IV. Martingales.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which produces a random object $\omega \in \Omega$, we are looking for quantities $M_n(\omega)$ which have a stochastic analogue of the property of conservation of energy in physics.

Moreover, we expect: decay of entropy $\to$ convergence and concentration of M_n.

$\rightsquigarrow$ theory of martingales.

Chapter IV Martingales

1. Definitions and examples.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$:

$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \ldots \subseteq \mathcal{F}_n \subseteq \ldots \subseteq \mathcal{F}$ with the $\mathcal{F}_n$ σ-subalgebras.

Intuitively: $\mathcal{F}_n = \left\{ \begin{array}{l} \text{events whose realisation can be decided by observations} \\ \text{up to time } n \end{array} \right\}$

<u>example</u> $(X_n)_{n\in\mathbb{N}}$ sequence of random variables $(\Omega, \mathcal{F}) \to (E, \mathcal{E})$

$\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$. canonical filtration attached to a random process.

<u>Definition</u>. A sequence of real random variables $(X_n)_{n\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a martingale with respect to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if:

1) The process is adapted: $\forall n \in \mathbb{N}$, X_n is $\mathcal{F}_n$-measurable.
2) The variables are integrable: $\forall n \in \mathbb{N}$, $X_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$.
3) characteristic property:

$$\forall n \geq 0, \quad \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n.$$

If 3) is replaced by $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n$ a.s., one has a submartingale.

by $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$ a.s, one has a supermartingale.

remarks: 1) by successive conditioning, if $(X_n)_{n\in\mathbb{N}}$ is a martingale, then $\mathbb{E}[X_{n+m} \mid \mathcal{F}_n] = X_n$.

2) also, $\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n \mid \mathcal{F}_0]] = \mathbb{E}[X_0]$ is a constant.

and we have similar statements for sub- and supermartingales.

3) $(X_n)_{n\in\mathbb{N}}$ is a submartingale iff $(-X_n)_{n\in\mathbb{N}}$ is a supermartingale

A martingale is a random process which is at the same time a sub- and a supermartingale.

examples: 1) $(\xi_n)_{n\geq 1}$ family of independent random variables with $\mathbb{E}[\xi_n]=0$

$\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$.

$X_n = \sum_{k=1}^{n} \xi_k$ random walk with steps ξ_k.

Then $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

particular case: ξ_n = Bernoulli ± 1 with probability $1/2$, independent.

$\longrightarrow$ symmetric simple random walk.

2) $(\mu_n)_{n\geq 1}$ family of positive independent random variables with $\mathbb{E}[\mu_n] = 1$; $\mathcal{F}_n = \sigma(\mu_1, \dots, \mu_n)$.

$X_n = \prod_{k=1}^{n} \mu_k$

Then, $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

particular case: ξ_n = Bernoulli ± 1 with $\mathbb{P}[\xi_n = 1] = p$, independent.
$\mathbb{P}[\xi_n = -1] = 1-p$.

$\mu_n = \left(\frac{1-p}{p}\right)^{\xi_n}$. Then $X_n = \left(\frac{1-p}{p}\right)^{\xi_1+\dots+\xi_n}$ is a martingale.

3) $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ some filtration.
$(X_n)_{n\in\mathbb{N}}$ defined by
$X_n = \mathbb{E}[X \mid \mathcal{F}_n]$.
$\Longrightarrow$ closed martingale.

4) Branching process and associated martingales.

The Ulam-Harris infinite tree is $\mathcal{U} = \bigsqcup_{r\geq 0} (\mathbb{N}^*)^r$. (set of all finite sequences of positive integers).

Consider a subset $T \subset \mathcal{U}$ with the following properties:

1) $\emptyset \in T$ (the root).
2) if $v = (v_1, \dots, v_r) \in T$ with $r \geq 1$, then its ascendant $(v_1, \dots, v_{r-1})$ also belongs to T.
3) conversely, if $v = (v_1, \dots, v_r) \in T$, then there exists $k(v) \in \mathbb{N}$ such that $w = (v_1, \dots, v_r, j) \in T \Longleftrightarrow 1 \leq j \leq k(v)$.
number of children

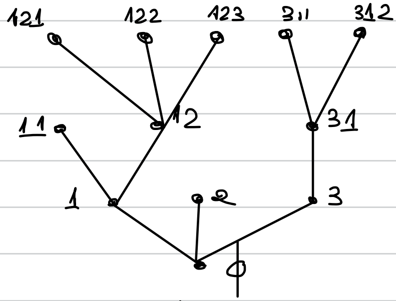
Then T is a (canonically labelled) rooted tree.
branching process: some random rooted tree $T \subset \mathcal{U}$.

We fix a probability measure μ on $\mathbb{N}$ and consider a family $(k(v))_{v\in\mathcal{U}}$ of independent variables with law μ, all defined on $(\Omega, \mathcal{F}, \mathbb{P})$

$$T = \left\{ v \in \mathcal{U} \mid v = (v_1, \dots, v_r) \text{ with } \begin{array}{l} v_1 \leq k(\emptyset) \\ v_2 \leq k(v_1) \\ v_3 \leq k(v_1, v_2) \\ \vdots \\ v_r \leq k(v_1, \dots v_{r-1}) \end{array} \right\}.$$

$\Longrightarrow$ random rooted tree, branching process with reproduction law μ.

etc.



$k(11) = 0,\ k(12) = 3,\ k(31) = 2$

$k(1) = 2,\ k(2) = 0,\ k(3) = 1$

$k(\phi) = 3$

$\mathcal{F}_n = \sigma(k(v),\ |v| < n)$ gives the structure of T up to depth n.

$Z_n = \mathrm{card}(\{v \in T \mid |v| = n\})$.

1) Z_n is $\mathcal{F}_n$-measurable:
$$Z_n = \sum_{v \in \mathbb{N}^n} \underbrace{\mathbb{1}_{(\underbrace{v_1 \le k(\phi), \dots, v_n \le k(v_1, \dots, v_{n-1})}_{\mathcal{F}_n\text{-measurable}})}}_{= B_v}$$

2) Z_n is integrable if μ has a first moment.

Obviously, each B_v is integrable. Let us compute $\mathbb{E}[B_v \mid \mathcal{F}_{n-1}]$:
$$\mathbb{E}[B_v \mid \mathcal{F}_{n-1}] = \mathbb{E}[\underbrace{B_{(v_1, \dots, v_{n-1})}}_{\mathcal{F}_{n-1}\text{ measurable}} \mathbb{1}_{v_n \le \underbrace{k(v_1, \dots, v_{n-1})}_{\text{independent from } \mathcal{F}_{n-1}}} \mid \mathcal{F}_{n-1}]$$
$$= B_{(v_1, \dots, v_{n-1})} \sum_{k = v_n}^{+\infty} \mu(k).$$

Therefore,
$$\mathbb{E}[B_v] = \mathbb{E}[\mathbb{E}[B_v \mid \mathcal{F}_{n-1}]]$$
$$= \mu([\![v_n, +\infty]\!])\ \mathbb{E}[B_{(v_1, \dots, v_{n-1})}].$$
$$\underset{(\text{induction})}{=} \prod_{j=1}^{n} \mu([\![v_j, +\infty]\!]).$$

By taking the sum over $v \in \mathbb{N}^n$,
$$\mathbb{E}[Z_n] = \sum_{v \in \mathbb{N}^n} \mathbb{E}[B_v] = \sum_{w \in \mathbb{N}^{n-1}} \mathbb{E}[B_w] \sum_{k=1}^{\infty} \mu([\![k, +\infty]\!])$$

$$= \mathbb{E}[Z_{n-1}] \sum_{k=1}^{\infty} \mu(k) k \longrightarrow m < +\infty.$$

$$= m^n.$$

3) The same computation yields:

$$\mathbb{E}[Z_n \mid \mathcal{F}_{n-1}] = \sum_{|v|=n} \mathbb{E}[B_v \mid \mathcal{F}_{n-1}]$$

$$= \sum_{|w|=n-1} B_w \sum_{k=1}^{\infty} \mu([\![k, +\infty]\!]) = m \sum_{|w|=n-1} B_w$$

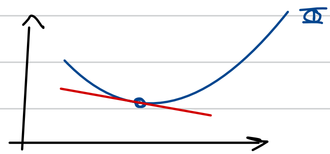
$$= m Z_{n-1}.$$

So, if $m < +\infty$, then $\left(X_n = \frac{Z_n}{m^n} \right)_{n \in \mathbb{N}}$ is a martingale w.r.t $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

2. Transformations

Lemma: if $(X_n)_{n \in \mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$, then
$\Phi : \mathbb{R} \to \mathbb{R}$ is a convex function
$\mathbb{E}[|\Phi(X_n)|] < +\infty \ \forall n$

$(\Phi(X_n))_{n \in \mathbb{N}}$ is a submartingale.



Proof: We have $\Phi(x) = \sup_{at+b \le \Phi(t)} (ax + b)$.

Therefore,

$$\mathbb{E}[\Phi(X_{n+1}) \mid \mathcal{F}_n] = \mathbb{E}\Big[\sup_{at+b \le \Phi(t)} aX_{n+1} + b \mid \mathcal{F}_n\Big]$$

$$\ge \sup_{at+b \le \Phi(t)} \mathbb{E}[aX_{n+1} + b \mid \mathcal{F}_n] = \sup_{at+b \le \Phi(t)} aX_n + b = \Phi(X_n).$$

$\square$

More importantly, consider a martingale $(X_n)_{n \in \mathbb{N}}$ w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and a locally bounded predictable process $(A_n)_{n \ge 1}$:
$\|A_n\|_\infty < +\infty \ \forall n \ge 1$ and A_n is $\mathcal{F}_{n-1}$ measurable.

Define the discrete stochastic integral:

$$(A.X)_n = \sum_{k=1}^{n} A_k (X_k - X_{k-1}) = \text{«} \int_0^n A_t \cdot dX_t \text{»}$$

Theorem: The random process $((A.X)_n)_{n\in\mathbb{N}}$ is again a martingale. Moreover, if $A_n \geq 0$ a.s., then sub- and supermartingales are also preserved.

Proof: $(A.X)_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ is easy.

Then, $\mathbb{E}[(A.X)_{n+1} \mid \mathcal{F}_n]$

$$= (A.X)_n + \mathbb{E}[A_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n]$$

$$= (A.X)_n + A_{n+1}(\underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - X_n}_{=0 \text{ for a martingale}})$$

If $A_{n+1} > 0$, the same computation applies to sub- and supermartingales.

$\square$.

application: $X_n - X_{n-1}$ = random result of a game at time n,
A_n = predictable choice of an amount of money gambled.
$\mathcal{F}_n = \sigma(X_1, \dots X_n)$, the $X_n - X_{n-1}$ independent.
$(A.X)_n$ = total win or loss at time n.

If the game is balanced: $\mathbb{E}[X_n - X_{n-1}] = 0$, then any predictable strategy yields:

$$\mathbb{E}[(A.X)_n] \underset{(\text{martingale})}{=} \mathbb{E}[(A.X)_0] = 0.$$

Definition A stopping time for the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a random variable $T: \Omega \to \mathbb{N} \sqcup \{+\infty\}$ such that:

$$\forall n \in \mathbb{N}, \{T = n\} \in \mathcal{F}_n.$$

example: $(X_n)_{n\in\mathbb{N}}$ adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$, with values in $(E, \mathcal{E})$. $A \in \mathcal{E}$.

$$T_A = \inf(\{n \in \mathbb{N} \mid X_n \in A\}) \quad (+\infty \text{ if } X_n \notin A \ \forall n \in \mathbb{N}).$$

This is a stopping time w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Indeed, $\{T_A = n\} = \{X_0 \notin A, X_1 \notin A, \dots, X_{n-1} \notin A, X_n \in A\}$

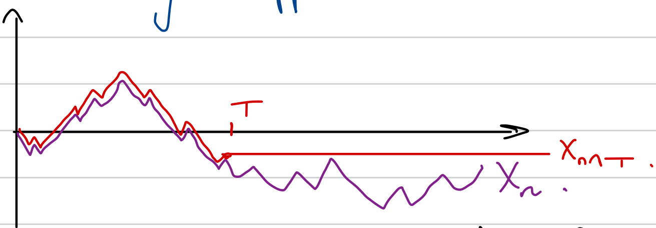
↳ intersection of events in $\mathcal{F}_n$.

<u>Theorem</u> If $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t to $(\mathcal{F}_n)_{n\in\mathbb{N}}$, then T is a stopping time

$$\underline{X_{n\wedge T}} = \begin{cases} X_T & \text{if } n > T \\ X_n & \text{if } n \leq T \end{cases}$$

defines a martingale. Idem with sub- and supermartingales.

↳ martingale stopped at time T.



Proof We can assume without loss of generality that $X_0 = 0$. a.s. (replace X_n by $X_n - X_0$).

Then,
$$X_n = \sum_{k=1}^{n} X_k - X_{k-1}.$$

$$X_{n\wedge T} = \sum_{k=1}^{n} \mathbb{1}_{k\leq T}(X_k - X_{k-1}) = (A.X)_n$$

with $A_k = \mathbb{1}_{k\leq T} = 1 - \mathbb{1}_{T\leq k-1}$ $\mathcal{F}_{k-1}$ measurable

$\square$.

<u>important remark</u>.

Therefore, if $(X_n)_{n\in\mathbb{N}}$ is a martingale, we have $\mathbb{E}[X_{n\wedge T}] = \mathbb{E}[X_0]$ $\forall n$.

If $T < +\infty$ a.s., then $\lim_{n\to+\infty} X_{n\wedge T} = X_T$ $\mathbb{P}$ almost surely.

But we need additional assumptions to ensure $\mathbb{E}[X_T] = \mathbb{E}[X_0]$!

e.g. dominated convergence, uniform integrability, etc.

ex: $\xi_{n\geq 1}$ iid Bernoulli ± 1, $\mathbb{P}[\xi_n = \pm 1] = \frac{1}{2}$.

$X_n = \xi_1 + \xi_2 + \dots + \xi_n$. $T = \inf(\{n \in \mathbb{N} \mid X_n = 1\})$.

$(X_{n\wedge T})_{n\in\mathbb{N}}$ is a martingale. We have $\{T < +\infty\} \supset \{\limsup X_n = +\infty\}$

this belongs to the tail σ-algebra!

Let us prove that $\mathbb{P}[\limsup X_n = +\infty] = 1$.

Note that since $X_n \geq -n$, $\limsup X_n = +\infty \iff \sup X_n = +\infty$.

Then, $\mathbb{P}[\sup_{n\in\mathbb{N}} X_n \neq +\infty] = \lim_{K\to+\infty} \uparrow \mathbb{P}[\forall n \in \mathbb{N}, X_n \leq K]$

$\leq \lim_{K\to+\infty} \mathbb{P}[X_{n_K} \leq K]$

for any sequence $(n_K)_{K\in\mathbb{N}}$. With $n_K = K^3$, by the central limit theorem,

$\mathbb{P}[X_{K^3} \leq K] \simeq \mathbb{P}[\mathcal{N}(0,1) \leq +\frac{1}{\sqrt{K}}] \simeq \frac{1}{2}$.

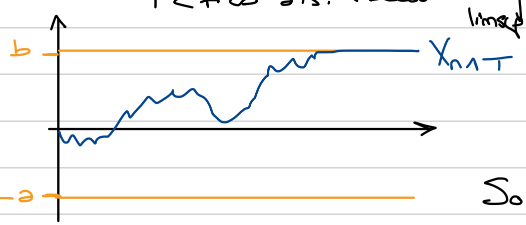
So $\mathbb{P}[\limsup X_n = +\infty] \geq \frac{1}{2}$, $\mathbb{P}[T < +\infty] = 1$.

By the stopping time theorem, $\mathbb{E}[X_{n\wedge T}] = 0$ $\forall n \in \mathbb{N}$.

But $X_T = 1$, so $\mathbb{E}[X_T] = 1 \neq 0$.

ex: In the same setting, set $T = T_{\{-a, b\}}$ with $a, b \geq 1$.

$T < +\infty$ a.s. (because $\liminf = -\infty$, $\limsup = +\infty$)



$(X_{n\wedge T})_{n\in\mathbb{N}}$ is dominated in absolute value by a, b.

So $\mathbb{E}[X_T] = \lim_{n\to+\infty} \mathbb{E}[X_{n\wedge T}] = 0$.

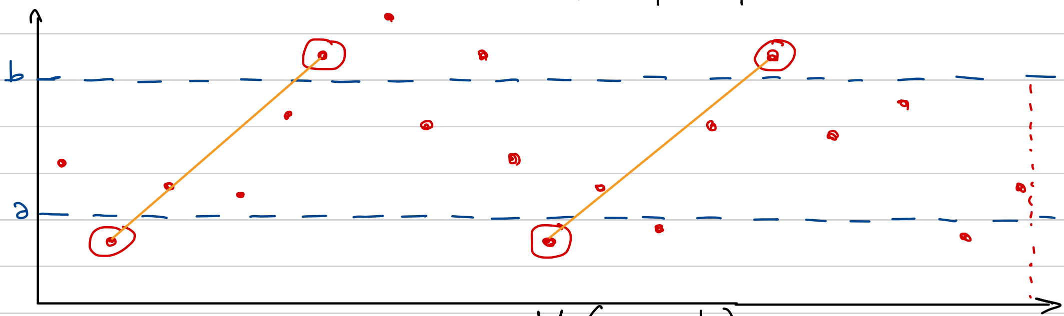
$0 = \mathbb{P}[X_T = -a](-a) + (1 - \mathbb{P}[X_T = -a])b$.

So $\mathbb{P}[X_T = -a] = \frac{b}{a+b}$.

3. Doob inequalities.

Given a sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $a<b$, the number of upcrossings of the sequence between the levels a and b and before time n is:

$$M_n(x_\cdot, a, b) = \sup\{k\in\mathbb{N} \mid \exists\, 0\le s_1<t_1<s_2<t_2 \dots < s_k < t_k \le n \text{ with } x_{s_i}\le a,\ x_{t_i}\ge b\}$$



ex: $M_n(x_\cdot, a, b) = 2$.

Theorem: If $(X_n)_{n\in\mathbb{N}}$ is a supermartingale, then

$$\mathbb{E}[M_n(x_\cdot, a, b)] \le \frac{\mathbb{E}[(X_n-a)_-]}{b-a}$$

$(X_n-a)_-$: negative part of $X_n - a$.

Proof: In order to compute the number of upcrossings, we can use optimal times S_i and T_i:

$$T_{-1} = 0.$$
$$S_k = \inf(\{n > T_{k-1} : X_n \le a\})$$
$$T_k = \inf(\{n > S_k : X_n \ge b\}).$$

stopping times w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$

Then, $M_n(X, a, b) = \sum_{k=1}^{\infty} \mathbb{1}_{T_k \le n}$ is a $\mathcal{F}_n$ measurable random variable.

Consider $A_n = \sum_{k=1}^{\infty} \mathbb{1}_{S_k < n \le T_k} = \mathbb{1}$ (we are along an upcrossing).

This is a predictable process, because

$$\mathbb{1}_{S_k < n \le T_k} = \mathbb{1}_{S_k \le n-1} \underbrace{(1 - \mathbb{1}_{T_k \le n-1})}_{\mathcal{F}_{n-1}\text{-measurable}}.$$

Therefore, $((A.X)_n)_{n\in\mathbb{N}}$ is a submartingale, and $\mathbb{E}[(A.X)_n] \le \mathbb{E}[(A.X)_0] = 0$.

What is $(A.X)_n$? There are two cases:

① $n \le S_{m+1}$, $m = M_n(X, a, b)$ (the $k+1$-th rise has not started).

Then, $(A.X)_n = \sum_{k=1}^{M_n(X,a,b)} X_{T_k} - X_{S_k} \ge (b-a) M_n(X,a,b)$

② $S_{m+1} < n < T_{m+1}$.

Then, $(A.X)_n = \sum_{k=1}^{M_n(X,a,b)} X_{T_k} - X_{S_k} + X_n - X_{S_{m+1}}$

$$\ge (b-a) M_n(X, a, b) - (X_{S_{m+1}} - X_n)$$

So, $\mathbb{E}[\mathbb{1}_{②} (X_{S_{m+1}} - X_n)] \ge (b-a)\mathbb{E}[M_n(X,a,b)]$.

$(X_n - a)_- = \max(0, a - X_n) \ge a - X_n \ge X_{S_{m+1}} - X_n$. $\square$.

Another important inequality due to Doob is:

<u>Theorem</u>: If $(X_n)_{n\in\mathbb{N}}$ is a submartingale, then for any $a>0$,

$$\mathbb{P}[\sup_{0\le k\le n} X_k \ge a] \le \frac{\mathbb{E}[(X_n)_+]}{a}.$$

(maximal inequality).

Proof Let $T = T_{[a,+\infty)}$. We have:

$$\mathbb{E}[(X_n)_+] = \mathbb{E}[\max(0, X_n)] \geq \mathbb{E}[\max(0, X_n) \mathbb{1}_{(T \leq n)}]$$
$$\geq \mathbb{E}[X_n \mathbb{1}_{(T \leq n)}]$$

and
$$\mathbb{E}[X_n \mathbb{1}_{(T\leq n)}] = \sum_{k=0}^{n} \mathbb{E}[X_n \mathbb{1}_{(T=k)}] = \sum_{k=0}^{n} \mathbb{E}[\mathbb{E}[X_n | \mathcal{F}_k] \mathbb{1}_{(T=k)}]$$
$$\geq \sum_{k=0}^{n} \mathbb{E}[X_k \mathbb{1}_{(T=k)}].$$
$$\geq a \sum_{k=0}^{n} \mathbb{P}[T=k] = a\,\mathbb{P}[X_n^* \geq a]$$

where $X_n^* = \sup_{0\leq k\leq n} X_k$ $\square$.

intuitively: the maximum of a (sub)-martingale at time n is controlled by X_n.

Corollary (Doob L^p inequality). Given a nonnegative submartingale $(X_n)_{n\in\mathbb{N}}$ and $p \in (1,+\infty)$, if the X_n belong to $L^p(\Omega, \mathcal{F}, \mathbb{P})$, then

$$\mathbb{E}[(X_n^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[(X_n)^p].$$

Proof The trick is to rewrite:

$$\mathbb{E}[(X_n^*)^p] = \mathbb{E}\left[p \int_0^{X_n^*} a^{p-1}\, da\right]$$
$$= p \int_0^{\infty} \mathbb{E}[\mathbb{1}_{a \leq X_n^*}\, a^{p-1}]\, da$$
$$\leq \int_0^{\infty} p a^{p-2}\, \mathbb{E}[X_n \mathbb{1}_{X_n^* \geq a}]\, da$$

by the proof of the maximal inequality.

$$\leq p\, \mathbb{E}\left[X_n \int_0^{X_n^*} a^{p-2}\, da\right]$$
$$\leq \frac{p}{p-1} \mathbb{E}[X_n (X_n^*)^{p-1}]$$

$$\leq_{(\text{Hölder})} \frac{p}{p-1} \mathbb{E}[(X_n)^p]^{\frac{1}{p}} \mathbb{E}[(X_n^*)^p]^{\frac{p-1}{p}}$$

$\square$

<u>Corollary</u> If $(X_n)_{n\in\mathbb{N}}$ is a martingale in $L^p(\Omega, \mathcal{F}, \mathbb{P})$, then

$$\mathbb{E}[(\sup_{k\leq n} |X_k|)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_n|^p].$$

4. Convergence of martingoles.

- Almost sure convergence.

<u>Theorem</u> Let $(X_n)_{n\in\mathbb{N}}$ be a supermartingale with

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|] = K < +\infty.$$

There exists $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \xrightarrow[\text{a.s.}]{} X$.

This is in particular true if $(X_n)_{n\in\mathbb{N}}$ is nonnegative, since in this case

$$\mathbb{E}[|X_n|] = \mathbb{E}[X_n] \leq \mathbb{E}[X_0] = K < +\infty.$$

<u>Proof</u> For any $a<b$ rational numbers,

$$\mathbb{E}[M_\infty(X,a,b)] = \lim_{n\to+\infty} \uparrow \mathbb{E}[M_n(X,a,b)] \leq \frac{K}{b-a} < +\infty$$

so $M_\infty(X,a,b) < +\infty$ a.s.

If X_n does not converge a.s., then $\liminf X_n < \limsup X_n$, so one can find two rational numbers in between such that $M_\infty(X,a,b) = +\infty$.

$$\Rightarrow \mathbb{P}[\liminf X_n = \limsup X_n] = \mathbb{P}[X_n(\omega) \text{ has a limit } X(\omega) \in \overline{\mathbb{R}}] = 1.$$

Denote $X = \lim X_n$ a.s. By Fatou's lemma,

$$\mathbb{E}[|X|] = \mathbb{E}[\liminf |X_n|] \leq \liminf \mathbb{E}[|X_n|] \leq K < +\infty$$

so $X \in L^1$ (in particular, $X \in \mathbb{R}$ a.s.) $\square$.

<u>example</u> : Consider an urn which contains at time $n=0$ 1 red ball
1 white ball.

At each time $n \geq 0$, there are $n+2$ balls in the urn: $n+2 = R_n + W_n$.
One chooses a random ball in the urn and replaces it by two balls with the same color. Concretely:

$(U_n)_{n\geq 1}$ sequence of independent random variables with $U_n \sim \mathrm{Unif}([0,1])$

If $U_{n+1} \leq \dfrac{R_n}{n+2}$: $R_{n+1} = R_n + 1$, $W_{n+1} = W_n$.

Otherwise : $R_{n+1} = R_n$, $W_{n+1} = W_n + 1$.

$$\mathcal{F}_n = \sigma(R_1, \dots, R_n) = \sigma(U_1, \dots, U_n).$$

Set $X_n = \dfrac{R_n}{n+2} \in [0,1] =$ proportion of red balls.

Then: $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$.
Indeed, $X_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ is obvious, and :

$$\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \frac{1}{n+3}\, \mathbb{E}[R_{n+1} \mid \mathcal{F}_n] \\
&= \frac{1}{n+3}\, \mathbb{E}\Big[R_n + \mathbb{1}\Big(U_{n+1} \leq \frac{R_n}{n+2}\Big) \,\Big|\, \mathcal{F}_n\Big] \\
&= \frac{1}{n+3}\Big(R_n + \frac{R_n}{n+2}\Big) = \frac{R_n}{n+2} = X_n.
\end{aligned}$$

So, $X_n \xrightarrow[a.s.]{} X$ limiting proportion. What is the law of X?

Note that for any $k \geq 1$, $(X_n)^k \in [0,1]$, so by dominated convergence,

$$\mathbb{E}[X^k] = \lim_{n\to+\infty} \mathbb{E}[(X_n)^k].$$

Set $$Y_{n,k} = \frac{R_n(R_n+1)\dots(R_n+k-1)}{(n+2)(n+3)\dots(n+k+1)} = \frac{R_n^{\uparrow k}}{(n+2)^{\uparrow k}} \in [0,1].$$

Then, $(Y_{n,k})_{n\in\mathbb{N}}$ is a martingale.
Indeed,

$$\mathbb{E}[Y_{n+1,k} \mid \mathcal{F}_n] = \frac{\mathbb{E}[R_{n+1}^{\uparrow k} \mid \mathcal{F}_n]}{(n+3)^{\uparrow k}}$$

$$= \frac{R_n}{n+2}\frac{(R_n+1)^{\uparrow k}}{(n+3)^{\uparrow k}} + \left(1-\frac{R_n}{n+2}\right)\frac{R_n^{\uparrow k}}{(n+3)^{\uparrow k}}$$

$$= \frac{R_n^{\uparrow k}}{(n+2)^{\uparrow k}}\left(\frac{R_n+k}{n+2+k} + \frac{n+2-R_n}{n+2+k}\right)$$

$$= \frac{R_n^{\uparrow k}}{(n+2)^{\uparrow k}} = Y_{n,k}.$$

So, for any $k \geq 1$, $Y_{n,k} \xrightarrow[a.s]{} Y_{\infty,k}$ with $\mathbb{E}[Y_{\infty,k}] \underset{\substack{\uparrow \\ \text{dominated convergence}}}{=} \mathbb{E}[Y_{0,k}] = \frac{1}{k+1}$.

However, $Y_{n,k} = (1+O(n^{-1}))((X_n)^k + O(n^{-1}))$ with deterministic $O(\cdot)$.
So, $\mathbb{E}[X^k] = \lim_{n\to+\infty} \mathbb{E}[(X_n)^k] = \lim_{n\to+\infty} \mathbb{E}[Y_{n,k}] = \frac{1}{k+1}$.

For any polynomial function f, $\mathbb{E}[f(X)] = \int_0^1 f(x)\,dx$.
By Stone Weierstrass, this extends to continuous functions $\rightsquigarrow X \sim \mathrm{Unif}([0,1])$.

- L^1 convergence.

Consider a martingale $(X_n)_{n\in\mathbb{N}}$ which is bounded in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. In general,
$\left(X = \lim_{n\to+\infty} X_n \text{ a.s.}\right)$

$$\mathbb{E}[X] \neq \lim_{n\to+\infty} \mathbb{E}[X_n] = \mathbb{E}[X_0]$$

Theorem: The following are equivalent:

1) $X_n \xrightarrow[L^1]{} X$.
2) $(X_n)_{n\in\mathbb{N}}$ is UI
3) There exists $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[Z \mid \mathcal{F}_n] = X_n$ a.s.

If they are satisfied, then one can assume Z measurable w.r.t. $\mathcal{F}_\infty = \sigma(\bigcup \mathcal{F}_n)$, and in this case, $Z = X = \lim_{n\to+\infty} X_n$ a.s.

<u>Proof</u>: ① $\iff$ ② is the general result on convergence in L^1 and uniform integrability.

③ $\implies$ ②. If $X_n = \mathbb{E}[Z \mid \mathcal{F}_n]$, then we have:

$$\mathbb{E}[|X_n| \underbrace{\mathbb{1}_{|X_n|\geq M}}_{\mathcal{F}_n\text{-measurable}}] \leq \mathbb{E}[\mathbb{E}[|Z| \mid \mathcal{F}_n] \mathbb{1}_{|X_n|\geq M}]$$

$$\leq \mathbb{E}[|Z| \mathbb{1}_{|X_n|\geq M}].$$

On the other hand, $\mathbb{P}[|X_n| \geq M] \leq \dfrac{\mathbb{E}[|X_n|]}{M} = \dfrac{\mathbb{E}[|\mathbb{E}[Z|\mathcal{F}_n]|]}{M}.$

$$\leq \frac{\mathbb{E}[|Z|]}{M}$$

So $\lim_{M\to+\infty} \sup \mathbb{E}[|X_n| \mathbb{1}_{|X_n|\geq M}] \leq \lim_{\substack{\varepsilon\to 0 \\ \mathbb{P}[A]\leq\varepsilon}} \mathbb{E}[|Z| \mathbb{1}_A] = 0.$

① $\implies$ ③. Set $Y_n = \mathbb{E}[X \mid \mathcal{F}_n]$, where $X = \lim_{n\to+\infty} X_n$ a.s and in L^1.

We have:
$$\mathbb{E}[|Y_n - X_n|] = \mathbb{E}[|\mathbb{E}[X - X_{m+n} \mid \mathcal{F}_n]|]$$
$$\leq \mathbb{E}[\mathbb{E}[|X - X_{m+n}| \mid \mathcal{F}_n]]$$
$$\leq \mathbb{E}[|X - X_{m+n}|] \quad \forall m \geq 0$$

So, $m \to +\infty$: $\mathbb{E}[|Y_n - X_n|] = 0$, $Y_n = X_n$ a.s.

For the last statement, X $\mathcal{F}_\infty$ measurable can be ensured by replacing X by $\mathbb{E}[X \mid \mathcal{F}_\infty]$, and then, one can prove for $\mathcal{F}_\infty$ measurable variables,

$$X = Z \iff \forall n \in \mathbb{N},\ \mathbb{E}[X \mid \mathcal{F}_n] = \mathbb{E}[Y \mid \mathcal{F}_n]. \qquad \square$$

Unfortunately, it is very difficult to check that a sequence is Ui, unless:
if $(X_n)_{n\in\mathbb{N}}$ bounded in $L^{p>1}$, then $(X_n)_{n\in\mathbb{N}}$ is Ui.

$$\mathbb{E}[|X_n| \mathbb{1}_{|X_n|\geq M}] \leq \frac{1}{M^{p-1}} \mathbb{E}[|X_n|^p].$$

- $L^{p>1}$ convergence.

Theorem: If $(X_n)_{n\in\mathbb{N}}$ is a martingale bounded in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ with $p>1$, then there exists $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $X_n \underset{\substack{a.s.\\ L^1, L^p}}{\longrightarrow} X$.

Proof: We already know that $X_n \underset{a.s.}{\longrightarrow} X$, and by Fatou's lemma, $X \in L^p$.

By Doob's inequality,

$$\mathbb{E}[\sup_{n\in\mathbb{N}} |X_n|^p] \leq \left(\frac{p}{p-1}\right)^p \sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|^p] \leq K < +\infty$$

so one can use dominated convergence to show that $\mathbb{E}[|X-X_n|^p] \to 0$.
Hence, $X_n \underset{L^p}{\longrightarrow} X$, and we have a fortiori $X_n \underset{L^1}{\longrightarrow} X$.

Note: one can give a simpler proof if $p=2$.
Indeed, note that the increments of a L^2 martingale are orthogonal:

$$\underset{n>m}{\mathbb{E}[(X_n - X_{n-1})(X_m - X_{m-1})]} = \mathbb{E}[(\mathbb{E}[X_n | \mathcal{F}_{n-1}] - X_{n-1})(X_m - X_{m-1})]$$
$$= 0.$$

Therefore, $(X_n)_{n\in\mathbb{N}}$ bounded in $L^2 \iff \sup_{n\in\mathbb{N}} \mathbb{E}[(X_n - X_0)^2] < +\infty$

$$\iff \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \mathbb{E}[(X_k - X_{k-1})^2] < +\infty$$

$\iff$ the series $\sum_{k\geq 1} \mathbb{E}[(X_k - X_{k-1})^2]$ converges

$\iff \sum_{k\geq 1} (X_k - X_{k-1})$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$

$\iff X_n$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

example Asymptotics of branching processes.

μ reproduction law with fixed moment. $\mathcal{F}_n = \sigma(k(v), |v| < n)$

$Z_n = \sum_{|v|=n} B_v$; $X_n = \frac{Z_n}{m^n}$ is a martingale $\left(m = \sum_{k=0}^{\infty} k\mu(k)\right)$.

It is convenient to introduce the generating series $G(s) = \sum_{k=0}^{\infty} \mu(k) s^k$, well defined at least for $s \in [0,1]$.

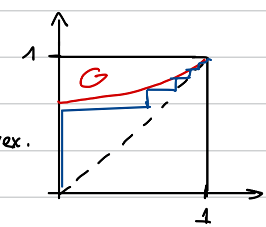
$$\begin{aligned}
\mathbb{E}[s^{Z_{n+1}} \mid \mathcal{F}_n] &= \mathbb{E}\Big[\prod_{|w|=n+1} s^{B_w} \mid \mathcal{F}_n\Big] \\
&= \mathbb{E}\Big[\prod_{|v|=n} \prod_{j=1}^{\infty} s^{B_v \mathbb{1}_{k(v)\geq j}} \mid \mathcal{F}_n\Big]. \\
&= \mathbb{E}\Big[\prod_{|v|=n} (s^{B_v})^{k(v)} \mid \mathcal{F}_n\Big] \\
&= \mathbb{E}\Big[\prod_{\substack{|v|=n \\ B_v = 1}} s^{k(v)} \mid \mathcal{F}_n\Big] = (G(s))^{Z_n}.
\end{aligned}$$

Consequently, $\mathbb{E}[s^{Z_n}] = G \circ G \circ \dots \circ G(s)$ (n times).

One of the main question is : what is $\mathbb{P}[\text{extinction}]$

$$= \mathbb{P}[\exists n \geq 1, Z_n = 0] = \lim_{n \to +\infty} \uparrow \mathbb{P}[Z_n = 0] \,?$$

Answer: this is $\lim_{n \to +\infty} G^{\circ n}(0)$!



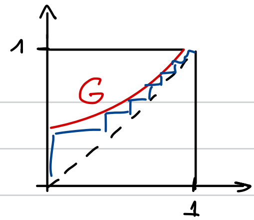
1st case : $m < 1$ subcritical case.

Note that $G(1) = 1$ and $G'(1) = m$; G is $\nearrow$, convex.

Then, $p_{ext} = 1$.

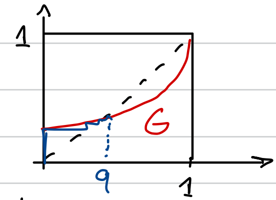
2nd case : $m = 1$ critical case

Then, unless $\mu = \delta_1$ and $G(s) = s$, $p_{ext} = 1$.



3rd case : $m > 1$ supercritical case.

There exists a unique other fixed point $q \in (0,1)$ with $G(q) = q$, and $p_{ext} = q$.



What happens if $Z_n \not\to 0$? This has probability $1-q$.

Suppose that $\sum k^2 \mu(k) < +\infty$. Set $\sigma^2 = \text{variance} = \sum k^2 \mu(k) - \left(\sum k \mu(k)\right)^2$.

Then, $(X_n)_{n \in \mathbb{N}}$ is bounded in L^2:

$$\mathbb{E}[(Z_{n+1})^2 \mid \mathcal{F}_n] = \sum_{|w| = |w'| = n+1} \mathbb{E}[B_w B_{w'} \mid \mathcal{F}_n]$$

$$= \sum_{|v| = |v'| = n} B_v B_{v'} \sum_{k, k' \geq 1} \mathbb{E}[\mathbb{1}_{k(v) \geq k} \mathbb{1}_{k(v') \geq k'}]$$

$$= \sum_{|v| = |v'| = n} B_v B_{v'} \underbrace{\mathbb{E}[k(v) k(v')]}$$

$= m^2$ if $v \neq v'$ and $m^2 + \sigma^2$ if $v = v'$

$$= m^2 \sum_{|v| = |v'| = n} B_v B_{v'} + \sigma^2 \sum_{|v| = n} B_v$$

$$= m^2 Z_n^2 + \sigma^2 Z_n.$$

$$\Longrightarrow \mathbb{E}[(Z_{n+1})^2] = m^2 \mathbb{E}[(Z_n)^2] + \sigma^2 m^n.$$

$$\mathbb{E}[(X_{n+1})^2] = \mathbb{E}[(X_n)^2] + \frac{\sigma^2}{m^{n+2}}$$

and $\mathbb{E}[(X_n)^2] \leq \sum_{n \geq 0} \frac{\sigma^2}{m^{n+2}} + 1 < +\infty$.

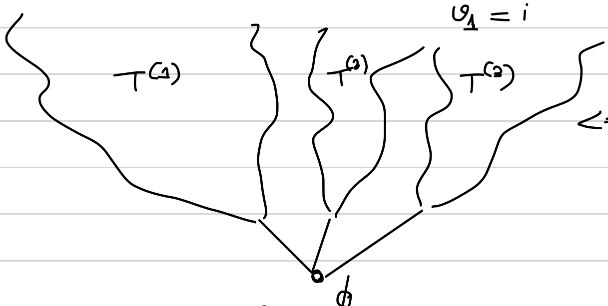
so $X_n \to X$ a.s, L^1, L^2; $\mathbb{E}[X] = \mathbb{E}[X_0] = 1$.

Therefore, $\mathbb{P}[X>0]>0$. Let us prove that $\mathbb{P}[X>0]=1-q$.
This will imply the alternative: $\underset{\mathbb{P}=q}{(Z_n \to 0)}$ or $\underset{\mathbb{P}=1-q}{(Z_n \to +\infty \text{ at speed } m^n)}$.

We compute $\mathbb{P}[X=0]$ by conditioning according to the number of children of the root. Note that:

$$\{X=0\} \text{ and } \{k(\phi)=k\} = \{X^{(1)}=0, \dots, X^{(k)}=0\} \text{ and } \{k(\phi)=k\}$$

where $X^{(i)} = \lim_{n\to+\infty}\left(\frac{1}{m^{n-1}} \sum_{\substack{|v|=n \\ v_1=i}} B_v\right)$.



subexponential growth of T $\iff$ subexponential growth of all the subtrees $T^{(1)} \dots T^{(k(\phi))}$.

$$p = \mathbb{P}[X=0] = \sum_{k=0}^{\infty} \mathbb{P}[k(\phi)=k \text{ and } X^{(1)}=0, \dots, X^{(k)}=0]$$

$$= \sum_{k=0}^{\infty} \mathbb{P}[k(\phi)=k]\, p^k = G(p).$$

As $p<1$, $p=q$ $\square$.

<u>remark</u>: we only need $\mathbb{E}[X]=1$, which is true if $X_n \xrightarrow[L^1]{} X$
$\iff (X_n)_{n\in\mathbb{N}}$ Ui

Kesten-Stigum: this is true iff

$$\sum_k k \log k \, p(k) < +\infty.$$

5. Backward martingales.

We consider a filtration labelled by negative integers:

$(\mathcal{F}_n)_{n\leq 0}$ with $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_{-1} \supset \dots \supset \mathcal{F}_{-n} \supset \dots$

Definition A backward martingale is a family $(X_n)_{n\leq 0}$ of variables in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ with X_n measurable w.r.t. $\mathcal{F}_n$ for all $n \leq 0$ and $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$ $\forall n \leq 0$.

Theorem Given a backward martingale $(X_n)_{n\leq 0}$, $X_n \xrightarrow{L^1, a.s} X$ as $n \to -\infty$, with $X = \mathbb{E}[X_0 \mid \mathcal{F}_{-\infty}]$, $\mathcal{F}_{-\infty} = \bigcap_{n\leq 0} \mathcal{F}_n$.

Proof: $(X_n)_{n\leq 0}$ is Ui and therefore bounded in L^1, because $X_n = \mathbb{E}[X_0 \mid \mathcal{F}_n]$ $\forall n \leq 0$.

Moreover, the inequality on upcrossings is true (same proof):

$$\mathbb{E}[M_{[-n,0]}(X, a, b)] \leq \frac{\mathbb{E}[(X_0 - a)_-]}{b-a}.$$

So $X_n \xrightarrow[a.s.]{} X$ as $n \to -\infty$, $X \in L^1$ by Fatou, X is $\mathcal{F}_{-\infty}$ measurable.

By uniform integrability, $X_n \xrightarrow[L^1]{} X$.

Finally, $\mathbb{E}[X_n \mid \mathcal{F}_{-\infty}] = \mathbb{E}[\mathbb{E}[X_0 \mid \mathcal{F}_n] \mid \mathcal{F}_{-\infty}] = \mathbb{E}[X_0 \mid \mathcal{F}_{-\infty}]$

So taking the limit, $X = \mathbb{E}[X \mid \mathcal{F}_{-\infty}] = \mathbb{E}[X_0 \mid \mathcal{F}_{-\infty}]$ $\square$

example: $(\xi_n)_{n\geq 1}$ sequence of iiD random variables in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

$$X_{-n} = \frac{\xi_1 + \xi_2 + \dots + \xi_n}{n}. \quad \mathcal{F}_{-n} = \sigma(X_{-n}, \xi_{n+1}, \xi_{n+2}, \dots).$$

We have indeed $\mathcal{F}_{-(n+1)} \subseteq \mathcal{F}_{-n}$, because

$$X_{-(n+1)} = \frac{\xi_1 + \dots + \xi_{n+1}}{n+1} = \frac{n}{n+1} X_{-n} + \frac{\xi_{n+1}}{n+1}.$$

$(X_n)_{n\leq -1}$ is a backward martingale.

$$\mathbb{E}[X_{-n} \mid \mathcal{F}_{-n-1}] = \mathbb{E}\left[\frac{(n+1)X_{-(n+1)} - \xi_{n+1}}{n} \,\Big|\, \mathcal{F}_{-n-1}\right]$$

$$= \frac{n+1}{n} X_{-(n+1)} - \frac{1}{n} \mathbb{E}[\xi_{n+1} \mid \mathcal{F}_{-n-1}].$$

<u>lemma</u> : let $\mathcal{G}$ and $\mathcal{H}$ be independent σ-algebras

$\mathcal{G}' = \sigma(\mathcal{G}, \mathcal{H})$

X integrable, independent from $\mathcal{H}$.

We have $\mathbb{E}[X \mid \mathcal{G}'] = \mathbb{E}[X \mid \mathcal{G}]$.

<u>Proof</u> : if $A \in \mathcal{G}$ and $B \in \mathcal{H}$, then

$$
\begin{aligned}
\mathbb{E}[X \mathbb{1}_A \mathbb{1}_B] &= \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}'] \mathbb{1}_A \mathbb{1}_B] \\
&= \mathbb{E}[X \mathbb{1}_A] \mathbb{E}[\mathbb{1}_B] \\
&= \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mathbb{1}_A] \mathbb{E}[\mathbb{1}_B] \\
&= \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mathbb{1}_A \mathbb{1}_B]
\end{aligned}
$$

and this extends to $\mathcal{G}'$ measurable events more general than $A \cap B$. $\square$.

Here, with $\mathcal{G} = \sigma(X_{-n-1})$ and $\mathcal{H} = \sigma(\xi_{n+2}, \dots)$, we get :

$$\mathbb{E}[\xi_{n+1} \mid \mathcal{F}_{-n-1}] = \mathbb{E}[\xi_{n+1} \mid X_{-n-1}] = (\text{exchangeability}) \ \frac{X_{-n-1}}{n+1}.$$

$\Rightarrow (X_n)_{n \le -1}$ backward martingale, converge a.s. and L^1 to a $\mathcal{F}_{-\infty}$ measurable variable, which is constant by the Kolmogorov 0-1 law.

$\longrightarrow$ new proof of the law of large numbers!