_0 = \mathrm{id}$ and $\gamma_1 = \beta^{k_1}\alpha^{k_1}$. Consider the piecewise geodesic ray which is geodesic between the consecutive points

$$p, q, \beta^{k_1} q, \beta^{k_1} p, \gamma_1 p, \dots, \gamma_n p, \gamma_n q, \gamma_n\beta^{k_{n+1}} q, \gamma_n\beta^{k_{n+1}} p, \gamma_{n+1} p, \dots\ .$$

Then if $A = \min\{L, L'_n, L''_n\ :\ n \in \mathbb{N}\}$ is big enough, as the comparison angles at the above points between the incoming and outgoing segments are at least $\pi/2$ by convexity, this piecewise geodesic ray is quasi-geodesic.



Hence, it stays at bounded distance (depending only on A) from a geodesic ray ρ^* starting from p. Note that, by convexity, the segments $[\gamma_n q, \gamma_n \beta^{k_{n+1}} q]$ and $[\gamma_n \beta^{k_{n+1}} p, \gamma_{n+1} p]$ (which are long if the L''_n, L'_n are big) are contained in images under Γ of C_0. The point at infinity ξ of ρ^* is in particular a conical limit point (since Γ_0 is convex-cocompact, there are points in one orbit under Γ that accumulate to ξ while staying at bounded distance from ρ^*). Up to taking the translation length of α, and hence A, big enough, the point ξ belongs neither to $\partial_\infty C_\infty$, nor to any $\gamma' \partial_\infty C_0$ for $\gamma' \in \Gamma$ (otherwise, two copies of C_0 would be close for a too long time, contradicting Lemma 2.1 (4)). Hence the approximation constant $c(\Gamma_\infty \xi)$ is well defined.

In order to apply Lemma 4.1, we fix $\epsilon > 0$. If the sequence $(k_n)_{n\in\mathbb{N}}$ tends to $+\infty$, then the geodesic ray ρ^* spends longer and longer time in the images by Γ of the ϵ-neighbourhood of C_0. Thus, $c(\Gamma_\infty \xi)$ is equal to 0, by Lemma 4.1.

To prove that 0 is not isolated, take the sequence $(k_n)_{n\in\mathbb{N}}$ to be constant, with k_1 big compared with $\kappa(\epsilon)$ (which has been defined in Lemma 2.1 (4)), L and the bounded distance between ρ^* and the above quasi-geodesic. In particular, ρ^* is (ϵ, g)-Liouville for g a constant map, having a big value if k_1 is big. By Lemma 2.1 (4), since ρ^* spends intervals of time of only bounded length outside $\Gamma \mathscr{N}_\epsilon C_0$, the geodesic ray ρ^* is not (ϵ, g')-Liouville for $g' > g$ a big enough constant map.

By Lemma 4.1, this implies that the approximation constant of (the image modulo Γ_∞ of) ξ is positive, and small if k_1 is big. □

Remark. Let us notice here that the approximation constants are generically equal to 0, hence that the nonvanishing of an approximation constant is a quite rare behaviour. We will make this explicit only in a particular case.

Assume that X is a Riemannian manifold and C_0 a geodesic line. For every $v \in \Gamma\backslash T^1X$, let $\xi_v \in \Gamma_\infty\backslash\partial_\infty X$ be the (orbit under Γ_∞ of the) endpoint of a geodesic line in X whose tangent vector at the origin maps to v by the quotient by Γ. (Several choices are possible, but they will give the same approximation constant.) Let μ be a (finite, positive, Borel) measure on $\Gamma\backslash T^1X$ invariant and ergodic under the quotient geodesic flow $(\phi_t)_{t\in\mathbb{R}}$. Assume that the support of μ contains the orbit under Γ of the lift of C_0 to T^1X by its unit tangent vector, and that the (measurable) subset of unit vectors v such that $\xi_v \in \Gamma_\infty\backslash\big((\partial_\infty X - \Lambda_c\Gamma) \cup \partial_\infty C_\infty\big)$ has measure 0. For instance, this is true if Γ has finite covolume, μ is the Liouville measure and C_∞ is a precisely invariant horoball, or if Γ is cocompact and μ is the maximal entropy measure, and C_∞ is the translation axis of a hyperbolic element. The ergodicity assumption implies that $\{\phi_t v\}_{t\in\mathbb{R}^+}$ is dense in the support of μ for almost every v. Recall that if two unit tangent vectors are very close, then the geodesic lines they define are close for a long time. Hence for μ-almost every v, we have $\xi_v \in \Gamma_\infty\backslash(\Lambda_c\Gamma - \partial_\infty C_\infty)$ and $c(\xi_v) = 0$ by Lemma 4.1.

4.2 The boundedness of the approximation spectra

If Γ is geometrically finite (see for instance [Bow]), then there exists a Γ-equivariant family $\mathscr{H}$ of horoballs centered at the parabolic fixed points of Γ, with pairwise disjoint interiors. There are many possible choices for such an $\mathscr{H}$ (though only one maximal one if Γ has only one orbit of parabolic fixed points). In the computations of Section 4.4, we will choose natural ones. We call $X_0 = \mathscr{C}\Gamma - \bigcup \mathscr{H}$ the *thick part* of $\mathscr{C}\Gamma$. Clearly, X_0 is Γ-invariant, and Γ acts isometrically on it. We call $\Gamma\backslash X_0$ the *thick convex core* of $\Gamma\backslash X$.

The next result gives a sufficient condition for the Hurwitz constant of $\mathscr{D}$ to be finite. In particular, this condition is satisfied when X is a Riemannian manifold and Γ has finite covolume. Recall that the Hurwitz constant of $\mathscr{D}$ is

$$K_{\mathscr{D}} = \sup \operatorname{Sp}(\mathscr{D}) \ \in [0, +\infty] \ .$$

Theorem 4.4 *If Γ is geometrically finite, then $\operatorname{Sp}(\mathscr{D})$ is bounded, hence $0 < K_{\mathscr{D}} < \infty$.*

The following consequence, amongst other similar ones, follows from Equation (- 9 -).

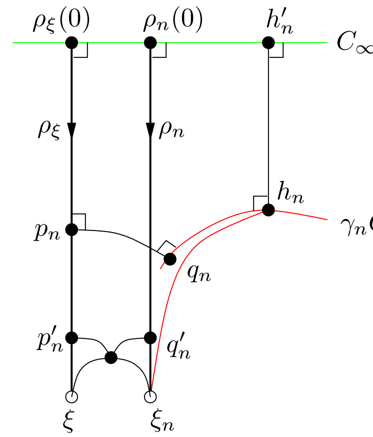
Corollary 4.5 *Let M be a geometrically finite complete connected Riemannian manifold with sectional curvature at most -1 and dimension at least 2. Let A_0 be a closed geodesic in M, and let A_∞ be a closed codimension 0 submanifold of M with smooth connected compact locally convex boundary, disjoint from A_0. Then the spiraling spectrum $\operatorname{Sp}_{A_\infty, A_0}(M)$ around A_0 is bounded.* □

Proof of Theorem 4.4. Let $\mathscr{H}$ be as above. Since $\Lambda\Gamma_0$ contains at least two points, C_0 is not contained in any element of $\mathscr{H}$, hence C_0 intersects X_0. Since Γ is geometrically finite, the diameter Δ of the quotient metric space $\Gamma\backslash X_0$ is finite. For every $\xi \in \Lambda_c\Gamma - (\partial_\infty C_\infty \cup \bigcup_{\gamma\in\Gamma} \gamma\partial_\infty C_0)$, let ρ_ξ be a geodesic ray starting from the closest point to ξ on C_∞ and converging to ξ. As ξ is a conical limit point, there exists a sequence of positive times $(t_n)_{n\in\mathbb{N}}$ converging to $+\infty$ such that $\rho_\xi(t_n) \in X_0$ for every n. Hence, there exists a sequence of elements $(\gamma_n)_{n\in\mathbb{N}}$ such that for every n in $\mathbb{N}$,

$$d(\rho_\xi(t_n), \gamma_n C_0) \leq \Delta \ .$$

For n big enough, the distance between the convex subsets C_∞ and $\gamma_n C_0$ is big. Indeed, if $d(C_\infty, \gamma_{n_k} C_0)$ is bounded for some subsequence $(n_k)_{k\in\mathbb{N}}$ tending to $+\infty$, then by Lemma 3.1 and up to extracting a subsequence, the double cosets $[\gamma_{n_k}] \in \Gamma_\infty\backslash\Gamma/\Gamma_0$ are constant. Since $\gamma_{n_k} C_0$ contains a point whose closest point on C_∞ is at bounded distance (at most Δ) from the point $\rho_\xi(0)$, up to extracting a subsequence and up to multiplying γ_{n_k} on the right by an element of Γ_0, we may assume that $(\gamma_{n_k})_{k\in\mathbb{N}}$ is constant. Since ρ_ξ converges to ξ, the construction of $(\gamma_n)_{n\in\mathbb{N}}$ implies that ξ belongs to the closed subset $\gamma_{n_0}\partial_\infty C_0$, a contradiction.

If ρ_ξ does not meet $\gamma_n C_0$, we denote by $[p_n, q_n]$ the shortest segment between ρ_ξ and $\gamma_n C_0$, with p_n in ρ_ξ (which exists since $\xi \notin \bigcup_{n\in\mathbb{N}} \gamma_n \partial_\infty C_0$). Otherwise, define $p_n = q_n$ to be the first intersection point of ρ_ξ with $\gamma_n C_0$. In particular, $d(p_n, q_n) \leq \Delta$. Let h_n be the point of $\gamma_n C_0$ the closest to C_∞, and h'_n its closest point on C_∞. Since Γ_0 is convex-cocompact, there exists a constant $c_1 \geq 0$ (depending only on Γ_0) such that the distance between q_n and a geodesic ray starting from h_n and staying inside $\gamma_n C_0$ is at most c_1, for every n in $\mathbb{N}$. Denote by ξ_n the point at infinity of this geodesic ray, which does not belong to $\partial_\infty C_\infty$. Let ρ_n be the geodesic ray starting from the closest point to ξ_n in C_∞ and converging to ξ_n.



By the distance formulae of the hyperbolic comparison quadrilateral with vertices corresponding to $\xi_n, \rho_n(0), h'_n$ and h_n, with one comparison angle 0 and the three others at least $\pi/2$ (see [Bea, 7.17]), if n (and hence $d(h_n, C_\infty)$) is big enough, then the distance between $\rho_n(0)$ and h'_n is at most 1. Therefore h_n is at distance at most a universal constant c_2 from ρ_n. Since $d(q_n, [h_n, \xi_n[) \leq c_1$, the point q_n is hence by convexity at distance at most $c_1 + c_2$ from a point x_n of ρ_n, so that

$$d(p_n, \rho_n) \leq d(p_n, x_n) \leq d(p_n, q_n) + d(q_n, x_n) \leq \Delta + c_1 + c_2 \ .$$

In particular, $d(\rho_n(0), x_n)$ tends to $+\infty$ as $n \to +\infty$. By the distance formulae of the hyperbolic comparison quadrilateral with vertices corresponding to $\rho_\xi(0), \rho_n(0), x_n, p_n$ (see [Bea, 7.17]), with two comparison angles at least $\pi/2$ and the length of the opposite segment, that is $d(p_n, x_n)$, bounded by a constant, if n is big enough, then $\rho_\xi(0)$ is at distance at most 1 from $\rho_n(0)$.

Let r_n be the class of γ_n in $\Gamma_\infty \backslash \Gamma / \Gamma_0$, so that $D(r_n) = d(h_n, C_\infty)$. Using the triangle inequality, we have

$$d(p_n, \rho_\xi(0)) \geq d(q_n, C_\infty) - d(q_n, p_n) \geq d(\gamma_n C_0, C_\infty) - d(q_n, p_n) \geq D(r_n) - \Delta \ .$$

Since $d(\rho_\xi(0), \rho_n(0)) \leq 1$, there exists a universal constant $c_3 \geq 0$ such that some point on the geodesic line between ξ and ξ_n is at distance at most c_3 from both a point p'_n of ρ_ξ and a point q'_n of ρ_n. In particular, $d(p'_n, q'_n) \leq 2\,c_3$. Since p_n is at bounded distance from ρ_n, there exists (by a geometric argument of geodesic triangles) a constant $c_4 \geq 0$ (which depends only on Δ) such that

$$d(p'_n, \rho_\xi(0)) \geq d(p_n, \rho_\xi(0)) - c_4 \ .$$

By the definition of the distance-like map d_{C_∞} and by the triangle inequality, we have

$$d_{C_\infty}(\xi, \xi_n) \leq e^{-\frac{1}{2}(d(\rho_\xi(0),\, p'_n) - c_3 + d(\rho_n(0),\, q'_n) - c_3)} \ .$$

By the triangle inequality, $d(\rho_n(0), q'_n) \geq d(\rho_\xi(0), p'_n) - d(\rho_\xi(0), \rho_n(0)) - d(p'_n, q'_n)$. By the previous inequalities, we hence have

$$d_{C_\infty}(\xi, \xi_n) \leq e^{-d(\rho_\xi(0),\, p'_n) + c_3 + \frac{d(p'_n,\, q'_n)}{2} + \frac{d(\rho_\xi(0),\, \rho_n(0))}{2}} \leq e^{\Delta + 2c_3 + c_4 + \frac{1}{2}}\ e^{-D(r_n)} \ .$$

The approximation constant $c(\xi)$ of ξ, for every ξ in $\Gamma_\infty\backslash(\Lambda_c\Gamma - (\partial_\infty C_\infty \cup \bigcup_{\gamma\in\Gamma} \gamma\Lambda\Gamma_0))$, is hence at most $e^{\Delta+2c_3+c_4+\frac{1}{2}}$, which proves the result. □

Remark. Note that this upper bound depends only on (X, Γ, C_∞), but not on C_0.

In special cases, the proof of Theorem 4.4 may be improved to give a simple explicit constant.

Proposition 4.6 *If Γ is geometrically finite, if Δ is the diameter of the thick convex core of $\Gamma\backslash\Gamma$, if C_∞ is a point in X or a horoball in X, and if C_0 is a geodesic line, then*

$$K_{\mathscr{D}} \leq (1+\sqrt{2})e^{\Delta}.$$

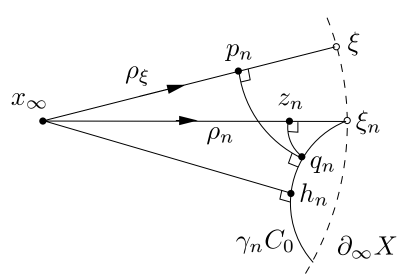
The following consequence follows from Equation (- 9 -).

Corollary 4.7 *Let M be a geometrically finite complete connected Riemannian manifold with sectional curvature at most -1 and dimension at least 2, and let Δ be the diameter of the thick convex core of M. Let A_0 be a closed geodesic in M, and let A_∞ be either a ball or a Margulis neighbourhood of a cusp of M. Then the spiraling spectrum $\mathrm{Sp}_{A_\infty,A_0}(M)$ around A_0 is contained in $[0,(1+\sqrt{2})e^{\Delta}]$.* □

Proof of Proposition 4.6. Assume first that $C_\infty = \{x_\infty\}$ with $x_\infty \in X$. For every ξ belonging to $\Lambda_c\Gamma - (\partial_\infty C_\infty \cup \bigcup_{\gamma\in\Gamma} \gamma\, \partial_\infty C_0)$, we define $\rho_\xi, \gamma_n, p_n, q_n, h_n$ as in the beginning of the proof of Theorem 4.4, so that $\rho_\xi(0) = x_\infty$. Let ξ_n be an endpoint of the geodesic line C_0 such that $q_n \in [h_n, \xi_n[$. Let ρ_n be the geodesic ray from x_∞ to ξ_n. Let $r_n = [\gamma_n] \in \Gamma_\infty\backslash\Gamma/\Gamma_0$, so that $d(x_\infty, q_n) \geq D(r_n)$. Let z_n be the closest point on ρ_n to q_n, which satisfies

$$d(z_n, q_n) \leq \delta\ ,$$

with $\delta = \log(1+\sqrt{2})$, by looking at the comparison triangle of the geodesic triangle with vertices x_∞, ξ_n, h_n (see the picture below).



By the triangle inequality, for all t big enough,

$$\begin{aligned}
&d(\rho_n(t), x_\infty) + d(\rho_\xi(t), x_\infty) - d(\rho_n(t), \rho_\xi(t))\\
&\geq \big(d(\rho_n(t), z_n) + d(z_n, x_\infty)\big) + \big(d(\rho_\xi(t), p_n) + d(p_n, x_\infty)\big) -\\
&\qquad \big(d(\rho_n(t), z_n) + d(z_n, p_n) + d(p_n, \rho_\xi(t))\big)\\
&= d(x_\infty, p_n) + d(x_\infty, z_n) - d(p_n, z_n)\\
&\geq d(x_\infty, q_n) - d(q_n, p_n) + d(x_\infty, q_n) - d(q_n, z_n) - \big(d(p_n, q_n) + d(q_n, z_n)\big)\\
&\geq 2D(r_n) - 2\Delta - 2\delta\ .
\end{aligned}$$

Hence,

$$d_{x_\infty}(\xi,\xi_n) = \lim_{t\to+\infty} e^{-\frac{1}{2}\big(d(\rho_n(t),\,x_\infty)+d(\rho_\xi(t),\,x_\infty)-d(\rho_n(t),\,\rho_\xi(t))\big)} \le e^{-D(r_n)+\Delta+\delta},$$

which proves that $c(\Gamma_\infty\xi) \le e^{\Delta+\delta}$.

If C_∞ is a horoball with point at infinity ξ_∞, the proof is similar, by replacing the geodesic rays starting from x_∞ by geodesic lines starting from ξ_∞ and meeting ∂C_∞ at time 0, and using the fact that $d(\rho_\xi(0),\rho_n(0))$ tends to 0 as $n\to+\infty$. □

Theorem 1.1 in the introduction follows from Corollary 4.7.

4.3 On the closedness of the approximation spectra

In this subsection, we prove that in the constant curvature manifold case, the spiraling spectrum around a closed geodesic is closed.

Theorem 4.8 *Let $\mathscr{D} = (X,\Gamma,\Gamma_0,C_\infty)$ be a quadruple of data such that X is the real hyperbolic n-space, Γ is geometrically finite, $C_0 = \mathscr{C}\Gamma_0$ is a geodesic line, and C_∞ is a horoball. Then $\mathrm{Sp}(\mathscr{D})$ is equal to the closure in $\mathbb{R}$ of the set of the approximation constants of the (orbits under Γ_∞ of the) fixed points of the hyperbolic elements of Γ (that are not conjugated to elements of Γ_0).*

In particular, the approximation spectrum $\mathrm{Sp}(\mathscr{D})$ is closed, the Hurwitz constant of $\mathscr{D}$ is the maximum of $\mathrm{Sp}(\mathscr{D})$, and the approximation constants of the (orbits under Γ_∞ of the) hyperbolic fixed points of Γ are dense in $\mathrm{Sp}(\mathscr{D})$.

Remark. The result is still true if C_0 is any totally geodesic subspace of dimension at least 1 and at most $n-1$; the adaptation of the proof below is left to the reader.

The following consequence follows from Equation (-9-), and proves the first claim in Theorem 1.2 in the Introduction.

Corollary 4.9 *Let M be a geometrically finite complete connected Riemannian manifold with constant sectional curvature -1 and dimension at least 2. Let A_0 be a closed geodesic in M, and let A_∞ be a Margulis neighbourhood of a cusp of M. Then the spiraling spectrum $\mathrm{Sp}_{A_\infty,A_0}(M)$ around A_0 is closed, and is equal to the closure of the set of spiraling constants of the geodesic lines spiraling around closed geodesics distinct from A_0.* □

Proof of Theorem 4.8. Let $\mathscr{L}_0$ be the set of images under Γ of the two oriented geodesics defined by C_0. Let $\tilde v$ be an element of T^1X, and $\tilde x$ be its base point. For every C in $\mathscr{L}_0$, define p_C to be the point of C the closest to $\tilde x$, which depends continuously on $\tilde v$; define θ_C to be the angle at p_C between the parallel transport along $[\tilde x, p_C]$ of $\tilde v$ and C, which depends continuously on $\tilde v$. Let

$$\tilde f(\tilde v) = \inf_{C\in\mathscr{L}_0} \cosh d(\tilde x, C) - \cos\theta_C\ .$$

Since $\mathscr{L}_0$ is locally finite and is preserved by Γ, the lower bound defining $\tilde f$ is locally a minimum over a finite set. Thus, the map $\tilde f : T^1X\to\mathbb{R}$ is continuous and invariant under Γ, and it defines a continuous map $f:\Gamma\backslash T^1X\to\mathbb{R}$. As the image of C_0 in $\Gamma\backslash X$ is compact and $\Gamma\backslash X$ is a proper metric space, the distance to this image is a proper map on $\Gamma\backslash X$. Hence f, which is at least $\cosh(0) - 1 = 0$, is proper. Let $(\phi^t : T^1X \to T^1X)_{t\in\mathbb{R}}$ be the geodesic flow of X, and denote again by $(\phi^t)_{t\in\mathbb{R}}$ its quotient flow under Γ.

We will use the following result of F. Maucourant [Mau, Theo. 2 (2)], whose main tool is Anosov's closing lemma (and which builds on a partial result of [HP2]). The result extends to our orbifold case.

Theorem 4.10 *Let V be a complete Riemannian manifold with sectional curvature at most -1, let $(\phi^t)_{t\in\mathbb{R}}$ be its geodesic flow, and let J_0 be the subset of T^1V which consists of periodic unit tangent vectors. If $f : T^1V \to \mathbb{R}$ is a proper continuous map, then*

$$\mathbb{R} \cap \{ \liminf_{t\to+\infty} f(\phi^t v) \ : \ v \in T^1V\} = \overline{\{ \min_{t\in\mathbb{R}} f(\phi^t v) \ : \ v \in J_0\}}\,. \quad \square$$

Assume that X is the upper halfspace model of $\mathbb{H}^n_\mathbb{R}$, and that C_∞ is centered at ∞. By the assumptions on the data $\mathscr{D}$, we are in the situation of Example 1 of Section 3. In the following, we use the notation of that example. Let ξ be the endpoint of the geodesic line defined by $\tilde v$, and note that this geodesic line is asymptotic to the geodesic line from ∞ to ξ. There are three cases to consider:

(1) If $\xi \in \Lambda_c\Gamma - \mathscr{R}_{\Gamma_0}$, then it follows from the definition of $\tilde f$ and from Equation (-11-) that $\liminf_{t\to+\infty} \tilde f(\phi^t\tilde v) = c(\Gamma_\infty\xi)$.

(2) If $\xi \in \mathscr{R}_{\Gamma_0}$, then $\liminf_{t\to+\infty} \tilde f(\phi^t\tilde v) = 0$, and we have already seen (in Proposition 4.2) that 0 belongs to $\mathrm{Sp}(\mathscr{D})$.

(3) If $\xi \in \partial_\infty X - \Lambda_c\Gamma$, then since Γ is geometrically finite, either ξ does not belong to the limit set, or ξ is a parabolic fixed point. In both cases, since f is proper, we have $\liminf_{t\to+\infty} \tilde f(\phi^t\tilde v) = +\infty$, which is not in $\mathbb{R}$.

These observations imply that $\mathbb{R} \cap \{ \liminf_{t\to+\infty} f(\phi^t v) \ : \ v \in \Gamma\backslash T^1X\}$ is contained in $\mathrm{Sp}(\mathscr{D}) = \{c(\Gamma_\infty\xi) \ : \ \xi \in \Lambda_c\Gamma - \mathscr{R}_{\Gamma_0}\}$.

By considering a vertical unit tangent vector $\tilde v$ ending at a given $\xi \in \Lambda_c\Gamma - \mathscr{R}_{\Gamma_0}$, the opposite inclusion also holds. If J_0' is the subset of vectors in J_0 that are not the image in $\Gamma\backslash T^1X$ of unit tangent vectors to C_0, then the set A of the approximation constants of the (orbits under Γ_∞ of the) points of $\partial_\infty X$ fixed by hyperbolic elements not conjugated to elements of Γ_0, is equal to $\{ \inf_{t\in\mathbb{R}} f(\phi^t v) \ : \ v \in J_0'\}$. Furthermore, the approximation constant of (the orbit under Γ_∞ of) a point of $\partial_\infty X$ fixed by a hyperbolic element conjugated to an element of Γ_0, is equal to 0. Hence, by Theorem 4.10, we have $\mathrm{Sp}(\mathscr{D}) = \overline{A \cup \{0\}}$. Since 0 is not isolated in $\mathrm{Sp}(\mathscr{D})$ (see Theorem 4.2), this implies that $\mathrm{Sp}(\mathscr{D}) = \overline{A}$. This proves Theorem 4.8. $\square$

4.4 Some upper bounds on the approximation spectra

We give estimates of the Hurwitz constants of data $\mathscr{D} = (X, \Gamma, \Gamma_0, C_\infty)$ in a number of arithmetically defined cases. The estimates are not likely to be very sharp, except for Proposition 4.11. In the following five examples, X is $\mathbb{H}^2_\mathbb{R}$, $\mathbb{H}^2_\mathbb{R}$, $\mathbb{H}^3_\mathbb{R}$, $\mathbb{H}^5_\mathbb{R}$ or $\mathbb{H}^2_\mathbb{C}$ respectively. These dimensions are chosen with number theoretical applications in mind, see Section 6. In the first four examples, the convex set C_∞ is the horoball centered at infinity consisting of the points with Euclidean height at least 1 in the upper half space model of $\mathbb{H}^n_\mathbb{R}$. The group Γ is specified in each example, and the subgroup Γ_0 is the stabilizer in Γ of any geodesic in X whose quotient in $\Gamma\backslash X$ is compact.

The following classical fact is used repeatedly in the examples: For every $\alpha \leq \pi/2$, the distance ℓ in $\mathbb{H}^2_{\mathbb{R}}$ between the points of angle α and of angle $\pi/2$ with respect to the real line $\partial_\infty \mathbb{H}^2_{\mathbb{R}} - \{\infty\}$ on any Euclidean circle centered at a point in $\partial_\infty \mathbb{H}^2_{\mathbb{R}} - \{\infty\}$ is

$$\ell = \log \cot \frac{\alpha}{2} = \log \frac{\sqrt{1+\tan^2\alpha}+1}{\tan\alpha} . \qquad (\text{-}\,15\,\text{-})$$

This equation is used to compute distances between points in isometrically embedded copies of $\mathbb{H}^2_{\mathbb{R}}$ in $\mathbb{H}^n_{\mathbb{R}}$ and $\mathbb{H}^2_{\mathbb{C}}$.

(1) Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. It is well known that the hyperbolic triangle F in $\mathbb{H}^2_{\mathbb{R}}$ with vertices at $\infty, e^{i\frac{\pi}{3}}$ and $e^{2i\frac{\pi}{3}}$ is a fundamental polygon for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The horoball C_∞ covers the maximal Margulis neighbourhood U of the (only) cusp of $M = \Gamma\backslash\mathbb{H}^2_{\mathbb{R}}$. The compact set $K = \overline{F - C_\infty}$ covers the complement of U in M. The symmetries imply that the diameter Δ of $M - U$ equals the distance between the cone points of M with angles π and $2\pi/3$. By Equation (-15-),

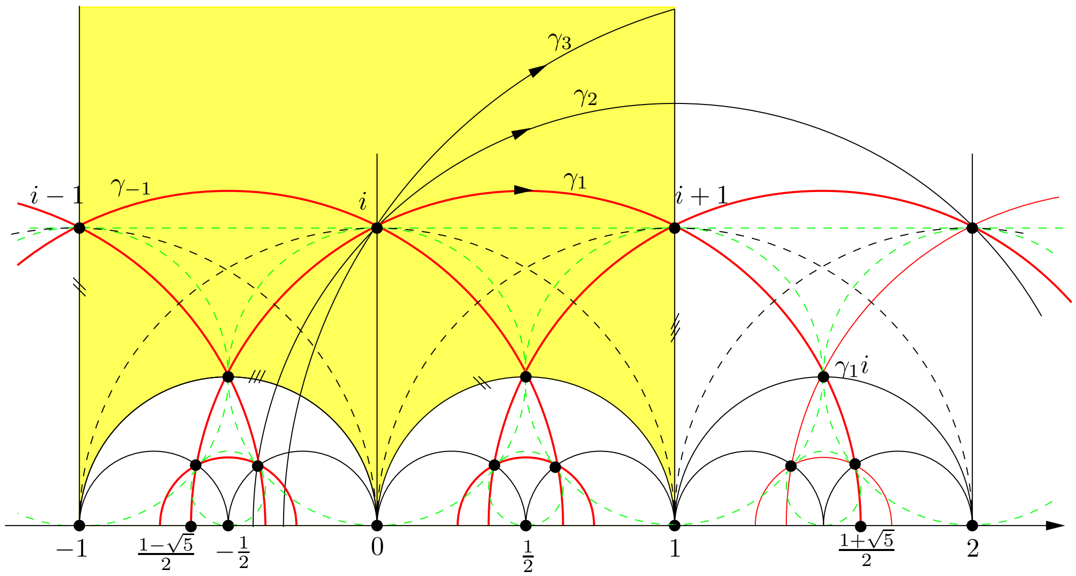
$$\Delta = d(i, e^{i\frac{\pi}{3}}) = \frac{1}{2}\log 3 \approx 0.55 .$$

By Proposition 4.6, we thus have $\mathrm{Sp}(\mathscr{D}) \subset [0, (1+\sqrt{2})\sqrt{3}] \subset [0, 4.19]$. Note that this upper bound is uniform amongst the subgroups Γ_0.

Let us give an exact computation of the Hurwitz constant in a particular case. Notice that the second assertion of the result below shows a different behaviour than the classical Lagrange spectrum.

Proposition 4.11 *Let Γ_0 be the cyclic subgroup of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ generated by $\gamma_1 = \pm\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and let $\mathscr{D} = (\mathbb{H}^2_{\mathbb{R}}, \Gamma, \Gamma_0, C_\infty)$. Then $K_{\mathscr{D}} = 1 - 1/\sqrt{5}$, and $K_{\mathscr{D}}$ is not isolated in the approximation spectrum* $\mathrm{Sp}(\mathscr{D})$.

Proof. The element γ_1 is hyperbolic, and its translation axis L_1 is the geodesic line in $\mathbb{H}^2_{\mathbb{R}}$ with endpoints at $(1 \pm \sqrt{5})/2$ (see the picture below). The translation length ℓ_1 of γ_1 satisfies $2\cosh(\ell_1/2) = 3$ (see [Bea, page 173]), and the translates of L_1 intersect in pairs at the orbit of i, and form a net (covering $\mathbb{H}^2_{\mathbb{R}} - \Gamma C_\infty$) of equilateral triangles (the images under Γ of the triangle with vertices $i, i+1, \frac{i+1}{2}$) as in the figure below. The edges of the triangles have length, by Equation (-15-), equal to $d(i, i+1) = \mathrm{argcosh}(3/2) = \ell_1/2$, and the (interior) angles $\theta \in [0, \frac{\pi}{2}]$ of the triangles satisfy $\cos\theta = 3/5$. These facts are easily seen by considering the 6-fold cover of M by the modular torus M' which is the quotient of $\mathbb{H}^2_{\mathbb{R}}$ by the commutator subgroup of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ (see (2) below). Notice that γ_1 and its translates by $z \mapsto z \pm 1$ are lifts of the three shortest periodic geodesics of M' which intersect at the three Weierstrass points of M' (see for instance [Sch, Theo. 2]).



Any geodesic line that connects ∞ with $\xi \in \mathbb{R} - \mathbb{Q}$ intersects infinitely many triangles. Let T be one of the triangles, and let γ be a geodesic line that intersects the interior of T. The points of intersection of γ with T and one of the vertices of T determine a triangle with angles θ, ϕ_1, ϕ_2. The supremum of $\min\{\cos\phi_1, \cos\phi_2\}$ over all nondegenerate hyperbolic triangles with angles θ, ϕ_1, ϕ_2 is obtained, by symmetry, when $\phi_1 = \phi_2$ and when the triangles become small, that is when they converge, after renormalization, to the Euclidean triangle with angles $\theta, \phi_1 = \phi_2$. Hence the above supremum is $\cos(\frac{\pi-\theta}{2}) = \sin\frac{\theta}{2} = \frac{1}{\sqrt{5}}$, and it is not attained, by the Gauss-Bonnet formula. Thus, the geodesic line from ∞ to ξ cuts a sequence of pairwise distinct Γ-translates of C_0, the cosine of the angle at each intersection point being at least $\frac{1}{\sqrt{5}}$. By Equation (-11-), this implies that for any $\xi \in \mathbb{R} - \mathbb{Q}$, we have $c(\xi) \leq 1 - 1/\sqrt{5}$.

For any nonzero integer n, consider the hyperbolic element $\gamma_n = \pm\begin{pmatrix} n^2+1 & n \\ n & 1 \end{pmatrix} \in \Gamma$. The fixed points of γ_n are $\frac{n}{2} \pm \sqrt{(\frac{n}{2})^2 + 1}$. Thus, the axis of γ_n is the intersection with the upper half plane of the Euclidean circle of center $n/2$ and radius $\sqrt{(\frac{n}{2})^2+1}$, which passes through the points i and $n+i$. The translation distance of γ_n, which is $2\,\mathrm{argcosh}(n^2/2+1)$ by [Bea, page 173], is twice the distance between the points i and $n+i$, by Equation (-15-). Thus, the translation axis of γ_n intersects, at each Γ-image of i on it, exactly two Γ-translates of the axis of γ_1, and always at the same angle in absolute value, with alternating signs. As $n \to \infty$, the smallest of the two positive angles approaches (while strictly increasing) the angle θ' between the (oriented) axis of γ_1 and the (upward oriented) imaginary axis at i, which satisfies $\cos\theta' = 1/\sqrt{5}$. Thus, by Equation (-11-), the approximation constants of the lines from ∞ to the fixed points of γ_n converge to $1-1/\sqrt{5}$ (while being different). The result follows. $\square$

(2) Let Γ be the commutator subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. It is well known (see for instance [Sch]) that Γ is a torsion-free subgroup of index 6 in $\mathrm{PSL}_2(\mathbb{Z})$, and that the quotient $\Gamma\backslash\mathbb{H}^2_{\mathbb{R}}$ is a punctured torus, called *the modular torus*.

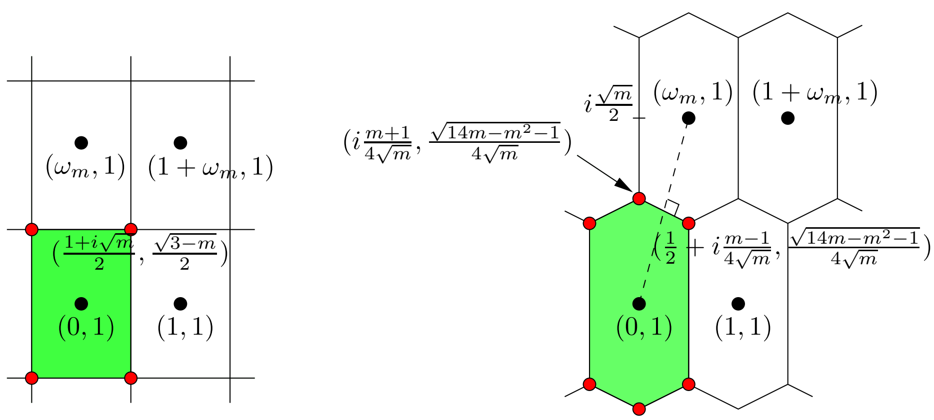
For every $k \in \mathbb{N}$, let H_k be the horoball centered at k with Euclidean height one. It is well known (see for instance [Coh, Sch]) that the modular torus is isometric to the quotient of the ideal hyperbolic square P with vertices $\infty, -1, 0, 1$ by the gluing of the opposite faces of P, such that the horoball C_∞ maps by the two gluings to the horoballs H_{-1} and H_1. In particular, C_∞ covers the maximal Margulis neighbourhood U of the cusp of $M = \Gamma\backslash\mathbb{H}^2_\mathbb{R}$.

Let T' be the closure of the relatively compact component of $\mathbb{H}^2_\mathbb{R} - (C_\infty \cup H_{-1} \cup H_0)$. Then the closure of $M - U$ is the union of the triangles with horocyclic sides T' and $T' + 1$, glued along their vertices. The diameter of T' for the induced distance of $\mathbb{H}^2_\mathbb{R}$ is, by a convexity argument, equal to $d(i, i+1) = \operatorname{argcosh}(3/2)$. Any point of $T' + 1$ is at distance at most $d(i, e^{i\frac{\pi}{3}}) = \frac{\log 3}{2}$ (by Equation (-15-)) from one vertex of $T' + 1$. Therefore

$$\Delta \leq \frac{\log 3}{2} + \operatorname{argcosh}(3/2)\ .$$

By Proposition 4.6, we thus have $\operatorname{Sp}(\mathscr{D}) \subset [0, (1+\sqrt{2})e^\Delta] \subset [0, 10.95]$.

(3) Let m be a positive squarefree integer, and let Γ be the *Bianchi group* $\operatorname{PSL}_2(\mathscr{O}_{-m})$, where $\mathscr{O}_{-m}$ is the ring of integers of $\mathbb{Q}(i\sqrt{m})$. All Bianchi groups contain the transformation $z \mapsto z + 1$, and thus, the interior of the horoball C_∞ is precisely invariant, by Shimizu's Lemma. Since $\iota : z \mapsto -\frac{1}{z}$ also belongs to Γ and since the horoballs C_∞ and ιC_∞ are tangent, the horoball C_∞ covers the maximal Margulis neighbourhood of the cusp of $M = \Gamma\backslash\mathbb{H}^3_\mathbb{R}$. Fundamental domains for the Bianchi groups have been determined in [Bia, Swa, EGM] and we will use the tables of [Hat, page 346].



$m = 1, 2$ $\qquad$ $m = 3, 7, 11$

In the above picture, ω_m is equal to $i\sqrt{m}$ if $m \equiv 1, 2 \mod 4$ and is equal to $\frac{1+i\sqrt{m}}{2}$ if $m \equiv 3 \mod 4$. The shaded area represents the vertical projection to $\mathbb{C}$ of the Ford fundamental domain at infinity F_m. The couples are the coordinates in $\mathbb{H}^3_\mathbb{R} \subset \mathbb{C} \times \mathbb{R}$ of the finite vertices of the polyhedron F_m projecting to these points, and of the center of the (unique) compact codimension 1 face of F_m.

Cases $m = 1, 2$ **:** A Ford fundamental domain of $\Gamma = \operatorname{PSL}_2(\mathscr{O}_{-m}) = \operatorname{PSL}_2\big(\mathbb{Z}[i\sqrt{m}]\big)$ is given by the polyhedron F_m with five vertices, one at ∞ and four finite ones at $(\pm\frac{1}{2} \pm \frac{\sqrt{m}}{2}i, \frac{\sqrt{3-m}}{2})$. The diameter Δ_m of the image of $K_m = \overline{F_m - C_\infty}$ in $\Gamma\backslash\mathbb{H}^3_\mathbb{R}$ satisfies, by the symmetries and Equation (-15-),

$$\Delta_m \leq d\Big((\frac{1}{2} + \frac{\sqrt{m}}{2}i, \frac{\sqrt{3-m}}{2}), (0,1)\Big) = \log\frac{2+\sqrt{1+m}}{\sqrt{3-m}}\ ,$$

which is $\log(1+\sqrt{2})$ if $m=1$ and $\log(2+\sqrt{3})$ if $m=2$. Now, as in (1) above, $\operatorname{Sp}(\mathscr{D})$ is contained in $[0,(1+\sqrt{2})^2] \subset [0,5.83]$ if $m=1$ and $[0,(1+\sqrt{2})(2+\sqrt{3})] \subset [0,9.01]$ if $m=2$.

Cases $m=3,7,11$ **:** A Ford fundamental domain of $\Gamma = \mathrm{PSL}_2(\mathscr{O}_{-m}) = \mathrm{PSL}_2\big(\mathbb{Z}[\frac{1+i\sqrt{m}}{2}]\big)$ is given by the polyhedron F_m with seven vertices at ∞, $\big(\pm\, i\frac{m+1}{4\sqrt{m}}, \frac{\sqrt{14\,m-m^2-1}}{4\sqrt{m}}\big)$ as well as $(\pm\frac{1}{2} \pm i\frac{m-1}{4\sqrt{m}}, \frac{\sqrt{14\,m-m^2-1}}{4\sqrt{m}})$. The diameter Δ_m of the image of $K_m = \overline{F_m - C_\infty}$ in $\Gamma\backslash\mathbb{H}^3_{\mathbb{R}}$ satisfies, by the symmetries and Equation (-15-),

$$\Delta_m \leq d\Big(\big(i\frac{m+1}{4\sqrt{m}}, \frac{\sqrt{14\,m-m^2-1}}{4\sqrt{m}}\big),(0,1)\Big) = \log\frac{4\sqrt{m}+m+1}{\sqrt{14m-m^2-1}}\,.$$

Now, as in (1) above, $\operatorname{Sp}(\mathscr{D})$ is contained in $\big[0,(1+\sqrt{2})\frac{4\sqrt{m}+m+1}{\sqrt{14m-m^2-1}}\big]$ which is for instance contained in $[0,4.664]$ if $m=3$.

(4) By the classification of the hyperbolic Coxeter simplices (see for instance [VS, page 207]), there exists one, called F thereafter, whose Coxeter diagram is

$$\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ\overset{4}{-\!\!\!-}\circ\!\!-\!\!\!-\!\!\circ.$$

Up to isometry of $\mathbb{H}^5_{\mathbb{R}} = \mathbb{R}^4\times\,]0,+\infty[$, we may assume that its ideal vertex is at infinity, and that the opposite face lies on the Euclidean unit sphere centered at 0.

Let $\Gamma = \Gamma_5$ be the group of isometries of $\mathbb{H}^5_{\mathbb{R}}$ generated by the reflexions on the codimension-one faces of F. The one-cusped orbifold $\Gamma_5\backslash\mathbb{H}^5_{\mathbb{R}}$ is the minimal volume cusped hyperbolic orbifold of dimension 5, see [Hil].

The horoball C_∞ is the maximal precisely invariant horoball centered at ∞, see [Hil, Prop. 5]. It is easy to see (see for instance [Hil, page 216]) that the vertical projection of F in $\mathbb{R}^4$ (which is a Euclidean Coxeter simplex with Coxeter diagram $\circ\!-\!\circ\!-\!\circ\overset{4}{-}\circ\!-\!\circ$ of type $\widetilde{F_4}$), has diameter $1/\sqrt{2}$. Thus the diameter of $\overline{F-C_\infty}$ is at most $2d_{\mathbb{H}^2_{\mathbb{R}}}(i,\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}) = 2\log(1+\sqrt{2})$, and $\operatorname{Sp}(\mathscr{D})$ is contained in $[0,(1+\sqrt{2})^3] \subset [0,14.08]$.

Let $\mathbb{H}$ be the skew field of Hamilton's quaternions. The Hurwitz ring $\mathscr{O}'$ consists of all quaternions in $\mathbb{H}$ of the form $\frac{1}{2}(a_0+a_1i+a_2j+a_3k)$ such that the coefficients $a_0,a_1,a_2,a_3\in\mathbb{Z}$ have equal parity. The *Hurwitz modular group* $\Gamma = \mathrm{PSL}_2(\mathscr{O}')$ (defined using the Dieudonné determinant) is, up to conjugation, the derived subgroup of Γ_5, which has index 4 in Γ_5, see [JW, page 186]. Since a fundamental domain for Γ can be built as the connected union of four copies of the fundamental domain F of Γ_5, the approximation spectrum $\operatorname{Sp}(\mathscr{D})$ is contained in $[0,(1+\sqrt{2})^9] \subset [0,2787]$, a very rough estimate.

(5) Before giving the fifth and last example, notice that the model of $\mathbb{H}^2_{\mathbb{C}}$ used therein will differ from the one used in Example 2 of Section 3 to facilitate references to [FP] on which the example is based: We will use the Siegel domain of the complex hyperbolic plane, whose underlying space is, as a subset of the complex projective plane $\mathbb{P}^2(\mathbb{C})$ with nonhomogeneous coordinates,

$$\mathbb{H}^2_{\mathbb{C}} = \big\{[W_0:W:1]\in\mathbb{P}^2(\mathbb{C})\;:\;2\operatorname{Re}W_0+|W|^2<0\big\}\,.$$

Consider the Hermitian form $q = Z_0\overline{Z_2}+Z_2\overline{Z_0}+Z_1\overline{Z_1}$ on $\mathbb{C}^3$, whose signature is $(1,2)$. The *Eisenstein-Picard modular group* $\Gamma = \mathrm{PU}_q(\mathscr{O}_{-3})$ is the projective unitary group of the form q with coefficients in $\mathscr{O}_{-3}$, acting projectively on $\mathbb{H}^2_{\mathbb{C}}$. Let

$$C_\infty = \big\{[W_0 : W : 1] \in \mathbb{P}^2(\mathbb{C}) \;:\; 2\,\mathrm{Re}\,W_0 + |W|^2 \leq -2\big\}\;,$$

which is a horoball centered at $\infty = [-1 : 0 : 0]$. Let $\omega = \frac{-1+i\sqrt{3}}{2}$ and

$$P = \begin{pmatrix} 1 & 1 & \omega \\ 0 & \omega & -\omega \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & \omega \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\;,$$

which define elements of Γ. A fundamental domain D for Γ is constructed in [FP, Theo. 4.15], as a simplex with one infinite vertex at ∞, which is the geodesic cone with cone point ∞ over a tetrahedron T_0 with four finite vertices

$$z_0 = [\bar\omega : 0 : 1], \quad z_1 = [-1 : -\omega : 1], \quad z_2 = [-1 : 1 : 1], \quad z_3 = [\omega : 0 : 1].$$

The images of the interior of D by P, Q, R are disjoint from D since D is a fundamental domain for Γ. By [FP, Prop. 4.6], the element R maps T_0 to itself, and the geodesic cones with vertex ∞ over the four faces of T_0 are paired, by PQ^{-1} and P.

The horoball C_∞, which is invariant by P, Q, and which meets its image by R only in $u_0 = [-1 : 0 : 1] \in T_0$, is hence the maximal precisely invariant horoball centered at ∞. As D is a cone with vertex ∞ and T_0 intersects C_∞ at u_0, the diameter of the complement of the horoball C_∞ in D is attained by two points of T_0. Note that (see [FP], Definition 4.5 and the claim before Proposition 4.6 therein) the faces of T_0 are foliated by geodesic arcs between points of the edges, and that the edges are geodesic arcs. Hence, by convexity of the distance map, the maximal distance between two points of T_0 is attained by a pair of vertices, that is by the maximal length of an edge of the tetrahedron T_0.

The intersection of $\mathbb{H}^2_{\mathbb{C}}$ with the complex lines of equations $W = 0$ and $W_0 = -1$ are totally geodesic, and are respectively a copy of the constant curvature -1 real hyperbolic left halfplane and disc of radius $\sqrt{2}$ and center 0 in $\mathbb{C}$ (and u_0 corresponds to the point $(-1, 0)$ in this left halfplane and to the center of this disc). Hence, for $i = 0$ and $i = 3$, we have $d_{\mathbb{H}^2_{\mathbb{C}}}(z_i, u_0) = \log(2 + \sqrt{3})$ by Equation (-15-). For $i = 1$ and $i = 2$, we have $d_{\mathbb{H}^2_{\mathbb{C}}}(z_i, u_0) = \log(3 + 2\sqrt{2})$. Thus for $0 \leq i, j \leq 3$, by the triangle inequality, we have

$$d_{\mathbb{H}^2_{\mathbb{C}}}(z_i, z_j) \leq d_{\mathbb{H}^2_{\mathbb{C}}}(z_i, u_0) + d_{\mathbb{H}^2_{\mathbb{C}}}(z_j, u_0) \leq 2\log(3 + 2\sqrt{2}) = 4\log(1 + \sqrt{2})\;.$$

Therefore, in the metric of $\mathbb{H}^2_{\mathbb{C}}$ with sectional curvature between -4 and -1, we have the estimate

$$\Delta \leq \max_{0 \leq i,j \leq 3} d'_{\mathbb{H}^2_{\mathbb{C}}}(z_i, z_j) = 2\log(1 + \sqrt{2})\;,$$

and a corresponding estimate on the approximation spectrum by Proposition 4.6

$$\mathrm{Sp}(\mathscr{D}) \subset [0, (1 + \sqrt{2})^3] \subset [0, 14.08]\;.$$

5 Hall rays in approximation spectra

In this section, for some quadruples of data $\mathscr{D} = (X, \Gamma, \Gamma_0, C_\infty)$, we will prove that the approximation spectrum $\mathrm{Sp}(\mathscr{D})$ contains a segment $[0, c]$ for some $c > 0$.

We start by recalling the following result from [PP3]. It says that given a family of almost disjoint neighbourhoods of geodesic lines, there exists a geodesic ray or line, with starting point (at infinity in the case of a line) any given point outside these neighbourhoods, that has a prescribed penetration in one given neighbourhood, and does not penetrate too much in the neighbourhoods thereafter. We refer to Section 2 for the definitions of the various penetration maps.

Theorem 5.1 *[PP3, Theorem 5.9] For every $\epsilon > 0$ and $\delta \geq 0$, there exists a positive constant h'_1 such that the following holds. Let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3. Let $(L_n)_{n\in\mathbb{N}}$ be a family of geodesic lines in X, such that $\operatorname{diam}(\mathscr{N}_\epsilon L_n \cap \mathscr{N}_\epsilon L_m) \leq \delta$ for all $n \neq m$ in $\mathbb{N}$. For every $\xi \in (X \cup \partial_\infty X) - (\mathscr{N}_\epsilon L_0 \cup \partial_\infty L_0)$, let $f_0 : T^1_\xi X \to [0, +\infty[$ be either $f_0 = \mathfrak{ftp}_{L_0}$, or $f_0 = \ell_{\mathscr{N}_\epsilon L_0}$ if X has constant curvature, or $f_0 = \mathfrak{crp}_{L_0}$ (in which case $\xi \in \partial_\infty X - \partial_\infty L_0$) if the metric spheres for the Hamenstädt distances (on $\partial_\infty X - \{\xi'\}$ for any $\xi' \in \partial_\infty X$) are topological spheres. Let $h \geq h'_1$.*

Then there exists a geodesic ray or line ρ starting from ξ and entering $\mathscr{N}_\epsilon L_0$ at time 0 with $f_0(\rho) = h$, such that $\ell_{\mathscr{N}_\epsilon L_n}(\rho) \leq h'_1$ for every $n \neq 0$ such that $\rho(]\delta, +\infty[)$ meets $\mathscr{N}_\epsilon L_n$. □

Note that the condition on the metric spheres of the Hamenstädt distance being topological spheres is satisfied by all negatively curved symmetric spaces.

The following result is an analog of Theorem 5.13 of [PP3], where we considered cusp excursions. It has as hypothesis the conclusion of the previous theorem. It says that if a given family of almost disjoint neighbourhoods of geodesic lines is rich enough, then we can find a geodesic line which has a prescribed upper asymptotic penetration in these neighbourhoods.

We first define what we mean precisely by this. Let X be a proper CAT(-1) space and let $\xi \in X \cup \partial_\infty X$. Let $\epsilon > 0$, $\delta, \kappa \geq 0$. Let $(C_\alpha)_{\alpha\in\mathscr{A}}$ be a family of convex subsets of X such that $\operatorname{diam}(\mathscr{N}_\epsilon C_\alpha \cap \mathscr{N}_\epsilon C_\beta) \leq \delta$ for all $\alpha \neq \beta$ in $\mathscr{A}$. For each $\alpha \in \mathscr{A}$ such that $\xi \notin C_\alpha \cup \partial_\infty C_\alpha$, let $f_\alpha : T^1_\xi X \to [0, +\infty]$ be a map such that $||f_\alpha - \ell_{\mathscr{N}_\epsilon C_\alpha}||_\infty \leq \kappa$. These assumptions guarantee that for every $\rho \in T^1_\xi X$, the set $\mathscr{E}_\rho$ of times $t \geq 0$ such that ρ enters in some C_α at time t with $f_\alpha(\rho) > \delta + \kappa$ is discrete in $[0, +\infty[$, and that such an α is then unique, denoted by α_t. Hence $\mathscr{E}_\rho = (t_i)_{i\in\mathscr{N}}$ for some initial segment $\mathscr{N}$ in $\mathbb{N}$, with $t_i < t_{i+1}$ for $i, i+1$ in $\mathscr{N}$. With $a_i(\rho) = f_{\alpha_{t_i}}(\rho)$, the (finite or infinite) sequence $\big(a_i(\rho)\big)_{i\in\mathscr{N}}$ will be called the *penetration sequence* of ρ with respect to $(\mathscr{N}_\epsilon C_\alpha, f_\alpha)_{\alpha\in\mathscr{A}}$ (and δ, κ). We will be interested in the possible values of $\limsup_{i\to+\infty} a_i(\rho)$, when $\mathscr{N} = \mathbb{N}$.

Theorem 5.2 *Let $\epsilon > 0$ and $\delta, \nu, \nu' \geq 0$. Let X be a proper CAT(-1) space, with $\partial_\infty X$ infinite, and let $\xi \in X \cup \partial_\infty X$. Let $(L_\alpha)_{\alpha\in\mathscr{A}}$ be a family of geodesic lines in X, such that $\operatorname{diam}(\mathscr{N}_\epsilon L_\alpha \cap \mathscr{N}_\epsilon L_\beta) \leq \delta$ for all $\alpha \neq \beta$ in $\mathscr{A}$. For every $\alpha \in \mathscr{A}$ such that $\xi \notin \mathscr{N}_\epsilon L_\alpha \cup \partial_\infty L_\alpha$, let f_α be either $\ell_{\mathscr{N}_\epsilon L_\alpha}$ or $\mathfrak{ftp}_{L_\alpha}$ or $\mathfrak{crp}_{L_\alpha}$, and in this last case, assume that $\xi \in \partial_\infty X$. Let κ be the upper bound of the $||f_\alpha - \ell_{\mathscr{N}_\epsilon L_\alpha}||_\infty$ for all α in $\mathscr{A}$ such that $\xi \notin \mathscr{N}_\epsilon L_\alpha \cup \partial_\infty L_\alpha$. Assume that $\bigcup_{\alpha\in\mathscr{A}} \partial_\infty L_\alpha$ is dense in $\partial_\infty X$. Assume that for every $h \geq \nu$ and $\alpha \in \mathscr{A}$ such that $\xi \notin \mathscr{N}_\epsilon L_\alpha \cup \partial_\infty L_\alpha$, there exists a geodesic ray or line ρ starting from ξ and entering $\mathscr{N}_\epsilon L_\alpha$ at time $t = 0$ with $f_\alpha(\rho) = h$, and with $f_\beta(\rho) \leq \nu'$ for every β in $\mathscr{A} - \{\alpha\}$ such that $\rho(]\delta, +\infty[)$ meets $\mathscr{N}_\epsilon L_\beta$. Let $\big(a_i(\rho')\big)_{n\in\mathscr{N}}$ be the penetration sequence of a geodesic ray or line ρ' with respect to $(\mathscr{N}_\epsilon L_\alpha, f_\alpha)_{\alpha\in\mathscr{A}}$ (and δ, κ).*

Then, there exists $h_* = h_*(\epsilon, \delta, \kappa, \nu, \nu') > 0$ such that for every $h \geq h_*$, there exists a geodesic ray or line ρ starting from ξ such that

$$\limsup_{i \to +\infty} a_i(\rho) = h \ .$$

Proof. We start by recalling two lemmas from [PP3], which explain the relative penetration behaviour of a pair of geodesic lines in the ϵ-neighbourhoods of geodesic lines.

Lemma 5.3 *[PP3, Lemma 2.3] Let C be a convex subset in X, let $\epsilon > 0$ and let $\xi_0 \in (X \cup \partial_\infty X) - (\mathscr{N}_\epsilon C \cup \partial_\infty C)$. If two geodesic rays or lines ρ, ρ' which start from ξ_0 intersect $\mathscr{N}_\epsilon C$, then the first intersection points x, x' of ρ, ρ' respectively with $\mathscr{N}_\epsilon C$ are at a distance at most $c_1'(\epsilon) = 2\,\mathrm{argsinh}(\coth \epsilon)$.* $\Box$

Lemma 5.4 *[PP3, Lemmas 2.5 and 2.6] For every $\epsilon, \eta > 0$, there exist (explicit) constants $c_2'(\epsilon), c_3'(\epsilon) > 0$ and $c(\epsilon, \eta) > 0$ such that the following holds. Let X be a $\mathrm{CAT}(-1)$ space, C a convex subset in X, $\xi_0 \in X \cup \partial_\infty X$, and ρ, ρ' two geodesic rays or lines starting from ξ_0. If ρ enters $\mathscr{N}_\epsilon C$ at a point $x \in X$ and exits $\mathscr{N}_\epsilon C$ at a point $y \in X$ such that $d(x, y) \geq c(\epsilon, \eta)$ and $d(y, \rho') \leq \eta$, then ρ' enters $\mathscr{N}_\epsilon C$ at a point $x' \in X$ such that $d(x, x') \leq c_2'(\epsilon)\, d(x, \rho')$ and exits $\mathscr{N}_\epsilon C$ at a point $y' \in X \cup \partial_\infty X$ such that*

$$d(y, y') \leq c_3'(\epsilon)\, d(y, \rho') \ \text{ or } \ d(x', y') > d(x, y) \ . \ \Box$$

Let $X, (L_\alpha, f_\alpha)_{\alpha \in \mathscr{A}}, \xi, c, c', \kappa$ be as in the statement of Theorem 5.2. Note that by the equations (-4-) and (-5-), we have

$$\kappa \leq 2\, c_1'(\epsilon) + 2\epsilon + 4 \log(1 + \sqrt{2}) \ .$$

In particular, κ is finite. We start the proof of this theorem by defining the constants that will be used therein. Let

$$c_* = \kappa + \max\left\{2(c_1'(\epsilon) + \delta),\ c(\epsilon, \delta + c_1'(\epsilon)),\ c_1'(\epsilon) c_2'(\epsilon) + c_3'(\epsilon)(c_1'(\epsilon) + \delta) + \nu' + \kappa\right\} \ ,$$

where the positive constants $c_i'(\cdot)$ for $i = 1, 2, 3$ and $c(\cdot, \cdot)$ are defined in the lemmas 5.3 and 5.4. Note that $c_* > \max\{\kappa + 2\delta, \nu'\}$, since $c_1'(\epsilon) \geq 1$ for all $\epsilon > 0$. Let

$$h_* = h_*(\epsilon, \delta, \kappa, \nu, \nu') = \max\{c_*, \nu\} \ . \tag{-16-}$$

Let $h \geq h_*$, and let $\alpha_0 \in \mathscr{A}$ be such that $\xi \notin \mathscr{N}_\epsilon L_{\alpha_0} \cup \partial_\infty L_{\alpha_0}$. The existence of such an index follows from the assumptions: Indeed, as $\partial_\infty X$ is (Hausdorff and) infinite, and by the density of $\bigcup_{\alpha \in \mathscr{A}} \partial_\infty L_\alpha$, the set $\mathscr{A}$ is infinite; note that $\partial_\infty L_\alpha \cap \partial_\infty L_\beta$ is empty if $\alpha \neq \beta$, otherwise, as geodesic rays converging to the same point at infinity become exponentially close, we would have $\mathrm{diam}(\mathscr{N}_\epsilon L_\alpha \cap \mathscr{N}_\epsilon L_\beta) = +\infty$; hence ξ belongs to at most one $\partial_\infty L_\alpha$ if $\xi \in \partial_\infty X$; if $\xi \in X$, then ξ belongs to at most finitely many $\mathscr{N}_\epsilon L_\alpha$ for $\alpha \in \mathscr{A}$, as X is proper and $\mathrm{diam}(\mathscr{N}_\epsilon L_\alpha \cap \mathscr{N}_\epsilon L_\beta) \leq \delta$ if $\alpha \neq \beta$.

As $h \geq h_* \geq \nu$, there exists, by the assumptions of Theorem 5.2, a geodesic ray or line ρ_0 starting from ξ, entering $\mathscr{N}_\epsilon L_{\alpha_0}$ at time $t = 0$, such that $f_{\alpha_0}(\rho_0) = h$, and $f_\alpha(\rho_0) \leq \nu'$ for every $\alpha \neq \alpha_0$ such that $\rho_0(]\delta, +\infty[)$ meets $\mathscr{N}_\epsilon L_\alpha$.

If a geodesic ray or line ρ starting from ξ meets $\mathscr{N}_\epsilon L_\alpha$ such that $\xi \notin \mathscr{N}_\epsilon L_\alpha \cup \partial_\infty L_\alpha$, let $t^-_\alpha(\rho)$ and $t^+_\alpha(\rho)$ be the entrance and exit times.

We construct, by induction, sequences $(\rho_k)_{k\in\mathbb{N}}$ of geodesic rays or lines starting from ξ, $(\alpha_k)_{k\in\mathbb{N}}$ of elements of $\mathscr{A}$, and $(t_k)_{k\in\mathbb{N}-\{0\}}$ of elements in $[0,+\infty[$ converging to $+\infty$, such that for every $k \in \mathbb{N}$,

(1) ρ_k enters the interior of $\mathscr{N}_\epsilon L_{\alpha_0}$ at time 0, with $d(\rho_k(0), \rho_{k-1}(0)) \leq \frac{1}{2^k}$ if $k \geq 1$;

(2) ρ_k enters $\mathscr{N}_\epsilon L_{\alpha_k}$, $\xi \notin \mathscr{N}_\epsilon L_{\alpha_k} \cup \partial_\infty L_{\alpha_k}$ and $f_{\alpha_k}(\rho_k) = h$;

(3) if $0 \leq j \leq k-1$, then $\rho_k(]0,+\infty[)$ enters the interior of $\mathscr{N}_\epsilon L_{\alpha_j}$ before entering $\mathscr{N}_\epsilon L_{\alpha_k}$ with $t^-_{\alpha_j}(\rho_k) < t_k = t^+_{\alpha_{k-1}}(\rho_k) < t^+_{\alpha_k}(\rho_k)$;

(4) if $k \geq 1$, then for every α such that $\rho_k(]0,+\infty[)$ meets $\mathscr{N}_\epsilon L_\alpha$, we have
   - $\big|f_\alpha(\rho_k) - f_\alpha(\rho_{k-1})\big| < \frac{1}{2^k}$ if $t^-_\alpha(\rho_k) < t_k$,
   - $f_\alpha(\rho_k) \leq c_*$ if $\alpha \neq \alpha_k$ and $t_k \leq t^-_\alpha(\rho_k) \leq t^-_{\alpha_k}(\rho_k) + \delta$,
   - $f_\alpha(\rho_k) \leq \nu'$ if $t^-_\alpha(\rho_k) > t^-_{\alpha_k}(\rho_k) + \delta$.

Let us first prove that the existence of such sequences implies Theorem 5.2. By the assertion (1), the sequence $\big(\rho_k(0)\big)_{k\in\mathbb{N}}$ stays at bounded distance from $\rho_0(0)$, by a geometric series argument. Hence as X is proper, up to extracting a subsequence, the sequence $(\rho_k)_{k\in\mathbb{N}}$ converges to a geodesic ray or line ρ_∞ starting from ξ, entering in $\mathscr{N}_\epsilon L_{\alpha_0}$ at time $t = 0$, by the continuity of the entering point in the interior of the ϵ-neighbourhood of a convex subset of X (see for instance [PP3, Lemma 3.1]). Let us prove that

$$\limsup_{i\to+\infty}\ a_i(\rho_\infty) = h\ .$$

The lower bound $\limsup_{i\to+\infty}\ a_i(\rho_\infty) \geq h$ is immediate by a semicontinuity argument. Indeed, for every $k > i$ in $\mathbb{N}$, we have by the assertions (2), (3) and (4),

$$\big|f_{\alpha_i}(\rho_k) - h\big| = \big|f_{\alpha_i}(\rho_k) - f_{\alpha_i}(\rho_i)\big| \leq \sum_{j=i}^{k-1}\big|f_{\alpha_i}(\rho_{j+1}) - f_{\alpha_i}(\rho_j)\big| \leq \sum_{j=i}^{k-1}\frac{1}{2^{j+1}} \leq \frac{1}{2^i}\ .$$

Hence by the continuity of f_{α_i} (see Section 2), we have the inequality $f_{\alpha_i}(\rho_\infty) \geq h - \frac{1}{2^i}$, whose right side converges to h as i tends to $+\infty$, which proves the lower bound, by the definition of κ and of the penetration sequence, as $h \geq h_* \geq c_* > \delta + \kappa$.

To prove the upper bound $\limsup_{i\to+\infty}\ a_i(\rho_\infty) \leq h$, assume by contradiction that there exists $\eta > 0$ such that for every $\lambda > 0$, there exists $\alpha = \alpha(\lambda) \in \mathscr{A}$ such that ρ_∞ enters $\mathscr{N}_\epsilon L_\alpha$ with $f_\alpha(\rho_\infty) \geq h + \eta$ and $t^-_\alpha(\rho_\infty) > \lambda + 2\,c'_1(\epsilon)$, where $c'_1(\epsilon)$ has been defined in Lemma 5.3. Take

$$\lambda_0 = \max\Big\{t_{i+1} : \frac{1}{2^i} \geq \frac{\eta}{2}\Big\}$$

and $\alpha = \alpha(\lambda_0)$.

By continuity of f_α, if k is big enough, we have $f_\alpha(\rho_k) \geq h + \frac{\eta}{2} > h_*$. In particular, $\alpha \neq \alpha_k$ by the assertion (2). Since

$$h_* \geq c_* \geq \kappa \geq |f_\alpha(\rho_k) - \ell_{\mathscr{N}_\epsilon L_\alpha}(\rho_k)|\ ,$$

the geodesic ρ_k meets $\mathscr{N}_\epsilon L_\alpha$. The entry time of ρ_k in $\mathscr{N}_\epsilon L_\alpha$ is positive, as $d(\rho_k(0), \rho_\infty(0)) \leq c_1'(\epsilon)$ and the entrance points of ρ_k and ρ_∞ in $\mathscr{N}_\epsilon L_\alpha$ are at distance at most $c_1'(\epsilon)$, both by Lemma 5.3, and as the entrance time of ρ_∞ in $\mathscr{N}_\epsilon L_\alpha$ is bigger than $2\,c_1'(\epsilon)$. Hence, since $\nu' \leq c_* \leq h_*$ by the definitions of c_* and of h_*, we have $t^-_\alpha(\rho_k) < t_k$, otherwise, by the assertion (4), $f_\alpha(\rho_k) \leq \max\{c_*, \nu'\} = c_* \leq h_*$, a contradiction. Let $i \leq k-1$ be the minimum element of $\mathbb{N}$ such that for $j = i, \dots, k-1$, the geodesic ρ_{j+1} meets $\mathscr{N}_\epsilon L_\alpha$ at a positive time with $t^-_\alpha(\rho_{j+1}) < t_{j+1}$. By the triangle inequality, we have

$$\big|t^-_\alpha(\rho_{i+1}) - t^-_\alpha(\rho_\infty)\big| \leq d\big(\rho_{i+1}(t^-_\alpha(\rho_{i+1})), \rho_\infty(t^-_\alpha(\rho_\infty))\big) + d\big(\rho_{i+1}(0), \rho_\infty(0)\big) \leq 2\,c_1'(\epsilon)\ ,$$

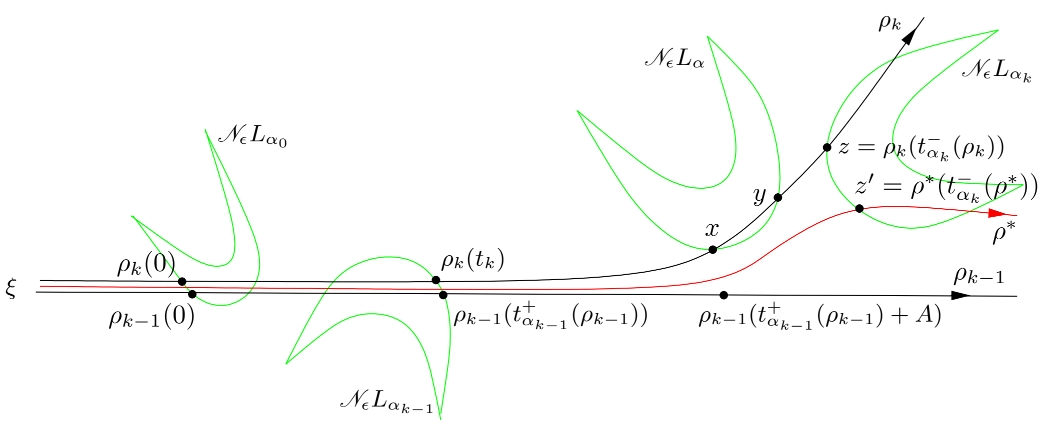
by applying twice Lemma 5.3. Hence, by the definition of i and of α,

$$t_{i+1} > t^-_\alpha(\rho_{i+1}) \geq t^-_\alpha(\rho_\infty) - 2\,c_1'(\epsilon) > \lambda_0 + 2\,c_1'(\epsilon) - 2\,c_1'(\epsilon) = \lambda_0.$$

By the definition of λ_0, we hence have $\frac{1}{2^i} < \frac{\eta}{2}$. By the definition of i and by the assertion (4), we have

$$\begin{aligned} f_\alpha(\rho_i) &= f_\alpha(\rho_k) + \sum_{j=i}^{k-1} \big(f_\alpha(\rho_j) - f_\alpha(\rho_{j+1})\big) \geq h + \frac{\eta}{2} - \sum_{j=i}^{k-1} \frac{1}{2^{j+1}} \\ &\geq h + \frac{\eta}{2} - \frac{1}{2^i} \geq h \geq h_*\ , \end{aligned}$$

and in particular by the same argument as for ρ_k above, ρ_i enters $\mathscr{N}_\epsilon L_\alpha$ at a positive time and $t^-_\alpha(\rho_i) < t_i$. This contradicts the minimality of i. This completes the proof of Theorem 5.2, assuming the existence of the sequences with the properties (1)–(4).



Let us now construct the sequences $(\rho_k)_{k\in\mathbb{N}}$, $(\alpha_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}-\{0\}}$. We already have defined ρ_0 and α_0, and they satisfy the properties (1)–(4). Let $k \geq 1$, and assume that ρ_{k-1}, α_{k-1}, as well as t_{k-1} if $k \geq 2$, have been constructed.

For every small $\mu > 0$ and big $A > 0$ (to be precised later on), consider the set $\mathscr{E}' = \mathscr{E}'_k$ of couples (α, ρ) where $\alpha \in \mathscr{A}$ and ρ is a geodesic ray or line, starting from ξ, entering $\mathscr{N}_\epsilon L_{\alpha_0}$ at time $t = 0$, which is μ-close to ρ_{k-1} on $[0, t^+_{\alpha_{k-1}}(\rho_{k-1}) + A]$, and which enters $\mathscr{N}_\epsilon L_\alpha$ with $t^-_\alpha(\rho) \geq t^+_{\alpha_{k-1}}(\rho_{k-1}) + \frac{A}{2}$. This set $\mathscr{E}'$ is not empty, as $\bigcup_{\alpha\in\mathscr{A}} \partial_\infty L_\alpha$ is dense in $\partial_\infty X$, and as the assumption that $\operatorname{diam}(\mathscr{N}_\epsilon L_\alpha \cap \mathscr{N}_\epsilon L_\beta) \leq \delta$ for all $\alpha \neq \beta$ in $\mathscr{A}$ implies that a compact subset of X meets only finitely many $\mathscr{N}_\epsilon L_\alpha$ for $\alpha \in \mathscr{A}$ (we may even find such a couple (α, ρ) with $\rho(+\infty) \in \partial_\infty L_\alpha$, which implies that a subray of ρ is contained in $\mathscr{N}_\epsilon L_\alpha$). Let (α_k, ρ^*) be an element of $\mathscr{E}'$ with $t^-_{\alpha_k}(\rho^*)$ minimal, which exists since the family $(\mathscr{N}_\epsilon L_\alpha)_{\alpha\in\mathscr{A}}$ is locally finite and by a continuity argument (when $\mathscr{A}$ is given the discrete topology, the subset $\mathscr{E}'$ is closed). Note that $\xi \notin \mathscr{N}_\epsilon L_{\alpha_k} \cup \partial_\infty L_{\alpha_k}$, as $t^-_{\alpha_k}(\rho^*) > 0$.

By the last hypothesis of Theorem 5.2, let ρ_k be a geodesic ray or line starting from ξ with $f_{\alpha_k}(\rho_k) = h$ and $f_\alpha(\rho_k) \leq \nu'$ for every $\alpha \in \mathscr{A}$ such that $\rho_k(]t^-_{\alpha_k}(\rho_k) + \delta, +\infty[)$ enters H_α. In particular, this proves the assertion (2) at rank k, since

$$h \geq h_* \geq c_* > \kappa \geq |f_{\alpha_k}(\rho_k) - \ell_{\mathscr{N}_\epsilon L_{\alpha_k}}(\rho_k)| \,. \qquad (-17-)$$

Let $z = \rho_k(t^-_{\alpha_k}(\rho_k))$ and $z' = \rho^*(t^-_{\alpha_k}(\rho^*))$ be the entering points of ρ_k and ρ^* in $\mathscr{N}_\epsilon L_{\alpha_k}$. By Lemma 5.3, we have $d(z, z') \leq c_1'(\epsilon)$. Hence, by hyperbolicity and as (α_k, ρ^*) is in $\mathscr{E}'$, if A is big enough, then ρ_k is (2μ)-close to ρ_{k-1} between ξ and $\rho_{k-1}(t^+_{\alpha_{k-1}}(\rho_{k-1}) + 1)$. In particular, if μ is small enough, and using properties (1) and (3) at rank $k-1$, we have the following properties.

- The geodesic ray or line ρ_k enters the interior of $\mathscr{N}_\epsilon L_{\alpha_0}$, at a time that we may assume to be 0, with $d(\rho_k(0), \rho_{k-1}(0)) \leq \frac{1}{2^k}$ (this proves the assertion (1) at rank k).
- For $0 \leq j \leq k-1$, the geodesic ray or line ρ_{k-1} meets the interior of $\mathscr{N}_\epsilon L_{\alpha_j}$ at a time strictly between 0 and $t^+_{\alpha_{k-1}}(\rho_{k-1})$, by the inductive assertions (3) if $k \neq 1$ and $j \leq k-2$, or (1) if $k = 1$ or (2) if $j = k-1$ (by Equation (-17-) where k has been replaced by $k-1$). Hence the geodesic ray ρ_k also meets the interior of $\mathscr{N}_\epsilon L_{\alpha_j}$ at a time strictly between 0 and $t^+_{\alpha_{k-1}}(\rho_{k-1})$. This allows, in particular, to define $t_k = t^+_{\alpha_{k-1}}(\rho_k)$, and proves the assertion (3) at rank k.
- For every α such that $\rho_k(]0, +\infty[)$ meets $\mathscr{N}_\epsilon L_\alpha$ and $t^-_\alpha(\rho_k) < t_k$, we may assume, by the continuity of f_α, up to taking μ small enough, that $\big|f_\alpha(\rho_k) - f_\alpha(\rho_{k-1})\big| < \frac{1}{2^k}$.

Hence (using also the construction of ρ_k), to prove the assertion (4) at rank k, we consider $\alpha \in \mathscr{A} - \{\alpha_k\}$ such that ρ_k meets $\mathscr{N}_\epsilon L_\alpha$ with $t_k \leq t^-_\alpha(\rho_k) \leq t^-_{\alpha_k}(\rho_k) + \delta$, and we prove that $f_\alpha(\rho_k) \leq c_*$.

Assume by absurd that $f_\alpha(\rho_k) > c_*$. In particular, $\ell_{\mathscr{N}_\epsilon L_\alpha}(\rho_k) \geq c_* - \kappa > 0$ (by the definition of c_*), so that ρ_k enters $\mathscr{N}_\epsilon L_\alpha$. Let $x = \rho_k(t^-_\alpha(\rho_k))$ be the entering point of ρ_k in $\mathscr{N}_\epsilon L_\alpha$. Note that

$$\ell_{\mathscr{N}_\epsilon L_{\alpha_k}}(\rho_k) \geq f_{\alpha_k}(\rho_k) - \kappa = h - \kappa \geq h_* - \kappa \geq c_* - \kappa \,,$$

by the definition of h_*. If $t^-_\alpha(\rho_k) \geq t^-_{\alpha_k}(\rho_k)$, then, since $t^-_\alpha(\rho_k) \leq t^-_{\alpha_k}(\rho_k) + \delta$ and

$$\min\{\ell_{\mathscr{N}_\epsilon L_\alpha}(\rho_k), \ell_{\mathscr{N}_\epsilon L_{\alpha_k}}(\rho_k)\} \geq c_* - \kappa > 2\delta$$

by the definition of c_*, this would imply that the intersection $\mathscr{N}_\epsilon L_\alpha \cap \mathscr{N}_\epsilon L_{\alpha_k}$ has diameter bigger than δ, which contradicts $\alpha \neq \alpha_k$. Hence $t^-_\alpha(\rho_k) < t^-_{\alpha_k}(\rho_k)$ (which implies that $x \in [\xi, z]$) and we have $t^+_\alpha(\rho_k) \leq t^-_{\alpha_k}(\rho_k) + \delta$, again since $\operatorname{diam}(\mathscr{N}_\epsilon L_\alpha \cap \mathscr{N}_\epsilon L_{\alpha_k}) \leq \delta$ and $\alpha \neq \alpha_k$. In particular, $y = \rho_k(t^+_\alpha(\rho_k))$ is a point in X.

We want to apply Lemma 5.4 with $\eta = \delta + c_1'(\epsilon)$, $\rho = \rho_k$, $\rho' = \rho^*$, $C = L_\alpha$ and $\xi_0 = \xi$. We first check the hypotheses of this lemma.

We do have that ρ_k enters $\mathscr{N}_\epsilon L_\alpha$ at x and exits it at y, and

$$d(x,y) \geq f_\alpha(\rho_k) - \kappa > c_* - \kappa \geq c(\epsilon,\eta)\ , \tag{-18-}$$

by the definition of c_*. As

$$d(\rho_k(t^-_{\alpha_k}(\rho_k)+\delta),\rho^*) \leq \delta + d(\rho_k(t^-_{\alpha_k}(\rho_k)),\rho^*) \leq \delta + d(z,z') \leq \eta\ ,$$

and by convexity, we have $d(y,\rho^*) \leq \eta$.

Hence we may indeed apply Lemma 5.4, and the geodesic ρ^* enters $\mathscr{N}_\epsilon L_\alpha$ at a point x' such that

$$d(x,x') \leq c_2'(\epsilon)\, d(x,\rho^*) \leq c_2'(\epsilon)\, d(z,\rho^*) \leq c_2'(\epsilon)\, c_1'(\epsilon)\ , \tag{-19-}$$

where the middle inequality holds by convexity, and the last one since $d(z,z') \leq c_1'(\epsilon)$.

Furthermore, the geodesic ρ^* exits $\mathscr{N}_\epsilon L_\alpha$ at a point y' (possibly at infinity) and, by the alternative at the end of Lemma 5.4 and the equations (-18-) and (-19-),

$$d(x',y') > d(x,y) > c_* - \kappa \geq \nu' + \kappa$$

or

$$\begin{aligned} d(x',y') &\geq d(x,y) - d(x,x') - d(y,y') \geq d(x,y) - d(x,x') - c_3'(\epsilon)\, d(y,\rho^*) \\ &> (c_* - \kappa) - c_2'(\epsilon)\, c_1'(\epsilon) - c_3'(\epsilon)\eta \geq \nu' + \kappa\ , \end{aligned}$$

by the definition of c_*. In both cases, $d(x',y') > \nu' + \kappa$.

Let us prove that

$$t^-_{\alpha_k}(\rho^*) > t^-_\alpha(\rho^*)\ . \tag{-20-}$$

Otherwise, the point z' belongs to $[\xi, x']$. Hence, with x'', z'' the closest points to x', z' respectively on ρ_k, we have $z'' \in [\xi, x'']$. Note that $x'' \in [\xi, y]$ since $d(x,x'') \leq d(x,x') \leq c_1'(\epsilon)$ and $d(x,y) > c_* - \kappa > c_1'(\epsilon)$ by Equation (-18-) and the definition of c_*.

Respectively by Equation (-18-), by the triangle inequality, since $z'' \in [\xi, x'']$ and $x'' \in [\xi, y]$, since $t^-_\alpha(\rho_k) \leq t^-_{\alpha_k}(\rho_k) + \delta$, since closest point maps do not increase distances, and by Lemma 5.3,

$$\begin{aligned} c_* - \kappa &\leq d(x,y) \leq d(x,x'') + d(x'',y) \leq d(x,x'') + d(z'',y) \leq d(x,x'') + d(z'',z) + \delta \\ &\leq d(x,x') + d(z',z) + \delta \leq 2\, c_1'(\epsilon) + \delta\ , \end{aligned}$$

which contradicts the definition of c_*.

Now, recall the constants $\mu > 0$ and $A \geq 0$ introduced in the definition of ρ^*. It follows from Equation (-20-), and from the minimality assumption in the definition of α_k, that we have

$$t^-_\alpha(\rho^*) < t^+_{\alpha_{k-1}}(\rho_{k-1}) + \frac{A}{2}\ .$$

Assume that μ is small enough and that A is big enough. Since $d(x',y') > \nu' + \kappa$, and as ρ^* is μ-close to ρ_{k-1} on $[0, t^+_{\alpha_{k-1}}(\rho_{k-1}) + A]$, this implies that ρ_{k-1} enters $\mathscr{N}_\epsilon L_\alpha$ at a point $x^\sharp$ close to x', and exits at a point $y^\sharp$ (possibly at infinity) such that $d(x^\sharp, y^\sharp) > \nu' + \kappa$. Hence

$$f_\alpha(\rho_{k-1}) \geq d(x^\sharp, y^\sharp) - \kappa > \nu'\ .$$

This implies that $t^-_\alpha(\rho_{k-1}) \le t^-_{\alpha_{k-1}}(\rho_{k-1})+\delta$, otherwise we have in particular that $\alpha \ne \alpha_{k-1}$ and by the assertion (4) at rank $k-1$ if $k \ge 2$ or by the construction of ρ_0 if $k=1$, we would have $f_\alpha(\rho_{k-1}) \le \nu'$. Hence

$$t^+_{\alpha_{k-1}}(\rho_{k-1}) - t^-_\alpha(\rho_{k-1}) \ge t^+_{\alpha_{k-1}}(\rho_{k-1}) - t^-_{\alpha_{k-1}}(\rho_{k-1}) - \delta = \ell_{\mathscr{N}_\epsilon L_{\alpha_{k-1}}}(\rho_{k-1}) - \delta$$
$$\ge f_{\alpha_{k-1}}(\rho_{k-1}) - \kappa - \delta = h - \delta - \kappa \ge h_* - \delta - \kappa \ge c_* - \delta - \kappa \;.$$

That is, ρ_{k-1} enters in $\mathscr{N}_\epsilon L_\alpha$ well before exiting $\mathscr{N}_\epsilon L_{\alpha_{k-1}}$, the amount of time being at least the constant $c_* - \delta - \kappa$ (which is positive by the definition of c_*). But since the entering points in $\mathscr{N}_\epsilon L_\alpha$, as well as the exiting points out of $\mathscr{N}_\epsilon L_{\alpha_{k-1}}$, of the geodesic rays or lines ρ_{k-1}, $\rho*$ and ρ_k are very close, this contradicts the fact that $t^-_\alpha(\rho_k) \ge t_k = t^+_{\alpha_{k-1}}(\rho_k)$.

This proves the result. □

Corollary 5.5 *Let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 3, such that the metric spheres for the Hamenstädt distances (on $\partial_\infty X - \{\xi'\}$ for any $\xi' \in \partial_\infty X$) are topological spheres. Let Γ be a discrete group of isometries of X with finite covolume, and let γ_0 be a hyperbolic element of Γ. Let $\xi_0 \in \partial_\infty X$ be a parabolic fixed point, and H_0 be a horosphere centered at ξ_0. For every $\xi \in \partial_\infty X$ which is not a fixed point of a conjugate of γ_0 or a parabolic fixed point, define*

$$c'(\xi) = \liminf \frac{d_{\xi_0,H_0}(\xi,\gamma_-)}{d_{\xi_0,H_0}(\gamma_+,\gamma_-)} \;,$$

where the lower limit is taken over the conjugates γ of γ_0 or its inverse, with fixed points γ_-, γ_+ and $d_{\xi_0,H_0}(\gamma_+,\gamma_-)$ tending to 0.

Then the subset of $\mathbb{R}$ consisting of the $c'(\xi)$ for $\xi \in \partial_\infty X$ which is neither a fixed point of a conjugate of γ_0 nor a parabolic fixed point, contains a segment $[0,c]$ for some $c>0$.

Proof. We will apply Theorem 5.2 with $(L_\alpha)_{\alpha\in\mathscr{A}}$ the family of translation axes of the conjugates of the element γ_0 (where each line appears exactly once), with $\xi = \xi_0$ and with $f_\alpha = \mathfrak{crp}_{L_\alpha}$ for every α in $\mathscr{A}$. Let $\kappa = 2\,c'_1(\epsilon) + 2\epsilon + 4\log(1+\sqrt{2})$, which satisfies $\|f_\alpha - \ell_{N_\epsilon L_\alpha}\|_\infty \le \kappa$ by Section 2.

For some positive ϵ and δ, this family satisfies the assumption that $\operatorname{diam}\big(\mathscr{N}_\epsilon L_\alpha \cap \mathscr{N}_\epsilon L_\beta\big) \le \delta$ for all $\alpha \ne \beta$ in $\mathscr{A}$. Otherwise, there would exist a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\Gamma - \Gamma_0$, where Γ_0 is the stabilizer in Γ of the translation axis L_0 of γ_0, such that $\operatorname{diam}\big(\mathscr{N}_\epsilon L_0 \cap \gamma_n \mathscr{N}_\epsilon L_0\big)$ converges to $+\infty$. Up to multiplying γ_n on the right and on the left by a power of γ_0 or its inverse, the element γ_n moves a point of L_0 less than a constant. Hence γ_n stays in a compact subset of the isometry group of X. By discreteness, up to extracting a subsequence, γ_n does not depend on n. But then L_0 and $\gamma_1 L_0$ are two distinct translation axes that meet at least in one point at infinity, which contradicts the discreteness of Γ.

As Γ has finite covolume, the set of fixed points of the conjugates of γ_0 is dense in $\partial_\infty X$, hence $\bigcup_{\alpha\in\mathscr{A}} \partial_\infty L_\alpha$ is dense in $\partial_\infty X$. The last hypothesis of Theorem 5.2 holds true by Theorem 5.1, with $\nu = h'_1$ and $\nu' = h'_1 + \kappa$ (by definition of κ). By Theorem 5.2, there exists $h_* > 0$ such that for every $h \ge h_*$, there exists a geodesic line ρ starting from ξ_0 such that $\limsup_{i\to+\infty} a_i(\rho) = h$ where $a_i(\rho)$ is the penetration sequence of ρ with respect to $(\mathscr{N}_\epsilon L_\alpha, f_\alpha)_{\alpha\in\mathscr{A}}$ (and δ,κ).

For every i in $\mathbb{N}$, let $\alpha_i \in \mathscr{A}$ be the unique element such that $a_i(\rho) = \mathfrak{crp}_{L_{\alpha_i}}(\rho)$. For every $\alpha \in \mathscr{A}$ such that ρ meets $\mathscr{N}_\epsilon L_\alpha$ at a time big enough with $\mathfrak{crp}_{L_\alpha}(\rho) > 0$, let $L_{\alpha,\pm}$

be the two endpoints of L_α such that, by the definition of the crossratio penetration map and by Equation (-3-),

$$\mathfrak{crp}_{L_\alpha}(\rho) = [\xi_0, L_{\alpha,-}, \rho(+\infty), L_{\alpha,+}] = \log \frac{d_{\xi_0,H_0}(L_{\alpha,+}, L_{\alpha,-})}{d_{\xi_0,H_0}(\rho(+\infty), L_{\alpha,-})} \;.$$

Only finitely many L_α's meet a given compact subset of X. Thus, for every subsequence $(i_k)_{k\in\mathbb{N}}$ such that the sequence $\big(\mathfrak{crp}_{L_{\alpha_{i_k}}}(\rho)\big)_{k\in\mathbb{N}}$ is bounded, since the entrance time of ρ in $\mathscr{N}_\epsilon L_{\alpha_{i_k}}$ tends to $+\infty$, the distance $d_{\xi_0,H_0}(L_{\alpha_{i_k},+}, L_{\alpha_{i_k},-})$ tends to 0 as $k\to+\infty$. Also note that, by definition of the penetration sequence, if $\alpha\in\mathscr{A}$ does not belong to $\{\alpha_i \;:\; i\in\mathbb{N}\}$, then either ρ does not meet $\mathscr{N}_\epsilon L_\alpha$ at a positive time, or $\mathfrak{crp}_{L_\alpha}(\rho)\le\delta+\kappa<h_*\le h$, see Equation (-16-).

Finally, if $(\gamma_k)_{k\in\mathbb{N}}$ is a sequence of conjugates of γ_0 or its inverse with fixed points $\gamma_{k,-},\gamma_{k,+}$ and with $d_{\xi_0,H_0}(\gamma_{k,-},\gamma_{k,+})$ tending to 0, such that the sequence $\frac{d_{\xi_0,H_0}(\rho(+\infty),\gamma_{k,-})}{d_{\xi_0,H_0}(\gamma_{k,-},\gamma_{k,+})}$ is bounded from above, then $\gamma_{k,-}$ tends to $\rho(+\infty)$. Hence, if

$$\liminf_{k\to+\infty}\;\frac{d_{\xi_0,H_0}(\rho(+\infty),\gamma_{k,-})}{d_{\xi_0,H_0}(\gamma_{k,-},\gamma_{k,+})}\le e^{-h},$$

then for every $\epsilon\in\,]0,h_*-\delta-\kappa[\,$, for k big enough, there exists $\alpha\in\mathscr{A}$ such that L_α is the translation axis of γ_k, and $\mathfrak{crp}_{L_\alpha}(\rho)\ge h-\epsilon\ge h_*-\epsilon>\delta+\kappa$ so that ρ meets $\mathscr{N}_\epsilon L_\alpha$, at a positive time. In particular α belongs to $\{\alpha_i \;:\; i\in\mathbb{N}\}$. Therefore, we have

$$\liminf \frac{d_{\xi_0,H_0}(\rho(+\infty),\gamma_-)}{d_{\xi_0,H_0}(\gamma_+,\gamma_-)} = e^{-h}\;,$$

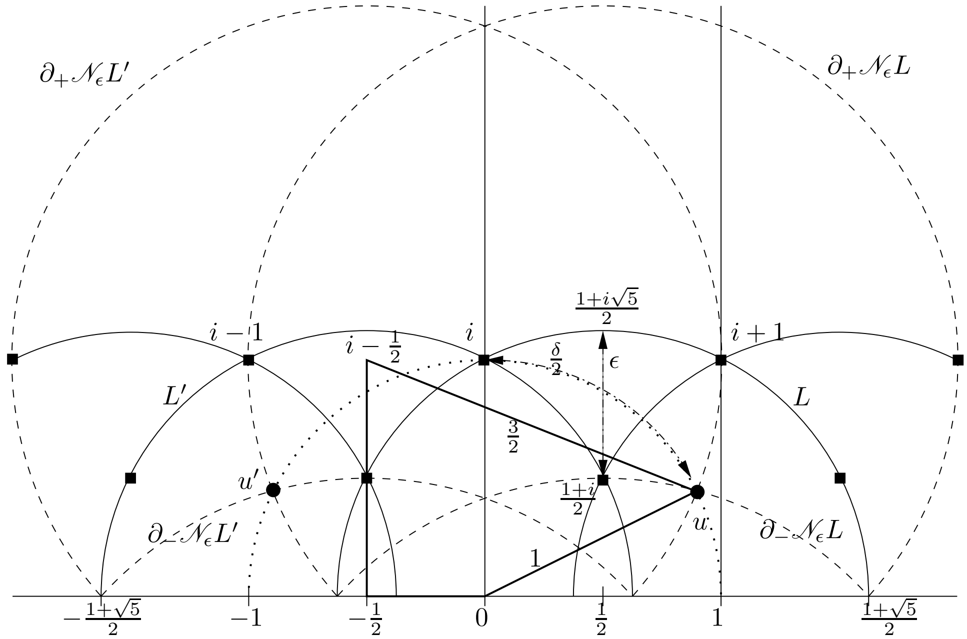
where the lower limit is taken as in the statement of the corollary. This proves the result, with $c=e^{-h_*(\epsilon,\delta,\kappa,h_1',h_1'+\kappa)}$. □

Specializing the above Corollary 5.5 to the particular cases of the real or complex hyperbolic space (see the examples at the end of Section 3), we have the following applications.

Corollary 5.6 *Let $n\ge3$, let Γ be a discrete group of isometries of $X=\mathbb{H}^n_\mathbb{R}$ with finite covolume, and let Γ_0 be the stabilizer in Γ of the translation axis of a hyperbolic element of Γ. Let C_∞ be a precisely invariant horoball centered at a parabolic fixed point of Γ, and $\mathscr{D}=(X,\Gamma,\Gamma_0,C_\infty)$. Then $\mathrm{Sp}(\mathscr{D})$ contains a segment $[0,c]$ for some $c>0$.* □

By the last equality in the proof of Corollary 5.5 (and since the constant h_1' appearing in Theorem 5.1 is explicited in [PP3]), if one wants in particular situations to be able to give an explicit (lower bound on the) constant c appearing in Corollary 5.6 (which is the same as in Corollary 5.5), one only needs to find explicit ϵ,δ such that $\mathrm{diam}\big(\mathscr{N}_\epsilon L_\alpha\cap\mathscr{N}_\epsilon L_\beta\big)\le\delta$ for all $\alpha\ne\beta$ in $\mathscr{A}$. We give such a computation in the following remark.

Remark 5.7 *In the real hyperbolic upper halfplane $\mathbb{H}^2_\mathbb{R}$, consider the geodesic line L with endpoints $\frac{1+\sqrt5}{2}$ and $\frac{1-\sqrt5}{2}$. Let $(L_\alpha)_{\alpha\in\mathscr{A}}$ be the family of the images of L by $\mathrm{PSL}_2(\mathbb{Z})$, acting by homographies on $\mathbb{H}^2_\mathbb{R}$, modulo the (global) stabilizer of L. Let $\epsilon=\frac{\log5}{2}$ and $\delta=2\log(2+\sqrt5)$. Then $\mathrm{diam}\big(\mathscr{N}_\epsilon L_\alpha\cap\mathscr{N}_\epsilon L_\beta\big)\le\delta$ for all $\alpha\ne\beta$ in $\mathscr{A}$.*



Proof. Call *Weierstrass points* the points of $\mathbb{H}^2_\mathbb{R}$ in the image of i by $\mathrm{PSL}_2(\mathbb{Z})$. Note that, by an easy computation, the ϵ-neighbourhood of L is the tubular neighbourhood of L of biggest radius such that the Weierstrass points in its interior lie on L. As seen by considering the fundamental domain for the integer horizontal translations in $\mathbb{H}^2_\mathbb{R}$, between the geodesic lines with endpoints $0,\infty$ and $1,\infty$ respectively, if two images of L by elements of $\mathrm{PSL}_2(\mathbb{Z})$ are disjoint, then the interiors of their ϵ-neighbourhoods do not meet. If two images of L by elements of $\mathrm{PSL}_2(\mathbb{Z})$ are distinct but meet, to prove that the diameter of the intersection of their ϵ-neighbourhoods is at most δ, we may assume that these images are L and the image L' of L by the translation by -1.

Recall that the boundary of the ϵ-neighbourhood of the geodesic line carried by the Euclidean circle of center $x\in\mathbb{R}$ and Euclidean radius r is the union of two arcs of circles between $x+r$ and $x-r$ that are invariant by reflection in the vertical line through x. Since the upper arc of circle $\partial_+\mathscr{N}_\epsilon L'$ of $\partial\mathscr{N}_\epsilon L'$ is tangent to the vertical line through the Weierstrass point $i+1$, its Euclidean center is the point $-\frac12+i$, and its radius is $\frac32$. Let L'' be the geodesic line with endpoints $-1,1$, which is a bisectrix of L and L'. Let u and u' be the intersection points of L'' with $\partial_-\mathscr{N}_\epsilon L$ and $\partial_-\mathscr{N}_\epsilon L'$ respectively. By considering the Euclidean quadrangle with vertices at $u, i-\frac12, -\frac12, 0$, we easily compute that the Euclidean height of u is $\frac{1}{\sqrt5}$. Using Formula (-15-), we hence have $d(i,u)=\log(2+\sqrt5)$. The distance between i and the intersection point of $\partial_+\mathscr{N}_\epsilon L'$ and $\partial_+\mathscr{N}_\epsilon L$ is $\log(1+\sqrt2)<\log(2+\sqrt5)$. Hence by convexity and symmetry arguments, the intersection $\mathscr{N}_\epsilon L'\cap\mathscr{N}_\epsilon L$ is contained in the hyperbolic ball of center i and radius $\log(2+\sqrt5)$. Therefore, the diameter of this intersection is $d(u,u')=2\log(2+\sqrt5)$. The result follows. □

The following result is a consequence of Corollary 5.6 and Equation (-9-), and proves the second claim of Theorem 1.2 in the Introduction.

Corollary 5.8 *Let M be a geometrically finite complete connected Riemannian manifold of constant sectional curvature -1 and of dimension at least 3. Let A_0 be a closed geodesic in M, and let A_∞ be a Margulis neighbourhood of a cusp of M. Then the spiraling spectrum $\mathrm{Sp}_{A_\infty,A_0}(M)$ around A_0 contains a segment $[0,c]$ for some $c>0$.* □

Using Equation (-13-), we obtain the following result in the complex hyperbolic case.

Corollary 5.9 *Let $n \geq 2$, let Γ be a discrete group of isometries of the Siegel domain model of $\mathbb{H}^n_{\mathbb{C}}$ with finite covolume, and let γ_0 be a hyperbolic element of Γ. Assume that the point ∞ is a parabolic fixed point of Γ. For every $\xi \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ which is neither a fixed point of a conjugate of γ_0 nor a parabolic fixed point, define*

$$c'(\xi) = \liminf \frac{d_{\rm Cyg}(\xi, \gamma_-)}{d_{\rm Cyg}(\gamma_+, \gamma_-)} ,$$

where the lower limit is taken over the conjugates γ of γ_0 or its inverse, with fixed points γ_-, γ_+ and $d_{\rm Cyg}(\gamma_+, \gamma_-)$ tending to 0.

Then the subset of $\mathbb{R}$ consisting of the $c'(\xi)$ for $\xi \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ which is neither a fixed point of a conjugate of γ_0 nor a parabolic fixed point, contains a segment $[0, c]$ for some $c > 0$. □

6 Applications to Diophantine approximation

In this section, we apply the results of Sections 4 and 5 to study the Diophantine approximation by quadratic irrational elements in $\mathbb{R}$, $\mathbb{C}$ and the Heisenberg group.

In order to obtain the Khinchin-type results in Subsection 6.1 and in Subsection 6.2, we will apply the following result, which follows as a slight extension of a particular case from [HP5, Theorem 4.6]. We refer for instance to [HP5] for the general definitions of the critical exponent $\delta = \delta_\Gamma \in [0, +\infty]$ and of the Patterson-Sullivan measure $\mu_{\xi_\infty, H_\infty}$ associated to a horosphere H_∞ with point at infinity ξ_∞, for a nonelementary discrete group of isometries Γ of a complete simply connected Riemannian manifold X with sectional curvature at most -1. In this paper, we will only be interested in the particular cases explained after the statement.

Theorem 6.1 *[HP5] Let X be a complete simply connected Riemannian manifold with sectional curvature at most -1 and dimension at least 2; let Γ be a discrete group of isometries of X with finite covolume and critical exponent δ; let γ_0 be a hyperbolic element of Γ and $\mathscr{R}_{\Gamma_0}$ be the set of points in $\partial_\infty X$ fixed by some conjugate of γ_0 in Γ; let ξ_∞ be a parabolic fixed point of Γ and H_∞ be a horosphere centered at ξ_∞; and let $f : [0, +\infty[\to]0, +\infty[$ be a slowly varying map (as defined in Section 4.1).*

If $\int_1^{+\infty} f(t)^\delta \, dt$ converges (resp. diverges), then $\mu_{\xi_\infty, H_\infty}$-almost no (resp. every) point of $\partial_\infty X - \{\xi_\infty\}$ belongs to infinitely many balls of center r and radius $f(D(r))e^{-D(r)}$ for the Hamenstädt distance d_{ξ_∞, H_∞}, where r ranges over $\mathscr{R}_{\Gamma_0}$. □

In our applications in Section 6.1 (resp. 6.2), X is the upper halfspace model of $\mathbb{H}^n_{\mathbb{R}}$ (resp. the Siegel domain model of $\mathbb{H}^n_{\mathbb{C}}$ as in Example 1 of Section 3) and H_∞ is the horosphere, centered at $\xi_\infty = \infty$, of the points at Euclidean height 1 (resp. $H_\infty = \{(w_0, w) \in \mathbb{H}^n_{\mathbb{C}} \;:\; 2\,{\rm Re}\, w_0 - |w|^2 = 2\}$). In this situation, since Γ has finite covolume,

- the critical exponent is $\delta = n - 1$ (resp. $\delta = 2n$, see for instance [CI, § 6]),
- the Hamenstädt distance d_{ξ_∞, H_∞} on $\partial_\infty X - \{\infty\}$ is the Euclidean distance, see [HP2, § 2.1] (resp. a multiple of the Cygan distance, see [HP3, § 3.11]), and

- the measure $\mu_{\xi_\infty,H_\infty}$ on $\partial_\infty X - \{\infty\}$ is the Hausdorff measure of d_{ξ_∞,H_∞}, which is the Lebesgue measure (resp. is in the same measure class as the Hausdorff measure of the Cygan distance, see [CI]).

6.1 Approximation in $\mathbb{R}$ and $\mathbb{C}$ by quadratic irrational elements

Let K be either the field $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$, and correspondingly, let $\widehat{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathscr{O}_K$ be the ring of integers of K. Denote by $K_{\rm quad}$ the set of quadratic irrational elements in $\widehat{K}$ over K. For every $\alpha \in K_{\rm quad}$, let α^σ be its Galois conjugate over K.

The group $\mathrm{PGL}_2(\widehat{K})$ acts on $\mathbb{P}^1(K) = \widehat{K} \cup \{\infty\}$ by homographies, and its subgroup $\mathrm{PGL}_2(\mathscr{O}_K)$ preserves K and $K_{\rm quad}$. Note that, for every $\alpha \in K_{\rm quad}$ and every $\gamma \in \mathrm{PGL}_2(\mathscr{O}_K)$, we have $(\gamma\cdot\alpha)^\sigma = \gamma\cdot(\alpha^\sigma)$.

Let us fix a finite index subgroup Γ of $\mathrm{PSL}_2(\mathscr{O}_K)$. An orbit of Γ in $K_{\rm quad}$ will be called a *congruence class* in $K_{\rm quad}$ under Γ. We are interested in Section 6.1 in the approximation of elements of $\widehat{K}$ by elements in the union of a fixed congruence class and of its Galois conjugate.

For every $\alpha \in K_{\rm quad}$, let

$$\mathscr{E}_{\alpha,\Gamma} = \Gamma\cdot\{\alpha,\alpha^\sigma\}\;,$$

endowed with its Fréchet filter, and let

$$h(\alpha) = \frac{2}{|\alpha-\alpha^\sigma|}\;.$$

Clearly, $h(\alpha)$ belongs to $]0,+\infty[$ (as $\alpha \neq \alpha^\sigma$), and $h(\alpha^\sigma) = h(\alpha)$. We will see in the proof of Theorem 6.4 that points $r \in \mathscr{E}_{\alpha,\Gamma}$ exit every finite subset of $\mathscr{E}_{\alpha,\Gamma}$ if and only if $h(r)$ tends to $+\infty$. Define the *quadratic Lagrange spectrum* relative to (α,Γ) by

$$\mathrm{Sp}_{\alpha,\Gamma} = \Big\{\; c_{\alpha,\Gamma}(\xi) = \liminf_{r\in\mathscr{E}_{\alpha,\Gamma}}\; h(r)\,|\xi - r| \;:\; \xi \in \widehat{K} - (K\cup\mathscr{E}_{\alpha,\Gamma})\Big\}\;.$$

The following result is very classical, its proof (given for the sake of completeness) was indicated to us by Y. Benoist.

Lemma 6.2 *Let $\alpha \in \widehat{K}$. Then α is quadratic irrational over K if and only if there exists a hyperbolic element γ in $\mathrm{PSL}_2(\mathscr{O}_K)$ having α as a fixed point, the other one then being α^σ.*

Proof. Let $\delta = \dim_{\mathbb{R}}\widehat{K}$. If $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ is a hyperbolic element in $\mathrm{SL}_2(\mathscr{O}_K)$, then its two fixed points (in $\partial_\infty\mathbb{H}^{\delta+1}_{\mathbb{R}} = \widehat{K}\cup\{\infty\}$) are distinct solutions of the quadratic equation $ax+b = x(cx+d)$ with coefficients in $\mathscr{O}_K$, and in particular they are quadratic irrational and Galois conjugated by σ.

Conversely, let $\alpha \in K_{\rm quad}$. We refer for instance to [Bor] for general information on linear algebraic groups. Let

$$T(\widehat{K}) = \{\gamma\in\mathrm{SL}_2(\widehat{K}) \;:\; \gamma\cdot\alpha = \alpha,\;\; \gamma\cdot\alpha^\sigma = \alpha^\sigma\}\;.$$

Since α and α^σ are two distinct points in the boundary of $\mathbb{H}^{\delta+1}_{\mathbb{R}}$, the subgroup $T(\widehat{K})$ is the set of $\widehat{K}$-points of an algebraic torus T in SL_2. This torus T is defined by a set of polynomial equations with coefficients in $K(\alpha)$, this set being invariant by the Galois group of $K(\alpha)$ over K. Hence T is defined over K. Notice that T does not split over K, as the eigenvectors of an element of T in the affine plane are $(\alpha, 1)$ and $(\alpha^\sigma, 1)$, that are not multiples of an element with coordinates in K. Hence, by Borel-Harish-Chandra's theorem (see for instance [BHC, Theo. 12.3]), the subgroup $T(\mathscr{O}_K)$ is a lattice in $T(\widehat{K})$, and in particular is not trivial. That is, there exists an element in $\mathrm{SL}_2(\mathscr{O}_K)$ having α (and α^σ) as fixed point. The result follows. $\Box$

Remark 1. Let $K = \mathbb{Q}$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Note that the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$ is in the same congruence class under Γ as its Galois conjugate $\frac{1-\sqrt{5}}{2} = -1/\phi$, hence $\mathscr{E}_{\phi,\Gamma} = \Gamma \cdot \phi$. On the other hand, $\frac{1+\sqrt{3}}{2}$ and $\frac{1-\sqrt{3}}{2}$ are Galois conjugate, but are not in the same congruence class under Γ, and this second example is more typical. Many papers have given necessary and sufficient condition for when a quadratic irrational element is in the same orbit under Γ as its Galois conjugate, see for instance [Sar, Lon, PR, Bur] and also [PP4, Prop. 5.3].

Remark 2. Let $K = \mathbb{Q}$. Let us give another expression of the approximation constants $c_{\alpha,\Gamma}(x)$.

For every $x \in \mathbb{P}^1(\mathbb{R})$, let Γ_x be the stabilizer of x in Γ. For every $\alpha \in \mathbb{Q}_{\rm quad}$, endow the infinite set Γ/Γ_α with its Fréchet filter. Denote by $N(\alpha) = \alpha\alpha^\sigma$ the *norm* of an element $\alpha \in \mathbb{Q}_{\rm quad}$. For every element γ in Γ, let $\gamma = \pm \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$.

Proposition 6.3 *Let* $\alpha \in \mathbb{Q}_{\rm quad}$ *and let* Γ *be a finite index subgroup of* $\mathrm{PSL}_2(\mathbb{Z})$. *For every* $x \in \mathbb{R} - (\mathbb{Q} \cup \mathscr{E}_{\alpha,\Gamma})$*, we have*

$$c_{\alpha,\Gamma}(x) = h(\alpha) \liminf_{(\gamma,\varepsilon)\in(\Gamma/\Gamma_\alpha)\times\{1,\sigma\}} \big|N\big(\alpha\, c(\gamma) + d(\gamma)\big)\big|\, \big|x - \gamma\cdot\alpha^\varepsilon\big| \;.$$

For instance, $\mathscr{E}_{\phi,\Gamma}$ is the set of real numbers whose continued fraction expansion is eventually constant equal to 1, and $\mathrm{Sp}_{\phi,\Gamma}$ is equal to

$$\Big\{\frac{2}{\sqrt{5}} \liminf_{a,b,c,d\in\mathbb{Z},\, ad-bc=1,\, d^2+dc-c^2\to+\infty} \big|d^2+dc-c^2\big|\, \Big|x - \frac{a\phi+b}{c\phi+d}\Big| \;:\; x \in \mathbb{R} - (\mathbb{Q}\cup\mathscr{E}_{\phi,\Gamma})\Big\} \;.$$

Proof. An easy computation shows that $|\gamma\cdot\alpha - \gamma\cdot\alpha^\sigma| = |\alpha-\alpha^\sigma|/N(c(\gamma)\alpha + d(\gamma))$, hence

$$h(\gamma\cdot\alpha) = N(c(\gamma)\alpha + d(\gamma))\; h(\alpha) \;.$$

The map $(\Gamma/\Gamma_\alpha)\times\{1,\sigma\} \to \mathscr{E}_{\alpha,\Gamma}$ defined by $(\gamma,\epsilon) \mapsto \gamma\alpha^\epsilon$ is a bijection if α and α^σ are not in the same congruence class, and is a 2-to-1 map otherwise. The result follows. $\Box$

Remark 3. Assume again that $K = \mathbb{Q}$. The quantity $h(\alpha)$ behaves in a very different way from the naive height $H(\alpha)$ of α (defined in the introduction). Clearly, $h(\alpha+n) = h(\alpha)$ for every n in $\mathbb{N}$, but $H(\alpha+n) \to \infty$ as $n \to \infty$. Hence there exists a sequence $(\alpha_i)_{i\in\mathbb{N}}$ in $\mathbb{Q}_{\rm quad}$ such that the ratio $\frac{h(\alpha_i)}{H(\alpha_i)}$ tends to 0 when $i \to +\infty$. But this ratio cannot tend to $+\infty$, as for every $\alpha \in \mathbb{Q}_{\rm quad}$, we have

$$\frac{h(\alpha)}{H(\alpha)} \le 2 \;.$$

Indeed, let aX^2+bX+c be a minimal polynomial of α over $\mathbb{Z}$, so that the naive height of α is $H(\alpha)=\max\{|a|,|b|,|c|\}$. Then

$$h(\alpha)=\frac{2}{\sqrt{(b/a)^2-4(c/a)}}=\frac{2|a|}{\sqrt{b^2-4ac}}\leq 2|a|\leq 2\,H(\alpha)\;,$$

There are many possibilities for the relative behaviour of $h(\alpha)$ and $H(\alpha)$. In particular, there exist sequences $(\alpha_i)_{i\in\mathbb{N}}$ in $\mathbb{Q}_{\rm quad}$ and constants $c,c',c''>0$ such that $h(\alpha_i)$ is equivalent as i tends to $+\infty$ either to $c\,H(\alpha_i)^{-\frac{1}{2}}$ (take $c=1$ and $\alpha_i=\sqrt{p_i}$ for p_i the i-th prime number), or to $c'\,H(\alpha_i)^{\frac{1}{2}}$ (take $c'=1$ and $\alpha_i=1/\sqrt{p_i}$ for p_i the i-th prime number) or to $c''\,H(\alpha_i)$ (take $c''=2/\sqrt{5}$ and $\alpha_i=\frac{3+2i+\sqrt{5}}{1+3i+i^2}$ for every $i\in\mathbb{N}$). This difference is good to bear in mind when comparing our results with for example the results of [DS, Spr, Bug] cited in the Introduction. We refer to [PP4, Lem. 5.2] for a treatment of the algebraic number theory aspects of $h(\alpha)$.

Theorem 6.4 *Let $K=\mathbb{Q}$ or $K=\mathbb{Q}(i\sqrt{m})$ where m is a squarefree positive integer, let $\widehat{K}=\mathbb{R}$ or $\widehat{K}=\mathbb{C}$ respectively, and let $\delta=\dim_{\mathbb{R}}\widehat{K}$. Let Γ be a finite index subgroup of $\mathrm{PSL}_2(\mathscr{O}_K)$.*

(1) For every $\alpha_0\in K_{\rm quad}$, the quadratic Lagrange spectrum $\mathrm{Sp}_{\alpha_0,\Gamma}$ is closed, and equal to the closure of the set of the approximation constants $c_{\alpha_0,\Gamma}(x)$ for x a quadratic irrational over K, not in $\mathscr{E}_{\alpha_0,\Gamma}$.

(2) There exists $C\geq 0$ such that for every $\alpha_0\in K_{\rm quad}$,

$$\max \mathrm{Sp}_{\alpha_0,\Gamma}\leq C\;.$$

(3) If $\widehat{K}=\mathbb{C}$, then for every $\alpha_0\in K_{\rm quad}$, there exists $c>0$ such that $\mathrm{Sp}_{\alpha_0,\Gamma}$ contains $[0,c]$.

(4) Let $\alpha_0\in K_{\rm quad}$ and $\varphi:\,]0,+\infty[\,\rightarrow\,]0,+\infty[$ be a map such that $t\mapsto\varphi(e^t)$ is slowly varying. If the integral $\int_1^{+\infty}\varphi(t)^\delta/t\;dt$ diverges (resp. converges), then for Lebesgue almost every $x\in\widehat{K}$,

$$\liminf_{r\in\mathscr{E}_{\alpha_0,\Gamma}}\frac{h(r)}{\varphi(h(r))}\;|x-r|=0\;(\text{resp.}=+\infty)\;.$$

Proof. Consider the data $\mathscr{D}=(X,\Gamma,\Gamma_0,C_\infty)$, where X is the upper halfspace model of $\mathbb{H}^{\delta+1}_{\mathbb{R}}$, Γ is as in the statement, Γ_0 is the stabilizer in Γ of the translation axis of a hyperbolic element of Γ one of whose fixed points is α_0, and C_∞ is the set of points in X with Euclidean height at least 1. Note that Γ_0 is nontrivial: By Lemma 6.2, the point α_0 is a fixed point of a hyperbolic element of $\mathrm{PSL}_2(\mathscr{O}_K)$, hence of a hyperbolic element of Γ, since Γ is a finite index subgroup of $\mathrm{PSL}_2(\mathscr{O}_K)$.

The data $\mathscr{D}$ satisfies the general assumptions of Example 1 of Section 3: Since $z\mapsto z+1$ belongs to $\mathrm{PSL}_2(\mathscr{O}_K)$, the point ∞ is fixed by a parabolic element of $\mathrm{PSL}_2(\mathscr{O}_K)$, hence of Γ; moreover, C_∞ is precisely invariant under the stabilizer of ∞ in $\mathrm{PSL}_2(\mathscr{O}_K)$, hence in Γ, by Shimizu's lemma; furthermore, the quotient of X by $\mathrm{PSL}_2(\mathscr{O}_K)$, hence by Γ, has finite volume.

Using the notations of Example 1 of Section 3, we easily check that $\mathscr{R}_{\Gamma_0} = \mathscr{E}_{\alpha_0,\Gamma}$. Furthermore, let $r \in \mathscr{E}_{\alpha_0,\Gamma}$ and let $\gamma_r \in \Gamma$ be an element such that $\gamma_r \gamma_0 \gamma_r^{-1}$ fixes r. Recall that the other fixed point of a hyperbolic element fixing r is the Galois conjugate of r over K, by Lemma 6.2. Hence, by Equation (-10-), we have

$$D([\gamma_r]) = \log h(r)\ . \qquad (-21-)$$

Therefore, by Lemma 3.1, points $r \in \mathscr{E}_{\alpha_0,\Gamma}$ exit every finite subset of $\mathscr{E}_{\alpha_0,\Gamma}$ if and only if $h(r)$ tends to $+\infty$. The set of parabolic fixed points of Γ is equal to the set of parabolic fixed point of $\mathrm{PSL}_2(\mathscr{O}_K)$, as Γ has finite index, hence it is equal to $K \cup \{\infty\}$. Therefore

$$\widehat{K} - (K \cup \mathscr{E}_{\alpha_0,\Gamma}) = \Lambda_c\Gamma - \mathscr{R}_{\Gamma_0}\ .$$

For every ξ in this set, we have

$$c(\Gamma_\infty \xi) = \liminf_{r \in \mathscr{E}_{\alpha_0,\Gamma}} \ h(r)|\xi - r|$$

by the first equality in Equation (-11-). Hence, the quadratic Lagrange spectrum $\mathrm{Sp}_{\alpha_0,\Gamma}$ coincides with the approximation spectrum $\mathrm{Sp}(\mathscr{D})$.

We can now conclude that the assertions (1), (2) and (3) follow, respectively, from Theorem 4.8, Proposition 4.6, and Corollary 5.6. In particular, in (2) we get an upper bound on $\mathrm{Sp}_{\alpha_0,\Gamma}$ which depends only on Γ, and not on the (congruence class under Γ) of α_0.

To prove the assertion (4), define $f : t \mapsto \varphi(e^t)$, which is slowly varying by the assumptions of (4). By an easy change of variable, the integral $\int_1^{+\infty} f(t)^\delta\, dt$ diverges if and only if $\int_1^{+\infty} \varphi(t)^\delta/t\ dt$ diverges. Hence by Theorem 6.1, by the comments following it, and by Equation (-21-), if $\int_1^{+\infty} \varphi(t)^\delta/t\ dt$ diverges, then for almost every (for the Lebesgue measure) point x in $\widehat{K}$,

$$\liminf_{r \in \mathscr{E}_{\alpha_0,\Gamma}} \frac{h(r)}{\varphi(h(r))}\ |x - r| \leq 1.$$

Replacing φ by $\frac{1}{k}\varphi$ and letting $k \in \mathbb{N}$ go to $+\infty$, this proves the divergence part of the assertion (4) in Theorem 6.4. The convergence part follows similarly. □

Remark. Replacing in the above proof, as in [PP3, PP1, PP2], X by $\mathbb{H}^5_{\mathbb{R}}$, δ by 4 and Γ by the image in the isometry group of X of a finite index subgroup of $\mathrm{SL}_2(\mathscr{O}')$ where $\mathscr{O}' = \mathbb{Z}(1+i+j+k)/2 + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$ is the Hurwitz order in the usual Hamilton quaternion algebra A over $\mathbb{Q}$ (using Dieudonné's determinant), we could get similar Diophantine approximation results of points in $A(\mathbb{R})$ by points in quadratic extensions of $A(\mathbb{Q})$.

To prove Theorem 1.3 of the Introduction, apply Theorem 6.4 with $K = \mathbb{Q}$ (so that $\delta = 1$), $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and $\varphi : t \mapsto t\psi(2/t)$, and notice that $\log(\varphi(e^t))$ is Lipschitz, and that $\int_1^{+\infty} \varphi(t)/t\, dt$ converges if and only if $\int_0^1 \psi(t)/t^2\, dt$ converges.

6.2 Approximation in the Heisenberg group

Let m be a squarefree positive integer, let K be the number field $\mathbb{Q}(i\sqrt{m})$, let $\mathscr{O}_K$ be its ring of integers, and let $K_{\rm quad}$ be the set of elements of $\mathbb{C}$ which are quadratic irrational over K.

Let $n \geq 2$, and let $(w', w) \mapsto w' \cdot \overline{w} = \sum_{i=1}^{n-1} w'_i \overline{w_i}$ be the usual Hermitian scalar product on $\mathbb{C}^{n-1}$. Consider the real Lie group

$$\mathrm{Heis}_{2n-1}(\mathbb{R}) = \{(w_0, w) \in \mathbb{C} \times \mathbb{C}^{n-1} \ : \ 2\,\mathrm{Re}\, w_0 - w \cdot \overline{w} = 0\}$$

whose law is

$$(w_0, w) \cdot (w'_0, w') = (w_0 + w'_0 + w' \cdot w, w + w')\ .$$

Endow it with the Cygan distance (see for instance [Gol]), which is the unique left-invariant distance such that

$$d_{\mathrm{Cyg}}((w_0, w), (0, 0)) = \sqrt{2|w_0|},$$

as well as with the *Cygan measure* (which is the Hausdorff measure of the Cygan distance). The Lie group $\mathrm{Heis}_{2n-1}(\mathbb{R})$ is isomorphic to the standard $(2n-1)$-dimensional Heisenberg group, that is for $n = 2$ to the Lie group $\Big\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \ : \ x, y, z \in \mathbb{R}\Big\}$.

We are interested in the Diophantine approximation of the points of $\mathrm{Heis}_{2n-1}(\mathbb{R})$ by elements all of whose coordinates are rational or quadratic over K, that is by elements of $\mathrm{Heis}_{2n-1}(\mathbb{R}) \cap (K \cup K_{\mathrm{quad}})^n$ (or nice subsets of it).

Let us identify $\mathrm{Heis}_{2n-1}(\mathbb{R})$ with its image in the projective space $\mathbb{P}_n(\mathbb{C})$ of $\mathbb{C}^{n+1}$ by the map $(w_0, w) \mapsto [w_0 : w : 1]$. Let q be the Hermitian form of signature $(1, n)$ on $\mathbb{C}^{n+1}$ with coordinates (z_0, z, z_n), defined by

$$q = -z_0 \overline{z_n} - z_n \overline{z_0} + z \cdot \overline{z}\ .$$

The induced action on $\mathbb{P}_n(\mathbb{C})$ of the special unitary group SU_q of q preserves (the image in $\mathbb{P}_n(\mathbb{C})$ of) $\mathrm{Heis}_{2n-1}(\mathbb{R}) \cup \{\infty\}$. Let $\mathrm{SU}_q(\mathscr{O}_K)$ be the arithmetic subgroup of SU_q which consists of matrices with coefficients in $\mathscr{O}_K$. Note that the action of $\mathrm{SU}_q(\mathscr{O}_K)$ on $\mathrm{Heis}_{2n-1}(\mathbb{R}) \cup \{\infty\}$ preserves both the set

$$\mathrm{Heis}_{2n-1}(\mathbb{Q}) \cup \{\infty\} = \big(\mathrm{Heis}_{2n-1}(\mathbb{R}) \cap K^n\big) \cup \{\infty\}$$

and the set $\big(\mathrm{Heis}_{2n-1}(\mathbb{R}) \cap K(\alpha)^n\big) \cup \{\infty\}$ for every $\alpha \in K_{\mathrm{quad}}$.

Let

$$\alpha_0 = \frac{i}{2}\big(\sqrt{m+4} - \sqrt{m}\,\big)\ ,$$

which is an element of K_{quad}, since it is a root of the quadratic polynomial $X^2 + i\sqrt{m}\,X + 1$ whose coefficients are in K, and it does not belong to K. (We could have taken many other examples, but α_0 is one of the simplest ones.) The Galois conjugate of α_0 over K is $\alpha_0^\sigma = \frac{i}{2}(-\sqrt{m+4} - \sqrt{m})$.

Let Γ be a finite index subgroup of $\mathrm{SU}_q(\mathscr{O}_K)$, and let $\mathscr{E}'_{\alpha_0,\Gamma} = \Gamma \cdot \{(\alpha_0, 0), (\alpha_0^\sigma, 0)\}$. For every $r \in \mathscr{E}'_{\alpha_0,\Gamma}$, let r^σ be the componentwise Galois conjugate of r, and let

$$h'(r) = \frac{1}{d_{\mathrm{Cyg}}(r, r^\sigma)}\ .$$

Endow $\mathscr{E}'_{\alpha_0,\Gamma}$ with its Fréchet filter. We will see in the proof below that points $r \in \mathscr{E}'_{\alpha_0,\Gamma}$ tend to infinity in $\mathscr{E}'_{\alpha_0,\Gamma}$ if and only if $h'(r)$ tends to $+\infty$.

In order to understand the Diophantine approximation of elements ξ of $\mathrm{Heis}_{2n-1}(\mathbb{R})$ by elements in the subset $\mathscr{E}'_{\alpha_0,\Gamma}$ of the set $\mathrm{Heis}_{2n-1}(\mathbb{R})\cap K(\alpha_0)^n$, we introduce the *approximation constant* of ξ, defined by

$$c'(\xi)=\liminf_{r\in\mathscr{E}'_{\alpha_0,\Gamma}} h'(r)\,d_{\mathrm{Cyg}}(\xi,r)\ ,$$

and the *quadratic Heisenberg-Lagrange spectrum* with respect to (α_0,Γ)

$$\mathrm{Sp}'_{\alpha_0,\Gamma}=\Big\{c'(\xi)\ :\ \xi\in\mathrm{Heis}_{2n-1}(\mathbb{R})-\big(\mathrm{Heis}_{2n-1}(\mathbb{Q})\cup\mathscr{E}'_{\alpha_0,\Gamma}\big)\Big\}\ .$$

Theorem 6.5 *If Γ is a finite index subgroup of $\mathrm{SU}_q(\mathscr{O}_K)$, then*

(1) the Heisenberg-Lagrange spectrum $\mathrm{Sp}'_{\alpha_0,\Gamma}$ is bounded;

(2) there exists $c>0$ such that $\mathrm{Sp}'_{\alpha_0,\Gamma}$ contains an interval $[0,c[$;

(3) let $\varphi:\,]0,+\infty[\;\rightarrow\;]0,+\infty[$ be a map such that $t\mapsto\varphi(e^t)$ is slowly varying, if the integral $\int_1^{+\infty}\varphi^{2n}(t)/t\ dt$ diverges (resp. converges), then for Cygan almost every $x\in\mathrm{Heis}_{2n-1}(\mathbb{R})$,

$$\liminf_{r\in\mathscr{E}'_{\alpha_0,\Gamma}}\frac{h'(r)}{\varphi(h'(r))}\,d_{\mathrm{Cyg}}(x,r)=0\ (\text{resp.} =+\infty)\ .$$

Proof. Let X be the Siegel domain model of the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$, as in Example 2 of Section 3, identified, as well as its boundary at infinity, with its image in $\mathbb{P}_n(\mathbb{C})$ by $(w_0,w)\mapsto[w_0:w:1]$. The group $\Gamma\subset\mathrm{SL}_{n+1}(\mathbb{C})$ acts (with finite kernel) on $X\subset\mathbb{P}_n(\mathbb{C})$ as a discrete group of isometries of X with finite covolume, by the restriction of the projective action. Note that ∞ (corresponding to $[1:0:0]$) is a parabolic fixed point of $\mathrm{SU}_q(\mathscr{O}_K)$, hence of Γ. Let

$$\gamma_0=\begin{pmatrix} m+1 & 0 & -i\sqrt{m}\\ 0 & I & 0\\ i\sqrt{m} & 0 & 1\end{pmatrix}\ .$$

It is easy to check that γ_0 is a hyperbolic element of $\mathrm{SU}_q(\mathscr{O}_K)$ whose fixed points in $\partial_\infty X$ are exactly $(\alpha_0,0)$ and $(\alpha_0^\sigma,0)$. Since Γ has finite index in $\mathrm{SU}_q(\mathscr{O}_K)$, there exists $k\in\mathbb{N}-\{0\}$ such that ${\gamma_0}^k\in\Gamma$. Let Γ_0 be the stabilizer in Γ of the translation axis of γ_0.

The horoball $\mathscr{H}_{2\sqrt{m}}$ is precisely invariant under the stabilizer of ∞ in $\mathrm{SU}_q(\mathscr{O}_K)$, hence under Γ_∞ in Γ, by the Kamiya-Parker inequality, see for instance [PP3, Lemma 6.4]. Thus, the data $\mathscr{D}=(X,\Gamma,\Gamma_0,\mathscr{H}_{2\sqrt{m}})$ satisfy the conditions of Example 2 of Section 3. Using the notation of this example, $\mathscr{E}'_{\alpha_0,\Gamma}$ is the set $\mathscr{R}_{\Gamma_0}$ of fixed points of the conjugates of γ_0^k in Γ. The set of parabolic fixed points of Γ is equal to the set of parabolic fixed points of $\mathrm{SU}_q(\mathscr{O}_K)$, which is exactly $\mathrm{Heis}_{2n-1}(\mathbb{Q})\cup\{\infty\}$ (see for instance [PP3, Sect. 6.3, Exam. 2]). In particular,

$$\Lambda_c\Gamma-\mathscr{R}_{\Gamma_0}=\mathrm{Heis}_{2n-1}(\mathbb{R})-\big(\mathrm{Heis}_{2n-1}(\mathbb{Q})\cup\mathscr{E}'_{\alpha_0,\Gamma}\big)\ .$$

For every r in $\mathscr{E}'_{\alpha_0,\Gamma}$, let $\gamma_r\in\Gamma$ be such that r is fixed by $\gamma_r\gamma_0^k\gamma_r^{-1}$. Note that if $r=\gamma_r(\alpha_0^\epsilon,0)$ for some $\epsilon\in\{1,\sigma\}$, then the other fixed point of $\gamma_r\gamma_0^k\gamma_r^{-1}$ is $\gamma_r((\alpha_0^\epsilon)^\sigma,0)$, which is equal to r^σ since SU_q acts projectively on $\mathrm{Heis}_{2n-1}(\mathbb{R})\cup\{\infty\}$. Since the Cygan distance and the modified Cygan distance are equivalent (see [PP3, Sect. 6.1]), there exists a constant $c_1 > 0$ such that for every r in $\mathscr{E}'_{\alpha_0,\Gamma}$, with $D([\gamma_r])$ computed in Lemma 3.4, we have

$$\mid D([\gamma_r]) - \log h'(r) \mid \leq c_1 \; . \qquad (\text{-}\,22\,\text{-})$$

Hence, it follows from Lemma 3.1 that points $r \in \mathscr{E}'_{\alpha_0,\Gamma}$ tend to infinity in $\mathscr{E}'_{\alpha_0,\Gamma}$ if and only if $h'(r)$ tends to $+\infty$.

Again, since the Cygan distance and the modified Cygan distance are equivalent, there exists a constant $c_2 > 0$ such that for every $\xi \in \Lambda_c\Gamma - \mathscr{R}_{\Gamma_0}$, we have

$$\liminf_{r\in\mathscr{E}'_{\alpha_0,\Gamma}} \frac{d_{\rm Cyg}(\xi,r)}{d_{\rm Cyg}(r,r^\sigma)} \leq c_2 \liminf_{r\in\mathscr{R}_{\Gamma_0}} \sqrt{2}\,\frac{d'_{\rm Cyg}(r,r^\sigma)\; d_{\rm Cyg}(\xi,r)}{d_{\rm Cyg}(r,r^\sigma)^2}\;.$$

The assertion (1) of Theorem 6.5 then follows from Equation (- 14 -) and from Proposition 4.6, and the assertion (2) follows from Corollary 5.9.

To prove the assertion (3) of Theorem 6.5, consider the map $f : t \mapsto \varphi(e^t)$, which is slowly varying. In particular, by Equation (- 22 -), there exists a constant $c_3 > 0$ such that for every $r \in \mathscr{E}'_{\alpha_0,\Gamma}$, we have

$$\frac{1}{c_3}\,\frac{\varphi(h'(r))}{h'(r)} \;\leq\; f\big(D([\gamma_r])\big)\; e^{-D([\gamma_r])} \;\leq\; c_3\,\frac{\varphi(h'(r))}{h'(r)}\;.$$

Hence the assertion (3) of Theorem 6.5 follows from Theorem 6.1, as in the proof of the assertion (4) in Theorem 6.4. □

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