A survey of some arithmetic applications of ergodic theory in negative curvature

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Abstract

This paper is a survey of some arithmetic applications of techniques in the geometry and ergodic theory of negatively curved Riemannian manifolds, focusing on the joint works of the authors. We describe Diophantine approximation results of real numbers by quadratic irrational ones, and we discuss various results on the equidistribution in $\mathbb{R}$, $\mathbb{C}$ and in the Heisenberg groups of arithmetically defined points. We explain how these results are consequences of equidistribution and counting properties of common perpendiculars between locally convex subsets in negatively curved orbifolds, proven using dynamical and ergodic properties of their geodesic flows. This exposition is based on lectures at the conference "Chaire Jean Morlet: Géométrie et systèmes dynamiques", at the CIRM, Luminy, 2014. We thank B. Hasselblatt for his strong encouragements to write this survey. [1]

1 Introduction

For several decades, tools from dynamical systems, and in particular ergodic theory, have been used to derive arithmetic and number theoretic, in particular Diophantine approximation results, see for instance the works of Furstenberg, Margulis, Sullivan, Dani, Kleinbock, Clozel, Oh, Ullmo, Lindenstrauss, Einsiedler, Michel, Venkatesh, Marklof, Green-Tao, Elkies-McMullen, Ratner, Mozes, Shah, Gorodnik, Ghosh, Weiss, Hersonsky-Paulin, Parkkonen-Paulin and many others, and the references [Kle2, Lin, Kle1, AMM, Ath, GorN, EiW, PaP5].

In Subsection 2.2 of this survey, we introduce a general framework of Diophantine approximation in measured metric spaces, in which most of our arithmetic corollaries are inserted (see the end of Subsection 2.2 for references concerning this framework). In order to motivate it, we first recall in Subsection 2.1 some very basic and classical results in Diophantine approximation (see for instance [Bug1, Bug2]). A selection (extracted from [PaP1, PaP4, PaP7]) of our arithmetic results are then stated in Subsections 2.3, 2.4 and 2.5, where we indicate how they fit into this framework: Diophantine approximation results (à la Khintchine, Hurwitz, Cusick-Flahive and Farey) of real numbers by quadratic

[1]**Keywords:** counting, equidistribution, mixing, decay of correlation, hyperbolic geometry, negative curvature, geodesic flow, geodesic arc, convexity, common perpendicular, ortholength spectrum, skinning measure, Gibbs measure, Bianchi group, Heisenberg group, diophantine approximation, Lagrange spectrum, approximation constant, Mertens formula, quadratic irrational, binary quadratic form. **AMS codes:** 11D85, 11E39, 11F06, 11J06, 11J17, 11N45, 20H10, 20G20, 30F40, 37A25, 37C35, 37D40, 53A35, 53C17, 53C22

irrational ones, equidistribution of rational points in $\mathbb{R}$ (for various height functions), in $\mathbb{C}$ and in the Heisenberg group, ...

We will explain in Subsection 4.2 the starting point of their proofs, using the geometric and ergodic tools and results previously described in Subsection 4.1, where we give an exposition of our work in [PaP6]: an asymptotic formula as $t \to +\infty$ for the number of common perpendiculars of length at most t between closed locally convex subsets D^- and D^+ in a negatively curved Riemannian orbifold, and an equidistribution result of the initial and terminal tangent vectors v_α^- and v_α^+ of the common perpendiculars α in the outer and inner unit normal bundles of D^- and D^+, respectively. Common perpendiculars have been studied, in various particular cases, sometimes not explicitly, by Basmajian, Bridgeman, Bridgeman-Kahn, Eskin-McMullen, Herrmann, Huber, Kontorovich-Oh, Margulis, Martin-McKee-Wambach, Meyerhoff, Mirzakhani, Oh-Shah, Pollicott, Roblin, Shah, the authors and many others (see the comments after Theorem 15 below, and the survey [PaP5] for references).

Section 3 presents the background notions on the geometry in negative curvature, describes various useful measures, and recalls the basic results about them, due to works of Patterson, Sullivan, Bowen, Margulis, Babillot, Roblin, Otal-Peigné, Kleinbock-Margulis, Clozel, Oh-Shah, Mohammadi-Oh and the authors (see for instance [Rob2, Bab, OP, MO, PaP3]). See [PaPS, BrPP] for extensions to manifolds with potentials and to trees with potentials.

Let us denote by ${}^cA$ the complementary subset of a subset A of any given set, by $\|\mu\|$ the total mass of a measure μ, by $\mathrm{Leb}_\mathbb{R}$ and $\mathrm{Leb}_\mathbb{C}$ the Lebesgue measures on $\mathbb{R}$ and $\mathbb{C}$, by Δ_x the unit Dirac mass at any point x in any topological space, and by $\overset{*}{\rightharpoonup}$ the weak-star convergence of measures on any locally compact space.

2 Arithmetic applications

2.1 Basic and classic Diophantine approximation

When denoting a rational number $\frac{p}{q} \in \mathbb{Q}$, we will assume that p and q are coprime and that $q > 0$. For every irrational real number $x \in \mathbb{R} - \mathbb{Q}$, let us define the *approximation exponent* $\omega(x)$ of x as

$$\omega(x) = \limsup_{\frac{p}{q} \in \mathbb{Q},\ q \to +\infty} \frac{-\ln \left|x - \frac{p}{q}\right|}{\ln q}\ .$$

The Dirichlet theorem implies that

$$\inf_{x \in \mathbb{R}-\mathbb{Q}} w(x) = 2\,,$$

which motivates the definition of the the *approximation constant* $c(x)$ of $x \in \mathbb{R} - \mathbb{Q}$ as

$$c(x) = \liminf_{\frac{p}{q} \in \mathbb{Q},\ q \to +\infty} q^2 \left|x - \frac{p}{q}\right|\ .$$

The convention varies, some other references consider $c(x)^{-1}$ or $(2c(x))^{-1}$ as the approximation constant. The *Lagrange spectrum* for the approximation of real numbers by rational ones is

$$\mathrm{Sp}_{\mathrm{Lag}} = \{c(x)\ :\ x \in \mathbb{R} - \mathbb{Q}\}\ .$$

Given $\psi : \mathbb{N} \to]0, +\infty[$, the set of *$\psi$-well approximable* real numbers by rational ones is

$$\mathrm{WA}_\psi = \Big\{x \in \mathbb{R} - \mathbb{Q} \;:\; \mathrm{Card}\Big\{\frac{p}{q} \in \mathbb{Q} \;:\; \Big|x - \frac{p}{q}\Big| \leq \psi(q)\Big\} = +\infty\Big\}\;.$$

Again the convention varies, some other references consider $q \mapsto \psi(q)/q$ or similar instead of ψ.

Denoting by $\dim_{\mathrm{Hau}}$ the Hausdorff dimension of subsets of $\mathbb{R}$, the Jarnik-Besicovich theorem says that, for every $c \geq 0$,

$$\dim_{\mathrm{Hau}}\{x \in \mathbb{R} - \mathbb{Q} \;:\; w(x) \geq 2 + c\} = \frac{2}{2+c}$$

Many properties of the Lagrange spectrum are known (see for instance [CuF]): $\mathrm{Sp}_{\mathrm{Lag}}$ is bounded, with maximum $\frac{1}{\sqrt{5}}$ known as the *Hurwitz constant* (Korkine-Zolotareff 1873, Hurwitz 1891); it is closed (Cusick 1975), and contains a maximal interval $[0, \mu]$ with $\mu > 0$ (Hall 1947), with in fact $\mu = 491993569/(2221564096 + 283748\sqrt{462}) \simeq 0.2208$ known as the *Freiman constant* (Freiman 1975).

The Khintchine theorem gives a necessary and sufficient criteria for the ψ-well approximable real number to have full or zero Lebesgue measure:

$$\begin{cases} \mathrm{Leb}_{\mathbb{R}}({}^c\,\mathrm{WA}_\psi) = 0 & \text{if } \sum_{q=1}^{+\infty} q\,\psi(q) = +\infty \\ \mathrm{Leb}_{\mathbb{R}}(\mathrm{WA}_\psi) = 0 & \text{if } \sum_{q=1}^{+\infty} q\,\psi(q) < +\infty\;. \end{cases}$$

Finally, the equidistribution of Farey fractions (which is closely related with the Mertens formula) is an equidistribution theorem of the rational numbers in $\mathbb{R}$ when their denominator tends to $+\infty$:

$$\frac{\pi^2}{12\,s} \sum_{\frac{p}{q} \in \mathbb{Q},\ |q| \leq s} \Delta_{\frac{p}{q}} \;\overset{*}{\rightharpoonup}\; \mathrm{Leb}_{\mathbb{R}}\;.$$

2.2 An approximation framework

The general framework announced in the introduction concerns the quantitative answers to how dense a given dense subset of a given topological set is.

Let (Y, d, μ) be a metric measured space (see for instance [Gro, Chap. $3\frac{1}{2}$] and [Hei] for generalities), with Y a subspace of a topological space X, let Z be a countable (to simplify the setting in this survey) subset of X whose closure contains Y (for instance a dense orbit of a countable group of homeomorphisms of X), and let $H : Z \to]0, +\infty[$ be a map called a *height function* which is proper (for every $r > 0$, the set $H^{-1}(]0, r])$ is finite). The classical Diophantine approximation problems of subsection 2.1 fit into this framework with $X = \mathbb{R}$, $Y = \mathbb{R} - \mathbb{Q}$, $Z = \mathbb{Q}$ and $H : \frac{p}{q} \mapsto q$ or $H : \frac{p}{q} \mapsto q^2$, considered modulo translation by integers. The height functions in question are invariant under translation by $\mathbb{Z}$, and the properness condition is satisfied in the quotient space $\mathbb{Z}\backslash\mathbb{R}$ or, equivalently, by restriction to the unit interval.

We endow Z with the Fréchet filter of the complementary subsets of its finite subsets: $z \in Z$ tends to infinity if and only if z leaves every finite subset of Z. Generalising the definitions of Subsection 2.1, for every $y \in Y$, we may define the *approximation exponent* $\omega(y) = \omega_{X,Y,Z,H}(y)$ of y by the elements of Z with height function H as

$$\omega(y) = \limsup_{z \in Z} \frac{-\ln d(y, z)}{\ln H(z)}\;,$$

and the *approximation constant* $c(y) = c_{X,\,Y,\,Z,\,H}(y)$ of y by the elements of Z with height function H as

$$c(y) = \liminf_{z \in Z}\ H(z)\, d(y,z)\ .$$

The *Lagrange spectrum* $\mathrm{Sp}_{\mathrm{Lag}} = \mathrm{Sp}_{X,\,Y,\,Z,\,H}$ for the approximation of the elements of Y by the elements of Z with height function H is

$$\mathrm{Sp}_{\mathrm{Lag}} = \{c(y)\ :\ y \in Y\}\ .$$

Its least upper bound will be called the *Hurwitz constant* for the approximation of the elements of Y by the elements of Z with height function H. Given $\psi :]0,+\infty[\, \to\,]0,+\infty[$, the set $\mathrm{WA}_\psi = \mathrm{WA}_{\psi,\,X,\,Y,\,Z,\,H}$ of *ψ-well approximable* elements of Y by the elements of Z with height function H is

$$\mathrm{WA}_\psi = \big\{y \in Y\ :\ \mathrm{Card}\big\{z \in Z\ :\ d(y,z) \leq \psi(H(z))\big\} = +\infty\big\}\ .$$

The approximation problems may be subdivided into several classes, as follows, possibly by taking the appropriate height function, for appropriate maps $\psi :]0,+\infty[\, \to\,]0,+\infty[$.

- The approximation exponent problem: study the map $y \mapsto \omega(y)$.
- The Lagrange problem: study the Lagrange spectrum for the approximation of the elements of Y by the elements of Z with height function H.
- The Jarnik-Besicovich problem: compute the Hausdorff dimension of the set of $y \in Y$ with $c(y) \geq \psi(H(y))$.
- The Khintchine problem: study whether μ-almost every (or μ-almost no) $y \in Y$ is ψ-well approximable.
- The counting problem: study the asymptotics as s tends to $+\infty$ of

$$\mathrm{Card}\{y \in Y\ :\ H(y) \leq s\}\ .$$

- The equidistribution problem: study the set of weak-star accumulation points as s tends to $+\infty$ of the probability measures

$$\frac{1}{\mathrm{Card}\{y \in Y\ :\ H(y) \leq s\}} \sum_{z \in Z,\ H(z) \leq s} \Delta_z\ .$$

The equidistribution problem, closely linked to the counting problem, is the one we will concentrate on in Subsections 2.3, 2.4 and 2.5.

This framework is not new (see for instance the works of Kleinbock), and many results have developped some aspect of it.

(1) For instance, X could be the boundary at infinity of a Gromov hyperbolic metric space, Y a (subset of) the limit set of a discrete group Γ of isometries of this hyperbolic space, Z could be the orbit under Γ of some point $x \in X$, and $H : Z \to\,]0,+\infty[$ could be $\gamma x \mapsto 1 + d_X(A_0, \gamma B_0)$ where A_0, B_0 are subsets of X, with B_0 invariant under the stabilizer of x in Γ. Numerous aspects of this particular case have been developped, by Patterson, Sullivan, Dani, Hill, Stratmann, Velani, Bishop-Jones, Hersonsky-Paulin, Parkkonen-Paulin, and the most complete and general version is due to Fishman-Simmons-Urbański [FSU], to which we refer in particular for their thorough long list of references.

(2) The case when $X = \mathbb{R}^N$, Y is a submanifold (or a more general subset) of X, $Z = \mathbb{Q}^N$ has been widely studied, under the name of Diophantine approximation on curves,

submanifolds and fractals subsets, by many authors, including Kleinbock-Margulis, Bernik, Dodson, Beresnevich, Velani, Kleinbock-Weiss and others, see for instance [BerD, BaBV] and their references.

(3) If $X = \underline{X}(\mathbb{R})$ is for instance the set of real points of an algebraic manifold defined over $\mathbb{Q}$, if $Z = \underline{X}(\mathbb{Q})$ is the set of rational points, if Y is the (Hausdorff) closure of Z in X or this closure minus Z, there are many results, in particular when X is homogeneous, on the above Diophantine approximation problems. Similarly, if $X = G/H$ is a homogeneous space of a semisimple connected Lie group G, if Z is the orbit in X of a lattice in G and if Y is the closure of Z in X, most of the above problems have been stated and studied for instance in [GGN3, GGN2, GGN1]. We refer to the works of Benoist, Browning, Colliot-Thélène, Duke, Einsiedler, Eskin, Gorodnik, Heath-Brown, Lindenstrauss, Margulis, Mozes, Oh, Ratner, Quint, Rudnick, Salberger, Sarnak, Shah, Tomanov, Ullmo, Venkatesh, Weiss and many others for equidistribution results in homogeneous spaces, see for instance [Bre, BF, Har, EiW, Serr, GreT, Kim, BenQ1, BenQ2].

In the next three subsections, we give a sample of the results obtained by the authors on the above problems.

2.3 Diophantine approximation in $\mathbb{R}$ by quadratic irrationals

Our first results concern the Diophantine approximation of real numbers, where we replace the approximating rationals by quadratic irrational numbers. We refer for instance to [Bug1] and its references for very different Diophantine approximation results by algebraic numbers.

We denote by α^σ the Galois conjugate of a real quadratic irrational α, and by $\mathtt{tr}\,\alpha = \alpha + \alpha^\sigma$ its trace. We approximate the real points by the elements of the orbit of a fixed quadratic irrational α_0 by homographies under $\mathrm{PSL}_2(\mathbb{Z})$ (and their Galois conjugates). We denote by $\gamma \cdot x$ the action by homography of $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ on $x \in \mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

Let $X = \mathbb{R}$ (with the standard Euclidean distance and Lebesgue measure), let $Y = \mathbb{R} - \mathbb{Q} - \mathrm{PSL}_2(\mathbb{Z}) \cdot \alpha_0$, let $Z = \mathrm{PSL}_2(\mathbb{Z}) \cdot \alpha_0$ and let $H : \alpha \mapsto \frac{2}{|\alpha - \alpha^\sigma|}$. It turns out that H is an appropriate height function on each $\mathrm{PSL}_2(\mathbb{Z})$-orbit under homographies (working modulo translation by $\mathbb{Z}$) of a given quadratic irrational. We refer to [PaP1, PaP2] for a proof of this, and for more algebraic expressions of this height function and its important differences with classical ones.

We consider in the first statement below the particular case when α_0 is the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$, and we refer to [PaP1] for the general version. One way this restriction simplifies the statement is that the Golden Ratio is in the same orbit under $\mathrm{PSL}_2(\mathbb{Z})$ as its Galois conjugate, which is not the case of every quadratic irrational (see for instance [Sar]). The next result is proven in [PaP1, Theo. 1.3, Prop. 1.4], with a mistake in the Hurwitz constant corrected in the erratum of loc. cit., thanks to Bugeaud, who gave another way to compute it in [Bug3], using only continued fractions techniques.

Theorem 1 (Parkkonen-Paulin) *With the notation X, Y, Z, H as above, the Lagrange spectrum $\mathrm{Sp}_{\mathrm{Lag}}$ is closed, bounded, with Hurwitz constant $\frac{3}{\sqrt{5}} - 1$. For every $\psi :]0, +\infty[\to]0, +\infty[$ such that $t \mapsto \ln \psi(e^t)$ is Lipschitz, we have*

$$\begin{cases} \mathrm{Leb}_{\mathbb{R}}({}^c\,\mathrm{WA}_\psi) = 0 & \text{if } \int_1^{+\infty} \psi(t)\,dt = +\infty \\ \mathrm{Leb}_{\mathbb{R}}(\mathrm{WA}_\psi) = 0 & \text{if } \int_1^{+\infty} \psi(t)\,dt < +\infty\,. \end{cases}$$

The exact value of the Hurwitz constant for the Diophantine approximation of real numbers by elements of an orbit under $\mathrm{PSL}_2(\mathbb{Z})$ (or a congruence subgroup) of a general quadratic irrational (and its Galois conjugate) is an interesting open problem.

Let α_0, β_0 be fixed integral quadratic irrationals, and let $R_{\alpha_0}, R_{\beta_0}$ be the regulators of the lattices $\mathbb{Z}+\mathbb{Z}\alpha_0, \mathbb{Z}+\mathbb{Z}\beta_0$ respectively. The integrality assumption is only present here in order to simplify the statements below in this survey, see [PaP4] for the general version.

The following result is an equidistribution result of the traces of the quadratic irrationals in a given orbit (by homographies) under $\mathrm{PSL}_2(\mathbb{Z})$ of a quadratic irrational, using the above height function $H : \alpha \mapsto \frac{2}{|\alpha-\alpha^\sigma|}$. We refer to [PaP4, Theo. 4.1] for a version with additional congruence assumptions, and to [PaP4, Theo. 4.2] and [PaP4, Theo. 4.4] for extensions to quadratic irrationals over an imaginary quadratic number field (using relative traces) or over a rational quaternion algebra.

Theorem 2 (Parkkonen-Paulin) *As $s \to +\infty$, we have*

$$\frac{\pi^2}{3\, R_{\alpha_0}\, s} \sum_{\alpha \in \mathrm{PSL}_2(\mathbb{Z})\cdot\alpha_0\,:\, H(\alpha)\leq s} \Delta_{\mathtt{tr}\,\alpha} \;\overset{*}{\rightharpoonup}\; \mathrm{Leb}_{\mathbb{R}}\ .$$

We introduced in [PaP4] another height function for the Diophantine approximation of real numbers by the elements of the orbit (by homographies) under $\mathrm{PSL}_2(\mathbb{Z})$ of β_0, which measures their relative complexity with respect to α_0. Let

$$[a,b,c,d] = \frac{(c-a)\,(d-b)}{(c-b)\,(d-a)}$$

be the standard crossratio of a quadruple (a,b,c,d) of pairwise distinct points in $\mathbb{R}$. For every $\beta \in \mathrm{PSL}_2(\mathbb{Z})\cdot\beta_0 - \{\alpha_0, \alpha_0^\sigma\}$, let

$$H_{\alpha_0}(\beta) = \frac{1}{\max\{|[\alpha_0,\beta,\alpha_0^\sigma,\beta^\sigma]|,\ |[\alpha_0,\beta^\sigma,\alpha_0^\sigma,\beta]|\}}\ .$$

We prove in [PaP4] that the map H_{α_0} is an appropriate height function modulo the action (by homographies) of the fixator $\mathrm{PSL}_2(\mathbb{Z})_{\alpha_0}$ of α_0 in $\mathrm{PSL}_2(\mathbb{Z})$. The following theorem is a counting result of quadratic irrationals relative to a given one.

Theorem 3 (Parkkonen-Paulin) *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\mathrm{Card}\{\beta \in \mathrm{PSL}_2(\mathbb{Z})_{\alpha_0}\backslash\, \mathrm{PSL}_2(\mathbb{Z})\cdot\beta_0,\ H_{\alpha_0}(\beta)\leq s\} = \frac{48\, R_{\alpha_0}\, R_{\beta_0}}{\pi^2}\ s + \mathrm{O}(s^{1-\kappa})\,.$$

We refer to [PaP4, Theo. 4.9] for a more general version, including additional congruence assumptions, and to [PaP4, Theo. 4.10] for an extension to quadratic irrationals over an imaginary quadratic extension of $\mathbb{Q}$. Theorem 3 fits into the framework of Subsection 2.2 with $X = Y = \mathrm{PSL}_2(\mathbb{Z})_{\alpha_0}\backslash(\mathbb{P}_1(\mathbb{R}) - \{\alpha_0, \alpha_0^\sigma\})$, $Z = \mathrm{PSL}_2(\mathbb{Z})_{\alpha_0}\backslash(\mathrm{PSL}_2(\mathbb{Z})\cdot\beta_0 - \{\alpha_0,\alpha_0^\sigma\})$ and $H : \mathrm{PSL}_2(\mathbb{Z})_{\alpha_0}\cdot\beta \mapsto H_{\alpha_0}(\beta)$.

2.4 Equidistribution of rational points in $\mathbb{R}$ and $\mathbb{C}$

In this section, we will consider equidistribution results in $\mathbb{R}$ and $\mathbb{C}$ of arithmetically defined points.

First, we would like to consider again the approximation of points in $\mathbb{R}$ by points in $\mathbb{Q}$, but to change the height function H, using an indefinite rational binary quadratic form Q which is not the product of two rational linear forms, by taking

$$H\big(\frac{p}{q}\big) = |Q(p,q)|.$$

It is easy to see that this is (locally) an appropriate height function outside the roots α, α^σ of $t \mapsto Q(t,1)$: for every $s \geq 0$, the number of $\frac{p}{q} \in \mathbb{Q}$ such that $H\big(\frac{p}{q}\big) \leq s$ is locally finite in $\mathbb{R} - \{\alpha, \alpha^\sigma\}$. This fits in the framework of Subsection 2.2 as follows. Let

$$\mathrm{SO}_Q(\mathbb{Z}) = \{g \in \mathrm{SL}_2(\mathbb{Z}) \;:\; Q \circ g = Q\}$$

be the integral group of automorphs of Q. Note that the subgroup $\mathrm{SO}_Q(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ injects into $\mathrm{PSL}_2(\mathbb{Z})$ if and only if $\mathtt{tr}\,\alpha \neq 0$. Let $\mathrm{PSO}_Q(\mathbb{Z})$ be the image of $\mathrm{SO}_Q(\mathbb{Z})$ in $\mathrm{PSL}_2(\mathbb{Z})$, which acts by homographies on $\mathbb{P}_1(\mathbb{R})$, fixing α and α^σ, acting properly discontinuously on $\mathbb{P}_1(\mathbb{R}) - \{\alpha, \alpha^\sigma\}$. We therefore study the Diophantine approximation problems with $X = \mathrm{PSO}_Q(\mathbb{Z})\backslash(\mathbb{P}_1(\mathbb{R}) - \{\alpha, \alpha^\sigma\})$ and Z the image of $\mathbb{P}_1(\mathbb{Q})$ by the canonical projection $(\mathbb{P}_1(\mathbb{R}) - \{\alpha, \alpha^\sigma\}) \to X$.

We only consider in this survey the particular case when Q is $Q(u,v) = u^2 - uv - v^2$ (closely related to the Golden Ratio, as the knowledgeable reader has already seen!). We refer to [PaP4, Theo. 5.10] for the general result, with error term estimates and additional congruence assumptions, and to [GorP] for an extension to n-ary norm forms for general $n > 2$.

Theorem 4 (Parkkonen-Paulin) *As $s \to +\infty$, we have*

$$\frac{\pi^2}{6\,s} \sum_{\frac{p}{q} \in \mathbb{Q},\ |p^2 - pq - q^2| \leq s} \Delta_{\frac{p}{q}} \;\overset{*}{\rightharpoonup}\; \frac{dt}{|t^2 - t - 1|}\,.$$

Note that the measure to which the rational points equidistribute when counted with this multiplicity is no longer the Lebesgue measure, but is the natural smooth measure invariant under the real group of automorphs of Q (unique up to multiplication by a locally constant positive function).

Now, let us turn to the Diophantine approximation of complex numbers by Gaussian rational ones. Every element of the imaginary quadratic field $K = \mathbb{Q}(i)$ of Gaussian rational numbers may and will be written $\frac{p}{q}$ with $p, q \in \mathbb{Z}[i]$ relatively prime Gaussian integers, and this writing is unique up to the multiplication of p and q by the same invertible Gaussian integer 1, -1, i or $-i$. In particular, the map

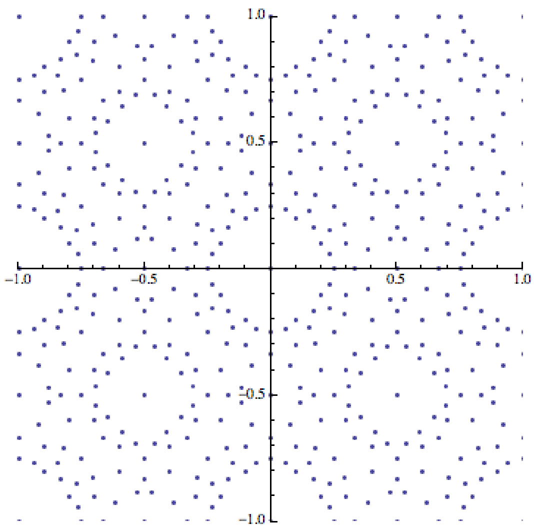
$$H\big(\frac{p}{q}\big) = |q|$$

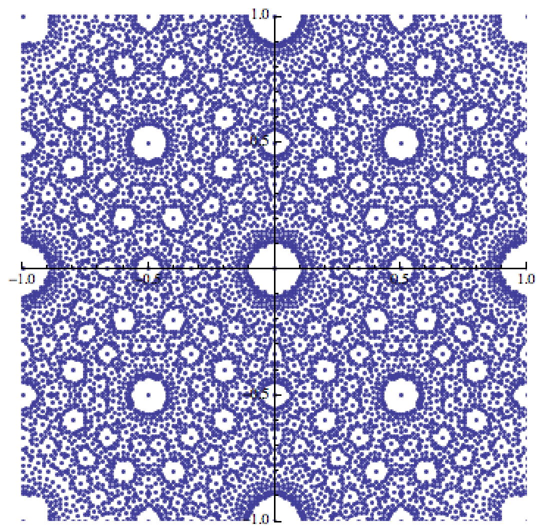
is well defined, and is clearly an appropriate height function on $\mathbb{Q}(i)$ modulo $\mathbb{Z}[i]$. We hence consider, with the notation of Subsection 2.2, the spaces $X = \mathbb{C}/\mathbb{Z}[i]$, $Z = \mathbb{Q}(i)/\mathbb{Z}[i]$, $Y = (\mathbb{C} - \mathbb{Q}(i))/\mathbb{Z}[i]$ and the above height function $H : Z \to [0, +\infty[\,$.

The following result (due to [Cos, Coro. 6.1] albeit in a less explicit form) is an equidistribution result of the Gaussian rational points in the complex field, analogous to the Mertens theorem on the equidistribution of Farey fractions in the real field. We denote, here and in Subsection 2.5, by $\mathscr{O}_K = \mathbb{Z}[i]$ the ring of integers of $K = \mathbb{Q}(i)$, by $\mathscr{O}_K^\times = \{1, -1, i, -i\}$ the group of invertible elements in $\mathscr{O}_K$, by $D_K = -4$ its discriminant, and by

$$\zeta_K : s \mapsto \sum_{\mathfrak{a} \text{ nonzero ideal in } \mathscr{O}_K} \frac{1}{N(\mathfrak{a})^s}$$

its Dedekind zeta function. The pictures below show the fractions $\frac{p}{q} \in \mathbb{Q}[i]$ in the square $[-1,1] \times [-1,1]$ where $p, q \in \mathbb{Z}[i]$ are relatively prime Gaussian integers with $|q| \leq 5$ and $|q| \leq 10$. The fact that there is a large white region around the fractions $\frac{p}{q} \in \mathbb{Q}[i]$ with $|q|$ small will be explained in Subsection 4.2.

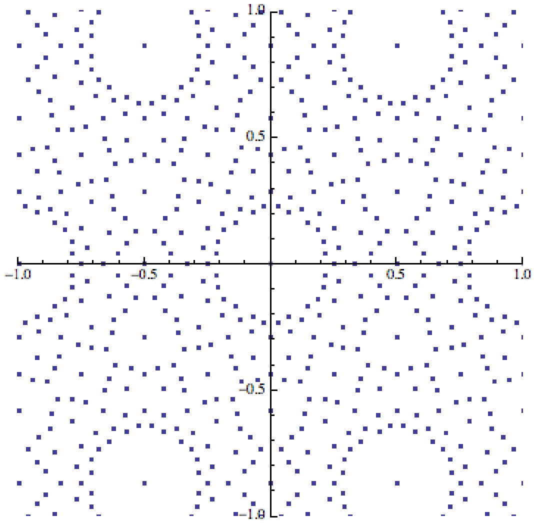


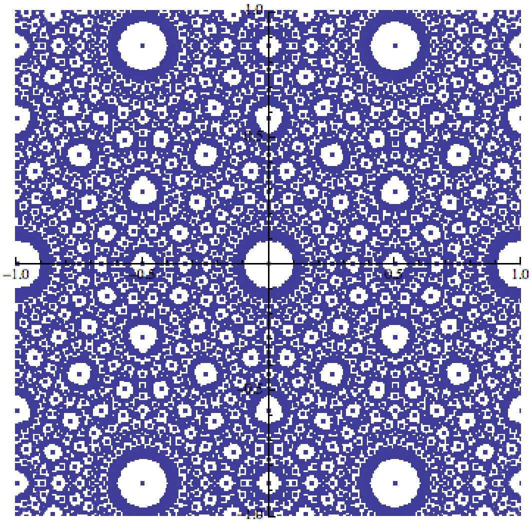


Theorem 5 (Cosentino, Parkkonen-Paulin) *As $s \to +\infty$,*

$$\frac{|\mathscr{O}_K^\times|\, |D_K|\, \zeta_K(2)}{2\pi\, s^2} \sum_{\frac{p}{q} \in K \,:\, |q| \leq s} \Delta_{\frac{p}{q}} \ \overset{*}{\rightharpoonup}\ \mathrm{Leb}_{\mathbb{C}}\ .$$

We refer to [PaP4, Theo. 1.1] for a version of this theorem valid for any imaginary quadratic number field (see below a plot in the square $[-1,1] \times [-1,1]$ of the fractions $\frac{p}{q} \in \mathbb{Q}[i\sqrt{3}]$ where $p, q \in \mathbb{Z}[e^{\frac{2\pi}{3}}]$ are relatively prime Eisenstein integers with $|q| \leq 5$ and $|q| \leq 10$), with additional congruence assumptions on u and v, to [PaP4, Theo. 1.2] for analogous results in Hamilton's quaternion division algebra, and to [Cos, PaP4] for error term estimates.





2.5 Equidistribution and counting in the Heisenberg group

In this section, we will consider Diophantine approximation results of elements of the Heisenberg group by arithmetically defined points.

Recall that the 3-dimensional *Heisenberg group* is the nilpotent real Lie group with underlying manifold

$$\mathrm{Heis}_3 = \{(w_0, w) \in \mathbb{C} \times \mathbb{C} \ : \ 2\mathrm{Re}\, w_0 = |w|^2\}$$

and law $(w_0, w)(w_0', w') = (w_0 + w_0' + w'\,\overline{w}, w + w')$. A standard model in control theory of Heis_3, with underlying manifold $\mathbb{R}^3$, may be obtained by the change of variables $(x, y, t) \in \mathbb{R}^3$ with $w = x + iy$ and $t = 2\,\mathrm{Im}\, w_0$. We endow Heis_3 with its Haar measure

$$d\,\mathrm{Haar}_{\mathrm{Heis}_3}(w_0, w) = d(\mathrm{Im}\, w_0)\, d(\mathrm{Re}\, w)\, d(\mathrm{Im}\, w)\,,$$

(that is $\frac{1}{2}\, dx\, dy\, dt$ in the above coordinates (x, y, t)) and with the (almost) distance d''_{Cyg} defined below.

The *Cygan distance* d_{Cyg} on Heis_3 (see [Gol, page 160], sometimes called the *Korányi distance* in sub-Riemannian geometry, though Korányi [Kor] attributes it to Cygan [Cyg]) is the unique left-invariant distance on Heis_3 such that the Cygan distance from $(w_0, w) \in \mathrm{Heis}_3$ to $(0, 0)$ is the *Cygan norm* $|(w_0, w)|_{\mathrm{Cyg}} = \sqrt{2|w_0|}$. Note that with the aforementioned change of variables ($x = \mathrm{Re}\, w, y = \mathrm{Im}\ w, t = 2\,\mathrm{Im}\ w_0$), we do recover the standard formulation of the Cygan norm on $\mathbb{R}^3$ (which is, by the way, equivalent to the Guivarc'h norm introduced much earlier in [Gui]):

$$|(x, y, t)|_{\mathrm{Cyg}} = \sqrt[4]{(x^2 + y^2)^2 + t^2}\ .$$

We will denote by $B_{\mathrm{Cyg}}(x, r)$ the ball for the Cygan distance of center x and radius r.

The *modified Cygan distance* d''_{Cyg} on Heis_3 is a minor variation of the Cygan distance, introduced in [PaP1, §4.4]: it is the unique left-invariant map d''_{Cyg} on $\mathrm{Heis}_3 \times \mathrm{Heis}_3$ such that

$$d''_{Cyg}((w_0, w), (0, 0)) = \frac{2|w_0|}{\sqrt{|w^2| + 2|w_0|}}\ .$$

Note that $\frac{1}{\sqrt{2}}\, d_{Cyg} \leq d''_{Cyg} \leq d_{Cyg}$. Though d''_{Cyg} might not be a distance, it is hence close to the Cygan distance, and it will allow error terms estimates in the following results.

We consider again the imaginary quadratic number field $K = \mathbb{Q}(i)$. The Heisenberg group Heis_3 is the set of real points of an algebraic group defined over $\mathbb{Q}$, with group of rational points equal to $\mathrm{Heis}_3(\mathbb{Q}) = \mathrm{Heis}_3 \cap (K \times K)$. It has also a natural $\mathbb{Z}$-structure, with the group of integral points equal to $\mathrm{Heis}_3(\mathbb{Z}) = \mathrm{Heis}_3 \cap (\mathscr{O}_K \times \mathscr{O}_K)$.

The following beautiful result (see [GaNT, Theo. 1.1] for a more general version), whose tools are in harmonic analysis, solves the analog in the Heisenberg group of the Gauss circle problem. It would be interesting to know if it is still valid for general imaginary quadratic number field K. Note that

$$\frac{1}{\mathrm{Haar}_{\mathrm{Heis}_3}(\mathrm{Heis}_3(\mathbb{Z})\backslash \mathrm{Heis}_3)} = \frac{2}{|D_K|} \tag{1}$$

(see Equation (24) in [PaP7], the volume $\mathrm{Vol}_{\mathrm{Heis}_3}$ used in this reference being $\mathrm{Vol}_{\mathrm{Heis}_3} = 8\,\mathrm{Haar}_{\mathrm{Heis}_3}$), and that $\mathrm{Haar}_{\mathrm{Heis}_3}(B_{\mathrm{Cyg}}(0,R)) = \mathrm{Haar}_{\mathrm{Heis}_3}(B_{\mathrm{Cyg}}(0,1))\, R^4$.

Theorem 6 (Garg-Nevo-Taylor) *As $R \to +\infty$, we have*

$$\mathrm{Card}\big(\,\mathrm{Heis}_3(\mathbb{Z}) \cap B_{\mathrm{Cyg}}(0,R)\big) \;=\; \frac{2\ \mathrm{Haar}_{\mathrm{Heis}_3}(B_{\mathrm{Cyg}}(0,1))}{|D_K|}\; R^4 + \mathrm{O}(R^2).$$

Any element in $\mathrm{Heis}_3(\mathbb{Q})$ may and will be written $(\frac{a}{c}, \frac{b}{c})$, where $a, b, c \in \mathscr{O}_K$ are relatively prime (that is, the ideal of $\mathscr{O}_K$ generated by a, b, c is equal to $\mathscr{O}_K$), and satisfy $c \neq 0$ and $2\,\mathrm{Re}(a\bar{c}) = |b|^2$. This writing is unique up to the multiplication of a, b, c by the same element of $\mathscr{O}_K^\times$. In particular, the map $H : \mathrm{Heis}_3(\mathbb{Q}) \to \mathbb{R}$

$$H\big(\frac{a}{c}, \frac{b}{c}\big) = |c|$$

is well defined, and is clearly an appropriate height function on $\mathrm{Heis}_3(\mathbb{Q})$ modulo the action of $\mathrm{Heis}_3(\mathbb{Z})$.

By studying the Diophantine approximation in the Heisenberg group by its rational points, we understand, as in example (3) at the end of Subsection 2.2, taking $X = \mathrm{Heis}_3(\mathbb{Z})\backslash \mathrm{Heis}_3$, $Z = \mathrm{Heis}_3(\mathbb{Z})\backslash \mathrm{Heis}_3(\mathbb{Q})$, $Y = \mathrm{Heis}_3(\mathbb{Z})\backslash(\mathrm{Heis}_3 - \mathrm{Heis}_3(\mathbb{Q}))$ and the above height function.

The following result is an equidistribution theorem of the set of rational points in Heis_3, analogous to the equidistribution of Farey fractions in the real field. We denote by ζ Riemann's zeta function. Note that the exponent 4 that appears below is the same as in the above theorem of Garg-Nevo-Taylor, it is the Hausdorff dimension of the Cygan distance.

Theorem 7 (Parkkonen-Paulin) *As $s \to +\infty$, we have*

$$\frac{\pi\, |\mathscr{O}_K^\times|\, |D_K|^{\frac{3}{2}}\, \zeta_K(3)}{\zeta(3)}\; s^{-4} \sum_{(\frac{a}{c}, \frac{b}{c}) \in \mathrm{Heis}_3(\mathbb{Q})\,:\, H\left(\frac{a}{c}, \frac{b}{c}\right) \leq s} \Delta_{(\frac{a}{c}, \frac{a}{c})} \;\overset{*}{\rightharpoonup}\; \mathrm{Haar}_{\mathrm{Heis}_3}\,.$$

We refer to [PaP7, Theo. 13] for a version of this theorem valid for any imaginary quadratic number field, with additional congruence assumptions, and with error term. The next result, analogous to the Mertens theorem in the real field, follows from the version with error term of Theorem 7, by Equation (1).

Corollary 8 (Parkkonen-Paulin) *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\operatorname{Card} \left\{ (\frac{a}{c}, \frac{b}{c}) \in \operatorname{Heis}_3(\mathbb{Z}) \backslash \operatorname{Heis}_3(\mathbb{Q}) \ : \ H(\frac{a}{c}, \frac{b}{c}) \leq s \right\} = \frac{\zeta(3)}{2\,\pi\,|\mathscr{O}_K^\times|\,|D_K|^{\frac{1}{2}}\,\zeta_K(3)}\, s^4 + \operatorname{O}(s^{4-\kappa})\,.$$

We now turn to equidistribution and counting results in the Heisenberg group of arithmetically defined topological circles, relating them to Diophantine approximation problems.

Let us consider the Hermitian form of signature $(1, 2)$ on $\mathbb{C}^3$ defined by

$$h : (z_0, z_1, z_2) \mapsto -z_0\overline{z_2} - z_2\overline{z_0} + |z_1|^2\,.$$

Using homogeneous coordinates in the complex projective plane $\mathbb{P}_2(\mathbb{C})$, *Poincaré's hypersphere*[2] is the projective isotropic locus of h

$$\mathscr{H}\mathscr{S} = \{[z_0 : z_1 : z_2] \in \mathbb{P}_2(\mathbb{C}) \ : \ h(z_0, z_1, z_2) = 0\}\,,$$

which is a real-analytic submanifold of $\mathbb{P}_2(\mathbb{C})$ diffeomorphic to the 3-sphere $\mathbb{S}^3$. The projective action on $\mathbb{P}_2(\mathbb{C})$ of the projective special unitary group PSU_h of h preserves $\mathscr{H}\mathscr{S}$. The Alexandrov compactification $\operatorname{Heis}_3 \cup \{\infty\}$ of the Heisenberg group Heis_3 identifies with Poincaré's hypersphere by mapping (w_0, w) to $[w_0 : w : 1]$ and ∞ to $[1 : 0 : 0]$. We identify Heis_3 with its image in $\mathbb{P}_2(\mathbb{C})$, called *Segre's hyperconic*, that we will think of as the projective model of the Heisenberg group.

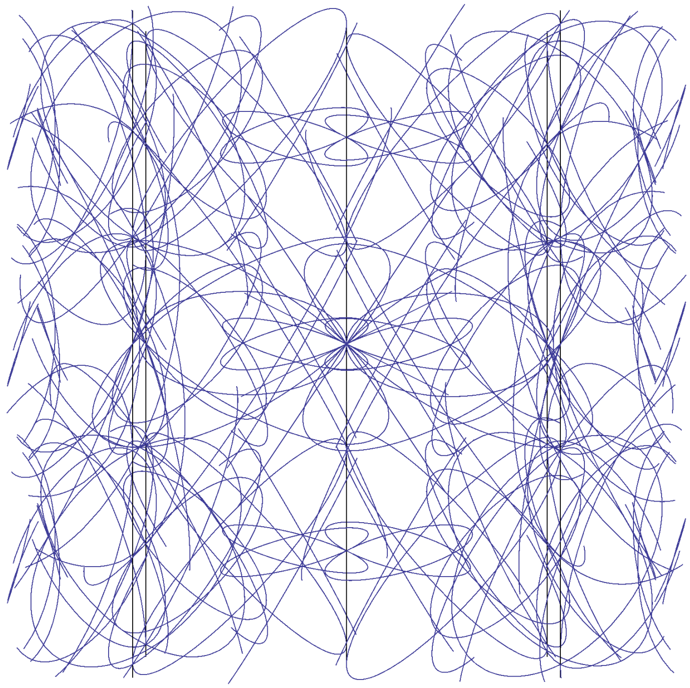
As defined by von Staudt,[3] a *chain* in Poincaré's hypersphere $\mathscr{H}\mathscr{S}$ is an intersection, nonempty and not reduced to a point, with $\mathscr{H}\mathscr{S}$ of a complex projective line in $\mathbb{P}_2(\mathbb{C})$. It is called *finite* if it does not contain $\infty = [1 : 0 : 0]$. A chain C separates the complex projective line containing it into two real discs $D_\pm(C)$, which we endow with their unique Poincaré metric (of constant curvature -1) invariant under the stabiliser of C in PSU_h.

If $\pi : \operatorname{Heis}_3 \to \mathbb{C}$ is the canonical Lie group morphism $(w_0, w) \mapsto w$, then the chains are the images, under the elements of PSU_h, of the *vertical chains*, that are the union with $\{\infty\}$ of the fibers of π. In particular, the finite chains are ellipses in (the aforementioned coordinates (x, y, t) of) Heis_3 whose images under π are Euclidean circles in $\mathbb{C}$. We refer for instance to [Gol, §4.3] for these informations and more on the chains.

A chain C will be called *arithmetic* (over $K = \mathbb{Q}[i]$) if its stabiliser $\operatorname{PSU}_h(\mathscr{O}_K)_C$ in the arithmetic group $\operatorname{PSU}_h(\mathscr{O}_K) = \operatorname{PSU}_h \cap \operatorname{PGL}_3(\mathscr{O}_K)$ has a dense orbit in C. It then turns out that $\operatorname{PSU}_h(\mathscr{O}_K)_C$ acts discretely with finite covolume on $D_\pm(C)$, and we denote by $\operatorname{Covol}(C)$ the (common) volume of $\operatorname{PSU}_h(\mathscr{O}_K)_C \backslash D_\pm(C)$. Note that the stabiliser $\operatorname{PSU}_h(\mathscr{O}_K)_\infty$ of $[1 : 0 : 0]$ in $\operatorname{PSU}_h(\mathscr{O}_K)$ preserves the d''_{Cyg}-diameters of the chains. The picture below shows part of an orbit of arithmetic chains under the arithmetic lattice $\operatorname{PSU}_h(\mathscr{O}_K)$.

[2] Actually, Poincaré in [Poi] was using another Hermitian form with signature $(1, 2)$.

[3] Though many references, including [Gol], attribute the notion of chains to E. Cartan, he himself attributes them to von Staudt in [Car, footnote 3)].

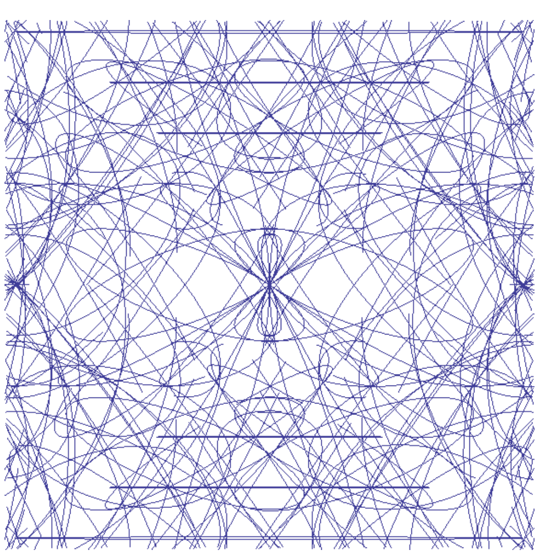


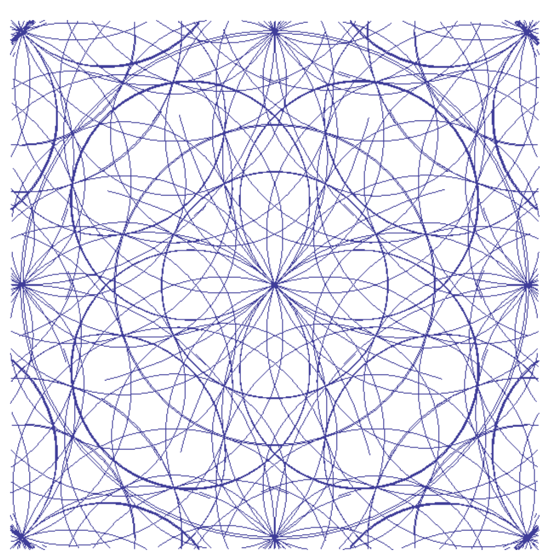
Theorem 9 (Parkkonen-Paulin) *Let C_0 be an arithmetic chain over K in the hypersphere $\mathscr{HS}$. Then there exists a constant $\kappa > 0$ such that, as $\epsilon > 0$ tends to 0, the number of chains modulo $\mathrm{PSU}_h(\mathscr{O}_K)_\infty$ in the $\mathrm{PSU}_h(\mathscr{O}_K)$-orbit of C_0 with d''_{Cyg}-diameter at least ϵ is equal to*

$$\frac{512\,\zeta(3)\ \mathrm{Covol}(C_0)}{|\mathscr{O}_K^\times|\,|D_K|^{\frac{3}{2}}\,\zeta_K(3)\,n_0}\ \epsilon^{-4}\big(1+\mathrm{O}(\epsilon^\kappa)\big)\ ,$$

where n_0 is the order of the pointwise stabiliser of C_0 in $\mathrm{PSU}_h(\mathscr{O}_K)$.

We refer to [PaP7, Theo. 19] for a version of this theorem valid for any imaginary quadratic number field (the pictures below represent two views of a part of an orbit of an arithmetic chain when $K = \mathbb{Q}(i\sqrt{2})$) and with additional congruence assumptions.





Given a complex projective line L in $\mathbb{P}_2(\mathbb{C})$, there is a unique order 2 complex projective map with fixed point set L, called the *reflexion* in L. Given a finite chain C, contained in the complex projective line $L(C)$, the *center* of C (see for instance [Gol, 4.3.3]), denoted by $\operatorname{cen}(C) \in \mathscr{H}\mathscr{S} - \{\infty\} = \operatorname{Heis}_3$, is the image of $\infty = [1:0:0]$ under the reflexion in $L(C)$. We also prove in [PaP7, Theo. 20] (a version valid for any imaginary quadratic number field and allowing additional congruence assumptions of) the following result saying that the centers of the finite arithmetic chains in a given $\operatorname{PSU}_h(\mathscr{O}_K)$-orbit of a given arithmetic chain C_0 with $d''_{\rm Cyg}$-diameter at least ϵ equidistribute in the Heisenberg group towards the normalised Haar measure $\frac{2}{|D_K|}\operatorname{Haar}_{\operatorname{Heis}_3}$.

Theorem 10 (Parkkonen-Paulin) *Let C_0 be an arithmetic chain over K. As $\epsilon > 0$ tends to 0, we have*

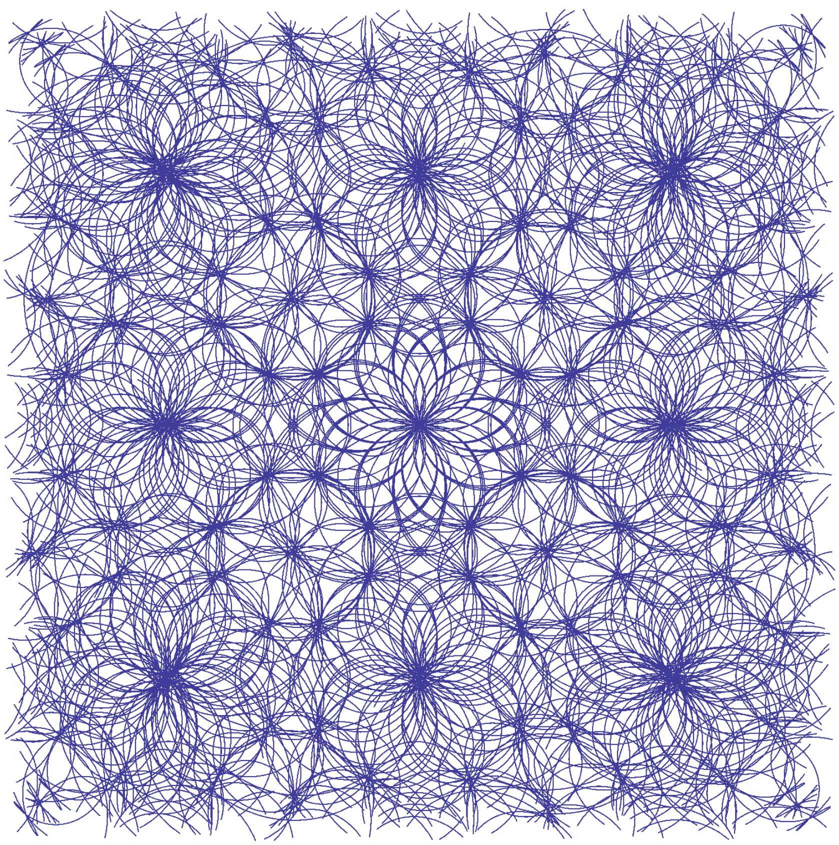
$$\frac{n_0\,(1+2\,\delta_{D_K,-3})\,|D_K|^{\frac{5}{2}}\,\zeta_K(3)}{128\,\zeta(3)\ \operatorname{Covol}(C_0)}\ \epsilon^4 \sum_{C\in\operatorname{PSU}_h(\mathscr{O}_K)\cdot C_0\,:\,\operatorname{diam}_{d''_{\rm Cyg}} C\geq\epsilon} \Delta_{\operatorname{cen}(C)} \overset{*}{\rightharpoonup} \operatorname{Haar}_{\operatorname{Heis}_3}\ .$$

The above result fits into the framework of Subsection 2.2 with

$$X = Y = \operatorname{Heis}_3(\mathbb{Z})\backslash\operatorname{Heis}_3, \quad Z = \{\gamma\operatorname{cen}(C_0)\ :\ \gamma\in\operatorname{PSU}_h(\mathscr{O}_K),\ \infty\notin\gamma C_0\}$$

and $H:\operatorname{cen}(\gamma C_0)\mapsto\operatorname{diam}_{d''_{\rm Cyg}}(\gamma C_0)$.

This equidistribution phenomenon can be understood by looking at the following picture, representing a different view of the same orbit of arithmetic chains as the one above Theorem 9.



3 Measures in negative curvature

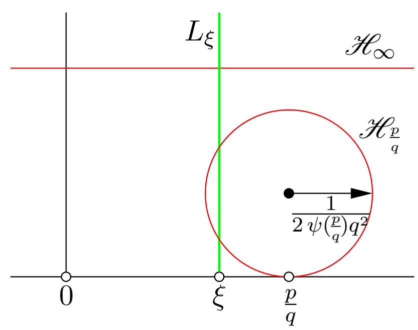
3.1 A classical link between basic Diophantine approximation and hyperbolic geometry

Let us briefly explain a well-known link between hyperbolic geometry and Diophantine approximation problems, which goes back to Gauss and Ford (see also [Seri]). This will start to explain why the proofs of Theorems 1, 2, 3, 4, 5, 7, 9 and 10 all use real or complex hyperbolic geometry.

For $n \geq 2$, let $\mathbb{H}^n_{\mathbb{R}}$ be the upper halfspace model of the real hyperbolic n-space, with underlying manifold $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,:\, x_n > 0\}$ and Riemannian metric $\frac{|dx|^2}{x_n^2}$.

Within the framework of example (1) of Subsection 2.2, let $\partial_\infty \mathbb{H}^2_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the boundary at infinity of $\mathbb{H}^2_{\mathbb{R}}$, $Y = \mathbb{R}$ and $Z = \mathbb{Q} = \Gamma \cdot \infty - \{\infty\}$, where $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ is the well-known *modular group*, which is a nonuniform lattice in $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}_0(\mathbb{H}^2_{\mathbb{R}})$, acting by homographies on $\mathbb{R}$.

Let $\psi : \mathbb{Q} \cup \{\infty\} \to [1, +\infty[$ be a map, let $\mathscr{H}_\infty = \{z \in \mathbb{C} \,:\, \mathrm{Im}\, z \geq \psi(\infty)\}$ and, for every $\frac{p}{q} \in \mathbb{Q}$, let $\mathscr{H}_{\frac{p}{q}}$ be the closed Euclidean disc of center $\frac{p}{q} + \frac{1}{2\,\psi(\frac{p}{q})q^2} i$ and radius $\frac{1}{2\,\psi(\frac{p}{q})q^2}$, with its tangency point to the horizontal axis removed. Then $(\mathscr{H}_x)_{x \in \mathbb{Q} \cup \{\infty\}}$ is a family of horodiscs in $\mathbb{H}^2_{\mathbb{R}}$, with pairwise disjoint interiors, centred at the parabolic fixed points of Γ. If ψ is constant, then this family is Γ-equivariant, and if $\psi = 1$, then it is maximal.



A link between Diophantine approximation of real numbers by rational ones and hyperbolic geometry is the following one: for every $\xi \in \mathbb{R}$, we have

$$\Big|\xi - \frac{p}{q}\Big| \leq \frac{1}{2\,\psi(\frac{p}{q})q^2}$$

if and only if the geodesic line L_ξ in $\mathbb{H}^2_{\mathbb{R}}$ from $\infty \in \partial_\infty \mathbb{H}^2_{\mathbb{R}}$ to $\xi \in \partial_\infty \mathbb{H}^2_{\mathbb{R}}$ meets the horodisc $\mathscr{H}_{\frac{p}{q}}$. Many Diophantine approximation properties of ξ may be explained by the behaviour of the image of the geodesic L_ξ in the modular curve $\Gamma \backslash \mathbb{H}^2_{\mathbb{R}}$. For instance, the coefficients of the continued fraction expansion of $\xi \in \mathbb{R} - \mathbb{Q}$ are bounded if and only if the positive subray of L_ξ has a bounded image in $\Gamma \backslash \mathbb{H}^2_{\mathbb{R}}$ (see for instance [Seri]).

Hence, it is useful for arithmetic applications to study the dynamical and ergodic properties of the geodesic flows in negative curvature, and we develop these topics in the following two sections.

3.2 Negative curvature background

We refer for instance to [BriH, Rob2, PaPS] for definitions, proofs and complements concerning this subsection.

Let $\widetilde{M}$ (for instance $\mathbb{H}^2_{\mathbb{R}}$) be a complete simply connected (smooth) Riemannian manifold with (dimension at least 2 and) pinched negative sectional curvature $-b^2 \leq K \leq -1$, and let $x_0 \in \widetilde{M}$ be a fixed basepoint. Let Γ (for instance $\mathrm{PSL}_2(\mathbb{Z})$) be a nonelementary (not virtually nilpotent) discrete group of isometries of $\widetilde{M}$, and let M be the quotient Riemannian orbifold $\Gamma\backslash\widetilde{M}$. See [Rob2, BrPP] to relax the pinching and manifold assumptions on $\widetilde{M}$.

We denote by $\partial_\infty\widetilde{M}$ the boundary at infinity of $\widetilde{M}$, that is the quotient space of the space of geodesic rays $\rho : [0, +\infty[\ \to \widetilde{M}$ in $\widetilde{M}$, two of them being equivalent if the Hausdorff distance between their images is finite. The class of a geodesic ray is called its *point at infinity*.

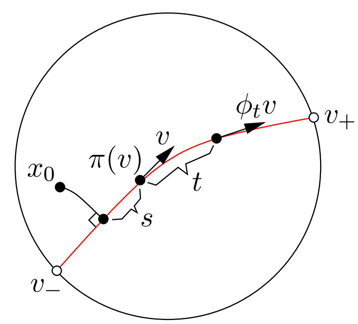
The *Busemann cocycle* of $\widetilde{M}$ is the map $\beta : \widetilde{M} \times \widetilde{M} \times \partial_\infty\widetilde{M} \to \mathbb{R}$ defined by

$$(x, y, \xi) \mapsto \beta_\xi(x, y) = \lim_{t\to+\infty} d(x, \rho(t)) - d(y, \rho(t)) ,$$

where ρ is any geodesic ray with point at infinity ξ. The above limit exists and is independent of ρ. The *horosphere* with center $\xi \in \partial_\infty\widetilde{M}$ through $x \in \widetilde{M}$ is $\{y \in \widetilde{M}\ :\ \beta_\xi(x, y) = 0\}$, and $\{y \in \widetilde{M}\ :\ \beta_\xi(x, y) \leq 0\}$ is the *horoball* centered at ξ bounded by this horosphere. For instance, in $\mathbb{H}^n_\mathbb{R}$, the horoballs are either the subspaces $\{(x_1, \dots, x_n) \in \mathbb{H}^n_\mathbb{R}\ :\ x_n \geq a\}$ for $a > 0$ or the closed Euclidean balls contained in the closure of $\mathbb{H}^n_\mathbb{R}$, tangent to the horizontal hyperplane, minus the tangency point.

Let $\widetilde{M} \cup \partial_\infty\widetilde{M}$ be the geometric compactification of $\widetilde{M}$, and let $\Lambda\Gamma = \overline{\Gamma x_0} - \Gamma x_0$ be the *limit set* of Γ. Let $\pi : T^1\widetilde{M} \to \widetilde{M}$ be the unit tangent bundle of $\widetilde{M}$.

For every $v \in T^1\widetilde{M}$, let $v_- \in \partial_\infty\widetilde{M}$ and $v_+ \in \partial_\infty\widetilde{M}$, respectively, be the endpoints at $-\infty$ and $+\infty$ of the geodesic line defined by v. Let $\partial^2_\infty\widetilde{M}$ be the subset of $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ which consists of pairs of distinct points at infinity of $\widetilde{M}$. *Hopf's parametrisation* of $T^1\widetilde{M}$ is the homeomorphism which identifies $T^1\widetilde{M}$ with $\partial^2_\infty\widetilde{M}\times\mathbb{R}$, by the map $v \mapsto (v_-, v_+, s)$, where s is the signed distance of the closest point to x_0 on the geodesic line defined by v to $\pi(v)$.



The geodesic flow is the smooth flow $(\phi_t)_{t\in\mathbb{R}}$ on $T^1\widetilde{M}$ defined, in Hopf's coordinates, by $\phi_t : (v_-, v_+, s) \mapsto (v_-, v_+, s + t)$.

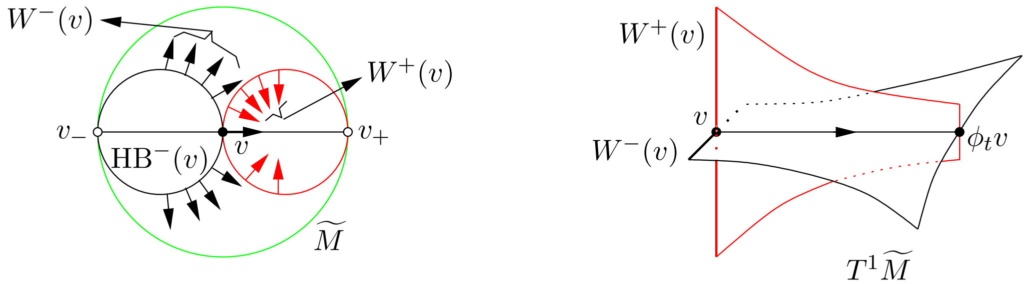
The *strong stable manifold* of $v \in T^1\widetilde{M}$ is

$$W^+(v) = \{v' \in T^1\widetilde{M}\ :\ \lim_{t\to+\infty} d(\pi(\phi_t v), \pi(\phi_t v')) = 0\},$$

and its *strong unstable manifold* is

$$W^-(v) = \{v' \in T^1\widetilde{M}\ :\ \lim_{t\to-\infty} d(\pi(\phi_t v), \pi(\phi_t v')) = 0\}.$$

The *stable/unstable manifold* of $v \in T^1\widetilde{M}$ is $W^{0\pm}(v) = \bigcup_{t\in\mathbb{R}} \phi_t W^\pm(v)$, which consists of the elements $v' \in T^1\widetilde{M}$ with $v'_\pm = v_\pm$. The map (t, w) from $\mathbb{R} \times W^\pm(v)$ to $W^{0\pm}(v)$ is a homeomorphism. The projections $\pi(W^+(v))$ and $\pi(W^-(v))$ in $\widetilde{M}$ of the strong stable and strong unstable manifolds of v are the horospheres through $\pi(v)$ centered at v_+ and v_-, respectively (see the picture below on the left hand side). These horospheres bound horoballs denoted by $\mathrm{HB}^\pm(v)$. The strong stable manifolds and strong unstable manifolds are the (smooth) leaves of topological foliations in $T^1\widetilde{M}$ that are invariant under the geodesic flow and the group of isometries of $\widetilde{M}$, denoted by $\mathscr{W}^+$ and $\mathscr{W}^-$ respectively.



For every $v \in T^1\widetilde{M}$, let $d_{W^-(v)}$ and $d_{W^+(v)}$ be *Hamenstädt's distances* on the strong unstable and strong stable leaf of v, defined as follows (see [HP3, Appendix] and compare with [Ham]): for all $w, z \in W^{\pm}(v)$, let

$$d_{W^{\pm}(v)}(w, z) = \lim_{t \to +\infty} e^{\frac{1}{2} d(\pi(\phi_{\mp t} w),\, \pi(\phi_{\mp t} z)) - t} \, .$$

The above limit exists, and Hamenstädt's distance induces the original topology on $W^{\pm}(v)$, though it has fractal properties: the Hausdorff dimension of $d_{W^+(v)}$ is in general bigger than the topological dimension of $W^{\pm}(v)$. For all $w, z \in W^{\pm}(v)$ and $t \in \mathbb{R}$, and for every isometry γ of $\widetilde{M}$, we have

$$d_{W^{\pm}(\gamma v)}(\gamma w, \gamma z) = d_{W^{\pm}(v)}(w, z) \quad \text{and} \quad d_{W^{\pm}(\phi_t v)}(\phi_t w, \phi_t z) = e^{\mp t} d_{W^{\pm}(v)}(w, z) \, .$$

These dilation/contraction properties of Hamenstädt's distances under the geodesic flow are a strengthening of the Anosov property of the geodesic flow (see the above picture on the right hand side).

The *Poincaré series* of Γ is the map $P_\Gamma : \mathbb{R} \to \,]0, +\infty]$ defined by

$$P_\Gamma(s) = \sum_{\gamma \in \Gamma} \; e^{-s\, d(\gamma x_0,\, x_0)} \, .$$

The *critical exponent* of Γ is

$$\delta_\Gamma = \lim_{n \to +\infty} \frac{1}{n} \ln \; \mathrm{Card}\{\gamma \in \Gamma,\; d(x_0,\, \gamma x_0) \leq n\} \, .$$

The above limit exists, is independent of x_0, we have $\delta_\Gamma \in \,]0, +\infty[$ and the Poincaré series $P_\Gamma(s)$ of Γ converges if $s > \delta_\Gamma$ and diverges if $s < \delta_\Gamma$, see for instance [Rob1, Rob2], as well as [PaPS] for versions with potential).

3.3 The various measures

We refer for instance to [Rob2, PaP3, PaPS, BrPP] for definitions, proofs and complements concerning this subsection. We introduce here the various measures that will be useful for our ergodic study in Section 4

A *Patterson-Sullivan measure* is a family $(\mu_x)_{x \in \widetilde{M}}$ of finite nonzero measures on $\partial_\infty \widetilde{M}$, whose support is the limit set $\Lambda\Gamma$ of Γ, such that, for all $\gamma \in \Gamma$, $x, y \in \widetilde{M}$ and $\xi \in \partial_\infty \widetilde{M}$,

$$\gamma_* \mu_x = \mu_{\gamma x} \quad \text{and} \quad d\mu_x(\xi) = e^{-\delta_\Gamma\, \beta_\xi(x,\, y)}\, d\mu_y(\xi) \, .$$

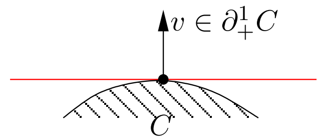
Such a family exists, and if $P_\Gamma(\delta_\Gamma)=+\infty$, then, for all $x\in\widetilde{M}$,

$$\mu_x=\lim_{s\rightarrow\delta_\Gamma{}^+}\frac{1}{P_\Gamma(s)}\sum_{\gamma\in\Gamma}\;e^{-s\,d(x,\gamma x_0)}\;\Delta_{\gamma x_0}$$

for the weak-star convergence of measures (see for instance [Rob2]).

Let C be a nonempty proper closed convex subset of $\widetilde{M}$, with stabiliser Γ_C in Γ, such that the family $(\gamma C)_{\gamma\in\Gamma/\Gamma_C}$ of subsets of $\widetilde{M}$ is locally finite.

The *inner* (respectively *outer*) *unit normal bundle* ∂^1_-C (respectively ∂^1_+C) of C is the topological submanifold of $T^1\widetilde{M}$ consisting of the unit tangent vectors $v\in T^1\widetilde{M}$ such that $\pi(v)\in\partial C$, v is orthogonal to a contact hyperplane to C and points towards (respectively away from) C (see [PaP3] for more precisions, and note that the boundary ∂C of C is not necessarily C^1, hence may have more than one contact hyperplane at some point, and that it is not necessarily true that $\exp(tv)$ belongs (respectively does not belong) to C for $t>0$ small enough).



The endpoint map $w\mapsto w_\pm$ from $\partial^1_\pm C$ into $\partial_\infty\widetilde{M}-\partial_\infty C$ is a homeomorphism. Using this homeomorphism, we defined in [PaP3] the *outer* (respectively *inner*) *skinning measure* on $T^1\widetilde{M}$ as the measure $\widetilde{\sigma}^+_C$ (respectively $\widetilde{\sigma}^-_C$) with support in ∂^1_+C (respectively ∂^1_-C) given by, for all $w\in\partial^1_\pm C$,

$$d\widetilde{\sigma}^\pm_C(w)=e^{-\delta_\Gamma\beta_{w_\pm}(\pi(w),\,x_0)}\,d\mu_{x_0}(w_\pm)\,. \tag{2}$$

This measure is independent of x_0 and satisfies $\gamma_*\widetilde{\sigma}^\pm_C=\widetilde{\sigma}^\pm_{\gamma C}$ for every $\gamma\in\Gamma$. Hence the measure $\sum_{\gamma\in\Gamma/\Gamma_C}\widetilde{\sigma}^\pm_{\gamma C}(w)$ is a Γ-invariant locally finite measure on $T^1\widetilde{M}$, therefore defining (see for instance [PaP6, §2.4] for details) a (locally finite) measure on $T^1M=\Gamma\backslash T^1\widetilde{M}$, called the *outer* (respectively *inner*) *skinning measure* of C on T^1M, and denoted by σ^+_C (respectively σ^-_C).

Note that the measure $\widetilde{\sigma}^+_{\mathrm{HB}^-(w)}$ (respectively $\widetilde{\sigma}^-_{\mathrm{HB}^+(w)}$) coincides with the Margulis measure (see for instance [Mar3, Rob2]) on the strong unstable leaf $W^-(w)$ (respectively strong stable leaf $W^+(w)$), for every $w\in T\widetilde{M}$.

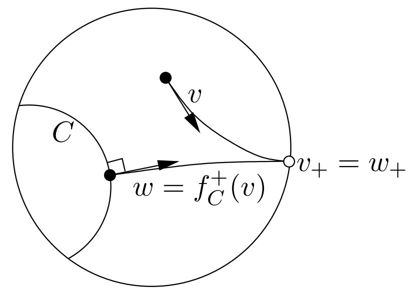
When $\widetilde{M}$ has constant curvature and Γ is geometrically finite, when C is an ball, horoball or totally geodesic submanifold, the (outer) skinning measure of C has been introduced by Oh and Shah [OhS3, OhS2], who coined the term, with beautiful applications to circle packings, see also [HP2, Lemma 4.3] for a closely related measure. The terminology comes from McMullen's proof of the contraction of the skinning map (capturing boundary information for surface subgroups of 3-manifold groups) introduced by Thurston to prove his hyperbolisation theorem.

The *Bowen-Margulis measure* on $T^1\widetilde{M}$ (associated to a given Patterson-Sullivan measure) is the measure $\widetilde{m}_{\mathrm{BM}}$ on $T^1\widetilde{M}$ given by the density

$$d\widetilde{m}_{\mathrm{BM}}(v)=e^{-\delta_\Gamma(\beta_{v_-}(\pi(v),x_0)\,+\,\beta_{v_+}(\pi(v),x_0))}\;d\mu_{x_0}(v_-)\,d\mu_{x_0}(v_+)\,ds \tag{3}$$

in Hopf's parametrisation. The Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ is independent of x_0, and it is invariant under the actions of the group Γ and of the geodesic flow. Thus (see for instance [PaP6, §2.4]), it defines a measure $m_{\rm BM}$ on T^1M which is invariant under the quotient geodesic flow, called the *Bowen-Margulis measure* on T^1M. We refer for instance to [Led, PaPS] for the extensions to the case with potential of the Patterson-Sullivan and Bowen-Margulis measures (the latter becomes the Gibbs measure), and to [PaP6] for the extensions to the case with potential of the skinning measures.

Let C be a nonempty proper closed convex subset of $\widetilde{M}$.



Let

$$U^\pm_C = \{v \in T^1\widetilde{M} \;:\; v_\pm \notin \partial_\infty C\}\;,$$

and let $f_C : U^\pm_C \to \partial^1_\pm C$ be the map sending v to the unique $w \in \partial^1_\pm C$ such that $w_\pm = v_\pm$. It is a topological fibration, whose fiber over $w \in \partial^1_\pm C$ is its stable/unstable leaf $W^{0\pm}(w)$.

Given $w \in T^1\widetilde{M}$, since $\partial^1_\mp \operatorname{HB}^\pm(v) = W^\pm(v)$ and using the homeomorphism $(t, v) \mapsto \phi_t v$ from $\mathbb{R} \times W^\pm(w)$ to $W^{0\pm}(w)$, we define a measure on $T^1\widetilde{M}$ with support contained in $W^{0\pm}(w)$ by, with $v \in W^\pm(w)$ and $t \in \mathbb{R}$,

$$d\mu_{W^{0\pm}(w)}(\phi_t v) = dt\; d\widetilde{\sigma}^\mp_{\operatorname{HB}^\pm(w)}(v)\;.$$

Note that the measure $d\mu_{W^{0\pm}(w)}$ depends on $W^\pm(w)$, not only on $W^{0\pm}(w)$.

The following result (see [PaP3, Prop. 8]) says that the Bowen–Margulis measure disintegrates over the skinning measure of C. When $\widetilde{M}$ has constant curvature and Γ is torsionfree and for special convex sets C, this result is implicit in [OhS2].

Proposition 11 (Parkkonen-Paulin) *The restriction to $U^\pm_C$ of the Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ disintegrates by the fibration $f^\pm_C : U^\pm_C \to \partial^1_\pm C$, over the skinning measure $\widetilde{\sigma}^\pm_C$ of C, with conditional measure $\mu_{W^{0\pm}(w)}$ on the fiber $W^{0\pm}(w)$ of $f^\pm_C$ over $w \in \partial^1_\pm C$: for $v \in U^\pm_C$,*

$$d\widetilde{m}_{\rm BM}(v) = \int_{w\in\partial^1_\pm C} d\mu_{W^{0\pm}(w)}(v)\; d\widetilde{\sigma}^\pm_C(w)\;.$$

We summarize in the following statement the ergodic properties of the Bowen-Margulis measure that we will use in Section 4.

Theorem 12 *Assume that $m_{\rm BM}$ is finite.*

(1) **(Patterson, Sullivan, Roblin)** *We have $P_\Gamma(\delta_\Gamma) = +\infty$ and the Patterson-Sullivan measure is unique up to a multiplicative constant; hence the Bowen-Margulis measure $m_{\rm BM}$ is uniquely defined, up to a multiplicative constant.*

(2) **(Bowen, Margulis, Otal-Peigné)** *When normalised to be a probability measure, the Bowen-Margulis measure on T^1M is the unique measure of maximal entropy of the geodesic flow.*

(3) **(Babillot)** *If the set of lengths of closed geodecics in M generates a dense subgroup of $\mathbb{R}$, then $m_{\rm BM}$ is mixing under the geodesic flow.*

(4) **(Kleinbock-Margulis, Clozel)** *If $\widetilde{M}$ is a symmetric space and Γ an arithmetic lattice, then there exist $c,\kappa>0$ and $\ell\in\mathbb{N}$ such that for all $\phi,\psi\in\mathscr{C}_c^\ell(T^1M)$ and all $t\in\mathbb{R}$, we have*

$$\Big|\int_{T^1M}\phi\circ g^{-t}\,\psi\;dm_{\rm BM}-\frac{1}{\|m_{\rm BM}\|}\;\int_{T^1M}\phi\;dm_{\rm BM}\int_{T^1M}\psi\;dm_{\rm BM}\;\Big|\le c\,e^{-\kappa|t|}\;\|\psi\|_\ell\;\|\phi\|_\ell\;.$$

Here are a few comments on these results. The Bowen-Margulis measure $m_{\rm BM}$ is finite for instance when M is compact, or when Γ is geometrically finite and its critical exponent is strictly bigger than the critical exponents of its parabolic subgroups (as it is the case when M is locally symmetric), by [DOP].

For the second assertion, we refer to [Mar2] and [Bow] when M is compact, and to [OP] under the weaker assumption that $m_{\rm BM}$ is finite. We refer to [Rob2] for a proof of the first assertion, to [Bab, Thm. 1] for the third one, and to [PaPS] for the extensions of the first three assertions to the case with potential. The assumption of Assertion (3), called *non-arithmeticity of the length spectrum* holds for instance, by [Dal1, Dal2], when M is locally symmetric or 2-dimensional, or when Γ contains a parabolic element, or when $\Lambda\Gamma$ is not totally disconnected.

In Assertion (4), called the *exponential decay of correlation* property of the Bowen-Margulis measure, we denote by $\|\cdot\|_\ell$ the Sobolev norm of regularity ℓ. We refer to [KM1] for a proof of this last assertion, with the help of [Clo, Theorem 3.1] to check its spectral gap property and of [KM2, Lemma 3.1] to deal with finite cover problems. Note that the spectral gap property has been checked by [MO] if $\widetilde{M}=\mathbb{H}^n_{\mathbb{R}}$ and Γ is only assumed to be geometrically finite with $\delta_\Gamma\ge n-2$ if $n\ge 3$ and $\delta_\Gamma\ge\frac{1}{2}$ if $n=2$, thus providing the first infinite volume examples for which Assertion (4) holds.

When M is locally symmetric with finite volume, the Bowen-Margulis measure $\widetilde{m}_{\rm BM}$ coincides, up to a multiplicative constant, with the Liouville measure, that is the Riemannian measure $\operatorname{Vol}_{T^1\widetilde{M}}$ of Sasaki's metric on $T^1\widetilde{M}$. See for instance [PaP5, §7] when M is real hyperbolic. In particular, the measure is finite in this case. More precisely (see [PaP5, §7] and [PaP6, §6]), if $\widetilde{M}=\mathbb{H}^n_{\mathbb{R}}$ and if M has finite volume, normalizing the Patterson-Sullivan measure so that its total mass is the volume of the $(n-1)$-sphere with its standard spherical metric (so that $\|\mu_{x_0}\|=\operatorname{Vol}(\mathbb{S}^{n-1})$), we have, denoting by Vol_N the Riemannian volume of any Riemannian manifold N,

- $\widetilde{m}_{\rm BM}=2^{n-1}\operatorname{Vol}_{T^1\widetilde{M}}$;
- if C is a horoball, then $\widetilde{\sigma}^{\pm}_D=2^{n-1}\operatorname{Vol}_{\partial^1_\pm D}$;
- if C is totally geodesic, then $\widetilde{\sigma}^{+}_D=\widetilde{\sigma}^{-}_D=\operatorname{Vol}_{\partial^1_\pm D}$.

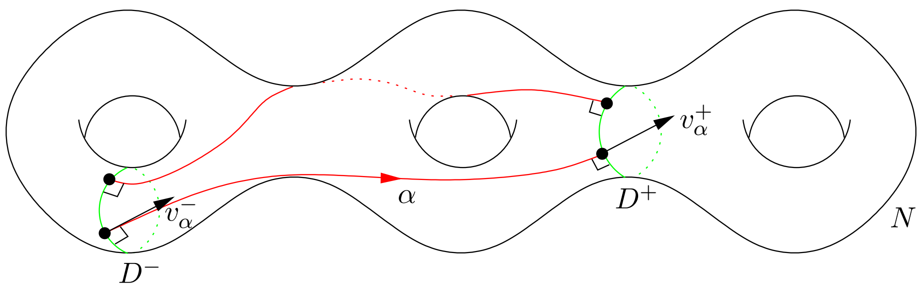
4 Geometric equidistribution and counting

In this section, we will link the Diophantine approximation problems of Subsections 2.3, 2.4 and 2.5 to general geometric equidistribution and counting problems on the common perpendiculars between two locally convex subsets in a negatively curved Riemannian orbifold.

Assume for simplicity in the introduction of this Section 4 (thus avoiding problems of regularity, multiplicities and finiteness), that N is a compact negatively curved Riemannian manifold, and that D^- and D^+ are proper nonempty disjoint closed locally convex subsets of M with smooth boundaries. A *common perpendicular* from D^- to D^+ is a locally geodesic path in N starting perpendicularly from D^- and arriving perpendicularly to D^+.

There is exactly one such common perpendicular in every homotopy class of paths starting from D^- and ending in D^+, where during the homotopy the origin of the path remains in D^- and its terminal point remains in D^+. In particular, there are at most countably many such common perpendiculars, and at most finitely many when their length is bounded.

Even when N is a closed hyperbolic surface and D^-, D^+ are simple closed geodesic (see the picture below), the result (see Equation (4)) was not known before appearing in [PaP6].



We give in Subsection 4.1 (refering to [PaP6] for complete statements and proofs) an asymptotic formula as $t \to +\infty$ for the number of common perpendiculars of length at most t from D^- to D^+, and an equidistribution result as $t \to +\infty$ of the initial and terminal tangent vectors v_α^- and v_α^+ of these common perpendiculars α in the outer and inner unit normal bundles of D^- and D^+, respectively. Although we do use Margulis's mixing ideas, major new techniques needed to be developed to treat the problem in the generality considered in [PaP6], some of them we will indicate in Subsection 4.1.

Here is a striking corollary of Theorem 15 in a very different context, that apparently does not involve negative curvature dynamics or geometry. Let Γ be a geometrically finite discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ (acting by homographies on $\mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$). Assume that Γ does not contain a quasifuchsian subgroup with index at most 2, and that its limit set $\Lambda\Gamma$ is bounded and not totally disconnected in $\mathbb{C}$. These assumptions are only here to ensure that the domain of discontinuity $\Omega\Gamma = (\mathbb{C} \cup \{\infty\}) - \Lambda\Gamma$ of Γ has infinitely many connected components (only one of them unbounded). The following result gives a precise asymptotic as ϵ tends to 0 on the counting function of the (finite) number of these connected components whose diameter are at least ϵ.

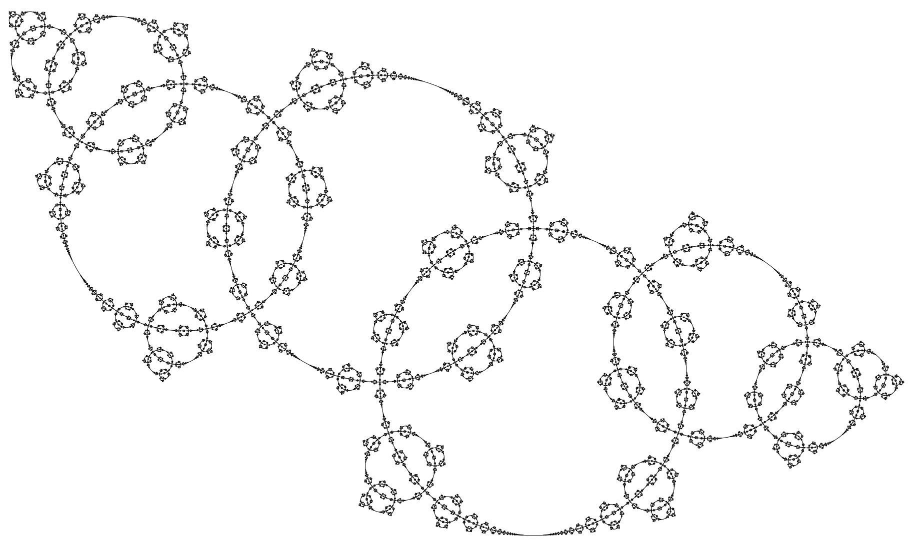
The multiplicative constant has an explicit value, that requires some more notation, and does involve hyperbolic geometry. We denote by $(\Omega_i)_{i\in I}$ a family of representatives, modulo the action of Γ, of the connected components of $\Omega\Gamma$ whose stabilisers have infinite index in Γ. For every $i \in I$, let $\mathscr{C}\Omega_i$ be the convex hull of Ω_i in the upper-half space model of the 3-dimensional real hyperbolic space $\mathbb{H}^3_\mathbb{R}$, and let $\sigma^-_{\mathscr{C}\Omega_i}$ be the (inner) skinning measure of $\mathscr{C}\Omega_i$ for Γ. We also denote by HB_∞ the horoball in $\mathbb{H}^3_\mathbb{R}$ consisting of points with vertical coordinates at least 1, and by $\sigma^+_{\mathrm{HB}_\infty}$ its (outer) skinning measure for Γ.

Corollary 13 (Parkkonen-Paulin) *Let Γ be a geometrically finite discrete group of $\mathrm{PSL}_2(\mathbb{C})$, with bounded and not totally disconnected limit set in $\mathbb{C}$, which does not contain a quasifuchsian subgroup with index at most 2. Assume that the Hausdorff dimension δ of the limit set of Γ is at least $\frac{1}{2}$. Then there exists $\kappa > 0$ such that the number of connected components of the domain of discontinuity $\Omega\Gamma$ of Γ with diameter at least ϵ is equal, as*

$\epsilon \to 0$, *to*

$$\frac{2^\delta\,\|\sigma^+_{\mathrm{HB}_\infty}\|\,\sum_{i\in I}\|\sigma^-_{\mathscr{C}\Omega_i}\|}{\delta\,\|m_{\mathrm{BM}}\|}\,\epsilon^{-\delta}+\mathrm{O}(\epsilon^{-\delta+\kappa})\ .$$

We refer to [PaP6, Coro. 25] for a proof of this result, with the error term coming from [PaP6, Theo. 28] and [MO], as explained in the discussion of Assertion (4) of Theorem 12. This corollary largely extends the result of Oh-Shah [OhS1] when all the connected components of the domain of discontinuity are assumed to be round discs. Note that the fractal geometry of the boundary of general convex hulls is an important feature that needs to be adressed by non-homogeneous dynamics arguments.



4.1 Equidistribution and counting of common perpendicular

Let $(\widetilde{M},\Gamma,M)$ be as in the beginning of Subsection 3.2. A *common perpendicular* from a closed convex subset A^- in $\widetilde{M}$ to a closed convex subset A^+ in $\widetilde{M}$ is a geodesic arc $\widetilde{\alpha}$ in $\widetilde{M}$ whose initial tangent vector $v^-_{\widetilde{\alpha}}$ belongs to $\partial^1_+A^-$ and terminal tangent vector $v^+_{\widetilde{\alpha}}$ belongs to $\partial^1_-A^+$. Note that there exists such a common perpendicular if and only if A^- and A^+ are nonempty, proper, with disjoint closures in $\widetilde{M}\cup\partial_\infty\widetilde{M}$. It is then unique, and its length is positive.

Let $C^\pm$ be nonempty proper closed convex subsets of $\widetilde{M}$, with stabiliser $\Gamma_{C^\pm}$ in Γ, such that the family $(\gamma C^\pm)_{\gamma\in\Gamma/\Gamma_{C^\pm}}$ of subsets of $\widetilde{M}$ is locally finite. We denote by $\mathrm{Perp}(C^-,C^+)$ the set of images in $M=\Gamma\backslash\widetilde{M}$ of the common perpendiculars from γ^-C^- to γ^+C^+ as $\gamma^\pm$ ranges over Γ, and, for every $t>0$, by $\mathrm{Perp}(C^-,C^+,t)$ the subset of the ones with length at most t. For every $\alpha\in\mathrm{Perp}(C^-,C^+)$, we denote by v^-_α and v^+_α its initial tangent vector and terminal tangent vector, which belong to the image in T^1M of respectively $\partial^1_+C^-$ and $\partial^1_-C^+$.

Since Γ might have torsion, and since $\Gamma_{C^\pm}\backslash C^\pm$ does not necessarily embed in $M=\Gamma\backslash\widetilde{M}$, each element α of $\mathrm{Perp}(C^-,C^+,t)$ comes with a natural multiplicity $m(\alpha)$. Denote by $\widetilde{\alpha}$ any common perpendicular in $\widetilde{M}$ between translates of C^- and C^+ with image α in

M and by $\Gamma_{\widetilde{\alpha}}$ its pointwise stabiliser in Γ. Then

$$m(\alpha) = \frac{\operatorname{Card}\{\gamma^- \in \Gamma/\Gamma_{C^-} \;:\; v^-_{\widetilde{\alpha}} \in \gamma^- \partial^1_+ C^-\} \operatorname{Card}\{\gamma^+ \in \Gamma/\Gamma_{C^+} \;:\; v^+_{\widetilde{\alpha}} \in \gamma^+ \partial^1_- C^+\}}{\operatorname{Card} \Gamma_{\widetilde{\alpha}}} \;.$$

Note that the numerator and the denominator are finite by the local finiteness of the families $(\gamma C^\pm)_{\gamma\in\Gamma/\Gamma_{C^\pm}}$ and the discreteness of Γ, and they depend only on the orbit of $\widetilde{\alpha}$ under Γ. This multiplicity is indeed natural. Concerning the denominator, in any counting problem of objects possibly having symmetries, the appropriate counting function consists in taking as the multiplicity of an object the inverse of the cardinality of its symmetry group. The numerator is here in order to take into account the fact that the elements $\gamma^\pm$ in Γ such that $\widetilde{\alpha}$ is a common perpendicular from $\gamma^- C^-$ to $\gamma^+ C^+$ are not necessarily unique, even modulo $\Gamma_{C^\pm}$. The natural counting function of the common perpendiculars between the images of $C^\pm$ is then the map

$$t \mapsto \mathscr{N}_{C^-,\,C^+}(t) = \sum_{\alpha\in\operatorname{Perp}(C^-,\,C^+,\,t)} m(\alpha) \;.$$

The reader can assume for simplicity that Γ is torsionfree and that $\Gamma_{C^\pm}\backslash C^\pm$ embeds in $M = \Gamma\backslash\widetilde{M}$ by the map induced by the inclusion of $C^\pm$ in M, in which case all multiplicities are 1.

Below, we state our equidistribution result, in the outer and inner unit normal bundles of the images in M of C^- and C^+, of the initial and terminal tangent vectors of the common perpendiculars between the images of C^- and C^+, as well as our asymptotic formula as $t \to +\infty$ for the number of common perpendiculars of length at most t between the images of C^- and C^+. We refer respectively to [PaP6, Theo. 14, 28] and [PaP6, Coro. 20, 28] for more general versions, involving more general locally finite families of convex subsets, versions with weights coming from potentials (the Bowen-Margulis measure being then replaced by the Gibbs measure of [PaPS]), and for version with error terms under an additional assumption of exponential decay of correlations (see Theorem 12 (4)).

Theorem 14 (Parkkonen-Paulin) *Assume that $m_{\rm BM}$ is finite and mixing under the geodesic flow, and that the skinning measures $\sigma^\pm_{C^\mp}$ are finite. For the weak-star convergence of measures on $T^1M \times T^1M$, we have*

$$\lim_{t\to+\infty} \delta_\Gamma \,\|m_{\rm BM}\|\, e^{-\delta_\Gamma t} \sum_{\alpha\in\operatorname{Perp}(C^-,\,C^+,\,t)} m(\alpha)\; \Delta_{v^-_\alpha} \otimes \Delta_{v^+_\alpha} \;=\; \sigma^+_{C^-} \otimes \sigma^-_{C^+} \;.$$

Theorem 15 (Parkkonen-Paulin) *Assume that $m_{\rm BM}$ is finite and mixing under the geodesic flow, and that the skinning measures $\sigma^\pm_{C^\mp}$ are finite and nonzero. Then, as $t \to +\infty$,*

$$\mathscr{N}_{C^-,\,C^+}(t) \sim \frac{\|\sigma^+_{C^-}\|\,\|\sigma^-_{C^+}\|}{\|m_{\rm BM}\|}\,\frac{e^{\delta_\Gamma\, t}}{\delta_\Gamma} \;.$$

The counting function $\mathscr{N}_{C^-,\,C^+}(t)$ has been studied in various special cases since the 1950's and in a number of recent works, sometimes in a different guise, see the survey [PaP5] for more details. A number of special cases were known before our result:

- C^- and C^+ are reduced to points, by for instance [Hub], [Mar1] and [Rob2],

- C^- and C^+ are horoballs, by [BeHP], [HP1], [Cos] and [Rob2] without an explicit form of the constant in the asymptotic expression,
- C^- is a point and C^+ is a totally geodesic submanifold, by [Her], [EsM] and [OhS1] in constant curvature,
- C^- is a point and C^+ is a horoball, by [Kon] and [KonO] in constant curvature, and [Kim] in rank one symmetric spaces,
- C^- is a horoball and C^+ is a totally geodesic submanifold, by [OhS3] and [PaP2] in constant curvature, and
- C^- and C^+ are (properly immersed) locally geodesic lines in constant curvature and dimension 3, by [Pol].

As a new particular case, if M has constant curvature -1, if the images in M of C^- and C^+ are closed geodesics of lengths ℓ_- and ℓ_+, respectively, then the number of common perpendiculars (counted with multiplicity) from the image of C^- to the image of C^+ of length at most s satisfies, as $s \to +\infty$,

$$\mathscr{N}_{C^-,\,C^+}(s) \sim \frac{\pi^{\frac{n}{2}-1}\Gamma(\frac{n-1}{2})^2}{2^{n-2}(n-1)\Gamma(\frac{n}{2})}\;\frac{\ell_-\ell_+}{\operatorname{Vol}(M)}\;e^{(n-1)s}\;. \tag{4}$$

When M is a closed hyperbolic surface and $C^- = C^+$, this formula (4) has been obtained by Martin-McKee-Wambach [MMW] by trace formula methods, though obtaining the case $C^- \neq C^+$ seems difficult by these methods.

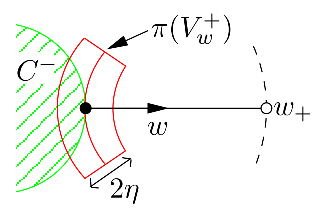
The family $(\ell(\alpha))_{\alpha\in\operatorname{Perp}(C^-,C^+)}$ (with multiplicities) will be called the *marked ortholength spectrum* from C^- to C^+. The set of lengths (with multiplicities) of elements of $\operatorname{Perp}(C^-, C^+)$ will be called the *ortholength spectrum* of C^-, C^+. This second set has been introduced by Basmajian [Bas] (under the name "full orthogonal spectrum") when M has constant curvature, and the images in M of C^- and C^+ are disjoint or equal embedded totally geodesic hypersurfaces or embedded horospherical cusp neighbourhoods or embedded balls (see also [BriK] when M is a compact hyperbolic manifold with totally geodesic boundary and the images in M of C^- and C^+ are exactly ∂M). The two results above are hence major contributions to the asymptotics of marked ortholength spectra.

Let us give a brief sketch of the proof of Theorem 15, refering to [PaP6, §4.1] for a full proof.

Step 1. In this technical step, we start by constructing dynamical neighbourhoods and test functions around the outer/inner unit normal bundles of our convex sets, that will be appropriately pushed forward/backwards by the geodesic flow (using the nice contraction/dilation properties of Hamenstädt's distances compared to the ones of the induced Riemannian metric in variable curvature). We fix $R > 0$ big enough, and we will let $\eta > 0$, a priori small enough, tend to 0.

For all $w \in T^1\widetilde{M}$, let V_w^+ be the open ball of center w and radius R for Hamenstädt's distance on the strong stable leaf $W^+(w)$ of w. For every $\eta > 0$, let

$$\mathscr{V}^+(C^-) = \bigcup_{w\in\partial_+^1 C^-,\; s\in\,]-\eta,\eta[} \phi_s V_w^+\;,$$



which is a neighbourhood of $\partial_+^1 C^-$.

Let $h^- : T^1\widetilde{M} \to]0, +\infty]$ be the measurable and $\widetilde{m}_{\rm BM}$-almost everywhere finite map (since its denominator is positive if $w^- \in \Lambda\Gamma$) defined by

$$h^-(w) = \frac{1}{2\eta\, \widetilde{\sigma}^-_{{\rm HB}^+(w)}(V^+_w)}\,.$$

Let us denote by $\mathbb{1}_A$ the characteristic function of a subset A. Consider the map $\widetilde{\psi}^-_\eta : T^1\widetilde{M} \to [0, +\infty]$ defined by

$$\widetilde{\psi}^-_\eta : v \mapsto \sum_{\gamma \in \Gamma/\Gamma_{C^-}} h^- \circ f^+_{\gamma C^-}(v)\ \ \mathbb{1}_{\mathscr{V}^+(\gamma C^-)}(v)\,. \tag{5}$$

This map is Γ-invariant and measurable, hence it defines a measurable map $\psi^-_\eta : T^1M \to [0, +\infty]$, with support in a neighbourhood of the image of $\partial^1_+ C^-$ in T^1M. We define similarly $\psi^+_\eta : T^1M \to [0, +\infty]$ with support in a neighbourhood of the image of $\partial^1_- C^+$ in T^1M.

By the disintegration result of Proposition 11, the functions $\psi^\mp_\eta$ are integrable and

$$\int_{T^1M} \psi^\mp_\eta\ dm_{\rm BM} = \|\sigma^\pm_{C^\mp}\|\,. \tag{6}$$

Step 2. In this step, we use the mixing property of the geodesic flow, as first introduced by Margulis in his thesis (see for instance [Mar3]). Due to the symmetry of the problem, a one-sided pushing of the geodesic flow, as in all the previous works using Margulis's ideas, is not sufficient, and we need to push simultaneously the outer and inner unit normal vectors to the convex sets in opposite directions.

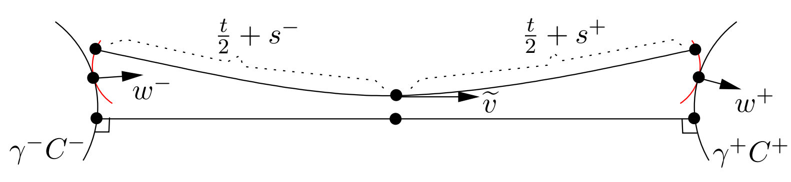
For all $t \geq 0$, let

$$a_\eta(t) = \int_{T^1M} \psi^-_\eta \circ \phi_{-t/2}\ \ \psi^+_\eta \circ \phi_{t/2}\ dm_{\rm BM}\,.$$

Then the mixing hypothesis of the geodesic flow and Equation (6) imply that

$$\lim_{t\to+\infty} a_\eta(t) = \frac{\|\sigma^+_{C^-}\|\ \|\sigma^-_{C^+}\|}{\|m_{\rm BM}\|}\,.$$

Step 3. In this step, we give another estimate of $a_\eta(t)$, relating it to the counting of common perpendiculars. One of the main new ideas in the proof (see [PaP6, §2.3] for a complete version) is an effective study of the geometry and the dynamics of the accidents that occur around midway of the pushing by the geodesic flow, yielding an effective statement of creation of common perpendiculars.



In order to give an idea of this phenomenon, assume that $v \in T^1M$ belongs to the support of the function $\psi_\eta^- \circ \phi_{-t/2} \;\; \psi_\eta^+ \circ \phi_{t/2}$ whose integral is $a_\eta(t)$. This is equivalent to assuming that $\phi_{\pm t/2} v$ belongs to the support of $\psi_\eta^\pm$. If $\widetilde{v}$ is a lift of v to $T^1\widetilde{M}$, by the definition of the maps $\psi_\eta^\mp$, this is equivalent to asking that there exist $\gamma^\pm \in \Gamma$, $w^\pm \in \partial^1_\mp C^\pm$ and $s^\pm \in \,]-\eta, \eta[$ such that $\phi_{\pm(t/2+s^\pm)}\widetilde{v}$ belongs to Hamenstädt's balls $V^\mp_{w^\pm}$. If R is fixed, when $\eta > 0$ is small enough and t is big enough, negative curvature estimates say that v is very close to the tangent vector at the midpoint of a common perpendicular from $\gamma^- C^-$ to $\gamma^+ C^+$ whose length is close to t (see [PaP6, §2.3]).

To obtain a precise estimate on $a_\eta(t)$, given a fundamental domain $\mathscr{F}$ for the action of Γ on $T^1\widetilde{M}$, we apply Fubini's theorem, as in Sarnak's "unfolding technique":

$$\int_{\mathscr{F}} \sum_{\gamma^- \in \Gamma/\Gamma_{C^-}} \; \sum_{\gamma^+ \in \Gamma/\Gamma_{C^+}} = \sum_{\gamma^- \in \Gamma/\Gamma_{C^-}} \; \sum_{\gamma^+ \in \Gamma/\Gamma_{C^+}} \int_{\mathscr{F}} ,$$

as well as a fine analysis (especially refined for the error term estimates) of the intrinsic geometry in variable curvature (almost everywhere defined second fundamental form, ...) of the outer/inner unit normal bundle pushed a long time by the geodesic flow. We then conclude by a Cesaro-type of argument in order to consider all common perpendiculars with length at most T, as T tends to $+\infty$, and by letting η tend to 0.

4.2 Towards the arithmetic applications

As we already hinted to, Theorems 2, 3, 4, 5, 7, 9 and 10 all follow from Theorem 14 or Theorem 15, though many more tools and ideas are needed, in particular volume computations of arithmetic orbifolds.

We only indicate the very beginning of the proof of these theorems, giving a bit more details on Theorems 1 to 4.

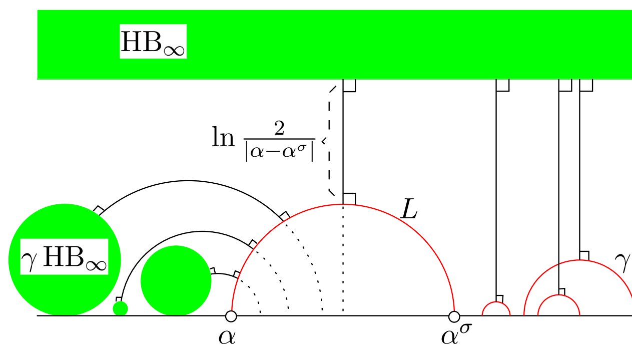
To prove Theorem 5, we apply Theorem 14 with $\widetilde{M} = \mathbb{H}^3_{\mathbb{R}}$, Γ the Bianchi group $\mathrm{PSL}_2(\mathscr{O}_K)$, and $C^- = C^+$ any horoball centered at ∞ in the upper halfspace model of $\mathbb{H}^3_{\mathbb{R}}$. Note that the cusps of the noncompact finite volume hyperbolic orbifold $\mathrm{PSL}_2(\mathscr{O}_K)\backslash\mathbb{H}^3_{\mathbb{R}}$ correspond to the ideal classes of $\mathscr{O}_K$ (in particular if $K = \mathbb{Q}(i)$, there is only one cusp), see for instance [ElGM]. Keeping the same C^- and taking C^+ centered at a parabolic fixed point defining the cusp allows version of Theorem 5 when p and q are varying in a given fractional ideal of $\mathscr{O}_K$ (when the class number of the imaginary quadratic number field K is larger than 1).

To prove Theorem 7, we consider the Hermitian form $h : (z_0, z_1, z_2) \mapsto -z_0\overline{z_2} - z_2\overline{z_0} + |z_1|^2$ on $\mathbb{C}^3$ whose signature is $(1, 2)$. We apply Theorem 14 with $\widetilde{M}$ the projective model $\{[z_0 : z_1 : z_2] \in \mathbb{P}_2(\mathbb{C}) \;:\; h(z_0, z_1, z_2) < 0\}$ of the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$, with Γ the Picard group $\mathrm{PSU}_h(\mathscr{O}_K) = \mathrm{PSU}_h \cap \mathrm{PGL}_3(\mathscr{O}_K)$, and with $C^- = C^+$ any horoball centered at $\infty = [1 : 0 : 0]$. See [PaP7, Theo. 12, 13] for the other ingredients of the proof. The reason why large white regions appear around the points $\frac{p}{q} \in K$ with $|q|$ small in the figures before and after Theorem 5 is that the horoballs in the Γ-orbit of C^- centered at these points have large Euclidean radius, hence it is (quadratically) difficult to fit disjoint horoballs in this orbit below them.

Replacing in the above data C^+ by the convex hull in $\mathbb{H}^2_{\mathbb{C}}$ of an arithmetic chain C_0, applying Theorem 15 and 14 is the very first step for proving Theorem 9 and 10, respectively. See [PaP7, Theo. 19, 20] for the other ingredients of the proof.

To prove Theorems 2, 3 and 4, we apply Theorem 14 or Theorem 15 with $\widetilde{M}$ the upper halfplane model of $\mathbb{H}^2_{\mathbb{R}}$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ (or appropriate finite index subgroups when we want versions with additional congruence assumptions). Note that the modular curve $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$, being arithmetic hyperbolic, has its Bowen-Margulis measure proportional to its Liouville measure, hence finite, and its geodesic flow is mixing, with exponential decay of correlation. We then take

- C^- a horoball centered at ∞ and C^+ the geodesic line $]\alpha_0, \alpha_0^\sigma[$ in $\mathbb{H}^2_{\mathbb{R}}$ with points at infinity $\alpha_0, \alpha_0^\sigma$ for Theorem 2,
- C^- the geodesic line $]\alpha_0, \alpha_0^\sigma[$ and C^+ the geodesic line $]\beta_0, \beta_0^\sigma[$ for Theorem 3,
- C^- the geodesic line $]\alpha_0, \alpha_0^\sigma[$ and C^+ a horoball centered at ∞ for Theorem 4.



The key input, also crucial to prove Theorem 1, is the well-known hyperbolic geometry understanding of quadratic irrationals. A real number α is a quadratic irrational if and only if it is fixed by (the action by homography of) an element $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ with $|\operatorname{tr}\,\gamma| > 2$. Then α^σ is the other fixed point of γ, and the geodesic line $L_\alpha =]\alpha, \alpha^\sigma[$ maps to a closed geodesic in the hyperbolic orbifold $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2_{\mathbb{R}}$. In particular, the image of $\partial^1_\pm L_\alpha$ in $\mathrm{PSL}_2(\mathbb{Z})\backslash T^1\mathbb{H}^2_{\mathbb{R}}$ is compact, and the skinning measures $\sigma^\pm_{L_\alpha}$ are positive and finite.

The first hint that there is a connection between quadratic irrationals and common perpendiculars is the following one. Let HB_∞ be the horoball centered at ∞, consisting of the points of $\mathbb{H}^2_{\mathbb{R}}$ with Euclidean height at least 1. Note that its stabiliser in $\mathrm{PSL}_2(\mathbb{Z})$ acts cocompactly on $\partial^1_\pm \mathrm{HB}_\infty$, hence the skinning measures $\sigma^\pm_{\mathrm{HB}_\infty}$ are positive and finite. Then, by an easy computation in hyperbolic geometry, the common perpendicular between HB_∞ and the geodesic line $]\alpha, \alpha^\sigma[$ (assuming that they are disjoint) has length $\ln H(\alpha)$, where

$$H(\alpha) = \frac{2}{|\alpha - \alpha^\sigma|}\ .$$

Another important observation to prove Theorem 4 (taking $\alpha = \frac{1+\sqrt{5}}{2}$ the Golden Ratio, $C^- = L_\alpha$ and $C^+ = \mathrm{HB}_\infty$) is that since the modular curve has finite volume, the skinning measure on $\partial^1_+ C^-$ is homogeneous. Hence, on each of the two connected components of $\partial^1_+ C^-$ which are naturally parametrised by $\mathbb{R}$, it is proportional to the Lebesgue measure, and this Lebesgue measure projects by the negative endpoint map $v \mapsto v^-$ to a measure on $\mathbb{R} - \{\alpha, \alpha^\sigma\}$ proportional to $\frac{d\,\mathrm{Leb}_{\mathbb{R}}(t)}{|t^2-t-1|}$, explaining the limit in Theorem 4. See [PaP4, Theo. 6] for a complete proof.

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