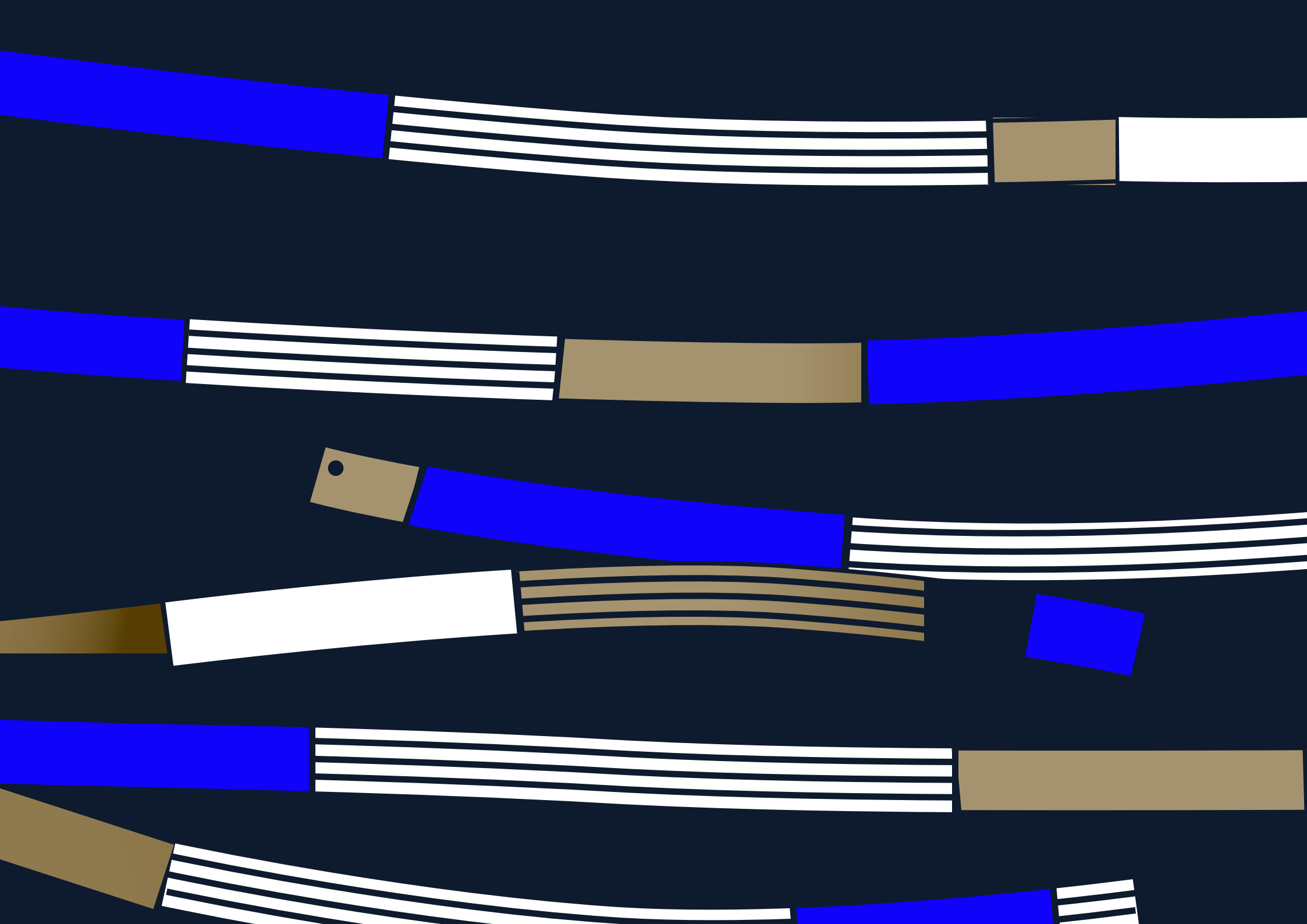


Convergence of martingales.

$(\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_n)_{n\in\mathbb{N}}$.

- number of upcrossings: $(X_n)_{n\in\mathbb{N}}$ supermartingale

$$\mathbb{E}[M_n(X, a, b)] \leq \frac{\mathbb{E}[(X_n - a)_-]}{b-a}.$$

- maximal inequality: $(X_n)_{n\in\mathbb{N}}$ submartingale.

$$X_n^* = \max\{X_0, X_1, \dots, X_n\}, \quad a > 0$$

$$\mathbb{P}[X_n^* \geq a] \leq \frac{\mathbb{E}[(X_n)_+]}{a}.$$

↳ $T = \inf\{n \in \mathbb{N} : X_n \geq a\}$ stopping time

$$\{X_n^* \geq a\} = \{T \leq n\}.$$

$$\mathbb{P}[T \le n] = \sum_{k=0}^{n} \mathbb{P}[T = k]$$

$$\le \sum_{k=0}^{n} \mathbb{E}\left[\mathbb{1}_{T=k} \frac{X_k}{a} \right]$$

$$\le \sum_{k=0}^{n} \mathbb{E}\left[\mathbb{1}_{(T=k)} \frac{\mathbb{E}[X_n \mid \mathcal{F}_k]}{a} \right].$$

$$\le \frac{1}{a} \sum_{k=0}^{n} \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{(T=k)} X_n \mid \mathcal{F}_k] \right]$$

$$\le \frac{1}{a} \sum_{k=0}^{n} \mathbb{E}[\mathbb{1}_{(T=k)} X_n] = \frac{1}{a} \mathbb{E}[\mathbb{1}_{(T \le n)} X_n]$$

$$\le \frac{1}{a} \mathbb{E}[X_n^+].$$

Corollary (Doob L^p inequalities).

$(X_n)_{n \in \mathbb{N}}$ non-negative submartingale; $p > 1$

$\mathbb{E}[(X_n)^p] < +\infty \quad \forall n \in \mathbb{N}$.

$\mathbb{E}[(X_n^*)^p] \le C_p \cdot \mathbb{E}[(X_n)^p]$; $C_p = \left(\frac{p}{p-1}\right)^p$.

Proof: $\mathbb{E}[(X_n^*)^p] = p\mathbb{E}\left[\int_0^{X_n^*} a^{p-1}\,da\right]$.

$= p\int_0^\infty \mathbb{E}[\mathbb{1}_{X_n^* \geq a}]\, a^{p-1}\,da$.

$\leq p\int_0^\infty \mathbb{E}[\mathbb{1}_{X_n^* \geq a}\, X_n]\, a^{p-2}\,da$.

$\leq \frac{p}{p-1}\,\mathbb{E}\left[(p-1)\int_0^{X_n^*} X_n\, a^{p-2}\,da\right]$

$\leq \frac{p}{p-1}\,\mathbb{E}[X_n\,(X_n^*)^{p-1}]$.

Hölder inequality: $\leq \frac{p}{p-1}\,\mathbb{E}[(X_n)^p]^{\frac{1}{p}}\,\mathbb{E}[(X_n^*)^p]^{1-\frac{1}{p}}$.

□.

- **Almost sure convergence**.

$(X_n)_{n\in\mathbb{N}}$ supermartingale / $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Hypothesis : $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n|] = K < +\infty$. (bounded in L^1).

$\exists\, X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_n \xrightarrow[a.s.]{} X$.

Particular case : if $X_n \geq 0$ a.s., then the hypothesis holds:

$$\mathbb{E}[|X_n|] = \mathbb{E}[X_n] \leq \mathbb{E}[X_0] = K < +\infty.$$

Proof. $\forall a, b \in \mathbb{Q}$, $a < b$:

$$\mathbb{E}[M_n(X, a, b)] \leq \frac{\mathbb{E}[(X_n - a)_-]}{b-a} \leq \frac{K + |a|}{b-a}.$$

$M_\infty(X, a, b) = \lim_{n \to +\infty} \uparrow M_n(X, a, b)$ is in L^1, is finite a.s.

$\mathbb{P}[\, \forall a < b$ rational numbers, $M_\infty(X, a, b) < +\infty\,] = 1$.

If $X_n(\omega)$ does not have a limit : $\limsup_{n \to +\infty} X_n(\omega) > \liminf_{n \to +\infty} X_n(\omega)$

Then, $\forall a<b$ in between, $M_\infty(\chi, a, b) = +\infty$.

$\mathbb{P}[\limsup X_n = \liminf X_n] = 1$.

Therefore, $X_n \xrightarrow[a.s.]{} X \in \mathbb{R} \cup \{\pm\infty\}$.

Moreover, $\mathbb{E}[|X|] \leq \liminf_{n\in\mathbb{N}} \mathbb{E}[|X_n|]$ (Fatou lemma)

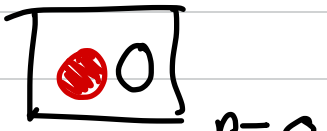
$\leq K$.

X is in L^1, is finite a.s. □.

<u>example</u> : Box which contains at time $n=0$

A_1 = Red, A_2 = White; $R_0 = 1$; $W_0 = 1$.

At each time $n \geq 0$:



$n=0$

- one picks a ball uniformly at random:

$$A_{n+3} = \begin{cases} \text{Red if the ball picked is red.} \\ \text{White} \quad \text{———} \quad \text{is white.} \end{cases}$$

Then, $(R_{n+1}, W_{n+1}) = (R_n, W_n) + \begin{pmatrix} (1,0) \\ \text{or } (0,1) \end{pmatrix}$.

At time n, the box contains $n+2$ balls.

$(U_n)_{n \geq 1}$ sequence of independent random variables.

$U_n \sim \text{Unif}(\llbracket 1, n+2 \rrbracket)$.

A_{n+3} = color of the ball with index U_{n+1}.

$X_n = \dfrac{R_n}{n+2}$ = ratio of red balls at time n.

Q: asymptotic behavior of $(X_n)_{n \in \mathbb{N}}$?.

$\mathcal{F}_n = \sigma(U_1, U_2, \dots, U_n)$; R_n, W_n and X_n are $\mathcal{F}_n$ measurable.

claim : $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

$X_n \in [0,1]$, so it is integrable.

$$\mathbb{E}[R_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[R_n + \mathbb{1}_{U_{n+1} \in \underbrace{\{\text{red balls at time } n\}}_{C_n^{red}}} \mid \mathcal{F}_n]$$

$[\![1, n+2]\!] = C_n^{red} \sqcup C_n^{white}$; the subsets C_n^{red} and C_n^{white} are $\mathcal{F}_n$ measurable.

$$\mathbb{E}[R_{n+1} \mid \mathcal{F}_n] = R_n + \mathbb{E}[\mathbb{1}_{U_{n+1} \in C_n^{red}} \mid \mathcal{F}_n]$$

$$= R_n + \sum_{C \subset [\![1,n+2]\!]} \mathbb{E}[\mathbb{1}_{U_{n+1} \in C}\, \mathbb{1}_{C = C_n^{red}} \mid \mathcal{F}_n].$$

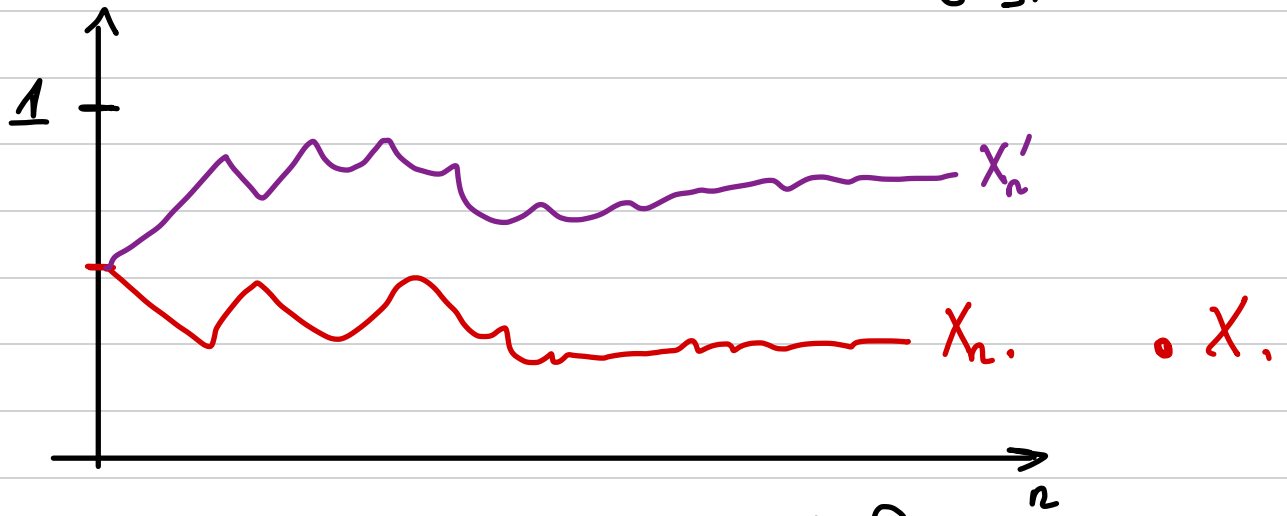
$$= R_n + \sum_{C \subset [\![1,n+2]\!]} \mathbb{1}_{C = C_n^{red}}\, \mathbb{P}[U_{n+1} \in C] \quad (U_{n+1} \text{ indep. from } \mathcal{F}_n)$$

$$= R_n + \sum_{C \subset [\![1,n+2]\!]} \mathbb{1}_{C = C_n^{red}} \frac{|C|}{n+2}$$

$$= R_n + \frac{|C_n^{red}|}{n+2} = R_n + \frac{R_n}{n+2} = \frac{n+3}{n+2} R_n.$$

$\Rightarrow \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ by dividing by $n+3$.

$(X_n)_{n \in \mathbb{N}}$ is a martingale; $X_n \xrightarrow{a.s.}$ some random variable X.



What is the distribution of the limit? $X \in [0,1]$.

$\mathbb{E}[X] = \lim_{n \to +\infty} \mathbb{E}[X_n]$ by the DCT + a.s. convergence

$= \frac{1}{2}$.

$k \geq 1$. $Y_{n,k} = \dfrac{R_n(R_n+1)\dots(R_n+k-1)}{(n+2)(n+3)\dots(n+k+1)}$ $\quad (Y_{n,1} = X_n)$.

$$= \frac{R_n^{\uparrow k}}{(n+2)^{\uparrow k}}$$

(rising factorials : $x^{\uparrow k} = x(x+1)\dots(x+k-1)$.

$\forall k \geq 1$, $(Y_{n,k})_{n \in \mathbb{N}}$ is a martingale.

① and ② are obvious ; $0 \leq Y_{n,k} \leq 1 \quad \forall n, \forall k \geq 1$.

③ $\mathbb{E}[(R_{n+1})^{\uparrow k} \mid \mathcal{F}_n] = \mathbb{E}[R_n^{\uparrow k} \mathbb{1}_{V_{n+1} \notin C_n^{red}} + (R_n+1)^{\uparrow k} \mathbb{1}_{V_{n+1} \in C_n^{red}} \mid \mathcal{F}_n]$.

$$= R_n^{\uparrow k} \frac{n+2-R_n}{n+2} + (R_n+1)^{\uparrow k} \frac{R_n}{n+2}.$$

$$= R_n^{\uparrow k} \left(\frac{n+2-R_n}{n+2}\right) + \frac{(R_n)^{\uparrow k}(R_n+k)}{n+2} = R_n^{\uparrow k} \frac{n+2+k}{n+2}.$$

$$\mathbb{E}[Y_{n+1,k} \mid \mathcal{F}_n] = \frac{n+2+k}{(n+2)(n+3)^{\uparrow k}} \qquad R_n^{\uparrow k} = \frac{R_n^{\uparrow k}}{(n+2)^{\uparrow k}} = Y_{n,k}.$$

$$\forall k,\ Y_{n,k} \xrightarrow[a.s.]{} Y_k; \qquad Y_1 = X.$$

Actually, $Y_k = X^k$ a.s.

$$Y_{n,k} = \frac{R_n^{\uparrow k}}{(n+2)^{\uparrow k}} = \frac{(n+2)^k}{(n+2)^{\uparrow k}} \cdot \frac{1}{(n+2)^k}\left(R_n^k + \sum_{j=0}^{k-1} c_{k,j} R_n^j\right).$$

$$= (1+o(1))\left(X_n^k + \sum_{j=0}^{k-1} c_{k,j} \frac{X_n^j}{(n+2)^{k-j}}\right).$$

$$\leq \frac{C_k}{n} \text{ a.s.}$$

$$Y_k = \lim_{a.s.} Y_{n,k} = \lim_{a.s.} (X_n)^k = X^k.$$

$$\mathbb{E}[X^k] = \lim_{(DCT)}{}^{n\to+\infty} \mathbb{E}[Y_{n,k}] = \mathbb{E}[Y_{0,k}] = \frac{k!}{(k+1)!} = \frac{1}{k+1}.$$

X is a random variable in $[0,1]$ such that $\mathbb{E}[X^k] = \frac{1}{k+1}$ $\forall k \geq 1$.

$\Longrightarrow$ $\boxed{X \sim \mathrm{Unif}([0,1])}$

$\int_0^1 x^k \, dx = \frac{1}{k+1}$.

$\forall P \in \mathbb{R}[X]$, $\mathbb{E}[P(X)] = \int_0^1 P(x) \, dx$.

$\Longrightarrow$ $\forall f \in C([0,1])$, $\mathbb{E}[f(X)] = \int_0^1 f(x) \, dx$. $\Longrightarrow$ determines the law.

- **Convergence in L^1**

$(X_n)_{n \in \mathbb{N}}$ martingale. bounded in L^1 : $\sup \mathbb{E}[|X_n|] < +\infty$.

$X_n \xrightarrow{a.s.} X$; do we also have $X_n \xrightarrow{L^1} X$?

Theorem The following are equivalent :

1) $X_n \xrightarrow[L^1, \text{a.s.}]{} X$, $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

2) $(X_n)_{n \in \mathbb{N}}$ is uniformly integrable.

comes from the general theory of convergent sequences

3) $\exists Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\forall n$, $X_n = \mathbb{E}[Z | \mathcal{F}_n]$ (closed martingale).

→ we can assume that Z is measurable w.r.t $\mathcal{F}_\infty = \sigma\left(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n\right)$.

by replacing Z by $\mathbb{E}[Z | \mathcal{F}_\infty]$:

if $X_n = \mathbb{E}[Z | \mathcal{F}_n]$, then $X_n = \mathbb{E}[\mathbb{E}[Z | \mathcal{F}_\infty] | \mathcal{F}_n]$.

Then, $Z = X = \lim_{\text{a.s.}} X_n$.

Proof 3) $\Rightarrow$ 2) $\mathbb{E}[|X_n| \mathbb{1}_{|X_n| \geq M}]$

$= \mathbb{E}[|\mathbb{E}[Z | \mathcal{F}_n]| \mathbb{1}_{|X_n| \geq M}] = \mathbb{E}[|\mathbb{E}[Z \mathbb{1}_{|X_n| \geq M} | \mathcal{F}_n]|]$

$$\leq \mathbb{E}\left[\mathbb{E}\left[|Z|\mathbb{1}_{|X_n|\geq M} \mid \mathcal{F}_n\right]\right] \quad : \quad |\mathbb{E}[Y|\mathcal{G}]| \leq \mathbb{E}[|Y||\mathcal{G}].$$

$$\leq \mathbb{E}\left[|Z|\mathbb{1}_{|X_n|\geq M}\right]$$

$$\mathbb{P}[|X_n|\geq M] = \mathbb{E}\left[\mathbb{1}_{|X_n|\geq M}\right] \leq \mathbb{E}\left[\mathbb{1}_{|X_n|\geq M}\frac{|X_n|}{M}\right]$$

$$\leq \frac{\mathbb{E}\left[\mathbb{1}_{|X_n|\geq M}|Z|\right]}{M} \leq \frac{\mathbb{E}[|Z|]}{M} \xrightarrow[M\to+\infty]{} 0.$$

$$\lim_{M\to+\infty} \sup_{n\in\mathbb{N}} \mathbb{E}\left[|X_n|\mathbb{1}_{|X_n|\geq M}\right] \leq \lim_{\varepsilon\to 0} \sup_{A \mid \mathbb{P}[A]\leq\varepsilon} \mathbb{E}\left[|Z|\mathbb{1}_A\right] = 0,$$

because $\{Z\}$ is UI.

① $\Rightarrow$ ③ : $X_n \xrightarrow[\substack{a.s.\\ L^1}]{} X$; $Y_n = \mathbb{E}[X|\mathcal{F}_n]$; X is $\mathcal{F}_\infty$ measurable.

$$\mathbb{E}[|X_n - Y_n|] = \mathbb{E}\left[\left|\mathbb{E}[X_{n+m}|\mathcal{F}_n] - \mathbb{E}[X|\mathcal{F}_n]\right|\right]$$

$$\leq \mathbb{E}\left[|X_{n+m} - X|\right] \qquad \forall m\geq 0.$$

Making $n \to +\infty$: $\mathbb{E}[|X_n - Y_n|] \leq 0$; $X_n = Y_n = \mathbb{E}[X|\mathcal{F}_n]$ a.s.

If $X_n \to X$ and X_n is closed by Z, Z and X $\mathcal{F}_\infty$ measurable.

$X_n = \mathbb{E}[Z|\mathcal{F}_n] = \mathbb{E}[X|\mathcal{F}_n]$.

X and Z have the same conditional expectations $\Rightarrow$ $X = Z$ a.s. because $\mathcal{F}_\infty = \sigma(\bigcup \mathcal{F}_n)$.

(idea : $\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[Z \mathbb{1}_A]$ $\forall A \in \mathcal{F}_\infty$). □.

- convergence in $L^{p>1}$.

Theorem : $(X_n)_{n\in\mathbb{N}}$ martingale with $\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|^p] < +\infty$.

Then, $\exists X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ | $X_n \to X$ a.s., L^1, L^p.

Proof: bounded in L^p $\Rightarrow$ bounded in L^1 $\Rightarrow$ a.s. convergence to some X.

with Fatou's lemma, $X \in L^p(\Omega, \mathcal{F}, P)$.

$$E\left[\sup_{n \le N} |X_n|^p\right] \le \left(\frac{p}{p-1}\right)^p E\left[|X_N|^p\right] \le K \text{ constant.}$$

so $R = \sup_{n \in \mathbb{N}} |X_n|^p$, then $E[R] \le K$ and R dominates X_n^p.

$$\Rightarrow X_n \xrightarrow[L^p]{} X. \qquad \square.$$