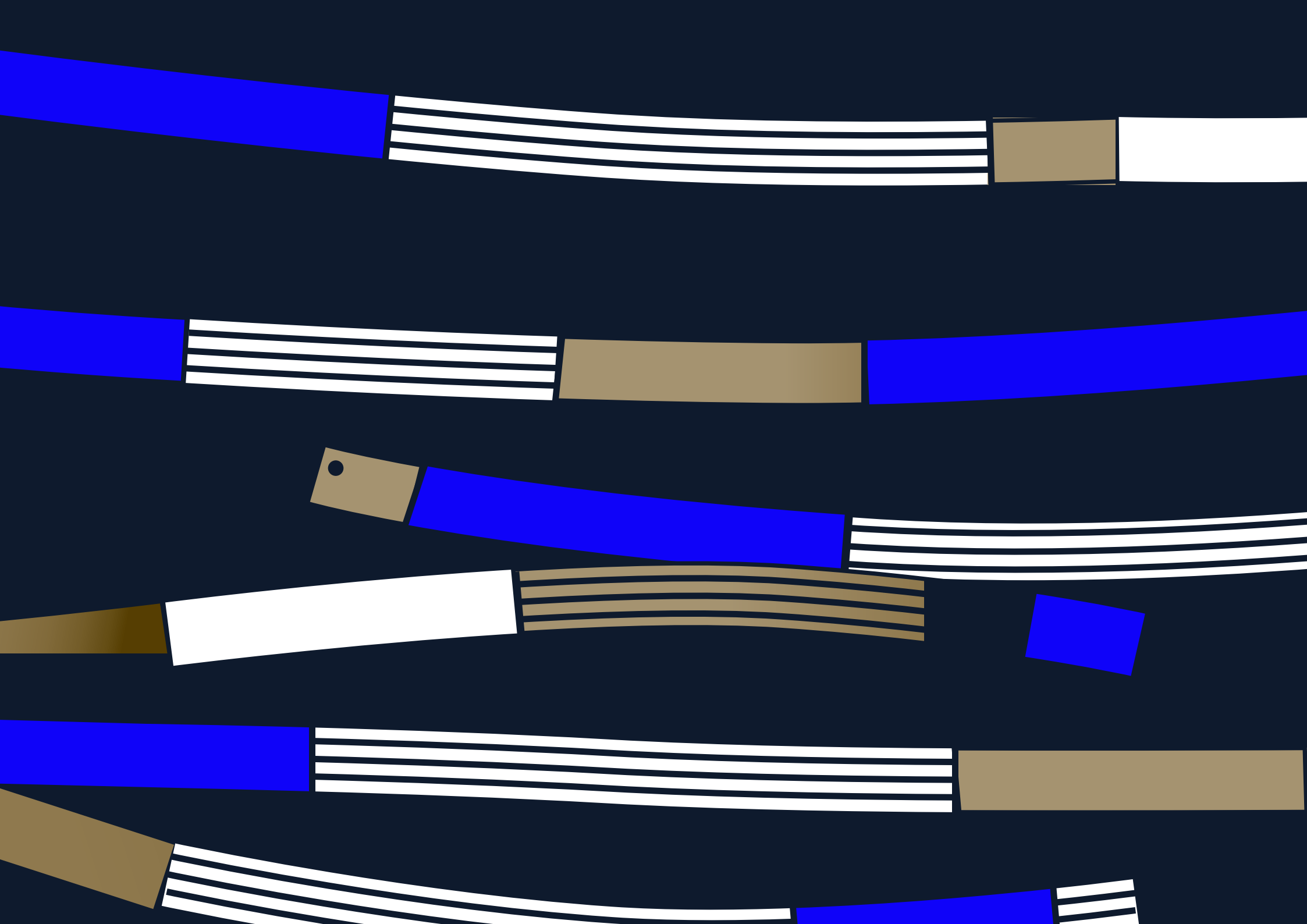


Martingales.

$(\Omega, \mathcal{F}, \mathbb{P})$ a random element.

$X_n(\omega)$: ① a kind of energy, at time n, with a conservation property.

② information known about ω at time n.

1. Definitions and examples

$(\Omega, \mathcal{F}, \mathbb{P})$

Definition A filtration of $(\Omega, \mathcal{F}, \mathbb{P})$ is a sequence

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_n \subset \dots \subset \mathcal{F}$$ of subσ-algebras.

$\mathcal{F}_n = \{$ events that can be decided at time $n\}$.

In particular: $(X_n)_{n \in \mathbb{N}}$ sequence of r.v. defined on $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$ is the canonical filtration associated to the random process $(X_n)_{n\in\mathbb{N}}$.

Definition $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ filtration.

A martingale is a sequence of real r.v. $(X_n)_{n\in\mathbb{N}}$ such that:

1) $(X_n)_{n\in\mathbb{N}}$ is adapted: $\forall n\in\mathbb{N}$, X_n is $\mathcal{F}_n$-measurable.

2) $\forall n$, $X_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

3) $\forall n$, $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$.

If instead of 3) we have $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n$ a.s., then we have a submartingale.

$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$ a.s. $\Rightarrow$ supermartingale

Remarks: ① $(X_n)_{n\in\mathbb{N}}$ submartingale $\Leftrightarrow$ $(-X_n)_{n\in\mathbb{N}}$ supermartingale.

A martingale is at the same time a sub- and a supermartingale.

② $\forall n, m \geq 0$, $\mathbb{E}[X_{n+m} \mid \mathcal{F}_n] = \mathbb{E}[\dots \mathbb{E}[\mathbb{E}[X_{n+m} \mid \mathcal{F}_{n+m-1}] \mid \mathcal{F}_{n+m-2}] \dots \mid \mathcal{F}_n]$

$= X_n$.

for a martingale.

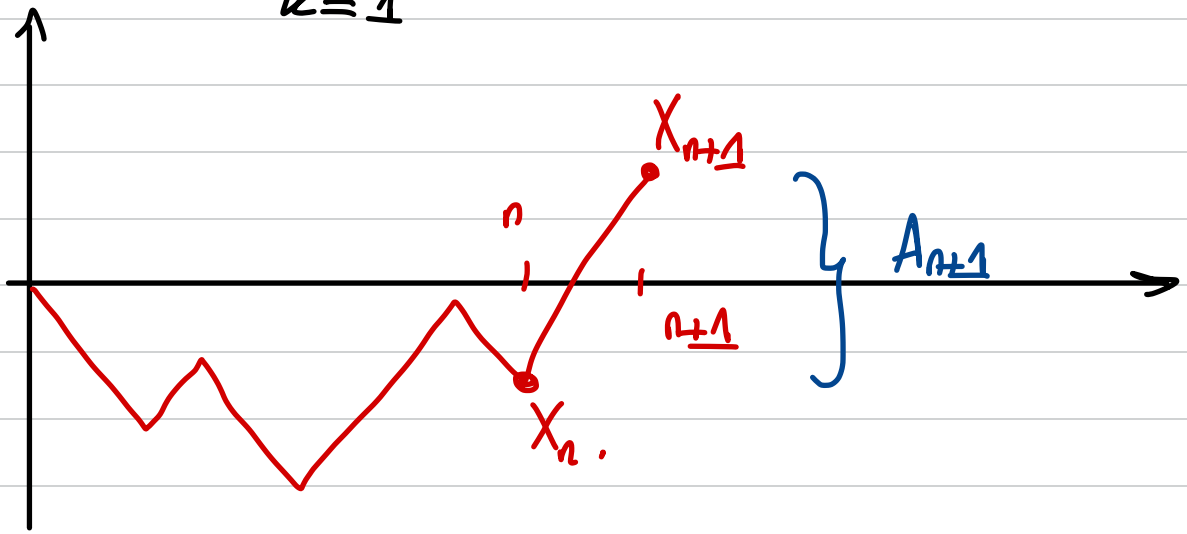
③ $\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n \mid \mathcal{F}_0]] = \mathbb{E}[X_0]$

The sequence of expectations is constant.

<u>Examples</u> : ① $(A_n)_{n \geq 1}$ sequence of independent variables, in L^1,

with $\mathbb{E}[A_n] = 0$.

$X_n = \sum_{k=1}^{n} A_k$ random walk with steps $A_{k \geq 1}$



$(X_n)_{n \in \mathbb{N}}$ is a martingale w.r.t.

$\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$

$= \sigma(A_1, \dots, A_n)$.

Conditions : 1) is automatic when working with the canonical filtration.

2) $X_n \in L^1$ because $X_n = \sum_{k=1}^{n} A_k$, the $A_k \in L^1$.

3) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n + A_{n+1} \mid \mathcal{F}_n]$.

X_n : $\mathcal{F}_n$ measurable

A_{n+1} : independant from $\mathcal{F}_n$

$= X_n + \mathbb{E}[A_{n+1}] = X_n + 0 = X_n$.

in particular, if $A_n \sim 2\text{Bernoulli}_{\pm 1}\left(\frac{1}{2}\right)$, then the symmetric random walk on $\mathbb{Z}$ $(X_n)_{n \in \mathbb{N}}$ is a martingale.

② $(B_n)_{n \geq 1}$ sequence of independant non negative r.v. with $\mathbb{E}[B_n] = 1$.

$X_n = \prod_{k=1}^{n} B_k$. is a martingale w.r.t. $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$.

$= \sigma(B_1, \dots, B_n)$.

1) OK. 2) $\mathbb{E}[X_n] = \prod_{k=1}^{n} \mathbb{E}[B_k] = \prod_{k=1}^{n} 1 = 1 < +\infty$.

so each $X_n \in L^1$.

3) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_n B_{n+1} \mid \mathcal{F}_n]$.

(X_n: $\mathcal{F}_n$-measurable; B_{n+1}: $\mathcal{F}_n$-independent.)

$= X_n \mathbb{E}[B_{n+1} \mid \mathcal{F}_n] = X_n \underbrace{\mathbb{E}[B_{n+1}]}_{=1} = X_n$.

A_n independent $\sim$ Bernoulli$_{\pm 1}(p)$:

$\mathbb{P}[A_n = 1] = p$; $\mathbb{P}[A_n = -1] = 1-p$; p fixed in $(0,1)$.

- if $p = \frac{1}{2}$, $X_n = \sum_{k=1}^{n} A_k$ is a martingale.
- if $p \neq \frac{1}{2}$; $Y_n = \left(\frac{1-p}{p}\right)^{X_n} = \left(\frac{1-p}{p}\right)^{\sum_{k=1}^{n} A_k}$ is a martingale

$= \prod_{k=1}^{n} B_k$; $B_k = \left(\frac{1-p}{p}\right)^{A_k}$.

$\mathbb{E}[B_k] = p \frac{1-p}{p} + (1-p) \frac{p}{1-p} = 1-p+p = 1$.

③ $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_n)_{n \in \mathbb{N}}$ filtration ; $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

$X_n = \mathbb{E}[X \mid \mathcal{F}_n]$ is a martingale closed by X.

1) X_n is $\mathcal{F}_n$ measurable and integrable by definition of $\mathbb{E}[\cdot \mid \mathcal{F}_n]$.

+2)

3) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] = \mathbb{E}[X \mid \mathcal{F}_n] = X_n$.

④ martingales associated to a branching process.

Definition: The infinite Ulam-Harris tree $\mathcal{U}$ is $\bigcup_{n \geq 0} (\mathbb{N}^*)^2$.

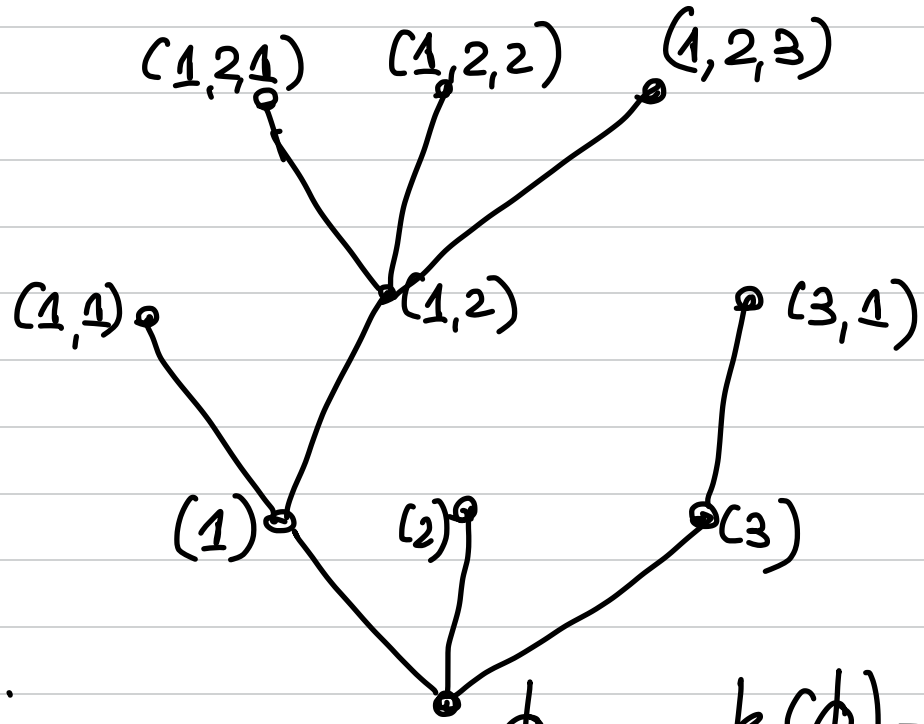
$(1, 3, 2, 2) \in \mathcal{U}$, $\phi \in \mathcal{U}$.

ascendency relation: $(v_1, \dots, v_n)$ is a child of $(v_1, \dots, v_{n-1})$.

A labelled tree is a subset $T \subset \mathcal{U}$ with 3 properties:

- $\phi \in T$.
- if $(v_1, \dots, v_n) \in T$, its parent $(v_1, \dots, v_{n-1}) \in T$.
- if $(v_1, \dots, v_{n-1}) \in T$, then $\exists\, k(v_1, \dots, v_{n-1}) \in \mathbb{N}$ such that:

$(v_1, \dots, v_n) \in T \iff v_n \leq k(v_1, \dots, v_{n-1}) =$ number of children of $(1, \dots v_{n-1})$.



T is a labelled tree.

$\xi(4)$ is not used, because $\xi_\phi = 3$

$k(1) = 2$
$k(2) = 0$
$k(3) = 1$.
$k(1,2) = 3$.

$k(\phi) = 3$

$|v| = |(v_1, \dots, v_n)| = n$ = depth of v in $\mathcal{V}$.

$$Z_n(T) = \sum_{|v|=n} \mathbb{1}_{(v \in T)}$$

<u>Definition</u> (Branching process ; Galton-Watson random tree).

μ law on $\mathbb{N}$: $\sum_{k=0}^{+\infty} \mu(k) = 1$. $(\xi_v)_{v \in \mathcal{V}}$ family of independent

random variables with law μ.

The branching process with reproduction law μ is the random tree $T \subset \mathcal{U}$ defined by: $T = \{ v \in \mathcal{U} \mid$ "$(v_1, \dots, v_n)$" $v_n \leq \xi_{(v_1 \dots v_{n-1})}$, $v_{n-1} \leq \xi_{(v_1, \dots, v_{n-2})}$, $\dots$, $v_1 \leq \xi_\phi \}$.

in other words: T is the tree with $k(v) = \xi_v$.

T is indeed a (random) tree:
1) $\phi \in T$
2) if $(v_1, \dots, v_n) \in T$, then $(v_1, \dots, v_{n-1}) \in T$
3) is true with $k(v_1, \dots, v_{n-1}) = \xi_{v_1, \dots, v_{n-1}}$

$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which the $\xi_{v \in \mathcal{U}}$ are defined.

$\mathcal{F}_n = \sigma(\xi_v, |v| < n) \rightsquigarrow$ gives the structure of the tree up to level n.

We assume $\mathbb{E}[\xi_v] = m < +\infty$; $m = \sum_{k=0}^{+\infty} k \mu(k)$.

$$Z_n = \sum_{|v|=n} \mathbb{1}_{v \in T} = \sum_{|v|=n} \mathbb{1}_{v_n \le \xi(v_1,\dots,v_{n-1});\ v_{n-1} \le \xi(v_1,\dots,v_{n-2});\ \dots;\ v_1 \le \xi(\emptyset)}$$

1) Z_n is adapted : it is a sum of $\mathcal{F}_n$ measurable variables.

$\downarrow$ B_v.

2) Z_n is integrable :

$$\mathbb{E}[B_v \mid \mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{1}_{v_n \le \xi(v_1,\dots,v_{n-1})} B_{(v_1,\dots,v_{n-1})} \mid \mathcal{F}_{n-1}].$$

$(v_1,\dots,v_n)$

$\mathbb{1}_{v_n \le \xi(v_1,\dots,v_{n-1})}$: independent from $\mathcal{F}_{n-1}$.

$B_{(v_1,\dots,v_{n-1})}$: $\mathcal{F}_{n-1}$ measurable.

$$= B_{(v_1,\dots,v_{n-1})} \sum_{k=v_n}^{+\infty} p(k)$$

$$= B_{(v_1,\dots,v_{n-1})}\ p([\![v_n, +\infty]\!]).$$

$$\mathbb{E}[B_v] = \mathbb{E}[B_{(v_1,\dots,v_{n-1})}]\ p([\![v_n, +\infty]\!]).$$

$$\mathbb{E}[Z_n] = \sum_{|v|=n} \mathbb{E}[B_v] = \sum_{|w|=n-1,\ k\ge 1} \mathbb{E}[B_w]\ p([\![k, +\infty]\!])$$

$$= \left(\sum_{|w|=n-1} \mathbb{E}[B_w] \right) \left(\sum_{k\ge 1} p([\![k, +\infty]\!]) \right).$$

$$= \mathbb{E}[Z_{n-1}] \left(\sum_{k \geq 1} k \mu(k) \right) = m \, \mathbb{E}[Z_{n-1}].$$

So $Z_n \in L^1$, $\mathbb{E}[Z_n] = m^n$.

3) $X_n = \dfrac{Z_n}{m^n}$ is a martingale.

$$\mathbb{E}[Z_n \mid \mathcal{F}_{n-1}] = \sum_{|v|=n} \mathbb{E}[B_v \mid \mathcal{F}_{n-1}]$$

$$= \sum_{\substack{|w|=n-1 \\ k \geq 1}} B_w \, \mu([\![k, +\infty[\![) = Z_{n-1} \cdot m.$$

$$\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}.$$

Another construction of martingale: generating series.

$$\mu : \sum_{k=0}^{+\infty} \mu(k) = 1 \; ; \qquad m = \sum_{k=0}^{+\infty} k \mu(k) < +\infty$$

$$G(s) = \sum_{k=0}^{+\infty} \mu(k) \, s^k = \mathbb{E}[s^{\sigma_v}] \qquad \forall v \in \mathcal{V}.$$

$$s \in [0, 1].$$

$Y_n = s^{Z_n}$; $(Y_n)_{n\in\mathbb{N}}$ is adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$, integrable

$$\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] = \mathbb{E}\left[s^{\sum_{|w|=n} B_w} \mid \mathcal{F}_{n-1} \right]$$

$$= \mathbb{E}\left[s^{\sum_{|w|=n-1,\, k\geq 1} B_w \mathbb{1}_{k\leq \xi(w)}} \mid \mathcal{F}_{n-1} \right].$$

$$= \mathbb{E}\left[s^{\sum_{|w|=n-1} B_w \xi(w)} \mid \mathcal{F}_{n-1} \right].$$

$$= \mathbb{E}\left[\prod_{|w|=n-1} (s^{\xi(w)})^{B_w} \mid \mathcal{F}_{n-1} \right].$$

$$\overset{?}{=} \prod_{|w|=n-1} (G(s))^{B_w} = G(s)^{\sum_{|w|=n-1} B_w} = (G(s))^{Z_{n-1}}.$$

$$\square = \sum_{(b_w)_{|w|=n-1}} \mathbb{1}_{\forall w,\, B_w = b_w} \prod_{|w|} (s^{\xi(w)})^{b_w}.$$

$\mathbb{1}_{\forall w,\, B_w = b_w}$: $\mathcal{F}_{n-1}$ measurable.

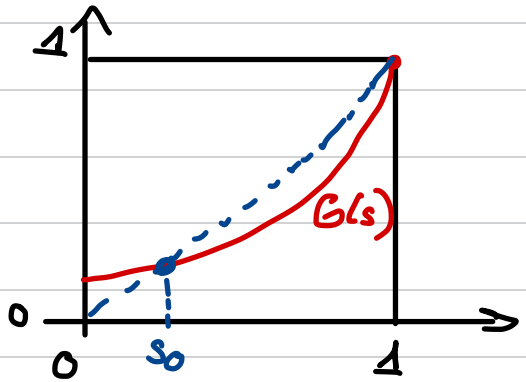
$\prod_{|w|} (s^{\xi(w)})^{b_w}$: independent from $\mathcal{F}_{n-1}$

$$\mathbb{E}[- \mid \mathcal{F}_{n-1}] = \sum_{(b_w)_{|w|=n-1}} \mathbb{1}_{\forall w,\, B_w = b_w} \prod_{|w|=n-1} G(s)^{b_w}.$$

$$= \prod_{|w|=n-1} (G(s))^{B_w}.$$

conclusion: $\mathbb{E}[s^{Z_n} \mid \mathcal{F}_{n-1}] = (G(s))^{Z_{n-1}}$.

if $s = G(s)$, then $(s^{Z_n})_{n\in\mathbb{N}}$ is a martingale. $G(s) = \sum_{k=0}^{+\infty} \mu(k) s^k$



$G'(1) = m$.

$G(s_0) = s_0$

$\Rightarrow (s_0^{Z_n})$ = martingale.

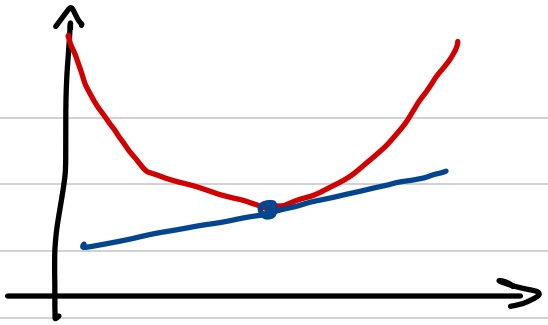
A useful transformation property:

$(X_n)_{n\in\mathbb{N}}$ martingale; $\phi : \mathbb{R} \to \mathbb{R}$ convex.

$Y_n = \phi(X_n)$; if Y_n is integrable $\forall n \in \mathbb{N}$, then $(\phi(X_n))_{n\in\mathbb{N}}$ is a submartingale.

$/(\mathcal{F}_n)$ $/(\mathcal{F}_n)$

Proof: $\Phi(t) = \sup\{ax+b : \forall x,\ ax+b \le \Phi(x)\}$



$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n]$$

$$= \mathbb{E}\Big[\sup_{\{(a,b)\mid \dots\}} aX_{n+1} + b \mid \mathcal{F}_n\Big]$$

$$\geq \sup_{\{(a,b)\mid \dots\}} \mathbb{E}[aX_{n+1} + b \mid \mathcal{F}_n]$$

$$\geq \sup \; aX_n + b = \Phi(X_n) = Y_n \qquad \square.$$