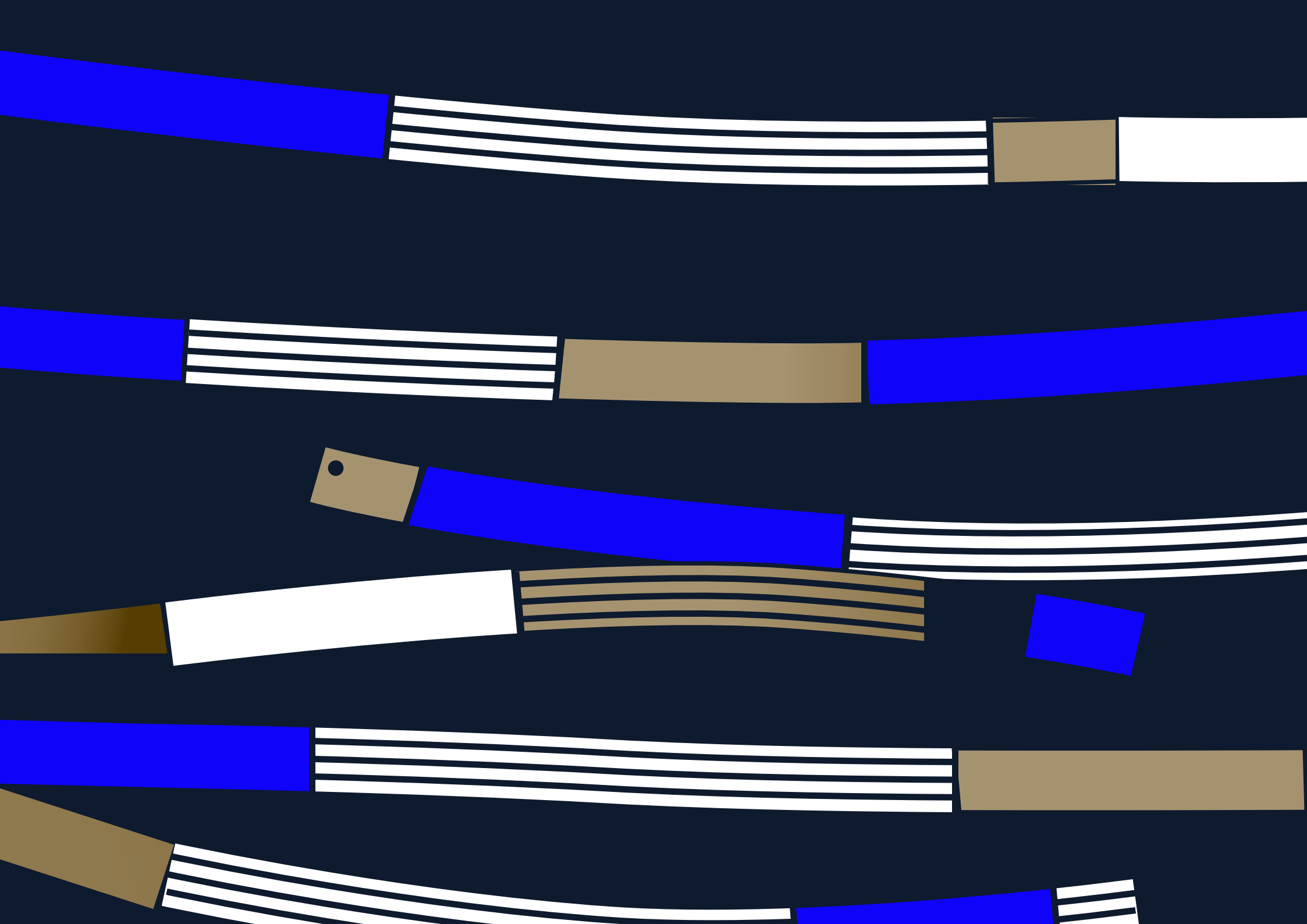


length of the cycle containing 1 in a random permutation $\sigma \in S(n)$?

$n = 3$: 6 permutations :

	k
id	1
$(1,2)$	2
$(1,3)$	2
$(2,3)$	1
$(1,2,3)$	3
$(1,3,2)$	3

2 such that $s_1(\sigma) = 1$

2 —— $s_1(\sigma) = 2$

2 —— $s_1(\sigma)$ 3

⋮

Martingales : $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

1) X_n is $\mathcal{F}_n$ measurable ; $(X_n)_{n\in\mathbb{N}}$ adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.
2) $X_n \in L^1$
3) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$.

2. Transformation of martingales, and inequalities.

Definition: $(A_n)_{n\geq 1}$ is a predictable process if A_n is $\mathcal{F}_{n-1}$ measurable for any $n \geq 1$.

$(A_n)_{n\geq 1}$ predictable

$(X_n)_{n\geq 0}$ adapted

$$(A \cdot X)_n = \sum_{k=1}^{n} A_k (X_k - X_{k-1})$$

= discrete stochastic integral.

= « $\int_{t=0}^{n} A_t \cdot dX_t$ ».

Lemma: $(A_n)_{n\geq 1}$ predictable, with $\forall n,\ |A_n| \leq_{a.s.} C_n < +\infty$.
$(X_n)_{n\geq 0}$ martingale.

$((A.X)_n)_{n\in\mathbb{N}}$ is again a martingale.

If $A_n \geq_{a.s.} 0$, then sub- and supermartingales are also conserved.

Proof 1) $A.X_n = \sum_{k=1}^{n} A_k (X_k - X_{k-1})$ is $\mathcal{F}_n$ measurable.

2) integrability: $A_k \in L^\infty$, $X_k - X_{k-1} \in L^1$: $A_k(X_k - X_{k-1}) \in L^1$.

3) $\mathbb{E}[(A.X)_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[(A.X)_n + A_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n]$

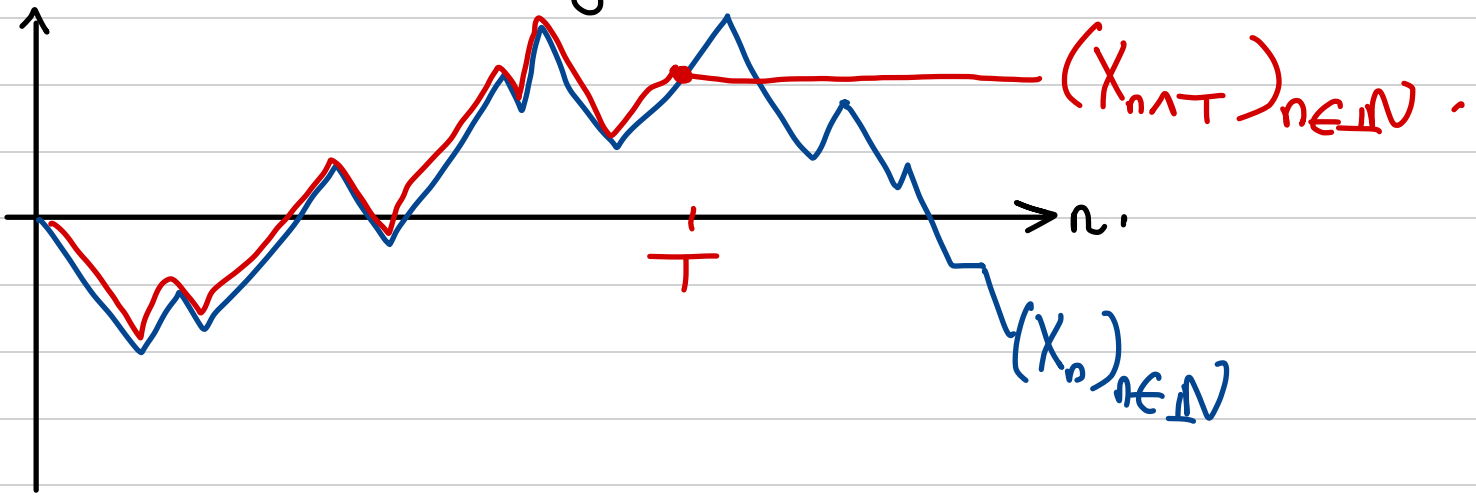
$= (A.X)_n + A_{n+1}(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - X_n)$.

$= 0$ if $(X_n)_{n\in\mathbb{N}}$ is a martingale.

$= (A.X)_n$.

□.

First application : stopping time theorem



$T \in \mathbb{N} \cup \{+\infty\}$: $X_{n\wedge T} = \begin{cases} X_n \text{ if } n \le T \\ X_T \text{ if } n > T \end{cases}$

Definition A stopping time T w.r.t $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ is a random variable T with values in $\mathbb{N} \cup \{+\infty\}$ s.t.

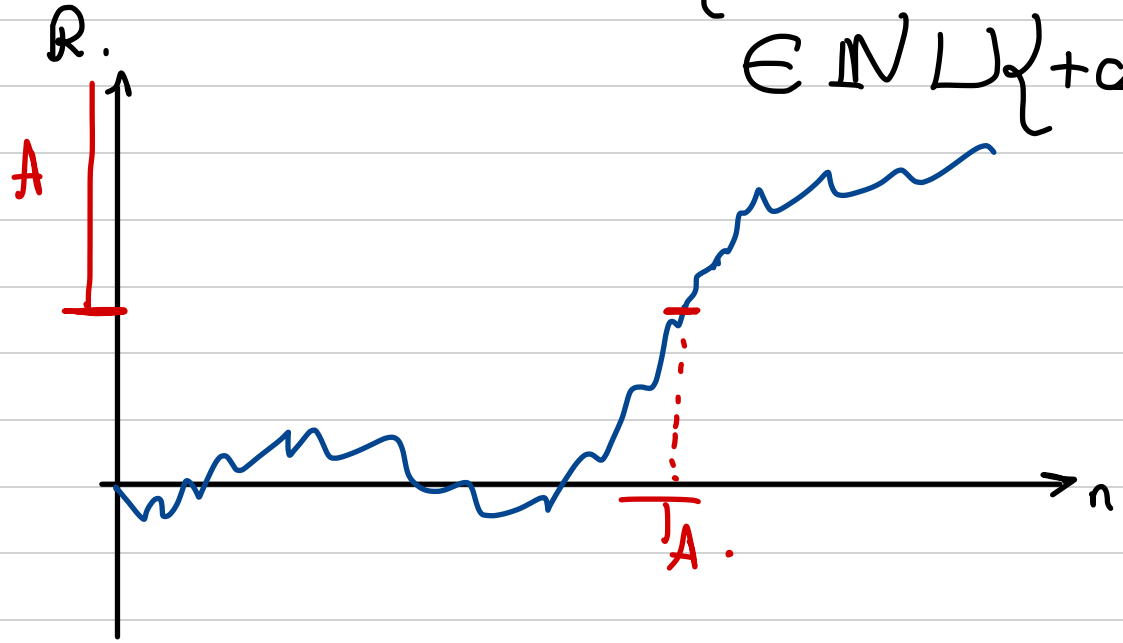
$\forall n \in \mathbb{N}$, $\{T = n\} \in \mathcal{F}_n$.

Example : $(X_n)_{n\in\mathbb{N}}$ adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$, with values in $\mathbb{R}$.

$A \in \mathcal{B}(\mathbb{R})$; T_A = hitting time of A

$= \inf(\{n \in \mathbb{N} \mid X_n \in A\})$

$\in \mathbb{N} \cup \{+\infty\}$; $T_A = +\infty$ when $X_n \notin A$ $\forall n \in \mathbb{N}$.



Then, T_A is a stopping time :

$\{T_A = n\} = \{X_0 \notin A \text{ and } X_1 \notin A, \dots, X_{n-1} \notin A, X_n \in A\}$

$=$ intersection of events in $\mathcal{F}_n$ $\Rightarrow \{T_A = n\} \in \mathcal{F}_n$.

$\{T_A = +\infty\} = \bigcap_{k \geq 0} \{X_k \notin A\} \in \mathcal{F}$.

$$T_A = \sum_{n \in \mathbb{N} \cup \{+\infty\}} n \cdot \mathbb{1}_{\{T_A = n\}}$$ is indeed a random variable, and a stopping time.

Doob stopping time theorem :

If T is a stopping time and $(X_n)_{n\in\mathbb{N}}$ is a martingale, $(X_{n\wedge T})_{n\in\mathbb{N}}$ is a martingale.

The conservation also holds for sub- and supermartingales.

Proof: We can assume that $X_0 = 0$: $(X_n)_{n\in\mathbb{N}}$ martingale $\Leftrightarrow (X_n - X_0)_{n\in\mathbb{N}}$ martingale if $X_0 \in L^1(\Omega, \mathcal{F}_0, \mathbb{P})$.

Then, $X_{n\wedge T} = \sum_{k=1}^{n} \mathbb{1}_{k\le T}(X_k - X_{k-1}) = (A.X)_n$

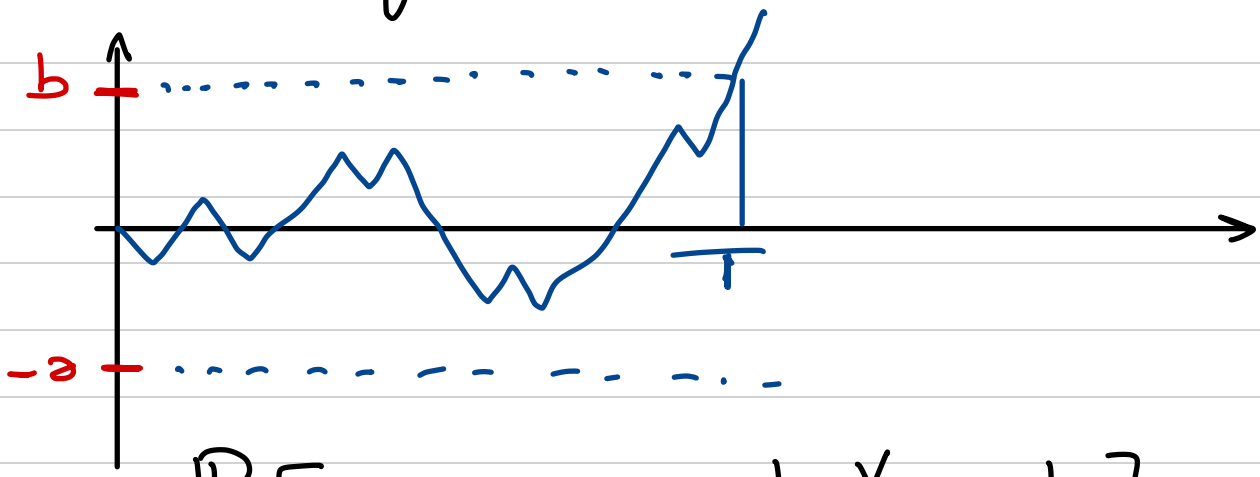
with $A_k = \mathbb{1}_{k\le T} = 1 - \mathbb{1}_{T\le k-1} = \underbrace{1 - \sum_{j=0}^{k-1} \mathbb{1}_{T=j}}_{\mathcal{F}_{k-1} \text{ measurable.}}$

≥ 0, predictable. □.

example: 1) $X_n = \sum_{k=1}^{n} \xi_k$, $\xi_k \sim \text{Bernoulli}_{\pm 1}\left(\frac{1}{2}\right)$.

the ξ_k independant.

$(X_n)_{n \in \mathbb{N}}$ symmetric random walk on $\mathbb{Z}$.



$a, b \in \mathbb{N}$.

$T = T_{\{-a, b\}}$.

$\mathbb{P}[T < +\infty \text{ and } X_T = b] = ?$.

$(X_{n \wedge T})_{n \in \mathbb{N}}$ is a martingale: ($\forall n \in \mathbb{N}$, $\mathbb{E}[X_{n \wedge T}] = \mathbb{E}[X_{0 \wedge T}]$

$= \mathbb{E}[X_0] = 0$.

We will show that $T < +\infty$ a.s. (proof in 5 minutes).

Then, $X_{n \wedge T} \xrightarrow[a.s.]{} X_T$, by the dominated convergence theorem,

$$\mathbb{E}[X_T] = \lim_{n\to+\infty} \mathbb{E}[X_{n\wedge T}] = 0. \qquad (|X_{n\wedge T}| \le \max(a,b)\ \forall n\in\mathbb{N})$$

$$\mathbb{P}[X_T = b]\, b + (1 - \underbrace{\mathbb{P}[X_T = b]}_{\mathbb{P}[X_T=-a]})(-a).$$

So, $\mathbb{P}[X_T = b] = \dfrac{a}{a+b}$.

2) same random process; $S = T_1 = \inf\{n\in\mathbb{N} \mid X_n = 1\}$.

Let us prove that $T_1 < +\infty$ a.s.

$$\{T_1 < +\infty\} \supset \underbrace{\{\limsup_{n\in\mathbb{N}} X_n = +\infty\}}_{E} = \{\sup_{n\in\mathbb{N}} X_n = +\infty\}.$$

has probability 1? This event belongs to the tail σ-algebra

$$\mathcal{G} = \bigcap_{n\ge 0} \sigma(\xi_{n+1}, \xi_{n+2}, \dots, \xi_{n+k}, \dots).$$

$$\mathbb{P}[E] = \mathbb{P}[\sup_{n\in\mathbb{N}} X_n = +\infty]$$

$$\mathbb{P}\left[\sup_{n\in\mathbb{N}} X_n \neq +\infty\right] = \lim_{K\to+\infty} \mathbb{P}[\forall n,\ X_n \le K].$$

$$\le \lim_{K\to+\infty} \mathbb{P}[X_{n_K} \le K] \quad \text{for any sequence } (n_K).$$

$$= \frac{1}{2} \text{ if } n_K = K^3.$$

Let us take $n_K = K^3$.

We know that $\dfrac{X_{K^3}}{K^{3/2}} \xrightarrow[K\to+\infty]{} \mathcal{N}(0,1)$. (central limit theorem).

$$\mathbb{P}[X_{K^3} \le K] = \mathbb{P}\left[\frac{X_{K^3}}{K^{3/2}} \le \frac{1}{\sqrt{K}}\right] \le \mathbb{P}\left[\frac{X_{K^3}}{K^{3/2}} \le \varepsilon\right]$$

$\forall \varepsilon > 0$, if K is large enough.

$$\le \underbrace{\frac{1}{2} + O(\varepsilon)}_{F_{\mathcal{N}(0,1)}(\varepsilon)} + o_K(1)$$

central limit theorem.

Conclusion : $\mathbb{P}[\sup X_n \neq +\infty] \leq \frac{1}{2}$.

$\mathbb{P}[\sup X_n = +\infty] \geq \frac{1}{2}$, so it must be $= 1$.

$\Rightarrow$ T_1 is finite (and in fact, $T_k < +\infty$ a.s. $\forall k \in \mathbb{Z}$).

$(X_{n \wedge T_1})$ is a martingale.

$\mathbb{E}[X_{n \wedge T_1}] = \mathbb{E}[X_0] = 0$.

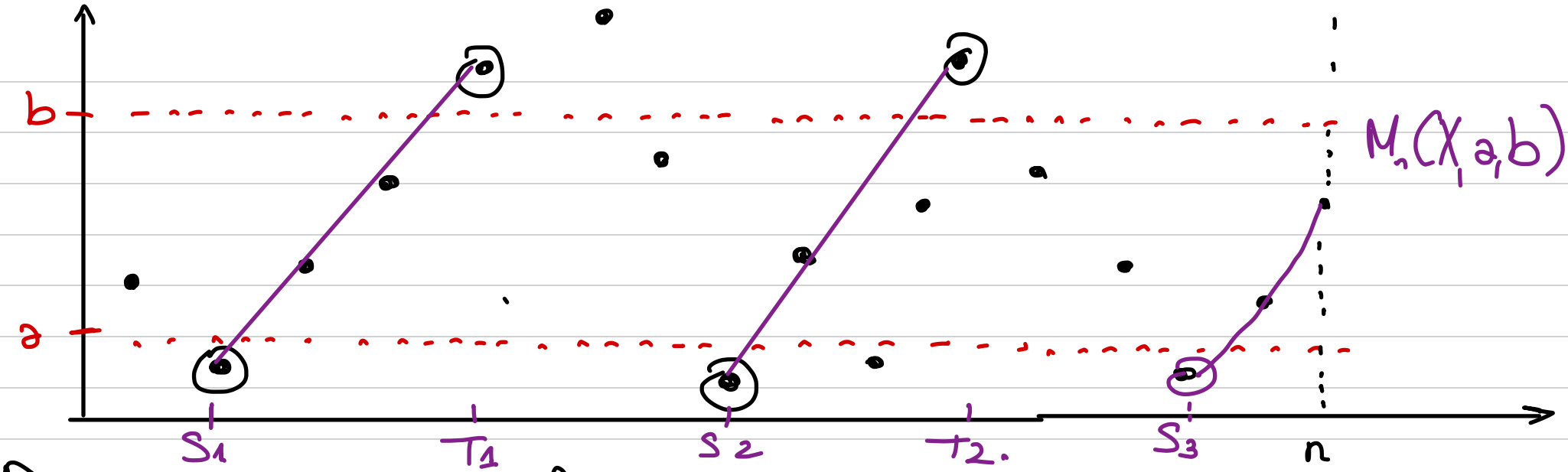
We do not have $\mathbb{E}[X_{T_1}] = 0$!.

$X_{T_1} = 1$ a.s. $\mid \mathbb{E}[X_{T_1}] = 1$.

We have $X_{n \wedge T_1} \xrightarrow[\text{a.s.}]{} X_{T_1} = 1$, but not in L^1.

3. Doob inequalities

$(X_n)_{n \in \mathbb{N}}$ adapted process.



Definition: (number of upcrossings / rises)

$$M_n(X,a,b) = \sup\{ k \geq 0 \mid \exists\, 0 \leq s_1 < t_1 < s_2 < t_2 \dots < s_k < t_k \leq n \mid X_{s_i} < a,\ X_{t_i} > b \}$$

Theorem if $(X_n)_{n\in\mathbb{N}}$ is a supermartingale ($\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$),

then $$\mathbb{E}[M_n(X,a,b)] \leq \frac{\mathbb{E}[(X_n - a)_-]}{(b-a)}$$

where $y_- = \max(0, -y)$.

Proof: $T_0 = -1$.

$S_k = \inf\{n > T_{k-1} \mid X_n < a\}$.) optimal times such that

$(T_k = \inf\{n > S_k \mid X_n > b\}$. $X_{S_i} < a$, $X_{T_i} > b$.

S_k and T_k are stopping times $\forall k \geq 1$.

$M_n(X, a, b) = \sum_{k=1}^{+\infty} \mathbb{1}_{T_k \leq n}$; $M_n(X, a, b)$ is $\mathcal{F}_n$ measurable.

$A_n = \sum_{k=1}^{+\infty} \underbrace{\mathbb{1}_{S_k < n \leq T_k}}_{\mathbb{1}} = \mathbb{1}$ (we are along one of the rise of the random process X).

$\in \{0, 1\}$.

$$\mathbb{1}_{S_k \leq n-1}\left(1 - \mathbb{1}_{T_k \leq n-1}\right)$$

$$= \left(\sum_{j=0}^{n-1} \mathbb{1}_{S_k = j}\right)\left(1 - \sum_{j=0}^{n-1} \mathbb{1}_{T_k = j}\right) \text{ is } \mathcal{F}_{n-1} \text{ measurable.}$$

$((A \cdot X)_n)_{n \in \mathbb{N}}$ is a supermartingale!.

$$0 = \mathbb{E}[(A.X)_0] \geq \mathbb{E}[(A.X)_n].$$

$n = M_n(X, a, b)$. there are two cases:

1) $n \leq S_{m+1}$. $(A.X)_n = \sum_{k=1}^{m} X_{T_k} - X_{S_k}$.

2) $S_{m+1} < n < T_{m+1}$ $|$. $(A.X)_n = \sum_{k=1}^{m} X_{T_k} - X_{S_k} + X_n - X_{S_{m+1}}$.

$$(A.X)_n \geq (b-a) M_n(X, a, b) - \mathbb{1}_{S_{m+1} < n < T_{m+1}} (X_n - a)_-.$$

then, take expectations!. □,

Another inequality due to Doob:

<u>Theorem</u>: $(X_n)_{n \in \mathbb{N}}$ submartingale w.r.t. $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

$X_n^* = \max\{X_0, X_1, \dots, X_n\}$.

$\forall a > 0$, $\mathbb{P}[X_n^* \geq a] \leq \dfrac{\mathbb{E}[(X_n)_+]}{a}$, $(X_n)_+ = \max(0, X_n)$.

Proof $T = T_{[a, +\infty)}$ hitting time.

$\{X_n^* \geq a\} \Leftrightarrow \{T \leq n\}$.

$$\mathbb{P}[T \leq n] = \sum_{k=0}^{n} \mathbb{P}[T = k] \leq \sum_{k=0}^{n} \mathbb{E}[\mathbb{1}_{\{T=k\}} \frac{X_k}{a}]$$

$$\leq \sum_{k=0}^{n} \mathbb{E}[\mathbb{1}_{\{T=k\}} \frac{\mathbb{E}[X_n | \mathcal{F}_k]}{a}].$$

$$\leq \frac{1}{a} \sum_{k=0}^{n} \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{T=k\}} X_n | \mathcal{F}_k]].$$

$$\leq \frac{1}{a} \sum_{k=0}^{n} \mathbb{E}[\mathbb{1}_{\{T=k\}} X_n] = \frac{1}{a} \mathbb{E}[\mathbb{1}_{(T \leq n)} X_n]$$

$$\leq \frac{1}{a} \mathbb{E}[\mathbb{1}_{(T \leq n)} (X_n)_+] \leq \frac{\mathbb{E}[(X_n)_+]}{a}.$$

□.