4. Asymptotics of the Plancherel measures

X, Y Schur positive specialisations of Sym.

Schur measure $\mathbb{S}_{X,Y}[\lambda] = \dfrac{s_\lambda(X)\, s_\lambda(Y)}{\exp\left(\sum_{k=1}^{\infty} \frac{p_k(X)\, p_k(Y)}{k}\right)}$ probability measure on $\mathcal{Y} = \bigsqcup_{n \in \mathbb{N}} \mathcal{Y}(n)$.

important example: $X = Y = \sqrt{\theta}\, E \implies$ Poissonised Plancherel measure
$$\mathrm{PPl}_\theta[\lambda] = e^{-\theta} \frac{\theta^n}{n!} \frac{|\mathrm{ST}(\lambda)|^2}{n!}$$

Theorem (Okounkov, 2000): If $\lambda \sim \mathbb{S}_{X,Y}$, then its descent set is a determinantal point process (DPP) on $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$.

$$\mathbb{P}\left[\{a_1, a_2, \ldots, a_n\} \subset \left\{\lambda_i - i + \tfrac{1}{2}\right\}_{i \geq 1}\right] = \det\left(K_{X,Y}(a_i, a_j)\right)_{1 \leq i,j \leq n}.$$

with
$$\sum_{x,y \in \mathbb{Z}'} K_{X,Y}(x,y)\, z^x w^{-y} = \exp\left(\sum_{k \geq 1} \frac{p_k(X)}{k}(z^k - w^k) - \sum_{k \geq 1} \frac{p_k(Y)}{k}(z^{-k} - w^{-k})\right) \cdot \frac{\sqrt{zw}}{z-w}.$$
$$|z| > |w|$$

1. Determinantal point processes.

- $D(\lambda) = \left\{\lambda_i - i + \frac{1}{2}\right\}_{i \geq 1}$ is a random subset of $\mathbb{Z}'$.

More generally, let us explain how to deal with general random discrete subsets of an ambient space $\mathfrak{X}$.

$\mathfrak{X}$ = complete metric space, separable, locally compact (e.g. $\mathbb{Z}^d$ or $\mathbb{R}^d$).

$\mathcal{M}_{+,\mathrm{loc}} = \{$positive locally finite Borelian measures on $\mathfrak{X}\}$ endowed with the topology which makes continuous the evaluation maps
$$\mu \mapsto \mu(f) = \int_{\mathfrak{X}} f(x)\, \mu(dx) \quad \text{with } f \in \mathcal{C}_c(\mathfrak{X}).$$

$\mathcal{M}_{\mathrm{atomic}}(\mathfrak{X})$ = closed subset formed by the integer-valued measures in $\mathcal{M}_{+,\mathrm{loc}}$
$$\iff \mu = \sum_{i \in I} \delta_{x_i} \quad \text{with } (x_i)_{i \in I} \text{ discrete subset of } \mathfrak{X}.$$

A point process on $\mathfrak{X}$ is a random variable with values in $\mathcal{M}_{\mathrm{atomic}}(\mathfrak{X})$.

convergence in distribution of random point processes :

$$M_n \to M \iff \forall f \in C_c(\mathcal{X}),\ M_n(f) \to M(f)$$

$$\iff \forall f \in C_c(\mathcal{X}),\ E[e^{-M_n(f)}] \xrightarrow[n\to+\infty]{} \mathbb{E}[e^{-M(f)}]$$

$$\iff \forall B_1, \dots, B_k \text{ with } M(\partial B_i) = 0 \text{ a.s.},$$
$$(M_n(B_1), \dots, M_n(B_k)) \to (M(B_1), \dots, M(B_k)).$$

It is not entirely obvious, but: if M is a random point process on $\mathcal{X}$, there exists

- $(X_i)_{i\geq 1}$ sequence of random variables with values in $\mathcal{X} \sqcup \{\dagger\}$
- N with values in $\mathbb{N} \cup \{+\infty\}$

s.t. $X_i = \dagger$ iff $i > N$, and $M = \sum_{i=1}^{N} \delta_{X_i}$.

- correlation functions.

$$M^{\downarrow n} = \sum_{i_1 \neq i_2 \neq \dots \neq i_n} \delta_{(X_{i_1}, \dots, X_{i_n})}.$$ factorial powers

$$\rho^{\downarrow n} = \mathbb{E}[M^{\downarrow n}].$$ factorial moments.

Usually, $\mathcal{X}$ is endowed with a reference measure λ (e.g. counting measure on $\mathbb{Z}^d$
Lebesgue measure on $\mathbb{R}^d$)

Definition: The k-th correlation function of M is

$$\rho_k(x_1, \dots, x_k) = \frac{d\rho^{\downarrow k}}{d\lambda^{\otimes k}}(x_1, \dots, x_k).$$

examples: ① $\rho_1(x)$ is the density of the point process:

$$\mathbb{E}[M(B)] = \rho^{\downarrow 1}(B) = \int_B \rho_1(x)\, \lambda(dx).$$

② if $\mathcal{X}$ is discrete endowed with the counting measure, then

$$\rho_k(x_1, \dots, x_k) = \mathbb{E}[\#\{k\text{-tuples of indices with } X_{i_1} = x_1, \dots X_{i_k} = x_k\}]$$

$$= \mathbb{P}[\{x_1, \dots, x_k\} \subset M].$$

if the point process is simple.

One can show that the correlation functions determine the distribution of M.

- among the random point processes, an important class consists in those whose correlation functions are given by determinants of a kernel.

$\forall f,\ \langle f | Af \rangle \geq 0.$

<u>setting</u>: $A : L^2(\mathcal{X}, \lambda) \to L^2(\mathcal{X}, \lambda)$ linear operator, auto-adjoint, positive

locally trace class: $\forall B \subset \mathcal{X}$ relatively compact, $\mathbb{1}_B A \mathbb{1}_B$ is trace class on $L^2(B, \lambda_{|B})$.

Then, there exists $K : \mathcal{X}^2 \to \mathbb{R}$ such that

$$Af(x) = \int_{\mathcal{X}} K(x,y)\, f(y)\, \lambda(dy).$$

We consider a RPP with $\rho_k(x_1, x_2, \dots, x_k) = \det(K(x_i, x_j))_{1 \leq i,j \leq k}$.

Then, notice that $\mathbb{E}\left[\prod_{i \geq 1} 1 + f(X_i)\right] = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathcal{X}^k} f(x_1) \dots f(x_k) \det K(x_i, x_j)_{i,j \leq k}\, \lambda(dx_1) \dots \lambda(dx_k)$

In particular: $f = (z-1)\mathbb{1}_B$ with B relatively compact.

$$\mathbb{E}[z^{M(B)}] = \sum_{k=0}^{\infty} \frac{(z-1)^k}{k!} \int_{B^k} \det K(x_i, x_j)\, \lambda(dx_1) \dots \lambda(dx_k)$$

$$= \det(\mathrm{id}_B + (z-1)\mathbb{1}_B A \mathbb{1}_B) \quad \text{Fredholm determinant}$$

$$= \prod_{n \geq 1} (1 + (z-1)\lambda_{n,B}), \quad (\lambda_{n,B})_{n \geq 1} \text{ spectrum of } \mathbb{1}_B A \mathbb{1}_B$$

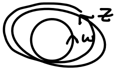
$$\Longrightarrow M(B) = \sum_{n \geq 1} \mathrm{Ber}(\lambda_{n,B}); \quad \text{spectrum} \subset [0,1].$$

For determinantal point processes, the local uniform convergence of the kernels implies the convergence in law.

2. Saddle point analysis of the Plancherel measures.

$$X = Y = \sqrt{\theta}\, E; \quad \mathcal{J}_{X,Y}(z,w) = \frac{\sqrt{zw}}{z-w} \exp\left(\sqrt{\theta}\left((z - z^{-1}) - (w - w^{-1})\right)\right).$$

$$K(x,y) = \frac{1}{(2i\pi)^2} \oint\oint \frac{1}{z-w}\, z^{-\frac{1}{2}-x}\, w^{-\frac{1}{2}+y} \exp\left(\sqrt{\theta}(z - z^{-1}) - (w - w^{-1})\right) dz\, dw.$$



Theorem: fix $x_0 \in [-2,2]$, s.t. $x_0\sqrt{\theta} \in \mathbb{Z}$.

$$K(x_0\sqrt{\theta} + x, x_0\sqrt{\theta} + y) \xrightarrow[\theta \to +\infty]{} \frac{\sin \phi_0 (x-y)}{\pi (x-y)} \quad \text{with } \phi_0 = \arccos \frac{x_0}{2}.$$

local convergence in the bulk towards the discrete sine kernel DPP.

Before proving this, let us remark that in particular,

$$K(x_0\sqrt{\theta}, x_0\sqrt{\theta}) \xrightarrow[\theta \to +\infty]{} \frac{\phi_0}{\pi} = \frac{1}{2} - \frac{1}{\pi} \arcsin \frac{x_0}{2} = \text{local density of the descent set}$$

$$\iff \Omega'(s) = \frac{2}{\pi} \arcsin\left(\frac{s}{2}\right).$$

Set $F(z,t) = z - z^{-1} - t \log z$.

$$K_\theta^{x_0}(x,y) = K(x_0\sqrt{\theta} + x, x_0\sqrt{\theta} + y) = \frac{1}{(2i\pi)^2} \oint\oint \frac{1}{(z-w)\sqrt{zw}} \exp\left(\sqrt{\theta}\left(F\left(z, x_0 + \frac{x}{\sqrt{\theta}}\right) - F\left(w, x_0 + \frac{y}{\sqrt{\theta}}\right)\right)\right) dz\, dw.$$

1st case: with $\frac{1}{z-w\sqrt{zw}}$ replaced by an analytic function $g(z,w)$.

We modify the contours $\Gamma_z \supset \Gamma_w$ in order to make them go through the critical points of the action $F(z,t)$.

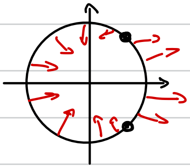
$$F'(z,t) = 1 + \frac{1}{z^2} - \frac{t}{z} = 0 \iff z^2 - tz + 1 = 0, \quad z = \frac{t \pm i\sqrt{4-t^2}}{2}$$

$$= e^{\pm i\phi} \text{ with } \phi = \arccos \frac{t}{2}$$

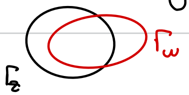
if $|t|<2$.

$\Rightarrow$ the critical points are on the unit circle. Moreover:

- if $z \in \mathbb{T}$, then $\mathrm{Re}\, F(z,t) = 0$.
- we also have $\nabla \mathrm{Re}\, F(e^{i\psi}, t) = 2(\cos\psi - \cos\phi)\, u_\psi$.



We fix $\Gamma_z = \mathbb{T}$ and we deform Γ_w to make it go through the critical points w.r.t. $t = x_0 + \frac{y}{\sqrt{\theta}}$.

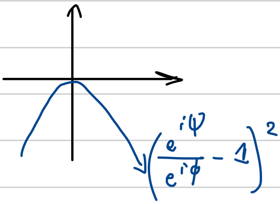


Then, $\underbrace{\mathrm{Re}\, F\left(z, x_0 + \frac{x}{\sqrt{\theta}}\right)}_{=0} - \mathrm{Re}\, F\left(w, x_0 + \frac{y}{\sqrt{\theta}}\right) < 0$ unless $w = e^{\pm i\phi}$.

Therefore, the main contribution of the integral comes from what happens in the vicinity of the critical points $w = e^{\pm i\phi}$.

$$\underset{\theta \text{ close to } \phi}{F(e^{i\psi}, t)} \cong F(e^{i\phi}, t) + \underbrace{\frac{1}{2} F''(e^{i\phi}, t)\left(e^{i\psi} - e^{i\phi}\right)^2}_{i \sin\phi \left(\frac{e^{i\psi}}{e^{i\phi}} - 1\right)^2}$$

So $-\mathrm{Re}\, F(e^{i\psi}, t) \sim \sin\phi \; \mathrm{Im}\left(\frac{e^{i\psi}}{e^{i\phi}} - 1\right)^2$



and the integral has a main contribution of order

$$\int e^{-\sqrt{\theta} K x^2} dx = O(\theta^{-1/4}) \longrightarrow 0.$$

So the kernel goes to 0 ??

<u>2nd case</u>: $g(z,w) = \frac{1}{z - w\sqrt{zw}}$ <u>is not</u> analytic.

We pick up a residue when Γ_w crosses Γ_z.

$$K_\theta^{x_0}(x,y) \cong \frac{1}{2\pi} \int_{\circlearrowleft e^{\pm i\phi_0}} \frac{1}{z} \exp\Big(\sqrt{\theta}\,\big(\underbrace{F\big(z, x_0 + \tfrac{x}{\sqrt{\theta}}\big) - F\big(z, x_0 + \tfrac{y}{\sqrt{\theta}}\big)}_{\rightsquigarrow\; -\frac{x-y}{\sqrt{\theta}} \log z}\big)\Big) dz$$

$\phi_0 = \arccos \frac{x_0}{2}$

$$\cong \frac{1}{2i\pi} \int_{\circlearrowleft e^{\pm i\phi}} \frac{dz}{z^{x-y+1}} = \frac{\sin \phi_0 (x-y)}{\pi (x-y)}.$$

If $x_0 > 2$, the critical points are $z = e^{\pm h}$, $h = \operatorname{arccosh} \frac{t}{2}$ $t \sim x_0$.

On the unit circle, we can show that $\nabla \mathrm{Re}\, F(e^{i\theta}, t) = (2\cos\theta - t)\, u_\theta$

$\Longrightarrow$ We take $\Gamma_z = \mathbb{T}$, $\Gamma_w = e^{-h}\mathbb{T}$.

$\Longrightarrow$ the kernel goes to 0

$\Longrightarrow$ the DPP —— the empty point process.

If $x_0 < -2$, the critical points are $-e^{\pm h}$, $h = \operatorname{arccosh} -\frac{t}{2}$, $t \sim x_0$.
We take $\Gamma_z = \mathbb{T}$, $\Gamma_w = -e^{h}\,\mathbb{T}$. The exchange of the circles yields a residue

$$\frac{1}{2i\pi}\oint \frac{dz}{z^{x-y+1}} = \mathbb{1}_{x=y} \implies \text{full point process as a limit.}$$

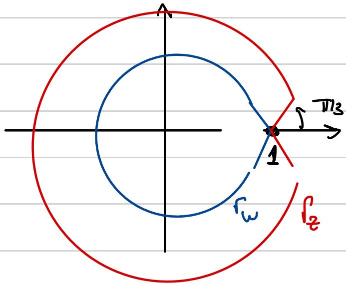
3. Saddle point analysis at the edge.

We look at the DPP around $2\sqrt{\theta}$. We look for a renormalisation t such that

$$t\,K(2\sqrt{\theta}+tx,\, 2\sqrt{\theta}+ty) \xrightarrow[\theta\to+\infty]{} K_{\text{limit}}(x,y).$$

This will imply the convergence of the $\underbrace{\text{edge of the}}_{\frac{\lambda_1-2\sqrt{\theta}}{t},\ \frac{\lambda_2-2\sqrt{\theta}}{t},\ \dots}$ DPP towards a DPP on $\mathbb{R}$ with kernel $K(x,y)$.

A Taylor expansion gives $F(z,2) = \frac{1}{3}(z-1)^3 + o\left((z-1)^4\right)$.
adequate contours:



We take $t = \theta^{1/6}$. $z = 1 + t^{-1}z'$. Then, $F\left(z, 2+\frac{tx}{\sqrt{\theta}}\right) = \left(\frac{1}{3}z'^3 - xz'\right)\frac{1}{t^3} + O\left(\frac{1}{t^4}\right)$

$$\implies K_{\text{limit}}(x,y) = \frac{1}{(2i\pi)^2}\iint_{\Gamma_{w'} \times \Gamma_{z'}} \frac{1}{z'-w'} \exp\left(\frac{1}{3}z'^3 - xz' - \frac{1}{3}w'^3 + yw'\right) dz'\,dw'$$

$$= \frac{Ai(x)\,Ai'(y) - Ai(y)\,Ai'(x)}{x-y} \qquad \text{with } Ai(x) = \frac{1}{2i\pi}\int e^{-\frac{1}{3}z'^3 + xz'}\,dz'$$

convergence towards the Airy process.