8. Classification of the central measures

We go back to the classification of central measures on partitions:

- A central measure on partitions is a family of probability measures $(\mathbb{P}_n)_{n\geq 1}$ on the sets $\mathbb{Y}(n)$ which satisfy the recurrence relation:

$$\frac{\mathbb{P}_n(\lambda)}{\dim \lambda} = \sum_{\Lambda : \lambda \nearrow \Lambda} \frac{\mathbb{P}_{n+1}(\Lambda)}{\dim \Lambda} \qquad \forall \lambda \in \mathbb{Y}(n).$$

- The central measures are the marginal distributions of the infinite random tableaux $T = \emptyset \nearrow \lambda_1 \nearrow \dots \nearrow \lambda_n \nearrow \dots$ with the property of conditional uniformity:

- conditionally to $\lambda_n \nearrow \lambda_{n+1} \nearrow \dots$, the distribution of the beginning of the tableau is uniform over $ST(\lambda_n)$.

$\Longleftrightarrow$ the sequence $(\lambda_{-n} = \lambda_n)_{-n\leq 0}$ is Markovian with respect to the kernels $p_{-n}^{-n-1}(\Lambda, \lambda) = \dfrac{\dim \lambda}{\dim \Lambda}$.

- The extremal central measures are in bijection with morphisms of algebra $\mathrm{Sym} \to \mathbb{R}$ which are Schur positive and normalised:

$\mathbb{P}_n(\lambda) = \dim \lambda \cdot s_\lambda(X)$ with $s_\lambda(X) \geq 0 \ \forall \lambda$
$p_1(X) = 1$.

- Among those Schur-positive normalised morphisms are those indexed by the Thoma simplex:

$$\Omega = \left\{ (\alpha, \beta) \ \middle|\ \begin{array}{l} \alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_i \geq \dots \geq 0) \\ \beta = (\beta_1 \geq \beta_2 \geq \dots \geq \beta_i \geq \dots \geq 0) \end{array}, \ \sum_{i=1}^{\infty} \alpha_i + \beta_i = 1 - \gamma \leq 1 \right\}$$

$$p_1(\alpha, \beta) = 1$$
$$p_{k\geq 2}(\alpha, \beta) = \sum_{i=1}^{\infty} (\alpha_i)^k + (-1)^{k-1} (\beta_i)^k.$$

We want to prove that these are the only Schur-positive normalised morphisms...

1. Central measures and characters of $\mathfrak{S}(\infty)$.

A normalised positive trace on a group G is a map $\tau : G \to \mathbb{C}$ such that:

1) $\tau(e_G) = 1$
2) $\forall g_1, \dots, g_n \in G$, $(\tau(g_i g_j^{-1}))_{1 \le i, j \le n}$ is Hermitian positive.
3) $\tau(gh) = \tau(hg)$.

<u>Proposition</u>: If G is finite, then a normalised positive trace is a barycenter of the normalised irreducible characters of G.

<u>Proof</u>: The 3rd condition is equivalent to: $\tau = \sum_{\lambda \in \hat{G}} c_\lambda \chi^\lambda$ for some $c_\lambda \in \mathbb{C}$

Indeed, $(ch^\lambda)_{\lambda \in \hat{G}}$ is a basis of $Z(\mathbb{C}G) = \{\text{functions such that } f(ghg^{-1}) = f(h)\}$

$= \{\text{functions such that } f(gh) = f(hg)\}$

The 1st condition is then equivalent to $\sum_{\lambda \in \hat{G}} c_\lambda = 1$.

So, we have to show that 2) $\iff$ the c_λ's are in $\mathbb{R}_+$.

$\Leftarrow$: it suffices to prove that ch^λ satisfies 2). However:

$(ch^\lambda(g_i g_j^{-1}))$ Hermitian positive

$\iff \forall a_1, \dots, a_n \in \mathbb{C}$, $\overline{(a_1, \dots, a_n)} \, (ch^\lambda(g_i g_j^{-1}))_{1 \le i, j \le n} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \ge 0$

$\iff \forall a_1, \dots, a_n \in \mathbb{C}$, $tr(U^* U) \ge 0$ with $U = \sum_{i=1}^{n} a_i \rho^\lambda(g_i^{-1})$.

this is true.

$\Rightarrow$ Let $x = \sum a_g g$ be the element of $Z(\mathbb{C}G)$ which is sent by the Fourier transform to (id_{V^λ}) (and 0 in the other representation spaces)

Then, $0 \leq \sum_{g,h} a_g \overline{a_h}\, \tau(gh^{-1}) = \sum_{\lambda} c_\lambda \dfrac{\mathrm{tr}\left(\hat{x}(\lambda)\, \hat{x}^*(\lambda)\right)}{\dim \lambda} = q$. $\square$.

Given a normalised positive trace τ its spectral measure is the set of coefficients $(c_\lambda)_{\lambda \in \hat{G}}$: this generalises the definition of spectral measure of a representation of a finite group.

$$\mathbb{E}_{spec(\tau)}[\chi^\lambda(g)] = \sum c_\lambda \chi^\lambda(g) = \tau(g).$$

<u>Theorem</u> A central measure $(\mathbb{P}_n)_{n \geq 1}$ is the set of spectral measures of the restrictions $\tau_{|\mathfrak{S}(n)}$ of a trace τ on $\mathfrak{S}(\infty) = \bigcup_{n\geq 1}^{\uparrow} \mathfrak{S}(n)$.

<u>Proof</u>: A trace on $\mathfrak{S}(\infty)$ determines a sequence of compatible traces $\tau_{|\mathfrak{S}(n)}$. The compatibility condition reads as follows:

$$\sum_{\lambda \in \mathcal{Y}(n)} \mathbb{P}_n(\lambda)\, \chi^\lambda = \tau_{|\mathfrak{S}(n)} = \left(\tau_{|\mathfrak{S}(n+1)}\right)_{|\mathfrak{S}(n)}$$

$$= \sum_{\Lambda \in \mathcal{Y}(n+1)} \mathbb{P}_{n+1}(\Lambda)\, (\chi^\Lambda)_{|\mathfrak{S}(n)} = (*)$$

By Frobenius reciprocity and the Frobenius-Schur isomorphism:

$$\langle V^\lambda \mid \mathrm{Res}_{\mathfrak{S}(n)}^{\mathfrak{S}(n+1)} V^\Lambda \rangle = \langle \mathrm{Ind}_{\mathfrak{S}(n)}^{\mathfrak{S}(n+1)} V^\lambda \mid V^\Lambda \rangle$$

$$= \langle p_1 s_\lambda \mid s_\Lambda \rangle = \mathbb{1}_{\lambda \nearrow \Lambda}.$$

so $(*) = \sum_{\substack{\lambda, \Lambda \\ \lambda \nearrow \Lambda}} \mathbb{P}_{n+1}(\Lambda) \dfrac{\dim \lambda}{\dim \Lambda} \chi^\lambda \implies (\mathbb{P}_n)_{n\geq 1}$ satisfies the induction relation of central measures

$\square$.

2. Backwards martingales associated to central measures.

Let T be a random infinite tableau with the conditional uniformity property, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space on which T is defined, $\mathbb{P}_n = \lambda_{n*}(\mathbb{P})$ the associated control measures.

We denote for $n \geq 0$: $\mathcal{F}_{-n} = \sigma(\lambda_n, \lambda_{n+1}, \dots)$
$(\mathcal{F}_n)_{n \leq 0}$ is a filtration indexed by negative integers.

Fix $k \geq 1$, $\mu \in \mathbb{Y}(k)$ and $\sigma_\mu \in \mathfrak{S}(k)$ with cycle type μ.

__Lemma:__ $(\chi^{\lambda_{-n}}(\sigma_\mu))_{n \leq -k}$ is a (backwards) martingale w.r.t. $(\mathcal{F}_n)_{n \leq -k}$.

__Proof:__ the measurability and integrability is obvious.

$$\mathbb{E}[\chi^{\lambda_{-n}}(\sigma_\mu) \mid \mathcal{F}_{n-1}] = \sum_{\lambda \in \mathbb{Y}(n)} P(\lambda_{-n-1}, \lambda)\, \chi^\lambda(\sigma_\mu)$$

$$P(\lambda_{-n-1}, \lambda) = \frac{\dim \lambda}{\dim \lambda_{-n-1}} \mathbb{1}_{\lambda_{-n-1} \nearrow \lambda}$$

$$= \chi^{\lambda_{-n-1}}(\sigma_\mu). \qquad \square.$$

Therefore, there exists for any σ_μ an almost sure limit $\chi^{\lambda_\infty}(\sigma_\mu) = \lim_{n \to -\infty} \chi^{\lambda_{-n}}(\sigma_\mu)$.

$\mathbb{E}[\chi^{\lambda_\infty}(\sigma_\mu)] = \mathbb{E}[\chi^{\lambda_{-k}}(\sigma_\mu)] = \tau(\sigma_\mu)$, τ character of $\mathfrak{S}(\infty)$ associated to T / $\mathbb{P}_{n \geq 1}$.

__Lemma:__ the character is extremal if and only if there exists a normalised Schur positive specialisation of Sym such that $\tau(\sigma_\mu) = p_\mu(X)$.

__Proof:__ For an extremal control measure associated to the specialisation X,

$$\tau(\sigma_\mu) = \sum_{\lambda \in \mathbb{Y}(n)} \mathbb{P}_n(\lambda)\, \chi^\lambda(\sigma_\mu) = \sum_{\lambda \in \mathbb{Y}(n)} s_\lambda(X)\, ch^\lambda(\sigma_\mu) = p_\mu(X)$$

by the Frobenius-Schur formula $\square$.

3. Extraction of the limiting frequencies.

For an extremal system:

$$\frac{\Sigma_\mu(\lambda_n)}{n^k} \xrightarrow[a.s.]{} \chi^{\lambda_\infty}(\sigma_\mu).$$

$\Sigma_\mu = p_\mu$ + terms with lower degree: $\frac{p_\mu(\lambda_n)}{n^k} \longrightarrow \chi^{\lambda_\infty}(\sigma_\mu)$.

We have $\mathbb{E}[\chi^{\lambda_\infty}(\sigma_\mu)] = \tau(\sigma_\mu)$

$$\mathbb{E}[\chi^{\lambda_\infty}(\sigma_\mu)^2] = \lim_{n\to+\infty} \mathbb{E}\left[\frac{p_\mu(\lambda_n)^2}{n^{2k}}\right] = \mathbb{E}[\chi^{\lambda_\infty}(\sigma_{\mu\sqcup\mu})]$$

$$= \tau(\sigma_{\mu\sqcup\mu}) = \tau(\sigma_\mu)^2 \text{ if the system is extremal!}$$

So, $\chi^{\lambda_\infty}(\sigma_\mu) = \text{constant} = p_\mu(X)$.

We therefore have, $\forall k \geq 1$, $\frac{p_k(\lambda_n)}{n^k} = \sum \left(\frac{a_{i,n}}{n}\right)^k + (-1)^{k-1}\left(\frac{b_{i,n}}{n}\right)^k$.

We endow the scaled Frobenius coordinates of $\lambda \in \mathbb{Y}(n)$ in Ω:

$$\omega_\lambda = \left(\frac{a_{i,n}}{n}\right)_{i\geq 1}, \left(\frac{b_{i,n}}{n}\right)_{i\geq 1}.$$

<u>Theorem:</u> The map
$$\begin{aligned} \Omega &\longrightarrow \mathcal{M}^1([-1,1]) \\ (\alpha,\beta) &\longmapsto \sum_{i\geq 1} \alpha_i \delta_{\alpha_i} + \beta_i \delta_{-\beta_i} + \gamma \delta_0 \end{aligned}$$
is a homeomorphism towards a closed subset of the set of probability measures on $[-1,1]$.

Then, the moments of $\mu_{\omega_{\lambda_n}}$ converge a.s; so there exists $\mu_{\alpha,\beta}$ |

$$\mu_{\omega_{\lambda_n}} \xrightarrow[n]{} \mu_{\alpha,\beta}$$
$$\omega_{\lambda_n} \xrightarrow[n]{} (\alpha,\beta). \qquad \square.$$