Lecture Notes
Distributions and Partial Differential Equations

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Introduction : why distributions ?

The notion of a distribution arose during the twentieth century as a powerful tool in the study of partial differential equations. Important contributors are J. Leray, S. Sobolev and L.Schwartz, who gave the formal definition. Here are three simple motivations for introducing distributions.

Deriving non differentiable functions

In differential calculus, one encounters immediately the unpleasant fact that not every function is differentiable. The purpose of distribution theory is to remedy this flaw. Recall that, given a function f on an open interval I of $\mathbb{R}$, the derivative function f' is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

as soon as this limit exists at every point $x \in I$. Let us mention a classical elementary application of this notion. If f' exists, f is a nondecreasing function on I if and only its derivative $f'(x) \geq 0$ for every x in I. However, there are many examples of non decreasing functions such that f' cannot be defined on the whole of I. A typical example is the Heaviside function,

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Of course, H admits a derivative at every point $x \neq 0$, and this derivative is 0. But this is not sufficient to conclude that H is nondecreasing, since the derivative of $-H$ enjoys the same property ! It is therefore necessary to extend the derivative in a way which takes into account the discontinuity of H or $-H$ at $x = 0$. In fact, to every locally integrable function f, we shall associate a mathematical object — a distribution — called its derivative, with the property that f is nondecreasing if and only if its derivative is nonnegative as a distribution .

Fourier series

In his famous memoir *Théorie analytique de la chaleur* (1822), Joseph Fourier introduced, for every "reasonable" 2π periodic function f , the coefficients

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, \mathrm{e}^{-inx}\, dx \ , \ n \in \mathbb{Z}$$

and stated that

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n(f)\, \mathrm{e}^{inx} \ .$$

Throughout the nineteenth and the early twentieth century, many mathematicians tried to give a general meaning to this statement. Though it has been proved by Dirichlet if f is a C^1 function, a striking observation of Kolmogorov is that there exists locally integrable functions f such that the above Fourier series is divergent for every $x \in \mathbb{R}$! However we shall see that the above series is convergent is a different sense. In fact, we shall define, for every general mathematical object called a 2π– periodic distribution u, a sequence of Fourier coefficients such that the corresponding series is always convergent in the sense of distributions, and so that its sum is always u. Furthermore, the derivative of u can be obtained by summing the series of derivatives, which is the reason why Fourier introduced this remarkable tool.

Electrostatics

If f is a function on $\mathbb{R}^3$ which represents a charge distribution, the Poisson equation

$$-\Delta u = f$$

is satisfied by the corresponding electric potential u. If f is smooth enough – say C^1— and small enough at infinity, — say it cancels outside a ball — one can prove that there exists a unique C^2 solution u of this equation which goes to 0 at infinity. Moreover, u is given by the formula

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\, dy \ .$$

However, the latter formula has a meaning for instance if f is just bounded and cancels outside a ball, though u is nomore C^2 in that case. It is therefore tempting to look for an interpretation of the Poisson equation in this case. Furthermore, in Physics, charge distributions can be more general, say supported by surfaces, curves, points, dipoles... In all these cases, the latter formula makes sense, so what is the meaning of the Poisson equation ? In fact it is possible to interpret Poisson equation in all these cases, in the sense of new mathematical objects called distributions. This context of Electrostatics is precisely the historical origin of the word “distributions”.

Chapter 1

Distributions in one space dimension

This chapter is devoted to basic calculus of distributions. We have chosen to begin with distributions of one variable to explore the basic ideas of the theory. That way, the reader has not to cope at the same time with several variables calculus.

1.1 Background on differential calculus

We recall here elementary facts about smooth functions of one real variable.

A smooth function on an open interval $I \subset \mathbb{R}$ is a function $\varphi : I \to \mathbb{C}$ whose derivatives of any order $\varphi', \varphi'', \ldots, \varphi^{(k)}, \ldots$ exist and are continuous on I. A linear combination of smooth functions is a smooth function, and we denote $\mathcal{C}^\infty(I)$ the vector space of smooth functions.

If $\varphi \in \mathcal{C}^\infty(I)$, and J is an open subset of I, the function defined on J by $x \mapsto \varphi(x)$ is a smooth function on J, that we denote $\varphi_{|_J}$. This function is called the restriction of φ to J.

It is true that the product of two smooth functions is smooth, and we have the so-called Leibniz formula

$$(\varphi_1\varphi_2)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} \varphi_1^{(j)} \varphi_2^{(k-j)}.$$

A smooth function $\varphi : I \to \mathbb{C}$ satisfies the Taylor formula

$$\forall x_0 \in I \;,\; \varphi(x) = \sum_{k=0}^{m} \frac{(x-x_0)^k}{k!}\varphi^{(k)}(x_0) + \frac{(x-x_0)^{m+1}}{m!}\int_0^1 (1-s)^m \varphi^{(m+1)}(x_0 + s(x-x_0))ds,$$

that one can prove integrating by parts the last term on the right.

Exercise 1.1.1 Prove Hadamard's lemma: if $\varphi \in \mathcal{C}^\infty(I)$ satisfies $\varphi(x_0) = 0$, there exists a function $\psi \in \mathcal{C}^\infty(I)$ such that $\varphi(x) = (x - x_0)\psi(x)$ for any $x \in I$.

1.1.1 Support of a continuous function

If f is continuous on I, and J is an open subset of I, we say that f vanishes on J if it vanishes at any point of J, or, equivalently, if $f_{|J}$ is the null function.

Definition 1.1.2 Let $f : I \to \mathbb{C}$ a continuous function. The support of f is the complementary of the union of all the open sets in I where f vanishes. This set is denoted by supp f.

Note that the support of f is a closed set. It is also the closure of the set of $x \in I$ such that $f(x) \neq 0$. The following characterization is often useful:

$$x_0 \notin \text{supp}\, f \iff \exists V \text{ neighborhood of } x_0 \text{ such that } f_{|V} = 0.$$

Exercise 1.1.3 Show

$$\text{supp}(f_1 f_2) \subset \text{supp}\, f_1 \cap \text{supp}\, f_2.$$

Are these two sets equal?

Of course, if $\varphi \in \mathcal{C}^\infty(I)$ vanishes on an open set $J \subset I$, all its derivatives vanish as well on J, and, therefore, for all integer k,

$$\text{supp}\, \varphi^{(k)} \subset \text{supp}\, \varphi.$$

1.2 Test functions

Let I be an open interval of $\mathbb{R}$.

Definition 1.2.1 We denote by $\mathcal{D}(I) = \mathcal{C}_0^\infty(I)$ the vector space of functions which are $\mathcal{C}^\infty$ on I, and whose support is a compact subset of I. Equivalently, a smooth function belongs to $\mathcal{D}(I)$ if it vanishes outside some segment $[a, b] \subset I$.

If J is an open subinterval of I, one may identify a function $\varphi \in \mathcal{D}(J)$ with the function $\tilde{\varphi} \in \mathcal{D}(I)$, defined as

$$\tilde{\varphi}(x) = \varphi(x) \text{ for } x \in J, \quad \tilde{\varphi}(x) = 0 \text{ for } x \in I \setminus J.$$

Indeed, $\tilde{\varphi}$ is smooth and with compact support on I for $\varphi \in \mathcal{D}(J)$. On the other hand, a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, can be identified with its restriction $\varphi_{|I} \in \mathcal{D}(I)$ for any open interval I that contains its support.

Exercise 1.2.2 Let f and φ be two functions in $\in L^1(\mathbb{R})$. Show that the convolution $f * \varphi$ of f and φ, given by

$$f * \varphi(x) = \int f(x-y)\varphi(y)dy,$$

is defined almost everywhere, and is an L^1 function.

Suppose moreover that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, show that $f * \varphi$ is smooth. At last if f is also continuous, show that

$$\text{supp}\, f * \varphi \subset \text{supp}\, f + \text{supp}\, \varphi.$$

Answer: First of all, $f * \varphi$ is an almost everywhere defined and L^1 function. Indeed, using Fubini's theorem for non-negative functions,

$$\begin{aligned}\int |f * \varphi(x)|dx &\leq \iint |f(x-y)\varphi(y)|dydx \\ &\leq \int |\varphi(y)|\big(\int |f(x-y)|dx\big)dy \leq \|\varphi\|_{L^1}\|f\|_{L^1} < +\infty\end{aligned}$$

Therefore $f * \varphi$ is finite almost everywhere, and belongs to L^1.

Suppose now that φ is smooth with compact support. Changing variable we have

$$f * \varphi(x) = \int f(y)\varphi(x-y)dy.$$

Thus $f * \varphi$ is also smooth thanks to the Lebesgue theorem, since

- the function $x \mapsto f(y)\varphi(x-y)$ is smooth for all y, and

$$\partial_x^k(f(y)\varphi(x-y)) = f(y)\varphi^{(k)}(x-y),$$

- we have the domination

$$|f(y)\varphi^{(k)}(x-y)| \leq |f(y)| \sup |\varphi^{(k)}| \in L^1.$$

Eventually suppose that f is continuous. If $x \notin \text{supp}\, f + \text{supp}\, \varphi$, then for any $y \in \text{supp}\, f$, $x - y$ does not belong to $\text{supp}\, \varphi$, thus

$$f * \varphi(x) = \int f(y)\varphi(x-y)dy = 0.$$

It is not immediately clear that $\mathcal{D}(I)$ is not reduced to the null function. One knows for example that the only compactly supported analytic function on $\mathbb{R}$ is this null function, due to the principle of isolated zeroes. However,

Proposition 1.2.3 The set $\mathcal{D}(I)$ is not trivial. More precisely, for every $x_0 \in I$ and $r > 0$ such that $[x_0 - r, x_0 + r] \subset I$, there exists $\varphi_{x_0,r} \in \mathcal{C}^\infty(I)$ such that $\text{supp}(\varphi_{x_0,r}) = [x_0 - r, x_0 + r]$.

Proof.— First of all, the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(t) = \begin{cases} e^{-1/t} & \text{as } t > 0, \\ 0 & \text{as } t \leq 0, \end{cases}$$

is smooth on $\mathbb{R}$. Indeed, it is easily seen to be $\mathcal{C}^\infty$ on $\mathbb{R}^*$, with $\varphi^{(k)}(t) = 0$ for $t < 0$. On the other hand, for $t > 0$, one can prove by induction that

$$\forall k \in \mathbb{N},\ \varphi^{(k)}(t) = P_k\left(\frac{1}{t}\right) e^{-1/t}$$

where P_k is a polynomial of degree $2k$. Therefore, for any $k \geq 1$, $\varphi^{(k)}(t) \to 0$ as $t \to 0^+$, so that $\varphi^{(k-1)}(t)$ is $\mathcal{C}^1$ on $\mathbb{R}$. Now let $x_0 \in I$, and $r > 0$ such that $[x_0 - r, x_0 + r] \subset I$. We denote $\varphi_{x_0,r} : I \to \mathbb{R}^+$ the function given by

$$\varphi_{x_0,r}(x) = \varphi(r^2 - |x - x_0|^2).$$

This function is smooth, with supp $\varphi_{x_0,r} = [x_0 - r, x_0 + r]$. In particular it belongs to $\mathcal{D}(I)$. □

As a matter of fact, we can even prove the

Proposition 1.2.4 Let $I \subset \mathbb{R}$ an open set, and a segment $[a, b] \subset I$. There exists a function $\psi \in \mathcal{D}(I)$ such that

i) $\psi = 1$ on $[a, b]$,

ii) $\forall x \in I\ , \psi(x) \in [0, 1]$.

Such a function is called a "plateau function" - from the French word plateau, which means a flat, high region. The usual English name for such functions is "cut-off" functions.

Proof.— First of all, we prove that, for every $\alpha < \beta$, there exists $\chi_{\alpha,\beta} \in \mathcal{C}^\infty(\mathbb{R})$ such that

i) $\chi_{\alpha,\beta} = 0$ on $]-\infty, \alpha]$,

ii) $\forall x \in \mathbb{R}\ , \chi_{\alpha,\beta}(x) \in [0, 1]$,

iii) $\chi_{\alpha,\beta} = 1$ on $[\beta, +\infty[$.

Let $\varphi_{x_0,r} \in \mathcal{C}^\infty(\mathbb{R})$ be the function defined above with

$$x_0 = \frac{\alpha + \beta}{2}\ ,\ r = \frac{\beta - \alpha}{2}\ .$$

Then it is easy to check that

$$\chi_{\alpha,\beta}(x) = \frac{\int_{-\infty}^{x} \varphi_{x_0,r}(t)\, dt}{\int_{-\infty}^{+\infty} \varphi_{x_0,r}(t)\, dt}$$

satisfies the required properties.
Coming back to $[a, b] \subset I$, select $a', b' \in I$ such that $a' < a < b < b'$. Then just choose

$$\psi(x) = \chi_{a',a}(x)(1 - \chi_{b,b'}(x))$$

□

Proposition 1.2.5 (Partitions of unity) Let $\varphi \in \mathcal{C}_0^\infty(I)$. Assume $I_1, \dots, I_n$ are open subintervals of I such that

$$\text{supp}(\varphi) \subset I_1 \cup \dots \cup I_n \ .$$

Then there exist $\varphi_1 \in \mathcal{C}_0^\infty(I_1), \dots, \varphi_n \in \mathcal{C}_0^\infty(I_n)$, such that

$$\varphi = \varphi_1 + \dots + \varphi_n \ .$$

Proof.— We start with an elementary observation.

Lemma 1.2.6 ("Shrinking lemma") Under the assumptions of Proposition 1.2.5, for every compact subset K of I such that

$$K \subset I_1 \cup \dots \cup I_n \ ,$$

there exist segments $[a_1, b_1] \subset I_1, \dots, [a_n, b_n] \subset I_n$ such that

$$K \subset]a_1, b_1[\cup \dots \cup]a_n, b_n[\ .$$

We proceed by induction on n. For $n = 1$, K is a compact subset of the open interval I_1, hence there exists a segment $[\alpha_1, \beta_1] \subset I_1$ such that $K \subset [\alpha_1, \beta_1]$. For instance, one may choose $\alpha_1 = \min \text{supp}(\varphi)$, $\beta_1 = \max \text{supp}(\varphi)$. Then just choose $a_1, b_1 \in I$ such that $a_1 < \alpha_1 < \beta_1 < b_1$.
Assume the result is true for $n - 1$ with $n \geq 2$, and prove it for n. Consider

$$K' = K \setminus I_n \ .$$

Then K' is closed and included in K, hence it is a compact subset of I, and

$$K' \subset I_1 \cup \dots \cup I_{n-1} \ .$$

Applying the induction assumption, there exists segments $[a_1, b_1] \subset I_1, \dots, [a_{n-1}, b_{n-1}] \subset I_{n-1}$ such that

$$K' \subset]a_1, b_1[\cup \dots \cup]a_{n-1}, b_{n-1}[\ .$$

Now consider the compact subset

$$K'' = K \setminus (]a_1, b_1[\cup \dots \cup]a_{n-1}, b_{n-1}[) \subset I_n \ .$$

Applying the result for $n = 1$, there exists a segment $[a_n, b_n] \subset I_n$ such that

$$K'' \subset]a_n, b_n[\ .$$

This yields

$$K \subset]a_1, b_1[\cup \dots \cup]a_n, b_n[\ ,$$

as announced.

Let us come back to the proof of Proposition 1.2.5. Let $[a_1, b_1], \dots, [a_n, b_n]$ be as in the lemma. Choose plateau functions $\psi_1 \in \mathcal{C}_0^\infty(I_1), \dots, \psi_n \in \mathcal{C}_0^\infty(I_n)$ such that $\psi_1 = 1$ on $[a_1, b_1]$,...,$\psi_n = 1$ on $[a_n, b_n]$. We define

$$\begin{aligned} \varphi_1 &= \varphi\psi_1 \ , \\ \varphi_2 &= \varphi\psi_2(1-\psi_1) \ , \\ &\dots \\ \varphi_n &= \varphi\psi_n(1-\psi_{n-1})\dots(1-\psi_1) \ . \end{aligned}$$

Then $\varphi_1 \in \mathcal{C}_0^\infty(I_1), \dots, \varphi_n \in \mathcal{C}_0^\infty(I_n)$,, and

$$\begin{aligned} \varphi - (\varphi_1 + \dots + \varphi_n) &= \varphi(1 - \psi_1 - \psi_2(1-\psi_1) - \dots - \psi_n(1-\psi_{n-1})\dots(1-\psi_1)) \\ &= \varphi(1-\psi_1)(1 - \psi_2 - \dots - \psi_n(1-\psi_{n-1})\dots(1-\psi_2)) \\ &= \varphi(1-\psi_1)\dots(1-\psi_n) \\ &= 0 \ . \end{aligned}$$

Indeed, by construction, $(1-\psi_1)\dots(1-\psi_n)$ is identically 0 on $[a_1, b_1] \cup \dots \cup [a_n, b_n]$, which contains the support of φ. □

1.2.1 Convergence in $\mathcal{D}(I)$

The natural notion of convergence for continuous functions is that of uniform convergence, since it is the simplest one for which the limit of a sequence of continuous functions is continuous. For smooth, compactly supported functions, the correct notion is given by the next definition.

Definition 1.2.7 Let (φ_j) be a sequence of functions in $\mathcal{D}(I)$, and $\varphi \in \mathcal{D}(I)$. We say that (φ_j) tends to φ in $\mathcal{D}(I)$ (or in the $\mathcal{D}(I)$-sense), when

i) There exits a segment $[a, b] \subset I$ such that supp $\varphi_j \subset [a, b]$ for all j.

ii) For all $k \in \mathbb{N}$, $\|\varphi_j^{(k)} - \varphi^{(k)}\|_\infty :=$ sup $|\varphi_j^{(k)} - \varphi^{(k)}| \to 0$ as $j \to +\infty$.

In that case we may write

$$\varphi = \mathcal{D} - \lim_{j \to +\infty} \varphi_j .$$

Remark 1.2.8 Notice that, under the conditions of the above definition, we always have supp$(\varphi) \subset [a, b]$.

Exercise 1.2.9 Let $\varphi \in \mathcal{D}(\mathbb{R})$, and, for $t \neq 1$, denote by $\psi_t \in \mathcal{D}(\mathbb{R})$ the function given by

$$\psi_t(x) = \frac{\varphi(tx) - \varphi(x)}{t - 1} .$$

Show that the family (ψ_t) converges in $\mathcal{D}(\mathbb{R})$ as t tends to 1.

1.3 Definitions and examples

1.3.1 Definitions

Definition 1.3.1 Let $I \subset \mathbb{R}$ an open subset, and T a complex valued linear form on $\mathcal{D}(I)$. One says that T is a distribution on I if, for every segment $[a,b] \subset I$,

$$\exists C > 0, \exists m \in \mathbb{N}, \forall \varphi \in \mathcal{C}_0^\infty(I) \text{ with supp}\, \varphi \subset [a,b]\ ,\ |T(\varphi)| \leq C \sum_{\alpha \leq m} \sup |\varphi^{(\alpha)}|.$$

We denote by $\mathcal{D}'(I)$ the set of distributions on I, and for $T \in \mathcal{D}'(I)$, $\varphi \in \mathcal{D}(I)$, we denote $\langle T, \varphi \rangle := T(\varphi)$.

Proposition 1.3.2 A linear form T on $\mathcal{D}(I)$ is a distribution on I if and only if $T(\varphi_j) \to T(\varphi)$ for any sequence (φ_j) of functions in $\mathcal{D}(I)$ that converges to φ in the $\mathcal{D}(I)$-sense.

Proof.— Let T be a distribution on I, and (φ_j) a sequence in $\mathcal{D}(I)$ which converges to φ in $\mathcal{D}(I)$. There is a segment $[a,b] \subset I$ such that supp $\varphi_j \subset [a,b]$ for all $j \in \mathbb{N}$, and supp $\varphi \subset [a,b]$. There exist $C > 0$ and $m \in \mathbb{N}$ such that

$$\forall \psi \in \mathcal{C}^\infty(I),\ \text{ with supp}\, \psi \subset [a,b],\ |T(\psi)| \leq C \sum_{\alpha \leq m} \sup |\psi^{(\alpha)}|.$$

In particular, for any $j \in \mathbb{N}$,

$$|T(\varphi_j) - T(\varphi)| = |T(\varphi_j - \varphi)| \leq C \sum_{\alpha \leq m} \sup |\varphi_j^{(\alpha)} - \varphi^{(\alpha)}|.$$

Therefore $T(\varphi_j) \to T(\varphi)$ as $j \to +\infty$, and we have proved the only if part of the proposition.

Suppose now that for any sequence (φ_j) of functions which converges in $\mathcal{D}(I)$, we have $T(\varphi_j) \to T(\varphi)$, where $\varphi = \mathcal{D} - \lim \varphi_j$. Suppose that the linear form T is *not* a distribution, that is

$$\exists [a,b] \subset I, \forall C > 0, \forall m \in \mathbb{N}, \exists \varphi \in \mathcal{C}_0^\infty(\Omega) \text{ with supp}\, \varphi \subset [a,b] \ \text{ such that } \ |T(\varphi)| > C \sum_{\alpha \leq m} \sup |\varphi^{(\alpha)}|.$$

Then for any $j \in \mathbb{N}$, choosing $C = m = j$, there is a function $\varphi_j \in \mathcal{C}_0^\infty(I)$ such that supp $\varphi_j \subset [a,b]$ and

$$|T(\varphi_j)| > j \sum_{\alpha \leq j} \sup |\varphi_j^{(\alpha)}|.$$

Let $\psi_j \in \mathcal{C}_0^\infty(I)$ given by $\psi_j = \varphi_j / |T(\varphi_j)|$. One has $|T(\psi_j)| = 1$, with supp $\psi_j \subset [a,b]$ and

$$\mathcal{D} - \lim \psi_j = 0,$$

since for all $j \geq \alpha$,

$$\sup |\psi_j^{(\alpha)}| \leq \sum_{\alpha \leq j} \sup |\psi_j^{(\alpha)}| < \frac{1}{j}.$$

This is a contradiction, since one should have $T(\psi_j) \to 0$. □

Remark 1.3.3 As a set of continuous linear forms, $\mathcal{D}'(I)$ is of course a vector space on $\mathbb{C}$: if $T_1, T_2 \in \mathcal{D}'(I)$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1 T_1 + \lambda_2 T_2$ is the distribution given by

$$\langle \lambda_1 T_1 + \lambda_2 T_2, \varphi \rangle := \lambda_1 \langle T_1, \varphi \rangle + \lambda_2 \langle T_2, \varphi \rangle.$$

For $T \in \mathcal{D}'(I)$, we may also denote $\bar{T}$ or T^* the distribution given by

$$\langle \bar{T}, \varphi \rangle = \overline{\langle T, \overline{\varphi} \rangle}.$$

Then, any distribution T can be written $T = T_1 + iT_2$ where T_1 and T_2 are real distributions, that is such that $\langle T, \varphi \rangle \in \mathbb{R}$ for any real-valued function φ. Indeed, this relation holds with

$$T_1 = \frac{1}{2}(T + \bar{T}) = \text{Re}(T) \text{ and } T_2 = \frac{1}{2i}(T - \bar{T}) = \text{Im}(T).$$

1.3.2 Distributions defined by locally integrable functions

If $f \in L^1_{\text{loc}}(I)$, the function $f\varphi$ is integrable for any $\varphi \in \mathcal{D}(I)$, and

$$T_f : \varphi \to \int_I f(x)\varphi(x)\,dx$$

is a linear form. Moreover, if $[a, b] \subset \Omega$ is a segment of I, for any $\varphi \in \mathcal{C}_0^\infty(I)$ supported in $[a, b]$, one has

$$|T_f(\varphi)| \leq \|f\|_{L^1([a,b])} \sup |\varphi|,$$

which shows that T_f is a distribution on I.

As a matter of fact, one can identify $L^1_{\text{loc}}(I)$ to a part of $\mathcal{D}'(I)$, that is identify f with T_f. This is the content of the

Proposition 1.3.4 The linear map

$$f \in L^1_{\text{loc}}(I) \longmapsto T_f \in \mathcal{D}'(I)$$

is one to one.

Remark 1.3.5 The statement $f = g$ for $f, g \in L^1_{\text{loc}}(I)$ should be understood in the sense of classes for the equivalence on functions : $f \sim g$ if $f(x) = g(x)$ a.e. In other words, if f is a locally integrable function, $T_f = 0$ if and only if $f = 0$ almost everywhere. However, in the special case $f \in \mathcal{C}^0(I)$, this is equivalent to $f = 0$ everywhere. Indeed, the only open set with zero Lebesgue measure is the empty set.

Proof.— Assume $T_f = 0$. It is enough to prove that $f = 0$ on every segment of I, or equivalently that $\psi f = 0$ for every $\psi \in \mathcal{C}_0^\infty(I)$. Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$, non negative, supported in $[-1, 1]$, and such that

$$\int_{\mathbb{R}} \rho(x)\, dx = 1 \ .$$

Consider, for every $\varepsilon > 0$,

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right) \ ,$$

so that the integral of ρ_ε is 1, and $\rho_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R})$ is supported in $[-\varepsilon, \varepsilon]$.

Lemma 1.3.6 Let $g \in L^1(\mathbb{R})$. Then the convolution product

$$g_\varepsilon = g * \rho_\varepsilon$$

converges to g in $L^1(\mathbb{R})$ as ε tends to 0.

Indeed, we have

$$\begin{aligned} |g_\varepsilon(x) - g(x)| &\leq \left| \int g(y)\rho_\varepsilon(x-y)dy - g(x) \right| \\ &\leq \left| \int g(x-\varepsilon z)\rho(z)dz - g(x) \right| \leq \int |g(x-\varepsilon z) - g(x)|\rho(z)dz \end{aligned}$$

Then by Fubini-Tonelli,

$$\|g_\varepsilon - g\|_{L^1} \leq \int \big(\int |g(x-\varepsilon z) - g(x)|\rho(z)dz\big)dx \leq \int \rho(z)\|\tau_{\varepsilon z}g - g\|_{L^1}dz,$$

where $\tau_a g$ denotes the translation of the function f given by $\tau_a g(x) = g(x-a)$. The result is then a direct consequence of the dominated convergence theorem, taking into account the continuity of the translations in the L^p spaces for $p \in [1, \infty[$, that is, here,

$$\|\tau_{\varepsilon z}g - g\|_{L^1} \to 0 \text{ as } \varepsilon \to 0 \ .$$

Indeed we have the domination

$$\|\tau_{\varepsilon z}g - g\|_{L^1}\rho(z) \leq 2\|g\|_{L^1}\rho(z).$$

Coming back to the proof of the proposition, we have

$$(\psi f) * \rho_\varepsilon(x) = \int f(y)\psi(y)\rho_\varepsilon(x-y)dy = \langle T_f, \psi\rho_\varepsilon(x-.)\rangle = 0,$$

since $y \mapsto \psi(y)\rho_\varepsilon(x-y)$ belongs to $\mathcal{C}_0^\infty(I)$. Applying the lemma to $g = \psi f \in L^1$, we conclude $\psi f = 0$ in L^1, which completes the proof. □

1.3.3 The Dirac mass

For $x_0 \in I$, we denote by $\delta_{x_0} : \mathcal{D}(I) \to \mathbb{C}$ the linear form given by

$$\delta_{x_0}(\varphi) = \varphi(x_0).$$

For any function $\varphi \in \mathcal{C}_0^\infty(I)$, one has

$$|\delta_{x_0}(\varphi)| \leq \sup \ |\varphi|,$$

so that δ_{x_0} is a distribution on I. It is called the Dirac mass at x_0.

The Dirac mass at x_0 cannot be defined by a locally integrable function f. Indeed, otherwise we would have, for any function $\varphi \in \mathcal{D}(I)$ such that $x_0 \notin \operatorname{supp} \varphi$,

$$\varphi(x_0) = 0 = \int f(x)\varphi(x)dx.$$

so that $f = 0$ a.e. by Proposition 1.3.4. In particular, for a plateau function ψ such that $\psi(x_0) = 1$, we would have

$$1 = \psi(x_0) = \langle T_f, \psi \rangle = \int f\psi = 0,$$

which is a contradiction.

1.3.4 A distribution involving an infinite number of derivatives

Let $I =]0,1[$. Consider a sequence (x_n) of points of $]0,1[$ converging to 0 as n tends to infinity. Let (a_n) a sequence of complex numbers. For every $\varphi \in \mathcal{C}_0^\infty(I)$, we claim that

$$\langle T, \varphi \rangle = \sum_{n=0}^{\infty} a_n \varphi^{(n)}(x_n)$$

defines a distribution on I. Indeed, let $a < b$ be points in $]0,1[$ such that $\operatorname{supp}(\varphi) \subset [a,b]$. Then there exists an integer N such that, for every $n > N$, $x_n < a$, so that

$$\sum_{n=0}^{\infty} a_n \varphi^{(n)}(x_n) = \sum_{n=0}^{N} a_n \varphi^{(n)}(x_n)$$

is well defined. For the same reason, if φ_k tends to φ in $\mathcal{C}_0^\infty(I)$, then, for N large enough,

$$\langle T, \varphi_k \rangle = \sum_{n=0}^{N} a_n \varphi_k^{(n)}(x_n) \to \sum_{n=0}^{N} a_n \varphi^{(n)}(x_n) = \langle T, \varphi \rangle \ .$$

Notice that it is crucial that (x_n) does not have an accumulation point in I, otherwise sequence (a_n) should satisfy quite stringent conditions !

1.3.5 Cauchy's principal value of $1/x$

Let us consider the linear form $T : \mathcal{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ given by

$$T(\varphi) = \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx.$$

This limit exists for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Indeed, for such a function, we have

$$\begin{aligned} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx &= \int_\varepsilon^{+\infty} \frac{\varphi(x)}{x}\, dx + \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\, dx \\ &= \int_\varepsilon^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x}\, dx \underset{\varepsilon\to 0}{\longrightarrow} \int_0^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x}\, dx\ . \end{aligned}$$

where we have used that the function

$$x \in \mathbb{R} \mapsto \frac{\varphi(x)-\varphi(-x)}{x}$$

is C^∞ and compactly supported.
Now we show that the well-defined linear form T is a distribution on $\mathbb{R}$. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Then

$$\left|\frac{\varphi(x)-\varphi(-x)}{x}\right| \le 2 \sup |\varphi'|$$

and, if $\operatorname{supp} \varphi \subset [-A, A]$,

$$\left|\int_0^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x}\, dx\right| \le 2A \sup |\varphi'|\ .$$

This shows that T satisfies the estimate in Definition 1.3.1. It is important to notice that the constant C in the estimate is here $2A$, that does indeed depend on the support of φ.

This distribution is called (Cauchy's) principal value of $1/x$, [1] and we denote it by

$$\langle \mathrm{pv}\left(\frac{1}{x}\right), \varphi\rangle = \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx = \int_0^{+\infty} \frac{\varphi(x)-\varphi(-x)}{x}\, dx\ .$$

There are many other examples of extensions of functions near a singular point (Hadamard finite parts,...)

1.4 Derivatives of distributions

This section introduces one of the most important operation on distributions, namely the derivative. In order to understand the definition below, let us notice that, if $f \in \mathcal{C}^1(I)$, a simple integration by parts yields

$$\forall \varphi \in \mathcal{C}_0^\infty(I), \langle T_{f'}, \varphi\rangle = \int_I f'(x)\varphi(x)dx = -\int_I f(x)\varphi'(x)dx = -\langle T_f, \varphi'\rangle\ .$$

This suggest the following definition.

1. In French, it is called "valeur principale de $1/x$", hence denoted as vp $\left(\frac{1}{x}\right)$.

Proposition 1.4.1 Let $T \in \mathcal{D}'(I)$. The linear form on $\mathcal{D}(I)$ defined by

$$\varphi \mapsto -\langle T, \varphi' \rangle$$

is a distribution, that we call the derivative of T, and that we denote by T'.

Proof.— Let $[a, b] \subset I$ be a segment, and $C > 0$, $m \in \mathbb{N}$ the constants given by the fact that $T \in \mathcal{D}'(I)$. For $\varphi \in \mathcal{C}_0^\infty(I)$, supported in $[a, b]$, we have

$$|\langle T', \varphi \rangle| = |\langle T, \varphi' \rangle| \leq C \sum_{\alpha \leq m} \sup |\varphi^{(1+\alpha)}| \leq C \sum_{\alpha \leq m+1} \sup |\varphi^{(\alpha)}|,$$

which shows that T' is a distribution. □

Of course the above calculation gives immediately

Proposition 1.4.2 If $T = T_f$ with $f \in \mathcal{C}^1(I)$, then $T' = T_{f'}$.

The next examples are much more interesting.

Example 1.4.3 Let $H : \mathbb{R} \to \mathbb{C}$ denote the Heaviside function, namely $H(x) = \mathbf{1}_{\mathbb{R}^+}(x)$. The function H belongs to $L^1_{\text{loc}}(\mathbb{R})$, and we can denote $T = T_H$ the associated distribution. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = -\int_0^{+\infty} \varphi'(x) dx = \varphi(0),$$

so that $T'_H = \delta_0$.

Example 1.4.4 Let $f \in L^1_{\text{loc}}(I)$ and $a \in I$. Consider

$$F(x) = \int_a^x f(t)\, dt \ .$$

A classical application of the dominated convergence theorem claims that F is continuous on I. In particular, F is locally integrable. We claim that

Proposition 1.4.5

$$T'_F = T_f \ .$$

Proof.— Introduce, for every $t \in I$,

$$I_{<t} = \{x \in I : x < t\} \ , \ I_{>t} = \{x \in I : x > t\} \ .$$

Let us compute, for every $\varphi \in \mathcal{C}_0^\infty(I)$,

$$\begin{aligned}\langle T_F', \varphi\rangle &= -\langle T_F, \varphi'\rangle = -\int_I \left(\int_a^x f(t)\,dt\right)\varphi'(x)\,dx \\ &= \int_{I_{<a}} \left(\int_{]x,a[} f(t)\,dt\right)\varphi'(x)\,dx - \int_{I_{>a}} \left(\int_{]a,x[} f(t)\,dt\right)\varphi'(x)\,dx\ .\end{aligned}$$

Let us apply the Fubini theorem to both integrals in the right hand side. Notice that, if the support of φ is included into $[\alpha, \beta]$ with $a \in [\alpha, \beta]$, the integrand of both integrals is supported by $[\alpha, \beta] \times [\alpha, \beta]$ and is bounded by

$$|f(t)||\varphi'(x)|$$

which is integrable on $[\alpha, \beta] \times [\alpha, \beta]$. Hence the Fubini theorem allows us to interchange the orders of integration. This yields

$$\begin{aligned}\langle T_F', \varphi\rangle &= \int_{I_{<a}} f(t)\left(\int_{I_{<t}} \varphi'(x)\,dx\right)\,dt - \int_{I_{>a}} f(t)\left(\int_{I_{>t}} \varphi'(x)\,dx\right)\,dt \\ &= \int_{I_{<a}} f(t)\varphi(t)\,dt + \int_{I_{>a}} f(t)\varphi(t)\,dt \\ &= \int_I f(t)\varphi(t)\,dt = \langle T_f, \varphi\rangle\ .\end{aligned}$$

□

Remark 1.4.6 (Going further than Lebesgue !) Lebesgue proved that F is almost everywhere derivable, with a derivative equal to f. However, the almost everywhere derivation does not allow to recover the initial function. There exist continuous functions with almost everywhere derivative equal to 0 on $[0, 1]$, which are not constants on $[0, 1]$. This uncomfortable phenomenon does not happen with distributional derivatives, as we shall see below in Proposition 1.4.8.

Example 1.4.7 Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be the function given by $f(x) = \ln(|x|)$, and $T = T_f \in \mathcal{D}'(\mathbb{R})$ be the associated distribution. We want to compute T'; it seems reasonable that T' is related to the function $x \mapsto 1/x$, but this one is not in $L^1_{\mathrm{loc}}(\mathbb{R})$. However, let $\varphi \in \mathcal{C}_0^\infty([-A, A])$. We compute

$$\begin{aligned}\langle T', \varphi\rangle &= -\int_{\mathbb{R}} \ln|x|\ \varphi'(x)\,dx = -\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \varphi'(x)\ln(|x|)\,dx \\ &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx + \lim_{\varepsilon\to 0}(\varphi(\varepsilon) - \varphi(-\varepsilon))\ln(\varepsilon)\end{aligned}$$

and the second limit in the right hand side is 0, since φ is smooth. We conclude that

$$T' = \mathrm{pv}(1/x),$$

as introduced in the previous paragraph.

The only functions with an identically null derivative on an interval are constant functions. For distributions, the same result holds true.

Proposition 1.4.8 If $T \in \mathcal{D}'(I)$ satisfies $T' = 0$, then T is associated with a constant function.

Proof.— We start the proof by a remark that may be useful in other contexts. A function $\varphi \in \mathcal{C}_0^\infty(I)$ is the derivative of a function $\psi \in \mathcal{C}_0^\infty(I)$ if and only if $\int_I \varphi = 0$. Indeed, if $\varphi = \psi'$ for ψ supported in $[a, b] \subset I$, then

$$\int_I \varphi(x)dx = \int_a^b \psi'(x)dx = [\psi(x)]_a^b = 0 \ .$$

Conversely, if $\int_I \varphi = 0$, the function $\psi : x \mapsto \int_{I\cap]-\infty, x[} \varphi(t)dt$ is compactly supported, with support included in any compact interval containing that of φ, and satisfies $\psi' = \varphi$.

Now let $\chi \in \mathcal{C}_0^\infty(I)$ a function such that $\int_I \chi = 1$. For $\varphi \in \mathcal{C}_0^\infty(I)$, we have

$$\varphi = \varphi_1 + \varphi_2 \ \text{ with } \ \varphi_1 = \varphi - \left(\int_I \varphi\right)\chi, \text{ and } \varphi_2 = \left(\int_I \varphi\right)\chi,$$

and $\langle T, \varphi\rangle = \langle T, \varphi_1\rangle + \langle T, \varphi_2\rangle$. Since $\int_I \varphi_1 = 0$, there exists $\psi \in \mathcal{C}_0^\infty(I)$ such that $\varphi_1 = \psi'$. Thus

$$\langle T, \varphi\rangle = \langle T, \psi'\rangle + \left(\int_I \varphi\right)\langle T, \chi\rangle = -\langle T', \psi\rangle + \left(\int_I \varphi\right)\langle T, \chi\rangle = C\int_I \varphi = \langle T_C, \varphi\rangle,$$

where $C = \langle T, \chi\rangle$ is a constant, independent of φ. □

Note that Proposition 1.4.2 says in particular that the derivative of a regular distribution associated to a constant function is null. Therefore, the above statement is an equivalence. Another important consequence is

Corollary 1.4.9 Let $f \in L^1_{\text{loc}}(I)$. The distributions $T \in \mathcal{D}'(I)$ such that

$$T' = T_f$$

are given by $T = T_F$, where

$$F(x) = \int_a^x f(t)\, dt + c \ , \ a \in I \ , \ c \in \mathbb{C} \ .$$

Proof.— Let $a \in I$ and

$$F_1(x) = \int_a^x f(t)\, dt \ .$$

By Proposition 1.4.5, $T'_{F_1} = T_f = T'$. Then just apply Proposition 1.4.9 to $T - T_{F_1}$. □

Using arguments from the proof of Proposition 1.4.8, we can establish the surjectivity of the mapping $T \mapsto T'$ on $\mathcal{D}'(I)$.

Proposition 1.4.10 If $S \in \mathcal{D}'(I)$, there exists $T \in \mathcal{D}'(I)$ such that $T' = S$.

Proof.— The identity $T' = S$ is equivalent to

$$\forall \varphi \in \mathcal{D}(I) \ , \ \langle T, \varphi' \rangle = -\langle S, \varphi \rangle \ .$$

In view of the beginning of the proof of Proposition 1.4.8, this imposes the value of T on test functions ψ on I such that

$$\int_I \psi \, dx = 0 \ .$$

Fix $\chi \in \mathcal{D}(I)$ such that

$$\int_I \chi \, dx = 1 \ .$$

Given $\psi \in \mathcal{D}(I)$, we define

$$P(\psi)(x) = \int_{t<x} \left[\psi(t) - \left(\int_I \psi(s) \, ds \right) \chi(t) \right] \, dt \ .$$

This defines a linear mapping

$$P : \mathcal{D}(I) \to \mathcal{D}(I) \ ,$$

such that

$$\text{supp}[P(\psi)] \subset \text{supp}(\psi) \cup \text{supp}(\chi) \ .$$

Consequently, if ψ_n converges to ψ in $\mathcal{D}(I)$, then $P(\psi_n)$ converges to $P(\psi)$ in $\mathcal{D}(I)$. Therefore we can define $T \in \mathcal{D}'(I)$ by

$$\forall \psi \in \mathcal{D}(I) \ , \ \langle T, \psi \rangle = -\langle S, P(\psi) \rangle \ .$$

Since $P(\varphi') = \varphi$ for every $\varphi \in \mathcal{D}(I)$, this completes the proof. □

Since T' is a distribution, it can be derivated. Iterating this idea, one can define the successive derivatives $T^{(k)}$ of T for all $k \in \mathbb{N}$. This means that distributions can be derivated at all order, with the formula

$$\langle T^{(k)}, \varphi \rangle = (-1)^k \langle T, \varphi^{(k)} \rangle.$$

For instance,

$$\langle \delta_{x_0}^{(k)}, \varphi \rangle = (-1)^k \varphi^{(k)}(x_0) \ .$$

1.5 Product by a smooth function

Following the same approach as for defining the derivative of a distribution, we introduce the following definition.

Proposition 1.5.1 Let $T \in \mathcal{D}'(I)$, and $f \in \mathcal{C}^\infty(I)$. The linear form

$$\varphi \mapsto \langle T, f\varphi \rangle$$

is a distribution, and we denote it fT.

Proof.— Suppose (φ_j) is a sequence of functions in $\mathcal{C}_0^\infty(I)$, that converges to 0 in the $\mathcal{D}(I)$-sense. There is a segment $[a,b] \subset\subset I$ such that supp $\varphi_j \subset [a,b]$ for all j, which implies supp $f\varphi_j \subset [a,b]$ for all j. Moreover, for any $\alpha \in \mathbb{N}$,

$$(f\varphi)^{(\alpha)} = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} f^{(\beta)} \, \varphi^{(\alpha-\beta)}.$$

so that if we denote

$$M = \max_{\beta \leq \alpha} \sup_{[a,b]} |f^{(\beta)}|,$$

we see that

$$\sup |(f\varphi_j)^{(\alpha)}| \leq M \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sup |\varphi_j^{(\alpha-\beta)}|.$$

Since each of the terms in the sum tends to 0 as $j \to +\infty$, we have $(f\varphi_j)^{(\alpha)} \to 0$ in $\mathcal{D}(I)$. Therefore, since T is a distribution, $\langle T, f\varphi_j \rangle \to 0$. Proposition 2.3.4 shows that fT is a distribution on I. □

Exercise 1.5.2 For $f \in \mathcal{C}^\infty(\mathbb{R}^d)$, show that $f\delta_0 = f(0)\delta_0$.

Exercise 1.5.3 For $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $g \in L^1_{\text{loc}}(\mathbb{R}^d)$, show that $fT_g = T_{fg}$.

Exercise 1.5.4 Show that $x\,\text{pv}(1/x) = 1$.

Exercise 1.5.5 For $f, g \in \mathcal{C}^\infty(\mathbb{R}^d)$, show that $f(gT) = (fg)T$.

As for the product of two smooth functions, there is a Leibniz formula for the product of a distribution by a function. We leave the proof of the following result to the reader.

Proposition 1.5.6 For $f \in \mathcal{C}^\infty(I)$, and $T \in \mathcal{D}'(I)$, we have

$$\forall \alpha \in \mathbb{N},\ (fT)^{(\alpha)} = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} f^{(\beta)} \, T^{(\alpha-\beta)}.$$

Our next statement in this subsection is an important converse of the elementary identity

$$(x - x_0)\delta_{x_0} = 0 \ .$$

Proposition 1.5.7 Let $x_0 \in I$ and let $T \in \mathcal{D}'(I)$ such that

$$(x - x_0)T = 0 \ .$$

Then there exists a constant $c \in \mathbb{C}$ such that $T = c\delta_{x_0}$.

Proof.— Let $\chi \in \mathcal{C}_0^\infty(I)$ such that $\chi(x_0) = 1$. For every $\varphi \in \mathcal{C}_0^\infty(I)$, we write, by Hadamard's lemma,

$$\begin{aligned} partitionsofunity\varphi(x) &= \varphi(x_0) + (x - x_0)\psi(x) \ , \ \psi \in \mathcal{C}^\infty(I) \ , \\ &= \varphi(x_0)\chi(x) + (x - x_0)\left(\frac{1 - \chi(x)}{x - x_0}\varphi(x_0) + \psi(x)\right) \\ &= \varphi(x_0)\chi(x) + (x - x_0)\theta(x) \ , \end{aligned}$$

where $\theta \in \mathcal{C}^\infty(I)$ from the explicit expression, and θ is compactly supported since $\varphi - \varphi(x_0)\chi$ is. We infer

$$\langle T, \varphi\rangle = \langle T, \chi\rangle\varphi(x_0) + \langle (x - x_0)T, \theta\rangle = \langle T, \chi\rangle\varphi(x_0) \ ,$$

whence $T = c\delta_{x_0}$ with $c := \langle T, \chi\rangle$. □

Example 1.5.8 Here is an example of a linear first order differential equation with non regular solutions. Let us look for solutions $T \in \mathcal{D}'(\mathbb{R})$ of

$$xT' + T = 0 \ .$$

By the Leibniz formula, this equation is equivalent to

$$(xT)' = 0 \ .$$

Using Proposition 1.4.8, this is equivalent to

$$xT = T_{c_1}$$

for some $c_1 \in \mathbb{C}$. Since $c_1 \,\mathrm{pv}(1/x)$ is a solution of this equation, we get equivalently

$$x\left(T - c_1 \,\mathrm{pv}\left(\frac{1}{x}\right)\right) = 0 \ .$$

Finally, using Proposition 1.5.7, we obtain the general solution of this differential equation,

$$T = c_1 \,\mathrm{pv}\left(\frac{1}{x}\right) + c_2\delta_0 \ , \ (c_1, c_2) \in \mathbb{C}^2 \ .$$

Notice that existence of non smooth solutions — in this case, all of them except 0 — is due to the fact that the coefficient of the highest derivative cancels at some point of the interval.

Finally, we use elements of the proof of Proposition 1.5.15 to prove the surjectivity of the mapping $T \mapsto (x - x_0)T$ on $\mathcal{D}'(I)$.

Proposition 1.5.9 Given $S \in \mathcal{D}'(I)$ and $x_0 \in I$, there exists $T \in \mathcal{D}'(I)$ such that

$$(x - x_0)T = S \ .$$

Remark 1.5.10 Of course, if $x_0 \notin I$, the statement trivially holds.

Proof.— We proceed similarly to the proof of Proposition 1.4.10. Indeed, $(x - x_0)T = S$ reads

$$\forall \varphi \in \mathcal{D}(I) \ , \ \langle T, (x - x_0)\varphi \rangle = \langle S, \varphi \rangle \ ,$$

which, in view of Hadamard's lemma, imposes the value of T on test functions which vanish at x_0. As in the proof of Proposition 1.5.15, let $\chi \in \mathcal{C}_0^\infty(I)$ such that $\chi(x_0) = 1$. For every $\psi \in \mathcal{D}(I)$, we define

$$Q(\psi)(x) = \frac{\psi(x) - \psi(x_0)\chi(x)}{x - x_0} = \int_0^1 [\psi'(tx + (1-t)x_0) - \psi(x_0)\chi'(tx + (1-t)x_0)] \, dt \ .$$

This defines a linear mapping $Q : \mathcal{D}(I) \to \mathcal{D}(I)$, with the properties

$$\text{supp}\, Q(\psi) \subset \text{supp}(\psi) \cup \text{supp}(\chi) \ ,$$

and, if ψ_n converges to ψ in $\mathcal{D}(I)$, then $Q(\psi_n)$ converges to $Q(\psi)$ in $\mathcal{D}(I)$. We can then define $T \in \mathcal{D}'(I)$ by

$$\forall \psi \in \mathcal{D}(I) \ , \ \langle T, \psi \rangle = \langle S, Q(\psi) \rangle \ .$$

Since, for every $\varphi \in \mathcal{D}(I)$,

$$Q((x - x_0)\varphi) = \varphi \ ,$$

the proof is complete. □

Exercise 1.5.11 Prove that, for every $S \in \mathcal{D}'(I)$, there exists $T \in \mathcal{D}'(I)$ such that

$$xT' + T = S \ .$$

Corollary 1.5.12 If $x_0 \in I$ and m is a positive integer, the mapping $T \in \mathcal{D}'(I) \mapsto (x - x_0)^m T \in \mathcal{D}'(I)$ is surjective, and its kernel is the m-dimensional vector space generated by $\delta_{x_0}^{(j)}$, for $j = 0, 1, \dots, m-1$.

Proof.— We proceed by induction on m. The case $m = 1$ has been solved in Propositions 1.5.9 and 1.5.7. Assume $m \geq 2$ and that the statement is true for $m - 1$. The surjectivity of the mapping $T \in \mathcal{D}'(I) \mapsto (x - x_0)^m T \in \mathcal{D}'(I)$ follows immediately from the similar property for $m - 1$ and from Proposition 1.5.9. As for the kernel, the identity

$$(x - x_0)^m T = 0$$

is equivalent to

$$(x - x_0)T = \sum_{j=0}^{m-2} c_j \delta_{x_0}^{(j)} \ .$$

Let us find an explicit solution to the above equation. For this, we need the following lemma.

Lemma 1.5.13

$$(x - x_0)\delta_{x_0}^{(k)} = -k\delta_{x_0}^{(k-1)} \ .$$

Indeed, for every test function φ, we have

$$\langle (x-x_0)\delta_{x_0}^{(k)}, \varphi\rangle = (-1)^k((x-x_0)\varphi)^{(k)}(x_0) = (-1)^k k\varphi^{(k-1)}(x_0) = -k\langle \delta_{x_0}^{(k-1)}, \varphi\rangle \ .$$

Coming back to the proof of Proposition 1.5.12, we have

$$(x-x_0)T = -(x-x_0)\sum_{j=0}^{m-2} \frac{c_j}{j+1}\delta_{x_0}^{(j+1)}$$

and applying Proposition 1.5.7 to the distribution

$$T + \sum_{j=0}^{m-2} \frac{c_j}{j+1}\delta_{x_0}^{(j+1)}$$

completes the proof. □

Using the above corollary, one easily obtains the following result.

Corollary 1.5.14 Let P be a non identically zero polynomial function. Then the mapping

$$T \in \mathcal{D}'(I) \mapsto PT \in \mathcal{D}'(I)$$

is surjective, and its kernel is the vector space generated by

$$\delta_a^{(j)}$$

where a describes the zeroes of P inside I, and $j = 0, 1, \dots, m(a) - 1$, $m(a)$ denoting the multiplicity of a as a zero of P.

We leave the easy proof to the reader.

Finally, we solve a (very simple) differential equation in $\mathcal{D}'(\mathbb{R})$.

Proposition 1.5.15 Let $I \subset \mathbb{R}$ be an open interval, and $a \in \mathcal{C}^\infty(I)$. The distributions in $\mathcal{D}'(I)$ that satisfy the differential equation

$$T' + aT = 0$$

are exactly the $\mathcal{C}^1$ solutions, that is the regular distributions T_f with $f : x \mapsto Ce^{-A(x)}$, for some constant $C \in \mathbb{C}$, where A is primitive of a in I.

In other words, the distribution solutions of this differential equation coincide with the classical solutions. Notice that this is not always the case if the coefficients of the highest derivative vanishes — see example 1.5.8 below.

Proof.— Let A be a primitive of a on I. For $T \in \mathcal{D}'(I)$, we have, using Leibniz formula,

$$(e^A T)' = ae^A T + e^A T' = e^A(T' + aT).$$

Thus

$$T' + aT = 0 \iff (e^A T)' = 0 \iff e^A T = T_C \iff T = e^{-A} T_C = T_{Ce^{-A}}.$$

□

Exercise 1.5.16 Solve in $\mathcal{D}'(I)$ the inhomogeneous equation $T' + aT = f$, for $f \in L^1_{\text{loc}}(I)$. Furthermore, prove that, if $S \in \mathcal{D}(I)$, there exists $T \in \mathcal{D}'(I)$ such that

$$T' + aT = S\ .$$

1.6 Restriction and support

1.6.1 Definitions

Definition 1.6.1 Let $T \in \mathcal{D}'(I)$, and $J \subset I$ an open subinterval of I. The restriction of T to J is the distribution $T_{|J} \in \mathcal{D}'(J)$ defined as

$$\forall \varphi \in \mathcal{D}(J)\ ,\ \langle T_{|J}, \varphi \rangle = \langle T, \underline{\varphi} \rangle\ ,$$

where $\underline{\varphi}$ denotes the extension of φ by 0 on $I \setminus J$.

We say that T vanishes in J if $T_{|J} = 0$.

Let us state without proof the following elementary properties of restriction with respect to derivation and multiplication by a smooth function.

$$(T_{|J})' = T'_{|J}\ , (fT)_{|J} = f_{|J} T_{|J}\ .$$

As a first application of this important notion, let us state a very useful result.

Proposition 1.6.2 Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of intervals in I such that $I_n =]a_n, b_n[$, $[a_n, b_n] \subset I_{n+1}$ for every n, and

$$\bigcup_{n\in\mathbb{N}} I_n = I\ .$$

Suppose we are given for every $n \in \mathbb{N}$, a distribution T_n on I_n such that

$$\forall n \in \mathbb{N}\ ,\ T_{n+1|I_n} = T_n\ .$$

Then there exists a unique distribution T on I such that

$$\forall n \in \mathbb{N}\ ,\ T_{|I_n} = T_n\ .$$

Proof.— The key of the proof is the following elementary fact. For every segment $[a, b]$ included in I, there exists n such that $[a, b] \subset I_n$. Indeed, from the assumptions, the sequence (a_n) is decreasing, tending to the left extremity of I, while the sequence (b_n) is increasing, tending to the right extremity of I. This fact is also a consequence of the Borel–Lebesgue property for the compact set $[a, b]$.

If such a distribution T exists, for every $\varphi \in \mathcal{D}(I)$, there exists n such that supp$(\varphi) \subset I_n$, and therefore we should have

$$\langle T, \varphi \rangle = \langle T_n, \varphi \rangle \ .$$

This proves the uniqueness of T, and suggests a way of constructing it. Indeed, given $\varphi \in \mathcal{D}(I)$, the assumption $T_{n+1|I_n} = T_n$ implies that, for every n such that supp$(\varphi) \subset I_n$, the quantity $\langle T_n, \varphi \rangle$ does not depend on n. We can therefore define

$$\langle T, \varphi \rangle = \langle T_n, \varphi \rangle$$

for every such n. This clearly defines a linear form on $\mathcal{D}(I)$. Furthermore, if φ_j converges to φ in $\mathcal{D}(I)$, then there exist n such that,

$$\forall j \ , \ \text{supp}(\varphi_j) \subset I_n \ .$$

Therefore

$$\langle T, \varphi_j \rangle = \langle T_n, \varphi_j \rangle$$

and, because T_n is a distribution on I_n, $\langle T, \varphi_j \rangle$ tends to $\langle T, \varphi \rangle$. □

Let us state a first consequence of Proposition 1.6.2.

Corollary 1.6.3 Let $f \in \mathcal{C}^\infty(I)$ admitting only finite order zeroes in I, namely, if $f(a) = 0$, there exists $m \geq 1$ such that $f^{(m)}(a) \neq 0$. Then the mapping $T \in \mathcal{D}'(I) \mapsto fT \in \mathcal{D}'(I)$ is surjective.

Proof.— If $f(a) = 0$ and m is the smallest integer such that $f^{(m)}(a) \neq 0$, the Taylor formula yields

$$f(x) = \frac{(x-a)^m}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(a + t(x-a))\, dt \ .$$

From the continuity of $f^{(m)}$, we infer that there exists an open interval $J_a \subset I$ containing a such that $f(x) \neq 0$ for every $x \in J_a \setminus \{a\}$. If $[\alpha, \beta]$ is a segment of I, we conclude that the compact set of zeroes of f in $[\alpha, \beta]$ is covered by the family of open intervals J_a's, hence – from the Borel–Lebesque property — by a finite subfamily of J_a's, hence this set is finite. Consequently, in $]\alpha, \beta[$, we can write

$$f = Pg \ ,$$

where P is a polynomial and g is a smooth function which does not have any zero in $]\alpha, \beta[$. Consequently, the mapping

$$T \in \mathcal{D}'(]\alpha, \beta[) \mapsto fT \in \mathcal{D}'(]\alpha, \beta[)$$

is surjective, and its kernel consists of finite linear combinations of Dirac masses in zeros of f in $]\alpha, \beta[$ and of some of their derivatives. In particular, any distribution in this kernel is the restriction of some distribution on I, the product of which with f is 0 on I.

Let us now construct a sequence $(I_n)_{n\in\mathbb{N}}$ as in Proposition 1.6.2. Let $S \in \mathcal{D}'(I)$. In view of the above observation, there exists $T_0 \in \mathcal{D}'(I_0)$ such that $f_{|I_0}T_0 = S_{|I_0}$, and there exists $\tilde{T}_1 \in \mathcal{D}'(I_1)$ such that $f_{|I_1}\tilde{T}_1 = S_{|I_1}$. Denote by $\tilde{T}_{1,0}$ the restriction of $\tilde{T}_1$ to I_0. Of course,

$$f_{|I_0}(\tilde{T}_{1,0} - T_0) = 0 \ ,$$

and from the above observation, there exists $U_1 \in \mathcal{D}'(I_1)$ such that $U_{1|I_0} = \tilde{T}_{1,0} - T_0$ and $f_{|I_1}U_1 = 0$. If we set

$$T_1 = \tilde{T}_1 - U_1 \in \mathcal{D}'(I_1) \ ,$$

we observe that

$$T_{1|I_0} = T_0 \ , \ f_{|I_1}T_1 = S_{|I_1} \ .$$

By induction, we can therefore construct, for every $n \in \mathbb{N}$, a distribution T_n on I_n such that

$$T_{n+1|I_n} = T_n \ , \ f_{|I_n}T_n = S_{|I_n} \ .$$

Applying Proposition 1.6.2, there exists $T \in \mathcal{D}'(I)$ such that, for every n, $T_{|I_n} = T_n$. Consequently,

$$\forall n \in \mathbb{N} \ , (fT)_{|I_n} = f_{|I_n}T_n = S_{|I_n} \ ,$$

hence, by the uniqueness part of Proposition 1.6.2, we conclude that

$$fT = S \ .$$

□

We now introduce a crucial notion.

Definition 1.6.4 The support of a distribution $T \in \mathcal{D}'(I)$ is the complement of the union of all the open subintervals on which the restriction of T is 0. We denote it by $\operatorname{supp} T$.

Notice that $\operatorname{supp} T$ is closed, and the following characterizations are convenient.

- $x_0 \notin \operatorname{supp} T$ if and only if there is an open neighborhood J of x_0 such that $T_{|J} = 0$.
- $x_0 \in \operatorname{supp} T$ if and only if for any open neighborhood J of x_0, one can find $\varphi \in \mathcal{C}_0^\infty(J)$ such that $\langle T, \varphi \rangle \neq 0$.

Example 1.6.5 *i)* Let $T = \delta_0$. If J is an open interval that does not contain $\{0\}$, then $\langle T, \varphi \rangle = \varphi(0) = 0$ for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \varphi \subset J$. Thus $\operatorname{supp} T \subset \{0\}$. On the other hand, if J is an open subinterval that contains 0, we can find a plateau function ψ over $[-r, r]$ in $\mathcal{C}_0^\infty(J)$, and for this function we have $\langle T, \psi \rangle = \psi(0) = 1$. Therefore we have $0 \in \operatorname{supp} T$ and finally $\operatorname{supp} T = \{0\}$.

ii) If $T = T_f$ for some $f \in \mathcal{C}^0(I)$, with I an open interval of $\mathbb{R}$, we have

$$\text{supp}\, T = \text{supp}\, f = \overline{\{x \in I, f(x) \neq 0\}}\ .$$

Indeed, suppose that $x_0 \notin \text{supp}\, f$. There is an open neighborhood J of x_0 such that $f_{|J} = 0$. For $\varphi \in \mathcal{C}_0^\infty(J)$, we have thus $\langle T_f, \varphi\rangle = 0$, so that T_f vanishes on J, and $x_0 \notin \text{supp}\, T_f$. Conversely, if $x_0 \notin \text{supp}\, T_f$, there is a neighborhood J of x_0 such that, for all $\varphi \in \mathcal{C}_0^\infty(J)$, we have $\int f\varphi dx = \langle T_f, \varphi\rangle = 0$. We have seen in Proposition 1.3.4 that this implies $f = 0$ in J, thus $x_0 \notin \text{supp}\, f$.

iii) If $T = T_f$ for some $f \in L^1_{\text{loc}}(I)$, the same argument leads to

$$\text{supp}(f) = \{x_0 \in I, \forall r > 0, \text{meas}\{x \in [x_0 - r, x_0 + r], f(x) \neq 0\} > 0\}\ .$$

1.6.2 Some properties

Lemma 1.6.6 Let $I \subset \mathbb{R}$ be an open set, and $T \in \mathcal{D}'(I)$.

i) For $k \in \mathbb{N}$, $\text{supp}\, T^{(k)} \subset \text{supp}\, T$.

ii) For $f \in \mathcal{C}^\infty(I)$, $\text{supp}\, fT \subset \text{supp}\, f \cap \text{supp}\, T$.

Proof.— Let $x_0 \notin \text{supp}\, T$. There exists a neighborhood V of x_0 such that for all $\psi \in \mathcal{C}_0^\infty(V)$, $\langle T, \psi\rangle = 0$. But if $\varphi \in \mathcal{C}_0^\infty(V)$, $\psi = \varphi^{(k)} \in \mathcal{C}_0^\infty(V)$, thus

$$\langle T^{(k)}, \varphi\rangle = (-1)^k \langle T, \varphi^{(k)}\rangle = 0.$$

Therefore $x_0 \notin \text{supp}\, T^{(k)}$, which proves *i)*. Now we prove the second point. If $x_0 \notin \text{supp}\, f \cap \text{supp}\, T$, x_0 either belongs to $(\text{supp}\, f)^c$ or to $(\text{supp}\, T)^c$. In the first case, there exists a neighborhood V of x_0 such that $f_{|V} = 0$. For $\varphi \in \mathcal{C}_0^\infty(V)$, we have $\langle fT, \varphi\rangle = \langle T, f\varphi\rangle = 0$, thus $x_0 \notin \text{supp}(fT)$. In the latter case, there exists a neighborhood V of x_0 such that, for all $\psi \in \mathcal{C}_0^\infty(V)$, $\langle T, \psi\rangle = 0$. For $\varphi \in \mathcal{C}_0^\infty(V)$, $f\varphi$ is a smooth function, which vanishes out of a compact set included in V. Thus $\langle fT, \varphi\rangle = \langle T, f\varphi\rangle = 0$, and $x_0 \notin \text{supp}\, fT$. □

The following result is fundamental.

Proposition 1.6.7 Let $\varphi \in \mathcal{C}_0^\infty(I)$ and $T \in \mathcal{D}'(I)$. If $\text{supp}\,\varphi \cap \text{supp}\, T = \emptyset$, then $\langle T, \varphi\rangle = 0$.

Proof.— Let $x \in \text{supp}\,\varphi$. We have, by assumption, $x \notin \text{supp}\, T$, thus there is an open subinterval J_x containing x on which T vanishes. From the covering of the compact subset $\text{supp}\,\varphi$ with the open sets J_x, one can extract a finite covering

$$\text{supp}\,\varphi \subset \bigcup_{j=1}^{n} J_{x_j}.$$

By Proposition 1.2.5, one can find $\varphi_j \in \mathcal{C}_0^\infty(J_{x_j})$ for $j = 1, \dots, n$ such that

$$\varphi = \varphi_1 + \cdots + \varphi_n \ .$$

Therefore

$$\langle T, \varphi \rangle = \sum_{j=1}^{n} \langle T, \varphi_j \rangle = 0 \ .$$

□

Caution: one may have $\varphi = 0$ on supp T and $\langle T, \varphi \rangle \neq 0$. For example, this is the case for $T = \delta_0'$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\varphi(0) = 0$, $\varphi'(0) = 1$.

An immediate consequence of Proposition 1.6.7 is the following important fact.

Corollary 1.6.8 The only distribution with empty support is 0.

We conclude this section by the important characterisation of distributions $T \in \mathcal{D}'(I)$ supported by a point.

Proposition 1.6.9 Let $T \in \mathcal{D}'(I)$, with $T \neq 0$, and $x_0 \in I$. If supp $T \subset \{x_0\}$, there exists $m \in \mathbb{N}$ and $m + 1$ complex numbers a_k for $0 \leq k \leq m$ such that

$$T = \sum_{k=0}^{m} a_k \delta_{x_0}^{(k)}.$$

Proof.— We write the proof for $x_0 = 0$. Let $[a, b]$ be a segment of I containing 0 in its interior, and $\chi \in \mathcal{C}_0^\infty(]a, b[)$ a plateau function above a smaller segment containing 0 in its interior. Since T is a distribution, there exist $C > 0$, $m \in \mathbb{N}$ such that

$$\forall \psi \in \mathcal{C}_0^\infty(I),\ \text{supp}\, \psi \subset [a, b] \Rightarrow |\langle T, \psi \rangle| \leq \ C \sum_{|\alpha| \leq m} \sup |\psi^{(\alpha)}|,$$

and from now on we denote by m the smallest integer for which this property holds. For any function $\varphi \in \mathcal{C}_0^\infty(I)$, since supp$(\varphi - \chi\varphi) \cap$ supp $T = \emptyset$, we have

$$\langle T, \varphi \rangle = \langle T, \chi\varphi \rangle + \langle T, \varphi - \chi\varphi \rangle = \langle T, \chi\varphi \rangle.$$

Since supp$(\chi\varphi) \subset [a, b]$, we have

$$|\langle T, \varphi \rangle| \leq |\langle T, \chi\varphi \rangle| \leq C \sum_{|\alpha| \leq m} \sup |(\chi\varphi)^{(\alpha)}|.$$

At last, the Leibniz formula gives

$$\sup |(\chi\varphi)^{(\alpha)}| = \sup_{[a,b]} |(\chi\varphi)^{(\alpha)}| \leq C \sum_{\beta \leq \alpha} \sup_{[a,b]} |\varphi^{(\beta)}|,$$

We first claim that, if $\varphi \in C_0^\infty(I)$ satisfies $\varphi^{(k)}(0) = 0$ for every $k \leq m$, then $\langle T, \varphi \rangle = 0$. Indeed, let χ be a plateau function on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, with supp$(\chi) \subset [-1, 1]$. For $\varepsilon > 0$ small enough, set

$$\chi_\varepsilon(x) = \chi\left(\frac{x}{\varepsilon}\right) .$$

Since $\varphi - \chi_\varepsilon\varphi$ vanishes in a neighborhood of 0, we have

$$\langle T, \varphi \rangle = \langle T, \chi_\varepsilon \varphi \rangle .$$

By the Taylor formula and the assumption on φ

$$\varphi(x) = x^{m+1}\psi(x)$$

for some $\psi \in \mathcal{C}_0^\infty(I)$. Hence

$$\chi_\varepsilon\varphi(x) = \chi\left(\frac{x}{\varepsilon}\right) x^{m+1}\psi(x) = \varepsilon^{m+1}\rho\left(\frac{x}{\varepsilon}\right)\psi(x) ,$$

where we introduced the notation $\rho(y) = y^{m+1}\chi(y)$. By the Leibniz formula, we have

$$\sup_{k \leq m} \sup_x \left| \left(\rho\left(\frac{x}{\varepsilon}\right)\psi(x)\right)^{(k)} \right| \leq \frac{B}{\varepsilon^m}$$

and therefore

$$|\langle T, \varphi \rangle| = |\langle T, \chi_\varepsilon\varphi \rangle| \leq C \sup_{k \leq m} \sup |\chi_\varepsilon \varphi^{(k)}| \leq CB\varepsilon .$$

As ε tends to 0, we obtain

$$\langle T, \varphi \rangle = 0 .$$

Let us now study the general case of $\varphi \in \mathcal{C}_0^\infty(I)$. Applying the Taylor formula, we have

$$\varphi(x) = \sum_{k \leq m} \frac{x^k}{k!}\varphi^{(k)}(0)\chi_{\varepsilon_0}(x) + r(x),$$

where $\varepsilon_0 > 0$ is small enough so that $[-\varepsilon_0, \varepsilon_0] \subset I$, and $r \in \mathcal{C}_0^\infty(I)$ satisfies

$$r^{(\alpha)}(0) = 0 , \ \alpha \leq m .$$

We can therefore apply the above statement to r, and get

$$\langle T, \varphi \rangle = \sum_{k \leq m} \varphi^{(k)}(0) \langle T, \frac{x^k}{k!}\chi_{\varepsilon_0}(x) \rangle.$$

This is precisely what we have claimed, if we set $a_k = \langle T, \frac{x^k}{k!}\chi_{\varepsilon_0} \rangle$. □

1.7 Sequences of distributions

1.7.1 Convergence in $\mathcal{D}'$

Definition 1.7.1 Let (T_j) be a sequence of distributions in $\mathcal{D}'(I)$. We say that (T_j) converges to $T \in \mathcal{D}'(I)$ when, for any function $\varphi \in \mathcal{C}_0^\infty(I)$, the sequence of complex numbers $(\langle T_j, \varphi\rangle)$ converges to $\langle T, \varphi\rangle$. In this case, we write $T_j \to T$ in $\mathcal{D}'(I)$.

Example 1.7.2 If (f_j) is a sequence of locally integrable functions on I such that, for some $f \in L^1_{\text{loc}}(I)$,

$$\forall [a,b] \subset I \ , \ \int_a^b |f_j(x) - f(x)|\, dx \to 0 \ ,$$

then

$$T_{f_j} \to T_f$$

in $\mathcal{D}'(I)$. Indeed, for every test function φ on I with $\operatorname{supp}\varphi \subset [a,b]$,

$$\langle T_{f_j}, \varphi\rangle - \langle T_f, \varphi\rangle = \int_I f_j(x)\varphi(x)\, dx - \int_I f(x)\varphi(x)\, dx$$

so that

$$\big|\langle T_{f_j}, \varphi\rangle - \langle T_f, \varphi\rangle\big| \le \sup|\varphi(x)| \int_a^b |f_j(x) - f(x)|\, dx$$

which tends to 0 as j tends to the infinity.

Notice that a special case of this is the following situation. The sequence f_j converges to f almost everywhere on I, and there exists a locally integrable function h on I such that $\sup_j |f_j| \le h$ almost everywhere on I. Indeed, the connection to the above condition of L^1 convergence is provided by the dominated convergence theorem.

Example 1.7.3 Let $\rho \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} \rho(x)\, dx = 1$, and (ρ_ε) the sequence of functions defined by

$$\rho_\varepsilon(x) = \varepsilon^{-1}\rho(\frac{x}{\varepsilon}).$$

We also denote $(T_\varepsilon) = (T_{\rho_\varepsilon})$ the associated family of distributions. For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we have

$$\langle T_\varepsilon, \varphi\rangle = \int_{\mathbb{R}} \rho_\varepsilon(x)\varphi(x) dx = \int_{\mathbb{R}} \rho(y)\varphi(\varepsilon y)\, dy.$$

By the dominated convergence theorem, $\langle T_\varepsilon, \varphi\rangle \to \varphi(0)$ as $\varepsilon \to 0$. Therefore the sequence (T_{ρ_ε}) converges to δ_0 in $\mathcal{D}'(\mathbb{R})$.

Exercise 1.7.4 Let (f_j) be a sequence of L^1 functions on I such that

$$\sup_j \int_I |f_j|\, dx < \infty \ , \ \int_I f_j dx \to c \ , \ \operatorname{supp}(T_{f_j}) \subset [x_0 - \varepsilon_j, x_0 + \varepsilon_j] \ , \ \varepsilon_j \to 0 \ .$$

Prove that

$$T_{f_j} \to c\delta_{x_0} \ .$$

Exercise 1.7.5 Let

$$f_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon^2} & \text{if } x \in]0, \varepsilon[\\ \frac{-1}{\varepsilon^2} & \text{if } x \in]-\varepsilon, 0[\\ 0 & \text{if } |x| > \varepsilon \end{cases}$$

Prove that

$$T_{f_\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} -\delta_0' \ .$$

Exercise 1.7.6 Let

$$f_\varepsilon(x) = \frac{x}{x^2 + \varepsilon} \ , \ \varepsilon > 0 \ .$$

Prove that

$$T_{f_\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} \text{pv}\left(\frac{1}{x}\right)$$

in $\mathcal{D}'(\mathbb{R})$.

The operations that we have defined on distributions are continuous with respect to the notion of convergence. More precisely,

Proposition 1.7.7 If (T_j) converges to T in $\mathcal{D}'(I)$, then

i) For any $k \in \mathbb{N}$, $(T_j^{(k)})$ converges to $T^{(k)}$.

ii) For any $f \in \mathcal{C}^\infty(I)$, (fT_j) converges to fT.

iii) For any open subinterval $J \subset I$, $(T_j)|J$ converges to $T_{|J}$.

Proof.— Let $\varphi \in \mathcal{C}_0^\infty(I)$. We have clearly

$$\langle \partial^\alpha T_j, \varphi \rangle = (-1)^{|\alpha|} \langle T_j, \partial^\alpha \varphi \rangle \to (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle = \langle \partial^\alpha T, \varphi \rangle,$$

and

$$\langle fT_j, \varphi \rangle = \langle T_j, f\varphi \rangle \to \langle T, f\varphi \rangle = \langle fT, \varphi \rangle.$$

The last statement is trivial. □

Exercise 1.7.8 Show that if (f_j) converges to f in $\mathcal{C}^\infty(I)$, in the sense of the uniform convergence of all derivatives on every compact subset, then (f_jT) converges to fT in $\mathcal{D}'(I)$.

1.7.2 The Uniform Boundedness Principle

Let us motivate the main statement of this paragraph by the following questions.

- Assume that, for a sequence (T_j) of distributions, and for every $\varphi \in \mathcal{D}(I)$, the sequence $(\langle T_j, \varphi \rangle)$ has a limit $L(\varphi)$. This clearly defines a linear form L on $\mathcal{D}(I)$. Is it true that L is a distribution ?

- Assume $T_j \to T$ in $\mathcal{D}'(I)$ and $f_j \to f$ in $\mathcal{C}^\infty(I)$ in the sense of the uniform convergence of all derivatives on every compact subset, does it imply that $f_jT_j \to fT$? In order to to prove such a result, we write

$$\langle f_jT_j, \varphi\rangle - \langle fT, \varphi\rangle = \langle T_j, (f_j - f)\varphi\rangle + \langle T_j, f\varphi\rangle - \langle T, f\varphi\rangle \ .$$

  The second term in the right hand side tends to 0 in view of the assumption. In the first term, we notice that $\psi_j = (f_j - f)\varphi$ converges to 0 in $\mathcal{D}(I)$, but it is seems difficult to conclude that $\langle T_j, \psi_j\rangle \to 0$ without a bound on the action of T_j which is uniform with respect to j.

We state below, without proof, an important theoretical result, which is a ”distributional version” of the Banach-Steinhaus theorem, and which provides such a uniform bound.

Proposition 1.7.9 Let (T_j) be a sequence in $\mathcal{D}'(I)$, and $[a, b] \subset I$ a segment. If, for any function $\varphi \in \mathcal{C}_0^\infty(I)$ with support in $[a, b]$,

$$\sup_j |\langle T_j, \varphi\rangle| < +\infty,$$

then there exists $C > 0$ and $m \in \mathbb{N}$, independent of j, such that

$$\forall \varphi \in \mathcal{C}_0^\infty(I),\ \operatorname{supp} \varphi \subset [a, b] \Rightarrow |\langle T_j, \varphi\rangle| \le C \sum_{|\alpha| \le m} \sup |\varphi^{(\alpha)}|.$$

As a consequence, we get the following somewhat surprising result.

Corollary 1.7.10 Let (T_j) be a sequence of distributions on I. If, for all functions $\varphi \in \mathcal{C}_0^\infty(I)$, the sequence $(\langle T_j, \varphi\rangle)$ converges in $\mathbb{C}$, then there exists a distribution $T \in \mathcal{D}'(I)$ such that $T_j \to T$ in $\mathcal{D}'(I)$.

Proof.— Let $T : \mathcal{C}_0^\infty(I) \to \mathbb{C}$ be the linear form given by

$$T(\varphi) = \lim_{j \to +\infty} \langle T_j, \varphi\rangle.$$

We want to show that T is a distribution. So pick a segment $[a, b] \subset I$. Proposition 1.7.9 ensures that there is a constant $C > 0$ and a natural number m such that, for any $\varphi \in \mathcal{C}_0^\infty(I)$ with $\operatorname{supp} \varphi \subset [a, b]$, we have

$$|\langle T_j, \varphi\rangle| \le C \sum_{|\alpha| \le m} \sup |\varphi^{(\alpha)}|.$$

Then we can pass to the limit $j \to +\infty$, and we get the required estimate. □

The second motivation can also be addressed successfully. The proof is left to the reader.

Corollary 1.7.11 Let (T_j) be a sequence of distributions on I which converges to T in $\mathcal{D}'(I)$. Then, for every sequence (ψ_j) which converges to ψ in $\mathcal{D}(I)$, we have $\langle T_j, \psi_j\rangle \to \langle T, \psi\rangle$. In particular, if $f_j \to f$ in $\mathcal{C}^\infty(I)$ in the sense of the uniform convergence of all derivatives on every compact subset, then $f_j T_j \to fT$.

A proof of the uniform boundedness principle as well as other applications will be given in the next chapter.

1.8 An introduction to Sobolev spaces

In this section, we give a short overview of some functional spaces, introduced by the Russian mathematician Sergei Sobolev (1908-1989), which allowed to solve differential equations in a wide context. Here we concentrate on a very special case in one variable. A more general presentation will be provided in the next chapters.

1.8.1 Generalised derivatives.

Definition 1.8.1 Let I be an open interval of $\mathbb{R}$, and let $u \in L^1_{\text{loc}}(I)$. We shall say that u admits a generalised derivative if there exists $f \in L^1\text{loc}(I)$ such that, in the sense of distributions in I,

$$T'_u = T_f \ .$$

The element $f \in L^1_{\text{loc}}(I)$ is then unique, is called the generalised derivative of u, and is denoted by $f = u'$.

Notice that uniqueness of f immediately follows from Proposition 1.3.4, and that the notation $f = u'$ is precisely made for leading to following generalised integration by parts formula, which is nothing but a reformulation of $T'_u = T_f$,

$$\forall \varphi \in \mathcal{C}^\infty_0(I) \ , \ \int_I u(x)\varphi'(x)\,dx = -\int_I u'(x)\varphi(x)\,dx \ .$$

From Corollary 1.4.9, we infer the following useful statement.

Proposition 1.8.2 Assume $u \in L^1_{\text{loc}}(I)$ admits a generalised derivative $u' \in L^1_{\text{loc}}(I)$. Let $x_0 \in I$. Then there exists $c \in \mathbb{C}$ such that

$$u(x) = \int_{x_0}^{x} u'(t)\,dt + c \quad a.e.$$

In particular, u is almost everywhere equal to a continuous function.

Notice that, as a consequence of the Proposition 1.8.2, if the generalised derivative u' of u turns out to be a continuous function, then u is a $\mathcal{C}^1$ function, and u' is merely its derivative in the usual sense.

1.8.2 The Sobolev space $H^1(I)$.

Definition 1.8.3 Let I be an open interval of $\mathbb{R}$. We denote by $H^1(I)$ the subspace of $u \in L^2(I)$ which admit a generalised derivative $u' \in L^2(I)$.

We endow the space $H^1(I)$ with the following inner product,

$$(u, v)_{H^1} = \int_I u'(x)\overline{v'}(x)\, dx + \int_I u(x)\overline{v}(x)\, dx \ , \ u \in H^1(I) \ , \ v \in H^1(I) \ . \tag{1.8.1}$$

The following proposition shows how the notion of convergence in $\mathcal{D}'(I)$ is useful in this context.

Proposition 1.8.4 Endowed with the inner product (1.8.1), the space $H^1(I)$ is a Hilbert space.

Proof.— Let (u_j) be a Cauchy sequence in $H^1(I)$ for the norm associated to (1.8.1). This precisely means that (u_j) and (u_j') are Cauchy sequences of $L^2(I)$. Since $L^2(I)$ is a Hilbert space, there exist u, v in $L^2(I)$ such that

$$u_j \to u \ , \ u_j' \to v$$

in $L^2(I)$. Since the convergence in $L^2(I)$ implies the convergence on $L^1(]a, b[)$ for every segment $[a, b] \subset I$, we infer that, in $\mathcal{D}'(I)$,

$$T_{u_j} \to T_u \ , \ T_{u_j'} \to T_v \ .$$

But, by definition of a generalised derivative, and using the continuity of derivation for the convergence of distributions,

$$T_{u_j'} = T_{u_j}' \to T_u' \ .$$

This implies $T_u' = T_v$, in other words u admits v as a generalised derivative, $u' = v \in L^2(I)$, which means that $u \in H^1(I)$ and

$$u_j \to u \ , \ u_j' \to u' \ .$$

This precisely means that u_j tends to u in $H^1(I)$. Hence $H^1(I)$ is a complete normed vector space.
□

1.8.3 The special case of a bounded interval

In this section, $I =]a, b[, a, b \in \mathbb{R}$, is an open bounded interval. In this case, we get more information from Proposition 1.8.2 .

Proposition 1.8.5 If $u \in H^1(]a,b[)$, then u is almost everywhere equal to a continuous function on $[a,b]$, and we have the Sobolev inequality,

$$\|u\|_{L^\infty} \le C\|u\|_{H^1} \ ,$$

where C is a constant depending only on $b-a$.

Proof.— We start from the identity in Proposition 1.8.2,

$$u(x) = \int_{x_0}^x u'(t)\, dt + c \ \ a.e.$$

Since $L^2(]a,b[) \subset L^1(]a,b[)$, we infer that the right hand side has a limit as x tends to a and as x tends to b. Hence u is almost everywhere equal to a continuous function on $[a,b]$. Furthermore, by the Cauchy-Schwarz inequality,

$$\left|\int_y^x u'(t)\, dt\right| \le \sqrt{b-a}\|u'\|_{L^2} \ .$$

Consider

$$m_u = \frac{1}{b-a}\int_a^b u(y)\, dy \ .$$

By the Cauchy–Schwarz inequality,

$$|m_u| \le \frac{1}{\sqrt{b-a}}\|u\|_{L^2} \ .$$

On the other hand,

$$u(x) - m_u = \frac{1}{b-a}\int_a^b (u(x)-u(y))\, dy = \frac{1}{b-a}\int_a^b \left(\int_y^x u'(t)\, dt\right)\, dy \ .$$

Therefore

$$|u(x) - m_u| \le \frac{1}{b-a}\int_a^b \left|\int_y^x u'(t)\, dt\right|\, dy \le \sqrt{b-a}\|u'\|_{L^2} \ .$$

Summing up, we obtain

$$\|u\|_{L^\infty} \le \|u - m_u\|_{L^\infty} + |m_u| \le (b-a)^{-1/2}\|u\|_{L^2} + (b-a)^{1/2}\|u'\|_{L^2} \le C\|u\|_{H^1} \ .$$

□

Our next step is the introduction of an important closed subspace of $H^1(]a,b[)$.

Definition 1.8.6 We denote by $H^1_0(]a,b[)$ the closure of $\mathcal{C}^\infty_0(]a,b[)$ in $H^1(I)$.

Recall from Proposition 1.8.5 that $u \mapsto u(a)$ and $u \mapsto u(b)$ are continuous linear forms on $H^1(]a,b[)$. A remarkable fact is that $H^1_0(]a,b[)$ can be characterised by these linear forms.

Proposition 1.8.7 Given $u \in H^1(]a,b[)$, u belongs to $H_0^1(]a,b[)$ if and only if $u(a) = u(b) = 0$.

Proof.—Since elements of $H^1(]a,b[)$ are continuous functions on $[a,b]$ and the H^1 norm controls the L^∞ norm on $[a,b]$ from Proposition 1.8.5, any element u of the H^1 closure of test functions can be uniformly approximated on $[a,b]$ by test functions. This immediately implies $u(a) = u(b) = 0$.

Conversely, consider $u \in H^1(]a,b[)$ such that $u(a) = u(b) = 0$. For $\varepsilon > 0$ small enough, we consider a function $\chi_\varepsilon \in C_0^\infty(I)$, supported into $[a+\varepsilon, b-\varepsilon]$, and such that $\chi_\varepsilon = 1$ on $[a+2\varepsilon, b-2\varepsilon]$, with the estimate

$$\|\chi_\varepsilon'\|_{L^\infty} = O\left(\frac{1}{\varepsilon}\right) \ ,$$

as $\varepsilon \to 0$. An example of such a family of functions is

$$\chi_\varepsilon(x) = \chi\left(\frac{x-a}{\varepsilon}\right)\chi\left(\frac{b-x}{\varepsilon}\right),$$

where $\chi \in \mathbb{C}^\infty(\mathbb{R})$ satisfies

$$\chi(t) = \begin{cases} 0 \text{ if } & t \leq 1 \\ 1 \text{ if } & t \geq 2. \end{cases}$$

Then we claim that $\chi_\varepsilon u \to u$ in $H^1(]a,b[)$ as $\varepsilon \to 0$. Indeed, $\chi_\varepsilon u \to u$ in $L^2(]a,b[)$ by dominated convergence, and

$$(\chi_\varepsilon u)' = \chi_\varepsilon u' + \chi_\varepsilon' u \ .$$

We are therefore reduced to proving that $\chi_\varepsilon' u \to 0$ in $L^2(]a,b[)$. We have

$$\|\chi_\varepsilon' u\|_{L^2} \leq O\left(\frac{1}{\varepsilon}\right)\left(\int_a^{a+2\varepsilon} |u(x)|^2\,dx + \int_{b-2\varepsilon}^b |u(x)|^2\,dx\right)^{\frac{1}{2}} \ ,$$

and, for $a \leq x \leq a + 2\varepsilon$,

$$\begin{aligned} |u(x)| &= \left|u(a) + \int_a^x u'(t)\,dt\right| = \left|\int_a^x u'(t)\,dt\right| \\ &\leq \sqrt{2\varepsilon}\left(\int_a^{a+2\varepsilon} |u'(x)|^2\,dx\right)^{\frac{1}{2}} \ , \end{aligned}$$

with a similar inequality on $[b-2\varepsilon, b]$. Finally,

$$\|\chi_\varepsilon' u\|_{L^2} \leq O(1)\left(\int_a^{a+2\varepsilon} |u'(x)|^2\,dx + \int_{b-2\varepsilon}^b |u'(x)|^2\,dx\right)^{\frac{1}{2}} \to 0 \ .$$

Summing up we have proved that u can be approximated in H^1 by compactly supported elements. The proof is completed by observing that any compactly supported element of $H^1(]a,b[)$ can be approximated by a sequence of $\mathcal{C}_0^\infty(]a,b[)$. This relies on the following regularisation argument. Let $v \in H^1(]a,b[)$ such that $\text{supp}(v) \subset [\alpha,\beta] \subset]a,b[$. We extend v by 0 as an element of $H^1(\mathbb{R})$ supported in $[\alpha,\beta]$. Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$ of integral 1, and

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right) \ .$$

Then $\rho_\varepsilon * v \in \mathcal{C}_0^\infty(\mathbb{R})$, is compactly supported in $]a,b[$ if ε is small enough, and $\rho_\varepsilon * v \to v$ in L^2 as ε tends to 0. Furthermore, by the Leibniz rule,

$$(\rho_\varepsilon * v)'(x) = \int_{\mathbb{R}} \rho_\varepsilon'(x-y)\, v(y)\, dy = \int_{\mathbb{R}} -\frac{d}{dy}[\rho_\varepsilon(x-y)]\, v(y)\, dy = \int_{\mathbb{R}} \rho_\varepsilon(x-y)\, v'(y)\, dy\ ,$$

by definition of the generalised derivative. In other words,

$$(\rho_\varepsilon * v)' = \rho_\varepsilon * v'$$

and therefore $(\rho_\varepsilon * v)' \to v'$ in L^2 as ε tends to 0. Summing up, we have proved that $\rho_\varepsilon * v \to v$ in H^1. □

Proposition 1.8.7 implies an important inequality for elements of $H_0^1(]a,b[)$.

Proposition 1.8.8 (The Poincaré inequality) For every $u \in H^1(]a,b[)$ such that $u(a) = 0$ or $u(b) = 0$, we have

$$\|u\|_{L^2} \le (b-a)\|u'\|_{L^2}\ . \qquad (1.8.2)$$

In particular,

$$(u,v)_{H_0^1} = \int_a^b u'(x)\, \overline{v'}(x)\, dx$$

is an inner product on $H_0^1(]a,b[)$, which is equivalent to the H^1 inner product.

Proof.— Again we refer to Proposition 1.8.2,

$$u(x) = \int_{x_0}^x u'(t)\, dt + c\ .$$

Assume for instance $u(a) = 0$. Then making x tend to a, we infer

$$c = \int_a^{x_0} u'(t)\, dt$$

and consequently

$$u(x) = \int_a^x u'(t)\, dt\ .$$

Inequality (1.8.2) then follows from the Cauchy–Schwarz inequality. In view of Proposition 1.8.7, inequality (1.8.2) applies in particular to elements of $H_0^1(]a,b[)$, so that

$$\|u\|_{L^2}^2 \le (b-a)^2 (u,u)_{H_0^1}\ .$$

Since $(u,u)_{H^1} = (u,u)_{H_0^1} + \|u\|_{L^2}^2$, we conclude that the inner products $(u,v)_{H^1}$ and $(u,v)_{H_0^1}$ are equivalent on H_0^1. □

The Hilbert structure on H_0^1 leads to a very efficient strategy for solving second order linear differential equations with homogeneous boundary conditions.

Theorem 1.8.9 Let $q \in L^1(]a,b[)$ such that $q \geq 0$, and let $f \in L^1(]a,b[)$. There exists a unique $u \in H^1(]a,b[)$ such that u' admits a generalised derivative u'' satisfying

$$-u'' + qu = f \ , \ u(a) = u(b) = 0 \ .$$

Proof.— Notice that qu is well defined in L^1 since $q \in L^1$ and $u \in L^\infty$. Decomposing f into its real and imaginary parts, we may assume f is real valued, so that u is to be real valued as well. So we shall work in the real Hilbert space made of real valued elements of H^1. This simple reduction allows to avoid the complex conjugation in the inner product, so that the connection with the distribution bracket is clearer.

In view of Proposition 1.8.7, the above problem is equivalent to

$$-T'_{u'} + T_{qu} = T_f \ , \ u \in H^1_0(]a,b[) \ ,$$

or, for every $\varphi \in \mathcal{C}^\infty_0(]a,b[)$,

$$\int_a^b u'\varphi'\,dx + \int_a^b qu\varphi\,dx = \int_a^b f\varphi\,dx \ , \ u \in H^1_0(]a,b[) \ .$$

Since the left hand side and the right hand side of the above identity are continuous linear forms of φ for the H^1 norm, and since H^1_0 is the closure of $\mathcal{C}^\infty_0$ in H^1, we infer that the problem is equivalent to finding $u \in H^1_0(]a,b[)$ such that

$$\forall v \in H^1_0(]a,b[) \ , \ \int_a^b u'v'\,dx + \int_a^b quv\,dx = \int_a^b fv\,dx \ .$$

Now observe that the left hand side is an inner product on the real space H^1_0, which is equivalent to the H^1 inner product. Indeed,

$$\begin{aligned}(u,u)_{H^1_0} \leq \int_a^b (u')^2\,dx + \int_a^b qu^2\,dx \quad &\leq \quad \int_a^b (u')^2\,dx + \left(\int_a^b q\,dx\right) \|u\|^2_{L^\infty} \\ &\leq \quad \max\left(1, C^2\left(\int_a^b q\,dx\right)\right)(u,u)_{H^1}\end{aligned}$$

in view of Proposition 1.8.5. Hence, from Proposition 1.8.2, the real space $H^1_0(]a,b[)$ endowed with

$$(u,v)_q = \int_a^b u'v'\,dx + \int_a^b quv\,dx$$

is a (real) Hilbert space. Since, in view of Proposition 1.8.5, the linear form

$$v \mapsto \int_a^b fv\,dx$$

is continuous on this Hilbert space, the theorem follows from the Riesz representation theorem. □

Remark 1.8.10 Since u'' belongs to $L^1(]a,b[)$, we infer that u' extends as a continuous function on $[a,b]$. Furthermore, if q, f are continuous functions on $[a,b]$, then u'' is continuous on $[a,b]$, which means that u is $\mathcal{C}^2$ on $[a,b]$, so that the differential equation is satisfied in the usual sense.

1.9 Further properties

1.9.1 Characterisation of Lipschitz functions

Definition 1.9.1 Let k be a positive number and I be an open interval. A function $u : I \to \mathbb{C}$ is k–Lipschitz if, for every $x, y \in I$,

$$|u(x) - u(y)| \leq k|x - y| \ .$$

A typical example of a k–Lipschitz function is a C^1 function u such $\|u'\|_\infty \leq k$. Of course, there are Lipschitz functions which are not derivable, like function $x \mapsto |x|$. The following result give a complete description of k–Lipschitz functions.

Theorem 1.9.2 A function $u : \mathbb{R} \to \mathbb{C}$ is k–Lipschitz if and only if there exists $f \in L^\infty(\mathbb{R})$ such that

$$\forall x \in \mathbb{R} \ , \ u(x) = u(0) + \int_0^x f(t)\, dt \ . \tag{1.9.3}$$

Remark 1.9.3 A similar result holds on any open interval I, from an adaptation of the proof below on $\mathbb{R}$.

Proof.— If $\forall x \in \mathbb{R} \ , \ u(x) = u(0) + \int_0^x f(t)\, dt$, we have

$$|u(x) - u(y)| = \left| \int_y^x f(t)\, dt \right| \leq \|f\|_\infty |x - y| \ .$$

Conversely, let u be a k–Lipschitz function. We are going to prove that u admits a generalised derivative f with $\|f\|_\infty \leq k$. According to Proposition 1.8.2, this will imply property (1.9.3).
As a first step, we are going to prove the following inequality,

$$\forall \varphi \in \mathcal{C}_0^\infty(I) \ , \ |\langle T_u', \varphi \rangle| \leq k \|\varphi\|_{L^1} \ . \tag{1.9.4}$$

Indeed, we have,

$$\begin{aligned} \langle T_u', \varphi \rangle &= -\int_{\mathbb{R}} u(x)\varphi'(x)\, dx = -\int_{\mathbb{R}} u(x) \lim_{h \to 0} \frac{\varphi(x+h) - \varphi(x)}{h}\, dx \\ &= -\lim_{h \to 0} \int_{\mathbb{R}} u(x) \frac{\varphi(x+h) - \varphi(x)}{h}\, dx \ , \end{aligned}$$

by using either dominated convergence or uniform convergence. Next we decompose the integral as follows,

$$\begin{aligned}\int_{\mathbb{R}} u(x)\frac{\varphi(x+h)-\varphi(x)}{h}\,dx &= \frac{1}{h}\left(\int_{\mathbb{R}} u(x)\varphi(x+h)\,dx - \int_{\mathbb{R}} u(x)\varphi(x)\,dx\right)\\ &= \frac{1}{h}\left(\int_{\mathbb{R}} u(y-h)\varphi(y)\,dy - \int_{\mathbb{R}} u(x)\varphi(x)\,dx\right)\\ &= \int_{\mathbb{R}} \frac{u(y-h)-u(y)}{h}\varphi(y)\,dy\ ,\end{aligned}$$

and, by the Lipschitz property of u, the modulus of the latter expression is bounded by $k\|\varphi\|_{L^1}$. This leads to inequality (1.9.4). At this stage we appeal to the following functional analytic lemma.

Lemma 1.9.4 Let $\mathcal{E}$ be a normed vector space, and let $\mathcal{D}$ be a dense vector subspace of $\mathcal{E}$. Let $L : \mathcal{D} \to \mathbb{C}$ be a linear form, which is continuous for the norm on $\mathcal{E}$. Then L admits a unique continuation $\tilde{L}$ as a continuous linear form on $\mathcal{E}$, and $\|\tilde{L}\| = \|L\|$.

Applying Lemma 1.9.4 to $\mathcal{E} = L^1(\mathbb{R})$, $\mathcal{D} = \mathcal{D}(\mathbb{R})$ and $L = T'_u$, we infer that T'_u extends as a continuous linear form on $L^1(\mathbb{R})$, of norm at most k. From the $L^1 - L^\infty$ duality theorem, there exists $f \in L^\infty(\mathbb{R})$ such that

$$\forall \varphi \in \mathcal{C}_0^\infty(I)\ ,\ \langle T'_u, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)\,dx\ ,$$

and $\|f\|_\infty \le k$. This precisely means that f is the generalised derivative of u, whence (1.9.3). □

1.9.2 Derivation and integration under the bracket

This paragraph provides two very useful rules of calculations which turn out be crucial when dealing with convolution of distributions — see the next paragraph.

Proposition 1.9.5 (Derivation under the bracket) Let I, J be open intervals, $\psi : I \times J \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function such that there exists a segment $[a, b] \subset I$ for which

$$\forall z \in J\ ,\ \mathrm{supp}\,\psi(.,z) \subset [a,b]\ ,$$

where we have set $\psi(., z) : x \in I \to \psi(x, z) \in \mathbb{C}$. Define

$$g(z) = \langle T, \psi(.,z)\rangle\ ,\ z \in J\ .$$

Then $g \in \mathcal{C}^\infty(J)$ and

$$\forall z \in J\ ,\ g'(z) = \left\langle T, \frac{\partial\psi}{\partial z}(.,z)\right\rangle\ .$$

Proof.— Let us first prove that g is derivable on J. We calculate, for $z \in J$ and $h \neq 0$ small enough,

$$\frac{g(z+h)-g(z)}{h} = \left\langle T, \frac{\psi(.,z+h)-\psi(.,z)}{h}\right\rangle\ .$$

Now we observe that, since $\psi \in \mathcal{C}^\infty(I \times J)$,

$$\forall x \in I \ , \ \frac{\psi(x, z+h) - \psi(x, z)}{h} \underset{h \to 0}{\longrightarrow} \frac{\partial \psi}{\partial z}(x, z) \ .$$

We claim that the convergence takes place in $\mathcal{C}_0^\infty(I)$. Indeed, we already know that the left hand side is a smooth function of x which is supported in $[a, b]$. We claim that it converges uniformly on I to the right hand side. Indeed,

$$\frac{\psi(x, z+h) - \psi(x, z)}{h} = \int_0^1 \frac{\partial \psi}{\partial z}(x, z+th)\, dt \ .$$

Fix $\varepsilon_0 > 0$ such that $|h| \leq \varepsilon_0$. Then the continuous function $\frac{\partial \psi}{\partial z}$ is uniformly continuous on the compact subset $[a, b] \times [z - \varepsilon_0, z + \varepsilon_0]$. Consequently,

$$\sup_{x \in [a,b]} \left| \int_0^1 \frac{\partial \psi}{\partial z}(x, z+th)\, dt - \frac{\partial \psi}{\partial z}(x, z) \right| \underset{h \to 0}{\longrightarrow} 0 \ .$$

The same argument holds for every derivative in x,

$$\frac{\psi^{(k)}(x, z+h) - \psi^{(k)}(x, z)}{h} \underset{h \to 0}{\longrightarrow} \frac{\partial \psi^{(k)}}{\partial z}(x, z)$$

uniformly for $x \in I$.
Using the continuity property of T, we infer

$$\frac{g(z+h) - g(z)}{h} \underset{h \to 0}{\longrightarrow} \left\langle T, \frac{\partial \psi}{\partial z}(., z) \right\rangle \ ,$$

which shows that g is derivable with the claimed formula for $g'(z)$. Finally, by applying this result, one easily proves by induction on n that g is n times derivable with

$$g^{(n)}(z) = \left\langle T, \frac{\partial^n \psi}{\partial z^n}(., z) \right\rangle \ .$$

This completes the proof. □

Proposition 1.9.6 (Integration under the bracket) Let I, J be open intervals, $\psi : I \times J \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function such that there exists segments $[a, b] \subset I$ and $[c, d] \subset J$ for which

$$\forall (x, z) \notin [a, b] \times [c, d] \ , \ \psi(x, z) = 0 \ .$$

Then

$$\int_J \langle T, \psi(., z) \rangle \ dz = \left\langle T, \int_J \psi(., z)\, dz \right\rangle .$$

Proof.— Consider

$$\tilde{\psi}(x, z) = \int_{t \in J, t < z} \psi(x, t)\, dt \ .$$

Then $\tilde{\psi}$ satisfies the assumptions of Proposition 1.9.5, and

$$\frac{\partial\tilde{\psi}}{\partial z}(x,z)=\psi(x,z)\ .$$

Applying this proposition leads to

$$\frac{d}{dz}\langle T,\tilde{\psi}(.,z)\rangle=\langle T,\psi(.,z)\rangle\ .$$

We integrate both sides on J. This gives

$$\langle T,\tilde{\psi}(.,d)\rangle-\langle T,\tilde{\psi}(.,c)\rangle=\int_J\langle T,\psi(.,z)\rangle\,dz\ .$$

In view of the assumptions on ψ, we have

$$\langle T,\tilde{\psi}(.,d)\rangle=\left\langle T,\int_J\psi(.,z)\,dz\right\rangle\ ,\ \langle T,\tilde{\psi}(.,c)\rangle=0\ .$$

This completes the proof. □

1.9.3 Convolution and regularisation

In this section, we generalise the convolution with a test function to a distribution.

Definition 1.9.7 Let $T\in\mathcal{D}'(\mathbb{R})$ and $\varphi\in\mathcal{D}(\mathbb{R})$. For every $x\in\mathbb{R}$, we denote by $\varphi(x-.)$ the element of $\mathcal{D}(\mathbb{R})$ which maps $y\in\mathbb{R}$ to $\varphi(x-y)\in\mathbb{C}$. We set

$$T*\varphi(x)=\langle T,\varphi(x-.)\rangle\ .$$

Notice that, if $T=T_f$ for $f\in L^1_{\text{loc}}(\mathbb{R})$, we have

$$T_f*\varphi(x)=\int_{\mathbb{R}}f(y)\varphi(x-y)\,dy=f*\varphi(x)\ ,$$

in the usual sense of convolution of functions.

Proposition 1.9.8 For every $(T,\varphi)\in\mathcal{D}'(\mathbb{R})\times\mathcal{D}(\mathbb{R})$, $T*\varphi\in\mathcal{C}^\infty(\mathbb{R})$, and

$$(T*\varphi)'=T*\varphi'=T'*\varphi\ .$$

Proof.— It is enough to prove that $T*\varphi$ is smooth on every finite open interval $]\alpha,\beta[$. If supp $\varphi\subset[a,b]$ and $x\in]\alpha,\beta[$, we have supp $\varphi(x-.)\subset[\alpha-b,\beta-a]$, so that the assumptions of Proposition 1.9.5 are fulfilled with $I=\mathbb{R},J=]\alpha,\beta[$ and

$$\psi(y,x)=\varphi(x-y)\ .$$

Consequently, $T * \varphi \in \mathcal{C}^\infty(\mathbb{R})$, and

$$(T * \varphi)'(x) = \langle T, \partial_x \varphi(x - .)\rangle = \langle T, \varphi'(x - .)\rangle = T * \varphi'(x) .$$

Furthermore, we have

$$\varphi'(x - y) = -\partial_y \varphi(x - y) ,$$

so that

$$(T * \varphi)'(x) = \langle T, -\partial_y \varphi(x - .)\rangle = \langle T', \varphi(x - .)\rangle = T' * \varphi(x) .$$

□

Let us come to regularisation. Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$ with

$$\int_{\mathbb{R}} \rho(x)\, dx = 1 .$$

For every $\varepsilon > 0$, we set

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right) .$$

It is well known that, if $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\rho_\varepsilon * \varphi \underset{\varepsilon \to 0}{\longrightarrow} \varphi$$

in $\mathcal{D}(\mathbb{R})$. We set $f_\varepsilon = T * \rho_\varepsilon$.

Proposition 1.9.9 For every $T \in \mathcal{D}'(\mathbb{R})$,

$$T_{f_\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} T .$$

Proof.— We have to prove that

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) ,\ \int_{\mathbb{R}} f_\varepsilon(x)\varphi(x)\, dx \underset{\varepsilon \to 0}{\longrightarrow} \langle T, \varphi\rangle . \tag{1.9.5}$$

Note that

$$\int_{\mathbb{R}} f_\varepsilon(x)\varphi(x)\, dx = \int_{\mathbb{R}} \langle T, \rho_\varepsilon(x - .)\rangle \varphi(x)\, dx = \int_{\mathbb{R}} \langle T, \rho_\varepsilon(x - .)\varphi(x)\rangle\, dx ,$$

and that

$$\psi(y, x) := \rho_\varepsilon(x - y)\varphi(x)$$

satisfies the assumption of Proposition 1.9.6 on $\mathbb{R} \times \mathbb{R}$. Indeed, if $\operatorname{supp} \rho \subset [-C, C]$, $\operatorname{supp} \varphi \subset [a, b]$, then

$$\forall (y, x) \notin [a - \varepsilon C, b + \varepsilon C] \times [a, b] ,\ \psi(y, x) = 0 .$$

Applying Proposition 1.9.6, we obtain

$$\int_{\mathbb{R}} \langle T, \rho_\varepsilon(x - .)\varphi(x)\rangle\, dx = \left\langle T, \int_{\mathbb{R}} \rho_\varepsilon(x - .)\varphi(x)\, dx \right\rangle = \langle T, \tilde{\rho}_\varepsilon * \varphi\rangle$$

where $\tilde{\rho}(z) = \rho(-z)$. Since $\tilde{\rho}$ satisfies the same assumption as ρ, we have $\tilde{\rho}_\varepsilon * \varphi \underset{\varepsilon \to 0}{\longrightarrow} \varphi$ in $\mathcal{D}(\mathbb{R})$, and consequently

$$\int_{\mathbb{R}} f_\varepsilon(x)\varphi(x)\, dx \underset{\varepsilon \to 0}{\longrightarrow} \langle T, \varphi\rangle .$$

□

1.9.4 Positive distributions and increasing functions

Definition 1.9.10 A distribution T on I is said positive (we note $T \geq 0$) if, for every $\varphi \in \mathcal{D}(I)$ valued in $[0, +\infty[$, we have $\langle T, \varphi \rangle \in [0, +\infty[$.

For instance, it is easy to check that, if $f \in L^1_{\text{loc}}(I)$, $T_f \geq 0$ if and only $f \geq 0$ a.e. Another example is $T = c\delta_{x_0}$ with $c \geq 0$ and $x_0 \in I$. On the other hand, whatever $c \neq 0$ is, it is clear that $c\delta'_{x_0}$ cannot be positive. In fact, we have the following general result.

Proposition 1.9.11 If T is a positive distribution, then T has order 0, namely

$$\forall [a, b] \subset I \ , \ \exists C > 0 \ , \ \forall \varphi \in \mathcal{D}(I), \text{supp}\, \varphi \subset [a, b] \Rightarrow |\langle T, \varphi \rangle| \leq C \|\varphi\|_\infty \ .$$

Proof.— Let $[a, b] \subset I$ and $\chi \in \mathcal{D}(I)$ be a plateau function on $[a, b]$. Let $\varphi \in \mathcal{D}(I)$ be real valued and such that supp $\varphi \subset [a, b]$. Then

$$-\chi \|\varphi\|_\infty \leq \varphi \leq \chi \|\varphi\|_\infty \ .$$

Consequently, the positivity of T implies $\langle T, \varphi \rangle \in \mathbb{R}$ and

$$-\langle T, \chi \rangle \|\varphi\|_\infty \leq \langle T, \varphi \rangle \leq \langle T, \chi \rangle \|\varphi\|_\infty \ ,$$

and therefore

$$|\langle T, \varphi \rangle| \leq \langle T, \chi \rangle \|\varphi\|_\infty \ .$$

If φ is complex valued with supp $\varphi \subset [a, b]$, we decompose

$$\varphi = \text{Re}(\varphi) + i\text{Im}(\varphi)$$

and we conclude

$$|\langle T, \varphi \rangle| \leq 2\langle T, \chi \rangle \|\varphi\|_\infty \ .$$

□

Remark 1.9.12 Since any compactly supported (positive) continuous function on I can be approximated uniformly by a sequence of (positive) test functions supported in a fixed segment of I, the above proposition implies that any positive distribution extends as a positive linear form on compactly supported continuous functions. Hence, according to the Riesz representation theorem, there exists a positive Borel measure μ on I, finite on segments, such that

$$\forall \varphi \in \mathcal{D}(I) \ , \ \langle T, \varphi \rangle = \int_I \varphi(x)\, d\mu(x) \ .$$

Let us come to the main result of this paragraph, which characterises increasing functions. For simplicity, we state and prove this result on $\mathbb{R}$, but a similar result holds on any open interval. Recall that an increasing function on $\mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\forall x \geq y \ , \ f(x) \geq f(y) \ .$$

(sometimes such functions are called nondecreasing functions).

Theorem 1.9.13 If $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, then $T_f' \geq 0$. Conversely, if $T \in \mathcal{D}'(\mathbb{R})$ is such that $T' \geq 0$ and $\langle T, \varphi \rangle \in \mathbb{R}$ for every real valued test function φ, then there exists an increasing function f on $\mathbb{R}$ such that $T_f = T$.

Proof.— Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Note that f is bounded on every segment, hence it is locally integrable, and it makes sense to consider T_f. Let $\varphi \in \mathcal{D}(\mathbb{R})$, valued in $[0, +\infty[$. Let us calculate

$$\begin{aligned} \langle T_f', \varphi \rangle &= -\int_{\mathbb{R}} u(x)\varphi'(x)\, dx = -\int_{\mathbb{R}} f(x) \lim_{h\to 0} \frac{\varphi(x+h)-\varphi(x)}{h}\, dx \\ &= -\lim_{h\to 0} \int_{\mathbb{R}} f(x) \frac{\varphi(x+h)-\varphi(x)}{h}\, dx \\ &= \lim_{h\to 0} \int_{\mathbb{R}} \frac{f(y-h)-f(y)}{h} \varphi(y)\, dy \ , \end{aligned}$$

by the same arguments as in subsection 1.9.1. It yields $\langle T_f', \varphi \rangle \geq 0$, hence T_f' is positive.
Conversely, let $T \in \mathcal{D}'(\mathbb{R})$ such that $T' \geq 0$. Choose $\rho \in \mathcal{D}(\mathbb{R})$, supported in $\mathbb{R}_+$, satisfying

$$0 \leq \rho \ , \ \int_{\mathbb{R}} \rho(x)\, dx = 1.$$

Recall that, from Proposition 1.9.9, the smooth function

$$f_\varepsilon = T * \rho_\varepsilon$$

is such that $T_{f_\varepsilon} \to T$ in $\mathcal{D}'(\mathbb{R})$. Notice moreover that, since T takes real values on real valued test functions, f_ε is real valued. Our strategy is to prove that, for an appropriate choice of ρ, f_ε is an increasing function and converges for the simple convergence to a function f, with local dominated convergence. This will imply that $T_{f_\varepsilon} \to T_f$, hence $T = T_f$.
Let us first prove that f_ε is increasing. We know that

$$f_\varepsilon' = T' * \rho_\varepsilon$$

which is ≥ 0 because $T' \geq 0$ and $\rho_\varepsilon \geq 0$. Therefore f_ε is an increasing function on $\mathbb{R}$.
We claim that moreover f_ε is an decreasing function of ε. In fact, applying Proposition 1.9.5 of derivation under the bracket, we have that $f_\varepsilon(x)$ is a $\mathcal{C}^\infty$ function of $\varepsilon > 0$, and that

$$\frac{d}{d\varepsilon} f_\varepsilon(x) = \langle T, \frac{\partial}{\partial \varepsilon} \rho_\varepsilon(x - .) \rangle \ .$$

Observe that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\rho_\varepsilon(x-y) &= \frac{\partial}{\partial\varepsilon}\left[\frac{1}{\varepsilon}\rho\left(\frac{x-y}{\varepsilon}\right)\right]\\ &= -\frac{1}{\varepsilon^2}\left[\rho\left(\frac{x-y}{\varepsilon}\right)+\frac{x-y}{\varepsilon}\rho'\left(\frac{x-y}{\varepsilon}\right)\right]\\ &= \frac{1}{\varepsilon^2}\frac{\partial}{\partial y}\left[(x-y)\rho\left(\frac{x-y}{\varepsilon}\right)\right]\end{aligned}$$

Consequently,

$$\frac{d}{d\varepsilon}f_\varepsilon(x) = -\frac{1}{\varepsilon}\left\langle T', \frac{(x-.)}{\varepsilon}\rho\left(\frac{x-.}{\varepsilon}\right)\right\rangle \leq 0\ ,$$

because $T' \geq 0$ and $z\rho(z) \geq 0$ in view of the assumptions on ρ.
Now we claim that, as $\varepsilon \to 0$, for every $x \in I$, $f_\varepsilon(x)$ is bounded from above. Indeed, consider $\psi \in \mathcal{D}(I)$, valued in $[0,+\infty[$, supported in $[x,+\infty[$, and such that

$$\int_{\mathbb{R}} \psi(y)\, dy = 1\ .$$

Then, because $f_\varepsilon(y)$ is an increasing function of y,

$$f_\varepsilon(x) = f_\varepsilon(x)\int_{\mathbb{R}}\psi(y)\, dy \leq \int_{\mathbb{R}} f_\varepsilon(y)\psi(y)\, dy \underset{\varepsilon\to 0}{\longrightarrow} \langle T, \psi\rangle$$

because of Proposition 1.9.9. Since $f_\varepsilon(x)$ is increasing as ε decreases to 0, we conclude that

$$f_\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} f(x)$$

where f is an increasing function on $\mathbb{R}$. Furthermore, if $x \in [a,b] \subset \mathbb{R}$

$$f(x) \geq f_\varepsilon(x) \geq f_\varepsilon(a) \underset{\varepsilon\to 0}{\longrightarrow} f(a)\ .$$

Therefore we can apply the dominated convergence theorem and conclude that, for every $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\int_{\mathbb{R}} f_\varepsilon(x)\varphi(x)\, dx \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}} f(x)\varphi(x)\, dx\ .$$

Summing up, applying again Proposition 1.9.9, we have proved

$$\langle T, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)\, dx\ ,$$

hence $T = T_f$. □

Remark 1.9.14 The first point of Theorem 1.9.13 says that, if f is increasing, T'_f is a positive distribution, hence is given by a positive Borel measure, finite on segments. This measure is called the Stieltjes measure associated to f, and the formula for the integral of a compactly supported continuous function with respect to this measure is an extension of the definition of the usual integral of a continuous function by using Riemann sums. More precisely, if supp $\varphi \subset [a,b]$,

$$\langle T'_f, \varphi\rangle = \lim_{N\to\infty}\sum_{j=0}^{N-1}\varphi\left(a+j\frac{b-a}{N}\right)\left[f\left(a+(j+1)\frac{b-a}{N}\right)-f\left(a+j\frac{b-a}{N}\right)\right]\ .$$

For instance, if $f : \mathbb{R} \to \mathbb{R}$ is the repartition function of a random variable on $\mathbb{R}$, then T'_f is nothing but the law of this random variable.

1.9.5 The structure of distributions

Recall that, by Proposition 1.4.10, every distribution T has a primitive S. Furthermore, as can be observed from the proof of 1.4.10, if T is real, then one can choose S real. Now, if T is positive, then S satisfies the assumption of Theorem 1.9.13, and there exists an increasing function f such that $T_f = S$, or $T = T_f'$. In other words, any positive distribution is the derivative of T_f, where f is increasing. This statement extends to any distribution of order 0 as follows.

Theorem 1.9.15 Any distribution of order 0 on $\mathbb{R}$ is of the form T_f', where $f \in L^\infty_{\text{loc}}(\mathbb{R})$.

Proof.— Let us come back to the proof of Proposition 1.4.10. Let $\chi \in \mathcal{D}(\mathbb{R})$ supported in $]-1,1[$ and satisfying

$$\int_{\mathbb{R}} \chi(y)\,dy = 1\ .$$

For every $\psi \in \mathcal{D}(\mathbb{R})$, we set

$$P(\psi)(x) = \int_{-\infty}^{x} \left[\psi(t) - \chi(t)\int_{\mathbb{R}} \psi(y)\,dy\right]\,dt\ .$$

Notice that $P(\psi) \in \mathcal{D}(\mathbb{R})$ and, if $\text{supp}(\psi) \subset]-n,n[$ for some $n \geq 1$, then $\text{supp}(P(\psi)) \subset]-n,n[$. Since $P(\varphi') = \varphi$, we infer that the distribution S defined by

$$\langle S, \psi\rangle = -\langle T, P(\psi)\rangle$$

satisfies $S' = T$. Since T is of order 0, for every $n \geq 1$, there exists $C_n > 0$ such that

$$\forall \varphi \in \mathcal{D}(]-n,n[)\ ,\ |\langle T, \varphi\rangle| \leq C_n \|\varphi\|_\infty\ .$$

Consequently, if $\psi \in \mathcal{D}(]-n,n[)$,

$$|\langle S, \psi\rangle| = |\langle T, P(\psi)\rangle| \leq C_n \|P(\psi)\|_\infty \leq C_n B \|\psi\|_{L^1}\ .$$

Arguing as in proof of Theorem 1.9.2, we infer that there exists $f_n \in L^\infty(]-n,n[)$ such that

$$S_{|]-n,n[} = T_{f_n}\ .$$

Therefore we have constructed a sequence $(f_n)_{n\geq 1}$ of functions $f_n \in L^\infty(]-n,n[)$ such that

$$T_{f_{n+1}}{}_{|]-n,n[} = (S_{|]-n-1,n+1[})_{|]-n,n[} = S_{|]-n,n[} = T_{f_n}\ .$$

Hence $f_{n+1}{}_{|]-n,n[} = f_n$ and we infer that there exists $f \in L^\infty_{\text{loc}}(\mathbb{R})$ such that

$$\forall n \geq 1\ ,\ f_{|]-n,n[} = f_n\ .$$

Since

$$S_{|]-n,n[} = T_{f_{|]-n,n[}}$$

for every $n \geq 1$, we conclude $S = T_f$, and $T = S' = T_f'$. □

Let us now consider the case of distributions of finite order. Let m be a nonnegative integer. We shall say that $T \in \mathcal{D}'(I)$ is of order $\leq m$ if

$$\forall [a,b] \subset I \ , \ \exists C > 0 \ , \ \forall \varphi \in \mathcal{D}(I) \ , \ \mathrm{supp}(\varphi) \subset [a,b] \Rightarrow |\langle T, \varphi \rangle| \leq C \sup_{0 \leq k \leq m} \|\varphi^{(k)}\|_\infty \ .$$

Theorem 1.9.16 *i)* If $T \in \mathcal{D}'(\mathbb{R})$ is of order $\leq m$, there exists $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ such that $T = T_f^{(m+1)}$.

ii) If $T \in \mathcal{D}'(\mathbb{R})$ is arbitrary, there exists a sequence $(f_n)_{n \geq 1}$ of $L^\infty(\mathbb{R})$ such that on every segment of $\mathbb{R}$, f_n vanishes for n large enough, and

$$T = \sum_{n=1}^{\infty} T_{f_n}^{(n)} \ .$$

Proof.— The first statement follows from an induction argument on m, based on Theorem 1.9.15 and on the following lemma.

Lemma 1.9.17 If $T \in \mathcal{D}'(\mathbb{R})$ is of order $\leq m$ with $m \geq 1$, there exists $S \in \mathcal{D}'(\mathbb{R})$ of order $\leq m-1$ such that $S' = T$.

The proof of this lemma is straightforward, taking into account the formula $\langle S, \psi \rangle = -\langle T, P(\psi) \rangle$ and the fact that, for every $m \geq 1$ and $\psi \in \mathcal{D}(]-n,n[)$,

$$\sup_{0 \leq k \leq m} \|P(\psi)^{(k)}\|_\infty \leq B\|\psi\|_{L^1} + \sup_{0 \leq k \leq m-1} \|\psi^{(k)}\|_\infty \leq B_n \sup_{0 \leq k \leq m-1} \|\psi^{(k)}\|_\infty \ .$$

Let us prove statement ii). For every integer $j \geq 1$, there exists an integer m_j such that $T_{|]-j-1/2, j+1/2[}$ is of order $\leq m_j$. Furthermore, we may impose without loss of generality that $m_{j+1} > m_j$.
We are going to construct a sequence $(g_j)_{j \geq 1}$ of L^∞ functions on $\mathbb{R}$ such that, for every $j \geq 1$, supp$(T_{g_j}) \subset [-j, -j+1] \cup [j-1, j]$, and

$$\left(T - \sum_{\ell=1}^{j} T_{g_\ell}^{(m_\ell + 1)} \right)_{|]-j,j[} = 0 \ .$$

Let us first construct g_1. Since T is of order $\leq m_1$ on $]-3/2, 3/2[$, property i) (adapted to an interval) implies that there exists $h_1 \in L^\infty_{\mathrm{loc}}(]-3/2, 3/2[)$ such that

$$T = T_{h_1}^{(m_1+1)} \ .$$

Then $g_1 := \mathbf{1}_{]-1,1[} h_1 \in L^\infty(\mathbb{R})$ and satisfies

$$\left(T - T_{g_1}^{(m_1+1)}\right)_{|]-1,1[} = 0 \ , \ \text{supp}(T_{g_1}) \subset [-1,1] \ .$$

Assuming $g_1, \dots, g_j$ are constructed, let us construct g_{j+1}. Since the sequence m_j is increasing, the distribution

$$\tilde{T}_{j+1} = \left(T - \sum_{\ell=1}^{j} T_{g_\ell}^{(m_\ell+1)}\right)_{|]-j-3/2,j+3/2[}$$

is of order $\leq m_{j+1}+1$. Furthermore, the restriction of $\tilde{T}_{j+1}$ to $]-j,j[$ is 0. Therefore, using again property i), there exists $h_{j+1} \in L^\infty_{\text{loc}}(]-j-3/2, j+3/2[)$ such that

$$\tilde{T}_{j+1} = T_{h_{j+1}}^{(m_{j+1}+1)} \ .$$

Furthermore, the restriction of $T_{h_{j+1}}^{(m_{j+1}+1)}$ to $]-j,j[$ is 0, which implies from Corollary 1.4.9 that h_{j+1} coincides with a polynomial function p_{j+1} on $]-j,j[$. We consider

$$g_{j+1} := \mathbf{1}_{|]-j-1,j+1[}(h_{j+1} - p_{j+1}) \ .$$

Then $g_{j+1} \in L^\infty(\mathbb{R})$, $T_{g_{j+1}}$ is supported in $[-j-1,-j] \cup [j,j+1]$, and

$$\left(\tilde{T}_{j+1} - T_{g_{j+1}}^{(m_{j+1}+1)}\right)_{|]-j-1,j+1[} = 0 \ .$$

Coming back to the expression of $\tilde{T}_{j+1}$, we have checked the induction assumption at rank $j+1$, so that the sequence $(g_j)_{j\geq 1}$ is constructed by induction on j.
In view of the properties of the supports of T_{g_j}, for every j, only the terms of rank $\ell \leq j$ of the series

$$\sum_{\ell=1}^{\infty} T_{g_\ell}^{(m_\ell+1)}$$

have a nonzero restriction to $]-j,j[$. This proves that this series is convergent in $\mathcal{D}'(\mathbb{R})$. Furthermore, by the construction of the g_j, the sum of this series coincides with T on every interval $]-j,j[$, therefore

$$T = \sum_{\ell=1}^{\infty} T_{g_\ell}^{(m_\ell+1)} \ .$$

Setting $f_n := g_\ell$ if there exists $\ell \geq 1$ such that $n = m_\ell + 1$, and $f_n := 0$ otherwise, we end up with

$$T = \sum_{n=1}^{\infty} T_{f_n}^{(n)} \ .$$

□

Chapter 2

Distributions in several variables

2.1 A brief review of several variables differential calculus

2.1.1 Scalar product, norm, distance, topology

From now on we shall work on the vector space $\mathbb{R}^d$, with typical elements

$$x = \begin{pmatrix} x_1 \\ . \\ . \\ x_d \end{pmatrix}$$

with $x_1, \dots, x_d \in \mathbb{R}$. We denote by $(e_1, \dots, e_d)$ the canonical basis, so that

$$x = \sum_{j=1}^{d} x_j e_j \ .$$

This canonical basis is an orthonormal basis for the canonical scalar product

$$x.y = \sum_{j=1}^{d} x_j y_j \ ,$$

defining the Euclidean norm

$$|x| = \sqrt{x.x} = \left(\sum_{j=1}^{d} x_j^2 \right)^{\frac{1}{2}} \ .$$

This norm classically defines the distance

$$d(x, y) = |x - y| \ ,$$

and a topology —independent of the chosen norm — for which a set Ω is open if and only if for every $a \in \Omega$, there exists $r > 0$ such that $B(a, r) \subset \Omega$. Here, $B(a, r)$ denotes the open ball

$$\{x \in \mathbb{R}^d, d(x, a) < r\} \ .$$

As in every metric space, a compact subset K of $\mathbb{R}^d$ can be equivalently defined by the Borel–Lebesgue covering property, or by the Bolzano–Weierstrass extraction property for every sequence in K. Since we are in a finite dimensional vector space, compact subsets of $\mathbb{R}^d$ coincide with closed bounded subsets.

If F is a nonempty closed subset of $\mathbb{R}^d$, we shall often use the distance function to F,

$$d(x,F) = \inf_{z\in F} d(x,z)\ .$$

Notice that $d(x,F)=0$ if and only if $x\in F$, and that, by the Bolzano–Weierstrass property for closed bounded subsets, this infimum is attained at some $z\in F$. Furthermore, the function $d(.,F)$ is continuous on $\mathbb{R}^d$. In fact, by the triangle inequality, it is 1–Lipschitz continuous,

$$|d(x,F)-d(y,F)|\le d(x,y)\ .$$

If K is a compact subset of $\mathbb{R}^d$ and $\delta>0$, the set

$$K_\delta = \{x\in \mathbb{R}^d, d(x,K)\le \delta\}$$

is a compact subset containing K in its interior. The following lemma will be of constant use.

Lemma 2.1.1 If K is a compact subset of an open subset Ω of $\mathbb{R}^d$, then

$$\inf_{x\in K} d(x,\Omega^c) = \delta_0 > 0\ .$$

For every $\delta \in]0,\delta_0[$, K_δ is contained in Ω.

Proof.— The first statement follow from the continuity of $d(.,\Omega^c)$, which consequently attains its minimum on K, and from the assumption $K\cap \Omega^c=\emptyset$. The second statement is an elementary consequence of the triangle inequality, since, for every $z\in\Omega^c$, $y\in K$,

$$d(x,z)\ge |d(y,z)-d(x,y)|$$

hence, if $x\in K_\delta$, choosing $y\in K$ such that $d(x,y)=d(x,K)$, $d(x,\Omega^c)\ge \delta_0-\delta>0$. □

Finally, we denote by $\mathcal{C}^0(\Omega)$ the space of continuous functions $f:\Omega\to\mathbb{C}$.

2.1.2 Partial derivative, C^1 functions, differential, gradient

Definition 2.1.2 Given $f:\Omega\to\mathbb{C}$ and $a\in\Omega$, $j\in\{1,\dots,d\}$, we say that f admits a j-th partial derivative at a if the function $t\mapsto f(a+te_j)$, locally defined for t in a neighbourhood of 0 in $\mathbb{R}$, has a derivative at $t=0$. We set

$$\frac{d}{dt}f(a+te_j)_{|t=0} = \frac{\partial f}{\partial x_j}(a) = \partial_j f(a)\ .$$

We say that f is a C^1 function on Ω if it admits a j–th partial derivative for every $j\in\{1,\dots,d\}$ at every point $a\in\Omega$, and if the functions $\partial_j f$ are continuous on Ω. We denote by $\mathcal{C}^1(\Omega)$ the space of C^1 functions on Ω.

If $f \in \mathcal{C}^1(\Omega)$, one can prove that f is differentiable at every point a ; using the mean value theorem,

$$f(a+h) = f(a) + L_a(h) + o(|h|) \text{ as } h \to 0$$

where

$$L_a(h) = \sum_{j=1}^{d} \partial_j f(a) h_j \ .$$

The linear map $L_a : \mathbb{R}^d \to \mathbb{C}$ is called the differential of f at point a, and usually denoted as

$$L_a = d_a f \ .$$

If f is real valued, $d_a f$ is a linear form on the Euclidean space $\mathbb{R}^d$, hence it can be represented by the scalar product with a vector, called the gradient of f at a, and denoted by $\nabla f(a)$,

$$d_a f(h) = \nabla f(a).h \ , \ \nabla f(a) = \begin{pmatrix} \partial_1 f(a) \\ .. \\ \partial_d f(a) \end{pmatrix} \ .$$

Notice that the mapping $\nabla f : \Omega \to \mathbb{R}^d$ is continuous.

2.1.3 The Chain rule

Proposition 2.1.3 Let $\Omega \subset \mathbb{R}^d$, $\Omega' \subset \mathbb{R}^p$ be open sets, and $\psi : \Omega \to \Omega'$, $\psi = (\psi_1, \psi_2, \dots, \psi_p)$, a function of class $\mathcal{C}^m$, $m \geq 1$. Let $f : \Omega' \to \mathbb{C}$ be a $\mathcal{C}^m$ function.

Then $f \circ \psi$ is $\mathcal{C}^m$ on Ω. Moreover

$$d_x(f \circ \psi) = d_{\psi(x)} f . d_x \psi = \nabla \psi(x) \cdot \nabla f(\psi(x)),$$

or, for any $j \in \{1, \dots, n\}$,

$$\partial_j (f \circ \psi)(x) = \sum_{k=1}^{p} \partial_k f(\psi(x)) \partial_j \psi_k(x).$$

2.1.4 Higher order partial derivatives

More generally, for $m \geq 2$, we denote by $\mathcal{C}^m(\Omega)$ the vector space of functions $f \in C^1(\Omega)$ whose partial derivatives $\partial_1 f, \partial_2 f, \dots, \partial_d f$ belong to $\mathcal{C}^{m-1}(\Omega)$. Moreover, $\mathcal{C}^\infty(\Omega)$ is the intersection of all $\mathcal{C}^m(\Omega)$.

In general, the order in which one computes repeated partial derivatives matters, but this is not the case for $\mathcal{C}^2$ functions:

Proposition 2.1.4 (Schwarz Lemma) If $f \in \mathcal{C}^2(\Omega)$, then, for any $j, k \in \{1, \dots, d\}$,

$$\partial_j(\partial_k f) = \partial_k(\partial_j f)$$

In particular for $\mathcal{C}^\infty$ functions, one can compute partial derivatives of f in any order. It is therefore very convenient to use multi–indices.

2.1.5 Multi–indices

Let $f \in \mathcal{C}^\infty(\Omega)$, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$ a multiindex. We denote $\partial^\alpha f$ the function

$$\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} f.$$

The number

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d,$$

is the order of the partial derivative, and it is called the length of α. The context usually avoids any confusion with the Euclidean norm ! We also set

$$\alpha! = \alpha_1! \alpha_2! \dots \alpha_d!$$

and, for $\beta \in \mathbb{N}^d$ such that $\beta_j \le \alpha_j$ for all j, which we will write $\beta \le \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\dots\binom{\alpha_d}{\beta_d}.$$

With these notations, the Leibniz formula for the derivatives of a product of functions easily extends to the case of partial derivatives of functions of several variables. Its proof is exactly the same.

Proposition 2.1.5 Let f and g be functions in $\mathcal{C}^\infty(\Omega)$, and $\alpha \in \mathbb{N}^d$ a multiindex. We have

$$\partial^\alpha(fg) = \sum_{\beta \in \mathbb{N}^d,\ \beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta f\ \partial^{\alpha-\beta} g.$$

Proof.— We prove the result by induction over $|\alpha|$. If $|\alpha| = 1$, $\partial^\alpha = \partial_j$ for some $j \in \{1, \dots, d\}$, and

$$\partial_j(fg) = (\partial_j f)g + f(\partial_j g),$$

which is the above formula. Suppose then that the formula is true for all multiindices of length $\le m$. Let $\alpha \in \mathbb{N}^d$ such that $|\alpha| = m+1$. There exists $j \in \{1, \dots, d\}$ et $\beta \in \mathbb{N}^d$ of length m such that

$$\alpha = \beta + 1_j,$$

where $1_j = (0, \dots, 0, 1, 0, \dots, 0)$ with a 1 as j-th coordinate. With these notations

$$\partial^\alpha(fg) = \partial^{\beta+1_j}(fg) = \partial^\beta(\partial_j(fg)) = \partial^\beta((\partial_j f)g) + \partial^\beta(f(\partial_j g)).$$

Since β is of length m, the induction assumption gives

$$\begin{aligned}
\partial^\alpha(fg) &= \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma(\partial_j f)\ \partial^{\beta-\gamma} g + \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma f\ \partial^{\beta-\gamma}(\partial_j g)\\
&= \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^{\gamma+1_j} f\ \partial^{\beta-\gamma} g + \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma f\ \partial^{\beta+1_j-\gamma} g\\
&= \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^{\gamma+1_j} f\ \partial^{\alpha-(\gamma+1_j)} g + \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma f\ \partial^{\alpha-\gamma} g
\end{aligned}$$

We change the multiindex in the first sum: $\gamma \leftarrow \gamma + 1_j$, and we get

$$\begin{aligned}\partial^\alpha(fg) &= \sum_{1_j\le\gamma\le\alpha}\binom{\beta}{\gamma-1_j}\partial^\gamma f\,\partial^{\alpha-\gamma}g + \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma f\,\partial^{\alpha-\gamma}g\\ &= \partial^\alpha f\,g + \sum_{1_j\le\gamma\le\beta}(\binom{\beta}{\gamma-1_j}+\binom{\beta}{\gamma})\partial^\gamma f\,\partial^{\alpha-\gamma}g + f\partial^\alpha g\\ &= \partial^\alpha f\,g + \sum_{1_j\le\gamma\le\beta}\binom{\beta+1_j}{\gamma}\partial^\gamma f\,\partial^{\alpha-\gamma}g + f\partial^\alpha g,\end{aligned}$$

which is the required result. □

We can continue the analogy with the 1 variable case: if $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_d^{\alpha_d}\ .$$

Then we can show, the same way as for the Leibniz formula, that for $x, y \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$,

$$(x+y)^\alpha = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}x^\beta y^{\alpha-\beta}.$$

With these notations, Taylor's formula can be written as

Proposition 2.1.6 Let $f : \Omega \subset \mathbb{R}^d \to \mathbb{C}$ a function of class $\mathcal{C}^{m+1}$. Let $a, b \in \Omega$, such that the segment $[a, b]$ is included in Ω. We have

$$f(b) = \sum_{|\alpha|\le m}\frac{(b-a)^\alpha}{\alpha!}\partial^\alpha f(a) + (m+1)\sum_{|\alpha|=m+1}\frac{(b-a)^\alpha}{\alpha!}\int_0^1(1-t)^m\partial^\alpha f(a+t(b-a))dt.$$

Proof.— We have already seen that if $\varphi : \mathbb{R} \to \mathbb{C}$ is smooth, we have

$$\varphi(1) = \sum_{k=0}^m\frac{1}{k!}\varphi^{(k)}(0) + \frac{1}{m!}\int_0^1(1-s)^m\varphi^{(m+1)}(s)ds.$$

We shall use this result for the function $\varphi : t \mapsto f(a + t(b-a))$. Notice that

$$\varphi'(t) = \sum_{j=1}^d(b-a)_j(\partial_j f)(a+t(b-a))$$

and more generally that

$$\varphi^{(k)}(t) = \sum_{j_1,j_2,\dots,j_k=1}^d(b-a)_{j_1}\dots(b-a)_{j_k}(\partial_{j_1}\dots\partial_{j_k}f)((a+t(b-a)).$$

This sum only contains terms of the form $(b-a)^\alpha(\partial^\alpha f)(a+t(b-a))$ with $\alpha \in \mathbb{N}^d$ of length $|\alpha| = k$. Therefore we can write, for some coefficients $c_\alpha \in \mathbb{R}$,

$$\sum_{j_1,j_2,\dots,j_k=1}^{d} (b-a)_{j_1}\dots(b-a)_{j_k}(\partial_{j_1}\dots\partial_{j_k} f)((a+t(b-a)) = \sum_{|\alpha|=k} c_\alpha (b-a)^\alpha(\partial^\alpha f)(a+t(b-a)).$$

Denoting $x = b-a$, this equality between two polynomials in x implies that the c_α are given by

$$c_\alpha = \#\{(j_1, j_2, \dots, j_k) \in \{1,\dots,d\}^k,\ x_1^{\alpha_1} x_2^{\alpha_2}\dots x_d^{\alpha_d} = x_{j_1}\dots x_{j_k}\}.$$

thus

$$c_\alpha = \binom{k}{\alpha_1}\binom{k-\alpha_1}{\alpha_2}\dots\binom{k-\alpha_1-\dots-\alpha_{n-1}}{\alpha_n} = \frac{k!}{\alpha!}.$$

Indeed, one has to choose first α_1 numbers among $j_1, \dots j_k$ that should be 1, then α_2 among the $k-\alpha_1$ numbers left that should be 2, …

Since $a+t(b-a)_{|t=0} = a$, and $a+t(b-a)_{|t=1} = b$, we obtain the stated formula. □

Exercise 2.1.7 Show that, for $k \in \mathbb{N}$ and $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we have

$$(x_1 + x_2 + \dots + x_d)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\, x^\alpha.$$

Applying the Taylor formula, we immediately obtain.

Corollary 2.1.8 (Hadamard's formula) Let $f \in \mathcal{C}^{m+1}(\Omega)$, where Ω is a convex open subset of $\mathbb{R}^d$. If $a \in \Omega$ and $f(a) = 0$, there exist d functions g_1, g_2,…g_d of class $\mathcal{C}^m(\Omega)$ such that

$$f(x) = \sum_{j=1}^{d}(x_j - a_j)g_j(x).$$

Notice that the convexity assumption of Ω plays a role, since we need that $[a,x] \subset \Omega$. In practice, we shall use this lemma when Ω is a ball.

2.2 Test functions

2.2.1 Definitions. Examples

If $f \in \mathcal{C}^0(\Omega)$, the support of f is the closure of $\{x \in \Omega, f(x) \neq 0\}$ for the topology induced on Ω. This is a closed subset of Ω, denoted by supp(f). If $m \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{C}_0^m(\Omega)$ the space of functions in $\mathcal{C}^m(\Omega)$ with a compact support. Notice that a compact set for the induced topology in Ω is also a compact subset of $\mathbb{R}^d$ contained in Ω. In particular, elements of $\mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$ are called test functions on Ω. As in one space dimension, if $V \subset \Omega$ is an open subset, every element φ of $\mathcal{D}(V)$ can be extended as an element $\underline{\varphi}$ of $\mathcal{D}(\Omega)$ by setting $\underline{\varphi} = 0$ in $\Omega \setminus V$. This allows to identify $\mathcal{D}(V)$ as the subspace of $\mathcal{D}(\Omega)$ defined by supp$(\varphi) \subset V$. We shall often make this identification.

Proposition 2.2.1 *i)* For every $a \in \mathbb{R}^d$, for every $r > 0$, there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi(x) \geq 0$ for every $x \in \mathbb{R}^d$ and $\operatorname{supp}(\varphi) = \overline{B(a,r)}$.

ii) For every compact subset K of Ω, there exists $\chi \in \mathcal{D}(\Omega)$, valued in $[0,1]$, such that $\chi = 1$ on K.

Proof.— The proof of the first statement is quite similar to its one dimensional analogue. Recall that the function f defined by

$$f(t) = \begin{cases} \mathrm{e}^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

is a $\mathcal{C}^\infty$ function. Then the function φ defined by

$$\varphi(x) = f(r^2 - |x-a|^2)$$

satisfies the requirements.

The second statement requires a little more work. Let φ be as in the first statement with $a = 0$ and $r = 1$. Since $\varphi \geq 0$ and is not identically 0, its integral on $\mathbb{R}^d$ is > 0. Up to dividing φ by this number, we may assume that

$$\int_{\mathbb{R}^d} \varphi(x)\, dx = 1 \ .$$

Then, by Lemma 2.1.1, we may choose $\delta > 0$ such that $K_{2\delta} \subset \Omega$. Set

$$\chi(x) = \int_{K_\delta} \varphi\left(\frac{x-y}{\delta}\right) \frac{dy}{\delta^d} \ .$$

By the Leibniz rule of derivation under the $\int$ sign, $\chi \in \mathcal{C}^\infty(\mathbb{R}^d)$. Since $\varphi \geq 0$, we have $\chi \geq 0$ and

$$\chi(x) \leq \int_{\mathbb{R}^d} \varphi\left(\frac{x-y}{\delta}\right) \frac{dy}{\delta^d} = \int_{\mathbb{R}^d} \varphi(z)\, dz = 1 \ ,$$

after performing the change of variable $y = x - \delta z$. Hence χ is valued in $[0,1]$. If $x \in K$ and $y \in (K_\delta)^c$, then $|x-y| > \delta$, hence $(x-y)/\delta \notin \operatorname{supp}(\varphi)$ and the integrand cancels at y. Consequently,

$$\forall x \in K \ , \ \chi(x) = \int_{\mathbb{R}^d} \varphi\left(\frac{x-y}{\delta}\right) \frac{dy}{\delta^d} = 1 \ .$$

Finally, if $x \notin K_{2\delta}$, $y \in K_\delta$, $|x-y| > \delta$, and the integrand identically vanishes, so that $\chi(x) = 0$. Hence the support of χ is included in $K_{2\delta}$, which is contained in Ω, so that $\chi \in \mathcal{C}_0^\infty(\Omega)$. □

Remark 2.2.2 Functions χ as in Proposition 2.2.1 are called *plateau* functions on K in Ω. It may happen that we need plateau functions on a compact neighborhood of K in Ω, for instance K_ε for $\varepsilon > 0$ enough, according to Lemma 2.1.1. We shall call these functions *plateau functions on a neighborhood of* K in Ω.

We now come to the contruction of partitions of unity, which turns out to be particularly useful in several variables, through the so–called "gluing principle".

2.2.2 Partitions of unity

Proposition 2.2.3 (Partition of unity) Let K be a compact covered by a finite collection $\Omega_1, \dots, \Omega_n$ of open subsets of Ω. There exist $\chi_1 \in \mathcal{C}_0^\infty(\Omega_1), \dots, \chi_n \in \mathcal{C}_0^\infty(\Omega_n)$, valued in $[0,1]$, such that

$$\chi_1 + \cdots + \chi_n = 1 \text{ on } K \ .$$

In particular, if $\varphi \in \mathcal{C}_0^\infty(\Omega)$ is supported in $\Omega_1 \cup \cdots \cup \Omega_n$, then one can find functions $\varphi_1 \in \mathcal{C}_0^\infty(\Omega_1), \dots, \varphi_n \in \mathcal{C}_0^\infty(\Omega_n)$, such that

$$\varphi_1 + \dots \varphi_n = \varphi \ .$$

Proof.— It is very similar to the one dimensional analogue. First, one proves the

Lemma 2.2.4 (Shrinking Lemma) If $K \subset \Omega_1 \cup \dots \cup \Omega_n$, there exists open subsets $U_1, \dots, U_n$ such that, for every $j \in \{1, \dots, n\}$, $\overline{U}_j \subset \Omega_j$ is compact, and

$$K \subset U_1 \cup \dots \cup U_n \ .$$

As in the one dimensional case, the proof proceeds by induction on $n \le 1$. The case $n = 1$ follows from Lemma 2.1.1.
Once the shrinking lemma is proved, set

$$\chi_1 = \psi_1, \chi_2 = \psi_2(1 - \psi_1) \ , \dots \ , \chi_n = \psi_n(1 - \psi_{n-1}) \dots (1 - \psi_1) \ ,$$

where, for every $j \in \{1, \dots, n\}$, ψ_j s a plateau function on $\overline{U}_j$ in Ω_j.
Finally, the last assertion follows by writing $\varphi_j = \chi_j \varphi$, where the χ_j's are associated to the covering of supp(φ) by $\Omega_1, \dots, \Omega_n$. □

2.3 Distributions on an open subset of $\mathbb{R}^d$

2.3.1 Définitions. Examples

Definition 2.3.1 Let (φ_j) be a sequence of functions in $\mathcal{D}(\Omega)$, and $\varphi \in \mathcal{D}(\Omega)$. We say that (φ_j) tends to φ in $\mathcal{D}(\Omega)$ (or in the $\mathcal{D}(\Omega)$-sense), when

i) There exits a compact subset $K \subset \Omega$ such that supp $\varphi_j \subset K$ for all j.

ii) For all $\alpha \in \mathbb{N}^d$, sup $|\partial^\alpha \varphi_j - \partial^\alpha \varphi| \to 0$ as $j \to +\infty$.

In that case we may write

$$\varphi = \mathcal{D} - \lim_{j \to +\infty} \varphi_j.$$

Remark 2.3.2 Notice that, under the conditions of the above definition, we have supp$(\varphi) \subset K$.

Definition 2.3.3 Let $\Omega \subset \mathbb{R}^d$ an open subset, and T a complex valued linear form on $\mathcal{D}(\Omega)$. One says that T is a distribution on Ω if, for every compact set $K \subset \Omega$,

$$\exists C > 0, \exists m \in \mathbb{N}, \forall \varphi \in \mathcal{C}_0^\infty(\Omega) \text{ with supp}\, \varphi \subset K \ , \ |T(\varphi)| \leq C \sum_{|\alpha| \leq m} \text{sup}\, |\partial^\alpha \varphi| = C\|\varphi\|_{C^m}.$$

We denote by $\mathcal{D}'(\Omega)$ the set of distributions on Ω, and for $T \in \mathcal{D}'(\Omega)$, $\varphi \in \mathcal{D}(\Omega)$, we set $\langle T, \varphi \rangle := T(\varphi)$.

Proposition 2.3.4 A linear form T on $\mathcal{D}(\Omega)$ is a distribution on Ω if and only if $T(\varphi_j) \to T(\varphi)$ for any sequence (φ_j) of functions in $\mathcal{D}(\Omega)$ that converges to φ in the $\mathcal{D}(\Omega)$-sense.

The proof is similar to the one dimensional case, as well as the following examples.

Locally integrable functions.

Given $f \in L^1_{loc}(\Omega)$, the formula

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\, \varphi(x)\, dx$$

defines a distribution on Ω. Furthermore, the linear mapping

$$f \in L^1_{loc}(\Omega) \mapsto T_f \in \mathcal{D}'(\Omega)$$

is one to one. In the sequel, we shall identify f to T_f.

Dirac masses.

Given $a \in \Omega$, the formula

$$\langle \delta_a, \varphi \rangle = \varphi(a)$$

defines a distribution δ_a on Ω, called the Dirac mass at a. It is not of the form f for $f \in L^1_{loc}(\Omega)$.

As in the previous chapter, one can define finite order distributions, and the corresponding notion of order. In particular, distributions of order $\leq m$ can be extended to continuous linear forms on $\mathcal{C}^m(\Omega)$, non-negative distributions are of order 0 — hence are positive measures, and distributions of order 0 are finite linear combinations of non-negative distributions.

We now come to the important notion of support, for which we shall be a little more specific than we were in Chapter 1.

2.3.2 Restriction and support

Definition 2.3.5 Let $T \in \mathcal{D}'(\Omega)$, and $V \subset \Omega$ an open subset. The restriction of T to V is the distribution $T_{|V} \in \mathcal{D}'(V)$ defined as

$$\forall \varphi \in \mathcal{D}(V)\ ,\ \langle T_{|V}, \varphi \rangle = \langle T, \underline{\varphi} \rangle\ ,$$

where $\underline{\varphi}$ denotes the extension of φ by 0 on $\Omega \setminus V$.

We say that T vanishes in V if $T_{|V} = 0$.

Definition 2.3.6 The support of a distribution $T \in \mathcal{D}'(\Omega)$ is the complement of the union of all the open subsets where T vanishes. We denote it by $\operatorname{supp} T$.

Notice that $\operatorname{supp} T$ is closed, and the following characterizations are convenient.

- $x_0 \notin \operatorname{supp} T$ if and only if there is an open neighborhood V of x_0 such that $T_{|V} = 0$.
- $x_0 \in \operatorname{supp} T$ if and only if for any open neighborhood V of x_0, one can find $\varphi \in \mathcal{C}_0^\infty(V)$ such that $\langle T, \varphi \rangle \neq 0$.

As in the one dimensional case, one can characterize distributions supported in one point a, as given by

$$\langle T, \varphi \rangle = \sum_\alpha c_\alpha \partial^\alpha \varphi(a)\ ,$$

where $(c_\alpha)_{\alpha \in \mathbb{N}^d}$ is a family of complex numbers which vanish except for a finite set of multi–indices.

Proposition 2.3.7 Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. If $\operatorname{supp} \varphi \cap \operatorname{supp} T = \emptyset$, then $\langle T, \varphi \rangle = 0$. In particular, if $\operatorname{supp}(T) = \emptyset$, then $T = 0$.

In view of Lemma 2.2.3, the proof is similar to the one dimensional case. We now emphasize the following corollary, which is very useful.

Corollary 2.3.8 (The gluing principle) Assume

$$\Omega = \bigcup_{j \in J} \Omega_j$$

for some collection $(\Omega_j)_{j\in J}$ of open subsets. Suppose we are given, for every $j \in J$, a distribution T_j on Ω_j, so that we have the following property : for every $j, k \in J$ such that $\Omega_j \cap \Omega_k \neq \emptyset$, then

$$(T_j)_{|\Omega_j\cap\Omega_k} = (T_k)_{|\Omega_j\cap\Omega_k} .$$

Then there exists a unique $T \in \mathcal{D}'(\Omega)$ such that, for every $j \in J$, $T_{|\Omega_j} = T_j$.

Proof.— The uniqueness of T follows from Proposition 2.3.7. Let us prove the existence of T. Given $\varphi \in \mathcal{D}(\Omega)$, we want to define $\langle T, \varphi\rangle$. From Proposition 2.2.3, one can decompose

$$\varphi = \sum_{j\in J} \varphi_j$$

where $\varphi_j \in \mathcal{D}(\Omega_j)$ and $\varphi_j = 0$ except for a finite set of j's. In such a situation, if T exists, one must have

$$\langle T, \varphi\rangle = \sum_{j\in J} \langle T, \varphi_j\rangle = \sum_{j\in J} \langle T_j, \varphi_j\rangle .$$

Therefore it is natural to establish the following lemma.

Lemma 2.3.9 For every decomposition

$$\varphi = \sum_{j\in J} \varphi_j$$

where $\varphi_j \in \mathcal{D}(\Omega_j)$ and $\varphi_j = 0$ except for a finite set of j's, the complex number

$$\sum_{j\in J} \langle T_j, \varphi_j\rangle$$

only depends on φ, not on the decomposition.

Let us prove the lemma. Consider the set

$$K = \bigcup_{j\in J} \text{supp}(\varphi_j) .$$

Since supp(φ_j) is compact and is empty except for a finite set of j's, K is a compact subset of Ω. Apply Proposition 2.2.3 to a finite covering

$$K \subset \bigcup_{r\in R} \Omega_r$$

extracted from the covering of K by the Ω_j's. The family $(\chi_r)_{r\in R}$ satisfies

$$\text{supp}(\chi_r) \subset \Omega_r , \ \sum_{r\in R} \chi_r = 1 \text{ on } K .$$

In particular, for every $j \in J$, $\varphi_j = \sum_{r\in R} \chi_r \varphi_j$, therefore

$$\langle T_j, \varphi_j\rangle = \sum_{r\in R} \langle T_j, \chi_r\varphi_j\rangle \ .$$

For any $r \in R$, we claim that $\langle T_j, \chi_r\varphi_j\rangle = \langle T_r, \chi_r\varphi_j\rangle$. Indeed, either $\Omega_j \cap \Omega_r = \emptyset$, and $\chi_r\varphi_j = 0$, so both sides of this equality cancel. Or $\Omega_j \cap \Omega_r \neq \emptyset$, and, since supp$(\chi_r\varphi_j) \subset \Omega_j \cap \Omega_r$, the assumption merely says that

$$\langle T_j, \chi_r\varphi_j\rangle = \langle T_r, \chi_r\varphi_j\rangle.$$

Consequently,

$$\begin{aligned}\sum_{j\in J} \langle T_j, \varphi_j\rangle &= \sum_{j\in J}\sum_{r\in R} \langle T_r, \chi_r\varphi_j\rangle = \sum_{r\in R} \langle T_r, \chi_r(\sum_{j\in J} \varphi_j)\rangle \\ &= \sum_{r\in R} \langle T_r, \chi_r\varphi\rangle \ .\end{aligned}$$

This proves the lemma. Indeed, given any other decomposition $\varphi = \sum_{j\in J} \tilde\varphi_j$, just apply the above construction with

$$K = \bigcup_{j\in J} \text{supp}(\varphi_j) \cup \bigcup_{j\in J} \text{supp}(\tilde\varphi_j) \ ,$$

and we see that

$$\sum_{j\in J} \langle T_j, \varphi_j\rangle = \sum_{r\in R} \langle T_r, \chi_r\varphi\rangle = \sum_{j\in J} \langle T_j, \tilde\varphi_j\rangle.$$

Let us complete the proof of Proposition 2.3.8. For every decomposition $\varphi = \sum_{j\in J} \varphi_j$ as in the lemma, we define

$$\langle T, \varphi\rangle = \sum_{j\in J} \langle T_j, \varphi_j\rangle \ .$$

It is clear that T is a linear form on $\mathcal{D}(\Omega)$. To check it is a distribution, let K a compact subset of Ω and $(\chi_r)_{r\in R}$ be a partition of unity associated to a finite covering of K extracted from $K \subset \cup_{j\in J}\Omega_j$. Then, for every test function φ supported in K, we have, from the lemma,

$$\langle T, \varphi\rangle = \sum_{r\in R} \langle T_r, \chi_r\varphi\rangle \ .$$

Since $T_r \in \mathcal{D}'(\Omega_r)$, we have

$$|\langle T, \chi_r\varphi\rangle| \leq C_r \|\chi_r\varphi\|_{C^{m_r}} \leq C'_r \|\varphi\|_{C^{m_r}} \ .$$

Then, with $m = \max_{r\in R} m_r$, we obtain

$$|\langle T, \varphi\rangle| \leq \left(\sum_{r\in R} C'_r\right) \|\varphi\|_{C^m} \ ,$$

so that $T \in \mathcal{D}'(\Omega)$. Finally, if $\varphi \in \mathcal{D}(\Omega_j)$ for some $j \in J$, we can write the decomposition $\varphi = \sum_{k\in J} \varphi_k$, with $\varphi_j = \varphi$ and $\varphi_k = 0$ if $k \neq j$. Using the lemma, we infer $\langle T, \varphi\rangle = \langle T_j, \varphi\rangle$. In other words, $T_{|\Omega_j} = T_j$. □

Let us close this paragraph by a number of additional remarks concerning Proposition 2.3.8.

i) If every T_j is a nonnegative distribution, then so is T. Indeed, the elements χ_r of the partitions of unity can be chosen to be nonnegative.

ii) If every T_j is a $\mathcal{C}^m$ function, then so is T.

2.3.3 Multiplication by a smooth function

Definition 2.3.10 Given $q \in \mathcal{C}^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, we define $qT \in \mathcal{D}'(\Omega)$ by

$$\forall \varphi \in \mathcal{D}(\Omega) \ , \ \langle qT, \varphi \rangle = \langle T, q\varphi \rangle \ .$$

Of course, if $T \in L^1_{loc}(\Omega)$, the definition coincides with the usual product. Let us just mention the following generalization of Proposition 1.5.7 in Chapter 1.

Proposition 2.3.11 Let $T \in \mathcal{D}'(\Omega)$ and $a \in \Omega$ such that

$$\forall j \in \{1 \dots, d\} \ , \ (x_j - a_j)T = 0 \ .$$

Then there exists $c \in \mathbb{C}$ such that $T = c\delta_a$.

Proof.— In view of what we did in Chapter 1, the proof reduces to the following lemma.

Lemma 2.3.12 (Hadamard lemma on an arbitrary open set) If $\varphi \in \mathcal{C}_0^\infty(\Omega)$ satisfies $\varphi(a) = 0$, there exists $\psi_1, \dots, \psi_d$ in $\mathcal{C}_0^\infty(\Omega)$ such that

$$\varphi(x) = \sum_{j=1}^d (x_j - a_j)\psi_j(x) \ .$$

Let us prove the lemma. Let $r > 0$ such that $B(a,r) \subset \Omega$, and let χ be a plateau function on $\overline{B}(a, r/2)$, supported in $B(a,r)$. From Corollary 2.1.8 — Hadamard's formula on a convex subset — we can write, for $x \in B(a,r)$,

$$\varphi(x) = \sum_{j=1}^d (x_j - a_j) f_j(x) \ ,$$

with $f_j \in \mathcal{C}^\infty(B(a,r)$ for every j. This implies $\chi\varphi = \sum_{j=1}^d (x_j - a_j)\chi f_j$. On the other hand,

$$(1 - \chi(x))\varphi(x) = \sum_{j=1}^d (x_j - a_j) \frac{(x_j - a_j)(1 - \chi(x))}{|x - a|^2} \varphi(x) \ .$$

Summing both above identities, the lemma follows with

$$\psi_j(x) = \chi(x) f_j(x) + \frac{(x_j - a_j)(1 - \chi(x))}{|x - a|^2} \varphi(x) \ .$$

□

2.3.4 Derivation

As in one space dimension, we first consider the case of a C^1 function. In this case, we have the following elementary lemma.

Lemma 2.3.13 Let $f \in \mathcal{C}^1(\Omega)$, $\varphi \in \mathcal{C}^1_0(\Omega)$, and $j \in \{1, \dots, d\}$.

$$\int_\Omega \partial_j f(x)\, \varphi(x)\, dx = -\int_\Omega f(x)\, \partial_j \varphi(x)\, dx\ .$$

Proof.— Let χ be a plateau function on a neighborhood of the support of φ. It is easy to check that

$$\partial_j(\chi f)\varphi = \partial_j f\, \varphi\ ,\ \chi f \partial_j \varphi = f \partial_j \varphi\ .$$

Furthermore, χf can be extended as a C^1 function on $\mathbb{R}^d$, vanishing outside of Ω. Therefore we are reduced to prove the lemma with $\Omega = \mathbb{R}^d$. This is an immediate consequence of the Fubini theorem and of integration by parts in the x_j variable. □

On the basis of Lemma 2.3.13, we introduce the following definition.

Definition 2.3.14 Let $T \in \mathcal{D}'(\Omega)$ and $j \in \{1, \dots, d\}$. We define $\partial_j T \in \mathcal{D}'(\Omega)$ by

$$\langle \partial_j T, \varphi \rangle = -\langle T, \partial_j \varphi \rangle\ ,\ \varphi \in \mathcal{D}(\Omega)\ .$$

Similarly, for every $\alpha \in \mathbb{N}^d$, we define $\partial^\alpha T \in \mathcal{D}'(\Omega)$ by

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle\ ,\ \varphi \in \mathcal{D}(\Omega)\ .$$

At this stage, it is natural to ask for the multidimensional analogue of the identity $H' = \delta_0$ proved in Chapter 1. A natural statement would be a formula for $\partial_j(\mathbf{1}_U)$, where U is an open subset of Ω and $\mathbf{1}_U$ is the indicator – or characteristic — function of U, equal to 1 on U and to 0 outside of U. However, open subsets in $\mathbb{R}^d$ are complicated enough so that no such formula exists without additional assumptions on U. A relatively general formula of this kind will be the purpose of the next section, devoted to superficial measures and to the jump formula. At this stage, let us just consider two very simple examples in $\mathbb{R}^2$.

i) Half–spaces. Consider $H_a = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > a\} \subset \mathbb{R}^2$ for some $a \in \mathbb{R}$. Then an elementary calculation gives

$$\langle \partial_1(\mathbf{1}_{H_a}), \varphi \rangle = \int_{\mathbb{R}} \varphi(a, x_2)\, dx_2\ ,\ \partial_2(\mathbf{1}_{H_a}) = 0\ .$$

ii) The unit disc. Consider $D = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^2$. Then using polar coordinates, one easily checks that

$$\langle \partial_1(\mathbf{1}_D), \varphi \rangle = -\int_0^{2\pi} \cos\theta\, \varphi(\cos\theta, \sin\theta)\, d\theta\ ,\ \langle \partial_2(\mathbf{1}_D), \varphi \rangle = -\int_0^{2\pi} \sin\theta\, \varphi(\cos\theta, \sin\theta)\, d\theta\ .$$

Notice that, in all the above examples, $\partial_j(\mathbf{1}_U)$ is a distribution of order 0, supported by the boundary ∂U of U. This fact will be generalized in the next section.

2.3.5 Convergence

Definition 2.3.15 A sequence (T_n) of elements of $\mathcal{D}'(\Omega)$ converges to $T \in \mathcal{D}'(\Omega)$ if

$$\forall \varphi \in \mathcal{D}(\Omega)\ ,\ \langle T_n, \varphi\rangle \to \langle T, \varphi\rangle\ .$$

If $f \in L^1(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} f(x)\, dx = 1\ ,$$

then

$$\frac{1}{\varepsilon^d} f\left(\frac{x}{\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} \delta_0\ .$$

This observation leads to the following notion of regularization. Fix $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, supported in the unit ball, such that

$$\int_{\mathbb{R}^d} \rho(x)\, dx = 1\ ,$$

and consider

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)\ .$$

Given an open subset Ω in $\mathbb{R}^d$, set

$$\Omega^\varepsilon := \{x \in \Omega, d(x, \Omega^c) > \varepsilon\}\ .$$

If $x \in \Omega^\varepsilon$, the function

$$\rho_\varepsilon(x - .) : y \mapsto \rho_\varepsilon(x - y)$$

is supported in the closed ball of radius ε centered at x, which is included in Ω. Hence this function belongs to $\mathcal{C}_0^\infty(\Omega)$, and we may define

$$F^\varepsilon(x) = \langle T, \rho_\varepsilon(x - .)\rangle\ .$$

Proposition 2.3.16 The function F^ε is smooth on Ω^ε, with

$$\partial^\alpha F^\varepsilon(x) = \langle T, \partial^\alpha \rho_\varepsilon(x - .)\rangle\ ,\ \alpha \in \mathbb{N}^d\ ,$$

and, for every $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega F^\varepsilon(x)\, \varphi(x)\, dx \underset{\varepsilon\to 0}{\longrightarrow} \langle T, \varphi\rangle\ .$$

Notice that, given a compact subset K in Ω, the integral $\int_\Omega F^\varepsilon(x)\,\varphi(x)\,dx$ is well defined for ε small enough and every φ supported in K. The proof of Proposition 2.3.16 is a consequence of the following two results, which will be frequently used throughout the course, and are generalisations to several variables of Propositions 1.9.5 and 1.9.6 from Chapter 1.

Proposition 2.3.17 (Derivation under the bracket) Let $\Omega \subset \mathbb{R}^d$, $Z \subset \mathbb{R}^p$ be open sets, and $T \in \mathcal{D}'(\Omega)$. Let also $\varphi \in \mathcal{C}_0^\infty(\Omega \times Z)$. The function

$$G : z \in Z \mapsto \langle T, \varphi(\cdot, z)\rangle$$

is $\mathcal{C}^\infty$, and, for $\alpha \in \mathbb{N}^p$,

$$\partial^\alpha G(z) = \langle T, \partial_z^\alpha \varphi(\cdot, z)\rangle$$

Remark 2.3.18 *i)* We have written $\langle T, \varphi(\cdot, z)\rangle$ in place of $\langle T, \varphi_z\rangle$, where $\varphi_z \in \mathcal{C}_0^\infty(\Omega)$ is the function given by $\varphi_z(x) = \varphi(x, z)$.

ii) If $T = f \in L^1_{loc}(\Omega)$, we have $G(z) = \int_\Omega f(x)\varphi(x, z)dx$, so that, under the above assumptions, we get $G \in \mathcal{C}^\infty(Z)$ and we recover the Leibniz rule of derivation under the $\int$ sign,

$$\partial^\alpha G(z) = \int_\Omega f(x)\partial_z^\alpha \varphi(x, z)dx.$$

Proof.— Denote by K the projection of the support of φ onto Ω. Let $z_0 \in Z$ and $x \in \Omega$. For $h \in \mathbb{R}^q$, Taylor's formula at order 1 gives

$$\varphi(x, z_0 + h) = \varphi(x, z_0) + \sum_{j=1}^{p} \partial_{z_j}\varphi(x, z_0)h_j + r(x, z_0, h),$$
$$\text{with } r(x, z_0, h) = 2 \sum_{|\alpha|=2} \frac{h^\alpha}{\alpha!} \int_0^1 (1-t)\partial_z^\alpha \varphi(x, z_0 + th)dt.$$

Since $x \mapsto r(x, z_0, h))$ is $\mathcal{C}^\infty$ with support in K,

$$|\langle T, r(x, z_0, h)\rangle| \le C \sum_{|\beta|\le k} \sup |\partial_x^\beta r(x, z_0, h)|$$

for a constant $C > 0$ and an integer $k \in \mathbb{N}$ independent of z_0 and h. But for $|h| \le 1$,

$$|\partial_x^\beta r(x, z_0, h)| \le 2 \sum_{|\alpha|\le 2} \frac{h^\alpha}{\alpha!} \int_0^1 (1-t)\partial_x^\beta \partial_z^\alpha \varphi(x, z_0 + th)dt \le C|h|^2 \sum_{|\alpha|\le 2} \sup_{K\times \overline{B}(0,1)} |\partial_x^\beta \partial_z^\alpha \varphi(x, z)|,$$

Thus

$$|\langle T, r(x, z_0, h)\rangle| = \mathcal{O}(|h|^2).$$

and,

$$G(z_0 + h) = G(z_0) + \sum_{j=1}^{q} \langle T, \partial_{z_j}\varphi(x, z_0)\rangle h_j + O(|h|^2),$$

which shows that G is differentiable at z - in particular G is continuous, and that

$$\partial_j G(z) = \langle T, \partial_{z_j}\varphi(x,z)\rangle.$$

Then one can replace $\varphi(x,z)$ by $\partial_{z_j}\varphi(x,z)$ in the above discussion. We see that for all j, $\partial_j\varphi$ is differentiable, thus in particular continuous. So G is $\mathcal{C}^1$, and the statement of the proposition is true for any $|\alpha| = 1$. One can easily get the general case by induction. □

Remark 2.3.19 In view of the proof above, the assumption $\varphi \in \mathcal{C}_0^\infty(\Omega \times Z)$ is too strong. What we need is indeed that

$$\mathrm{supp}(\varphi) \subset K \times Z$$

for some compact subset K of Ω.

Proposition 2.3.20 (Integration under the bracket) Let $\Omega \subset \mathbb{R}^d$ be an open set, and $T \in \mathcal{D}'(\Omega)$. Let also $\varphi \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R}^p)$. Then

$$\int_{\mathbb{R}^p} \langle T, \varphi(\cdot,z)\rangle dz = \left\langle T, \int_{\mathbb{R}^p} \varphi(\cdot,z)dz \right\rangle$$

Proof.— We start with the case $p = 1$. Let $\varphi \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R})$. We choose $A > 0$ and a compact set $K \subset \Omega$ such that $\mathrm{supp}\,\varphi \subset K \times [-A, A]$. We denote by $\psi : \Omega \times \mathbb{R} \to \mathbb{C}$ the function given by

$$\psi(x,z) = \int_{t<z} \varphi(x,t)dt.$$

The function ψ belongs to $\mathcal{C}^\infty(\Omega \times \mathbb{R})$, and for any z, $\mathrm{supp}(\psi)$ is included into $K \times \mathbb{R}$. Therefore Proposition 2.3.17— with Remark 2.3.19— applies. The function

$$G(z) = \langle T, \psi(x,z)\rangle = \left\langle T, \int_{t<z} \varphi(x,t)dt \right\rangle$$

is smooth, and

$$G'(z) = \langle T, \partial_y \psi(x,z)\rangle = \langle T, \varphi(x,z)\rangle.$$

Integrating, we get

$$\left\langle T, \int_{t<z} \varphi(x,t)dt \right\rangle = G(z) = \int_{t<z} G'(t)dt = \int_{t<z} \langle T, \varphi(x,t)\rangle dt\ .$$

This gives the proposition, taking $z = A$ for example.

For $p > 1$, we proceed by induction on p. Let $\varphi \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R}^p)$. Using the result in the case $p = 1$, we get, for every $z' \in \mathbb{R}^{p-1}$,

$$\left\langle T, \int_{\mathbb{R}} \varphi(x,z',t)dt \right\rangle = \int_{\mathbb{R}} \langle T, \varphi(x,z',t)\rangle dt.$$

It remains to apply the induction assumption to $\tilde{\varphi} \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R}^{p-1})$ defined by

$$\tilde{\varphi}(x,z') = \int_{\mathbb{R}} \varphi(x,z',t)\,dt\ .$$

The proof is completed by using the Fubini theorem. □

Finally we come back to the proof of Proposition 2.3.16. Fix $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Applying Proposition 2.3.17 — derivation under the bracket— to

$$\varphi(x)F^\varepsilon(x) = \langle T, \varphi(x)\rho_\varepsilon(x - .)\rangle \ ,$$

we infer $\varphi F^\varepsilon \in \mathcal{C}^\infty(\Omega)$ and

$$\forall \alpha \in \mathbb{N}^d \ , \ \partial^\alpha(\varphi F^\varepsilon)(x) = \langle T, \partial_x^\alpha(\varphi(x)\rho_\varepsilon(x - .))\rangle \ .$$

Choosing φ a plateau function near a point $x_0 \in \Omega$, we get the first statement of the proposition,

$$\forall \alpha \in \mathbb{N}^d \ , \ \partial^\alpha F^\varepsilon(x_0) = \langle T, \partial^\alpha \rho_\varepsilon(x_0 - .))\rangle \ .$$

As for the second statement, we apply Proposition 2.3.20 — integration under the bracket — to obtain

$$\int_\Omega \varphi(x)F^\varepsilon(x)\, dx = \langle T, \varphi_\varepsilon\rangle \ , \ \varphi_\varepsilon(y) := \int_{\mathbb{R}^d} \varphi(x)\rho_\varepsilon(x - y)\, dx \ .$$

Notice that a simple change of variables $x = y + \varepsilon z$ provides

$$\varphi_\varepsilon(y) = \int_{\mathbb{R}^d} \varphi(y + \varepsilon z)\rho(z)\, dz \ .$$

Since ρ is supported in the unit ball, we observe that φ_ε is supported in a fixed compact subset of Ω is ε is small enough. Furthermore, the standard derivation under the integral yields $\varphi_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ and

$$\forall \alpha \in \mathbb{N}^d \ , \ \partial^\alpha \varphi_\varepsilon(y) = \int_{\mathbb{R}^d} \partial^\alpha \varphi(y + \varepsilon z)\rho(z)\, dz \ .$$

Passing to the limit as ε tends, to 0, $\partial^\alpha \varphi_\varepsilon$ converges uniformly to $\partial^\alpha \varphi$. We conclude that φ_ε converges to φ in $\mathcal{C}_0^\infty(\Omega)$. Consequently,

$$\langle T, \varphi_\varepsilon\rangle \underset{\varepsilon \to 0}{\longrightarrow} \langle T, \varphi\rangle \ .$$

The proof of Proposition 2.3.16 is complete. □

Corollary 2.3.21 Every distribution on Ω is the limit of a sequence of test functions in $\mathcal{D}'(\Omega)$.

Proof.— For every $j \geq 1$, define

$$K_j = \left\{x \in \Omega : d(x, \Omega^c) \geq \frac{1}{j} \ , \ |x| \leq j\right\} \ .$$

Then K_j is a compact subset pof Ω, $K_j \subset \mathring{K}_{j+1}$ and

$$\bigcup_{j \geq 1} K_j = \Omega \ .$$

Choose $\chi_j \in \mathcal{C}_0^\infty(\mathring{K}_{j+1})$ a plateau function near K_j and consider a sequence (ε_j) tending to 0 such that

$$(K_{j+1})_{\varepsilon_j} \subset \Omega \ .$$

Then define

$$F_j(x) = \chi_j(x)\langle T, \rho_{\varepsilon_j}(x - .)\rangle \ .$$

By Proposition 2.3.16 and by the above support information, we know that $F_j \in \mathcal{C}_0^\infty(\Omega)$. Furthermore, if $\varphi \in \mathcal{C}_0^\infty(\Omega)$, the support of φ is covered by a finite union of the open subsets $\mathring{K}_j$'s, hence is contained into some K_{j_0}. Then, for $j \geq j_0$,

$$\int_\omega \varphi(x) F_j(x)\, dx = \int_\Omega \varphi(x)\langle T, \rho_{\varepsilon_j}(x - .)\rangle\, dx \ ,$$

which converges to $\langle T, \varphi\rangle$ by Proposition 2.3.16. Therefore $F_j \to T$. □

Corollary 2.3.22 Let $T \in \mathcal{D}'(\Omega)$ such that, for every $j \in \{1, \dots, d\}$, $\partial_j T \in \mathcal{C}^0(\Omega)$. Then $T \in \mathcal{C}^1(\Omega)$.

Remark 2.3.23 The proof below is significantly more intricate than its analogue in dimension 1. Indeed, the analogue of the integral formula used in Corollary 1.4.9 in Chapter 1 is much more complicated in several space dimensions, so we prefer to proceed differently. Moreover, let us mention that, on the contrary to the one dimensional case, the assumptions $\partial_j T \in L^1_{loc}(\Omega), j = 1, \dots, d$, do not lead to $T \in \mathcal{C}^0(\Omega)$, but only to $T \in L^p_{loc}(\Omega)$ for $p = \frac{d}{d-1}$ (Sobolev estimates).

Let us prove Corollary 2.3.22. By the gluing principle, it is enough to prove it locally, so we may assume that Ω is a ball B. For every j, denote by $f_j \in \mathcal{C}^0(B)$ the continuous function defining $\partial_j T$. As above, we define, on the ball B^ε, the smooth function

$$F^\varepsilon(x) = \langle T, \rho_\varepsilon(x - .)\rangle \ .$$

Notice that

$$\partial_j F^\varepsilon(x) = \langle T, \partial_{x_j}\rho_\varepsilon(x - .)\rangle = -\langle T, \partial_{y_j}(\rho_\varepsilon(x - .))\rangle = \langle \partial_j T, \rho_\varepsilon(x - .)\rangle = \int_\Omega f_j(y)\rho_\varepsilon(x - y)\, dy \ .$$

Fix any ball $B' \subset\subset B$ and $\varepsilon_0 > 0$ such that $B' \subset B^{\varepsilon_0}$. Then the family $(\partial_j F^\varepsilon)_{\varepsilon < \varepsilon_0}$ converges uniformly to f_j on B' as ε tends to 0. Furthermore, picking $\chi \in \mathcal{C}_0^\infty(B')$ of integral 1, we have

$$F^\varepsilon(x) - \int_B \chi(y) F^\varepsilon(y)\, dy = \int_B \chi(y)(F^\varepsilon(x) - F^\varepsilon(y))\, dy = \sum_{j=1}^d \int_0^1 \int_B \chi(y)(x_j - y_j)\partial_j F^\varepsilon(y + t(x - y))\, dy\, dt,$$

and the right hand side converges uniformly on B'. On the other hand, by Proposition 2.3.16,

$$\int_B \chi(y) F^\varepsilon(y)\, dy \underset{\varepsilon \to 0}{\longrightarrow} \langle T, \chi\rangle \ ,$$

so we infer that F^ε converges uniformly on B' to some function F. Since each $\partial_j F^\varepsilon$ is uniformly convergent on B', we conclude that $F \in \mathcal{C}^1(B')$. But we also know by Proposition 2.3.16 that $T_{|B'} = F$. Since B' is arbitrary, this completes the proof. □

A very useful property of the convergence of distributions is the following lemma, which is a consequence of the principle of uniform boundedness which is proved at the end of this chapter.

Lemma 2.3.24 (Bicontinuity of the bracket) Let (T_n) be sequence in $\mathcal{D}'(\Omega)$ and (φ_n) be a sequence in $\mathcal{D}(\Omega)$. We assume that $T_n \to T$ in $\mathcal{D}'(\Omega)$ and $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$. Then

$$\langle T_n, \varphi_n \rangle \to \langle T, \varphi \rangle \ .$$

We close this subsection by a useful remark on the gluing principle for convergence of sequences of distributions.

Proposition 2.3.25 Let (T_n) be a sequence of distributions on Ω. Assume

$$\Omega = \bigcup_{j \in J} \Omega_j$$

for some collection $(\Omega_j)_{j\in J}$ of open subsets, and that, for every $j \in J$, $(T_n)_{|\Omega_j}$ converges to some $T^{(j)} \in \mathcal{D}'(\Omega_j)$. Then T_n is convergent in $\mathcal{D}'(\Omega)$.

Proof.— Let K be a compact subset of Ω. From Proposition 2.2.3 let $(\chi_j)_{j\in J}$ be a family of test functions, which are identically 0 except for a finite set of indices j, and such that

$$\text{supp}(\chi_j) \subset \Omega_j \ , \ \sum_{j \in J} \chi_j = 1 \text{ on } K \ .$$

Then, for every test function φ supported in K, we have

$$\langle T_n, \varphi \rangle = \sum_{j \in J} = \langle T_n, \chi_j \varphi \rangle \to \sum_{j \in J} \langle T^{(j)}, \chi_j \varphi \rangle \ .$$

Since each $T^{(j)}$ is a distribution, the right hand side is estimated by $C\|\varphi\|_{C^m}$ for some $C > 0$ and $m \in \mathbb{N}$ only depending on K. Hence T_n is convergent in $\mathcal{D}'(\Omega)$. □

2.4 Superficial measures and the jump formula

2.4.1 Motivation

In this section, we come back to the problem of identifying the partial derivatives of the characteristic function of an open set U. It is easy to check that these derivatives are distributions supported by the boundary ∂U of U. Under suitable assumptions on U, we shall show that these distributions are of order 0, and can be expressed in terms of a positive measure supported by ∂U, called the superficial measure of ∂U as an hypersurface. Thus we first need to define the superficial measure on an hypersurface, which is the purpose of the next subsection. An intuitive way to define a superficial measure of a subset of an hypersurface is to thicken this subset with a thickness ε, and to to take the limit, as ε tends to 0, of the ratio to ε of the Lebesgue measure of this thickened subset. The next construction tries to make rigorous this intuitive definition.

2.4.2 Preliminaries on non–negative distributions

Definition 2.4.1 We say that $T \in \mathcal{D}'(\Omega)$ is a non-negative distribution when $\langle T, \varphi \rangle \in \mathbb{R}^+$ for any function $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with values in $[0, +\infty)$.

Proposition 2.4.2 If $T \in \mathcal{D}'(\Omega)$ is non-negative, then it is of order 0.

Proof.— Let $K \subset I$ be a compact subset, and $\chi \in \mathcal{C}_0^\infty(I)$ a plateau function above K. For $\varphi \in \mathcal{C}_0^\infty(\Omega)$ supported in K with real values, one has

$$\forall x \in \Omega,\ -\chi \sup|\varphi| \leq \varphi(x) \leq \chi \sup|\varphi|,$$

thus

$$\langle T, \varphi + \chi \sup|\varphi| \rangle \geq 0 \text{ and } \langle T, \chi \sup|\varphi| - \varphi \rangle \geq 0.$$

This gives

$$|\langle T, \varphi \rangle| \leq \langle T, \chi \rangle \sup|\varphi|.$$

If $\varphi \in \mathcal{C}_0^\infty(\Omega)$ is supported in K and is complex valued, we write $\varphi = \varphi_1 + i\varphi_2$ with φ_1, φ_2 real-valued, and, with what we have seen before,

$$|\langle T, \varphi \rangle| = |\langle T, \varphi_1 + i\varphi_2 \rangle| \leq |\langle T, \varphi_1 \rangle| + |\langle T, \varphi_2 \rangle| \leq C \sup|\varphi_1| + C \sup|\varphi_2| \leq C \sup|\varphi|,$$

and this concludes the proof of the proposition. □

From Proposition 2.4.2 — and the fact that convenient regularisation processes conserve non-negative functions —, non-negative distributions therefore extends as non-negative linear forms on $\mathcal{C}_0^0(\Omega)$. By the Riesz representation theorem, non-negative linear forms on $\mathcal{C}_0^0(I)$ correspond to positive Borel measures on I which are finite on compact subsets. In other words,

Theorem 2.4.3 For every non-negative distribution T on I there exists such a unique Borel measure μ, finite on compact subsets, such that

$$\forall \varphi \in \mathcal{C}_0^0(\Omega)\ ,\ \langle T, \varphi \rangle = \int_\Omega \varphi \, d\mu\ .$$

2.4.3 The measure $\delta(f)$ and the superficial measure associated to $\{f = 0\}$.

Let Ω be open subset of $\mathbb{R}^d$, and let $f : \Omega \to \mathbb{R}$ be a $\mathcal{C}^1$ function. We assume the following property,

$$\forall x \in \Omega\ ,\ f(x) = 0 \Rightarrow \nabla f(x) \neq 0\ . \tag{2.4.1}$$

We denote by Σ the set of points $x \in \Omega$ such that $f(x) = 0$.

Theorem 2.4.4 There exists a positive distribution $\delta(f)$, supported by Σ, such that, for every compactly supported function $h \in \mathcal{C}^0(\mathbb{R})$ such that

$$\int_{\mathbb{R}} h(z)\, dz = 1 \ ,$$

the family of functions μ_ε defined on Ω by

$$\mu_\varepsilon(x) := \frac{1}{\varepsilon} h\left(\frac{f(x)}{\varepsilon}\right)$$

converges in $\mathcal{D}'(\Omega)$ to $\delta(f)$ as ε tends to 0. Furthermore, if $g : \Omega \to \mathbb{R}$ is another $\mathcal{C}^1$ function which satisfies (2.4.1) and $\Sigma = \{x \in \Omega, g(x) = 0\}$, then

$$|\nabla g|\, \delta(g) = |\nabla f|\, \delta(f) \ . \tag{2.4.2}$$

Proof.— By the gluing principle for convergence of distributions — Proposition 2.3.25—, it is enough to prove that every point a of Ω has an open neighborhood on which μ_ε has a limit in the sense of distributions. Moreover, the gluing principle shows that the positivity of this limit near each point a implies that this limit is a positive distribution on Ω — see the remark after Proposition 2.3.8.
Let $a \in \Omega$. If $f(a) \neq 0$, then the continuity of f implies that $|f| \geq c > 0$ on some neighborhood of a, so that, since h is compactly supported in $\mathbb{R}$, μ_ε is identically zero on this neighborhood if ε is small enough. Hence μ_ε tends to 0 in a neighborhood of a.
Now assume $f(a) = 0$. By assumption (2.4.1), $\nabla f(a) \neq 0$, so there exists $j \in \{1, \dots, d\}$ such that $\partial_j f(a) \neq 0$. Let us assume for instance that $j = 1$, and write elements of $\mathbb{R}^d$ as $x = (x_1, y)$, with $x_1 \in \mathbb{R}$ and $y \in \mathbb{R}^{d-1}$. Then, by the implicit function theorem, there exists an open interval $I \subset \mathbb{R}$ and an open subset $W \subset \mathbb{R}^{d-1}$ such that $a = (a_1, b) \in I \times W \subset \Omega$, and a $\mathcal{C}^1$ function $q : W \to I$ such that

$$\forall (x_1, y) \in I \times W \ , \ f(x_1, y) = 0 \iff x_1 = q(y) \ .$$

By the Taylor formula in the x_1 variable, we can write

$$\forall (x_1, y) \in I \times W \ , \ f(x_1, y) = f(q(y), y) + (x_1 - q(y)) \int_0^1 \partial_1 f(q(y) + t(x_1 - q(y), y)\, dt = m(x_1, y)(x_1 - q(y)) \ ,$$

with $m_1(q(y), y) = \partial_1 f(q(y), y) \neq 0$. Given $\varphi \in \mathcal{C}_0^0(I \times W)$, we compute

$$\int_{I \times W} \varphi(x)\, \mu_\varepsilon(x)\, dx = \int_W \int_I \varphi(x_1, y)\, \frac{1}{\varepsilon} h\left(\frac{m(x_1, y)(x_1 - q(y))}{\varepsilon}\right)\, dx_1\, dy \ .$$

Extending φ by 0 to $\mathbb{R} \times W$, we can write the inner integral as

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(x_1, y)\, \frac{1}{\varepsilon} h\left(\frac{m(x_1, y)(x_1 - q(y))}{\varepsilon}\right)\, dx_1 &= \int_{\mathbb{R}} \varphi(q(y) + \varepsilon z, y)\, h(m(q(y) + \varepsilon z, y) z)\, dz \\
&\to \left(\int_{\mathbb{R}} h(m(q(y), y) z)\, dz\right) \varphi(q(y), y) = \frac{\varphi(q(y), y)}{|m(q(y), y)|} \\
&= \frac{\varphi(q(y), y)}{|\partial_1 f(q(y), y)|} \ .
\end{aligned}$$

Passing to the uniform limit under the integral in the y variable, we infer

$$\int_{I\times W}\varphi(x)\,\mu_\varepsilon(x)\,dx \to \int_W \frac{\varphi(q(y),y)}{|\partial_1 f(q(y),y)|}\,dy\ .$$

Furthermore, notice that the denominator of the above integrand can be rewritten in terms of $|\nabla f|$ as follows. Taking the derivative with respect to $y\in W$ of the equation $f(q(y),y)=0$, we obtain, by the chain rule,

$$\nabla_y f(q(y),y)+\partial_1 f(q(y),y)\nabla q(y)=0\ ,$$

so that

$$\nabla f(q(y),y)=\partial_1 f(q(y),y)\begin{pmatrix}1\\ -\nabla q(y)\end{pmatrix}\ ,\ |\nabla f(q(y),y)|=|\partial_1 f(q(y),y)|\left(1+|\nabla q(y)|^2\right)^{\frac12}\ .$$

Finally, we obtain

$$\int_{I\times W}\varphi(x)\,\mu_\varepsilon(x)\,dx\to\langle\delta(f),\varphi\rangle=\int_W\varphi(q(y),y)\frac{\left(1+|\nabla q(y)|^2\right)^{\frac12}}{|\nabla f(q(y),y)|}\,dy\ .$$

Thus on W we get a positive distribution $\delta(f)$, hence a distribution of order 0, so we can multiply it by the continuous function $|\nabla f|$, and obtain, if φ is supported into $I\times W$,

$$\langle|\nabla f|\,\delta(f),\varphi\rangle=\int_W\varphi(q(y),y)\left(1+|\nabla q(y)|^2\right)^{\frac12}\,dy\ .$$

If g is another $\mathcal{C}^1$ function which satisfies (2.4.1) and $\Sigma=\{x\in\Omega, g(x)=0\}$, we take the derivative of $g(q(y),y)=0$, and obtain

$$\nabla g(q(y),y)=\partial_1 g(q(y),y)\begin{pmatrix}1\\ -\nabla q(y)\end{pmatrix}\ ,\ |\nabla g(q(y),y)|=|\partial_1 g(q(y),y)|\left(1+|\nabla q(y)|^2\right)^{\frac12}\ ,$$

so that the same computation as above yields

$$\langle|\nabla g|\,\delta(g),\varphi\rangle=\int_W\varphi(q(y),y)\left(1+|\nabla q(y)|^2\right)^{\frac12}\,dy\ .$$

In particular, $|\nabla g|\,\delta(g)=|\nabla f|\,\delta(f)$ in $I\times W$, and, by the gluing principle,

$$|\nabla g|\,\delta(g)=|\nabla f|\,\delta(f)$$

in Ω. □

From this proposition, we infer that the positive distribution $|\nabla f|\,\delta(f)$ does not depend on the choice of the function f satisfying (2.4.1) and such that $\Sigma=\{f=0\}$. Therefore this positive distribution is intrinsically associated to Σ.

Definition 2.4.5 The superficial measure of Σ is the positive distribution

$$\sigma=|\nabla f|\,\delta(f)\ ,$$

where f is any $\mathcal{C}^1$ function satisfying (2.4.1) and such that $\Sigma=\{f=0\}$.

Let us retain from the above definition and from the proof of Theorem 2.4.4 the following two important facts about the superficial measure on Σ.

i) If $\Sigma = \{f = 0\}$ where f satisfies (2.4.1), then, for every $\varphi \in \mathcal{C}_0^0(\Omega)$,

$$\int_\Omega \frac{1}{\varepsilon} h\left(\frac{f(x)}{\varepsilon}\right) \varphi(x)\, dx \to \int_\Omega \varphi(x)\, \frac{d\sigma(x)}{|\nabla f(x)|} \ .$$

ii) If $\Sigma \cap (I \times W) = \{(q(y), y), y \in W\}$ where $q : W \to I$ is a $\mathcal{C}^1$ function, then, for every continuous function φ supported into $I \times W$,

$$\int_{I\times W} \varphi(x)\, d\sigma(x) = \int_W \varphi(q(y), y) \left(1 + |\nabla q(y)|^2\right)^{\frac{1}{2}} \, dy \ . \tag{2.4.3}$$

Example 2.4.6 (The superficial measure on a sphere) Let $r > 0$. Then the function

$$\begin{array}{rcl} f_r : \mathbb{R}^d \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & |x| - r \end{array}$$

is $\mathcal{C}^1$ and $f_r(x) = 0$ is the equation of the sphere S_r of radius r centered at 0. Notice that

$$\nabla f_r(x) = \frac{x}{|x|} \ ,$$

hence $|\nabla f_r(x)| = 1$. For instance, the intersection of S_r with the open set $]0, +\infty[\times \mathbb{R}^{d-1}$ is given by the equation $x_1 = q(y)$, where

$$q(y) = \sqrt{r^2 - |y|^2} \ , \ |y| < r \ .$$

Notice that

$$\nabla q(y) = \frac{-y}{\sqrt{r^2 - |y|^2}} \ , \ 1 + |\nabla q(y)|^2 = \frac{r^2}{r^2 - |y|^2} \ ,$$

so that the superficial measure σ_r on S_r is given by $\sigma_r = \delta(f_r)$, and, if φ is compactly supported in $]0, +\infty[\times \mathbb{R}^{d-1}$,

$$\int_{]0,+\infty[\times \mathbb{R}^{d-1}} \varphi(x)\, d\sigma(x) = \int_{|y|<r} \varphi\left(\sqrt{r^2 - |y|^2}, y\right) \, \frac{r dy}{\sqrt{r^2 - |y|^2}} \ .$$

Notice that we have a simple connection between σ_r and σ_1,

$$\forall \varphi \in \mathcal{C}_0^0(\mathbb{R}^d) \ , \ \int_{\mathbb{R}^d} \varphi(x)\, d\sigma_r(x) = r^{d-1} \int_{\mathbb{R}^d} \varphi(ry)\, d\sigma_1(y) \ . \tag{2.4.4}$$

Indeed, using the definition,

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)\, d\sigma_r(x) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} h\left(\frac{|x| - r}{\varepsilon}\right) \varphi(x)\, dx \\ &= \lim_{\varepsilon\to 0} \frac{r}{\varepsilon} r^{d-1} \int_{\mathbb{R}^d} h\left(\frac{r(|y| - 1)}{\varepsilon}\right) \varphi(ry)\, dy \\ &= r^{d-1} \int_{\mathbb{R}^d} \varphi(ry)\, d\sigma_1(y) \ , \end{aligned}$$

using the change of variables $x = ry$ in $\mathbb{R}^d$ and the new small parameter $\varepsilon' = \frac{\varepsilon}{r}$.

Finally, it is easy to check that, if $d = 2$,

$$\int_{\mathbb{R}^2} \varphi(x)\, d\sigma_r(x) = r \int_0^{2\pi} \varphi(r \cos\theta, r \sin\theta)\, d\theta \ .$$

2.4.4 Integration on level sets : the smooth coarea formula

The above definition of the superficial measure easily leads to an important formula of integral calculus. Let $f = \Omega \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

(2.4.5) $$\forall x \in \Omega \ , \ \nabla f(x) \neq 0 \ .$$

It is easy to prove that the set $f(\Omega) := \{f(x), x \in \Omega\}$ is an open subset of $\mathbb{R}$ – hence is an open interval if Ω is connected. Indeed, if $t_0 = f(a)$ with $a \in \Omega$, and if, say, $\partial_1 f(a) \neq 0$, then, on a small neighborhood $I \times W$ of $a = (a_1, b)$, we have $\partial_1 f \neq 0$, hence the function $x_1 \in I \mapsto f(x_1, b)$ is strictly monotone, and consequently its range is an open interval of $\mathbb{R}$.
For every $t \in f(\Omega)$, denote by Σ_t the set of equation $f(x) = t$, and by σ_t the superficial measure on Σ_t.

Proposition 2.4.7 For every $\varphi \in \mathcal{C}_0^0(\Omega)$,

$$\int_\Omega \varphi(x)\, dx = \int_{f(\Omega)} \left(\int_{\Sigma_t} \varphi(x)\, \frac{d\sigma_t(x)}{|\nabla f(x)|} \right) dt \ .$$

Proof.— Fix $h \in \mathcal{C}_0^0(\mathbb{R})$ of integral 1. For every $x \in \Omega$, we observe that, for every $\varepsilon > 0$,

$$1 = \frac{1}{\varepsilon} \int_{\mathbb{R}} h\left(\frac{f(x) - t}{\varepsilon} \right) dt \ .$$

We plug this identity in the x integral and apply the Fubini theorem,

$$\int_\Omega \varphi(x)\, dx = \int_{\mathbb{R}} \left(\int_\Omega \frac{1}{\varepsilon} h\left(\frac{f(x) - t}{\varepsilon} \right) \varphi(x)\, dx \right) dt \ .$$

We now pass to the limit inside the inner integral as ε tends to 0. We obtain

$$I_\varepsilon(t) = \int_\Omega \frac{1}{\varepsilon} h\left(\frac{f(x) - t}{\varepsilon} \right) \varphi(x)\, dx \to \begin{cases} 0 & \text{if } t \notin f(\Omega) \\ \int_{\Sigma_t} \varphi(x)\, \frac{d\sigma_t(x)}{|\nabla f(x)|} & \text{if } t \in f(\Omega) \ . \end{cases}$$

Then the formula follows from the interversion of the integral in t and of the limit as ε tends to 0. In order to justify this interversion, we apply the dominated convergence theorem. First we notice that, because φ and h are compactly supported, $I_\varepsilon(t)$ is supported into a compact subset of $\mathbb{R}$. So we just have to prove that $I_\varepsilon(t)$ is uniformly bounded. By the gluing principle — partitions of unity again —, it is enough to assume that the test function φ is supported in a small open subset V of Ω. Assuming that $\partial_1 f \neq 0$ on V, the implicit function theorem shows that, if $V = I \times W$ is small enough, the equation $f(x_1, y) = t$ in V, for t close to some value t_0, is equivalent to $x_1 = q(y, t)$, where q is a $\mathcal{C}^1$ function. Then we can reproduce the proof of Theorem 2.4.4 with the additional parameter t, writing

$$\begin{aligned} I_e(t) &= \int_W \int_I \frac{1}{\varepsilon} h\left(\frac{m(x_1, y, t)(x_1 - q(y, t))}{\varepsilon} \right) \varphi(x_1, y)\, dy\, dx_1 \\ &= \int_W \int_{\mathbb{R}} h(m(q(y, t) + \varepsilon z, y) z)\, \varphi(q(y, t) + \varepsilon z, y)\, dz\, dy \ , \end{aligned}$$

which is uniformly bounded since φ is compactly supported, and $|m| \geq c > 0$. This completes the proof. □

Corollary 2.4.8 (Integration on spherical level sets) For every $\varphi \in \mathcal{C}_0^0(\mathbb{R}^d \setminus \{0\})$, we have

$$\int_{\mathbb{R}^d \setminus \{0\}} \varphi(x)\, dx = \int_0^\infty \int_{S_1} \varphi(r\omega)\, d\sigma_1(\omega)\, r^{d-1} dr \ .$$

Proof.— Combine Proposition 2.4.7 applied to $f : x \in \mathbb{R}^d \setminus \{0\} \mapsto |x|$ with the change of variable formula (2.4.4). □

2.4.5 Superficial measure on a closed hypersurface and the jump formula for a regular open subset

We first need a number of definitions.

Definition 2.4.9 Let Ω be an open subset of $\mathbb{R}^d$. A closed $\mathcal{C}^1$ hypersurface of Ω is a closed subset $\Sigma \subset \Omega$ with the following property : every $a \in \Sigma$ has an open neighborhood V in Ω such that there exists a $\mathcal{C}^1$ function $f : V \to \mathbb{R}$ with

$$\Sigma \cap V = \{x \in V, f(x) = 0\} \ , \ \forall x \in \Sigma \cap V, \nabla f(x) \neq 0 \ .$$

Essentially this definition is the one we used in the previous subsection, except that one does not impose that the existence of a *global* equation f for Σ. This naturally leads to the following

Proposition 2.4.10 Let Σ be a closed $\mathcal{C}^1$ hypersurface of Ω. There exists a unique positive distribution σ supported in Σ such that, for every open subset $V \subset \Omega$ anf $f \in \mathcal{C}^1(V, \mathbb{R})$ such that

$$\Sigma \cap V = \{x \in V, f(x) = 0\} \ , \ \forall x \in \Sigma \cap V, \nabla f(x) \neq 0 \ ,$$

we have, in V,

$$|\nabla f|\, \delta(f) = \sigma \ .$$

Furthermore, $\operatorname{supp} \sigma = \Sigma$.

Proof.— Existence and uniqueness of σ follow from Theorem 2.4.4 and from the gluing principle. Indeed, given $a \in \Sigma$, consider $V_a \subset \Omega$ and $f_a : V_a \to \mathbb{R}$ as in the definition above. We define $\sigma_a \in \mathcal{D}'(V_a)$ as

$$\sigma_a = |\nabla f_a|\, \delta(f_a) \ .$$

Moreover, we consider $\tilde{\sigma} = 0$ in $\mathcal{D}'(\Omega \setminus \Sigma)$. By Theorem 2.4.4 and the gluing principle applied to the open covering

$$\Omega = (\Omega \setminus \Sigma) \cup \bigcup_{a \in \Sigma} V_a \ ,$$

we infer the existence of $\sigma \in \mathcal{D}'(\Omega)$. Uniqueness also follows from the gluing principle and Theorem 2.4.4. The last statement follows from the local expression of σ given in (2.4.3). Notice that the closedness of Σ is crucial in this proof. □

Definition 2.4.11 Measure σ is called the superficial measure on the closed hypersurface Σ.

We now come to the important notion of a regular open set.

Definition 2.4.12 Let U be an open subset of Ω. We say that U is a regular open subset of Ω of class $\mathcal{C}^1$ if every point $a \in \partial U$ has an open neighborhood V in Ω such that there exists a $\mathcal{C}^1$ function $f : V \to \mathbb{R}$ with

$$U \cap V = \{x \in V, f(x) > 0\} \ , \ \forall x \in \partial U \cap V, \nabla f(x) \neq 0 \ .$$

In such a situation, it easy to check that ∂U is a closed $\mathcal{C}^1$ hypersurface of Ω. Indeed, with the notation of the above definition, one verifies that

$$\partial U \cap V = \{x \in V, f(x) = 0\} \ .$$

However, a regular open set is not only an open set whose boundary is a closed hypersurface. The above definition also imposes that the open subset U is locally *on the same side* of this hypersurface ! For instance $U = \mathbb{R}^d \setminus \{x_1 = 0\}$ is not a regular subset of $\mathbb{R}^d$, though its boundary is the closed hypersurface $\{x_1 = 0\}$.

Definition 2.4.13 Let Ω be an open subset of $\mathbb{R}^d$.

- If Σ is a closed $\mathcal{C}^1$ hypersurface of Ω and $a \in \Sigma$, the normal line to Σ at a is the line $\mathbb{R}\nabla f(a)$, where f is any function as in Definition 2.4.9.
- If U is a regular open subset of Ω of class $\mathcal{C}^1$, the inward unit normal vector to U at $a \in \partial U$ is the unit vector
$$N^{\text{int}}(a) = \frac{\nabla f(a)}{|\nabla f(a)|} \ ,$$
where f is any function as in Definition 2.4.12. The outward unit normal vector to U at $a \in \partial U$ is
$$N^{\text{ext}}(a) = -N^{\text{int}}(a) \ .$$

These definitions make sense because of the independence of the objects with respect to function f. In the case of a closed hypersurface, we already checked, in the proof of Theorem 2.4.4, that the vectors $\nabla f(a)$ are all proportional, so they define the same line. In the case of a regular open subset, there exist only two unit vectors on the normal line to ∂U at $a \in \partial U$, which can be identified as follows. For $\varepsilon > 0$ small enough, $a + \varepsilon N^{\text{int}}(a) \in U$, while $a + \varepsilon N^{\text{ext}}(a) \in (\overline{U})^c$.
Finally, notice that the mapping $N^{\text{int}} : \partial U \to \mathbb{R}^d$ is continuous,. We call it the *inward unit normal vector field to* U. A similar definition holds for the *outward unit normal vector field to* U.

Theorem 2.4.14 (The jump formula) Let U be a regular open subset of Ω of class $\mathcal{C}^1$. Denote by σ the superficial measure on the closed hypersurface ∂U, and by $N^{\text{int}} : \partial U \to \mathbb{R}^d$ the inward unit normal vector field to U. Then, in $\mathcal{D}'(\Omega)$,

$$\forall j \in \{1, \dots, d\} \ , \ \partial_j(\mathbf{1}_U) = N_j^{\text{int}} \sigma \ .$$

Proof.— By the gluing principle, it is enough to prove this identity near every point $a \in \Omega$. If $a \notin \partial U$, then $\mathbf{1}_U$ is a constant function in a neighborhood of a, so that $\partial_j(\mathbf{1}_U) = 0$ in this neighborhood, which is also the case of σ.
Assume $a \in \partial U$, and let $f : V \to \mathbb{R}$ be a function near a as in Definition 2.4.12. Denote by $\chi : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^1$ function such that

$$\chi(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1 & \text{if } z \geq 1 \ . \end{cases}$$

Then it is easy to check that

$$\forall x \in V \ , \ \mathbf{1}_U(x) = \lim_{\varepsilon \to 0} \chi\left(\frac{f(x)}{\varepsilon}\right) \ .$$

Furthermore, since the right hand side is uniformly bounded, the dominated convergence theorem implies that this convergence also holds in $\mathcal{D}'(V)$. Therefore, in $\mathcal{D}'(V)$, we have

$$\partial_j(\mathbf{1}_U) = \lim_{\varepsilon \to 0} \partial_j \left(\chi\left(\frac{f(x)}{\varepsilon}\right)\right) = \lim_{\varepsilon \to 0} \frac{\partial_j f}{\varepsilon} \chi'\left(\frac{f(x)}{\varepsilon}\right) \ .$$

The function $h = \chi'$ belongs to $\mathcal{C}_0^0(\mathbb{R})$ and satisfies

$$\int_{\mathbb{R}} h(z)\, dz = \int_0^\infty \chi'(z)\, dz = 1 \ .$$

Consequently, for every $\varphi \in \mathcal{C}_0^0(V)$,

$$\int_\Omega \varphi(x) \frac{1}{\varepsilon} \chi'\left(\frac{f(x)}{\varepsilon}\right) dx \to \langle \delta(f), \varphi \rangle \ ,$$

so that

$$\partial_j(\mathbf{1}_U) = \partial_j f \delta(f) = \frac{\partial_j f}{|\nabla f|} \sigma = N_j^{\text{int}} \sigma \ .$$

□

Corollary 2.4.15 (The Gauss–Green formula) Let U be regular open subset of class $\mathcal{C}^1$ in Ω. For every $\varphi \in \mathcal{C}^1_0(\Omega)$,

$$\forall j \in \{1, \dots, d\} \ , \ \int_U \partial_j \varphi(x)\, dx = \int_{\partial U} N_j^{\text{ext}}(x)\varphi(x)\, d\sigma(x) \ .$$

Proof.— First assume $\varphi \in \mathcal{C}^\infty_0(\Omega)$. In $\mathcal{D}'(\Omega)$, we apply the jump formula and get

$$\int_U \partial_j \varphi(x)\, dx = -\langle \partial_j(\mathbf{1}_U), \varphi\rangle = -\langle N_j^{\text{int}}\sigma, \varphi\rangle = \int_{\partial U} N_j^{\text{ext}}(x)\varphi(x)\, d\sigma(x) \ .$$

The general case $\varphi \in \mathcal{C}^1_0(\Omega)$ follows by approximation. □ Let us come to the special case where U is a bounded regular open subset $\mathbb{R}^d$. In this case, we denote by $\mathcal{C}^1(\overline{U})$ the space of restrictions to U of $\mathcal{C}^1$ functions on $\mathbb{R}^d$.

Corollary 2.4.16 (The Gauss–Green formula, compact form) Let U be a regular open subset of class $\mathcal{C}^1$ in $\mathbb{R}^d$ such that $\overline{U}$ is compact . For every $\varphi \in \mathcal{C}^1(\overline{U})$,

$$\forall j \in \{1, \dots, d\} \ , \ \int_U \partial_j \varphi(x)\, dx = \int_{\partial U} N_j^{\text{ext}}(x)\varphi(x)\, d\sigma(x) \ .$$

Proof.—
If $\varphi \in \mathcal{C}^1(\overline{U})$, let $\tilde{\varphi} \in \mathcal{C}^1(\tilde{U})$ be an extension of φ to an open neighborhood $\tilde{U}$ of $\overline{U}$. Let χ be a plateau function on $\overline{U}$ compactly supported in $\tilde{U}$. The first formula follows from applying the Gauss–Green formula to $\chi\tilde{\varphi} \in \mathcal{C}^1_0(\mathbb{R}^d)$. □

Remark 2.4.17 Both corollaries above combined with the Leibniz formula imply the following multidimensional integration by parts formula.

$$\forall j \in \{1, \dots, d\} \ , \ \int_U \partial_j \varphi(x)\, \psi(x)\, dx = \int_{\partial U} N_j^{\text{ext}}(x)\varphi(x)\, \psi(x)\, d\sigma(x) - \int_U \varphi(x)\partial_j\psi(x)\, dx \ .$$

2.5 Sobolev spaces in an open set

We recall that, from now on, we identify a locally integrable function f with the associated distribution T_f.

2.5.1 Definition and general facts

Definition 2.5.1 Let $\Omega \subset \mathbb{R}^d$ be an open set, and $s \in \mathbb{N}$. A distribution $u \in \mathcal{D}'(\Omega)$ belongs to $H^s(\Omega)$ if for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \le s$, it holds that $\partial^\alpha u \in L^2(\Omega)$. We denote by $(\cdot|\cdot)_{H^s}$ the sesquilinear form defined on $H^s(\Omega) \times H^s(\Omega)$ by

$$(u|v)_{H^s} = \sum_{|\alpha| \le s} (\partial^\alpha u | \partial^\alpha v)_{L^2} \ .$$

We also introduce the associated H^s norm

$$\|u\|_{H^s} = (u|u)^{\frac{1}{2}} = \left(\sum_{|\alpha| \leq s} \|\partial^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}},$$

so that convergence of u_n to u in H^s is equivalent to the convergence of $\partial^\alpha u_n$ to $\partial^\alpha u$ in L^2 for every $|\alpha| \leq s$.

Notice that $H^0(\Omega) = L^2(\Omega)$. The structure of Hilbert space of $L^2(\Omega)$ is transferred to $H^s(\Omega)$, as shown by the next proposition.

Proposition 2.5.2 The sesquilinear form $(\cdot, \cdot)_{H^s}$ is a Hermitian scalar product, which makes $H^s(\Omega)$ a Hilbert space.

Proof.— The only non trivial fact is completeness of $H^s(\Omega)$ with respect to the H^s norm. Let (u_j) be a Cauchy sequence in $H^s(\Omega)$. For all $|\alpha| \leq s$, the sequence $(\partial^\alpha u_j)$ is Cauchy in L^2, thus converges to a $v_\alpha \in L^2$. In particular, $u_j \to v_0$ in $\mathcal{D}'(\Omega)$, so that $\partial^\alpha u_j \to \partial^\alpha v_0 = v_\alpha \in L^2(\Omega)$, and $(u_j) \to v_0$ in $H^s(\Omega)$. □

Notice that, for $s \geq 1$, $u \in H^s(\Omega)$ if and only if $u \in H^1(\Omega)$ and, for every $j = 1, \dots, d-1$, $\partial_j u \in H^{s-1}(\Omega)$. By induction on s, this reduces many properties of $H^s(\Omega)$ to the special case $s = 1$, on which we are going to focus in what follows.

2.5.2 Variational formulation of some elliptic problems

Definition 2.5.3 Given an arbitrary open subset Ω of $\mathbb{R}^d$, we denote by $H^1_0(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^1(\Omega)$.

Note that $H^1_0(\Omega)$ is a closed subspace of the Hilbert space $H^1(\Omega)$, hence it si a Hilbert space with the inner product $(\,.|\,.)_{H^1}$ defined in athe previous section.

In what follows, we are going to use the Laplacian differential operator, defined on $\mathcal{D}'(\Omega)$ by

$$\Delta T = \sum_{j=1}^{d} \partial_j^2 T\ .$$

Theorem 2.5.4 Let Ω be an arbitrary open subset of $\mathbb{R}^d$. For every $f \in L^2(\Omega)$, there exists $u \in H_0^1(\Omega)$ unique such that

$$-\Delta u + u = f \tag{2.5.6}$$

in $\mathcal{D}'(\Omega)$.

Proof.— Equation (2.5.6) in $\mathcal{D}'(\Omega)$ exactly means

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega)\ ,\ \langle -\Delta u + u, \varphi\rangle = \int_\Omega f\varphi\, dx\ .$$

Since we are looking for $u \in H_0^1(\Omega)$, we know in particular that $u \in L^2(\Omega)$ and $\partial_j u \in L^2(\Omega)$ for $j = 1, \dots, d$. Therefore the left hand side reads

$$\langle -\Delta u + u, \varphi\rangle = \sum_{j=1}^d \int_\Omega \partial_j u \partial_j \varphi\, dx + \int_\Omega u\varphi\, dx = (\varphi|\overline{u})_{H^1}\ .$$

Summing up, we are looking for some $u \in H_0^1(\Omega)$ such that

$$\forall \varphi \in \mathcal{C}_0^\infty(\Omega)\ ,\ (\varphi|\overline{u})_{H^1} = \int_\Omega f\varphi\, dx\ .$$

Since both sides of the above equation are linear forms of φ which are continuous for the H^1 norm, the problem is equivalent to finding $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega)\ ,\ (v|\overline{u}) = \int_\Omega fv\, dx\ .$$

By the Riesz representation theorem, the continuous linear form

$$v \in H_0^1(\Omega) \mapsto \int_\Omega fv\, dx$$

can be represented by the inner product with a unique element $w \in H_0^1(\Omega)$. Therefore the problem is solved by setting $u = \overline{w}$ — notice that the conjugate of a function in $H_0^1(\Omega)$ is still in $H_0^1(\Omega)$. □

Let us complete this subsection by some remarks about the possible extensions of Theorem 2.5.4

Remark 2.5.5 *i)* The L^2 function f can be replaced more generally by any distribution which extends as a continuous linear form on $H_0^1(\Omega)$. The space of such distributions is denoted by $H^{-1}(\Omega)$.

ii) If $q \in L^\infty(\Omega)$ and there exists $m > 0$ such that

$$q(x) \geq m$$

almost everywhere on Ω, the sesquilinear form

$$(u|v)_q = \sum_{j=1}^{d} (\partial_j u|\partial_j v)_{L^2} + \int_\Omega q u\overline{v}\, dx \tag{2.5.7}$$

is an inner product on $H^1(\Omega)$, inducing a norm which is equivalent to the H^1 norm. Therefore $H^1(\Omega)$ and $H^1_0(\Omega)$ are Hilbert spaces for this new inner product, and one can solve similarly the equation

$$-\Delta u + qu = f \tag{2.5.8}$$

in $\mathcal{D}'(\Omega)$, for every $f \in H^{-1}(\Omega)$.

iii) The above assumption on q can be relaxed for special cases of open sets Ω, in particular if Ω is bounded.

Lemma 2.5.6 Let Ω be a bounded open subset of $\mathbb{R}^d$. For every $j \in \{1, \dots, d\}$, there exists $C > 0$ such that, for every $\varphi \in H^1_0(\Omega)$,

$$\|\varphi\|_{L^2} \le C\|\partial_j \varphi\|_{L^2} \ .$$

By density, it is enough to prove this inequality for $\varphi \in \mathcal{C}^\infty_0(\Omega)$. Let us prove Lemma 2.5.6 with $j = 1$, say. We write $x = (x_1, y) \in \mathbb{R} \times \mathbb{R}^d$ and

$$|\varphi(x_1, y)|^2 = -\int_{x_1}^{\infty} \partial_t(|\varphi(t, y)|^2)\, dt = -2\int_{x_1}^{\infty} \mathrm{Re}(\partial_t\varphi(t, y)\overline{\varphi}(t, y))\, dt \ .$$

Using the Cauchy–Schwarz inequality, we infer

$$|\varphi(x_1, y)|^2 \le 2\left(\int_{\mathbb{R}} |\partial_t\varphi(t, y)|^2\, dt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\varphi(t, y)|^2\, dt\right)^{\frac{1}{2}} \ .$$

Assuming $\Omega \subset]a, a + L[\times\mathbb{R}^{d-1}$, we conclude, again by Cauchy–Schwarz, that

$$\int_{\mathbb{R}\times\mathbb{R}^{d-1}} |\varphi(x_1, y)|^2\, dx_1\, dy \le 2L\, \|\partial_t\varphi\|_{L^2}\|\, \varphi\|_{L^2} \ .$$

This completes the proof of the lemma, with $C = 2L$.

In view of Lemma 2.5.6, if Ω is bounded, the sesquilinear form $(\ .|\ .)_q$ defined in (2.5.7) is still an inner product on $H^1_0(\Omega)$, with a norm equivalent to the H^1 norm, if $q \in L^\infty(\Omega)$ is just non–negative — including q identically 0. Hence, under this more general assumption, it still possible to solve equation (2.5.8) on Ω bounded. In particular, if Ω is bounded, for every $f \in H^{-1}(\Omega)$, there exists $u \in H^1_0(\Omega)$ unique such that

$$-\Delta u = f$$

in $\mathcal{D}'(\Omega)$. We will come back to this equation at the end of this paragraph, in the particular case where Ω is regular. We will see that, in this case, the subspace $H^1_0(\Omega)$ of $H^1\Omega)$ can be described in a more explicit way.

2.5.3 Approximation by smooth functions

Let us start with the special case $\Omega = \mathbb{R}^d$.

Proposition 2.5.7 $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense into $H^1(\mathbb{R}^d)$. In other words, $H_0^1(\mathbb{R}^d) = H^1(\mathbb{R}^d)$.

Proof.— We proceed by cut off and regularisation. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x) = 1$ on the unit ball. For $n \geq 1$, set

$$\chi_n(x) = \chi\left(\frac{x}{n}\right) .$$

Given $v \in L^2(\mathbb{R}^d)$, we already know that $\chi_n v$ tends to v in L^2 as $n \to \infty$. Let $u \in H^1(\mathbb{R}^d)$. Let us prove that $\chi_n u \to u$ in H^1 as $n \to \infty$. It is equivalent to establish that $\chi_n u \to u$ in L^2, which is laready known, and that for every $j = 1, \dots, d$, $\partial_j(\chi_n u) \to \partial_j u$. From the Leibniz formula for distributions,

$$\partial_j(\chi_n u) = \chi_n \partial_j u + (\partial_j \chi_n) u ,$$

so we just have to prove that $(\partial_j \chi_n) u \to 0$ in L^2. This follows from $u \in L^2$ and

$$\|\partial_j \chi_n\|_\infty = O\left(\frac{1}{n}\right) .$$

As a second step, we prove that, if $u \in H^1(\mathbb{R}^d)$ is compactly supported, then u can be approximated by a sequence of compactly supported smooth functions. Let $\rho \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, supported in the unit ball B, and such that

$$\int_{\mathbb{R}^d} \rho(z)\, dz = 1 .$$

Consider, for $\varepsilon > 0$,

$$u_\varepsilon(x) = \rho_\varepsilon * u(x) = \int_{\mathbb{R}^d} \frac{1}{\varepsilon^d} \rho\left(\frac{x-y}{\varepsilon}\right) u(y)\, dy .$$

Then $u_\varepsilon \in C_0^\infty(\mathbb{R}^d)$, with supp$(u_\varepsilon) \subset$ supp$(u) + \varepsilon B$. We already know that $u_\varepsilon \to u$ in L^2 as $\varepsilon \to 0$. Furthermore, for $j = 1, \dots, d$,

$$\partial_j u_\varepsilon(x) = \int_{\mathbb{R}^d} \partial_j \rho_\varepsilon(x-y)\, u(y)\, dy = -\int_{\mathbb{R}^d} \partial_{y_j} [\rho_\varepsilon(x-y)]\, u(y)\, dy = \rho_\varepsilon * \partial_j u(x) .$$

Therefore $\partial_j u_\varepsilon \to \partial_j u$ as $\varepsilon \to 0$. Summing up, $u_\varepsilon \to u$ in H^1 as $\varepsilon \to 0$, and the proof is complete. □ Notice that the second step of the above proof implies the following useful result.

Proposition 2.5.8 Every compactly supported element u of $H^1(\Omega)$ is the limit of a sequence of $\mathcal{C}_0^\infty(\Omega)$ supported in an arbitrarily small neighbourhood of supp(u).

Next we deal with the main result of this subsection, which concerns the special case of a bounded regular open subset of $\mathbb{R}^d$.

Proposition 2.5.9 Let Ω be a bounded regular open subset of $\mathbb{R}^d$ with a $\mathcal{C}^\infty$ boundary. Then $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$.

Proof.— Given $a \in \partial\Omega$, there exists an open neighbourhood V_a of a such that $\Omega \cap V_a$ is the strict epigraph of a smooth function. Since the V_a cover the compact subset $\partial\Omega$ of : R^d, we may extract a finite covering

$$\partial\Omega \subset \bigcup_{k=1}^{N} V_k$$

and we consider a partition of unity $\chi_0, \dots, \chi_N$ associated to the covering

$$\overline{\Omega} \subset \Omega \cup \bigcup_{k=1}^{N} V_k \ ,$$

so that $\chi_0 \in \mathcal{C}_0^\infty(\Omega)$, $\chi_k \in \mathcal{C}_0^\infty(V_k$ for $k = 1, \dots, N$, and

$$\chi_0 + \chi_1 + \dots + \chi_N = 1$$

on $\overline{\Omega}$. In particular, on Ω, we have

$$u = \chi_0 u + \sum_{k=1}^{N} \chi_k u \ .$$

We may apply Proposition 2.5.8 to $\chi_0 u$. Therefore it remains to prove that, for every $k = 1, \dots, N$, $\chi_k u$ can be approximated in $H^1(\Omega)$ by a sequence of $\mathcal{C}^\infty(\overline{\Omega})$.

We drop the index k for simplicity, and we consider an open subset $V = I \times W$ of $\mathbb{R}^d = \mathbb{R}_{x_1} \times \mathbb{R}_y^{d-1}$ and a smooth function $q : W \to \mathbb{R}$ such that

$$V \cap \Omega = \{(x_1, y) \in I \times W, x_1 > q(y) \ \} \ ,$$

and $\chi \in \mathcal{C}_0^\infty(V)$. For $(z, y) \in]0, \infty[\times W$, we set

$$v(z, y) = (\chi u)(z + q(y), y) \ .$$

Notice that, for some compact subset K of W and some $a > 0$,

$$\text{supp}(v) \subset]0, a] \times K \ .$$

We need a couple of results concerning this function v.

Lemma 2.5.10 The function v belongs to $H^1(]0, \infty[\times W)$, with

$$\partial_z v(z, y) = \partial_z(\chi u)(z+q(y), y) \ , \ \partial_{y_j} v(z, y) = \partial_{y_j}(\chi u)(z+q(y), y) + \partial_{y_j} q(y) \partial_z(\chi u)(z+q(y), y) \ .$$

Let us prove Lemma 2.5.10. First of all, $v \in L^2(]0,\infty[\times W)$, since, by the change of variables formula,

$$\int_W \int_0^\infty |v(z,y)|^2 \, dz \, dy = \int_W \int_{q(y)}^\infty |(\chi u)(x_1,y)|^2 \, dx_1 \, dy \ < +\infty \ .$$

Let $\varphi \in \mathcal{C}_0^\infty(]0,\infty[\times W)$. Let us calculate, using the fact that $\chi u \in H^1(\Omega)$,

$$\begin{aligned}
-\int_W \int_0^\infty v(z,y)\partial_z \varphi(z,y) \, dz \, dy &= -\int_W \int_{q(y)}^\infty (\chi u)(x_1,y)\partial_z \varphi(x_1 - q(y), y) \, dx_1 \, dy \\
&= -\int_\Omega (\chi u)(x_1,y) \frac{\partial}{\partial x_1}[\varphi(x_1 - q(y), y)] \, dx_1 \, dy \\
&= \int_\Omega \frac{\partial}{\partial x_1}[(\chi u)(x_1,y)] \varphi(x_1 - q(y), y) \, dx_1 \, dy \\
&= \int_W \int_0^\infty \partial_{x_1}(\chi u)(z + q(y), y) \, \varphi(z,y) \, dz \, dy \ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
-\int_W \int_0^\infty v(z,y)\partial_{y_j} \varphi(z,y) \, dz \, dy &= -\int_W \int_{q(y)}^\infty (\chi u)(x_1,y)\partial_{y_j} \varphi(x_1 - q(y), y) \, dx_1 \, dy \\
&= -\int_\Omega (\chi u)(x_1,y) \frac{\partial}{\partial y_j}[\varphi(x_1 - q(y), y)] \, dx_1 \, dy \\
&\quad -\int_\Omega (\chi u)(x_1,y) \partial_{y_j} q(y) \frac{\partial}{\partial x_1}[\varphi(x_1 - q(y), y)] \, dx_1 \, dy \\
&= \int_\Omega [\partial_{y_j} + \partial_{y_j} q(y) \partial_{x_1}](\chi u)(x_1,y) \varphi(x_1 - q(y), y) \, dx_1 \, dy \\
&= \int_W \int_0^\infty [\partial_{y_j} + \partial_{y_j} q(y) \partial_{x_1}](\chi u)(z + q(y), y) \, \varphi(z,y) \, dz \, dy \ .
\end{aligned}$$

Since the functions

$$\begin{aligned}
&(z,y) \in]0,\infty[\times W \mapsto \partial_{x_1}(\chi u)(z + q(y), y) \\
&(z,y) \in]0,\infty[\times W \mapsto [\partial_{y_j} + \partial_{y_j} q(y) \partial_{x_1}](\chi u)(z + q(y), y)
\end{aligned}$$

belong to $L^2(]0,\infty[\times W)$ by change of variables again, this completes the proof of Lemma 2.5.10.

Lemma 2.5.11 Given $v \in H^1(]0,\infty[\times W)$, define $\tilde{v} \in L^2(\mathbb{R} \times W)$ by

$$\tilde{v}(z,y) = \begin{cases} v(z,y) & \text{if } z > 0 \\ v(-z,y) & \text{if } z < 0 \end{cases}$$

Then $\tilde{v} \in H^1(\mathbb{R} \times W)$.

Let us prove Lemma 2.5.11. First we observe that

$$\|\tilde{v}\|^2_{L^2(\mathbb{R}\times W)} = 2\|\tilde{v}\|^2_{L^2(]0,\infty[\times W)} < \infty$$

so $\tilde v \in L^2(\mathbb{R}\times W)$ indeed. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}\times W)$. Let us calculate

$$\begin{aligned}
-\int_{\mathbb{R}}\int_W \tilde v(z,y)\partial_{y_j}\varphi(z,y)\,dz\,dy &= -\int_0^{+\infty}\int_W v(z,y)\partial_{y_j}\varphi(z,y)\,dz\,dy - \int_{-\infty}^0\int_W v(-z,y)\partial_{y_j}\varphi(z,y)\,dz\,dy \\
&= -\int_0^\infty\int_W v(z,y)[\partial_{y_j}\varphi(z,y)+\partial_{y_j}\varphi(-z,y)]\,dz\,dy
\end{aligned}$$

In order to use the information $v\in H^1(]0,\infty[\times W)$, we need to reduce to test functions compactly supported in $]0,\infty[\times W$. We introduce $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $\chi(t)=0$ for $t\le \frac12$ and $\chi(t)=1$ for $t\ge 1$, and we set, for $\varepsilon>0$,

$$\chi_\varepsilon(z)=\chi\left(\frac{z}{\varepsilon}\right)\ .$$

The $\chi_\varepsilon(z)\to \mathbf{1}_{]0,\infty[}(z)$ as $\varepsilon\to 0$, and

$$\begin{aligned}
-\int_{\mathbb{R}}\int_W \tilde v(z,y)\partial_{y_j}\varphi(z,y)\,dz\,dy &= -\lim_{\varepsilon\to 0}\int_0^\infty\int_W v(z,y)\partial_{y_j}[\chi_\varepsilon(z)(\varphi(z,y)+\varphi(-z,y))]\,dz\,dy \\
&= \lim_{\varepsilon\to 0}\int_0^\infty\int_W \partial_{y_j}v(z,y)\chi_\varepsilon(z)(\varphi(z,y)+\varphi(-z,y))\,dz\,dy \\
&= \int_0^\infty\int_W \partial_{y_j}v(z,y)(\varphi(z,y)+\varphi(-z,y))\,dz\,dy \\
&= \int_{\mathbb{R}}\int_W f_j(z,y)\varphi(z,y)\,dz\,dy\ ,
\end{aligned}$$

where

$$f_j(z,y):=\begin{cases}\partial_{y_j}v(z,y) & \text{if } z>0\\ \partial_{y_j}v(-z,y) & \text{if } z<0\end{cases}$$

defines an L^2 function on $\mathbb{R}\times W$. Similarly,

$$\begin{aligned}
-\int_{\mathbb{R}}\int_W \tilde v(z,y)\partial_z\varphi(z,y)\,dz\,dy &= -\int_0^{+\infty}\int_W v(z,y)\partial_z\varphi(z,y)\,dz\,dy - \int_{-\infty}^0\int_W v(-z,y)\partial_z\varphi(z,y)\,dz\,dy \\
&= -\int_0^\infty\int_W v(z,y)\partial_z[\varphi(z,y)-\varphi(-z,y)]\,dz\,dy \\
&= -\lim_{\varepsilon\to 0}\int_0^\infty\int_W v(z,y)\partial_z[\chi_\varepsilon(z)(\varphi(z,y)-\varphi(-z,y))]\,dz\,dy \\
&\quad + \lim_{\varepsilon\to 0}\int_0^\infty\int_W v(z,y)\chi_\varepsilon'(z)(\varphi(z,y)-\varphi(-z,y))]\,dz\,dy \\
&= \lim_{\varepsilon\to 0}\int_0^\infty\int_W \partial_z v(z,y)\chi_\varepsilon(z)(\varphi(z,y)-\varphi(-z,y))\,dz\,dy \\
&\quad + \lim_{\varepsilon\to 0} O\left(\frac{1}{\varepsilon}\right)\int_{\frac{\varepsilon}{2}}^{\varepsilon}|v(z,y)|O(z)\,dz\,dy \\
&= \int_0^\infty\int_W \partial_z v(z,y)(\varphi(z,y)-\varphi(-z,y))\,dz\,dy \\
&= \int_{\mathbb{R}}\int_W g(z,y)\varphi(z,y)\,dz\,dy\ ,
\end{aligned}$$

where where

$$g(z,y):=\begin{cases}\partial_z v(z,y) & \text{if } z>0\\ -\partial_z v(-z,y) & \text{if } z<0\end{cases}$$

defines an L^2 function on $\mathbb{R}\times W$. Finally, we get $\tilde v\in H^1(\mathbb{R}\times W)$ with

$$\partial_{y_j}\tilde v=f_j\ ,\ \partial_z\tilde v=g\ .$$

This completes the proof of Lemma 2.5.11.
Let us complete the proof of Proposition 2.5.9. Applying Lemma 2.5.11 to

$$v(z,y)=(\chi u)(z+q(y),y)\ ,$$

we observe that $\tilde v\in H^1(\mathbb{R}\times W)$ and supp$(\tilde v)\subset[-a,a]\times K$, a compact subset of $\mathbb{R}\times W$. Now we appeal to Proposition 2.5.8 and obtain that there exist $\tilde v_\varepsilon\in\mathcal{C}_0^\infty(\mathbb{R}\times W)$ such that supp$(\tilde v_\varepsilon)\subset[-(a+\varepsilon,a+\varepsilon]\times K_\varepsilon$ and $\tilde v_\varepsilon\to\tilde v$ in $H^1(\mathbb{R}\times W)$. Then the restriction v_ε of $\tilde v_\varepsilon$ to $]0,+\infty[\times W$ converge to v in $H^1(]0,\infty[\times W)$, and

$$u_\varepsilon(x_1,y)=v_\varepsilon(x_1-q(y),y)$$

is a family of functions in $\mathcal{C}^\infty(\overline{\Omega})$ which, in view of the formulae for derivatives established in Lemma 2.5.10, converges to χu in $H^1(\Omega)$ as $\varepsilon\to0$. □

2.5.4 The trace theorem

Theorem 2.5.12 Let Ω be a bounded regular open subset of $\mathbb{R}^d$, with a $\mathcal{C}^\infty$ boundary. Denote by σ the superficial measure on $\partial\Omega$ and by N^{ext} the exterior unit normal on $\partial\Omega$. There exists a unique linear mapping

$$\gamma_0:H^1(\Omega)\to L^2(\partial\Omega,\sigma)$$

such that, for every $\varphi\in\mathcal{C}^\infty(\overline{\Omega})$,

$$\gamma_0\varphi=\varphi_{|\partial\Omega}\ .$$

Furthermore, γ_0 satisfies the following identity,
(2.5.9)

$$\forall j\in\{1,\dots,d\}\ ,\ \forall u,v\in H^1(\Omega)\ ,\ \int_\Omega u\,\partial_jv\,dx=\int_{\partial\Omega}\gamma_0u\,\gamma_0v\,N_j^{ext}\,d\sigma-\int_\Omega\partial_ju\,v\,dx\ .$$

Furthermore, Ker γ_0 is the closure $H_0^1(\Omega)$ of $\mathcal{C}_0^\infty(\Omega)$ in $H^1(\Omega)$.

Proof.— From Corollary 2.4.16 and Remark 2.4.17, we have, for every $\varphi,\psi\in\mathcal{C}^\infty(\overline{\Omega})$,

(2.5.10) $$\forall j\in\{1,\dots,d\}\ ,\ \int_\Omega\varphi\,\partial_j\psi\,dx=\int_{\partial\Omega}\varphi\,\psi\,N_j^{ext}\,d\sigma-\int_\Omega\partial_j\varphi\,\psi\,dx\ .$$

Consider the mapping

$$\gamma_0:\varphi\in\mathcal{C}^\infty(\overline{\Omega})\longmapsto\varphi_{|\partial\Omega}\in L^2(\partial\Omega,\sigma)\ .$$

We claim that this mapping is continuous if $\mathcal{C}^\infty(\overline{\Omega})$ is endowed with the H^1 norm. Indeed, this is equivalent to proving the estimate

(2.5.11) $$\|\varphi\|_{L^2(\partial\Omega,\sigma)}\le C\|\varphi\|_{H^1(\Omega)}\ .$$

Using a partition of unity associated to a finite open covering of the compact set $\partial\Omega$, it is enough to prove the above inequality when φ is supported in the intersection of Ω with a small neighbourhood of some point of $\partial\Omega$. On such a neighbourhood, we may assume that, for some j,

$$|N_j^{ext}| \geq c$$

for some $c > 0$. Then (2.5.11) follows from (2.5.10) with $\psi = \overline{\varphi}$ and the Cauchy–Schwarz inequality.

At this stage, we appeal to Proposition 2.5.9. Since γ_0 is a continuous linear mapping from a dense subspace of $H^1(\Omega)$ into the Banach space $L^2(\partial\Omega, \sigma)$, it admits a unique linear continuous extension from $H^1(\Omega)$ to $L^2(\partial\Omega, \sigma)$, still denoted by γ_0. Since both sides of (2.5.9) are continuous bilinear maps on $H^1(\Omega) \times H^1(\Omega)$ and coincide on the dense subspace $\mathcal{C}^\infty(\overline{\Omega}) \times \mathcal{C}^\infty(\overline{\Omega})$ in view of (2.5.10), we infer that (2.5.9) holds.
Finally, let us characterise the kernel of γ_0. If $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we have $\gamma_0\varphi = \varphi_{|\partial\Omega} = 0$, therefore Ker γ_0 contains the closure $H_0^1(\Omega)$ of $\mathcal{C}^\infty(\overline{\Omega})$ in $H^1(\Omega)$. Conversely, let us assume that $u \in$ Ker γ_0. For every $f \in L^2(\Omega)$, denote by $\underline{f} \in L^2(\mathbb{R}^d)$ the extension of f by 0 to $\mathbb{R}^d$. From identity (2.5.9) with $v = \varphi_{|\Omega}, \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, we infer, in $\mathcal{D}'(\mathbb{R}^d)$,

$$\partial_j(\underline{u}) = \underline{\partial_j u} + \gamma_0 u\, N_j^{int}\sigma = \underline{\partial_j u} \ .$$

Therefore $\underline{u} \in H^1(\mathbb{R}^d)$. Using the same partition of unity (χ_k) as in the proof of Proposition 2.5.9, we are reduced to prove that, for every $k \geq 1$,

$$\chi_k u \in H_0^1(\Omega) \ .$$

Notice that $\underline{\chi_k u} = \chi_k \underline{u} \in H^1(\mathbb{R}^d)$. Let us drop again the index k and consider

$$v(z, y) = (\chi u)(z + q(y), y) \ , \ (z, y) \in]0, \infty[\times W \ .$$

Then

$$\underline{\chi u}(z + q(y), y) = \underline{v}(z, y) \ ,$$

where $\underline{v}$ denotes the extension of v from$]0, \infty[\times W$ to $\mathbb{R} \times W$ by 0, and so we infer, by the same argument as in Lemma 2.5.10, that $\underline{v} \in H^1(\mathbb{R} \times W)$. Then we consider, for $\varepsilon > 0$,

$$v_\varepsilon(z, y) = \underline{v}(z - \varepsilon, y) \ .$$

Of course $v_\varepsilon \in H^1(\mathbb{R} \times W)$ and supp$(v_\varepsilon) \subset [\varepsilon, a + \varepsilon] \times W$, a compact subset of $]0, \infty[\times W$. Furthermore, the continuous action of translations on L^2 implies that

$$\|v_\varepsilon - v\|_{H^1}^2 = \int_0^\infty \int_W \left(|v(z - \varepsilon, y) - v(z, y)|^2 + \sum_j |\partial_j v(z - \varepsilon, y) - \partial_j v(z, y)|^2 \right) dz\, dy$$

tends to 0 as $\varepsilon \to 0$. We conclude that $v_\varepsilon \to v$ in $H^1(]0, \infty[\times W)$, and, again by Lemma 2.5.10, that the family u_ε defined by

$$u_\varepsilon(x_1, y) = v_\varepsilon(x_1 - q(y), y)$$

converges to χu in $H^1(\Omega)$. Since, for every $\varepsilon > 0$, u_ε is supported into a compact subset of Ω, Proposition 2.5.8 implies that $v_\varepsilon \in H_0^1(\Omega)$. Consequently, $\chi u \in H_0^1(\Omega)$. □

2.5.5 Solving the Dirichlet problem

Theorem 2.5.13 Let Ω be a bounded regular open subset of $\mathbb{R}^d$ with a $\mathcal{C}^\infty$ boundary. For every $f \in L^2(\Omega)$, there exists a unique $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega \ , \\ \gamma_0 u = 0 & \text{in } \partial\Omega \ . \end{cases}$$

Proof.— This is an immediate consequence of Remark 2.5.5 (iii) and of Theorem 2.5.12. □

Remark 2.5.14 One can prove in fact that, under the assumptions of Theorem 2.5.13, $u \in H^2(\Omega)$, so that the mapping

$$\Delta : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$$

is an isomorphism if Ω is a regular bounded open subset. Furthermore, more regularity on f implies more regularity on u, typically : if $f \in H^m(\Omega)$ for some $m \in \mathbb{N}$, then $u \in H^{m+2}(\Omega)$; if $f \in \mathcal{C}^\infty(\overline{\Omega})$, then $u \in \mathcal{C}^\infty(\overline{\Omega})$. The latter statement is a consequence of the previous one and of

$$\bigcap_{m \in \mathbb{N}} H^m(\Omega) = \mathcal{C}^\infty(\overline{\Omega}) \ .$$

2.6 The uniform boundedness principle

In this section, we prove an important result about families of distributions, which implies Lemma 2.3.24.

Theorem 2.6.1 Let (T_n) be a family if distributions on Ω such that, for every $\varphi \in \mathcal{D}(\Omega)$,

$$\sup_n |\langle T_n, \varphi \rangle| < +\infty \ .$$

Then, for every compact subset K of Ω, there exist $C > 0$ and $m \in \mathbb{N}$ such that, for every $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp} \varphi \subset K$,

$$\sup_n |\langle T, \varphi \rangle| \le C \|\varphi\|_{C^m} \ .$$

In other words, if for instance a sequence T_n is such that $\langle T_n, \varphi \rangle$ has a limit for every $\varphi \in \mathcal{D}(\Omega)$, one gets a uniform estimate on the action of the sequence T_n. Let us show how this implies Lemma 2.3.24. If $T_n \to T$, then the assumption of Theorem 2.6.1 is fulfilled. If $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$, let K be a compact subset which contains the support of φ_n for every n, and hence the support of φ. By Theorem 2.6.1, we have, for some C, m independent of n,

$$|\langle T_n, \varphi_n - \varphi \rangle| \le C \|\varphi_n - \varphi\|_{C^m} \to 0 \ .$$

Then

$$\langle T_n, \varphi_n \rangle = \langle T_n, \varphi_n - \varphi \rangle + \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \ .$$

This completes the proof of Lemma 2.3.24.

In the rest of this section, we give a proof of Theorem 2.6.1 in three steps. For every compact subset K of Ω, we denote by $\mathcal{D}_K$ the subspace of $\mathcal{D}(\Omega)$ of test functions φ such supp$(\varphi) \subset K$.

2.6.1 Step1. Realizing $\mathcal{D}_K$ as a complete metric space

Lemma 2.6.2 There exists a distance function d on $\mathcal{D}_K$ having the following properties.

i) If $\varphi_n, \varphi \in \mathcal{D}_K$ then $\varphi_n \to \varphi$ in $\mathcal{D}$ if and only if $d(\varphi_n, \varphi) \to 0$.

ii) The metric space $(\mathcal{D}_K, d)$ is complete.

iii) For every $\varphi, \psi, \theta \in \mathcal{D}_K$, $d(\varphi + \theta, \psi + \theta) = d(\varphi, \psi)$.

iv) For every $\varepsilon > 0$, there exist $m \in \mathbb{N}$ and $r > 0$ such that the set $\{\varphi \in \mathcal{D}_K : \|\varphi\|_m \leq r\}$ is contained into the closed ball for d centered at 0 and of radius ε.

Proof.— For $\varphi, \psi \in \mathcal{D}_K$, we set

$$d(\varphi, \psi) = \sum_{m=0}^{\infty} \min\left(2^{-m}, \|\varphi - \psi\|_{C^m}\right) \ .$$

It is clear that d takes values in $[0, \infty[$, and that $d(\varphi, \psi) = 0 \iff \varphi = \psi$. The symmetry $d(\varphi, \psi) = d(\psi, \varphi)$ is trivial, as well item iii) of the Theorem. Finally, the triangle inequality is a consequence of the elementary inequality

$$\min(a, x + y) \leq \min(a, x) + \min(a, y) \ , \ a \geq 0, x \geq 0, y \geq 0 \ .$$

Let us prove item i). If $\varphi_n, \varphi \in \mathcal{D}_K$, the statement $\varphi_n \to \varphi$ in $\mathcal{D}$ is equivalent to

$$\forall m \in \mathbb{N}, \|\varphi_n - \varphi\|_{C^m} \to 0 \ . \qquad (2.6.12)$$

Since $d(\varphi_n, \varphi) \geq \min\left(2^{-m}, \|\varphi_n - \varphi\|_{C^m}\right)$, it is clear that (2.6.12) is implied by $d(\varphi_n, \varphi) \to 0$. Conversely, (2.6.12) implies that every term of the series defining $d(\varphi_n, \varphi)$ tends to 0. Since this series is normally convergent, this implies $d(\varphi_n, \varphi) \to 0$.

Let us prove item ii). Let (φ_n) be a Cauchy sequence in the metric space $(\mathcal{D}_K, d)$,

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N} : \forall n, p \geq N(\varepsilon), d(\varphi_n, \varphi_p) \leq \varepsilon \ .$$

Let $m \in \mathbb{N}$ and $\varepsilon > 0$ be such that $\varepsilon < 2^{-m}$. Since $d(\varphi_n, \varphi_p) \geq \min\left(2^{-m}, \|\varphi_n - \varphi_p\|_{C^m}\right)$, we conclude that, for $n, p \geq N(\varepsilon)$, $\|\varphi_n - \varphi_p\|_{C^m} \leq \varepsilon$. In other words, (φ_n) is a Cauchy sequence in the Banach space $\mathcal{C}^m_K$ of $\mathcal{C}^m$ functions supported in K. Hence there exists $\varphi^{[m]} \in \mathcal{C}^m_K$ such that

$\|\varphi_n - \varphi^{[m]}\|_{C^m} \to 0$. Since this is true for every $m \in \mathbb{N}$, and since convergence in $\mathcal{C}_K^{m+1}$ implies convergence in $\mathcal{C}_K^m$, we conclude that

$$\varphi^{[m]} = \varphi \in \bigcap_{m\in\mathbb{N}} \mathcal{C}_K^m = \mathcal{D}_K \ ,$$

and that $\varphi_n \to \varphi$ in $\mathcal{D}$, which, by item i), means $d(\varphi_n, \varphi) \to 0$. Hence $(\mathcal{D}_K, d)$ is a complete metric space.

Finally, let us prove item iv). Let $m \in \mathbb{N}$ be such that $2^{-m} \le \frac{\varepsilon}{2}$, and let $r > 0$ be such that $(m+1)r \le \frac{\varepsilon}{2}$. If $\|\varphi\|_{C^m} \le r$, we have

$$\begin{aligned} d(\varphi, 0) &= \sum_{q=0}^{\infty} \min\left(2^{-q}, \|\varphi\|_{C^q}\right) \\ &\le \sum_{q=0}^{m} \|\varphi\|_{C^q} + \sum_{q=m+1}^{\infty} 2^{-q} \\ &\le (m+1)r + 2^{-m} \le \varepsilon \ . \end{aligned}$$

This completes the proof. □

2.6.2 Step 2. The Baire lemma

Lemma 2.6.3 Let $(\mathcal{E}, d)$ be a complete metric space. Every countable intersection of open dense subsets of $\mathcal{E}$ is dense in $\mathcal{E}$. Every countable union of closed subsets of $\mathcal{E}$ with empty interior has empty interior.

Proof.—The second statement is equivalent to the first one, since the complement of an open set is a closed set, and the complement of a dense set has an empty interior.
Let us prove the first statement. Denote by $B(\varphi, r)$ and $B_f(\varphi, r)$ respectively the open and the closed balls of radius r centered at $\varphi \in \mathcal{E}$. Let $(\mathcal{O}_k)_{k\in\mathbb{N}}$ be a sequence of dense open subsets of $\mathcal{E}$. Fix $\varphi_0 \in \mathcal{E}$ and $\varepsilon > 0$. We want to prove that

$$B(\varphi_0, \varepsilon) \cap \bigcap_{k\in\mathbb{N}} \mathcal{O}_k \neq \emptyset \ .$$

Since $\mathcal{O}_0$ is dense, $B(\varphi_0, \varepsilon) \cap \mathcal{O}_0$ contains some element φ_1. Since $B(\varphi_0, \varepsilon) \cap \mathcal{O}_0$ is open, there exists $r_1 > 0$ such that

$$B_f(\varphi_1, r_1) \subset B(\varphi_0, \varepsilon) \cap \mathcal{O}_0 \ ,$$

moreover we may assume that $r_1 \le \frac{\varepsilon}{2}$.
Since $\mathcal{O}_1$ is dense, $B(\varphi_1, r_1) \cap \mathcal{O}_1$ contains some element φ_2. Since $B(\varphi_1, r_1) \cap \mathcal{O}_1$ is open, there exists $r_2 > 0$ such that

$$B_f(\varphi_2, r_2) \subset B(\varphi_1, r_1) \cap \mathcal{O}_1 \ ,$$

moreover we may assume that $r_2 \leq \frac{\varepsilon}{2^2}$.
Continuing that way, we define a sequence $(\varphi_k)_{k\geq 1}$ of elements of $\mathcal{E}$ and a sequence $(r_k)_{k\geq 1}$ of positive numbers such that

$$\forall k \geq 1 \;\; B_f(\varphi_{k+1}, r_{k+1}) \subset B(\varphi_k, r_k) \cap \mathcal{O}_k \ , \ r_k \leq \frac{\varepsilon}{2^k} \ .$$

In particular, the sequence of closed balls $(B_f(\varphi_k, r_k))_{k\geq 1}$ is decreasing, with a radius tending to 0. Moreover,

$$B_f(\varphi_k, r_k) \subset \mathcal{O}_k \cap \dots \cap \mathcal{O}_0 \cap B(\varphi_0, \varepsilon) \ .$$

Hence we are reduced to prove that

$$\bigcap_{k=1}^{\infty} B_f(\varphi_k, r_k) \neq \emptyset \ .$$

We observe that $(\varphi_k)_{k\geq 1}$ is a Cauchy sequence in $\mathcal{E}$. Indeed, if $\ell \geq 0$, $\varphi_{k+\ell} \in B_f(\varphi_k, r_k)$, hence

$$d(\varphi_{k+\ell}, \varphi_k) \leq r_k \ ,$$

which tends to 0 as k tends to ∞. Since $\mathcal{E}$ is a complete metric space, φ_k has a limit $\varphi \in \mathcal{E}$. Passing to the limit as ℓ tends to infinity in the above inequality, we get, for every $k \geq 1$,

$$d(\varphi, \varphi_k) \leq r_k \ ,$$

so that

$$\varphi \in \bigcap_{k=1}^{\infty} B_f(\varphi_k, r_k).$$

This completes the proof. □

2.6.3 Step 3. Proof of Theorem 2.6.1

Let us use the notation of Theorem 2.6.1. We set, for every $k \in \mathbb{N}$,

$$\mathcal{F}_k = \{\varphi \in \mathcal{D}_K : \forall n, |\langle T_n, \varphi \rangle| \leq k\} \ .$$

Since each T_n is continuous on $\mathcal{D}_K$, $\mathcal{F}_k$ is a closed subset of $\mathcal{D}_K$. Furthermore, the assumption of Theorem 2.6.1 precisely means that

$$\bigcup_{k\in\mathbb{N}} \mathcal{F}_k = \mathcal{D}_K \ .$$

Let endow $\mathcal{D}_K$ with a distance function as in Lemma 2.6.2. By Baire's lemma, we infer that there exists k_0 such that $\mathcal{F}_{k_0}$ has a nonempty interior, which means that there exists some $\varphi_0 \in \mathcal{D}_K$ and $\varepsilon > 0$ such that

$$B_f(\varphi_0, \varepsilon) \subset \mathcal{F}_{k_0} \ .$$

By item iii) of Lemma 2.6.2, we know that

$$B_f(0, \varepsilon) = -\varphi_0 + B_f(\varphi_0, \varepsilon),$$

so that, for every $\psi \in B_f(0, \varepsilon)$, for every n,

$$|\langle T_n, \psi \rangle| \leq |T_n, -\varphi_0 \rangle| + k_0 \leq A \ ,$$

for some $A > 0$. Using item iv) of Lemma 2.6.2, we infer that there exists $m \in \mathbb{N}$ and $r > 0$ such that every $\psi \in \mathcal{D}_K$ such that $\|\psi\|_{C^m} \leq r$ belongs to $B_f(0, \varepsilon)$, hence satisfies

$$\sup_n |\langle T_n, \psi \rangle| \leq A \ .$$

Given $\varphi \in \mathcal{D}_K \setminus \{0\}$, we may apply this fact to

$$\psi = r \frac{\varphi}{\|\varphi\|_{C^m}} \ ,$$

and we conclude that

$$\sup_n |\langle T_n, \varphi \rangle| \leq \frac{A}{r} \|\varphi\|_{C^m} \ .$$

The proof is complete.

Remark 2.6.4 The above theorem is an adaptation of the Banach–Steinhaus theorem to the case of vector spaces admitting a distance enjoying Lemma 2.6.2. Such spaces are called Fréchet spaces.

Chapter 3

The Fourier Transformation

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is the function $\mathcal{F}(f) \in L^\infty(\mathbb{R}^d)$ given by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \text{ with } \|\mathcal{F}(f)\|_{L^\infty} \leq \|f\|_{L^1}.$$

The important role played by the Fourier transform in PDE's theory is mainly due to the fact that, when these objects are well-defined,

$$\mathcal{F}(\partial_j f)(\xi) = i\xi_j \mathcal{F}(f)(\xi).$$

Otherwise stated, $\mathcal{F}$ transforms the action of a differential operator with constant coefficients to that of the product by a polynomial. This would be worthless without an inversion formula giving back the function f in terms of $\mathcal{F}(f)$, as

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \mathcal{F}(f)(\xi) d\xi,$$

Unfortunately, this formula only makes sense when $\mathcal{F}(f) \in L^1(\mathbb{R}^d)$, and this is not the case in general for $f \in L^1$. In this chapter, we are going to introduce a subspace of distributions on $\mathbb{R}^d$, which contains $L^1(\mathbb{R}^d)$, and a Fourier transformation on this space which sastifies the two above identity.

3.1 The Schwartz space

3.1.1 Definitions and examples

Definition 3.1.1 A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be rapidly decreasing if, as $|x| \to \infty$,

$$\forall \alpha \in \mathbb{N}^d, x^\alpha f(x) \to 0\ .$$

Notice that, if f is rapidly decreasing and continuous, all the functions $x^\alpha f$ are bounded on $\mathbb{R}^d$.

Definition 3.1.2 We denote $\mathcal{S}(\mathbb{R}^d)$ the set of functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ which are rapidly decreasing as well as all their derivatives. In other words,

$$\forall(\alpha,\beta) \in \mathbb{N}^d, x^\alpha \partial^\beta \varphi(x) \to 0$$

as $|x| \to \infty$. The set $\mathcal{S}(\mathbb{R}^d)$ is a vector space; it is called the Schwartz space.

Example 3.1.3 *i)* $\mathcal{C}_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$.

ii) For $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$, the function$\varphi(x) = e^{-z|x|^2}$ belongs to $\mathcal{S}(\mathbb{R}^d)$.

iii) If $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $\varphi_1\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$.

iv) No rational function (even smooth ones) belongs to $\mathcal{S}(\mathbb{R}^d)$.

The topology on $\mathcal{S}(\mathbb{R}^d)$ we will work with is that given by the family of norms $(N_p)_{p\in\mathbb{N}}$ given by

$$N_p(\varphi) = \sup_{|\alpha|,|\beta| \le p} \sup |x^\alpha \partial^\beta \varphi(x)|.$$

It is clear that for $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$, we have the equivalence

$$\varphi \in \mathcal{S}(\mathbb{R}^d) \Longleftrightarrow \forall p \in \mathbb{N}, N_p(\varphi) < +\infty.$$

Proposition 3.1.4 If $\varphi \in \mathcal{S}(\mathbb{R}^d)$ then $x^\alpha \partial^\beta \varphi \in \mathcal{S}(\mathbb{R}^d)$ for every $\alpha, \beta \in \mathbb{N}^d$.

Proof.— This follows immediately from the fact that

$$N_p(x^\alpha \partial^\beta \varphi) = \sup_{|\lambda|,|\mu| \le p} \sup |x^\lambda \partial^\mu (x^\alpha \partial^\beta \varphi(x))| \le C_{p,\alpha,\beta} N_{p+q}(\varphi) \tag{3.1.1}$$

when $|\alpha|, |\beta| \le q$. □

3.1.2 Convergence in $\mathcal{S}(\mathbb{R}^d)$ and density results

Definition 3.1.5 Let (φ_n) be a sequence of functions in $\mathcal{S}(\mathbb{R}^d)$. One says that (φ_n) converges to φ in $\mathcal{S}(\mathbb{R}^d)$ when, for every $\alpha, \beta \in \mathbb{N}^d$, $x^\alpha \partial^\beta \varphi_n(x) \to x^\alpha \partial^\beta \varphi(x)$ uniformly in $x \in \mathbb{R}^d$. Equivalently, for all $p \in \mathbb{N}$,

$$N_p(\varphi_n - \varphi) \to 0 \text{ as } j \to +\infty.$$

Remark 3.1.6 Let $\alpha \in \mathbb{N}^d$. It follows from (3.1.1) that, if (φ_n) converges to φ in $\mathcal{S}(\mathbb{R}^d)$, then $(x^\alpha \varphi_n)$ converges to $x^\alpha \varphi$ and $(\partial^\alpha \varphi_n)$ converges to $(\partial^\alpha \varphi)$ in $\mathcal{S}(\mathbb{R}^d)$. Otherwise stated, multiplication by a polynomial and derivation are continuous operations in $\mathcal{S}(\mathbb{R}^d)$.

We already know that $\mathcal{S} \subset L^\infty$, with $\|\varphi\|_{L^\infty} = N_0(\varphi)$. The next proposition states a similar property for L^1.

Proposition 3.1.7 $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, and there exists $C_d > 0$ such that

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d) \ , \ \|\varphi\|_{L^1} \leq C_d N_{d+1}(\varphi) \ .$$

Proof.— We have

$$(1+|x|^2)^{d+1} = (1 + x_1^2 + \cdots + x_d^2)^{d+1} = \sum_{|\alpha| \leq d+1} c_{\alpha,d} (x^\alpha)^2 \ ,$$

hence

$$(1+|x|^2)^{d+1} |\varphi(x)|^2 \leq B_d N_{d+1}(\varphi)^2 \ .$$

Consequently,

$$\int_{\mathbb{R}^d} |\varphi(x)| \, dx \leq \sqrt{B_d} N_{d+1}(\varphi) \int_{\mathbb{R}^d} \frac{dx}{(1+|x|^2)^{(d+1)/2}} \leq C_d N_{d+1}(\varphi) \ ,$$

since $x \mapsto (1+|x|^2)^{-(d+1)/2}$ is integrable on $\mathbb{R}^d$. □

Remark 3.1.8 For every $q \in \mathbb{N}$, one can establish similarly

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d) \ , \ \forall x \in \mathbb{R}^d \ , \ (1 + |x|^q) |\varphi(x)| \leq C_{q,d} N_q(\varphi) \ .$$

Since $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in the $L^p(\mathbb{R}^d)$ spaces for $p \in [1, +\infty[$, we get the

Corollary 3.1.9 The set $\mathcal{S}(\mathbb{R}^d)$ is dense in all the $L^p(\mathbb{R}^d)$ for $p \in [1, +\infty[$.

We also have the important

Proposition 3.1.10 The space $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$: for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, there is a sequence (φ_n) of functions in $\mathcal{C}_0^\infty(\mathbb{R}^d)$ which converges to φ in $\mathcal{S}(\mathbb{R}^d)$.

Proof.— Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be a plateau function over $B(0,1)$. We set $\varphi_n(x) = \varphi(x)\chi(x/n)$. The functions φ_n are smooth with compact support, and equal φ in $B(0,n)$. By the Leibniz formula, we obtain

$$\partial^\beta (\varphi - \varphi_n)(x) = \partial^\beta \varphi(x)(1 - \chi(x/n)) + \sum_{|\gamma| \geq 1, \gamma \leq \beta} C_\beta^\gamma \frac{1}{n^{|\gamma|}} \partial^{\beta-\gamma} \varphi(x) (\partial^\gamma \chi)(\frac{x}{n}).$$

Thus, as $n \to +\infty$,

$$\|x^\alpha \partial^\beta (\varphi - \varphi_n)(x)\|_\infty \le \sup_{|x| \ge n} |x^\alpha \partial^\beta \varphi(x)| + \frac{C}{n} \sum_{\gamma \le \beta} \|x^\alpha \partial^{\beta - \gamma} \varphi\|_\infty \to 0.$$

□

3.2 The Fourier transformation in $\mathcal{S}(\mathbb{R}^d)$

3.2.1 Definition and first properties

For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have $\varphi \in L^1(\mathbb{R}^d)$, so that $\mathcal{F}(\varphi)$ is well defined ans belongs to $L^\infty(\mathbb{R}^d)$.

Definition 3.2.1 For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we denote $\hat{\varphi}$, $\mathcal{F}(\varphi)$ or even $\mathcal{F}_{x \to \xi}(\varphi(x))$ the function in $L^\infty(\mathbb{R}^d)$ given by

$$\hat{\varphi}(\xi) = \mathcal{F}(\varphi)(\xi) = \mathcal{F}_{x \to \xi}(\varphi(x)) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \varphi(x) dx.$$

The linear map $\mathcal{F} : \varphi \mapsto \hat{\varphi}$ is called the Fourier transformation.

Here follows some of the properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ that we have asked for in the introduction.

Proposition 3.2.2 Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$. More precisely,

i) For all $j \in \{1, \dots, n\}$ $\mathcal{F}_{x \to \xi}(x_j \varphi(x)) = i\partial_j \mathcal{F}(\varphi)(\xi)$.

ii) For all $j \in \{1, \dots, n\}$, we have $\mathcal{F}(\partial_j \varphi)(\xi) = i\xi_j \mathcal{F}(\varphi)(\xi)$.

iii) For $a \in \mathbb{R}^d$, $\mathcal{F}_{x \to \xi}(\varphi(x - a)) = e^{-ia\cdot\xi} \mathcal{F}(\varphi)(\xi)$.

iv) For $a \in \mathbb{R}^d$, $\mathcal{F}_{x \to \xi}(e^{ia\cdot x} \varphi(x)) = \mathcal{F}(\varphi)(\xi - a)$.

v) For all integrer $p \ge 0$, $N_p(\hat{\varphi}) \le C_{p,d} N_{p+d+1}(\varphi)$.

Proof.— i) The function $(x, \xi) \mapsto e^{-ix\cdot\xi} \varphi(x)$ is $\mathcal{C}^1$ on $\mathbb{R}^d$, et

$$|\partial_{\xi_j}(e^{-ix\cdot\xi} \varphi(x))| = |-ix_j e^{-ix\cdot\xi} \varphi(x)| = |x_j \varphi(x)| \in L^1(\mathbb{R}^d).$$

By Lebesgue theorem, we see that $\mathcal{F}(\varphi)$ is $\mathcal{C}^1$ and

$$\partial_{\xi_j} \mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^d} -ix_j e^{-ix\cdot\xi} \varphi(x) = \mathcal{F}(-ix_j \varphi(x)).$$

ii) Let us first write the proof for $j = 1$. Integrating by parts, we get

$$\int_{\mathbb{R}} \partial_1 \varphi(x) e^{-ix\cdot\xi} dx_1 = i\xi_1 \int_{\mathbb{R}} \varphi(x) e^{-ix\cdot\xi} dx_1$$

Now we integrate with respect to the variable x'. By Fubini, since $\varphi, \partial_1\varphi \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \partial_1 \varphi(x) e^{-ix\cdot\xi} dx = i\xi_1 \int_{\mathbb{R}^d} \varphi(x) e^{-ix\cdot\xi} dx.$$

To get (iv), we only have to write

$$\mathcal{F}_{x\to\xi}(e^{ia\cdot x}\varphi(x)) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{ia\cdot x} \varphi(x) dx = \int_{\mathbb{R}^d} e^{-i(\xi - a)\cdot x} \varphi(x) dx = \mathcal{F}(\varphi)(\xi - a).$$

Eventually, performing a change of variable, we have

$$\mathcal{F}_{x\to\xi}(\varphi(x-a)) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \varphi(x-a) dx = \int_{\mathbb{R}^d} e^{-i(x+a)\cdot\xi} \varphi(x) dx = e^{-ia\cdot\xi}\mathcal{F}(\varphi)(\xi),$$

and this is property (iii).

Iterating properties $i), ii)$, we obtain that $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$, and, using Proposition 3.1.7, that

$$\begin{aligned} N_p(\hat{\varphi}) &= \sup_{|\alpha|,|\beta|\le p} \|\xi^\alpha \partial^\beta \hat{\varphi}\|_{L^\infty} = \sup_{|\alpha|,|\beta|\le p} \|\mathcal{F}(\partial^\alpha(x^\beta \varphi))\|_{L^\infty} \\ &\le \sup_{|\alpha|,|\beta|\le p} \|\partial^\alpha(x^\beta\varphi)\|_{L^1} \le B_d \sup_{|\alpha|,|\beta|\le p} N_{d+1}(\partial^\alpha(x^\beta\varphi)) \\ &\le C_{p,d} N_{p+d+1}(\varphi) \ . \end{aligned}$$

□

Because of the presence of a factor $i = \sqrt{-1}$ in (i) and (ii), it is sometimes convenient to use the notation

$$D_j = \frac{1}{i}\partial_j.$$

Then, for example, (ii) becomes $\mathcal{F}(D_j\varphi) = \xi_j \mathcal{F}(\varphi)$, and (i) is $D_j\mathcal{F}(\varphi) = -\mathcal{F}(x_j\varphi)$. Summing up, we have

$$\begin{cases} \widehat{D_j\varphi} = \xi_j \widehat{\varphi}, \\ \widehat{x_j\varphi} = -D_j\widehat{\varphi}. \end{cases}$$

Notice also that, by (i), $\mathcal{F}(\varphi)$ is $\mathcal{C}^\infty$ since $x^\alpha\varphi \in \mathcal{S}$ for all $\alpha \in \mathbb{N}^d$.

3.2.2 Gaussians

For $\lambda > 0$, we set

$$G_\lambda(x) = \mathrm{e}^{-\lambda\frac{|x|^2}{2}} \ , \ x \in \mathbb{R}^d \ .$$

Proposition 3.2.3

$$\hat{G}_\lambda = \left(\frac{2\pi}{\lambda}\right)^{\frac{d}{2}} G_{\frac{1}{\lambda}} \ .$$

Proof.— Since $G_\lambda(x_1,\dots,x_d)=G_\lambda(x_1)\dots G_\lambda(x_d)$, it is enough to prove the case $d=1$, which we now assume. Then we observe that

$$G'_\lambda(x)=-\lambda x G_\lambda(x)\ ,$$

hence by properties i) and ii) of Proposition 3.2.2, we infer

$$i\xi\hat G_\lambda(\xi)=-i\lambda\frac{d}{d\xi}\hat G_\lambda(\xi)\ .$$

Solving this differential equation leads to $\hat G_\lambda(\xi)=\hat G_\lambda(0)\,\mathrm{e}^{-\frac{\xi^2}{2\lambda}}$, and the proof is completed by the Gauss integral identity

$$\hat G_\lambda(0)=\int_{\mathbb R}\mathrm{e}^{-\lambda\frac{x^2}{2}}\,dx=\sqrt{\frac{2\pi}{\lambda}}\ .$$

□

Exercise 3.2.4 Compute $\mathcal F_{x\to\xi}(e^{-\lambda x^2})$ by calculating $\int_{\mathrm{Im}\,\zeta=\xi}e^{-\lambda\zeta^2}d\zeta$ using the Cauchy integral formula for holomorphic functions.

3.2.3 The Inversion formula

Theorem 3.2.5 Let $\varphi\in\mathcal S(\mathbb R^d)$. Then

$$\varphi(x)=\frac{1}{(2\pi)^d}\int_{\mathbb R^d}\mathrm{e}^{ix.\xi}\hat\varphi(\xi)\,d\xi\ .$$

Proof.— We first observe that it is enough to prove this identity for $x=0$. Indeed, in view of property (iii) of Proposition 3.2.2, the general statement follows from applying the formula

$$\psi(0)=\frac{1}{(2\pi)^d}\int_{\mathbb R^d}\hat\psi(\xi)\,d\xi$$

to the function ψ_x defined by $\psi_x(y)=\varphi(x+y)$.
Consider the linear form

$$L:\psi\in\mathcal S(\mathbb R^d)\mapsto\int_{\mathbb R^d}\hat\psi(\xi)\,d\xi$$

We claim that, if $\psi(0)=0$, then $L(\psi)=0$. Indeed, applying the Hadamard lemma 2.3.12 and its proof, we infer

$$\psi(x)=\sum_{j=1}^d x_j\psi_j(x)\ ,$$

where $\psi_1,\dots,\psi_d\in\mathcal S(\mathbb R^d)$. Taking the Fourier transform of both sides and applying property (i) of Proposition 3.2.2, we infer

$$\hat\psi(\xi)=\sum_{j=1}^d i\partial_j\hat\psi_j(\xi)\ .$$

Integrating both sides on $\mathbb{R}^d$, we conclude

$$\int_{\mathbb{R}^d} \hat{\psi}(\xi)\, d\xi = 0 \ .$$

Hence we have proved that the kernel of the linear form $\psi \mapsto \psi(0)$ on $\mathcal{S}(\mathbb{R}^d)$ is contained in the kernel of L. Consequently, there exists $c \in \mathbb{C}$ such that, for every $\psi \in \mathcal{S}(\mathbb{R}^d)$,

$$L(\psi) = c\psi(0) \ .$$

Applying this identity to $\psi = G_1$, we infer, in view of Proposition 3.2.3,

$$c = (2\pi)^d \ .$$

□

In order to reformulate this important theorem, we introduce the symmetry $\sigma : \mathcal{S} \to \mathcal{S}$ defined by

$$\sigma\varphi(x) = \check{\varphi}(x) := \varphi(-x) \ ,$$

and observe that

$$\sigma \circ \mathcal{F} = \mathcal{F} \circ \sigma =: \check{\mathcal{F}} \ .$$

Corollary 3.2.6 The Fourier transformation is an isomorphism on the vector space $\mathcal{S}(\mathbb{R}^d)$. Its inverse $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1} = (2\pi)^{-d}\check{\mathcal{F}} \ .$$

Equivalently, we have

$$\mathcal{F} \circ \mathcal{F} = (2\pi)^d \sigma \ . \tag{3.2.2}$$

Remark 3.2.7 As an immediate consequence of (3.2.2), $\mathcal{F}^4 = (\mathcal{F} \circ \mathcal{F})^2 = (2\pi)^{2d} I$.

3.2.4 The "change of head" lemma

The following proposition is elementary but crucial for the whole chapter.

Proposition 3.2.8 Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \hat{\varphi}(\xi)\psi(\xi)\, d\xi = \int_{\mathbb{R}^d} \varphi(x)\hat{\psi}(x)\, dx \ .$$

Proof.— By Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\psi(\xi)\, d\xi &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathrm{e}^{-ix.\xi} \varphi(x)\, dx \right) \psi(\xi)\, d\xi \\ &= \int_{\mathbb{R}^d} \varphi(x) \left(\int_{\mathbb{R}^d} \mathrm{e}^{-ix.\xi} \psi(\xi)\, d\xi \right)\, dx = \int_{\mathbb{R}^d} \varphi(x)\hat{\psi}(x)\, dx \ . \end{aligned}$$

□

We now come to some important identities which easily follow from the inversion formula.

3.2.5 The Plancherel identity

Proposition 3.2.9 (Plancherel formula) Let φ and ψ be two functions in $\mathcal{S}(\mathbb{R}^d)$. Then

$$\int_{\mathbb{R}^d} \varphi(x)\overline{\psi}(x)\, dx = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\overline{\hat{\psi}}(\xi)\, d\xi$$

Proof.— We first observe the following identity,

$$\overline{\hat{\varphi}} = \sigma\left(\hat{\overline{\varphi}}\right) \ , \tag{3.2.3}$$

which is merely a reformulation of

$$\overline{\hat{\varphi}}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{ix.\xi}\overline{\varphi}(x)\, dx \ .$$

Then, applying Proposition 3.2.8, we obtain

$$\int_{\mathbb{R}^d} \hat{\varphi}\overline{\hat{\psi}} = \int_{\mathbb{R}^d} \varphi\hat{\overline{\hat{\psi}}} \ .$$

But, combining identity (3.2.3) with identity (3.2.2),

$$\hat{\overline{\hat{\psi}}} = \sigma\overline{\hat{\hat{\psi}}} = (2\pi)^d\sigma\overline{\sigma\psi} = (2\pi)^d\overline{\psi} \ .$$

This completes the proof. □

In terms of the Hermitian scalar product in $L^2(\mathbb{R}^d, \mathbb{C})$, the above identity can be written as

$$(\varphi, \psi)_{L^2} = \frac{1}{(2\pi)^d}(\hat{\varphi}, \hat{\psi})_{L^2}.$$

In particular, for $\psi = \varphi$, we obtain the famous Plancherel formula in $\mathcal{S}(\mathbb{R}^d)$:

Corollary 3.2.10 (Plancherel) For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, it holds that

$$\|\varphi\|_{L^2(\mathbb{R}^d)} = (2\pi)^{-d/2}\|\hat{\varphi}\|_{L^2(\mathbb{R}^d)}.$$

3.3 The space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions

3.3.1 Definition, examples

Definition 3.3.1 A distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ is said to be (a) tempered (distribution) when there exists $C > 0$ and $p \in \mathbb{N}$ such that, for all test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$|\langle T, \varphi\rangle \leq C N_p(\varphi).$$

Notation. We denote by $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions on $\mathbb{R}^d$.

Example 3.3.2 If $T \in \mathcal{E}'(\mathbb{R}^d)$, there is $C > 0$, $m \in \mathbb{N}$, such that for any $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$,

$$|\langle T, \varphi\rangle \leq C \sum_{|\alpha|\leq m} \sup |\partial^\alpha \varphi| \leq C N_p(\varphi).$$

Thus $\mathcal{E}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, that is compactly supported distributions are tempered.

Example 3.3.3 For $p \in [1, +\infty]$, we have $L^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$. Indeed, if $f \in L^p(\mathbb{R}^d)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, and for $q \in [1, +\infty]$ such that $1/p + 1/q = 1$, we have

$$|\langle T_f, \varphi\rangle| \leq |\int f(x)\varphi(x)dx| \leq \|f\|_{L^p}\|\varphi\|_{L^q} \leq C\|f\|_{L^p} N_{n+1}(\varphi),$$

thanks to Proposition 3.2.2. In particular, in view of Proposition 3.1.7, $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$.

Example 3.3.4 Let $f \in L^1_{loc}(\mathbb{R}^d)$ be such that, for some $C > 0$ and $p \in \mathbb{N}$,

$$\forall R > 0, \int_{|x|\leq R} |f(x)|\, dx \leq C(1+R)^p.$$

. Then we claim that f is a tempered distribution. Indeed, if $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} f\varphi\right| &\leq \int_{\mathbb{R}^d} |f(x)||\varphi(x)|\, dx = \int_{|x|<1} |f(x)||\varphi(x)|\, dx + \sum_{k=0}^{\infty} \int_{2^k \leq |x| < 2^{k+1}} |f(x)||\varphi(x)\, dx \\
&\leq 2^p C N_0(\varphi) + C \sum_{k=0}^{\infty} (1+2^{k+1})^p \sup_{2^k\leq|x|\leq 2^{k+1}} |\varphi(x)| \\
&\leq C' \left(N_0(\varphi) + \sum_{k=0}^{\infty} 2^{-k} \sup_{2^k\leq|x|\leq 2^{k+1}} |x|^{p+1}|\varphi(x)| \right) \\
&\leq C'' N_{p+1}(\varphi)\ .
\end{aligned}$$

Example 3.3.5 The previous example admits the following partial converse. If $f \in L^1_{loc}(\mathbb{R}^d)$ has nonnegative values and is a tempered distribution, then, for some $C > 0$ and $p \in \mathbb{N}$,

$$\forall R > 0, \int_{|x|\leq R} f(x)\, dx \leq C(1+R)^p.$$

Indeed, if f is tempered, we have, for some $A > 0$ et $p \in \mathbb{N}$,

$$\forall \varphi \in C_0^\infty(\mathbb{R}^d), \left| \int_{\mathbb{R}^d} f(x)\, \varphi(x)\, dx \right| \leq A\, N_p(\varphi) \ .$$

Let $\chi \geq 0$ be a plateau function over the unit ball centered at 0, supported into the ball of radius 2 centered at 0. Then

$$0 \leq \int_{|x| \leq R} f(x)\, dx \leq \int_{\mathbb{R}^d} f(x) \chi \left(\frac{x}{R} \right)\, dx \leq A\, N_p \left(\chi \left(\frac{\cdot}{R} \right) \right) \leq C(1+R)^p.$$

In particular, e^x is not tempered on $\mathbb{R}$, since, for every $p \in \mathbb{N}$,

$$R^{-p} \int_0^R e^x\, dx \underset{R \to \infty}{\longrightarrow} +\infty \ .$$

We conclude this first paragraph with a simple observation.

Proposition 3.3.6 If $T \in \mathcal{S}'(\mathbb{R}^d)$, then $x^\alpha \partial^\beta T \in \mathcal{S}'(\mathbb{R}^d)$ for all $\alpha, \beta \in \mathbb{N}^d$.

Proof.— It is sufficient to show that $x_j T$ and $\partial_j T$ are tempered distributions when T is. Since $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\langle x_j T, \varphi \rangle = \langle T, x_j \varphi \rangle$ and $\langle \partial_j T, \varphi \rangle = -\langle T, \partial_j \varphi \rangle$, the proposition follows directly from (3.1.1). □

Exercise 3.3.7 Show that the function $x \mapsto e^x e^{ie^x}$ is not bounded by a polynomial, but belongs to $\mathcal{S}'(\mathbb{R})$. Hint: it is the derivative of a tempered distribution.

The multiplication of a tempered distribution by a smooth function does not always yield a tempered distribution. However, it is the case when the function has moderate growth, in the following sense.

Definition 3.3.8 A function $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ has moderate growth when for any $\beta \in \mathbb{N}^d$, there is $C_\beta > 0$ and $m_\beta \in \mathbb{N}$ such that

$$|\partial^\beta f(x)| \leq C_\beta (1 + |x|)^{m_\beta}.$$

We denote $\mathcal{O}_M(\mathbb{R}^d)$ the set of such functions.

Proposition 3.3.9 If $T \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{O}_M(\mathbb{R}^d)$, then $fT \in \mathcal{S}'(\mathbb{R}^d)$.

Proof.— Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and $\alpha, \beta \in \mathbb{N}^d$. The Leibniz formula gives

$$|x^\alpha \partial^\beta (f\varphi)| \leq \sum_{\gamma \leq \alpha} \binom{\beta}{\gamma} |x^\alpha \partial^\gamma f|\, |\partial^{\beta-\gamma} \varphi| \leq C_\gamma \sum_{\gamma \leq \alpha} \binom{\beta}{\gamma} (1+|x|)^{m_\gamma} |\partial^{\beta-\gamma} \varphi|.$$

Then, as in (3.1.1), we obtain

$$N_p(f\varphi) \le CN_{p+M}(\varphi),$$

where $M = \max_{|\gamma|\le p} m_\gamma$. Thus, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, we obtain

$$|\langle fT, \varphi\rangle| = |\langle T, f\varphi\rangle| \le CN_p(f\varphi) \le C' N_{p+M}(\varphi),$$

and this means that fT is a tempered distribution. □

Exercise 3.3.10 Show that $\mathrm{vp}(1/x) \in \mathcal{S}'(\mathbb{R})$.

3.3.2 The fundamental characterization of $\mathcal{S}'$

Proposition 3.3.11 If T is a tempered distribution, then T extends in a unique way as a linear form $\widetilde{T}$ on $\mathcal{S}(\mathbb{R}^d)$ continuous in the following sense: if $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$, then $\langle \widetilde{T}, \varphi_n\rangle \to \langle \widetilde{T}, \varphi\rangle$.

Proof.— Let $\varphi \in \mathcal{S}(\mathbb{R})$. There is a sequence (φ_j) in $\mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$. The sequence $(\langle T, \varphi_j\rangle)_j$ is a Cauchy sequence in $\mathbb{C}$ since T is tempered:

$$|\langle T, \varphi_j - \varphi_k\rangle| \le CN_p(\varphi_j - \varphi_k). \tag{3.3.4}$$

Its limit does not depend on the choice of the sequence (φ_j), since when $\varphi_j, \psi_j \to \varphi$, we have

$$|\langle T, \varphi_j - \psi_j\rangle| \le CN_p(\varphi_j - \psi_j) \to 0.$$

Thus we can let $\tilde{T}$ be the linear form on $\mathcal{S}(\mathbb{R}^d)$ given by

$$\langle \tilde{T}, \varphi\rangle = \lim_{j\to+\infty} \langle T, \varphi_j\rangle,$$

where (φ_j) is any sequence in $\mathcal{C}_0^\infty(\mathbb{R}^d)$ which converges to φ. Since N_p is a continuous function on $\mathcal{S}$, passing to the limit in

$$|\langle T, \varphi_j\rangle| \le CN_p(\varphi_j),$$

we get

$$|\langle \tilde{T}, \varphi\rangle| \le CN_p(\varphi).$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. So $\tilde{T}$ is continuous.

Finally, if T_1 is another continuous extension of T to $\mathcal{S}(\mathbb{R}^d)$, we should have

$$\langle T_1, \varphi\rangle = \langle T_1, \varphi - \varphi_j\rangle + \langle T_1, \varphi_j\rangle \to 0 + \langle \tilde{T}, \varphi\rangle,$$

so that $T_1 = \tilde{T}$. □

Remark 3.3.12 For simplicity, we shall drop the $tilde$ in the sequel, and set T in place of $\tilde{T}$. Notice that, conversely, every continuous linear form on $\mathcal{S}(\mathbb{R}^d)$ satisfies a bound of the form

$$\forall \varphi \in \mathcal{S}, |\langle T, \varphi\rangle| \le C\, N_p(\varphi)$$

hence its restriction to $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is a tempered distribution. Summing up, tempered distributions identify to continuous linear forms on the Schwartz space $\mathcal{S}$. This explains the notation $\mathcal{S}'$. We shall use this identification systematically.

3.3.3 Convergence in $\mathcal{S}'(\mathbb{R}^d)$

Definition 3.3.13 Let (T_n) be a sequence of tempered distributions. One says that (T_n) tends to T in $\mathcal{S}'(\mathbb{R}^d)$ when for any function $\varphi \in \mathcal{S}(\mathbb{R}^d)$, it holds that $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$.

As it is the case in $\mathcal{D}'(\mathbb{R}^d)$, this notion of convergence, a weak one, implies a stronger one. We admit the following result.

Proposition 3.3.14 If $T_n \to T$ in $\mathcal{S}'$, there exists $C > 0$ and $p \in \mathbb{N}$ such that

$$\forall \varphi \in \mathcal{S}\ ,\ \forall n\ ,\ |\langle T_n, \varphi\rangle| \leq C\, N_p(\varphi)\ .$$

As we did for the convergence in $\mathcal{D}'$, the above proposition relies on a uniform boundedness principle, which follows from identifying the convergence on $\mathcal{S}$ as the one on a complete metric space.

Remark 3.3.15 When $T_n \to T$ in $\mathcal{S}'(\mathbb{R})$, it is true that $T_n \to T$ in $\mathcal{D}'(\mathbb{R})$ since $\mathcal{C}_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. The converse is not true in general, as shown by the following example: for any sequence (a_n) of complex numbers, the sequence $(a_n\delta_n)$ tends to 0 in $\mathcal{D}'(\mathbb{R})$. However it only converges (necessarily to 0) in $\mathcal{S}'(\mathbb{R}^d)$ if (a_n) has moderate growth, i.e. there is $C > 0$, $p \in \mathbb{N}$ such that

$$\forall n\ ,\ |a_n| \leq C(1+n)^p.$$

Remark 3.3.16 If $f_n \to f$ in $L^p(\mathbb{R}^d)$, then $f_n \to f$ in $\mathcal{S}'(\mathbb{R}^d)$. If $T_n \to T$ in $\mathcal{S}'(\mathbb{R}^d)$, then $fT_n \to fT$ in $\mathcal{S}'(\mathbb{R}^d)$ for all $f \in \mathcal{O}_M(\mathbb{R}^d)$.

3.4 The Fourier Transformation in $\mathcal{S}'(\mathbb{R}^d)$

3.4.1 Definition

Let $T \in \mathcal{S}'(\mathbb{R}^d)$ be a tempered distribution. The linear form on $\mathcal{S}(\mathbb{R}^d)$ given by $\varphi \mapsto \langle T, \hat{\varphi}\rangle$ is a tempered distribution since there exist $C > 0$ and $p \in \mathbb{N}$ such that

$$|\langle T, \hat{\varphi}\rangle| \leq C N_p(\hat{\varphi}) \leq C' N_{p+d+1}(\varphi),$$

thanks to Proposition 3.2.6.

Definition 3.4.1 For $T \in \mathcal{S}'(\mathbb{R}^d)$, we denote by $\hat{T} = \mathcal{F}(T)$ the tempered distribution given by

$$\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle,\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Example 3.4.2 *i)* For $f \in L^1$, we have by Fubini's theorem,

$$\begin{aligned}\langle \hat{f}, \varphi \rangle &= \int_{\mathbb{R}^d} f(x)\hat{\varphi}(x)\,dx = \int_{\mathbb{R}^d} f(x)\left(\int_{\mathbb{R}^d} \varphi(\xi)\,e^{-ix.\xi}\,d\xi\right)\,dx \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x)\,e^{-ix.\xi}\,dx\right)\varphi(\xi)\,d\xi = \int_{\mathbb{R}^d} \hat{f}(\xi)\varphi(\xi)d\xi,\end{aligned}$$

so that $\widehat{T_f} = T_{\hat{f}}$. The Fourier transformation on $\mathcal{S}'$ restricts to the classical Fourier transformation on L^1 (see also Proposition 3.4.5 below).

ii) For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\langle \widehat{\delta_0}, \varphi \rangle = \int \varphi(x)dx$, thus $\widehat{\delta_0} = 1$.

Proposition 3.4.3 The Fourier Transform $\mathcal{F}$ is an isomorphism on $\mathcal{S}'(\mathbb{R}^d)$. Its inverse is $\mathcal{F}^{-1} = (2\pi)^{-d}\check{\mathcal{F}}$. Moreover $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous on $\mathcal{S}'(\mathbb{R}^d)$, in the following sense: if $T_n \to T \in \mathcal{S}'(\mathbb{R}^d)$, then $\mathcal{F}(T_n) \to \mathcal{F}(T)$ in $\mathcal{S}'(\mathbb{R}^d)$.

These results follow immediatlely form the above definition and Proposition 3.2.6. The same way, transfering to $\mathcal{S}'(\mathbb{R}^d)$ the properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$, we obtain easily the following identities in $\mathcal{S}'(\mathbb{R}^d)$:

$$\mathcal{F}(D_jT) = \xi_j\hat{T},\ \mathcal{F}(x_jT) = -D_j\hat{T}, \mathcal{F}(\tau_aT) = e^{-ia.\xi}\mathcal{F}(T), \mathcal{F}\big(e^{ia.x}T\big) = \tau_a\mathcal{F}(T)\ .$$

Example 3.4.4 $\hat{1} = \mathcal{F}\circ\mathcal{F}(\delta_0) = (2\pi)^d\,\check{\delta_0} = (2\pi)^d\delta_0$.

3.4.2 The Fourier Transformation on L^1 and L^2

Here we briefly sum up the main properties of the Fourier transform of a tempered distribution given by an L^1 or an L^2 function.

Proposition 3.4.5 If $T = f \in L^1(\mathbb{R}^d)$, then $\mathcal{F}(T) = \hat{f}$. More precisely

i) $\mathcal{F}(T)$ is the continuous function given by $\mathcal{F}(T)(\xi) = \int e^{-ix\cdot\xi}f(x)dx$, and $\mathcal{F}(T)(\xi) \to 0$ when $|\xi| \to +\infty$.

ii) If moreover $\mathcal{F}(T)$ belongs to $L^1(\mathbb{R}^d)$, then $\mathcal{F}^{-1}(\mathcal{F}(T)) = T$ almost everywhere.

Proof.— The fact that $\hat{f}$ is a continuous function follows easily from Lebesgue theorem of continuity for integral with parameters,and the fact that $\hat{f}$ goes to 0 at infinity is called the Riemann-Lebesgue lemma. For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle = \int f(\xi)\hat{\varphi}(\xi)d\xi = \int f(\xi)\left(\int e^{-ix\cdot\xi}\varphi(x)dx\right)d\xi.$$

Since the function $(x, \xi) \mapsto f(\xi)e^{-ix\cdot\xi}\varphi(x)$ belongs to $L^1(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$, we have

$$\langle \hat{T}, \varphi\rangle = \int \varphi(x)(\int e^{-ix\cdot\xi} f(\xi)d\xi)dx = \langle \hat{f}, \varphi\rangle,$$

and this ends the proof of (i). We also know that $\mathcal{F}^{-1}(\hat{f}) = f$ in $\mathcal{S}'(\mathbb{R}^d)$. If $\hat{f} \in L^1(\mathbb{R}^d)$, we thus have $\mathcal{F}^{-1}(\hat{f}) = f$ in $\mathcal{D}'(\mathbb{R}^d)$, and this implies that $\mathcal{F}^{-1}(\hat{f}) = f$ almost everywhere. □

Proposition 3.4.6 The map $T \in \mathcal{S}'(\mathbb{R}^d) \mapsto (2\pi)^{-d/2}\mathcal{F}(T) \in \mathcal{S}'(\mathbb{R}^d)$ induces a bijective isometry on $L^2(\mathbb{R}^d)$.

Proof.— Let $f \in L^2(\mathbb{R}^d)$. For every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\langle \hat{f}, \varphi\rangle = \int_{\mathbb{R}^d} f(x)\hat{\varphi}(x)\, dx$$

and therefore, by the Cauchy–Schwarz inequality and the Plancherel identity (Corollary 3.2.10),

$$|\langle \hat{f}, \varphi\rangle| \leq \|f\|_{L^2}\|\hat{\varphi}\|_{L^2} = (2\pi)^{\frac{d}{2}}\|f\|_{L^2}\|\varphi\|_{L^2}.$$

By the Riesz representation theorem, this implies that $\hat{\in} L^2(\mathbb{R}^d)$ and

$$\|\hat{f}\|_{L^2} \leq (2\pi)^{\frac{d}{2}}\|f\|_{L^2}\ .$$

Applying this inequality to $\hat{f}$ and using the inversion formula from Proposition 3.4.3, we infer

$$(2\pi)^d\|f\|_{L^2} = \|\hat{\hat{f}}\|_{L^2} \leq (2\pi)^{\frac{d}{2}}\|\hat{f}\|_{L^2}$$

so that, eventually,

$$\|f\|_{L^2} = (2\pi)^{-\frac{d}{2}}\|\hat{f}\|_{L^2}\ .$$

Since it is clear from Proposition 3.4.3 that $\mathcal{F}$ is bijective on L^2, the proof is complete. □

Remark 3.4.7 There are functions f in $L^2(\mathbb{R}^d)$ such that $x \mapsto e^{-ix\cdot\xi}f(x)$ is not integrable whatever the value of ξ is (for example, for $d = 1$, $f(x) = (1 + |x|)^{-3/4}$). Nevertheless, for $R > 0$, the function g_R given by

$$g_R(\xi) = \int_{|x|<R} e^{-ix\cdot\xi} f(x)\, dx$$

tends to $\hat{f}$ in $L^2(\mathbb{R}^d)$ thanks to the Proposition 3.4.6, since $f1_{|x|<R} \to f$ in $L^2(\mathbb{R}^d)$. Thus, for $f \in L^2(\mathbb{R}^d)$,

$$\hat{f}(\xi) = \lim_{R\to+\infty} \int_{|x|<R} e^{-ix\cdot\xi} f(x)\, dx$$

in the L^2 sense.

3.4.3 The Fourier Transform of compactly supported distributions

The Fourier transformation exchanges the speed of decay at infinity of a function with the regularity of its image, as shown by example by the following inequality

$$\|D^\alpha \hat{\varphi}\|_{L^\infty} = \|\mathcal{F}(x^\alpha \varphi)\|_{L^\infty} \leq \|x^\alpha \varphi\|_{L^1},\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

The best speed of decay at infinity for a function is achieved for compactly supported ones ; the result of this subsection explores this phenomenon more precisely.

Proposition 3.4.8 If $T \in \mathcal{D}'(\mathbb{R}^d)$ is compactly supported, its Fourier transform $\mathcal{F}(T)$ is the smooth function on $\mathbb{R}^d$ given by

$$\mathcal{F}(T)(\xi) = \langle T_x, e^{-ix\cdot\xi}\rangle.$$

Moreover, there is an integer $m \in \mathbb{N}$ such that, for all $\alpha \in \mathbb{N}^d$, there is a $C_\alpha > 0$ satisfying

$$|\partial^\alpha \mathcal{F}(T)(\xi)| \leq C_\alpha (1 + |\xi|)^m.$$

Proof.— Differentiating under the bracket, we see that the function $v(\xi)$ given by

$$v(\xi) = \langle T_x, e^{-ix\cdot\xi}\rangle$$

is a $\mathcal{C}^\infty$ function (as a matter of fact, one should write $v(\xi) = \langle T_x, \chi(x)e^{-ix\cdot\xi}\rangle$, for some plateau function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ above the support of T, and then apply the theorem). We also have

$$\partial^\alpha v(\xi) = \langle T_x, (-ix)^\alpha e^{-ix\cdot\xi}\rangle,$$

where, for some constant $C > 0$, an integer $m \in \mathbb{N}$ and a compact set K that depend only on T,

$$|\partial^\alpha v(\xi)| \leq C \sum_{|\beta| \leq m} \sup_{x \in K} |\partial_x^\beta \big((-ix)^\alpha e^{-ix\cdot\xi}\big)| \leq C_\alpha (1 + |\xi|)^m$$

Last, we have $\hat{T} = v$. Indeed, for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, thanks to the result about integrating under the bracket,

$$\langle \hat{T}, \varphi\rangle = \langle T_x, \chi(x) \int e^{-ix\cdot\xi} \varphi(\xi) d\xi\rangle = \int \langle T_x, \chi(x) e^{-ix\cdot\xi}\rangle \varphi(\xi) d\xi,$$

where $\hat{T}(\xi) = \langle T_x, \chi(x)e^{-ix\cdot\xi}\rangle$. □

3.4.4 Fourier characterization of Sobolev spaces on $\mathbb{R}^d$

We recall that Sobolev spaces $H^s(\Omega)$ were introduced in Chapter 2 for $s \in \mathbb{N}$ and Ω an open subset of $\mathbb{R}^d$. In this subsection, we use the Fourier transform to provide more information about these spaces when $\Omega = \mathbb{R}^d$.

For $\xi \in \mathbb{R}^d$, we set $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. The function $\xi \mapsto \langle \xi \rangle$ is smooth on $\mathbb{R}^d$, and there is a constant $C > 0$ such that, for $|\xi| \geq 1$,

$$\frac{1}{C}|\xi| \leq \langle \xi \rangle \leq C|\xi|.$$

Thus, $\langle\xi\rangle$ is a regularized version of $|\xi|$, in the sense that it has the same behavior at infinity. Furthermore, since this function is smooth and since all its derivatives have growth at most polynomial at infinity, it acts on $\mathcal{S}'(\mathbb{R}^d)$ by multiplication.

Proposition 3.4.9 Let $s \in \mathbb{N}$ and $u \in \mathcal{S}'(\mathbb{R}^d)$. Then $u \in H^s(\mathbb{R}^d)$ if and only if $\langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^d)$. Furthermore, there exists $C_s > 0$ such that

$$\forall u \in H^s(\mathbb{R}^d)\ ,\ C_s^{-1}\|\langle\xi\rangle^s \hat{u}\|_{L^2} \le \|u\|_{H^s} \le C_s\|\langle\xi\rangle^s \hat{u}\|_{L^2}\ .$$

Proof.— If $u \in H^s(\mathbb{R}^d)$, then $\partial^\alpha u \in L^2(\mathbb{R}^d)$ for $|\alpha| \le s$. By the Plancherel theorem, this means

$$\xi^\alpha \hat{u} \in L^2(\mathbb{R}^d)\ ,\ |\alpha| \le s\ .$$

Now observe that

$$\langle\xi\rangle^{2s} = (1 + \xi_1^2 + \cdots + \xi_d^2)^s = \sum_{|\alpha| \le s} c_\alpha \xi^{2\alpha}$$

for some positive constants c_α. Consequently,

$$\langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 = \sum_{|\alpha| \le s} c_\alpha \xi^{2\alpha} |\hat{u}(\xi)|^2 \in L^1(\mathbb{R}^d)\ ,$$

whence $\langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^d)$.

Conversely, if $\langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^d)$, then, for $|\alpha| \le s$, $\xi^\alpha \langle\xi\rangle^{-s}$ is bounded on $\mathbb{R}^d$ and we have

$$\xi^\alpha \hat{u} = \xi^\alpha \langle\xi\rangle^{-s}\, \langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^d),$$

which precisely means that $\partial^\alpha u \in L^2(\mathbb{R}^d)$.

The inequality follows from these considerations and from the Plancherel formula. □

Proposition 3.4.9 suggests to extend the definition of $H^s(\mathbb{R}^d)$ to every real number s as follows.

Definition 3.4.10 Let $s \in \mathbb{R}$. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $H^s(\mathbb{R}^d)$ if $\langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^d)$.

Example 3.4.11 *i)* $\delta_0 \in H^s(\mathbb{R}^d)$ if and only if $s < \frac{-d}{2}$. Indeed $\hat{\delta}_0 = 1$, so that $\langle\xi\rangle^s \hat{\delta}_0 \in L^2(\mathbb{R}^d)$ if and only if $2s < -d$.

ii) Constant functions do not belong to $H^s(\mathbb{R}^d)$, since $\hat{1} = (2\pi)^d \delta_0$ is not in L^1_{loc}.

Here follows another illustration of the fact that H^s contains elements that are more and more singular as s decreases.

Proposition 3.4.12 Let $T \in \mathcal{E}'(\mathbb{R}^d)$ be a compactly supported distribution, with order $m \geq 0$. Then $T \in H^s(\mathbb{R}^d)$for any $s < -m - \frac{d}{2}$.

Proof.— For $T \in \mathcal{E}'(\mathbb{R}^d)$, we know that $\hat{T} \in \mathcal{C}^\infty$. Moreover

$$|\langle\xi\rangle^s \hat{T}(\xi)| = |\langle\xi\rangle^s \langle T_x, e^{-ix\cdot\xi}\rangle| \leq C\langle\xi\rangle^s \sum_{|\alpha|\leq m} \sup|\partial_x^\alpha(e^{-ix\cdot\xi})| \leq C\langle\xi\rangle^{s+m}$$

Thus $T \in H^s(\mathbb{R}^d)$ when $2(s+m) < -d$, as stated. □

We close this subsection by proving a connection between H^s regularity and $\mathcal{C}^k$ regularity.

We denote by $\mathcal{C}^k_{\to 0}(\mathbb{R}^d)$ the space of $\mathcal{C}^k$ functions on $\mathbb{R}^d$ that tends to 0 at infinity, as well as all their derivatives of order $\leq k$.

Proposition 3.4.13 (Sobolev embedding) If $s > \frac{d}{2}+k$, then every element of $H^s(\mathbb{R}^d)$ belongs to $\mathcal{C}^k_{\to 0}(\mathbb{R}^d)$, and the embedding $H^s(\mathbb{R}^d) \to \mathcal{C}^k_{\to 0}(\mathbb{R}^d)$ is continuous.

Proof.— Let $u \in H^s(\mathbb{R}^d)$. For $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leq k$, we have $\xi^\alpha \hat{u} \in L^1$. Indeed,

$$|\xi^\alpha \hat{u}(\xi)| \leq \frac{|\xi|^{|\alpha|}}{\langle\xi\rangle^s}\langle\xi\rangle^s|\hat{u}(\xi)| \leq \langle\xi\rangle^{k-s}\langle\xi\rangle^s|\hat{u}(\xi)|,$$

and $\langle\xi\rangle^{k-s} \in L^2(\mathbb{R}^d)$ since $-2(k-s) > d$. By the Cauchy-Schwarz inequality, we thus get

$$\|\xi^\alpha \hat{u}\|_{L^1} \leq C_{s,n}\|u\|_{H^s}. \tag{3.4.5}$$

Therefore $\partial^\alpha u = \mathcal{F}^{-1}((i\xi)^\alpha \hat{u}) \in \mathcal{C}^0_{\to 0}$ by Proposition 3.4.5, and the fact that the identity from $H^s(\mathbb{R}^d)$ to $\mathcal{C}^k_{\to 0}(\mathbb{R}^d)$ is continuous is just a way to read the inequalities

$$\forall |\alpha| \leq k,\ \|\partial^\alpha u\|_{L^\infty} \leq \|\xi^\alpha \hat{u}\|_{L^1} \leq C_{s,n}\|u\|_{H^s}.$$

□ Another interesting consequence of Proposition 3.4.9 is that we recover Theorem 2.5.4 on $\mathbb{R}^d$ in a very general way, with optimal regularity of the solution.

Proposition 3.4.14 For every $s \in \mathbb{R}^d$, the mapping $u \in H^s(\mathbb{R}^s) \mapsto u - \Delta u \in H^{s-2}(\mathbb{R}^d)$ is an isomorphism.

Proof.— Indeed, this mapping is conjugated through Fourier transform to the mapping

$$\hat{u} \mapsto \langle\xi\rangle^2 \hat{u}$$

so that the statement becomes trivial in view of Proposition 3.4.9 or the subsequent definition of $H^s(\mathbb{R}^d)$. □

Combining the above three propositions, we obtain he following extension of Theorem 1.9.16 to several dimensions.

Theorem 3.4.15 For every distribution T on $\mathbb{R}^d$ with compact support, there exist $p \in \mathbb{N}$ and a family $(f_\alpha)_{|\alpha|\leq p}$ of continuous functions on $\mathbb{R}^d$ such that

$$T = \sum_{|\alpha|\leq p} \partial^\alpha (f_\alpha) \ .$$

Proof.— From Proposition 3.4.12, we know that, for some $s \in \mathbb{R}$, $T \in H^s$ Let q be a positive integer such that

$$s + 2q > \frac{d}{2} \ .$$

By Proposition 3.4.14 applied q times, there exists $g \in H^{s+2q}$ unique such that

$$(I - \Delta)^q g = T \ .$$

By Proposition 3.4.13, we know that g is continuous. It remains to write

$$T = (I - \Delta)^q g = \left(1 - \sum_{j=1}^d \partial_j^2\right)^q g = \sum_{|\beta|\leq q} c_\beta \partial^{2\beta} g$$

and the proof is complete. □

Finally, we introduce spaces of distributions in an arbitrary open set of $\mathbb{R}^d$ which will be very useful in studying local regularity of solutions of partial differential equations.

Definition 3.4.16 Let Ω be an open subset of $\mathbb{R}^d$ and let $s \in \mathbb{R}$. We denote by $H^s_{\mathsf{loc}}(\Omega)$ the space of distributions $T \in \mathcal{D}'(\Omega)$ such that, for any $\chi \in \mathcal{C}_0^\infty(\Omega)$, the extension to $\mathbb{R}^d$ by 0 of the compactly supporetd distribution χu belongs to $H^s(\mathbb{R}^d$.

We notice the following important regularity theorem.

Proposition 3.4.17

$$\bigcap_{s\in\mathbb{R}} H^s_{\mathsf{loc}}(\Omega) = \mathcal{C}^\infty(\Omega) \ .$$

Proof.— If $u \in \mathcal{C}^\infty(\Omega)$, then, for any $\chi \in \mathcal{C}_0^\infty(\Omega)$, $\chi u \in \mathcal{C}_0^\infty(\Omega)$, so that its extension to $\mathbb{R}^d$ by 0 is such that $\widehat{\chi u}$ belongs to $\mathcal{S}(\mathbb{R}^d)$, and consequently $\chi u \in H^s$ for every s.
Conversely, if $u \in H^s_{\mathsf{loc}}(\Omega)$ for every $s \in \mathbb{R}$, then, any $\chi \in \mathcal{C}_0^\infty(\Omega)$, $\chi u \in H^s(\mathbb{R}^d)$ for every s, hence, from Proposition 3.4.13, $\chi u \in \mathcal{C}^k_{\to 0}(\mathbb{R}^d)$ for every k, in particular $\chi u \in C^\infty(\mathbb{R}^d)$. Since this holds for any $\chi \in \mathcal{C}_0^\infty(\Omega)$, this completes the proof. □

3.5 Some applications

3.5.1 Partial differential equations with constant coefficients

Let $p \in \mathbb{C}[X_1, \dots, X_d]$ be a polynomial of n variables with complex coefficients,

$$p(X) = \sum_{|\alpha| \le m} a_\alpha X^\alpha, \quad X \in \mathbb{R}^d,$$

with $a_\alpha \in \mathbb{C}$. The integer $m \in \mathbb{N}$ is the degree of P. Given an open subset Ω of $\mathbb{R}^d$, we denote by $p(\partial)$ the operator on $\mathcal{D}'(\Omega)$ given by

$$\mathcal{D}'(\Omega) \ni T \mapsto p(\partial)T = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha T \in \mathcal{D}'(\Omega).$$

Those operators are called linear partial differential operators with constant coefficients. The equation

$$p(\partial)u = f,$$

where $f \in \mathcal{D}'(\Omega)$ is given, and $u \in \mathcal{D}'(\Omega)$ is the unknown, is called a Partial Differential Equation (PDE) of order m with constant coefficients. When $f \neq 0$, it is said to be inhomogeneous, or with source term f.

Remark 3.5.1 For $d = 1$, the equation $p(\partial)u = f$ is a linear differential equation of order m with constant coefficients, that can be explicitly solved. For $d > 1$, the situation is drastically different and it may even be very difficult to show existence of solutions.

Example 3.5.2 If $p(X_1, \dots, X_d) = X_1^2 + \dots + X_d^2$, then

$$p(\partial)T = \partial_1^2 T + \dots + \partial_d^2 T = \Delta T \ .$$

We already encountered operator Δ which is called the Laplacean, and equation $\Delta u = f$ is called the Laplace (or Poisson) equation. A specificity of Δ is that it is invariant by the rotations of $\mathbb{R}^d$. Moreover, one can show that every linear partial differential operator with constant coefficients on $\mathbb{R}^d$ which is invariant by rotations, is of the form

$$P = Q(\Delta)$$

where Q is a polynomial of one variable. This invariance property explains why the Laplacean appears so frequently in many areas of Mathematical Physics.

The following proposition is an easy consequence of properties of the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$

Proposition 3.5.3 Let $p \in \mathbb{C}[X_1, \dots, X_d]$ and $P = p(\partial)$ be the corresponding differential operator. For every $u \in \mathcal{S}'(\mathbb{R}^d)$, we have

$$\widehat{Pu} = p(i\xi)\hat{u} \ .$$

The importance of the above proposition appears through the multiplicity of its consequences. We shall draw some of them in the next subsections.

3.5.2 Local results

Let $P = p(\partial)$ be a linear partial differential operators with constant coefficients. In this section, we study some properties of P on $\mathcal{D}'(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$. We start with a very natural statement.

Proposition 3.5.4 If P is a linear partial differential operators with constant coefficients of order $\leq m$ and $s \in \mathbb{R}$, then $P : H^s_{\mathrm{loc}}(\Omega) \to H^{s-m}_{\mathrm{loc}}(\Omega)$.

Proof.— It is based on the following lemma.

Lemma 3.5.5 If P is a linear partial differential operators with constant coefficients of order $\leq m$ and $\varphi \in \mathcal{C}_0^\infty(\Omega)$, there exists functions $\varphi_\beta \in \mathcal{C}_0^\infty(\Omega)$, $|\beta| \leq m-1$, such that, for every $u \in \mathcal{D}'(\Omega)$,

$$P(\varphi u) = \varphi P u + \sum_{|\beta| \leq m-1} \partial^\beta(\varphi_\beta u) \ .$$

Let us prove this lemma. By linearity, it is enough to prove it for $P = \partial^\alpha$. We proceed by induction on $|\alpha|$. If $\alpha = 0$, the formula trivially holds. Assume it is true for $|\alpha| \leq m-1$, and let us prove it for $|\alpha| = m$. By the Leibniz formula,

$$\partial^\alpha(\varphi u) = \varphi \partial^\alpha u + \sum_{\beta \leq \alpha, \beta \neq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\varphi\, \partial^\beta u \ .$$

Using the induction hypothesis, we may write each term in the second part of the right hand side as a sum of terms $\partial^\gamma(\psi_\gamma u)$, with $|\gamma| \leq m-1$. This completes the proof of Lemma 3.5.5.
Coming back to the proof of Proposition 3.5.4, we use Lemma 3.5.5 to write

$$P(\varphi u) = \varphi P u + \sum_{|\beta| \leq m-1} \partial^\beta(\varphi_\beta u) \ .$$

If $u \in H^s_{\mathrm{loc}}(\Omega)$, then $\varphi u, \varphi_\beta u \in H^s(\mathbb{R}^d)$ and thus

$$P(\varphi u) \in H^{s-m}(\mathbb{R}^d) \ , \ \partial^\beta(\varphi_\beta u) \in H^{s-m+1}(\mathbb{R}^d) \ .$$

This leads to $\varphi P u \in H^{s-m}(\mathbb{R}^d)$, so that $Pu \in H^{s-m}_{\mathrm{loc}}(\mathbb{R}^d)$. □

A classical problem is the local regularity of solutions $u \in \mathcal{D}'(\Omega)$ of equation $p(\partial)u = f$, knowing the local regularity of f. It is possible to give a general answer to this question under some assumption on p, which we now define.

Definition 3.5.6 Operator $p(\partial) = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ is called elliptic of order m if the highest homogeneous part $p_m = \sum_{|\alpha| = m} a_\alpha X^\alpha$ of polynomial p satisfies

$$\forall \xi \in \mathbb{R}^d \setminus \{0\} \ , \ p_m(\xi) \neq 0 \ .$$

For instance, the Laplace operator is elliptic, while the operator $\partial_1^2 - \partial_2^2$ is not.

Theorem 3.5.7 Let P be a linear partial differential operators with constant coefficients, elliptic of order m, and let Ω be an open subset of $\mathbb{R}^d$. If $u \in \mathcal{D}'(\Omega)$ satisfies $Pu \in H^s_{\text{loc}}(\Omega)$ for some $s \in \mathbb{R}$, then $u \in H^{s+m}_{\text{loc}}(\Omega)$. In particular, if $Pu \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

Proof.— Let us first prove the result under the additional assumption that $u \in H^\sigma_{\text{loc}}(\Omega)$ for some $\sigma \in \mathbb{R}$. Since p is elliptic and since the unit sphere is compact, there exists $c > 0$ such that, for every $\xi \in \mathbb{R}^d$ with $|\xi| = 1$,

$$|p_m(i\xi)| \geq c \ .$$

By homogeneity, this implies

$$\forall \xi \in \mathbb{R}^d \ , \ |p_m(\xi)| \geq c|\xi|^m \ .$$

Since

$$p(i\xi) - i^m p_m(i\xi) = O\left(|\xi|^{m-1}\right)$$

as $|\xi| \to \infty$, we infer that there exits $R > 0$ such that, for every $\xi \in \mathbb{R}^d$ with $|\xi| \geq R$,

(3.5.6) $$|p(i\xi)| \geq \frac{c}{2}|\xi|^m \ .$$

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a plateau function on the ball $B(0,R)$. The function

$$q(\xi) = \frac{1-\chi(\xi)}{p(i\xi)}$$

is C^∞ on $\mathbb{R}^d$ and satisfies, due to estimate (3.5.6),

$$|q(\xi)| = O(\langle \xi \rangle^{-m}) \ .$$

Therefore we can define an operator $Q : H^\tau(\mathbb{R}^d) \to H^{\tau+m}(\mathbb{R}^d)$ for every $\tau \in \mathbb{R}$ by

$$\widehat{Q(T)} = q\hat{T} \ .$$

Now let $u \in H^\sigma_{\text{loc}}(\Omega)$ such that $Pu \in H^s_{\text{loc}}(\Omega)$. For every $\varphi \in \mathcal{C}_0^\infty(\Omega)$, Lemma 3.5.5 yields

$$P(\varphi u) = \varphi P u + \sum_{|\alpha| \leq m-1} \partial^\alpha(\varphi_\alpha u) \ ,$$

where $\varphi_\alpha \in \mathcal{C}_0^\infty(\Omega)$. Consequently,

$$QP(\varphi u) = Q(\varphi P u) + \sum_{|\alpha| \leq m-1} Q(\partial^\alpha(\varphi_\alpha u)) \ .$$

In the right hand side, the first term belongs to $H^{s+m}(\mathbb{R}^d)$ while the other terms belong to $H^{\sigma+1}(\mathbb{R}^d)$. By Proposition 3.5.3, $QP(\varphi u) = \varphi u - R(\varphi u)$, where

$$\widehat{Rv} = \chi \hat{v} \ ,$$

so that $R(\varphi u) \in H^\tau(\mathbb{R}^d)$ for every $\tau \in \mathbb{R}$. We infer that $\varphi u \in H^{\min(s+m,\sigma+1)}(\mathbb{R}^d)$, and so

$$u \in H^{\min(s+m,\sigma+1)}_{\mathrm{loc}}(\Omega) .$$

If $\sigma+1 < s+m$, we iterate this result replacing σ by $\sigma+1$, and, after a sufficient number of iterations, we conclude

$$u \in H^{s+m}_{\mathrm{loc}}(\Omega) .$$

Now let $u \in \mathcal{D}'(\Omega)$ such that $Pu \in H^s_{\mathrm{loc}}(\Omega)$. Let V be an open subset of Ω such that $\overline{V} \subset \Omega$ is compact. If $\psi \in C_0^\infty(\Omega)$ is a plateau function on $\overline{V}$, Proposition 3.4.12 implies that there exists $\sigma \in \mathbb{R}$ such that $\psi u \in H^\sigma(\mathbb{R}^d)$. The following lemma implies that

$$u_{|V} \in H^\sigma_{\mathrm{loc}}(V) .$$

Lemma 3.5.8 If $\sigma \in \mathbb{R}$, $v \in H^\sigma(\mathbb{R}^d)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, then $\varphi v \in H^\sigma(\mathbb{R}^d)$.

Assuming this lemma, we can apply to $u_{|V}$ the first part of the proof and conclude that $u \in H^{s+m}_{\mathrm{loc}}(V)$. Since V is an arbitrary open subset of Ω such that $\overline{V} \subset \Omega$ is compact, we conclude that $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$.

Let us prove Lemma 3.5.8. We have, from Proposition 3.4.8,

$$\widehat{\varphi v}(\xi) = \langle v, \varphi \mathrm{e}^{-ix.\xi} \rangle.$$

Using the inversion formula and the properties of the Fourier transform on $\mathcal{S}$,

$$\langle v, \varphi \mathrm{e}^{-ix.\xi} \rangle = \langle \hat v, \hat\varphi(\xi - .) \rangle = \int_{\mathbb{R}^d} \hat v(\eta) \hat\varphi(\xi-\eta)\, d\eta .$$

This leads to

$$\langle \xi \rangle^\sigma \widehat{\varphi v}(\xi) = \langle \xi \rangle^\sigma \int_{\mathbb{R}^d} \hat v(\eta)\hat\varphi(\xi-\eta)\, d\eta = \langle \xi \rangle^\sigma \int_{\mathbb{R}^d} \hat v(\xi-\eta)\hat\varphi(\eta)\, d\eta,$$

and we have to prove that the right hand side is square integrable in ξ on $\mathbb{R}^d$. We decompose the integral into two parts, corresponding to the domains $|\eta| \le |\xi|/2$ and $|\eta| > |\xi|/2$. Using the rapid decay of $\hat\varphi$, the second integral is rapidly decreasing in ξ. As for the first integral, we apply the Cauchy–Schwarz inequality,

$$\langle \xi \rangle^{2\sigma} \left| \int_{|\eta|\le|\xi|/2} \hat v(\xi-\eta)\hat\varphi(\eta)\, d\eta \right|^2 \le \left(\int_{\mathbb{R}^d} |\hat\varphi(\eta)|\, d\eta \right) \left(\int_{|\eta|\le|\xi|/2} \langle \xi \rangle^{2\sigma} |\hat v(\xi-\eta)|^2 |\hat\varphi(\eta)|\, d\eta \right) .$$

Since $|\eta| \le |\xi|/2$, we have $\langle \xi \rangle \le A\langle \xi-\eta \rangle$ for some constant $A > 0$, and therefore the right hand side is bounded by

$$B \int_{\mathbb{R}^d} \langle \xi-\eta \rangle^{2\sigma} |\hat v(\xi-\eta)|^2 |\hat\varphi(\eta)|\, d\eta ,$$

whose integral with respect to ξ is finite by Fubini's theorem. This completes the proof. □

3.5.3 Global results

We start with solutions of homogeneous equations.

Corollary 3.5.9 With the notation of Proposition 3.5.3, we have

i) If $p(i\xi) \neq 0$ for every $\xi \in \mathbb{R}^d$, then the only tempered solution u of $Pu = 0$ is $u = 0$.

ii) If $p(i\xi) \neq 0$ for every $\xi \in \mathbb{R}^d \backslash \{0\}$, then every tempered solution of $Pu = 0$ is a polynomial function.

Proof.— If $u \in \mathcal{S}'$ satisfies $Pu = 0$, then $p(i\xi)\hat{u} = 0$, which implies that the distribution $\hat{u}$ is supported in the set

$$\{\xi \in \mathbb{R}^d, p(i\xi) = 0\}$$

If this set is empty, this implies $\hat{u} = 0$, hence $u = 0$, whence (i).

If this set is $\{0\}$, this implies, from the structure of the distributions supported at one point only, that $\hat{u}$ is a finite linear combination of derivatives of δ_0. Applying $\mathcal{F}^{-1}$, we conclude that u is a polynomial function, which is (ii). □

Remark 3.5.10 A consequence of Corollary 3.5.9 is that, under assumption ii), bounded solutions of $Pu = 0$ on $\mathbb{R}^d$ are constants. Indeed, since $L^\infty(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, this follows from the elementary fact claiming that a polynomial function which is bounded on $\mathbb{R}^d$ is constant. This is trivial if $d = 1$, but less trivial if $d \geq 2$. Let us sketch a proof of it. Let $p \in \mathbb{C}[X_1, \dots, X_d]$ be such that function $x \in \mathbb{R}^d \mapsto p(x) \in \mathbb{C}$ is bounded. For every $y \in \mathbb{R}^d$, consider

$$p_y(t) = p(ty) \ , \ t \in \mathbb{R} \ .$$

Function p_y is a one variable polynomial function, which is bounded, hence it is a constant. Writing

$$p(X) = \sum_{|\alpha| \leq m} a_\alpha X^\alpha,$$

this implies, for every $r = 1, \dots, m$, for every $y \in \mathbb{R}^d$,

$$\sum_{|\alpha| = r} a_\alpha y^\alpha = 0 \ .$$

Taking the ∂^α derivative of the left hand side, we infer $a_\alpha = 0$.

Example 3.5.11

- Case (i) is fulfilled by $P = \Delta + \lambda$ if $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$.
- Case (ii) is fulfilled by $P = \Delta$ on $\mathbb{R}^d$. Consequently, tempered harmonic functions on $\mathbb{R}^d$ are polynomial functions. In particular, we infer the strong Liouville theorem : any bounded harmonic function on $\mathbb{R}^d$ is a constant.

  Case (ii) is also fulfilled by $P = \partial_1 + i\partial_2$ on $\mathbb{R}^2$. In other words, the only tempered entire functions on $\mathbb{C}$ are polynomial, and again (Liouville), we recover that the only bounded entire functions are constant.

The second application concerns fundamental solutions.

Definition 3.5.12 Let $P = p(\partial)$ be a linear partial differential operators with constant coefficients. One says that $E \in \mathcal{D}'(\mathbb{R}^d)$ is a fundamental solution of P when $PE = \delta_0$.

Notice that in Physics, fundamental solutions are often called Green functions. The importance of fundamental solutions is provided by the following remark.

Proposition 3.5.13 If P has a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^d)$, then for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, the equation $Pu = \varphi$ has a solution given by $u = E * \varphi$.

Proof.— In chapter 2, we saw that $\partial_j(T * \varphi) = T * \partial_j \varphi = \partial_j T * \varphi$, hence

$$P(E * \varphi) = (PE) * \varphi = \delta * \varphi = \varphi,$$

so that $u = E * \varphi$ is a solution of $Pu = \varphi$. □

Using a convenient definition of convolution of distributions, it is in fact possible to extend the above proposition to any right hand side which is a compactly supported distribution. We shall see an example of this in Theorem 3.5.17 below.

B. Malgrange and L. Ehrenpreis have proved, independently in 1954/1955, that any non trivial linear partial differential operator with constant coefficients has a fundamental solution. It is fact possible to prove that one can choose this distribution to be tempered. The proofs of these results are beyond the scope of these lectures. In what follows, we rather study the important case of the Laplace operator.

Proposition 3.5.14 Let $d \geq 3$, and let $E_d \in \mathcal{D}'(\mathbb{R}^d)$ be the $L^1_{\text{loc}}(\mathbb{R}^d)$ function given by

$$E_d(x) = \frac{1}{(d-2)\sigma(S^{d-1})|x|^{d-2}}.$$

Then

$$-\Delta E_d = \delta_0 \ .$$

Proof.— We are going to use Fourier transformation. Indeed, by Proposition 3.5.3, in $\mathcal{S}'(\mathbb{R}^d)$, the equation

$$-\Delta E = \delta_0$$

is equivalent to

$$|\xi|^2 \hat{E} = 1 \ .$$

Since $d \geq 3$, the function $1/|\xi|^2$ is locally integrable in $\mathbb{R}^d$. More precisely, by decomposing it as the sum of the contributions for $|\xi| \leq 1$ and for $|\xi| > 1$, one shows that this function is the sum of an L^1 function and of an L^∞ function. As a consequence, $1/|\xi|^2$ belongs to $\mathcal{S}'(\mathbb{R}^d)$, and therefore

$$E_d := \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2}\right)$$

is a tempered fundamental solution of $-\Delta$. It remains to calculate E_d. Since $-\Delta E_d = 0$ on $\mathbb{R}^d \setminus \{0\}$, we know from Theorem 3.5.7 that E_d is a C^∞ function on $\mathbb{R}^d \setminus \{0\}$. We claim that this function is of the form

$$E_d(x) = \frac{c_d}{|x|^{d-2}}$$

for some constant c_d. This is equivalent to the following statement : for every rotation R of $\mathbb{R}^d$, for every $\lambda > 0$, for every $x \in \mathbb{R}^d \setminus \{0\}$,

$$E_d(\lambda R x) = \lambda^{2-d} E_d(x) \ ,$$

or equivalently, for every $\varphi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$,

$$\int_{\mathbb{R}^d} E_d(\lambda R x)\varphi(x)\, dx = \lambda^{2-d} \int_{\mathbb{R}^d} E_d(x)\varphi(x)\, dx \ . \tag{3.5.7}$$

Let $Ax := \lambda R x$ and $\varphi_A(x) := |\operatorname{det} A|^{-1}\varphi(A^{-1}x)$. We have

$$\begin{aligned}
\int_{\mathbb{R}^d} E_d(Ax)\varphi(x)\, dx &= \int_{\mathbb{R}^d} E_d(x)\varphi_A(x)\, dx \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\widehat{\varphi_A}(\xi)}{|\xi|^2}\, d\xi \\
\int_{\mathbb{R}^d} E_d(x)\varphi(x)\, dx &= (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{\varphi}(\xi)}{|\xi|^2}\, d\xi \ .
\end{aligned}$$

Furthermore,

$$\widehat{\varphi_A}(\xi) = \int_{\mathbb{R}^d} |\operatorname{det} A|^{-1}\varphi(A^{-1}x)\mathrm{e}^{-ix.\xi}\, dx = \int_{\mathbb{R}^d} \varphi(x)\mathrm{e}^{-iAx.\xi}\, dx = \hat{\varphi}({}^tA\xi) \ .$$

Consequently,

$$\int_{\mathbb{R}^d} \frac{\widehat{\varphi_A}(\xi)}{|\xi|^2}\, d\xi = \int_{\mathbb{R}^d} \frac{\varphi({}^tA\xi)}{|\xi|^2}\, d\xi = |\operatorname{det} A|^{-1} \int_{\mathbb{R}^d} \frac{\varphi(\xi)}{|{}^tA^{-1}\xi|^2}\, d\xi \ .$$

Since ${}^tA^{-1} = \lambda^{-1}R$, we conclude

$$\int_{\mathbb{R}^d} \frac{\widehat{\varphi_A}(\xi)}{|\xi|^2}\, d\xi = \lambda^{2-d} \int_{\mathbb{R}^d} \frac{\hat{\varphi}(\xi)}{|\xi|^2}\, d\xi$$

and identity (3.5.7) is proved.
In order to calculate c_d, it is enough to check against the Gaussian function G_1,

$$c_d \int_{\mathbb{R}^d} \frac{G_1(x)}{|x|^{d-2}}\, dx = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\hat{G}_1(\xi)}{|\xi|^2}\, d\xi \ .$$

Since $\hat{G}_1(\xi) = (2\pi)^{d/2} G_1(\xi)$, we infer

$$c_d \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-|x|^2/2}}{|x|^{d-2}}\, dx = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-|\xi|^2/2}}{|\xi|^2}\, d\xi \ .$$

Passing in spherical coordinates, this reads

$$c_d \int_0^\infty r\mathrm{e}^{-r^2/2}\, dr = (2\pi)^{-d/2} \int_0^\infty r^{d-3}\mathrm{e}^{-r^2/2}\, dr \ ,$$

or

$$c_d = (2\pi)^{-d/2} 2^{(d-4)/2} \int_0^\infty t^{(d-4)/2}\mathrm{e}^{-t}\, dt = \frac{1}{4}\pi^{-d/2}\Gamma\left(\frac{d}{2} - 1\right)$$

Recall that, by applying the integration formula on spherical level sets from Corollary 2.4.8 to a Gaussian function, one gets

$$\sigma(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \frac{4\pi^{d/2}}{(d-2)\Gamma(d/2-1)}$$

so that

$$c_d = \frac{1}{(d-2)\sigma(S^{d-1})} \ .$$

□

Remark 3.5.15 The value of c_d can also be determined by applying the Gauss–Green formula to the integral of $|x|^{2-d}\Delta\varphi(x)$ outside a ball of radius ε.

Remark 3.5.16 It is possible to extend Proposition 3.5.14 to $d = 1, 2$. For $d = 1$, it is easy to check that

$$E_1(x) = -\frac{1}{2}|x|$$

is a fundamental solution of $-\partial^2$. The case $d = 2$ is more delicate. The above approach by Fourier analysis has to be slightly modified, because $1/|\xi|^2$ is not locally integrable in $\mathbb{R}^2$. A good way to do this is to consider the operator

$$P = \partial_1 + i\partial_2 \ .$$

Since $1/(\xi_1 + i\xi_2) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, the distribution $E \in \mathcal{S}'(\mathbb{R}^2)$ such that

$$\hat{E} = \frac{1}{i(\xi_1 + i\xi_2)}$$

satisfies $PE = \delta_0$. Furthermore, from Theorem 3.5.7, E is a holomorphic function outside $\{0\}$, and the above proof shows that it is homogeneous of degree -1. This finally leads to

$$E(x) = \frac{1}{2\pi(x_1 + ix_2)}$$

and, observing that

$$E = (\partial_1 - i\partial_2)[(4\pi)^{-1}\log(x_1^2 + x_2^2)]$$

in $\mathcal{D}'(\mathbb{R}^2)$, we conclude that

$$E_2 = -\frac{1}{2\pi}\log|x|$$

is a fundamental solution of $-\Delta$ on $\mathbb{R}^2$.

Our next and final step is to study equation $-\Delta u = f$ when f is an arbitrary compactly supported distribution on $\mathbb{R}^2$. Notice that, from Theorem 3.5.7, every solution u of this equation is a C^∞ function on the complement of supp(f). For simplicity, we shall restrict ourselves to the case $d = 3$, for which this equation is the Poisson equation of electrostatics.

Theorem 3.5.17 For every compactly supported distribution f on $\mathbb{R}^3$, there exists a unique distribution u on $\mathbb{R}^3$ satisfying $-\Delta u = f$ and $u(x) \to 0$ as $x \to \infty$. Furthermore, for $x \notin$ supp(f), $u(x)$ is given by

(3.5.8)
$$u(x) = \left\langle f, \frac{\chi}{4\pi|x - .|} \right\rangle ,$$

for every $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ equal to 1 in a neighbourhood of supp(f) and equal to 0 near x.

Proof.— The uniqueness of u is immediate in view of the Liouville theorem for harmonic functions – see Example 3.5.11. For the existence, we look for u in $\mathcal{S}'(\mathbb{R}^3)$, and Proposition 3.5.3 leads to equation

$$|\xi|^2 \hat{u} = \hat{f} .$$

Since $\hat{f}$ is smooth on $\mathbb{R}^3$ with moderate growth and $1/|\xi|^2$ is the sum of an L^1 function near $\xi = 0$ and of an L^∞ function near infinity, we can define u by

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2} .$$

Our task is now to prove identity (3.5.8), since it implies that $u(x) \to 0$ as $x \to \infty$. As a first step, we are going to prove this identity if $f = \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$. In this case, denote by g the inverse Fourier transform of

$$\frac{\hat{\varphi}}{|\xi|^2} .$$

Since $\hat{g}$ is integrable on $\mathbb{R}^3$ and rapidly decaying at infinity, we know that g is smooth and tends to 0 at infinity. Consider the following smooth function ,

$$E_3 * \varphi(x) = \int_{\mathbb{R}^3} \frac{\varphi(y)}{4\pi|x - y|} \, dy .$$

Then

$$-\Delta(E_3 * \varphi) = \varphi = -\Delta g$$

and it is clear on the above expression that $E_3 * \varphi(x) \to 0$ as $x \to \infty$. Therefore, by the Liouville theorem, $g = E_3 * \varphi$.

Let us come to the general case of a compactly supported distribution f. Let $\rho \in \mathcal{C}^\infty(\mathbb{R}^3)$ with integral 1, and, for every $\varepsilon > 0$,

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^3} \rho\left(\frac{x}{\varepsilon}\right) .$$

Then $\rho_\varepsilon * f = f_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ and converges to f in $\mathcal{D}'(\mathbb{R}^3)$ as $\varepsilon \to 0$. In fact, as we already observed in similar cases, f_ε is supported into a small neighbourhood of supp(f), so that $\hat{f}_\varepsilon(\xi) \to \hat{f}(\xi)$ with a uniform moderate bound $(1+|\xi|)^m$ for some m. Let $u_\varepsilon = E_3 * f_\varepsilon$. We know from the above identity that

$$\hat{u}_\varepsilon = \frac{\hat{f}_\varepsilon}{|\xi|^2}$$

and the right hand side converges to $\hat{u}$ in $\mathcal{S}'(\mathbb{R}^3)$. Hence u_ε converges to u in $\mathcal{S}'(\mathbb{R}^3)$, and for every test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}
\langle u, \varphi \rangle &= \lim_{\varepsilon \to 0} \langle u_\varepsilon, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} E_3 * f_\varepsilon(x)\varphi(x)\, dx \\
&= \lim_{\varepsilon \to 0} \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} E_3(x-y) f_\varepsilon(y)\varphi(x)\, dx\, dy \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} f_\varepsilon(y) E_3 * \varphi(y)\, dy \\
&= \langle f, \chi(E_3 * \varphi) \rangle
\end{aligned}$$

for every $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ equal to 1 in a neighbourhood of supp(f). Notice that

$$\chi(y) E_3 * \varphi(y) = \int_{\mathbb{R}^3} \frac{\chi(y)}{4\pi|x-y|} \varphi(x)\, dx\ .$$

If supp$(\varphi) \cap$ supp$(\chi) = \emptyset$, we can apply Proposition 2.3.20 of integration under the bracket to obtain

$$\langle u, \varphi \rangle = \langle f, \chi(E_3 * \varphi) \rangle = \int_{\mathbb{R}^3} \varphi(x) \left\langle f, \frac{\chi}{4\pi|x - .|} \right\rangle dx$$

and this yields formula (3.5.8). □

Remark 3.5.18 Notice that formula (3.5.8) leads to a complete expansion of $u(x)$ as $x \to \infty$. In particular,

$$u(x) = \frac{q}{4\pi|x|} + O\left(\frac{1}{|x|^2}\right)$$

where $q = \langle f, \chi \rangle$ for every $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ equal to 1 in a neighbourhood of supp(f). This expansion supports the well known fact in Physics that any charge distribution f with nonzero total charge $(q = \langle f, 1 \rangle \neq 0)$ can be seen at infinity as a point distribution with charge q.

We conclude this subsection by a remark about the generalization of the above approach to partial differential equations with non constant coefficients.

Remark 3.5.19 There are several directions where Fourier transformation can be used to study solutions of partial differential equations with non constant coefficients.

i) The first and most direct topic concerns equations with affine coefficients. Indeed, the Fourier transform converts such equations into first order linear equations with polynomial coefficients, which can be solved in general. A famous example in dimension $d = 1$ is the Airy differential equation,

$$u''(x) = xu(x)$$

on $\mathbb{R}$. It is well known from the classical theory of differential equations that the space of solutions of this equation is contained into $\mathcal{C}^\infty(\mathbb{R})$ and is two–dimensional. Let us look for solutions u which are moreover tempered. Then the equation satisfied by $\hat{u}$ is

$$(i\xi)^2\hat{u} = i\frac{d}{d\xi}\hat{u},$$

or

$$\frac{d\hat{u}}{d\xi} = i\xi^2\hat{u} \ .$$

The solutions in $\mathcal{D}'(\mathbb{R})$ are given by

$$\hat{u}(\xi) = c\, \mathrm{e}^{i\frac{\xi^3}{3}} \ , \ c \in \mathbb{C} \ ,$$

which are indeed tempered distributions, since in $L^\infty(\mathbb{R})$. Consequently, the space of tempered solutions of the Airy differential equation is one dimensional, generated by the Airy function,

$$Ai(x) = \mathcal{F}^{-1}\left(\mathrm{e}^{i\frac{\xi^3}{3}}\right) \ .$$

ii) The second and much more general topic concerns the regularity theory of distribution solutions to partial differential equations with arbitrary smooth coefficients in arbitrary open subsets of $\mathbb{R}^d$, using cutoff functions in a smart way. This is the starting point of the theory of pseudo–differential operators, which is beyond the scope of this course.

3.6 Fourier series

In this subsection, we show how the theory of Fourier series can be considered as a special case of the Fourier transformation on tempered distributions. We start with a very natural definition. Recall that, for $a \in \mathbb{R}^d$ and $T \in \mathcal{D}'(\mathbb{R}^d)$, we have defined $\tau_a(T) \in \mathcal{D}'(\mathbb{R}^d)$ by

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d) \ , \ \langle \tau_a(T), \varphi \rangle = \langle T, \tau_{-a}(\varphi) \rangle \ ,$$

and $\tau_b(\varphi)(x) := \varphi(x-b)$.

Definition 3.6.1 A distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ is said to be $\mathbb{Z}^d$–periodic if

$$\forall \gamma \in \mathbb{Z}^d \ , \ \tau_\gamma(T) = T \ .$$

Notice that derivatives of $\mathbb{Z}^d$–periodic distributions are $\mathbb{Z}^d$–periodic, as well as their product with $\mathcal{C}^\infty$ $\mathbb{Z}^d$–periodic functions. A typical example of a $\mathbb{Z}^d$–periodic distribution is given by distributions defined by $\mathbb{Z}^d$–periodic locally integrable functions. Another example is

$$T = \sum_{\gamma \in \mathbb{Z}^d} \delta_\gamma \ .$$

The following proposition explains how to construct $\mathbb{Z}^d$–periodic locally integrable functions from L^1 functions on $\mathbb{R}^d$.

Proposition 3.6.2 Let $f \in L^1(\mathbb{R}^d)$. For almost every $x \in \mathbb{R}^d$, we have

$$\sum_{\gamma\in\mathbb{Z}^d} |f(x-\gamma)| < \infty \ ,$$

so that the formula

$$f^\sharp(x) := \sum_{\gamma\in\mathbb{Z}^d} f(x-\gamma)$$

defines a $\mathbb{Z}^d$–periodic locally integrable function on $\mathbb{R}^d$. Furthermore, if Q denotes the cube $[0,1[^d$, the following identity holds,

$$\int_Q f^\sharp(x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx \ . \tag{3.6.9}$$

If $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $\varphi^\sharp \in \mathcal{C}^\infty(\mathbb{R}^d)$, $\partial^\alpha\varphi^\sharp = (\partial^\alpha\varphi)^\sharp$ and

$$\|\partial^\alpha\varphi^\sharp\|_{L^\infty} \le C_d\, N_{d+1+|\alpha|}(\varphi) \ . \tag{3.6.10}$$

Furthermore, there exists $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\chi^\sharp = 1$.

Proof.— By the monotone convergence theorem,

$$\int_Q \sum_{\gamma\in\mathbb{Z}^d} |f(x-\gamma)|\,dx = \sum_{\gamma\in\mathbb{Z}^d}\int_Q |f(x-\gamma)|\,dx = \sum_{\gamma\in\mathbb{Z}^d}\int_{\gamma+Q} |f(y)|\,dy \ .$$

Since

$$\bigcup_{\gamma\in\mathbb{Z}^d} (\gamma+Q) = \mathbb{R}^d$$

and since the cubes $\gamma+Q$ are pairwise disjoint, we conclude

$$\sum_{\gamma\in\mathbb{Z}^d}\int_{\gamma+Q} |f(y)|\,dy = \int_{\mathbb{R}^d} |f(y)|\,dy < \infty \ .$$

Consequently, for almost every $x \in \mathbb{R}^d$,

$$\sum_{\gamma\in\mathbb{Z}^d} |f(x-\gamma)| < \infty \ ,$$

and so we can define $f^\sharp(x)$. Notice that, for every $\gamma_0 \in \mathbb{Z}^d$,

$$f^\sharp(x-\gamma_0) = \sum_{\gamma\in\mathbb{Z}^d} f(x-\gamma_0-\gamma) = \sum_{\beta\in\mathbb{Z}^d} f(x-\beta) = f^\sharp(x) \ .$$

Furthermore, by the above calculation,

$$\int_Q |f^\sharp(x)|\,dx \le \int_{\mathbb{R}^d} |f(x)|\,dx < \infty \ ,$$

so $f^\sharp$ is a $\mathbb{Z}^d$–periodic locally integrable function on $\mathbb{R}^d$. Identity (3.6.9) follows from integrating the series defining $f^\sharp$, and from using again the disjoint covering of $\mathbb{R}^d$ by the cubes $\gamma + Q$.

Let us suppose $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Recall the estimate of the proof of Proposition 3.1.7.

$$|\varphi(x)| \le \frac{A_d}{(1+|x|^2)^{\frac{d+1}{2}}}\, N_{d+1}(\varphi) \ .$$

Consequently, for every $\gamma \in \mathbb{Z}^d$, we have

$$\sup_{x\in\gamma+Q} |\varphi(x)| \le \frac{B_d}{(1+|\gamma|^2)^{\frac{d+1}{2}}}\, N_{d+1}(\varphi) \ .$$

Using for instance Proposition **??**, we have

$$\sum_{\gamma\in\mathbb{Z}^d} \frac{1}{(1+|\gamma|^2)^{\frac{d+1}{2}}} < \infty \ .$$

Hence the series defining $\varphi^\sharp(x)$ is uniformly convergent on compact subset of $\mathbb{R}^d$, and this also true for the series defining $(\partial^\alpha\varphi)^\sharp$. Consequently, $\varphi^\sharp$ is smooth, $\partial^\alpha\varphi^\sharp = (\partial^\alpha\varphi)^\sharp$ and (3.6.10) follows. Finally, choose $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $0 \le \chi_1(x) \le 1$ and $\chi_1(x) = 1$ on Q. Then $\chi^\sharp \in \mathcal{C}^\infty(\mathbb{R}^d)$ is $\mathbb{Z}^d$ –periodic, and $\chi_1^\sharp \ge 1$. Therefore

$$\chi := \frac{\chi_1}{\chi_1^\sharp} \in \mathcal{C}_0^\infty(\mathbb{R}^d)$$

satisfies $\chi^\sharp = 1$. □

Given $k \in (2\pi\mathbb{Z})^d$, notice that

$$e_k(x) := \mathrm{e}^{ik.x}$$

defines a $\mathcal{C}^\infty$ $\mathbb{Z}^d$ –periodic function. We say that a sequence $(c_k)_{k\in(2\pi\mathbb{Z})^d}$ of complex numbers has moderate growth if there exits $M, C > 0$ such that

$$\forall k \in (2\pi\mathbb{Z})^d \ ,\ |c_k| \le C(1+|k|)^M \ .$$

If (c_k) is such a sequence, the series

$$\sum_{k\in(2\pi\mathbb{Z})^d} c_k e_k$$

defines a tempered distribution. Indeed, if $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\hat\varphi(k) = O((1+|k|)^{-p})N_{p+d+1}(\varphi)$ for every p, so the series

$$\sum_{k\in(2\pi\mathbb{Z})^d} c_k \hat\varphi(-k)$$

is absolutely summable, and we can define a tempered distribution by

$$\forall\varphi \in \mathcal{S}(\mathbb{R}^d)\ ,\ \left\langle \sum_{k\in(2\pi\mathbb{Z})^d} c_k e_k, \varphi \right\rangle = \sum_{k\in(2\pi\mathbb{Z})^d} c_k \hat\varphi(-k) \ .$$

The next theorem claims that every $\mathbb{Z}^d$ –periodic distribution is of this form.

Theorem 3.6.3 Every $\mathbb{Z}^d$–periodic distribution T can be written uniquely

(3.6.11) $$T = \sum_{k\in(2\pi\mathbb{Z})^d} c_k(T)e_k \ , \ e_k(x) = \mathrm{e}^{ik.x} \ ,$$

where $(c_k(T))_{k\in(2\pi\mathbb{Z})^d}$ is a sequence of complex numbers with moderate growth. Furthermore, for every $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\chi^\sharp = 1$, we have

$$\forall k \in (2\pi\mathbb{Z})^d \ , \ c_k(T) = \langle T, \chi e_{-k}\rangle \ .$$

Proof.— First, we prove that every periodic distribution T is tempered. Indeed, if $\varphi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, let us check the formula

(3.6.12) $$\langle T, \varphi\psi^\sharp\rangle = \langle T, \varphi^\sharp\psi\rangle \ .$$

Since the series defining $\varphi^\sharp$ converges in $\mathcal{C}^\infty$, we have

$$\langle T, \varphi\psi^\sharp\rangle = \sum_{\gamma\in\mathbb{Z}^d} \langle T, \varphi\, \tau_\gamma\psi\rangle = \sum_{\gamma\in\mathbb{Z}^d} \langle T, \tau_{-\gamma}\varphi\, \psi\rangle = \langle T, \varphi^\sharp\psi\rangle \ .$$

Then, introducing $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\chi^\sharp = 1$, we have, for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\langle T, \varphi\rangle = \langle T, \varphi\chi^\sharp\rangle = \langle T, \varphi^\sharp\chi\rangle \ .$$

Since T is a distribution, we infer, for some $C > 0$ and some integer m,

$$|\langle T, \varphi\rangle| \le C \sup_{|\alpha|\le m} \|\partial^\alpha(\chi\varphi^\sharp)\|_{L^\infty} \le \tilde{C} N_{d+1+m}(\varphi) \ ,$$

using the Leibniz formula and estimate (3.6.10). Hence $T \in \mathcal{S}'(\mathbb{R}^d)$.

The next step is to characterize the structure of $\hat{T} \in \mathcal{S}'(\mathbb{R}^d)$. Using the identity

$$\widehat{\tau_a T} = \mathrm{e}^{-ia.\xi}\hat{T} \ ,$$

and the $\mathbb{Z}^d$–periodicity of T, we have

$$\forall \gamma \in \mathbb{Z}^d \ , \ \mathrm{e}^{-i\gamma.\xi}\hat{T} = \hat{T} \ .$$

Notice that the set where all the functions $\xi \mapsto (\mathrm{e}^{-i\gamma.\xi} - 1)$ vanish is $(2\pi\mathbb{Z})^d$, hence the support of $\hat{T}$ is included into $(2\pi\mathbb{Z})^d$. Let us describe in more detail the structure of $\hat{T}$ near each point $k \in (2\pi\mathbb{Z})^d$. Set

$$Q_k^* := k +]-2\pi, 2\pi[^d \ .$$

Then, for $j = 1, \dots, d$, the functions

$$\xi \mapsto \frac{\mathrm{e}^{i\xi_j} - 1}{\xi_j - k_j}$$

are C^∞ and do not cancel on Q_k^*. Consequently, on Q_k^*, the equations

$$\mathrm{e}^{-i\xi_j}\hat{T} = \hat{T}\ ,\ j = 1, \dots, d$$

are equivalent to

$$(x_j - k_j)\hat{T} = 0\ ,\ j = 1, \dots, d\ ,$$

and we know that this implies, by Proposition 2.3.11, that there exists $a_k \in \mathbb{C}$ such that

$$\hat{T}_{|Q_k^*} = a_k\delta_k\ .$$

Since the open sets Q_k^* cover $\mathbb{R}^d$, we infer, by the gluing principle, that

$$\hat{T} = \sum_{k\in(2\pi\mathbb{Z})^d} a_k\delta_k\ .$$

Furthermore, notice that the sequence (a_k) has moderate growth. Indeed, since $\hat{T}$ is tempered, we have, if χ_0 is a Plateau function above 0 and compactly supported in $]-2\pi, 2\pi[^d$,

$$|a_k| = |\langle T, \tau_k\chi_0\rangle| \le CN_p(\tau_k\chi_0) \le B(1 + |k|)^p\ .$$

Finally, applying the inverse Fourier transform, we obtain

$$T = (2\pi)^d \sum_{k\in(2\pi\mathbb{Z})^d} a_k\check{\mathcal{F}}(\delta_k) = (2\pi)^d \sum_{k\in(2\pi\mathbb{Z})^d} a_k e_k\ .$$

This proves the first part of the theorem, with $c_k := (2\pi)^d a_k$. Furthermore, given $\ell \in (2\pi\mathbb{Z})^d$, and χ a test function such that $\chi^\sharp = 1$,

$$\langle T, \chi e_{-\ell}\rangle = \sum_{k\in(2\pi\mathbb{Z})^d} c_k\langle e_k, \chi e_{-\ell}\rangle = \sum_{k\in(2\pi\mathbb{Z})^d} c_k\langle 1, \chi e_{k-\ell}\rangle\ .$$

But, by formula (3.6.9),

$$\langle 1, \chi e_{k-\ell}\rangle = \int_{\mathbb{R}^d} \chi e_{k-\ell} = \int_Q \chi^\sharp e_{k-\ell} = \int_Q e_{k-\ell} = \begin{cases} 0 & \text{if } k \ne \ell \\ 1 & \text{if } k = \ell. \end{cases}$$

Finally, we obtain

$$\langle T, \chi e_{-\ell}\rangle = c_\ell\ ,$$

as claimed, and this proves the uniqueness of the sequence (c_k). □

Remark 3.6.4 By deriving the Fourier decomposition (3.6.11), we have

$$c_k(\partial^\alpha T) = (ik)^\alpha c_k(T)\ ,\ k \in (2\pi\mathbb{Z})^d\ .$$

Corollary 3.6.5 Let f be a locally integrable $\mathbb{Z}^d$–periodic function on $\mathbb{R}^d$. Then, in $\mathcal{S}'(\mathbb{R}^d)$,

(3.6.13) $$f = \sum_{k\in(2\pi\mathbb{Z})^d} c_k(f)e^{ik.x}\ ,\ c_k(f) := \int_Q f(x)\mathrm{e}^{-ik.x}\,dx\ ,\ k \in (2\pi\mathbb{Z})^d\ .$$

Furthermore, if f is locally square integrable, the above series is convergent in $L^2(Q)$, with

$$\|f\|^2_{L^2(Q)} = \sum_{k\in(2\pi\mathbb{Z})^d} |c_k(f)|2 \ .$$

Finally, if f is smooth, the sequence $(c_k(f))$ is rapidly decreasing and the above series is convergent in $\mathcal{C}^\infty$.

Proof.— Let χ be a test function such that $\chi^\sharp = 1$. Then

$$c_k(f) = \langle f, \chi e_{-k}\rangle = \int_{\mathbb{R}^d} f(x)\chi(x)\mathrm{e}^{-ik.x}\,dx = \int_Q (f\chi e_{-k})^\sharp\,dx = \int_Q f e_{-k}\,dx \ .$$

Therefore the claim is a consequence of Theorem 3.6.3.

In the L^2 case, we observe that (e_k) is a orthonormal family of $L^2(Q)$ and that $c_k(f)$ is the inner product of f with e_k. From the identity (3.6.13), we observe that, if $c_k(f) = 0$ for every k, then $f = 0$. This implies that (e_k) is an orthonormal basis of $L^2(Q)$, and the assertion follows.

Finally, if f is smooth, for every α $\partial^\alpha f$ is locally square integrable, thus

$$\sum_k |k^\alpha c_k(f)|^2 = \sum_k |c_k(\partial^\alpha f)|^2 < \infty \ ,$$

therefore $(c_k(f))$ is rapidly decreasing and consequently the series and all its derivatives converge uniformly. □

Corollary 3.6.6 (The Poisson summation formula)

$$\sum_{\gamma\in\mathbb{Z}^d} \delta_\gamma = \sum_{k\in(2\pi\mathbb{Z})^d} \mathrm{e}^{ik.x} \ .$$

Proof.— Let χ be a test function such that $\chi^\sharp = 1$. Then

$$c_k\left(\sum_{\gamma\in\mathbb{Z}^d}\delta_\gamma\right) = \sum_{\gamma\in\mathbb{Z}^d}\langle\delta_\gamma, \chi e_{-k}\rangle = \sum_{\gamma\in\mathbb{Z}^d}\chi(\gamma) = \chi^\sharp(0) = 1 \ .$$

Therefore the claim is a consequence of Theorem 3.6.3. □

As an application, let us now consider periodic solutions of PDE with constant coefficients.

Corollary 3.6.7 Let $p \in \mathbb{C}[X_1, \dots, X_d]$ and $P = p(\partial)$ be the corresponding differential operator.

i) If $p(ik) \neq 0$ for every $k \in 2\pi\mathbb{Z}^d$, then the only $\mathbb{Z}^d$ periodic solution u of $Pu = 0$ is $u = 0$.

ii) If $p(ik) \neq 0$ for every $k \in 2\pi\mathbb{Z}^d \setminus \{0\}$, then every $\mathbb{Z}^d$–periodic solution of $Pu = 0$ is a constant.

As an illustration of Corollary 3.6.7, let us prove the following density theorem. Let $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}^d$ be such that the components are linearly independent on $\mathbb{Q}$. Then, for every $x \in \mathbb{R}^d$, for every $\varepsilon > 0$, there exist $t \in \mathbb{R}$ and $\gamma \in \mathbb{Z}^d$ such that

$$|x - t\omega - \gamma| < \varepsilon \ .$$

Indeed, the function u defined on $\mathbb{R}^d$ by

$$u(x) := \inf_{t\in\mathbb{R}} \inf_{\gamma\in\mathbb{Z}^d} |x - t\omega - \gamma|$$

is Lipschitz continuous, and is $\mathbb{Z}^d$ periodic. Furthermore, for every $t \in \mathbb{R}$,

$$u(x + t\omega) = u(x) \ .$$

This means

$$\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \ , \ \int_{\mathbb{R}^d} u(x + t\omega)\varphi(x)\, dx = \int_{\mathbb{R}^d} u(x)\varphi(x)\, dx \ .$$

Notice that the left hand side reads

$$\int_{\mathbb{R}^d} u(x + t\omega)\varphi(x)\, dx = \int_{\mathbb{R}^d} u(y)\varphi(y - t\omega)\, dy \ ,$$

and, taking the derivative of this quantity with respect to t at $t = 0$, we obtain

$$-\int_{\mathbb{R}^d} u(y) \sum_{j=1}^d \omega_j \partial_j \varphi(y)\, dy \ .$$

In other words, in the sense of distributions on $\mathbb{R}^d$,

$$\sum_{j=1}^d \omega_j \partial_j u = 0 \ .$$

Since, by the assumption on the components of ω, the polynomial function

$$p(x) = \sum_{j=1}^d \omega_j x_j$$

does not vanish at $k \in 2\pi\mathbb{Z}^d \setminus \{0\}$, we infer that u is constant. Since clearly $u(0) = 0$, this completes the proof.

3.7 An application to probability theory

Recall that a probability measure on $\mathbb{R}^d$ is a positive Borel measure μ on $\mathbb{R}^d$ of total mass $\mu(\mathbb{R}^d) = 1$. Such a measure defines a tempered distribution through he formula

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d), \langle \mu, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x) \ .$$

Using Fubini's theorem, we observe that

$$\langle \hat{\mu}, \varphi \rangle = \int_{\mathbb{R}^d} \hat{\mu}(\xi)\varphi(\xi)\, d\xi \ ,$$

where

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-ix.\xi} d\mu(x) \ .$$

Notice that $|\hat{\mu}(\xi)| \leq 1$ and that the function $\hat{\mu}$ is continuous, thanks to the dominated convergence theorem. We now introduce a standard definition in probability theory.

Definition 3.7.1 We say that a sequence (μ_n) of probability measures converges tightly to a probability measure μ if, for every bounded continuous function f on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} f\, d\mu_n \to \int_{\mathbb{R}^d} f\, d\mu \ .$$

Before stating the main result, let us make the connection between tight convergence and convergence in $\mathcal{S}'$.

Proposition 3.7.2 Let (μ_n) be a sequence of probability measures on $\mathbb{R}^d$, and let μ be a probability measure on $\mathbb{R}^d$. Then μ_n converges tightly to μ if and only if μ_n converges to μ in $\mathcal{S}'(\mathbb{R}^d)$.

Proof.— Since every function in the Schwartz class is bounded and continuous, the only non trivial part to be proven is that the convergence in $\mathcal{S}'$ of the probability measures μ_n to the probability measure μ implies the tight convergence of μ_n to μ. First of all, we prove that

$$\int_{\mathbb{R}^d} f\, d\mu_n \to \int_{\mathbb{R}^d} f\, d\mu \ ,$$

if f is continuous and compactly supported. Indeed, by regularization, f can be uniformly approximated by a sequence of elements of the Schwartz space. The uniform bound on $\mu_n(\mathbb{R}^d) = 1$ then allows to extend the convergence on Schwartz test functions to continuous compactly supported test functions. Next, let f be bounded continuous function. Let χ be a plateau function over the unit ball of $\mathbb{R}^d$, and define, for every $R > 0$,

$$\chi_R(x) = \chi\left(\frac{x}{R}\right) \ .$$

Then

$$\int_{\mathbb{R}^d} f\, d\mu_n - \int_{\mathbb{R}^d} f\, d\mu = \int_{\mathbb{R}^d} f\chi_R\, d\mu_n - \int_{\mathbb{R}^d} f\chi_R\, d\mu + \int_{\mathbb{R}^d} f(1-\chi_R)\, d\mu_n - \int_{\mathbb{R}^d} f(1-\chi_R)\, d\mu \ .$$

Since $f\chi_R$ is continuous and compactly supported, we have

$$\int_{\mathbb{R}^d} f\chi_R\, d\mu_n - \int_{\mathbb{R}^d} f\chi_R\, d\mu \underset{n\to\infty}{\longrightarrow} 0 \ .$$

Furthermore,

$$\left|\int_{\mathbb{R}^d} f(1-\chi_R)\, d\mu_n\right| \leq \|f\|_{L^\infty} \int (1-\chi_R)\, d\mu_n = \|f\|_{L^\infty} \left(1 - \int \chi_R\, d\mu_n\right) \underset{n\to\infty}{\longrightarrow} \|f\|_{L^\infty} \left(1 - \int \chi_R\, d\mu\right) .$$

Summing up, we have, for every $R > 0$,

$$\limsup_{n\to\infty} \left|\int_{\mathbb{R}^d} f\, d\mu_n - \int_{\mathbb{R}^d} f\, d\mu\right| \leq 2\|f\|_{L^\infty} \left(1 - \int \chi_R\, d\mu\right) .$$

Since μ is a probability measure, the right hand side tends to 0 as R tends to ∞, by the dominated convergence theorem. Hence the left hand side, which is independent of R, is 0. □

We now state the main result of this paragraph.

Theorem 3.7.3 (P. Lévy) Let (μ_n) be a sequence of probability measures on $\mathbb{R}^d$. The following statements are equivalent.

i) μ_n converges tightly to some probability measure.

ii) There exists a function $g : \mathbb{R}^d \to \mathbb{C}$, continuous at 0, such that,

$$\forall \xi \in \mathbb{R}^d \ , \ \hat{\mu}_n(\xi) \underset{n\to\infty}{\longrightarrow} g(\xi) \ .$$

Proof.— The first implication is trivial. Indeed, since, for every $\xi \in \mathbb{R}^d$, the function $f : x \mapsto e^{-ix.\xi}$ is continuous and bounded, the tight convergence of μ_n to μ implies the convergence of $\hat{\mu}_n(\xi)$ to $\hat{\mu}(\xi)$.

Let us prove that (ii) implies (i). Since $\hat{\mu}_n$ and continuous and bounded by 1, we already know that g is measurable and bounded by 1, hence it defines a tempered distribution. Furthermore, the dominated convergence theorem implies that $\hat{\mu}_n \to g$ in $\mathcal{S}'$. Introducing $T = \mathcal{F}^{-1}(g)$, we infer that μ_n converges to T in $\mathcal{S}'$. Since μ_n is a positive measure, it is a positive distribution, and so is T. Hence $T = \mu$ is a positive measure.

It remains to prove that μ is a probability measure. Let χ be a plateau function over the unit ball, and, for every $R > 0$,

$$\chi_R(x) = \chi\left(\frac{x}{R}\right) .$$

By Fatou's lemma,

$$\mu(\mathbb{R}^d) \leq \liminf_{R\to\infty} \int_{\mathbb{R}^d} \chi_R\, d\mu .$$

Since

$$\int_{\mathbb{R}^d} \chi_R\, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}^d} \chi_R\, d\mu_n \leq 1,$$

we first conclude that $\mu(\mathbb{R}^d) \leq 1$. Hence μ has finite total mass, and the dominated convergence theorem implies that

$$\mu(\mathbb{R}^d) = \lim_{R\to\infty} \int_{\mathbb{R}^d} \chi_R\, d\mu .$$

Let $\rho \in \mathcal{S}$ such that $\chi = \hat{\rho}$. Then $\chi_R = \hat{\rho}_R$, with

$$\rho_R(\xi) = R^d \rho(R\xi) \ .$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}^d} \chi_R \, d\mu &= \int_{\mathbb{R}^d} \hat{\rho}_R \, d\mu \\
&= \langle \mu, \hat{\rho}_R \rangle = \langle \hat{\mu}, \rho_R \rangle = \int_{\mathbb{R}^d} g(\xi) \rho_R(\xi) \, d\xi \\
&= \int_{\mathbb{R}^d} g\left(\frac{\xi}{R}\right) \rho(\xi) \, d\xi \ .
\end{aligned}$$

Since g is continuous at $\xi = 0$, we infer, from the dominated convergence theorem,

$$\int_{\mathbb{R}^d} g\left(\frac{\xi}{R}\right) \rho(\xi) \, d\xi \underset{R\to\infty}{\longrightarrow} g(0) \int_{\mathbb{R}^d} \rho(\xi) \, d\xi = g(0)\chi(0) = g(0) \ .$$

But $g(0) = \lim_{n\to\infty} \hat{\mu}_n(0) = 1$. Hence $\mu(\mathbb{R}^d) = 1$. □

Remark 3.7.4 Notice that the assumption of continuity of g at 0 is crucial, as shown by the following counterexample. Let $\rho \in L^1(\mathbb{R}^d)$ be nonnegative, with integral 1 on $\mathbb{R}^d$. For $n \geq 1$, consider the probability measure μ_n given by

$$d\mu_n = \rho_n(x) \, dx \ , \ \rho_n(x) := \frac{1}{n^d} \rho\left(\frac{x}{n}\right) \ .$$

Then, by the Riemann–Lebesgue lemma,

$$\hat{\mu}_n(\xi) = \hat{\rho}(n\xi) \underset{n\to\infty}{\longrightarrow} g(\xi) = \begin{cases} 0 \text{ if } & \xi \neq 0 \\ 1 \text{ if } & \xi = 0 \ . \end{cases}$$

Thus g is not continuous at 0. Indeed, one easily checks that μ_n converges to 0 in $\mathcal{S}'$, hence does not converge tightly.

We conclude by the classical application of P. Lévy's theorem to the central limit theorem.

Theorem 3.7.5 Let μ be a probability measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |x|^2 \, d\mu(x) < \infty \ , \ m = \int_{\mathbb{R}^d} x \, d\mu(x) \ .$$

Assume that no affine hyperplane H of $\mathbb{R}^d$ passing through m satisfies $\mu(H) = 1$. Then the symmetric matrix $A = (a_{jk})_{1\leq j,k\leq d}$ defined by

$$a_{jk} = \int_{\mathbb{R}^d} (x_j - m_j)(x_k - m_k) \, d\mu(x)$$

is positive definite and, for every bounded continuous function f on $\mathbb{R}^d$,

$$\int_{(\mathbb{R}^d)^n} f\left(\frac{x_1 + \cdots + x_n - nm}{\sqrt{n}}\right) d\mu(x_1) \dots d\mu(x_n) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-A^{-1}x.x/2}}{(2\pi)^{d/2}\sqrt{\det A}} f(x) \, dx \ .$$

Proof.— For every integer $n \geq 1$, the formula

$$\int_{\mathbb{R}^d} f\, d\mu_n = \int_{(\mathbb{R}^d)^n} f\left(\frac{x_1 + \cdots + x_n - nm}{\sqrt{n}}\right)\, d\mu(x_1)\ldots d\mu(x_n)$$

defines a probability measure on $\mathbb{R}^d$, and we have to prove that μ_n converges tightly to the probability measure of density

$$G_A(x) = \frac{\mathrm{e}^{-A^{-1}x.x/2}}{(2\pi)^{d/2}\sqrt{\det A}}$$

with respect to the Lebesgue measure. Notice that, after diagonalising A and using the Fubini theorem, a linear change of variables and Proposition 3.2.3,

$$\widehat{G}_A(\xi) = \mathrm{e}^{-A\xi.\xi/2}\ .$$

Therefore, by P. Lévy's theorem, it is enough to prove that

$$\forall \xi \in \mathbb{R}^d\ ,\ \hat{\mu}_n(\xi) \underset{n\to\infty}{\longrightarrow} \mathrm{e}^{-A\xi.\xi/2}\ .$$

From the above definition of μ_n and the Fubini theorem, we have

$$\hat{\mu}_n(\xi) = \left[\mathrm{e}^{i\xi.m/\sqrt{n}}\hat{\mu}\left(\frac{\xi}{\sqrt{n}}\right)\right]^n\ .$$

Because of the assumption

$$\int_{\mathbb{R}^d} |x|^2\, d\mu(x) < \infty\ ,$$

the function

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-ix.\xi}\, d\mu(x)$$

is C^2 and we can expand it near 0 by the Taylor formula, so that

$$\begin{aligned}
\mathrm{e}^{i\xi.m/\sqrt{n}}\hat{\mu}\left(\frac{\xi}{\sqrt{n}}\right) &= \left(1 + i\frac{\xi.m}{\sqrt{n}} - \frac{(\xi.m)^2}{2n} + o\left(\frac{1}{n}\right)\right)\left(1 - i\frac{\xi.m}{\sqrt{n}} - \frac{1}{2n}\int_{\mathbb{R}^d}(\xi.x)^2\, d\mu(x) + o\left(\frac{1}{n}\right)\right)\\
&= 1 - \frac{A\xi.\xi}{2n} + o\left(\frac{1}{n}\right)\ .
\end{aligned}$$

Consequently,

$$\hat{\mu}_n(\xi) = \left(1 - \frac{A\xi.\xi}{2n} + o\left(\frac{1}{n}\right)\right)^n\ ,$$

which —- by taking for instance the complex logarithm near 1 — converges to $\exp(-A\xi.\xi/2)$ as $n \to \infty$. □