# 7. The method of observables

We want to use the functions $\Sigma_\mu(\lambda) = \begin{cases} n^{\downarrow k}\, \chi^\lambda(\sigma_\mu) & \text{if } n=|\lambda| \geq |\mu| = k \\ 0 & \text{otherwise} \end{cases}$

in order to study the Plancherel measures, and more generally the central measures on integer partitions.

## 1. The Ivanov-Kerov algebra

<u>Def</u>: A partial permutation is a pair $(\sigma, A)$ with $\sigma \in \mathfrak{S}(A)$ and $A$ finite subset of $\mathbb{N}^*$. We can also view $\sigma$ as an element of $\mathfrak{S}(\infty) = \bigcup_{n\geq 1}^{\uparrow} \mathfrak{S}(n)$, which fixes every $k \notin A$.

The product of two partial permutations is:

$$(\sigma, A) \times (\tau, B) = (\sigma \circ \tau, A \cup B).$$

<u>ex</u>: $((1,2), \{1,2,5\}) \times ((2,3), \{2,3,5\}) = ((1,2,3), \{1,2,3,5\})$

We extend this product by linearity to the real vectorspace of all formal linear combinations of partial permutations which are bounded in degree:

$$\mathcal{P} = \Big\{ x = \sum c_{(\sigma,A)} (\sigma, A) \text{ with } c_{(\sigma,A)} = 0 \text{ if } |A| > d \text{ for some } d \in \mathbb{N} \Big\}$$

$d = \deg x$.

$\hookrightarrow$ the graded algebra of partial permutations.

$\deg(\sigma, A) = |A|$; $\deg((\sigma,A)\times(\tau,B)) \leq \deg(\sigma,A) + \deg(\tau,B)$.

We can define another gradation on $\mathcal{P}$: the weight.

$$\mathrm{wt}(\sigma, A) = |A| + \#\,\mathrm{Cycles}(\sigma, A).$$

<u>Lemma</u>: $\mathrm{wt}((\sigma,A)\times(\tau,B)) \leq \mathrm{wt}(\sigma,A) + \mathrm{wt}(\tau,B)$.

Indeed, we split the cycles of $\sigma \circ \tau$ on $A \cup B$ in three subsets:

1) those included in $A \setminus A\cap B$.
2) those included in $B \setminus A \cap B$.
3) the other cycles.

If $c$ is a cycle of type 1) in $A\setminus A\cap B$, then $c = c\circ \tau^{-1}$ is also a cycle of $\sigma$
2) $c = \sigma^{-1}\circ c$ ______ $\tau$.

So, $\#\text{cycles}(\sigma\tau, A\cup B) \leq \#\text{cycles}(\sigma, A) + \#\text{cycles}(\tau, B) + \#(3)$.
The cycles of the third type must intersect $A\cap B$, so $\#(3) \leq |A\cap B|$. We conclude that

$$\mathrm{wt}(\sigma\tau, A\cup B) \leq \#\text{cycles}(\sigma, A) + \#\text{cycles}(\tau, B) + \underbrace{|A\cup B| + |A\cap B|}_{|A| + |B|}$$
$$\leq \mathrm{wt}(\sigma, A) + \mathrm{wt}(\tau, B). \qquad \square$$

case of equality: $\deg((\sigma,A)\times(\tau,B)) = \deg(\sigma,A) + \deg(\tau,B)$ iff $A\cap B = \emptyset$
$\mathrm{wt}(\text{———}) = \mathrm{wt}(\text{—}) + \mathrm{wt}(\text{—})$ ______.

We make $\mathfrak{S}(\infty)$ act linearly on $\mathcal{P}$ by conjugation:
$$\tau\cdot(\sigma, A) = (\tau\sigma\tau^{-1}, \tau(A)).$$

**Def** The Ivanov-Kerov algebra is the graded subalgebra of invariants $\mathcal{O} = \mathcal{P}^{\mathfrak{S}(\infty)}$.

linear basis? two partial permutations $(\sigma_1, A_1)$ and $(\sigma_2, A_2)$ are conjugated if and only if $|A_1| = |A_2|$ and the cycle types of $\sigma_1 \in \mathfrak{S}(A_1)$, $\sigma_2 \in \mathfrak{S}(A_2)$ are the same.

$$\to \mathcal{O} = \mathrm{span}\Big( D_\mu = \sum_{(\sigma,A) \text{ with type } \mu} (\sigma, A),\ \mu \in \mathcal{Y} \Big).$$

$\deg D_\mu = |\mu|$; $\mathrm{wt}\, D_\mu = |\mu| + \ell(\mu)$.

It is convenient to work with certain renormalisations of the $D_\mu$'s.

$$\Sigma_\mu = \sum_{a_1 \neq a_2 \neq \cdots \neq a_k \in \mathbb{N}^*} \left(\sigma_\mu(a_1,\dots,a_k), \{a_1,\dots,a_k\}\right) = z_\mu D_\mu .$$

$k = |\mu|$

product rule for the symbols $\Sigma_\mu$:

The cycle type of $(\sigma, A) \times (\tau, B)$ only depends on which equalities $a_i = b_j$ hold.

$A = \{a_1,\dots,a_k\}$, $B = \{b_1,\dots,b_\ell\}$

If one knows the partial pairing of $[\![1,k]\!]$ and $[\![1,\ell]\!]$ describing the equalities $a_i = b_j$, then the cycle type of $\sigma_\mu(a_1\dots,a_k) \times \sigma_\nu(b_1,\dots,b_\ell)$ is known.

$$\longrightarrow \Sigma_\mu \times \Sigma_\nu = \sum_{\substack{P \text{ partial pairing} \\ \text{of } [\![1,k]\!] \text{ and } [\![1,\ell]\!]}} \Sigma_{\rho(\mu,\nu;P)} .$$

<u>example</u>:

$$\Sigma_3 \times \Sigma_2 = \Sigma_{(3,2)} \qquad \phi$$

$$+ 6\,\Sigma_4 \qquad \text{one equality}$$

$$+ 6\,\Sigma_{(2,1)} \qquad \text{two equalities}$$

## 2. Observables: algebra vs geometry.

For any $n \geq 1$, there is a morphism of algebras

$$\mathcal{P} \longrightarrow \mathbb{R}\mathfrak{S}(n)$$

$$(\sigma, A) \longmapsto \begin{cases} \sigma & \text{if } A \subset [\![1,n]\!], \\ 0 & \text{otherwise.} \end{cases}$$

It restricts to a morphism $\pi_n : \mathcal{O} \to Z(\mathbb{R}\mathfrak{S}(n))$.

$$\pi_n(\Sigma_\mu) = \frac{n^{\downarrow k}}{\operatorname{card} C_{\mu 1^{n-k}}} \qquad C_{\mu 1^{n-k}} = z_\mu \binom{m_1+n-k}{m_1} C_{\mu 1^{n-k}}$$

Link with the definition $\Sigma_\mu(\lambda) = n^{\downarrow k}\, \chi^\lambda(\sigma_\mu)$:

$\chi^\lambda : Z(\mathbb{R}\mathfrak{S}(n)) \longrightarrow \mathbb{R}$ is a morphism of algebras.

Therefore, $\mathcal{O} \longrightarrow \mathbb{R}$ is a morphism: it is the composition $\chi^\lambda \circ \pi_n$.

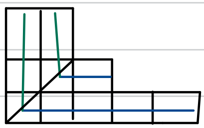
$$\Sigma_\mu \longmapsto \Sigma_\mu(\lambda)$$

We know that $\Sigma_k(\lambda) = \sum_{i=1}^{n} \mu_i^{\downarrow k} \prod_{j\neq i} \frac{\mu_i - \mu_j - k}{\mu_i - \mu_j}$. $\mu_i = \lambda_i + n - i$.

This can be rewritten as:

$$\Sigma_k(\lambda) = [z^{-1}]\left( -\frac{1}{k} z^{\downarrow k} \prod_{j=1}^{n} \frac{z-\mu_j-k}{z-\mu_j} \right).$$

Given an integer partition $\lambda$, its Frobenius coordinates are the lengths of its rows and columns, starting from the diagonal:



$\lambda = (5,3,2)$

$A(\lambda) = \left(\frac{9}{2}, \frac{3}{2}\right)$ ; $B(\lambda) = \left(\frac{5}{2}, \frac{3}{2}\right)$.

The Frobenius moments are: $p_k(\lambda) = \sum_{i=1}^{d} (a_i)^k + (-1)^{k-1}(b_i)^k$.

In particular, $p_1(\lambda) = |\lambda|$.

Set $G_\lambda(z) = \prod_{i=1}^{d} \frac{z+b_i}{z-a_i} = \exp\left( \sum_{k=1}^{\infty} \frac{p_k(\lambda)}{k} z^{-k} \right)$.

By turning $\lambda$ by 45 degrees, we see that $A(\lambda) = \mathbb{Z}'_+ \cap D(\lambda)$

$B(\lambda) = \mathbb{Z}'_- / (\mathbb{Z}'_- \cap D(\lambda))$.

$$\longrightarrow G_\lambda(z) = \prod_{i=1}^{\infty} \frac{z+i-1/2}{z-\lambda_i+i-1/2}.$$

$$\Longrightarrow \Sigma_k(\lambda) = [z^{-1}]\left( -\frac{1}{k}(z-1/2)^{\downarrow k} \frac{G_\lambda(z)}{G_\lambda(z-k)} \right)$$

$$= [t^{k+1}]\left( -\frac{1}{k} \prod_{i=1}^{k} \left(1-\left(i-\tfrac{1}{2}\right)t\right) \exp\left( \sum_{j=1} \frac{p_j(\lambda)}{j} t^j \left(1 - \frac{1}{(1-kt)^j}\right)\right)\right)$$

$$\Longrightarrow \Theta = \mathbb{C}[\Sigma_1, \Sigma_2, \dots] = \mathbb{C}[p_1, p_2, \dots]$$

$\Sigma_k = p_k +$ terms with lower degree.

# 3. End of the proof of LSKV

By induction on the Young diagram, one can show that if $x_0 < y_1 < x_1 < \dots < y_s < x_s$ are the interlaced coordinates of $\lambda$, then

$$\frac{\prod_{i=1}^{s} z - y_i}{\prod_{i=0}^{s} z - x_i} = \frac{1}{z} \exp\left( \sum_{j=1}^{\infty} \frac{\tilde{p}_j(\lambda)\, z^{-j}}{j} \right) = \frac{1}{z} \frac{G_\lambda(z - 1/2)}{G_\lambda(z + 1/2)}$$

This implies that $\mathcal{O} = \mathbb{R}[\tilde{p}_2, \tilde{p}_3, \dots]$ and by working with the change of basis formulae, that

$$\tilde{p}_k = \sum_{|\mu| + \ell(\mu) = k} \frac{k^{\downarrow \ell(\mu)}}{\prod_{i \geq 1} m_i(\mu)!} \Sigma_\mu + \text{terms with lower weight.}$$

Finally:

$$\mathbb{E}_{Pl_n}[\Sigma_\mu(\lambda)] = \mathbb{1}_{\mu = 1^k}\, n^{\downarrow k} = O(n^{\frac{wt(\mu)}{2}}).$$

$$\mathbb{E}_{Pl_n}[\tilde{p}_k(\lambda)] = \mathbb{1}_{k \text{ even}} \binom{k}{k/2} n^{k/2} + O(n^{\frac{k-1}{2}}).$$

$$\mathbb{E}[\tilde{p}_k(\omega_n)] = \mathbb{1}_{k \text{ even}} \binom{k}{k/2} + O(n^{-1/2}).$$

and since

$$\tilde{p}_k^{\,2} = \sum_{\substack{\mu, \nu \\ |\mu| + \ell(\mu) = |\nu| + \ell(\nu) = k}} \bullet\, \Sigma_{\mu \sqcup \nu} + \text{terms with lower weight},$$

we obtain similarly $\mathbb{E}[\tilde{p}_k^{\,2}(\omega_n)] = \mathbb{1}_{k \text{ even}} \binom{k}{k/2} + O(n^{-1/2})$,

so $\mathrm{var} \to 0$.

$\square$.