DETERMINANTAL POINT PROCESSES AND APPLICATIONS

PIERRE-LOÏC MÉLIOT

ABSTRACT. The purpose of this note is to present the theory of determinantal point processes, and two of its most classical applications: the study of eigenvalues of large random matrices, and the study of large random integer partitions.

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1. Kernels and determinantal point processes

In this first section, we develop the general theory of determinantal point processes, following [Sos00a; Joh05; Bor09] and [Hou+09, Chapter 4]. We voluntarily omit some technical details related to the theory of trace class operators.

1.1. Random point processes and correlation functions.

A *random point process* on a measurable space $\mathfrak{X}$ is a random sum or series $\sum_{i\in I}\delta_{X_i}$ of Dirac masses. To define this correctly, it is convenient to make some assumptions on the space $\mathfrak{X}$; thus, in the sequel, we shall assume that $\mathfrak{X}$ is a locally compact, complete and separable metric space, see for instance [Kal02, Chapter 12] for a general study of random measures in a slightly broader setting. In all the applications hereafter, $\mathfrak{X}$ will be a subset of some real vector space $\mathbb{R}^d$. We endow $\mathfrak{X}$ with its Borel σ-field $\mathscr{B}(\mathfrak{X})$. A (locally finite) *atomic measure* on $\mathfrak{X}$ is a positive measure $\mu : \mathscr{B}(\mathfrak{X}) \to \mathbb{N}\sqcup\{+\infty\}$ which takes integer values, and such that for any compact subset $K\subset\mathfrak{X}$, $\mu(K)<+\infty$. Then, there exists a countable family $(x_i)_{i\in I}$ of points in $\mathfrak{X}$ such that

$$\mu=\sum_{i\in I}\delta_{x_i},$$

and such that for any compact subset K, $\{i\,|\,x_i\in K\}$ is finite. We denote $\mathscr{M}^{\mathrm{atom}}(\mathfrak{X})$ the set of atomic measures on $\mathfrak{X}$, and we endow it with the smallest σ-field which makes measurable the maps $\mu\mapsto\mu(B)$ with $B\in\mathscr{B}(\mathfrak{X})$. Then, a random point process on $\mathfrak{X}$ is a random element in $\mathscr{M}^{\mathrm{atom}}(\mathfrak{X})$, that is to say a measurable map M from a probability space $(\Omega,\mathscr{F},\mathbb{P})$ to $\mathscr{M}^{\mathrm{atom}}(\mathfrak{X})$. By definition, all the quantities $M(B)$ with $B\in\mathscr{B}(\mathfrak{X})$ are then random variables with values in $\mathbb{N}\sqcup\{+\infty\}$.

Example 1.1. Suppose that $\mu:\mathscr{B}(\mathfrak{X})\to\mathbb{R}\sqcup\{+\infty\}$ is a locally finite positive Borel measure on $\mathfrak{X}$. A *Poisson point process* with intensity μ on $\mathfrak{X}$ is a random point process P_ν such that, for any family $(B_a)_{a\in A}$ of disjoint Borel subsets of $\mathfrak{X}$, $(P_\mu(B_a))_{a\in A}$ is a family of independent Poisson variables with parameters $\mu(B_a)$. Any locally finite positive Borel measure on $\mathfrak{X}$ gives rise to a Poisson point process, which is unique in law in $\mathscr{M}^{\mathrm{atom}}(\mathfrak{X})$.

In order to study a random point process M on a space $\mathfrak{X}$, it is natural to consider the joint moments of the associated positive random variables $M(B)$: they describe how many points fall in a given Borel subset B, and the correlations of these cardinalities for distinct Borel subsets $B_1,B_2,\dots,B_n$. The *factorial moment measures* of M will enable one to encode all these joint moments in a convenient way. For $n\geq 1$, we first define the n-th *factorial power* $M^{\downarrow n}$ of the point process M. Although this is not trivial, given a random point process $M:(\Omega,\mathscr{F},\mathbb{P})\to\mathscr{M}^{\mathrm{atom}}(\mathfrak{X})$ on a locally compact polish space, we can actually define on the same probability space some random variables $X_i:(\Omega,\mathscr{F},\mathbb{P})\to\mathfrak{X}\sqcup\{\dagger\}$ for $i\geq 1$, such that:

- $X_i=\dagger$ if and only if $M(\mathfrak{X})<+\infty$ and $i>M(\mathfrak{X})$;
- $M=\sum_{i=1}^{M(\mathfrak{X})}\delta_{X_i}$.

We then define $M^{\downarrow n}$ as the random point process on $\mathfrak{X}^n$ given by

$$M^{\downarrow n}=\sum_{\substack{i_1\neq i_2\neq\dots\neq i_n\\ 1\leq i_a\leq M(\mathfrak{X})}}\delta_{(X_{i_1},X_{i_2},\dots,X_{i_n})}.$$

This random point process count the sequences $(X_{i_1},X_{i_2},\dots,X_{i_n})$ of length n consisting in distinct points of the random point process M. Here the word "distinct" is a bit misleading, because if $X_i=X_j$ for $i\neq j$ (so, if M is not a *simple* random point process), then the pair (X_i,X_j) is allowed to appear in a sequence $(X_{i_1},X_{i_2},\dots,X_{i_n})$ counted by the factorial power $M^{\downarrow n}$. The terminology of factorial power is justified by the following computation: if B is a (relatively) compact subset of

$\mathfrak{X}$, then $M(B)$ is almost surely finite, and

$$\begin{aligned} M^{\downarrow n}(B^n) &= \text{number of } n\text{-sequences of distinct points in } B \\ &= M(B)(M(B)-1)\cdots(M(B)-n+1) = (M(B))^{\downarrow n}. \end{aligned}$$

The n-th factorial moment measure of M is the positive Borel measure $\mu_M^{\downarrow n}$ on $\mathfrak{X}^n$ defined by:

$$\mu_M^{\downarrow n}(B_1 \times B_2 \times \cdots \times B_n) = \mathbb{E}\big[M^{\downarrow n}(B_1 \times B_2 \times \cdots \times B_n)\big].$$

Example 1.2. For $n = 1$, $\mu_M^{\downarrow 1} = \mu_M$ is the intensity of the point process M, defined by $\mu_M(B) = \mathbb{E}[M(B)]$.

Example 1.3. Consider a Poisson point process P on $\mathfrak{X}$ with intensity μ, and some disjoint locally compact subsets $B_1, \ldots, B_n$ in $\mathfrak{X}$. A way to construct the restriction of the Poisson point process P to $B = \bigsqcup_{a=1}^n B_a$ is as follows: we first take a Poisson random variable N with parameter $\mu(B)$, and we then set

$$P_{|B} = \sum_{i=1}^{N} \delta_{X_i},$$

where the X_i's are independent random variables in B with law $\frac{\mu(\cdot)}{\mu(B)}$, and are independent of N. We then have:

$$\mu_P^{\downarrow n}(B_1 \times B_2 \times \cdots \times B_n) = \mathbb{E}\left[\sum_{\substack{i_1 \neq i_2 \neq \cdots \neq i_n \\ 1 \leq i_a \leq N}} \left(\prod_{a=1}^n 1_{X_{i_a} \in B_a}\right)\right] = \left(\prod_{a=1}^n \frac{\mu(B_a)}{\mu(B)}\right) \mathbb{E}\big[N^{\downarrow n}\big] = \prod_{a=1}^n \mu(B_a).$$

By additivity, we conclude that $\mu_P^{\downarrow n} = \mu^{\otimes n}$. This identity encodes the independence of the restrictions of the Poisson point process P to disjoint subsets.

Example 1.4. The factorial powers and the factorial moment measures are related to the computation of products

$$\prod_{i \in I} (1 + f(X_i)),$$

where $f : \mathfrak{X} \to \mathbb{C}$ is some bounded measurable function with compact support, and the random point process M is given by $M = \sum_{i \in I} \delta_{X_i}$, I being a random interval $[\![1, M(\mathfrak{X})]\!] \subset \mathbb{N}$. Indeed, since f is supported by a compact, we can assume without loss of generality that $M(\mathfrak{X})$ is finite almost surely, and then,

$$\prod_{i \in I} (1 + f(X_i)) = 1 + \sum_{n=1}^{\infty} \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq M(\mathfrak{X})} \prod_{a=1}^n f(X_{i_a}) \right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} M^{\downarrow n}(f^{\otimes n}).$$

In many cases, given a random point process M on a locally compact polish space $\mathfrak{X}$, there exists a reference Radon measure (locally finite Borel positive measure) λ on $\mathfrak{X}$ such that for any $n \geq 1$, the factorial moment measure $\mu_M^{\downarrow n}$ is absolutely continuous with respect to $\lambda^{\otimes n}$. In this situation, the density

$$\rho_n(x_1, \ldots, x_n) = \frac{d(\mu_M^{\downarrow n})}{d(\lambda^{\otimes n})}(x_1, \ldots, x_n)$$

is called the n-th *correlation function* of the random point process. In particular, if $\mathfrak{X} = \mathbb{Z}^d$ or $\mathfrak{X} = \mathbb{R}^d$, we shall take for reference measure λ the counting measure or the standard Lebesgue measure. The determinantal point processes will be the random point processes whose correlation functions write as

$$\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{1 \leq i,j \leq n}$$

for some adequate kernel K which does not depend on n.

Theorem 1.5 (Correlation functions)**.** *Let M be a random point process on a locally compact polish space $\mathfrak{X}$. We suppose that there exists a reference Radon measure λ on $\mathfrak{X}$ such that the correlation functions ρ_n with respect to M and λ are all well-defined. Then:*

(1) The correlation functions are symmetric: for any $\sigma \in \mathfrak{S}(n)$,

$$\rho_n(x_1, \dots, x_n) = \rho_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

(2) The correlation functions are positive, in the following sense: for any set $(\phi_0, \phi_1, \dots, \phi_N)$ of compactly supported measurable functions $\phi_k : \mathfrak{X}^k \to \mathbb{R}$ such that

$$\phi_0 + \sum_{k=1}^{N} \sum_{\substack{i_1 \neq i_2 \neq \cdots \neq i_k \\ 1 \leq i_a \leq N}} \phi_k(x_{i_1}, \dots, x_{i_k}) \geq 0,$$

we also have

$$\phi_0 + \sum_{k=1}^{N} \int_{\mathfrak{X}^k} \phi_k(x_1, \dots, x_k)\, \rho_k(x_1, \dots, x_k)\, \lambda^{\otimes k}(dx_1 \cdots dx_k) \geq 0.$$

Conversely, given a family of locally integrable positive functions $(\rho_n)_{n\in\mathbb{N}}$ which satisfy the two conditions above, one can define a random point process M on $\mathfrak{X}$ with these correlation functions. This random point process M is unique in law if and only if the random variables $M(B)$ with $B \in \mathscr{B}(\mathfrak{X})$ are determined by their moments.

This result is due to Lenard, see [Len73; Len75].

1.2. **Locally trace class Hermitian kernels.** In this paragraph, we fix a locally compact polish space $\mathfrak{X}$, and a reference Radon measure λ on $\mathfrak{X}$. We denote $\mathscr{L}^2(\mathfrak{X}, \lambda)$ the Hilbert space of square-integrable functions on $\mathfrak{X}$; by [Coh13, Proposition 3.4.5], $\mathscr{L}^2(\mathfrak{X}, \lambda)$ is separable. We recall that a *trace class operator* on a separable Hilbert space H is a bounded linear operator $A : H \to H$ such that, given an orthonormal basis $(e_i)_{i\in I}$ of H, we have

$$\sum_{i\in I} \left\langle e_i \,\middle|\, (A^*A)^{1/2}(e_i) \right\rangle_H < +\infty$$

the index set I being finite if H is finite-dimensional, and infinite countable if H is infinite-dimensional. The *trace* of the operator A is then defined by the aboslutely convergent series

$$\operatorname{tr}(A) = \sum_{i\in I} \langle e_i \mid A(e_i) \rangle_H ;$$

this does not depend on the choice of the orthonormal basis. In the same setting, a bounded linear operator $A : H \to H$ is called *Hilbert–Schmidt* if A^*A is a trace class operator; equivalently,

$$(\|A\|_{\mathrm{HS}})^2 = \operatorname{tr}(A^*A) = \sum_{i\in I} \langle A(e_i) \mid A(e_i) \rangle_H < +\infty$$

for any orthonormal basis $(e_i)_{i\in I}$ of H. We have the following inclusions of ideals of the Banach algebra of bounded linear operators on H:

$$\{\text{finite rank}\} \subset \{\text{trace class}\} \subset \{\text{Hilbert–Schmidt}\} \subset \{\text{compact}\}.$$

If $H = \mathscr{L}^2(\mathfrak{X}, \lambda)$, then its Hilbert–Schmidt operators are given by square-integrable kernels. Thus, if $\mathscr{K} : \mathscr{L}^2(\mathfrak{X}, \lambda) \to \mathscr{L}^2(\mathfrak{X}, \lambda)$ is Hilbert–Schmidt, then there exists a unique kernel $K \in \mathscr{L}^2(\mathfrak{X}^2, \lambda^{\otimes 2})$ such that

$$(\mathscr{K} f)(x) = \int_{\mathfrak{X}} K(x, y)\, f(y)\, \lambda(dy).$$

The map $K \mapsto \mathscr{K}$ is an isometry between $\mathscr{L}^2(\mathfrak{X}^2, \lambda^{\otimes 2})$ and the space of Hilbert–Schmidt operators $\mathrm{HS}(\mathscr{L}^2(\mathfrak{X}, \lambda))$ [GGK00, Chapter IV, Theorem 7.7]. Moreover, if $\mathfrak{X} = \mathbb{R}^d$, λ is the Lebesgue measure, $\mathscr{K}$ is a trace class operator and if K is continuous at (x, x) for λ-almost any x, then

$$\mathrm{tr}(\mathscr{K}) = \int_{\mathfrak{X}} K(x, x)\, \lambda(dx),$$

see [Bri91, Corollary 3.2]. In a more general setting, for instance if the kernel K is not continuous, then as an element of $\mathscr{L}^2(\mathfrak{X}^2, \lambda^{\otimes 2})$, it might not be well defined on the diagonal. However, there is a general averaging process which yields λ-almost everywhere a value of $K(x, x)$ such that the relation above holds; see again [Bri91].

Given a trace class operator A on a separable Hilbert space H, we can also define the *Fredholm determinant* $\det(I + A)$. Denote $\bigwedge^k H$ the k-th exterior power of H, which is the Hilbert completion of the algebraic k-th exterior power for the scalar product

$$\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k \mid w_1 \wedge w_2 \wedge \cdots \wedge w_k \rangle_{\bigwedge^k H} = \det(\langle v_i \mid w_j \rangle_H)_{1 \leq i,j \leq k}.$$

It is again a separable Hilbert space, and if $(e_i)_{i \in I}$ is an orthonormal basis of H with $I \subset \mathbb{N}$, then $(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k})_{i_1 < i_2 < \cdots < i_k \in I}$ is an orthonormal basis of $\bigwedge^k H$. Now, the k-th exterior power of A defined by extension of the rule

$$\left(\bigwedge^k A\right)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = A(v_1) \wedge A(v_2) \wedge \cdots \wedge A(v_k)$$

is again a trace class operator. Indeed, given A of trace class, consider an orthonormal basis $(e_i)_{i \in I}$ of diagonalisation of the compact self-adjoint operator $|A| = (A^* A)^{1/2}$, with $|A|(e_i) = \lambda_i\, e_i$. Each λ_i is a non-negative real number, and $\|A\|_1 = \mathrm{tr}(|A|) = \sum_{i \in I} \lambda_i$. We then have:

$$\begin{aligned}
&\sum_{i_1 < i_2 < \cdots < i_k} \left\langle e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \,\middle|\, \left|\bigwedge^k A\right| (e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) \right\rangle_{\bigwedge^k H} \\
&= \sum_{i_1 < i_2 < \cdots < i_k} \langle e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid |A|(e_{i_1}) \wedge |A|(e_{i_2}) \wedge \cdots \wedge |A|(e_{i_k}) \rangle_{\bigwedge^k H} \\
&= \sum_{i_1 < i_2 < \cdots i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} = \frac{1}{k!} \sum_{i_1 \neq i_2 \neq \cdots \neq i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \leq \frac{1}{k!} \left(\mathrm{tr}(|A|)\right)^k,
\end{aligned}$$

so $\bigwedge^k A$ is of trace class, with $\left\|\bigwedge^k A\right\|_1 \leq \frac{(\|A\|_1)^k}{k!}$. The Fredholm determinant is defined by:

$$\det(I + A) = 1 + \sum_{k=1}^{\infty} \mathrm{tr}\left(\bigwedge^k A\right);$$

by the previous calculation, the series is convergent and $|\det(I + A)| \leq \mathrm{e}^{\|A\|_1}$. We recover the traditional determinant when H is finite-dimensional. On the other hand, if $\mathscr{K}$ is a trace class operator on $\mathscr{L}^2(\mathfrak{X}, \lambda)$ associated to a kernel $K \in \mathscr{L}^2(\mathfrak{X}^2, \lambda^2)$, then its Fredholm determinant is given by:

$$\det(I + \mathscr{K}) = 1 + \sum_{k=1}^{\infty} \mathrm{tr}\left(\bigwedge^k \mathscr{K}\right) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathfrak{X}^k} \det(K(x_i, x_j))_{1 \leq i,j \leq k}\, \lambda(dx_1) \cdots \lambda(dx_k).$$

This is the *Fredholm formula*; as before, it is a bit ambiguous if for instance $\mathfrak{X} = \mathbb{R}^d$ and K is not a continuous kernel, but it can still be given a sense even in this case. We refer to [GGK00, Section 1.6] for a proof in the easy case where $\mathscr{K}$ has finite rank, and to [*loc. cit.*, Section 4.5] for an extension to general integral operators. This problem is also discussed in details at the beginning of [Sos00a] in the framework of determinantal point processes.

In the following, we consider a linear operator $\mathscr{K} : \mathscr{L}^2(\mathfrak{X}, \lambda) \to \mathscr{L}^2(\mathfrak{X}, \lambda)$ which is:

(1) *Hermitian and non-negative*: for $f, g \in \mathscr{L}^2(\mathfrak{X}, \lambda)$, $\langle f \mid \mathscr{K}(g)\rangle_{\mathscr{L}^2(\mathfrak{X},\lambda)} = \langle \mathscr{K}(f) \mid g\rangle_{\mathscr{L}^2(\mathfrak{X},\lambda)}$ and $\langle f \mid \mathscr{K}(f)\rangle_{\mathscr{L}^2(\mathfrak{X},\lambda)} \geq 0$.

(2) *locally of trace class*: for any relatively compact subset $B \subset \mathfrak{X}$, $\mathscr{K}_B = 1_B \mathscr{K} 1_B$ is a trace class operator on $\mathscr{L}^2(B, \lambda_{|B})$.

This implies the existence of a unique measurable function $K : \mathfrak{X}^2 \to \mathbb{C}$ such that:

- $K(x, y) = \overline{K(y, x)}$.
- $\det(K(x_i, x_j))_{1\leq i,j\leq n} \geq 0$ for $\lambda^{\otimes n}$-almost any family of points $x_1, \dots, x_n$.
- $\int_{B^2} |K(x, y)|^2 \, \lambda(dx) \, \lambda(dy) < +\infty$ for any relatively compact subset $B \subset \mathfrak{X}$.

Theorem 1.6 (Determinantal point process associated to a Hermitian kernel)**.** *Suppose that K is the kernel of a Hermitian non-negative locally trace class operator $\mathscr{K}$ on $\mathscr{L}^2(\mathfrak{X}, \lambda)$.*

(1) The spectrum of $\mathscr{K}$ (set of complex numbers such that $zI - \mathscr{K}$ is not invertible) is included in $[0, 1]$ if and only if, for any relatively compact subset $B \subset \mathfrak{X}$, the spectrum of the restricted operator $\mathscr{K}_B$ is included in $[0, 1]$.

(2) If the condition $\mathrm{Spec}(\mathscr{K}) \subset [0, 1]$ is satisfied, then there exists a random point process M on $\mathfrak{X}$ whose correlation functions with respect to λ are given by:

$$\rho_n(x_1, \dots, x_n) = \det(K(x_i, x_j))_{1\leq i,j\leq n}.$$

This random point process is unique in law; equivalently, all the random variables $M(B)$ with B relatively compact are determined by their moments (actually, they have subexponential tails).

(3) Suppose conversely given a determinantal point process M whose correlations are associated to the kernel K of a Hermitian non-negative locally trace class operator $\mathscr{K}$ on $\mathscr{L}^2(\mathfrak{X}, \lambda)$. Then, $\mathrm{Spec}(\mathscr{K}) \subset [0, 1]$.

We refer to [Sos00a, Theorem 3] for a proof of this result; see also [Hou+09, Section 4.5]. The determinantal point processes can be defined under weaker assumptions (for instance, with non-Hermitian kernels), but in the sequel we shall stick to the setting of Theorem 1.6. Note that in the first item of the theorem, a restricted operator $\mathscr{K}_B$ is trace class hence compact, so

$$\mathrm{Spec}(\mathscr{K}_B) \cup \{0\} = \{\text{eigenvalues of } \mathscr{K}_B\} \cup \{0\}.$$

The non-zero eigenvalues of $\mathscr{K}_B$ will be involved in a precise description of the law of the random variable $M(B)$, where M is a determinantal point process with kernel K.

1.3. **Observables of determinantal point processes.**

Given a determinantal point process M with kernel K, let us see how to use this kernel in order to obtain information on M.

Total number of particles. Consider as above a non-negative Hermitian and locally trace class operator $\mathscr{K} : \mathscr{L}^2(\mathfrak{X}, \lambda) \to \mathscr{L}^2(\mathfrak{X}, \lambda)$. One can define the trace of the whole operator $\mathscr{K}$ by taking the supremum of the traces of the restricted operators $\mathscr{K}_B$:

$$\mathrm{tr}(\mathscr{K}) = \left(\int_{\mathfrak{X}} K(x, x) \, \lambda(dx) \right) \in \mathbb{R}_+ \sqcup \{+\infty\}.$$

Suppose from now on that we have a determinantal point process M with such a kernel K. Then, $M(\mathfrak{X}) = +\infty$ with probability 0 if $\mathrm{tr}(\mathscr{K}) < +\infty$, and with probability 1 if $\mathrm{tr}(\mathscr{K}) = +\infty$ [Sos00a, Theorem 4]. In the first case, we have:

- $\mathbb{P}[M(\mathfrak{X}) \leq n] = 1$ if and only if $\mathrm{rank}(\mathscr{K}) \leq n$. If $\mathrm{rank}(\mathscr{K}) = n$, then there exists a family of n orthonormal functions $\psi_1, \dots, \psi_n$ in $\mathscr{L}^2(\mathfrak{X}, \lambda)$ and eigenvalues $\lambda_1, \dots, \lambda_n$ in

$(0,1]$ such that

$$K(x,y) = \sum_{i=1}^{n} \lambda_i \, \overline{\psi_i(x)} \, \psi_i(y).$$

- $\mathbb{P}[M(\mathfrak{X}) = n] = 1$ if and only if $\mathrm{rank}(\mathscr{K}) = n$ and $\mathscr{K}$ is the orthogonal projection on a vector space with rank n. Then, in the decomposition above, all the λ_i's are equal to 1.

Density of particles. In the same setting, the first correlation function $\rho_1(x) = K(x,x)$ gives the expected density of particles: for any relatively compact subset B,

$$\mathbb{E}[M(B)] = \int_B K(x,x)\, \lambda(dx).$$

Later, we shall consider sequences of determinantal point processes $(M_N)_{N\in\mathbb{N}}$ with $M_N(\mathfrak{X}) = N$ almost surely. In this setting, a very simple criterion on the Hermitian kernels K_N of the point processes M_N allows one to prove the convergence of the renormalised empirical densities of particles:

Proposition 1.7 (Limiting density of particles)**.** *We write $M_N = \sum_{i=1}^{N} \delta_{X_i}$, and we consider the random probability measures*

$$\nu_N = \frac{M_N}{N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i}.$$

We assume that:

(1) λ-almost everywhere, $\frac{K_N(x,x)}{N}$ converges to a density function $f(x)$ with $\int_{\mathfrak{X}} f(x)\, \lambda(dx) = 1$.

(2) the reference measure λ satisfies some mild regularity assumption: one can find a countable family $(B_j)_{j\in J}$ of relatively compact open subsets of $\mathfrak{X}$, such that any open subset $B \subset \mathfrak{X}$ writes as the union of some B_j's, and such that all the B_j's satisfy $\lambda(\partial B_j) = 0$.

Then, with respect to the topology of convergence in law on $\mathscr{M}^1(\mathfrak{X})$, the random distribution ν_N converges in probability towards the deterministic distribution $\nu = f(x)\, \lambda(dx)$.

Note that the Lebesgue measure on $\mathbb{R}^d$ trivially satisfies the regularity assumption. Besides, if $(B_j)_{j\in J}$ is an adequate countable family of relatively compact open subsets, then without loss of generality, we can assume that any open subset $B \subset \mathfrak{X}$ writes as $B = \bigcup_{n\in\mathbb{N}} B_{j_n}$ for some *increasing* sequence $(B_{j_n})_{n\in\mathbb{N}}$. Indeed, starting from an adequate family $(B_j)_{j\in J}$, the larger family formed by the finite unions of B_j's is also countable, and it also consists of relatively compact open subsets whose boundaries have a vanishing λ-measure. In the sequel, we add this property with increasing sequences to the definition of an adequate family.

Proof of Proposition 1.7. We fix:

- a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which all the random point processes M_N are defined;
- an adequate family $(B_j)_{j\in J}$ of relatively compact open subsets of $\mathfrak{X}$.

Suppose that we can prove that for any $j \in J$, $\nu_N(B_j)$ converges in probability towards $\nu(B_j)$. The convergence in probability is equivalent to the almost sure convergence of a subsubsequence of any given subsequence of the sequence of random variables. Therefore, if $(\nu_{\psi(N)})_{N\in\mathbb{N}}$ is a subsequence of $(\nu_N)_{N\in\mathbb{N}}$, then by diagonal extraction we can find a subsubsequence $(\nu_{\psi\circ\theta(N)})_{N\in\mathbb{N}}$ such that

$$\forall j \in J,\ \ \nu_{\psi\circ\theta(N)}(B_j) \to_{\text{a.s.}} \nu(B_j).$$

Then, with probability 1, we also have for any open subset $B \subset \mathfrak{X}$

$$\liminf_{N\to\infty} \nu_{\psi\circ\theta(N)}(B) \geq \sup_{n\in\mathbb{N}} \left(\liminf_{N\to\infty} \nu_{\psi\circ\theta(N)}(B_{j_n}) \right) = \sup_{n\in\mathbb{N}} \nu(B_{j_n}) = \nu(B).$$

Therefore, by the Portmanteau theorem [Bil99, Theorem 2.1], we have the almost sure convergence in law $\nu_{\psi\circ\theta(N)} \to_{\text{a.s.}} \nu$, and this implies the convergence in probability $\nu_N \to_{\text{probability}} \nu$.

So, it suffices to prove the following fact: for any B_j in an adequate family, $\nu_N(B_j) \to_{\text{probability}} \nu(B_j)$. This convergence in probability will be obtained by the second moment method. First, we have

$$\mathbb{E}[\nu_N(B_j)] = \frac{\mathbb{E}[M_N(B_j)]}{N} = \int_{B_j} \frac{K_N(x,x)}{N}\,\lambda(dx),$$

and by assumption $\frac{K_N(x,x)}{N}$ converges to $f(x)$. This implies by classical arguments the convergence in law of $\frac{K_N(x,x)}{N}\,\lambda(dx)$ towards the probability measure $\nu(dx) = f(x)\,\lambda(dx)$. As a consequence, since B_j is a continuity set for λ and ν,

$$\mathbb{E}[\nu_N(B_j)] \to_{N\to\infty} \nu(B_j).$$

On the other hand,

$$\mathbb{E}[(\nu_N(B_j))^2] = \frac{\mathbb{E}[(M_N(B_j))(M_N(B_j)-1)] + \mathbb{E}[M_N(B_j)]}{N^2} = \frac{\mathbb{E}[M_N^{\downarrow 2}(B_j\times B_j)]}{N^2} + O\left(\frac{1}{N}\right).$$

By definition of a determinantal point process, the leading term on the right-hand side is given by the integral

$$\begin{aligned}&\int_{(B_j)^2} \frac{K_N(x,x)K_N(y,y) - K_N(x,y)K_N(y,x)}{N^2}\,\lambda(dx)\,\lambda(dy)\\ &= (\mathbb{E}[\nu_N(B_j)])^2 - \int_{(B_j)^2}\left(\frac{|K_N(x,y)|}{N}\right)^2 \lambda(dx)\,\lambda(dy) \le (\mathbb{E}[\nu_N(B_j)])^2.\end{aligned}$$

Therefore, the variance of $(\nu_N(B_j))^2$ is a $O(N^{-1})$; together with the convergence of the mean, this proves the convergence in probability. □

Number of points in a relatively compact subset. If $B_1,\dots,B_n$ is a family of disjoint relatively compact subsets in $\mathfrak{X}$, the joint law of the random variables $M(B_1),\dots,M(B_n)$ with M determinantal point process with kernel K can be computed as follows. We consider the joint generating function of these variables:

$$\begin{aligned}\mathbb{E}\left[\prod_{a=1}^n (z_a)^{M(B_a)}\right] &= \mathbb{E}\left[\prod_{a=1}^n \sum_{m_a=0}^{M(B_a)} \binom{M(B_a)}{m_j} (z_a-1)^{m_a}\right]\\ &= 1 + \sum_{m=1}^\infty \sum_{m_1+\cdots+m_n=m} \mathbb{E}\left[\prod_{a=1}^n (M(B_a))^{\downarrow m_a}\right] \prod_{a=1}^n \frac{(z_a-1)^{m_a}}{(m_a)!}.\end{aligned}$$

The convergence of these series is ensured by the following identity: for any composition $m = m_1+\cdots+m_n$, setting $B = \bigsqcup_{a=1}^n B_a$, we have

$$\begin{aligned}\mathbb{E}\left[\prod_{a=1}^n (M(B_a))^{\downarrow m_a}\right] &= \mathbb{E}[M^{\downarrow m}((B_1)^{m_1}\times\cdots\times(B_n)^{m_n})]\\ &= \int_{(B_1)^{m_1}\times\cdots\times(B_n)^{m_n}} \det(K(x_i,x_j))_{1\le i,j\le m}\,\lambda(dx_1)\cdots\lambda(dx_m)\\ &= m!\,\mathrm{tr}\left((1_B\mathscr{K}1_{B_1})^{\wedge m_1}\wedge(1_B\mathscr{K}1_{B_2})^{\wedge m_2}\wedge\cdots\wedge(1_B\mathscr{K}1_{B_n})^{\wedge m_n}\right)\\ &\le (\mathrm{tr}(\mathscr{K}_B))^m.\end{aligned}$$

If $a = a_j$ is the index of the set B_a corresponding to the variable x_j, then the same computation shows that

$$\begin{aligned}
&\mathbb{E}\left[\prod_{a=1}^{n}(z_a)^{M(B_a)}\right] \\
&= 1 + \sum_{\substack{m_1+\cdots+m_n=m \\ m\geq 1}} \int_{\mathfrak{X}^m} \det(1_B(x_i)\,(z_{a_j}-1)\,K(x_i,x_j)\,1_{B_{a_j}}(x_j))_{1\leq i,j\leq m}\,\frac{\lambda(dx_1)\cdots\lambda(dx_m)}{(m_1)!\cdots(m_n)!} \\
&= 1 + \sum_{m=1}^{\infty}\frac{1}{m!}\int_{\mathfrak{X}^m}\det\left(\sum_{a=1}^{n}1_B(x_i)\,(z_{a_j}-1)\,K(x_i,x_j)\,1_{B_{a_j}}(x_j)\right)_{1\leq i,j\leq m}\lambda(dx_1)\cdots\lambda(dx_m).
\end{aligned}$$

The quantity that one obtains is a Fredholm determinant:

$$\mathbb{E}\left[\prod_{a=1}^{n}(z_a)^{M(B_a)}\right] = \det\left(I + \sum_{a=1}^{n}1_B\,(z_a-1)\,\mathscr{K}\,1_{B_a}\right).$$

Let us consider in particular the case where $n = 1$ and the trace class self-adjoint non-negative operator $\mathscr{K}_B$ has a countable family of eigenvalues $(\lambda_{B,i})_{i\in I}$ with $0 \leq \lambda_{B,i} \leq 1$ for any i. The Fredholm determinant is then given by:

$$\mathbb{E}[z^{M(B)}] = \det(I + (z-1)\mathscr{K}_B) = \prod_{i\in I}(1 + (z-1)\lambda_{B,i}).$$

This is precisely the generating series of the random variable $X = \sum_{i\in I}\mathrm{Ber}(\lambda_{B,i})$, where all the Bernoulli variables are assumed to be independent. This random series converges almost surely by the two-series Kolmogorov criterion, since $\mathrm{tr}(\mathscr{K}_B) = \sum_{i\in I}\lambda_{B,i} < +\infty$ by hypothesis. We thus have:

Proposition 1.8 (Marginales of a determinantal point process)**.** *Given a determinantal point process M associated to a Hermitian non-negative locally trace class operator with kernel K, for any relatively compact subset $B \subset \mathfrak{X}$, if $(\lambda_{B,i})_{i\in I}$ is the collection of eigenvalues of $\mathscr{K}_B$, then the law of $M(B)$ is the law of a random series of independent Bernoulli variables with parameter $\lambda_{B,i}$.*

In particular, if $\mathrm{tr}(\mathscr{K}) = +\infty$ and if we have an increasing sequence of subsets $(B_N)_{N\in\mathbb{N}}$ with

$$\mathrm{var}(M(B_N)) = \left(\int_{B_N}K(x,x)\,\lambda(dx) - \int_{(B_N)^2}|K(x,y)|^2\,\lambda(dx)\,\lambda(dy)\right) \to +\infty,$$

then we have the central limit theorem

$$\frac{M(B_N) - \mathbb{E}[M(B_N)]}{\sqrt{\mathrm{var}(M(B_N))}} \rightharpoonup_{N\to+\infty}\mathcal{N}(0,1).$$

This theorem appears for instance in [Sos00b]; see also [Sos00a, Theorem 8]. Note that the formula for $\mathrm{var}(M(B))$ shows that we always have $\mathrm{var}(M(B)) \leq \mathbb{E}[M(B)]$ for a determinantal point process M and a relatively compact subset B.

Remark 1.9. Suppose that $\mathscr{K}$ is the orthogonal projection from $\mathscr{L}^2(\mathfrak{X},\lambda)$ to a vector subspace with dimension N. Then, Proposition 1.8 ensures that the associated determinantal point process M satisfies $M(\mathfrak{X}) = N$ almost surely, since there are N non-zero eigenvalues equal to 1. Thus, the determinantal point processes with a fixed number of points are naturally associated to orthogonal projections, and later we shall see the connection with orthogonal polynomials.

Gap probabilities and simple determinantal point processes. A particular case of the Fredholm formula for the generating series $\mathbb{E}[z^{M(B)}]$ is with $z = 0$. We are then evaluating the *gap probability* to not having any point of M in B:

$$\mathbb{P}[M(B) = 0] = \det(I - \mathscr{K}_B) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int_{B^m} \det(K(x_i, x_j))_{1 \leq i,j \leq m} \, \lambda(dx_1) \cdots \lambda(dx_m).$$

The formula extends readily to any subset $B \subset \mathfrak{X}$ such that $\mathscr{K}_B$ is still a trace class operator. This formula will play an essential role in the study of extremal points of determinantal point processes. To conclude this section, let us give a simple criterion for a determinantal point process to be simple, that is to say that with probability 1, $M = \sum_{i \in I} \delta_{X_i}$ with $X_i \neq X_j$ for any $i \neq j$.

Proposition 1.10 (Particles are fermions)**.** *Suppose that the reference measure λ has no atom. Then, a determinantal point process M with Hermitian locally trace class operator associated to a kernel K is always a simple random point process.*

Proof. Consider a relatively compact subset B, and let us evaluate the expected number $X(B, \varepsilon)$ of ordered pairs of points of M that fall in B and are at distance smaller than ε. This is given by:

$$\mathbb{E}[X(B, \varepsilon)] = \int_{B^2} \det \begin{pmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{pmatrix} 1_{d(x,y) \leq \varepsilon} \, \lambda(dx) \, \lambda(dy).$$

As ε goes to zero, the locally integrable correlation function $\rho_2(x, y)$ yields an integral which goes to 0. Since $\mathbb{E}[X(B, \varepsilon)] \geq \mathbb{P}[X(B, \varepsilon) > 0] \geq \mathbb{P}[M \text{ has a multiple point in } B]$, we can conclude. □

2. Eigenvalues of large random matrices

We now present some classical models of random matrices whose eigenvalues yield determinantal point processes; and we explain how to use this structure in order to obtain information on the asymptotic behavior of these eigenvalues when the size of the matrices goes to infinity.

2.1. The Gaussian unitary ensemble.

We start with the most classical example, namely, Gaussian matrices chosen in the vector space $\mathrm{H}(N)$ of Hermitian $N \times N$ square matrices. A convenient framework is the following: H_N is a random matrix with size $N \times N$, whose diagonal coefficients are independent real Gaussian variables

$$(H_N)_{i,i} = \mathcal{N}_{\mathbb{R}}\left(0, \frac{1}{N}\right),$$

and whose off-diagonal coefficients are independent complex Gaussian variables:

$$(H_N)_{i,j} = \overline{(H_N)_{j,i}} = \mathcal{N}_{\mathbb{R}}\left(0, \frac{1}{2N}\right) + \mathrm{i}\mathcal{N}_{\mathbb{R}}\left(0, \frac{1}{2N}\right).$$

The density of the law of H_N is thus:

$$\frac{1}{Z_{N,\mathrm{GUE},1}} \, \mathrm{e}^{-\frac{N}{2} \operatorname{tr} H^2} \prod_{1 \leq i \leq N} dH_{i,i} \prod_{1 \leq i < j \leq N} d\mathrm{Re}(H_{i,j}) \, d\mathrm{Im}(H_{i,j}), \quad \text{with } Z_{N,\mathrm{GUE},1} = \sqrt{\frac{2^N \pi^{N^2}}{N^{N^2}}}.$$

Denote $x_{N,1} \geq x_{N,2} \geq \cdots \geq x_{N,N}$ the random real eigenvalues of H_N. Any Hermitian matrix H with size N writes as $H = UDU^*$, where $U \in \mathrm{U}(N)$ is a unitary matrix and $D = \mathrm{diag}(x_1, \ldots, x_N)$ is a diagonal matrix with non-increasing real entries. Moreover, up to multiplication by a diagonal matrix $\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_N})$, we can assume that all the diagonal entries of U belong to $\mathbb{R}_+$. Denote $\mathrm{U}(N)_{\mathrm{p.d.}}$ the subset of $\mathrm{U}(N)$ that consists in unitary matrices with positive diagonal entries, and

$\mathrm{C}(N)$ the Weyl chamber of decreasing sequences $(x_1 > x_2 > \cdots > x_N)$ of real numbers. The smooth map

$$\begin{aligned} \mathrm{U}(N)_{\mathrm{p.d.}} \times \mathrm{C}(N) &\to \mathrm{H}(N) \\ (U, D) &\mapsto UDU^* \end{aligned}$$

is injective, and the complement of its image has Lebesgue measure zero. Therefore, we can use a change of variable formula to obtain the following:

Theorem 2.1 (Joint distribution of eigenvalues)**.** *The ordered random sequence* $(x_{N,1} \geq x_{N,2} \geq \cdots \geq x_{N,N})$ *of eigenvalues of a random Gaussian Hermitian matrix* H_N *admits the following density in* $\overline{\mathrm{C}(N)}$*:*

$$\frac{1_{x_{N,1} \geq \cdots \geq x_{N,N}}}{Z_{N,\mathrm{GUE},2}} \, \mathrm{e}^{-\frac{N}{2} \sum_{i=1}^N (x_{N,i})^2} \prod_{1 \leq i < j \leq N} |x_{N,i} - x_{N,j}|^2 \prod_{1 \leq i \leq N} dx_{N,i},$$

with $Z_{N,\mathrm{GUE},2} = (2\pi)^{\frac{N}{2}} \, N^{-\frac{N^2}{2}} \, (N-1)!(N-2)! \cdots 1!$.

We refer to [AGZ10, Theorem 2.5.2] for a detailed proof of the change of variables which yields this joint distribution, in which the *Vandermonde determinant*

$$\Delta(x) = \prod_{1 \leq i < j \leq N} (x_{N,i} - x_{N,j}) = \det((x_{N,i})^{N-j})_{1 \leq i,j \leq N}$$

appears. In the sequel, we shall rather work with the unordered random sequence $(x_1, \ldots, x_N)$ with joint distribution

$$\frac{1}{Z_{N,\mathrm{GUE}}} \, \mathrm{e}^{-\frac{N}{2} \sum_{i=1}^N (x_i)^2} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \prod_{1 \leq i \leq N} dx_i$$

on $\mathbb{R}^N$, where $Z_{N,\mathrm{GUE}} = N! \, Z_{N,\mathrm{GUE},2} = (2\pi)^{\frac{N}{2}} \, N^{-\frac{N^2}{2}} \, N!(N-1)! \cdots 1!$. This unordered random sequence yields a random point process $M_N = \sum_{i=1}^N \delta_{X_i}$ which is called the *Gaussian unitary ensemble* (GUE), and which happens to be determinantal.

2.2. **Reproducing kernels and orthogonal polynomials.** The proof that M_N is determinantal relies mostly on algebraic calculations, which can be in fact be performed with more general unitary invariant probability measures on $\mathrm{H}(N)$. We start with the following:

Lemma 2.2 (Cauchy–Binet formula)**.** *Let* $(\phi_i, \psi_i)_{1 \leq i \leq N}$ *be a double family of square-integrable functions in* $\mathscr{L}^2(\mathfrak{X}, \lambda)$*. We have:*

$$\begin{aligned} &\det \left(\int_{\mathfrak{X}} \phi_i(x) \, \psi_j(x) \, \lambda(dx) \right)_{1 \leq i,j \leq N} \\ &= \frac{1}{N!} \int_{\mathfrak{X}^N} \det(\phi_i(x_j))_{1 \leq i,j \leq N} \, \det(\psi_i(x_j))_{1 \leq i,j \leq N} \, \lambda(dx_1) \cdots \lambda(dx_N). \end{aligned}$$

Proof. We compute the determinant $\det(\int_{\mathfrak{X}} \phi_i(x) \, \psi_j(x) \, \lambda(dx))_{1 \leq i,j \leq N}$ as follows:

$$\begin{aligned} &\sum_{\sigma \in \mathfrak{S}(N)} \varepsilon(\sigma) \prod_{1 \leq i \leq N} \left(\int_{\mathfrak{X}} \phi_i(x_i) \, \psi_{\sigma(i)}(x_i) \, \lambda(dx_i) \right) \\ &= \int_{\mathfrak{X}^N} \left(\sum_{\sigma \in \mathfrak{S}(N)} \varepsilon(\sigma) \prod_{1 \leq i \leq N} \phi_i(x_i) \, \psi_{\sigma(i)}(x_i) \right) \lambda(dx_1) \cdots \lambda(dx_N) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathfrak{X}^N} \det(\phi_i(x_i)\,\psi_j(x_i))_{1\le i,j\le N}\,\lambda(dx_1)\cdots\lambda(dx_N)\\
&= \int_{\mathfrak{X}^N} \left(\prod_{1\le i\le N}\phi_i(x_i)\right)\det(\psi_i(x_j))_{1\le i,j\le N}\,\lambda(dx_1)\cdots\lambda(dx_N)\\
&= \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}(N)}\int_{\mathfrak{X}^N} \left(\prod_{1\le i\le N}\phi_i(x_{\sigma(i)})\right)\det(\psi_i(x_{\sigma(j)}))_{1\le i,j\le N}\,\lambda(dx_1)\cdots\lambda(dx_N)\\
&= \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}(N)}\int_{\mathfrak{X}^N} \varepsilon(\sigma)\left(\prod_{1\le i\le N}\phi_i(x_{\sigma(i)})\right)\det(\psi_i(x_j))_{1\le i,j\le N}\,\lambda(dx_1)\cdots\lambda(dx_N)\\
&= \frac{1}{N!}\int_{\mathfrak{X}^N} \det(\phi_i(x_j))_{1\le i,j\le N}\,\det(\psi_i(x_j))_{1\le i,j\le N}\,\lambda(dx_1)\cdots\lambda(dx_N). \qquad \square
\end{aligned}
$$

In the sequel, we suppose that the ϕ_i's and ψ_j's are real-valued, and we set $A_{i,j} = \langle \psi_i \mid \phi_j\rangle_{\mathscr{L}^2(\mathfrak{X},\lambda)} = \int_{\mathfrak{X}} \psi_i(x)\,\phi_j(x)\,\lambda(dx)$. Suppose that the determinant in the Cauchy–Binet formula is positive; then, $(A_{i,j})_{1\le i,j\le N}$ is invertible. We then define a kernel

$$K_N(x,y) = \sum_{1\le i,j\le N}\psi_i(x)\,(A^{-1})_{i,j}\,\phi_j(y).$$

Without loss of generality, we can assume $\det A = 1$; this amounts to multiplying all the functions ϕ_i and ψ_j by a positive constant.

Theorem 2.3 (Determinantal point process associated to a finite rank reproducing kernel)**.** *Let $(\phi_i,\psi_i)_{1\le i\le N}$ be a family of real-valued functions in $\mathscr{L}^2(\mathfrak{X},\lambda)$, such that $\det(\langle\phi_i\mid\psi_j\rangle_{\mathscr{L}^2(\mathfrak{X},\lambda)})_{1\le i,j\le N} = 1$. The kernel K_N is a reproducing kernel:*

$$
\begin{aligned}
&\int_{\mathfrak{X}} K_N(x,x)\,\lambda(dx) = N;\\
&\int_{\mathfrak{X}} K_N(x,y)\,K_N(y,z)\,\lambda(dy) = K_N(x,z).
\end{aligned}
$$

Consider random variables $X_1,\dots,X_N$ with joint distribution

$$\frac{1}{N!}\,\det(\phi_i(x_j))_{1\le i,j\le N}\,\det(\psi_i(x_j))_{1\le i,j\le N}\,\lambda(dx_1)\cdots\lambda(dx_N),$$

and the associated random point process $M_N = \sum_{i=1}^N \delta_{X_i}$. The random point process M_N is determinantal on $(\mathfrak{X},\lambda)$, with kernel K_N.

Proof. The identities satisfied by K_N are obtained as follows:

$$
\begin{aligned}
\int_{\mathfrak{X}} K_N(x,x)\,\lambda(dx) &= \sum_{1\le i,j\le N}(A^{-1})_{i,j}\,\langle\phi_j\mid\psi_i\rangle_{\mathscr{L}^2(\mathfrak{X},\lambda)}\\
&= \sum_{1\le i,j\le N}(A^{-1})_{i,j}A_{j,i} = \sum_{1\le i\le N}1 = N;
\end{aligned}
$$

$$\int_{\mathfrak{X}} K_N(x,y)\,K_N(y,z)\,\lambda(dy) = \sum_{\substack{1\le i,j\le N\\ 1\le k,l\le N}} \psi_i(x)\,(A^{-1})_{i,j}\left(\int_{\mathfrak{X}} \phi_j(y)\psi_k(y)\,\lambda(dy)\right)(A^{-1})_{k,l}\,\phi_l(z)$$
$$= \sum_{\substack{1\le i,j\le N\\ 1\le k,l\le N}} \psi_i(x)\,(A^{-1})_{i,j}A_{j,k}(A^{-1})_{k,l}\,\phi_l(z)$$
$$= \sum_{1\le i,l\le N} \psi_i(x)\,(A^{-1})_{i,l}\,\phi_l(z) = K_N(x,z).$$

Let us now compute the correlation functions of M_N. We start by a general observation on random point processes $M=\sum_{i=1}^N \delta_{X_i}$ defined by a symmetric density function $f(x_1,\dots,x_N)$ on $\mathbb{R}^N$ (with respect to $\lambda^{\otimes N}$). For any family of measurable subsets $B_1,\dots,B_N$, we have

$$\mathbb{E}[M^{\downarrow N}(B_1,\dots,B_N)] = \sum_{\sigma\in\mathfrak{S}(N)} \mathbb{P}[\forall i\in[\![1,N]\!],\ X_{\sigma(i)}\in B_i]$$
$$= \int_{B_1\times\cdots\times B_N} f(x_1,\dots,x_N)\,\lambda(dx_1)\cdots\lambda(dx_N),$$

so the N-th correlation function of M_N is $\rho_N(x_1,\dots,x_N)=N!\,f(x_1,\dots,x_N)$. If $n>N$, then $M^{\downarrow n}=0$, so $\rho_n(x_1,\dots,x_n)=0$. Finally, if $n<N$, then we have the recurrence relations

$$M^{\downarrow n}(B_1,\dots,B_n) = \frac{1}{N-n}\,M^{\downarrow n+1}(B_1,\dots,B_n,\mathfrak{X});$$
$$\rho_n(x_1,\dots,x_n) = \frac{1}{N-n}\int_{\mathfrak{X}} \rho_{n+1}(x_1,\dots,x_{n+1})\,\lambda(dx_{n+1}),$$

so we have the following formula for correlation functions with $n<N$:

$$\rho_n(x_1,\dots,x_n) = \int_{\mathfrak{X}^{N-n}} N^{\downarrow n}\,f(x_1,\dots,x_N)\ \lambda(dx_{n+1})\cdots\lambda(dx_N).$$

We now use these observations in the case where $N!\,f(x_1,\dots,x_N)$ is the product of determinants of the statement of the theorem.

- If $n=N$, $\rho_N(x_1,\dots,x_N)=\det(\phi_i(x_j))_{1\le i,j\le N}\,\det(\psi_i(x_j))_{1\le i,j\le N}$, and on the other hand, we have

$$\det(K_N(x_i,x_j))_{1\le i,j\le N} = \det\left(\sum_{1\le k,l\le N}\psi_k(x_i)\,(A^{-1})_{k,l}\,\phi_l(x_j)\right)_{1\le i,j\le N}$$
$$= \det(\psi_k(x_i))_{1\le i,k\le N}\,\det A^{-1}\,\det(\phi_l(x_j))_{1\le i,j\le N},$$

  so $\rho_N(x_1,\dots,x_N)=\det(K_N(x_i,x_j))_{1\le i,j\le N}$ since $\det A=\det A^{-1}=1$.

- If $n>N$, then

$$\det(K_N(x_i,x_j))_{1\le i,j\le n} = \det\left(\sum_{1\le k,l\le N}\psi_k(x_i)\,(A^{-1})_{k,l}\,\phi_l(x_j)\right)_{1\le i,j\le n}$$

  is the determinant of the product of three matrices with respective sizes $n\times N$, $N\times N$ and $N\times n$. Therefore, its rank is smaller than N and the determinant vanishes. The same is true for the correlation functions $\rho_n(x_1,\dots,x_n)$ since M_N consists in N points.

- Finally, for $n<N$, it suffices to prove that the $n\times n$ determinants of the kernel K_N satisfy the same recurrence relation as the correlation functions ρ_n. For any $n\ge 0$, we expand

$$\det(K_N(x_i,x_j))_{1\le i,j\le n+1} = \sum_{\sigma\in\mathfrak{S}(n+1)}\varepsilon(\sigma)\prod_{\substack{c\text{ cycle of }\sigma\\ c=(k_1,k_2,\dots,k_l)}} K_N(x_{k_1},x_{k_2})\cdots K_N(x_{k_l},x_{k_1}).$$

For each permutation σ, consider the cycle C which contains $k_i = n+1$. We are going to use the identities established at the beginning of the proof. If the cycle C consists only in $n+1$, then

$$\begin{aligned}&\varepsilon(\sigma)\int_{\mathfrak{X}} \prod_{\substack{c \text{ cycle of } \sigma \\ c=(k_1,k_2,\dots,k_l)}} K_N(x_{k_1},x_{k_2})\cdots K_N(x_{k_l},x_{k_1})\,\lambda(dx_{n+1})\\ &= N\varepsilon(\sigma') \prod_{\substack{c \text{ cycle of } \sigma' \\ c=(k_1,k_2,\dots,k_l)}} K_N(x_{k_1},x_{k_2})\cdots K_N(x_{k_l},x_{k_1}),\end{aligned}$$

where σ' is the same permutation as σ, but considered in $\mathfrak{S}(n)$. Now, if the cycle C consists in more than one point, then the integration x_{n+1} replaces $K_N(x_k,x_{n+1})\,K_N(x_{n+1},x_l)$ by $K_N(x_k,x_l)$. This amounts to replace σ by the permutation $\sigma' \in \mathfrak{S}(n)$ where $n+1$ has been removed from C; $\varepsilon(\sigma') = -\varepsilon(\sigma)$ and therefore,

$$\begin{aligned}&\varepsilon(\sigma)\int_{\mathfrak{X}} \prod_{\substack{c \text{ cycle of } \sigma \\ c=(k_1,k_2,\dots,k_l)}} K_N(x_{k_1},x_{k_2})\cdots K_N(x_{k_l},x_{k_1})\,\lambda(dx_{n+1})\\ &= -\varepsilon(\sigma') \prod_{\substack{c \text{ cycle of } \sigma' \\ c=(k_1,k_2,\dots,k_l)}} K_N(x_{k_1},x_{k_2})\cdots K_N(x_{k_l},x_{k_1}).\end{aligned}$$

For any permutation σ' in $\mathfrak{S}(n)$, there are one way to obtain it in the first case, and n ways to obtain it in the second case. We conclude that

$$\begin{aligned}&\int_{\mathfrak{X}} \det(K_N(x_i,x_j))_{1\le i,j\le n+1}\,\lambda(dx_{n+1})\\ &= (N-n)\sum_{\sigma'\in\mathfrak{S}(n)}\varepsilon(\sigma') \prod_{\substack{c \text{ cycle of } \sigma' \\ c=(k_1,k_2,\dots,k_l)}} K_N(x_{k_1},x_{k_2})\cdots K_N(x_{k_l},x_{k_1})\\ &= (N-n)\det(K_N(x_i,x_j))_{1\le i,j\le n},\end{aligned}$$

and this is the same recurrence relation as for correlation functions. □

In the setting of Theorem 2.3, the trace class operator $\mathscr{K}$ associated to K_N is the orthogonal projection in $\mathscr{L}^2(\mathfrak{X},\lambda)$ onto the vector space with dimension N spanned by the functions ϕ_i (or by the functions ψ_i). An important particular case is when $\mathfrak{X}=\mathbb{R}$ and λ is a measure with moments of any order:

$$\forall n\in\mathbb{N},\quad \int_{\mathbb{R}} |x|^n\,\lambda(dx) < +\infty.$$

We can then take $\phi_i = \psi_i =$ i-th normalised *orthogonal polynomial* for the measure λ. Without loss of generality, we can assume that λ is a probability measure, and on the other hand we convene to start the indexation of the orthonormal polynomials at $i=0$, so the kernel K_N is redefined as

$$K_N(x,y) = \sum_{i=0}^{N-1}\phi_i(x)\,\phi_i(y).$$

Let us list some properties of the family of orthonormal polynomials $(\phi_i)_{i\ge0}$ (see [Sze39, Chapters 2 and 3]):

(1) The i-th polynomial ϕ_i has degree i, and $\int_{\mathfrak{X}}\phi_i(x)\,\phi_j(x)\,\lambda(dx) = 1_{i=j}$.

(2) Denote k_i the coefficient of x^i in $\phi_i(x)$. The orthonormal polynomials satisfy the three terms recurrence relation:

$$\phi_{i+1}(x) = (A_i x + B_i)\phi_i(x) - C_i\,\phi_{i-1}(x)$$

with $A_i = \frac{k_{i+1}}{k_i}$, $B_i = -\frac{k_{i+1}}{k_i}\,\langle x\,\phi_i \mid \phi_i\rangle_{\mathscr{L}^2(\mathbb{R},\lambda)}$ and $C_i = \frac{A_i}{A_{i-1}} = \frac{k_{i+1}k_{i-1}}{(k_i)^2}$.

(3) For $N \geq 1$, the kernel K_N is given by the *Christoffel–Darboux formula*:

$$K_N(x,y) = \frac{k_{N-1}}{k_N}\,\frac{\phi_N(x)\,\phi_{N-1}(y) - \phi_{N-1}(x)\,\phi_N(y)}{x-y}.$$

Indeed, we have

$$\phi_0 = k_0 = 1 \qquad ; \qquad \phi_1 = k_1(x-m) \qquad ; \qquad k_1 = \sigma^{-1},$$

where $m = \int_{\mathfrak{X}} t\,\lambda(dt)$ and $\sigma^2 = \int_{\mathfrak{X}}(t-m)^2\,\lambda(dt)$ are the mean and variance of λ. Therefore,

$$\frac{k_0}{k_1}\,\frac{k_1\left((x-m)*1 - 1*(y-m)\right)}{x-y} = 1 = K_1(x,y)$$

and the relation is true for $N = 1$. For $N \geq 2$, we use the recurrence relation and we get:

$$\begin{aligned}
&\frac{k_{N-1}}{k_N}\,\frac{\phi_N(x)\,\phi_{N-1}(y) - \phi_{N-1}(x)\,\phi_N(y)}{x-y}\\
&= \frac{k_{N-1}}{k_N}\,\frac{A_{N-1}(x-y)\,\phi_{N-1}(x)\phi_{N-1}(y) + C_{N-1}\left(\phi_{N-1}(x)\,\phi_{N-2}(y) - \phi_{N-2}(x)\,\phi_{N-1}(y)\right)}{x-y}\\
&= \phi_{N-1}(x)\,\phi_{N-1}(y) + \frac{k_{N-2}}{k_{N-1}}\,\frac{\phi_{N-1}(x)\,\phi_{N-2}(y) - \phi_{N-2}(x)\,\phi_{N-1}(y)}{x-y}
\end{aligned}$$

which is the same as for the sequence $(K_N(x,y))_{N\geq 1}$. With $x = y$, the Christoffel–Darboux formula degenerates into:

$$K_N(x,x) = \frac{k_{N-1}}{k_N}\left(\phi'_N(x)\,\phi_{N-1}(x) - \phi'_{N-1}(x)\,\phi_N(x)\right).$$

(4) The density function $f_N(x_1,\ldots,x_N)$ on $\mathbb{R}^N$ associated to the kernel K_N is

$$f_N(x_1,\ldots,x_N) = \frac{1}{N!}\left(\det(\phi_i(x_j))_{i\in[\![0,N-1]\!],\,j\in[\![1,N]\!]}\right)^2 = \frac{(k_0\,k_1\cdots k_{N-1})^2}{N!}\left(\Delta(x_1,\ldots,x_N)\right)^2$$

where $\Delta(x)$ is the Vandermonde determinant; this follows readily from operations on rows and columns of the matrix $(\phi_i(x_j))_{i\in[\![0,N-1]\!],\,j\in[\![1,N]\!]}$. Conversely, such a density function yields a determinantal point process with kernel K_N.

The next paragraph applies this theory to the case where $\lambda(dx) = \sqrt{\frac{N}{2\pi}}\,\mathrm{e}^{-\frac{Nx^2}{2}}\,dx$.

2.3. **Hermite polynomials and saddle point analysis.** Theorem 2.3 shows readily that the eigenvalues of a random matrix H_N of the Gaussian unitary ensemble form a determinantal point process associated to the kernel $K_N(x,y) = \sum_{i=0}^{N-1}\phi_{N,i}(x)\,\phi_{N,i}(y)$, where the $\phi_{N,i}$'s are the orthonormal polynomials for the scaled normal law

$$\lambda = \lambda_N = \mathcal{N}_{\mathbb{R}}\left(0, \frac{1}{N}\right).$$

Beware in the following that the reference measure λ_N also changes with N. Let us make the functions $\phi_{N,i}$ explicit. If λ is given by a density $\omega(x)\,dx$ where ω satisfies a first-order differential equation with coefficients that are rational functions, the so-called *Rodrigues' formulas* provide a general method in order to obtain an orthogonal family with respect to the measure λ. In the setting of the normal law, we shall look at the *Hermite polynomials*

$$H_i(x) = (-1)^i\,\mathrm{e}^{\frac{x^2}{2}}\,\frac{d^i}{dx^i}\left(\mathrm{e}^{-\frac{x^2}{2}}\right).$$

Each H_i is a monic polynomial with degree i, and if $j < i$, then an integration by parts shows that

$$\int_{\mathbb{R}} H_i(x)\, x^j\, \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx = \frac{(-1)^i}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{d^i}{dx^i}\left(\mathrm{e}^{-\frac{x^2}{2}}\right)\right) x^j\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\mathrm{e}^{-\frac{x^2}{2}}\right) \left(\frac{d^i}{dx^i}(x^j)\right)\, dx = 0,$$

so the H_i's are orthogonal with respect to the standard normal law. The same computation shows that

$$\int_{\mathbb{R}} (H_i(x))^2\, \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx = \int_{\mathbb{R}} H_i(x)\, (-x)^i\, \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx = \int_{\mathbb{R}} \left(\frac{d^i}{dx^i}(x^i)\right) \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx = i!\,.$$

It follows that the functions $\frac{H_i(x)}{\sqrt{i!}}$ are orthonormal with respect to the distribution $\mathcal{N}_{\mathbb{R}}(0,1)$, and that

$$\phi_{N,i}(x) = \frac{H_i(\sqrt{N}x)}{\sqrt{i!}} = \frac{(-1)^i}{\sqrt{i!\, N^i}}\, \mathrm{e}^{\frac{Nx^2}{2}}\, \frac{d^i}{dx^i}\left(\mathrm{e}^{-\frac{Nx^2}{2}}\right).$$

In particular, the leading coefficient k_i of $\phi_{N,i}(x)$ is $\sqrt{\frac{N^i}{i!}}$, and we recover

$$Z_{N,\mathrm{GUE}} = \left(\sqrt{\frac{2\pi}{N}}\right)^N \frac{N!}{(k_0 k_1 \cdots k_{N-1})^2} = (2\pi)^{\frac{N}{2}}\, N^{-\frac{N^2}{2}}\, N!(N-1)! \cdots 1!\,.$$

In the sequel, we shall use the following recurrence relation:

$$H_{i+1}(x) = -\,\mathrm{e}^{\frac{x^2}{2}}\, \frac{d}{dx}\left(\mathrm{e}^{-\frac{x^2}{2}}\, H_i(x)\right) = x\, H_i(x) - H_i'(x) = x\, H_i(x) - i\, H_{i-1}(x),$$

the last identity $H_i'(x) = i\, H_{i-1}(x)$ following from the formula $H_i(x) = \mathrm{e}^{-\frac{D^2}{2}}(x^i)$, where $D = \frac{d}{dx}$. This implies that the exponential generating series $H(x,z) = \sum_{i=0}^{\infty} \frac{z^i}{i!}\, H_i(x)$ of Hermite polynomials satisfies:

$$\frac{d}{dz}\, H(x,z) = (x-z)\, H(x,z) \qquad ; \qquad H(x,z) = \mathrm{e}^{zx - \frac{z^2}{2}}.$$

The explicit formula for the kernel of the GUE leads to precise asymptotic results, which we now detail. We start with the *global* statistical behavior of the eigenvalues:

Theorem 2.4 (Wigner). *Let $\nu_N = \frac{1}{N}\, M_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i}$ be the spectral measure of a random matrix of size $N \times N$ in the GUE. As N goes to infinity, ν_N converges in probability towards the Wigner semicircle law*

$$\nu(dx) = 1_{x \in [-2,2]}\, \frac{\sqrt{4-x^2}}{2\pi}\, dx.$$

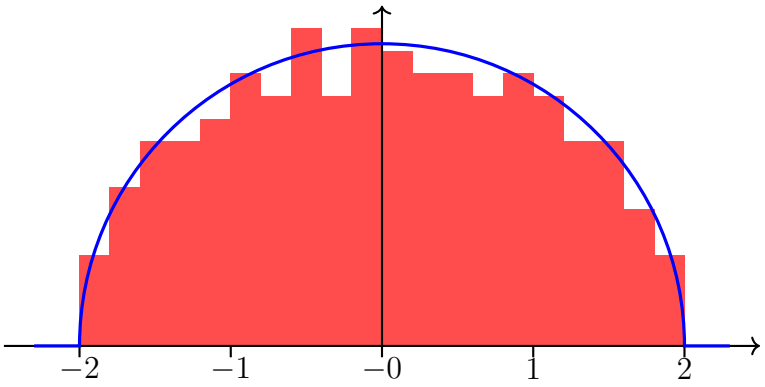

FIGURE 1. Empirical distribution of the eigenvalues of a matrix of the GUE, with $N = 200$.

This law of large numbers for eigenvalues is due to Wigner, see [Wig55]; the original proof is totally different from the one below. By Proposition 1.7, in order to prove the convergence in probability $\nu_N \to \nu$, it suffices to control the rescaled kernel $\frac{K_N(x,x)}{N}\sqrt{\frac{N}{2\pi}}\,\mathrm{e}^{-\frac{Nx^2}{2}}$. By the Christoffel–Darboux formula,

$$\begin{aligned}\frac{K_N(x,x)}{N} &= \frac{k_{N-1}}{N\,k_N}\left(\phi'_{N,N}(x)\,\phi_{N,N-1}(x) - \phi'_{N,N-1}(x)\,\phi_{N,N}(x)\right)\\ &= \frac{1}{N!}\left(H'_N\,H_{N-1} - H'_{N-1}\,H_N\right)\left(\sqrt{N}x\right)\\ &= \frac{1}{N!}\left((H_N)^2 - H_{N-1}\,H_{N+1}\right)\left(\sqrt{N}x\right)\end{aligned}$$

by using on the last line the formulas for $H_i(x)$ with $i \in \{N-1, N, N+1\}$. We therefore need to understand the behavior of $H_N(\sqrt{N}x)$ as N goes to infinity, x being a fixed parameter. The asymptotics of the classical orthogonal polynomials can be obtained by using *saddle point analysis*, which is a generalisation of the *steepest descent method*. Two excellent references for this method are [Bru10, Chapters 5-6] and [Won01, Section II.4]. We start from the following integral representation:

$$H_N(x) = [z^N](H(x,z)) = \frac{N!}{2\mathrm{i}\pi}\oint \frac{\mathrm{e}^{zx-\frac{z^2}{2}}}{z^{N+1}}\,dz,$$

where the path integral runs over an arbitrary smooth curve γ circling around 0. Since we are interested in the asymptotics of $H_N(x)$ when x is of size $\sqrt{N}$, it is convenient to make the changes of variables $\overline{x} = \sqrt{N}x$ and $\overline{z} = \sqrt{N}z$, so that

$$H_N(\sqrt{N}x) = \frac{N!}{2\mathrm{i}\pi\,N^{\frac{N}{2}}}\oint \frac{\mathrm{e}^{Nzx-N\frac{z^2}{2}}}{z^{N+1}}\,dz = \frac{N!}{2\mathrm{i}\pi\,N^{\frac{N}{2}}}\oint \mathrm{e}^{N\left(zx-\frac{z^2}{2}-\log z\right)}\,\frac{dz}{z},$$

where the logarithm is defined on the open subset $\mathbb{C}\setminus\mathbb{R}_-$ of the complex plane (removing the negative real numbers does not change the value of the path integral). Set

$$f(z) = zx - \frac{z^2}{2} - \log z.$$

In order to compute the asymptotics of an integral $I_N = \oint \mathrm{e}^{Nf(z)}k(z)\,dz$ with f analytic, we proceed as follows. Set $f(z) = g(z) + \mathrm{i}h(z)$, where the two real-valued functions g and h are harmonic and sastisfy the Cauchy–Riemann equations $\partial_x g = \partial_y h$ and $\partial_y g = -\partial_x h$. If a path γ is fixed, then we expect the main contribution to the integral $\int_\gamma \mathrm{e}^{Nf(z)}k(z)\,dz$ to be provided by the neighborhoods of points where $|\mathrm{e}^{f(z)}|$ is maximal, that is to say where $g(z) = \mathrm{Re}f(z)$ is maximal. Note however that if z_0 is a point around which $h(z)$ is not constant at first order, then the oscillating part $\mathrm{e}^{\mathrm{i}Nh(z)}$ might make

$$\int_{\text{neighborhood of } z_0} \mathrm{e}^{Nf(z)}k(z)\,dz \ll O(\mathrm{e}^{Ng(z_0)}),$$

so the contribution of the neighborhood of z_0 to the path integral is much smaller than expected. To avoid this phenomenon, we shall deform the path γ so that it crosses (some of) the *critical points* of the holomorphic function f, that is to say points z_0 such that $f'(z_0) = 0$. If z_0 is a critical point, then we have $0 = \partial_z f = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)(f)$, so

$$\partial_x g(z_0) = -\partial_y h(z_0) = -\partial_x g(z_0) \qquad ; \qquad \partial_y g(z_0) = \partial_x h(x_0) = -\partial_y g(z_0)$$

and z_0 is also a critical point of g and of h: $\nabla g(z_0) = \nabla h(z_0) = (0,0)$. Moreover, the Hessian of g satisfies

$$(\det \mathrm{Hess}\, g)(z) = \partial^2_{xx}g(z_0)\,\partial^2_{yy}g(z_0) - \left(\partial^2_{xy}g(z_0)\right)^2 = -\left(\partial^2_{xx}g(z_0)\right)^2 - \left(\partial^2_{xy}g(z_0)\right)^2 \le 0,$$

so z is a *saddle point* of g. This means that there are orthogonal directions that intersect at z_0, and such that at this critical point, g attains its local minimum along one direction and its local

maximum along the other direction. We are going to make the path γ cross z_0 along the direction which corresponds to a local maximum. Intuitively, it is convenient to imagine the function $|\mathrm{e}^{f(z)}|$ as a mountainous landscape, for which we try to find a path γ which is a closed loop in the holomorphy domain of f and k, and which stays at the lowest possible altitude. The path γ is then forced to cross some mountain passes, which are the critical points of f and are saddle points of g. Of course, we shall then cross these passes in directions which correspond to local maxima (we do not descend from a peak to attain the pass). We are indebted to [Bru10, Section 5.2] for this intuitive description. The specific prescribed way to cross the critical points is the following. We suppose that $f''(z_0) \neq 0$, and we consider a smooth path $\gamma : t \in \mathbb{R} \mapsto \gamma(t) \in \mathbb{C}$ such that $\gamma(0) = z_0$ and $\gamma'(0) \neq 0$; up to local reparametrisation, we may assume that we have a unitary tangent vector $\gamma'(0) = \mathrm{e}^{\mathrm{i}\phi}$. If $f''(z_0) = A\,\mathrm{e}^{\mathrm{i}\theta}$, then we have locally

$$g(\gamma(t)) = g(z_0) + \frac{1}{2}\operatorname{Re}(f''(z_0)\,(\gamma'(0)t)^2) + o(t^2) = g(z_0) + \frac{A\,t^2}{2}\,\cos(\theta + 2\phi) + o(t^2).$$

In particular, with $\phi = -\frac{\theta+\pi}{2}$, we obtain

$$g(\gamma(t)) = g(z_0) - \frac{A\,t^2}{2} + o(t^2),$$

so g decreases at the fastest possible quadratic speed around z_0 if one follows the path $\gamma(t)$. Note that if we replace $\gamma(t)$ by $\overline{\gamma}(t) = \gamma(-t)$, this does not change the result, as $\overline{\phi} = \phi + \pi$ and $\cos(\theta + 2\phi) = \cos(\theta + 2\overline{\phi}) = -1$; thus, if we make the path $\gamma(t)$ cross z_0 in the direction $\mathrm{e}^{\mathrm{i}\phi}$ or $-\mathrm{e}^{\mathrm{i}\phi}$, then g descends at the fastest speed. Now, the condition $\phi = \frac{-\theta\pm\pi}{2}$ is equivalent to the following requirement on $(h \circ \gamma)(t)$. If we use the Taylor series of h instead of g, we obtain

$$h(\gamma(t)) = h(z_0) + \frac{1}{2}\operatorname{Im}(f''(z_0)\,(\gamma'(0)t)^2) + o(t^2) = h(z_0) + \frac{A\,t^2}{2}\,\sin(\theta + 2\phi) + o(t^2) = h(z_0) + o(t^2).$$

This observation leads to the following:

Lemma 2.5 (Characterisation of the direction of steepest descent)**.** *Let f be an analytic function in an open domain $U \subset \mathbb{C}$, and z_0 be a critical point of f. If γ is a smooth path such that $\gamma(0) = z_0$ and $\operatorname{Im} f(\gamma(t)) = \operatorname{Im} f(z_0) + o(t^2)$, then $\operatorname{Re} f(\gamma(t))$ decreases at the fastest possible quadratic speed around z_0:*

$$\operatorname{Re} f(\gamma(t)) = \operatorname{Re} f(z_0) - \frac{|f''(z_0)|}{2}\,|\gamma'(0)|^2\,t^2 + o(t^2).$$

Actually, it is not necessary to cross a critical point z_0 of f exactly along the steepest descent direction: it is sufficient to do so in such a way that the angle between the steepest direction and the tangent vector of γ at z_0 is always strictly smaller than $\frac{\pi}{4}$ (see Figure 2). Indeed, if this is the case, then the expansion $g(\gamma(t)) = g(z_0) + (\operatorname{Re}(f''(z_0)\,(\gamma'(0))^2)\,t^2)/2 + o(t^2)$ still involves a negative quadratic term, so we are able to use the Laplace method in order to evaluate the integral. These observations are due to Perron, and they lead to:

Method 2.6 (Saddle point analysis)**.** *Let f and k be analytic functions on a domain $U \subset \mathbb{C}$, and $I_N = \oint \mathrm{e}^{Nf(z)}k(z)\,dz$ be a contour integral. To obtain the asymptotics of I_N as N goes to infinity, we deform the integration contour in order to obtain an explicit path γ with the following properties:*

(1) The path γ crosses one or several critical points $z_1, \ldots, z_l$ of the function f, in such a way that

$$\max_{z\in\gamma} \operatorname{Re} f(z) = \max_{i\in[\![1,l]\!]} \operatorname{Re} f(z_i).$$

(2) At each critical point z_i, $f''(z_i) \neq 0$, and the path γ follows around z_i an adequate direction (close to the steepest descent direction).

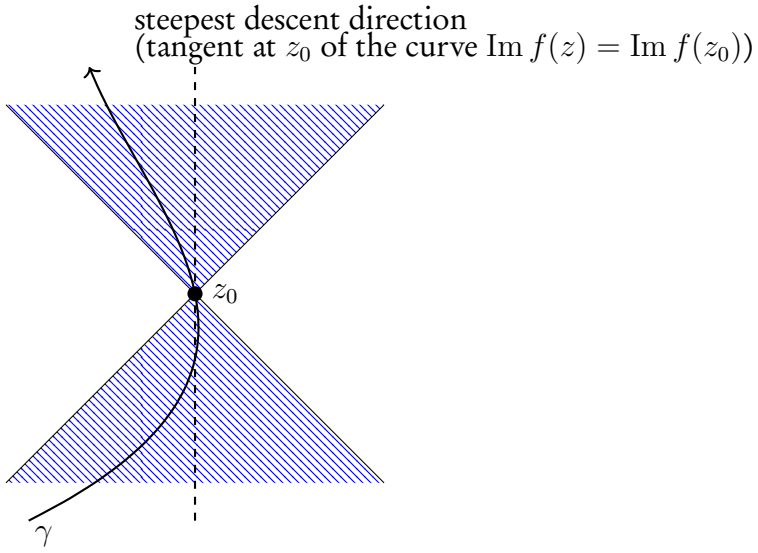

FIGURE 2. An adequate contour of integration for the saddle point analysis crosses the critical points with a direction close to the steepest descent direction.

If the hypotheses above are satisfied, then the Laplace method yields

$$I_N \simeq \sum_{i=1}^{l} \int_{\substack{\text{neighborhood of } t_i \\ \text{such that } \gamma(t_i) = z_i}} \mathrm{e}^{N\left(f(z_i) + \frac{f''(z_i)\,(\gamma'(t_i))^2\,t^2}{2}\right)}\, k(z_i)\,\gamma'(t_i)\,dt$$
$$\simeq \sum_{i=1}^{l} k(z_i)\,\gamma'(t_i)\,\mathrm{e}^{Nf(z_i)} \sqrt{\frac{2\pi}{-N\,(\gamma'(t_i))^2\,f''(z_i)}},$$

where the complex square root of $-(\gamma'(t_i))^2\,f''(z_i)$ is chosen with an argument between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. Assuming that $k(z_i) \neq 0$ for all the critical points $z_1, \ldots, z_l$, the asymptotic expansion above is valid up to a multiplicative $(1 + O(N^{-1}))$ for each term of the sum; on the other hand, the main contribution is of course provided by the z_i's which maximise $\operatorname{Re} f(z_i)$.

We now apply this technique to the asymptotic analysis of Hermite polynomials. The critical points of the analytic map $z \in \mathbb{C} \setminus \mathbb{R}_- \mapsto zx - \frac{z^2}{2} - \log z$ satisfy

$$0 = f'(z) = x - z - \frac{1}{z} \qquad ; \qquad z = \begin{cases} \frac{x \pm \sqrt{x^2-4}}{2} & \text{if } |x| \geq 2, \\ \frac{x \pm \mathrm{i}\sqrt{4-x^2}}{2} & \text{if } |x| < 2. \end{cases}$$

Since it suffices to prove the convergence of the scaled kernel almost everywhere with respect to the Lebesgue measure, the critical case $|x| = 2$ is not important for the global asymptotics (we shall look at this case later when studying the edge asymptotics of the spectrum). Notice on the other hand that $H_N(-x) = (-1)^N\,H_N(x)$, so it suffices to treat the case where x is positive. Suppose first that $x > 2$. The function f has then two real critical points, and its restriction to the set of positive real numbers is drawn in Figure 3.

We choose for γ the circle $\gamma(\phi) = r_0\,\mathrm{e}^{\mathrm{i}\phi}$, where $r_0 = \frac{x - \sqrt{x^2-4}}{2} = \frac{2}{x + \sqrt{x^2-4}}$ is the local minimal of the function f on $\mathbb{R}_+^*$. We then have

$$(g \circ \gamma)(\phi) = \frac{2x}{x + \sqrt{x^2-4}}\,\cos\phi - \frac{1}{2}\frac{x - \sqrt{x^2-4}}{x + \sqrt{x^2-4}}\,\cos 2\phi + \log\left(\frac{x + \sqrt{x^2-4}}{2}\right).$$

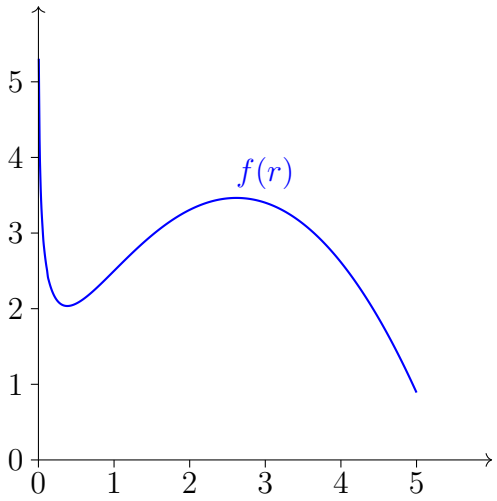

FIGURE 3. The map $r \in \mathbb{R}_+^* \mapsto f(r)$ when $x > 2$.

This function of ϕ is even and decreasing on $[0, \pi)$. Indeed, its derivative is

$$\begin{aligned}(g \circ \gamma)'(\phi) &= -\frac{2x}{x + \sqrt{x^2-4}} \sin\phi + \frac{x - \sqrt{x^2-4}}{x + \sqrt{x^2-4}} \sin 2\phi \\ &= -\frac{2\sin\phi}{x + \sqrt{x^2-4}} \left(x - \left(x - \sqrt{x^2-4} \right) \cos\phi \right) \leq 0.\end{aligned}$$

Therefore, for any angle ϕ, $(g \circ \gamma)(\phi) \leq (g \circ \gamma)(0)$. On the other hand, $f''(r_0)$ is real, so the path γ follows exactly at the critical point the steepest direction. By saddle point analysis and by using the Stirling estimates of $N!$, we conclude that if $x = 2\cosh t$ with $t > 0$, then $r_0 = \mathrm{e}^{-t}$ and

$$H_N(\sqrt{N}x) = N^{\frac{N}{2}}\, \mathrm{e}^{N(f(r_0)-1)} \sqrt{\frac{1}{f''(r_0)}}\, 1_{N^{-1}} = N^{\frac{N}{2}}\, \mathrm{e}^{N\left(\frac{\mathrm{e}^{-2t}}{2}+t\right)} \sqrt{\frac{1}{\mathrm{e}^{2t}-1}}\, 1_{N^{-1}},$$

where $1_{N^{-1}} = 1 + O(N^{-1})$. We obtain with the exact same method:

$$\begin{aligned}H_{N-1}(\sqrt{N}x) &= \frac{N!}{2\mathrm{i}\pi\, N^{\frac{N+1}{2}}} \oint \mathrm{e}^{N\left(zx - \frac{z^2}{2} - \log z\right)}\, dz = \frac{\mathrm{e}^{-t}}{\sqrt{N}}\, H_N(\sqrt{N}x)\, 1_{N^{-1}}; \\ H_{N+1}(\sqrt{N}x) &= \frac{(N+1)!}{2\mathrm{i}\pi\, N^{\frac{N+1}{2}}} \oint \mathrm{e}^{N\left(zx - \frac{z^2}{2} - \log z\right)}\, \frac{dz}{z^2} = \sqrt{N}\, \mathrm{e}^{t}\, H_N(\sqrt{N}x)\; 1_{N^{-1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{K_N(x,x)}{N} &= O\left(\frac{\mathrm{e}^{N\left(\mathrm{e}^{-2t}+1+2t\right)}}{N^{\frac{3}{2}}(\mathrm{e}^{2t}-1)} \right); \\ \frac{K_N(x,x)}{N} \sqrt{\frac{N}{2\pi}}\, \mathrm{e}^{-\frac{Nx^2}{2}} &= O\left(\frac{\mathrm{e}^{-N(\sinh(2t)-2t)}}{N(\mathrm{e}^{2t}-1)} \right) = O\left(\frac{\mathrm{e}^{-\frac{4Nt^3}{3}}}{Nt} \right),\end{aligned}$$

where the constant in the $O(\cdot)$ can depend on t (actually, it is not very hard to show that the estimate is uniform when t says in a compact subset of $(0, +\infty)$).

Proposition 2.7 (Asymptotics of the Hermite kernels for $|x| > 2$)**.** *Suppose $|x| > 2$. Then,*

$$\frac{K_N(x,x)}{N} \sqrt{\frac{N}{2\pi}}\, \mathrm{e}^{-\frac{Nx^2}{2}} \to_{N\to\infty} 0.$$

Let us now analyse the case where $0 < x < 2$. We set $x = 2\cos\theta$ with $\theta \in (0, \frac{\pi}{2})$; then, the critical points are $z_\pm = \mathrm{e}^{\pm \mathrm{i}\theta}$, and $f''(z) = \frac{1}{z^2} - 1$, so

$$f''(z_\pm) = \mathrm{e}^{\mp 2\mathrm{i}\theta} - 1 = 2\sin\theta\, \mathrm{e}^{\mp \mathrm{i}(\frac{\pi}{2}+\theta)}.$$

The steepest descent direction at $z_\pm$ is thus $\mathrm{e}^{\pm \mathrm{i}(\frac{\theta}{2}+\frac{\pi}{4})}$. We choose for γ the unit circle $\gamma(\phi) = \mathrm{e}^{\mathrm{i}\phi}$, $\phi \in (-\pi, \pi)$. The tangent vector of γ at $\phi = \pm\theta$ is $\mathrm{e}^{\pm \mathrm{i}(\theta+\frac{\pi}{2})}$. The two arguments

$$\pm\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \quad \text{and} \quad \pm\left(\theta + \frac{\pi}{2}\right)$$

indeed differ from less than $\frac{\pi}{4}$ for any $\theta \in (0, \frac{\pi}{2})$, so our path allows us to use saddle point analysis, provided that $g \circ \gamma$ attains its maximum at the two parameters $\phi = \pm\theta$. However,

$$(g \circ \gamma)(\phi) = \mathrm{Re}\left(2\cos\theta\, \mathrm{e}^{\mathrm{i}\phi} - \frac{\mathrm{e}^{2\mathrm{i}\phi}}{2}\right) = 2\cos\theta\cos\phi - \frac{\cos 2\phi}{2} = \frac{3}{2} - (\cos\theta - \cos\phi)^2,$$

which clearly attains its maximum when $\cos\phi = \cos\theta$, that is when $\phi = \pm\theta$. Thus, we get

$$\begin{aligned} H_N(\sqrt{N}x) &= N^{\frac{N}{2}} \left(\frac{\mathrm{e}^{N(f(z_+)-1)}}{\sqrt{-(\gamma'(\theta))^2 f''(z_+)}} + \frac{\mathrm{e}^{N(f(z_-)-1)}}{\sqrt{-(\gamma'(-\theta))^2 f''(z_-)}} \right) 1_{N^{-1}} \\ &= 2\,\frac{N^{\frac{N}{2}}\, \mathrm{e}^{N\frac{\cos 2\theta}{2}}}{\sqrt{2\sin\theta}}\, \cos\left(N\left(\frac{\sin 2\theta}{2} - \theta\right) + \frac{\theta}{2} - \frac{\pi}{4}\right) 1_{N^{-1}} \end{aligned}$$

with $x = 2\cos\theta$. The same technique yields:

$$\begin{aligned} H_{N-1}(\sqrt{N}x) &= \frac{N!}{2\mathrm{i}\pi\, N^{\frac{N+1}{2}}} \oint \mathrm{e}^{N\left(zx - \frac{z^2}{2} - \log z\right)}\, dz \\ &= N^{\frac{N-1}{2}} \left(\frac{\gamma(\theta)\, \mathrm{e}^{N(f(z_+)-1)}}{\sqrt{-(\gamma'(\theta))^2 f''(z_+)}} + \frac{\gamma(-\theta)\, \mathrm{e}^{N(f(z_-)-1)}}{\sqrt{-(\gamma'(-\theta))^2 f''(z_-)}} \right) 1_{N^{-1}} \\ &= 2\,\frac{N^{\frac{N-1}{2}}\, \mathrm{e}^{N\frac{\cos 2\theta}{2}}}{\sqrt{2\sin\theta}}\, \cos\left(N\left(\frac{\sin 2\theta}{2} - \theta\right) + \frac{3\theta}{2} - \frac{\pi}{4}\right) 1_{N^{-1}}; \\ H_{N+1}(\sqrt{N}x) &= \frac{(N+1)!}{2\mathrm{i}\pi\, N^{\frac{N+1}{2}}} \oint \mathrm{e}^{N\left(zx - \frac{z^2}{2} - \log z\right)}\, \frac{dz}{z^2} \\ &= N^{\frac{N+1}{2}} \left(\frac{\mathrm{e}^{N(f(z_+)-1)}}{\gamma(\theta)\sqrt{-(\gamma'(\theta))^2 f''(z_+)}} + \frac{\mathrm{e}^{N(f(z_-)-1)}}{\gamma(-\theta)\sqrt{-(\gamma'(-\theta))^2 f''(z_-)}} \right) 1_{N^{-1}} \\ &= 2\,\frac{N^{\frac{N+1}{2}}\, \mathrm{e}^{N\frac{\cos 2\theta}{2}}}{\sqrt{2\sin\theta}}\, \cos\left(N\left(\frac{\sin 2\theta}{2} - \theta\right) - \frac{\theta}{2} - \frac{\pi}{4}\right) 1_{N^{-1}}. \end{aligned}$$

As a consequence, setting $\alpha = N(\frac{\sin 2\theta}{2} - \theta) + \frac{\theta}{2} - \frac{\pi}{4}$, we have

$$\begin{aligned} \frac{(H_N(\sqrt{N}x))^2}{N!} &= \frac{2\,\mathrm{e}^{\frac{Nx^2}{2}}}{\sin\theta\sqrt{2\pi N}}\, \cos^2(\alpha)\, 1_{N^{-1}}; \\ \frac{H_{N-1}(\sqrt{N}x)\, H_{N+1}(\sqrt{N}x)}{N!} &= \frac{2\,\mathrm{e}^{\frac{Nx^2}{2}}}{\sin\theta\sqrt{2\pi N}}\, \cos(\alpha+\theta)\, \cos(\alpha-\theta)\, 1_{N^{-1}}; \\ \frac{K_N(x,x)}{N} &= \frac{2\sin\theta\, \mathrm{e}^{\frac{Nx^2}{2}}}{\sqrt{2\pi N}}\, 1_{N^{-1}}. \end{aligned}$$

Since $2\sin\theta = \sqrt{4 - x^2}$, we conclude:

Proposition 2.8 (Asymptotics of the Hermite kernels for $|x| < 2$)**.** *Suppose* $|x| < 2$*. Then,*

$$\frac{K_N(x,x)}{N}\sqrt{\frac{N}{2\pi}}\,\mathrm{e}^{-\frac{Nx^2}{2}} \to_{N\to\infty} \frac{\sqrt{4-x^2}}{2\pi}.$$

The reunion of Propositions 2.7 and 2.8 implies immediately Theorem 2.4.

2.4. **The sine and Airy kernels.** The saddle point analysis of Hermite polynomials also allows one to understand the *local* statistical behavior of the eigenvalues of a large random matrix in the GUE. To make sense of the results hereafter, we first need to introduce a notion of convergence for random (determinantal) point processes. Consider as in Section 1 a locally compact, complete and separable metric space $\mathfrak{X}$, and a sequence of random point processes $(M_N)_{N\in\mathbb{N}}$ on $\mathfrak{X}$. We say that M_N converges to a random point process M if the law of M_N as a random element of $\mathscr{M}^{\mathrm{atom}}(\mathfrak{X})$ converges to the law of M. By definition of the σ-field on $\mathscr{M}^{\mathrm{atom}}(\mathfrak{X})$, this means that for any family of measurable subsets $B_1,\dots,B_n \subset \mathfrak{X}$, we have the convergence in law $(M_N(B_1),\dots,M_N(B_n)) \rightharpoonup_{N\to\infty} (M(B_1),\dots,M(B_k))$. Suppose that the random point processes M_N and M are determinantal, with Hermitian kernels K_N and K with respect to a common reference measure λ on $\mathfrak{X}$. We assume that $K(x,y)$ is locally bounded, and that $K_N(x,y) \to K(x,y)$ locally uniformly in x and y. Then, the correlation functions also converge locally uniformly, and therefore, the joint moments of the vectors $(M_N(B_1),\dots,M_N(B_n))$ converge. As these moments determine the random point processes M_N and M, we thus have:

Proposition 2.9 (Convergence of determinantal point processes)**.** *Suppose that* M *is a determinantal point process on* $(\mathfrak{X},\lambda)$ *with locally bounded Hermitian kernel* $K(x,y)$*, and that* $(M_N)_{N\in\mathbb{N}}$ *is a sequence of determinantal point processes with Hermitian kernels* $K_N(x,y)$*. If* $K_N(x,y) \to K(x,y)$ *locally uniformly, then* $M_N \to M$ *as* N *goes to infinity.*

Fix a point $x_0 \in (-2,2)$. By Wigner's theorem, in a small interval $(x_0-\varepsilon, x_0+\varepsilon)$, one expects to see $N \times 2\varepsilon \times \frac{\sqrt{4-(x_0)^2}}{2\pi}$ eigenvalues of a random Hermitian matrix H_N of the GUE. Therefore, the distance between two consecutive eigenvalues in this interval is expected to be of order

$$\frac{2\pi}{\sqrt{4-(x_0)^2}\,N} = O\left(\frac{1}{N}\right).$$

We denote as before $x_{N,1} \geq x_{N,2} \geq \cdots \geq x_{N,N}$ the N eigenvalues of H_N. The previous estimate leads one to introduce the following scaling of eigenvalues:

$$y_{N,i} = \frac{N\sqrt{4-(x_0)^2}}{2\pi}\,(x_{N,i}-x_0).$$

and we set $M_N^{\mathrm{local},x_0} = \sum_{i=1}^N \delta_{y_{N,i}}$. This renormalised random point process is expected to have points spaced by a distance of order 1, and it encodes the behavior of the eigenvalues of H_N in the neighborhood of a parameter x_0 *in the bulk of the spectrum*, that is to say with $-2 < x_0 < 2$. We have

$$M_N^{\mathrm{local},x_0}(B) = M_N\left(x_0 + \frac{2\pi\,B}{N\sqrt{4-(x_0)^2}}\right),$$

so M_N^{local,x_0} is again a determinantal point process. If K_N is the Hermite kernel from the previous paragraph and

$$\overline{K}_N(a,b) = \sqrt{\frac{N}{2\pi}}\,\mathrm{e}^{-\frac{N(a^2+b^2)}{4}}\,K_N(a,b)$$

is the corresponding kernel with respect to the Lebesgue measure, then M_N^{local,x_0} has for kernel

$$K_N^{\mathrm{local},x_0}(x,y) = \frac{2\pi}{N\sqrt{4-(x_0)^2}}\,\overline{K}_N\left(x_0 + \frac{2\pi\,x}{N\sqrt{4-(x_0)^2}}, x_0 + \frac{2\pi\,y}{N\sqrt{4-(x_0)^2}}\right)$$

with respect to the Lebesgue measure. We denote a and b the two arguments of $\overline{K}_N$ in the above; let us then determine the limit of this kernel as N goes to infinity. By the Christoffel–Darboux formula,

$$\begin{aligned} K_N(a,b) &= \frac{k_{N-1}}{k_N}\,\frac{\phi_{N,N}(a)\,\phi_{N,N-1}(b) - \phi_{N,N-1}(a)\,\phi_{N,N}(b)}{a-b} \\ &= \frac{1}{\sqrt{N!\,(N-1)!}}\,\frac{H_N(\sqrt{N}a)\,H_{N-1}(\sqrt{N}b) - H_{N-1}(\sqrt{N}a)\,H_N(\sqrt{N}b)}{a-b}; \\ \overline{K}_N(a,b) &= \frac{1}{(N-1)!}\,\frac{\mathrm{e}^{-\frac{N(a^2+b^2)}{4}}}{\sqrt{2\pi}}\,\frac{H_N(\sqrt{N}a)\,H_{N-1}(\sqrt{N}b) - H_{N-1}(\sqrt{N}a)\,H_N(\sqrt{N}b)}{a-b}. \end{aligned}$$

If $c = 2\cos\phi_c$, then we have seen in the previous paragraph that

$$\begin{aligned} \left(\frac{N}{2\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{Nc^2}{4}}\,\frac{H_N(\sqrt{N}c)}{\sqrt{N!}} &= \frac{1}{\sqrt{\pi\,\sin\phi_c}}\,\cos\left(N\left(\frac{\sin 2\phi_c}{2} - \phi_c\right) + \frac{\phi_c}{2} - \frac{\pi}{4}\right) 1_{N^{-1}}; \\ \left(\frac{N}{2\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{Nc^2}{4}}\,\frac{H_{N-1}(\sqrt{N}c)}{\sqrt{(N-1)!}} &= \frac{1}{\sqrt{\pi\,\sin\phi_c}}\,\cos\left(N\left(\frac{\sin 2\phi_c}{2} - \phi_c\right) + \frac{3\phi_c}{2} - \frac{\pi}{4}\right) 1_{N^{-1}} \end{aligned}$$

where $1_{N^{-1}} = 1 + O(N^{-1})$. These estimates are uniform if c stays in a compact interval of $(-2,2)$; in particular, they are valid if we take c equal to a or b. In the sequel, we set $\phi = \arccos\frac{x_0}{2}$, $\phi_a = \arccos\frac{a}{2}$ and $\phi_b = \arccos\frac{b}{2}$. Given an angle ϕ_c, we write

$$\alpha_c = N\left(\frac{\sin 2\phi_c}{2} - \phi_c\right) + \frac{\phi_c}{2} - \frac{\pi}{4}.$$

Then,

$$K_N^{\mathrm{local},x_0}(x,y) = \frac{1_{N^{-1}}\,\cos(\alpha_a)\,\cos(\alpha_b+\phi_b) - 1_{N^{-1}}\,\cos(\alpha_a+\phi_a)\,\cos(\alpha_b)}{N\sin\phi\,\sqrt{\sin\phi_a\,\sin\phi_b}\,(a-b)}.$$

Notice that the angles ϕ, ϕ_a and ϕ_b all differ by a $O(\frac{1}{N})$, where the constant in the $O(\cdot)$ only depends on x_0, x and y. Indeed, we have

$$\phi_a = \phi - \frac{\pi x}{2N\,\sin^2\phi} + O\left(\frac{1}{N^2}\right) \qquad ; \qquad \phi_b = \phi - \frac{\pi y}{2N\,(\sin^2\phi)} + O\left(\frac{1}{N^2}\right).$$

This enables the following simplifications:

$$\begin{aligned} K_N^{\mathrm{local},x_0}(x,y) &= \frac{1_{N^{-1}}\,\cos(\alpha_a)\,\cos(\alpha_b+\phi_b) - 1_{N^{-1}}\,\cos(\alpha_a+\phi_a)\,\cos(\alpha_b)}{N\,\sin^2\phi\,(a-b)} \\ &= \frac{\cos(\alpha_a)\,\cos(\alpha_b+\phi_b) - \cos(\alpha_a+\phi_a)\,\cos(\alpha_b)}{\pi\,(\sin\phi)\,(x-y)} + O\left(\frac{1}{N}\right) \\ &= \frac{\cos(\alpha_a-\alpha_b-\phi_b) - \cos(\phi_a+\alpha_a-\alpha_b)}{2\pi\,(\sin\phi)\,(x-y)} + O\left(\frac{1}{N}\right) \\ &= \frac{\sin\left(\alpha_a-\alpha_b+\frac{\phi_a-\phi_b}{2}\right)\,\sin\left(\frac{\phi_a+\phi_b}{2}\right)}{\pi\,(\sin\phi)\,(x-y)} + O\left(\frac{1}{N}\right) \\ &= \frac{\sin\left(N\left(\frac{\sin 2\phi_a}{2} - \frac{\sin 2\phi_b}{2} + \phi_b - \phi_a\right)\right)}{\pi\,(x-y)} + O\left(\frac{1}{N}\right) \end{aligned}$$

since $\cos s \cos t = \frac{\cos(s+t)+\cos(s-t)}{2}$ and $\cos s - \cos t = -2\,\sin(\frac{s+t}{2})\,\sin(\frac{s-t}{2})$. Finally, we have

$$\begin{aligned}\frac{\sin 2\phi_a}{2} - \frac{\sin 2\phi_b}{2} + \phi_b - \phi_a &= \cos(\phi_a+\phi_b)\sin(\phi_a-\phi_b) + \phi_b - \phi_a \\ &= (\cos(\phi_a+\phi_b)-1)(\phi_a-\phi_b) + O\left(\frac{1}{N^2}\right) \\ &= 2\,\sin^2\left(\frac{\phi_a+\phi_b}{2}\right)\,(\phi_b-\phi_a) + O\left(\frac{1}{N^2}\right) \\ &= 2\,\sin^2\phi\,(\phi_b-\phi_a) + O\left(\frac{1}{N^2}\right) \\ &= \frac{\pi(x-y)}{N} + O\left(\frac{1}{N^2}\right)\end{aligned}$$

so we conclude that locally uniformly in x and y, $K_N^{\mathrm{local},x_0}(x,y) \to \frac{\sin(\pi(x-y))}{\pi(x-y)}$. We have thus established:

Theorem 2.10 (Gaudin–Mehta)**.** *For any parameter x_0 in the bulk of the spectrum, as N goes to infinity, the rescaled local random point process M_N^{local,x_0} converges towards the determinantal point process M whose kernel is the* sine kernel

$$K^{\mathrm{sine}}(x,y) = \frac{\sin\pi(x-y)}{\pi(x-y)},$$

the reference measure being the Lebesgue measure on $\mathbb{R}$.

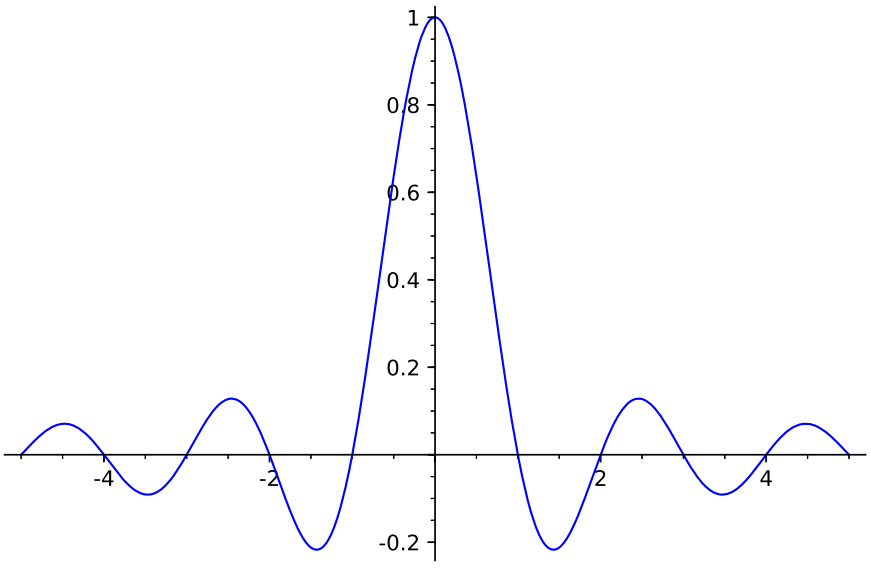

FIGURE 4. The two-point correlation function of the sine-kernel, as a function of $x-y$.

Let us make several remarks on the limiting determinantal point process. First, since $K^{\mathrm{sine}}(x,x) = 1$ for every $x \in \mathbb{R}$, we have $\mathbb{E}[M(B)] = \mathrm{Leb}(B)$ for any measurable subset $B \subset \mathbb{R}$; this is because we have rescaled the random point processes M_N in order to have an expected spacing of points equal to 1. On the other hand, $K^{\mathrm{sine}}(x,y)$ is invariant by translation of the variables x and y, so the law of M is translation invariant. Notice also that this limiting process M does not depend on the choice of the parameter x_0; so, the density of eigenvalues in a neighborhood of x_0 depends on x_0 *via* the semicircle function, but the local structure is independent of x_0 once this density has been taken into account. Finally, the Hermitian and locally trace class operator $\mathscr{K}$ associated to

the sine kernel can be made entirely explicit by using the Fourier transform of square integrable functions. Indeed,

$$(\mathscr{K}f)(x) = \int_{\mathbb{R}} \frac{\sin \pi(x-y)}{\pi(x-y)}\, f(y)\, dy = \big((\mathrm{sinc}(\pi\,\cdot)) * f\big)(x),$$

where $*$ denotes the convolution of functions. Going to the Fourier space, we obtain:

$$\widehat{\mathscr{K}f}(\xi) = \widehat{\mathrm{sinc}(\pi\,\cdot)}(\xi)\, \widehat{f}(\xi),$$

and the Fourier transform of the function $x \mapsto \mathrm{sinc}(\pi x)$ is $1_{|\xi|\leq\pi}$. Consequently, $\mathscr{K}$ is the orthogonal projection in $\mathscr{L}^2(\mathbb{R}, dx)$ onto the subspace of functions with Fourier transform compactly supported in $[-\pi, \pi]$.

A similar study can be performed at the edge of the spectrum, that is to say in the neighborhood of $x_0 = 2$ or -2; by symmetry, it suffices to look at the right-side edge. If $x = 2 - t$ with t small, then the density of eigenvalues at x is of order $\sqrt{t}$, so we can expect to see

$$O\left(N \int_0^t \sqrt{u}\, du\right) = O\left(Nt^{3/2}\right)$$

eigenvalues in the interval $(2-t, 2)$. To see a $O(1)$ number of eigenvalues, it is thus natural to choose $t = O(N^{-\frac{2}{3}})$; equivalently, we can expect the spacing of eigenvalues in the neighborhood of $x_0 = 2$ to be of order $N^{-\frac{2}{3}}$ (instead of N^{-1} in the bulk of the spectrum). We therefore rescale the eigenvalues of a matrix H_N of the GUE as follows: we set $z_i = N^{\frac{2}{3}}(x_{N,i} - 2)$ and $M_N^{\mathrm{edge}} = \sum_{i=1}^N \delta_{z_i}$. In other words,

$$M_N^{\mathrm{edge}}(B) = M_N\left(2 + N^{-\frac{2}{3}} B\right).$$

The reasoning that leads to the choice of the scaling $N^{\frac{2}{3}}$ is really not rigorous: as we shall see later, the z_i's are allowed to be negative or positive, whereas in our reasoning the non-vanishing density could only be considered with $t > 0$; however, the scaling order which we obtained is correct. The rescaled random point process M_N^{edge} is again a determinantal point process on $\mathbb{R}$; its kernel is

$$\begin{aligned} K_N^{\mathrm{edge}}(t,u) &= \frac{1}{N^{\frac{2}{3}}}\, \overline{K}_N\left(2 + \frac{u}{N^{\frac{2}{3}}}, 2 + \frac{u}{N^{\frac{2}{3}}}\right) \\ &= \frac{\mathrm{e}^{-\frac{N(x^2+y^2)}{4}}}{\sqrt{2\pi}\,(N-1)!}\, \frac{H_N(\sqrt{N}x)\, H_{N-1}(\sqrt{N}y) - H_{N-1}(\sqrt{N}x)\, H_N(\sqrt{N}y)}{t-u} \end{aligned}$$

where $x = 2 + tN^{-\frac{2}{3}}$, $y = 2 + uN^{-\frac{2}{3}}$ and the reference measure is the Lebesgue measure on $\mathbb{R}$. Denote

$$\psi_N(x) = \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{x^2}{4}}\, \frac{H_N(x)}{\sqrt{N!}}.$$

These normalised *oscillator wave-functions* form an orthonormal basis of $\mathscr{L}^2(\mathbb{R}, dx)$, and they satisfy the following differential equation:

$$\psi_N'(x) = -\frac{x}{2}\, \psi_N(x) + \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{x^2}{4}}\, \frac{N H_{N-1}(x)}{\sqrt{N!}} = -\frac{x}{2}\, \psi_N(x) + \sqrt{N}\, \psi_{N-1}(x).$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{e}^{-\frac{N(x^2+y^2)}{4}}}{\sqrt{2\pi}\,(N-1)!}\,\frac{H_N(\sqrt{N}x)\,H_{N-1}(\sqrt{N}y)-H_{N-1}(\sqrt{N}x)\,H_N(\sqrt{N}y)}{x-y}\\
&=\frac{\psi_N(\sqrt{N}x)\,\Big(\psi_N'(\sqrt{N}y)+\frac{\sqrt{N}y}{2}\,\psi_N(\sqrt{N}y)\Big)-\psi_N(\sqrt{N}y)\,\Big(\psi_N'(\sqrt{N}x)+\frac{\sqrt{N}x}{2}\,\psi_N(\sqrt{N}y)\Big)}{x-y}\\
&=\frac{\psi_N(\sqrt{N}x)\,\psi_N'(\sqrt{N}y)-\psi_N(\sqrt{N}y)\,\psi_N'(\sqrt{N}x)}{x-y}-\frac{\sqrt{N}}{2}\,\psi_N(\sqrt{N}x)\,\psi_N(\sqrt{N}y).
\end{aligned}$$

We are then led to the asymptotic study of $H_N(\sqrt{N}x)$ and $\psi_N(\sqrt{N}x)$ when x is very close to 2. This is the critical case of the saddle point analysis of the integral representations of the Hermite polynomials, and we shall modify accordingly the method in order to make it work also in this case. For a general treatment of the asymptotics of integrals with two critical points that depend on a parameter x and that coalesce at $x=x_0$, see [Won01, Section VII.4]. In the sequel, we fix $x=2+tN^{-\frac{2}{3}}$, where t is a real parameter. We have

$$\begin{aligned}
N^{\frac{1}{4}}\,\psi_N(\sqrt{N}x)=\left(\frac{N}{2\pi}\right)^{\frac{1}{4}}\mathrm{e}^{-\frac{Nx^2}{4}}\,\frac{H_N(\sqrt{N}x)}{\sqrt{N!}}&=\left(\frac{N}{2\pi}\right)^{\frac{1}{4}}\frac{\sqrt{N!}}{2\mathrm{i}\pi\,N^{\frac{N}{2}}}\oint\mathrm{e}^{N\left(zx-\frac{x^2}{4}-\frac{z^2}{2}-\log z\right)}\,\frac{dz}{z}\\
&=\frac{\sqrt{N}\,1_{N^{-1/3}}}{2\mathrm{i}\pi}\oint\mathrm{e}^{N^{\frac{1}{3}}t(z-1)+N\left(-\frac{3}{2}+2z-\frac{z^2}{2}-\log z\right)}\,\frac{dz}{z}
\end{aligned}$$

where $1_{N^{-1/3}}=1+O(N^{-\frac{1}{3}})$. The function $f(z)=2z-\frac{z^2}{2}-\log z$ has a unique critical point at $z=1$, and we have $f''(1)=0$, so in a neighborhood of the critical point, if $z=1+N^{-\frac{1}{3}}y$, then

$$p(z,N)=N^{\frac{1}{3}}t(z-1)+N\left(-\frac{3}{2}+2z-\frac{z^2}{2}-\log z\right)=-ty-\frac{y^3}{3}+o(y^3).$$

In this expansion, in order to make the term $-\frac{y^3}{3}$ decrease as fast as possible, we need to take $\arg(y)\in\{0,\frac{2\pi}{3},-\frac{2\pi}{3}\}$. Recall that for $x<2$, the contour chosen for the saddle point analysis was the unit circle.

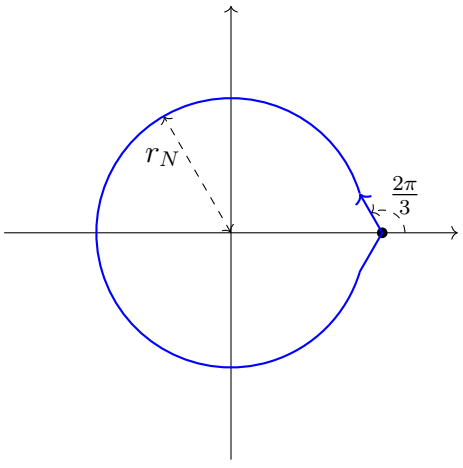

FIGURE 5. Deformation of the contour of integration for the saddle point analysis in the critical case $x=2+tN^{-\frac{2}{3}}$.

For $x\simeq 2$, we slightly deform this contour as follows:

- Around 1, we take the union of the two segments
$$z=1+N^{-\frac{1}{3}}\,\mathrm{e}^{\pm\frac{2\mathrm{i}\pi}{3}}\,u,\quad 0\leq u\leq N^{\varepsilon}$$
with $\frac{1}{9}<\varepsilon<\frac{1}{6}$.

- We join the endpoints of these two segments by the circle with center 0 and radius

$$r_N = \left|1 + N^{\varepsilon-\frac{1}{3}} \mathrm{e}^{\frac{2\mathrm{i}\pi}{3}}\right| = \sqrt{1 - N^{\varepsilon-\frac{1}{3}} + N^{2\varepsilon-\frac{2}{3}}} = 1 - \frac{1}{2}\, N^{\varepsilon-\frac{1}{3}} + O\Big(N^{2\varepsilon-\frac{2}{3}}\Big)\,.$$

We denote γ_1 and γ_2 the two parts of this new contour. Note that on the second part γ_2, writing $z = r_N\, \mathrm{e}^{\mathrm{i}\psi}$, we have

$$\mathrm{Re}\left(-\frac{3}{2} + 2z - \frac{z^2}{2} - \log z\right) = -(1 - r_N \cos\psi)^2 + \frac{(r_N)^2 - 1}{2} - \log r_N$$

so this quantity decreases with $\psi \in (0, \pi)$ and is always smaller than its value at $z = 1 + \mathrm{e}^{\frac{2\mathrm{i}\pi}{3}} N^{\varepsilon-\frac{1}{3}}$, which is

$$\frac{-2N^{\varepsilon-\frac{1}{3}} + N^{2\varepsilon-\frac{2}{3}} - 2\log(1 - N^{\varepsilon-\frac{1}{3}} + N^{2\varepsilon-\frac{2}{3}})}{4} = -\frac{N^{3\varepsilon-1}}{3} + o(N^{3\varepsilon-1}).$$

Therefore,

$$\log\left(\frac{1}{2\pi}\oint_{\gamma_2} |\mathrm{e}^{p(z,N)}|\,\frac{dz}{z}\right) \le 2|t|N^{1/3} - \frac{N^{3\varepsilon}}{3} + o(N^{3\varepsilon}) = -\frac{N^{3\varepsilon}}{3} + o(N^{3\varepsilon})$$

since $\varepsilon > \frac{1}{9}$. This implies that the contribution to the contour integral of γ_2 decreases as $\exp(-CN^{3\varepsilon})$, so it will be negligible. On the other hand, a change of variables yields

$$\oint_{\gamma_1} \mathrm{e}^{p(z,N)}\,\frac{dz}{z} = 1_{N^{\varepsilon-1/3}}\, N^{-\frac{1}{3}} \int_{\curlywedge} \mathrm{e}^{ty - \frac{y^3}{3}}\, dy,$$

where the path of integration on the right-hand side is the union of the two half-lines $\mathbb{R}_+\mathrm{e}^{\frac{2\mathrm{i}\pi}{3}}$ and $\mathbb{R}_+\mathrm{e}^{-\frac{2\mathrm{i}\pi}{3}}$. So, if $x = 2 + tN^{-\frac{2}{3}}$, then

$$N^{\frac{1}{4}}\,\psi_N(\sqrt{N}x) = N^{\frac{1}{6}}\left(\frac{1}{2\mathrm{i}\pi}\int_{\curlywedge} \mathrm{e}^{ty - \frac{y^3}{3}}\, dy\right) + O(N^{\varepsilon-\frac{1}{6}}).$$

The remainder is by construction a $o(1)$, and on the other hand, the path integral is the so-called *Airy function* $\mathrm{Ai}(t)$. This function satisfies the differential equation $\mathrm{Ai}''(t) - t\,\mathrm{Ai}(t) = 0$, and its graph is drawn in Figure 6. It can also be redefined as the real semi-convergent integral $\mathrm{Ai}(t) =$

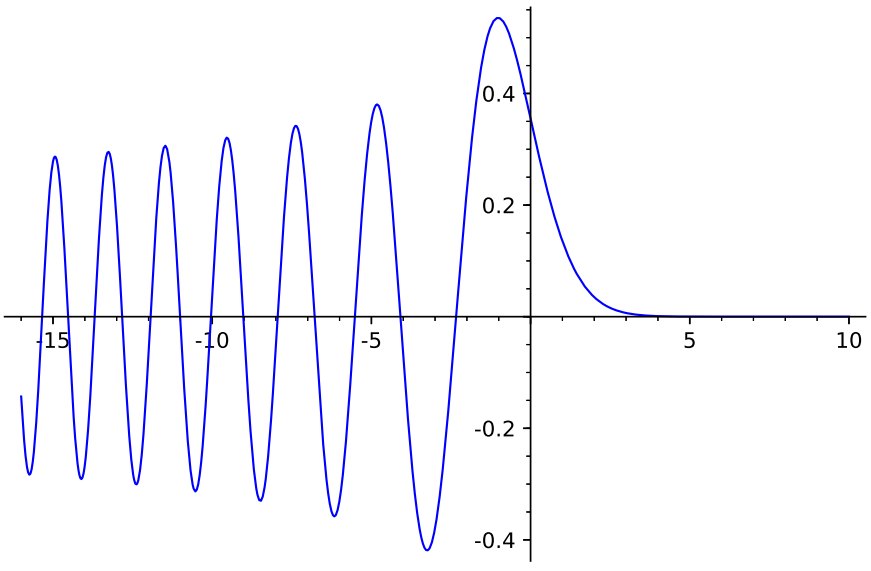

FIGURE 6. Graph of the Airy function.

$\frac{1}{\pi}\int_0^\infty \cos(ty+\frac{y^3}{3})\,dy$, and its Fourier transform is

$$\widehat{\mathrm{Ai}}(\xi) = \mathrm{e}^{\frac{(\mathrm{i}\xi)^3}{3}}.$$

We refer to [AGZ10, Section 3.7.3] and [Won01, Chapter II, example 7] for more details on this entire function. We have proved above that

$$N^{\frac{1}{12}}\,\psi_N\left(2N^{\frac{1}{2}}+tN^{-\frac{1}{6}}\right) \to_{N\to\infty} \mathrm{Ai}(t).$$

This estimate can be made locally uniform in t, and it can be considered as a result of convergence of holomorphic functions of the variable t. Denote the left-hand side of the asymptotic formula above by $\theta_N(t)$. We have on the one hand

$$\begin{aligned} K_N^{\mathrm{edge}}(t,u) &= \frac{\psi_N(\sqrt{N}x)\,\psi_N'(\sqrt{N}y)-\psi_N(\sqrt{N}y)\,\psi_N'(\sqrt{N}x)}{t-u} - \frac{1}{2N^{\frac{1}{6}}}\,\psi_N(\sqrt{N}x)\,\psi_N(\sqrt{N}y) \\ &= \frac{\theta_N(t)\,\theta_N'(u)-\theta_N(u)\,\theta_N'(t)}{t-u} - \frac{1}{2N^{\frac{1}{3}}}\,\theta_N(t)\,\theta_N(u), \end{aligned}$$

and on the other hand, θ_N and all its derivatives converge locally uniformly on the complex plane towards the Airy function and its derivatives. We have therefore established the following result:

Theorem 2.11 (Airy kernel)**.** *As N goes to infinity, the rescaled local random point process M_N^{edge} converges towards the determinantal point process whose kernel is the* Airy kernel

$$K^{\mathrm{Airy}}(t,u) = \frac{\mathrm{Ai}(t)\,\mathrm{Ai}'(u)-\mathrm{Ai}'(t)\,\mathrm{Ai}(u)}{t-u},$$

the reference measure being the Lebesgue measure on $\mathbb{R}$.

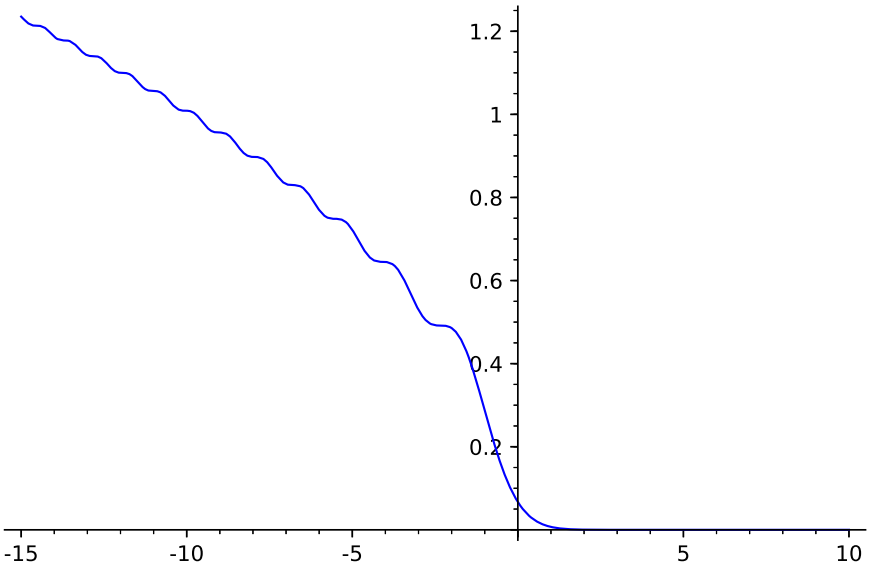

FIGURE 7. First correlation function for the Airy kernel.

In particular, the density (first correlation function) of the limiting determinantal point process is

$$\rho_1(t) = (\mathrm{Ai}'(t))^2 - \mathrm{Ai}(t)\,\mathrm{Ai}''(t) = (\mathrm{Ai}'(t))^2 - t\,(\mathrm{Ai}(t))^2,$$

which is drawn in Figure 7. The asymptotics of the Airy function and its derivative as t goes to $\pm\infty$ can again be derived by saddle point analysis of the integral representation of $\mathrm{Ai}(t)$. Hence,

we have:

$$\rho_1(t) \simeq_{t\to+\infty} \frac{\mathrm{e}^{-\frac{4}{3}t^{\frac{3}{2}}}}{8\pi t} \quad ; \quad \rho_1(t) \simeq_{t\to-\infty} \frac{\sqrt{|t|}}{\pi}.$$

The exponential decay of $\rho_1(t)$ for t positive leads one to guess that with high probability, there are very few eigenvalues of H_N larger than 2, and that the *largest eigenvalue* $x_{N,1}$ of H_N writes as $2+T_N\,N^{-\frac{2}{3}}$, where T_N converges in law as N goes to infinity. The limiting distribution of T_N will be detailed in the next paragraph.

Remark 2.12. The Airy kernel can be rewritten as

$$K^{\mathrm{Airy}}(t,u) = \int_0^\infty \mathrm{Ai}(s+t)\,\mathrm{Ai}(s+u)\,ds,$$

as can be seen by applying the differential operator $\frac{\partial}{\partial t}+\frac{\partial}{\partial u}$ and solving a differential equation; see [AGZ10, Lemma 3.9.33]. This implies in particular that the restriction of $\mathscr{K}^{\mathrm{Airy}}$ to any subspace $\mathscr{L}^2([M,+\infty),\,dx)$ is trace class, since the Airy function of a positive argument decreases exponentially fast.

To close this section, let us mention another classical model of random matrices where similar techniques can be used in order to compute the local asymptotics of eigenvalues. The *circular unitary ensemble* is the random point process of the eigenvalues of a random matrix U_N chosen in the unitary group $\mathrm{U}(N)$ of order N according to the Haar measure. The eigenvalues of a unitary matrix belong to the unit circle and write as

$$\left\{\mathrm{e}^{\mathrm{i}\theta_{N,1}}, \mathrm{e}^{\mathrm{i}\theta_{N,2}}, \dots, \mathrm{e}^{\mathrm{i}\theta_{N,N}}\right\}$$

with $0 \le \theta_{N,1} \le \cdots \le \theta_{N,N} \le 2\pi$. The Weyl integration formula states that if f is a function on $\mathrm{U}(N)$ which is conjugacy-invariant and hence only depends on the eigenvalues of the matrices, then the Haar integral $\int_{\mathrm{U}(N)} f(g)\,dg$ rewrites as an integral over the torus $\mathbb{T}^N = (\mathbb{R}/2\pi\mathbb{Z})^N$, with a Vandermonde determinant taking into account the change of variables:

$$\int_{\mathrm{U}(N)} f(g)\,dg = \frac{1}{(2\pi)^N\,N!}\int_{\mathbb{T}^N} f\big(\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1},\dots,\mathrm{e}^{\mathrm{i}\theta_N})\big)\left|\Delta(\mathrm{e}^{\mathrm{i}\theta_1},\dots,\mathrm{e}^{\mathrm{i}\theta_N})\right|^2 d\theta_1\cdots d\theta_N.$$

We are then exactly in the situation of Theorem 2.3, with the space $\mathscr{L}^2(\mathbb{R}/2\pi\mathbb{Z},\,\frac{d\theta}{2\pi})$ and the N orthogonal functions $\mathrm{e}^{\mathrm{i}k\theta}$ with $k \in [\![0, N-1]\!]$ (the only difference with what precedes is that we are dealing with complex-valued orthogonal functions, but this has no important consequence). So:

Proposition 2.13 (Circular unitary ensemble)**.** *The eigenvalues of a random unitary matrix $U_N \in \mathrm{U}(N)$ chosen according to the Haar measure form a determinantal point process M_N on $\mathbb{T}$ with kernel*

$$K_N(\theta,\phi) = \frac{\sin\left(\frac{N(\theta-\phi)}{2}\right)}{\sin\left(\frac{\theta-\phi}{2}\right)},$$

the reference measure being the Lebesgue measure $\frac{d\theta}{2\pi}$.

Indeed, one can compute

$$K_N(\theta,\phi) = \sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{i}k(\theta-\phi)} = \mathrm{e}^{\mathrm{i}\frac{N-1}{2}(\theta-\phi)}\,\frac{\sin\left(\frac{N(\theta-\phi)}{2}\right)}{\sin\left(\frac{\theta-\phi}{2}\right)},$$

and one sees easily that a determinant $\det(K(x_i,x_j))_{1\le i,j\le N}$ is invariant by multiplication of the kernel by $\mathrm{e}^{\mathrm{i}\frac{N-1}{2}(x-y)}$. Since $K_N(\theta,\theta) = N$ for any θ, we deduce immediately from Proposition 1.7 that the eigenvalue density $\frac{1}{N}\sum_{i=1}^N \delta_{\theta_{N,i}}$ converges in probability towards the uniform Lebesgue

measure on the torus. Moreover, if one looks at the neighborhood of an angle θ_0 and at the scaled local point process defined by

$$M_N^{\mathrm{local},\theta_0}(B) = M_N\left(\theta_0 + \frac{2\pi B}{N}\right)$$

for B subset of $\mathbb{R}$, then it converges towards the sine kernel as N goes to infinity:

$$K_N^{\mathrm{local},\theta_0}(x,y) = \frac{1}{N} K_N\left(\theta_0 + \frac{2\pi x}{N}, \theta_0 + \frac{2\pi y}{N}\right) = \frac{\sin(\pi(x-y))}{N \sin\left(\frac{\pi(x-y)}{N}\right)} \to_{N\to\infty} K^{\mathrm{sine}}(x,y).$$

So, the sine kernel also appears as the limit of the local statistics of the eigenvalues of a random unitary matrix.

2.5. **The Tracy–Widom distribution.** An important precision of Theorem 2.11 is the following result due to Tracy and Widom, see [TW94].

Theorem 2.14 (Tracy–Widom). *Let M be a determinantal point process with kernel K^{Airy}.*

(1) There is a largest random point $T \in M$, whose distribution writes as

$$\mathbb{P}[T \leq t] = F_2(t) = \exp\left(-\int_t^\infty (x-t)\,(q(x))^2\,dx\right),$$

where q is the unique solution of the Painlevé II equation $q''(x) = x\,q(x) + 2\,(q(x))^3$ such that $q(x) \simeq \mathrm{Ai}(x)$ as x goes to $+\infty$.

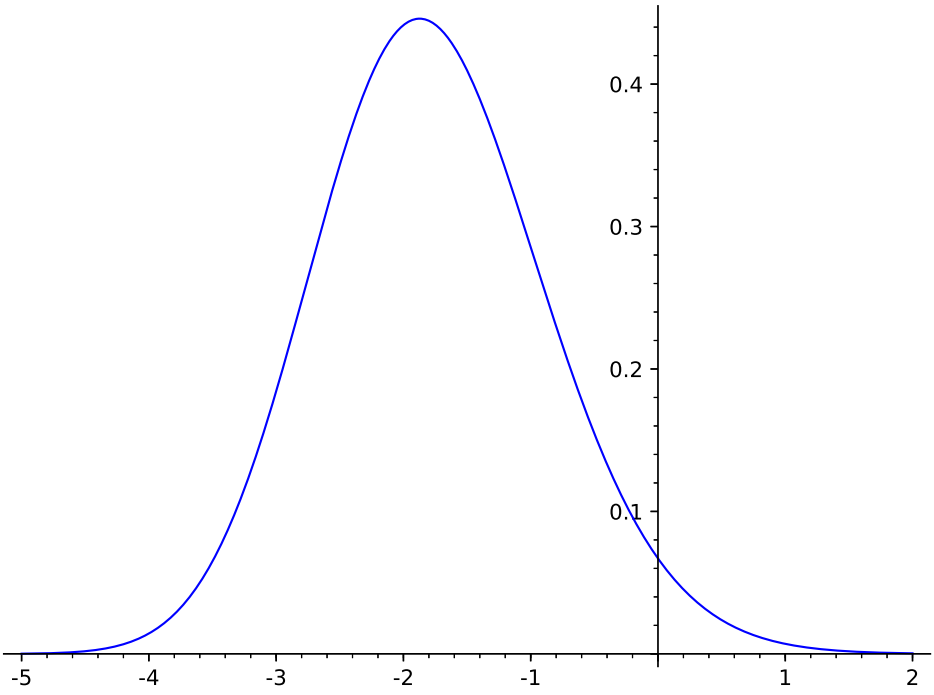

FIGURE 8. Density of the Tracy–Widom F_2 distribution.

(2) Let T_N be the scaled largest eigenvalue in the Gaussian unitary ensemble:

$$T_N = N^{\frac{2}{3}}\,(x_{N,1} - 2).$$

We have the convergence in law $T_N \rightharpoonup_{N\to+\infty} T$.

The cumulative distribution function F_2 is called the Tracy–Widom distribution, see Figure 8.

The general Fredholm formula for gap probabilities of determinantal point processes yields:

$$\mathbb{P}[T \leq t] = \mathbb{P}[M(t,+\infty) = 0] = \det\Big(I - \mathscr{K}^{\mathrm{Airy}}_{|(t,+\infty)}\Big)$$
$$= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int_{(t,+\infty)^m} \det(\mathrm{Ai}(x_i,x_j))_{1\leq i,j\leq m}\, dx_1 \cdots dx_m.$$

The connection between this Fredholm determinant of the Airy kernel and the Painlevé II equation is then detailed in [AGZ10, Section 3.8]; it is a bit mysterious.

3. Large random integer partitions

The sine and Airy kernels introduced in the previous section also drive the local asymptotics of large random integer partitions chosen according to certain probability measures which are related to the representation theory of the symmetric groups. This section explains this theory, and the asymptotic results for the so-called Plancherel measures.

3.1. **Plancherel and Schur measures.** Given a positive integer N, recall that an *integer partition* of size N is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ of positive integers which is non-increasing and such that $|\lambda| = \sum_{i=1}^{\ell} \lambda_i = N$. We shall denote $\ell = \ell(\lambda)$ the length of a partition, which is its number of non-zero parts. A partition is usually represented by its *Young diagram*, which is the array of boxes with λ_1 boxes on the first row, λ_2 boxes on the second row, *etc.*; see Figure 9 for an example. The integer partitions can be used to label many objects from algebra or combinatorics. In

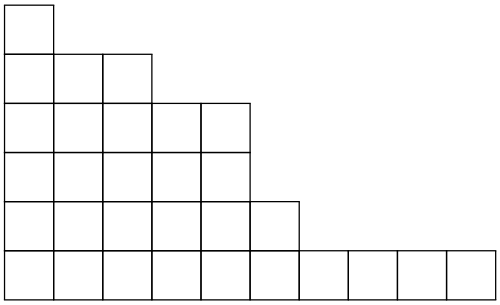

FIGURE 9. The Young diagram of the integer partition $\lambda = (10,6,5,5,3,1)$ with size $|\lambda| = 30$.

particular, the set $\mathfrak{Y}(N)$ of integer partitions of size N is in bijection with the conjugacy classes of permutations of size N; indeed, two permutations in $\mathfrak{S}(N)$ are conjugated if and only if they have the same cycle-type (sequence of the lengths of the disjoint cycles of the permutation). By standard results from the representation theory of finite groups, the isomorphism classes of (complex, linear) irreducible representations of $\mathfrak{S}(N)$ are also in bijection with $\mathfrak{Y}(N)$, and denoting V^λ the irreducible representation with label $\lambda \in \mathfrak{Y}(N)$ and $\dim \lambda = \dim_{\mathbb{C}}(V^\lambda)$, we have the isomorphism of $(\mathbb{C}\mathfrak{S}(N), \mathbb{C}\mathfrak{S}(N))$-bimodules

$$\mathbb{C}\mathfrak{S}(N) = \bigoplus_{\lambda\in\mathfrak{Y}(N)} V^\lambda \otimes_{\mathbb{C}} (V^\lambda)^*,$$

which leads to the combinatorial identity

$$N! = \sum_{\lambda\in\mathfrak{Y}(N)} (\dim\lambda)^2.$$

This formula has another explanation, related to the problem of the longest increasing subsequence of a permutation. Call *standard tableau of shape* λ a numbering of the $N = |\lambda|$ boxes of the Young

diagram of the partition λ by the integers of $[\![1, N]\!]$, such that each row and each column of the diagram is strictly increasing. For instance,

4			
2	5	8	
1	3	6	7

is a Young tableau of shape $(4, 3, 1)$. The study of the induction rules satisfied by the irreducible representations of the symmetric groups leads to the identity

$$\dim \lambda = \operatorname{card}(\mathrm{ST}(\lambda)),$$

where $\mathrm{ST}(\lambda)$ denotes the set of standard tableaux with shape λ; see for instance [Mél17, Proposition 3.8]. On the other hand, one can construct a bijection between:

- the permutations $\sigma \in \mathfrak{S}(N)$,
- and the pairs of standard tableaux (P, Q) with $\mathrm{shape}(P) = \mathrm{shape}(Q) = \lambda \in \mathfrak{Y}(N)$.

This bijection is known as the *Robinson–Schensted algorithm*; see [Mél17, Section 3.2]. It yields a combinatorial explanation of the formula $N! = \sum_{\lambda \in \mathfrak{Y}(N)} (\dim \lambda)^2$. The Robinson–Schensted algorithm has the following important property: if σ is a permutation and λ is the shape of the two associated standard tableaux P and Q, then for any $r \geq 1$,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = \max\{\ell(w_1) + \ell(w_2) + \cdots + \ell(w_r)\},$$

where the maximum runs over r-tuples of disjoint increasing subwords of the word of the permutation σ. In particular, the largest part λ_1 is the length of a longest increasing subword in σ.

The *Plancherel measure* of order N is the probability measure on $\mathfrak{Y}(N)$ given by

$$\mathbb{P}_N[\lambda] = \frac{(\dim \lambda)^2}{N!}.$$

By the previous discussion, it is also the image of the uniform probability measure on $\mathfrak{S}(N)$ by the Robinson–Schensted map. The Plancherel measure is the spectral measure of the regular trace on $\mathfrak{S}(N)$: if χ^λ is the normalised character of the irreducible representation V^λ, then for any permutation $\sigma \in \mathfrak{S}(N)$,

$$\sum_{\lambda \in \mathfrak{Y}(N)} \mathbb{P}_N[\lambda]\ \chi^\lambda(\sigma) = 1_{(\sigma = \mathrm{id})}.$$

In order to relate the study of random partitions to the theory of determinantal point processes, we shall need to consider Plancherel measures with a random size N, and more generally a family of probability measures on $\mathfrak{Y} = \bigsqcup_{N \in \mathbb{N}} \mathfrak{Y}(N)$ called *Schur measures*. Given an integer partition λ and variables $x_1, \ldots, x_M$ with $M \geq \ell(\lambda)$, we denote $s_\lambda(x_1, \ldots, x_M)$ the Schur polynomial with label λ, which is given by

$$s_\lambda(x_1, \ldots, x_M) = \frac{\det\left(x_i^{\lambda_j + M - j}\right)_{1 \leq i,j \leq M}}{\det\left(x_i^{M-j}\right)_{1 \leq i,j \leq M}}.$$

Here, it is understood that $\lambda_i = 0$ if $i \geq \ell(\lambda)$. This definition is compatible with the addition of a variable: $s_\lambda(x_1, \ldots, x_M, 0) = s_\lambda(x_1, \ldots, x_M)$. Therefore, we can consider s_λ as an element of the *algebra of symmetric functions*

$$\mathrm{Sym} = \varprojlim_{M \to \infty} \mathrm{Sym}^{(M)},$$

where $\mathrm{Sym}^{(M)}$ is the space of symmetric polynomials in M variables (say, with real coefficients). The projective limit is taken in the category of graded real algebras, and a *Schur function* $s_\lambda \in \mathrm{Sym}$

can be considered as a homogeneous symmetric polynomial of degree $|\lambda|$ with an infinite alphabet of variables $X = \{x_{i\geq 1}\}$. The Schur functions satisfy the *Cauchy identity* [Mél17, Theorem 2.18]:

$$\sum_{\lambda\in\mathfrak{Y}} s_\lambda(X)\, s_\lambda(Y) = \prod_{i,j} \frac{1}{1-x_i y_j} = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(X)\, p_k(Y)}{k}\right),$$

where $p_k(X) = \sum_i (x_i)^k$ is the k-th power sum. Suppose in particular that $X = \{x_{i\geq 1}\}$ and $Y = \{y_{j\geq 1}\}$ are two sets of non-negative real numbers such that the infinite product $\prod_{i,j\geq 1}(1-x_i y_j)$ converge. Then, the formula

$$\mathbb{P}_{X,Y}[\lambda] = \left(\prod_{i,j\geq 1}(1-x_i y_j)\right) s_\lambda(X)\, s_\lambda(Y)$$

defines a probability on $\mathfrak{Y}$ called the Schur measure with parameters X and Y. Actually, one can consider more general parameters by using specialisations of the algebra Sym which are non-negative on the basis of Schur functions. These *positive specialisations* have been classified by Thoma [Tho64], in connection with the representation theory of the infinite symmetric group $\mathfrak{S}(\infty)$; see [Mél17, Chapter 11] for details.

Theorem 3.1 (Thoma). *A morphism of algebras $f \in \mathrm{Sym} \to f(X) \in \mathbb{R}$ takes non-negative values on any Schur functions if and only if there exists $\gamma \geq 0$ and two non-increasing summable sequences $A = \{\alpha_{i\geq 1}\}$ and $B = \{\beta_{i\geq 1}\}$ of non-negative real numbers with*

$$p_1(X) = \gamma + \sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i;$$
$$p_{k\geq 2}(X) = \sum_{i=1}^{\infty} (\alpha_i)^k + (-1)^{k-1} \sum_{i=1}^{\infty} (\beta_i)^k.$$

Since $(p_k)_{k\geq 1}$ is an algebraic basis of Sym*, these formulas entirely determine the morphism of algebras, which is then denoted by the formal alphabet $X = A + B + \gamma E$.*

Example 3.2. The exponential alphabet E sends p_1 to 1 and all the other power sums $p_{k\geq 2}$ to 0. On the basis of Schur functions, one obtains

$$s_\lambda(E) = \frac{\dim \lambda}{|\lambda|!},$$

by using the Frobenius–Schur formula, which relates the Schur functions to the power sums and to the characters of the irreducible representations of the symmetric groups (see Theorem 2.32 in [Mél17]).

Given two formal alphabets $X = A_X + B_X + \gamma_X E$ and $Y = A_Y + B_Y + \gamma_Y E$ corresponding to positive specialisations of Sym, the Schur measure with parameters X and Y is the probability measure on $\mathfrak{Y}$ given by

$$\mathbb{P}_{X,Y}[\lambda] = \exp\left(-\sum_{k=1}^{\infty} \frac{p_k(X)\, p_k(Y)}{k}\right) s_\lambda(X)\, s_\lambda(Y).$$

Note that if one replaces X by tX and Y by $t^{-1}Y$ with $t \in \mathbb{R}_+^*$, then this does not change the probability measure $\mathbb{P}_{X,Y}$. An essential property of Schur measures is that they define random point processes on $\mathbb{Z} + \frac{1}{2}$ which are determinantal; see Theorem 3.8 hereafter.

Example 3.3. Take $X = \theta E$ and $Y = E$. Then, if $|\lambda| = N$, one obtains

$$\mathbb{P}_{X,Y}[\lambda] = \left(\mathrm{e}^{-\theta}\,\frac{\theta^N}{N!}\right)\left(\frac{(\dim\lambda)^2}{N!}\right) = \left(\mathrm{e}^{-\theta}\,\frac{\theta^N}{N!}\right)\mathbb{P}_N[\lambda],$$

which is a Poissonised Plancherel measure, with a size N distributed according to the Poisson distribution with parameter θ.

3.2. **Point processes and continuous curves associated to a partition.** Given any probability measure on $\mathfrak{Y} = \bigsqcup_{N\in\mathbb{N}} \mathfrak{Y}(N)$, a random integer partition λ chosen under $\mathbb{P}$ can be seen as a random point process on $\mathbb{Z} + \frac{1}{2}$ by using the *Russian convention* in order to draw its Young diagram. One rotates the Young diagram of λ by 45 degrees, and one draws it in such a way that the cells of the Young diagram have their area equal to 2. One also adds the half-lines $y = \pm x$ to this drawing, and one projects to the x-axis the coordinates of the middles of the decreasing segments of the upper boundary of the shape thus obtained; see Figure 10. The configuration of point $M_\lambda \subset \mathbb{Z} + \frac{1}{2}$ is called the set of *descent coordinates* of the partition λ, and it is given by the formula

$$M_\lambda = \left\{\lambda_i - i + \frac{1}{2}\right\}_{i\geq 1}.$$

The possible sets of descent coordinates are characterised by the following property: if $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2} = \mathbb{Z}'_- \sqcup \mathbb{Z}'_+$, then

$$\mathrm{card}\left(\mathbb{Z}'_- \setminus \left(M_\lambda \cap \mathbb{Z}'_-\right)\right) = \mathrm{card}\left(M_\lambda \cap \mathbb{Z}'_+\right) < +\infty.$$

In the next section, we shall see that the random point process M_λ with λ chosen according to a Schur measure is determinantal on $\mathbb{Z}'$.

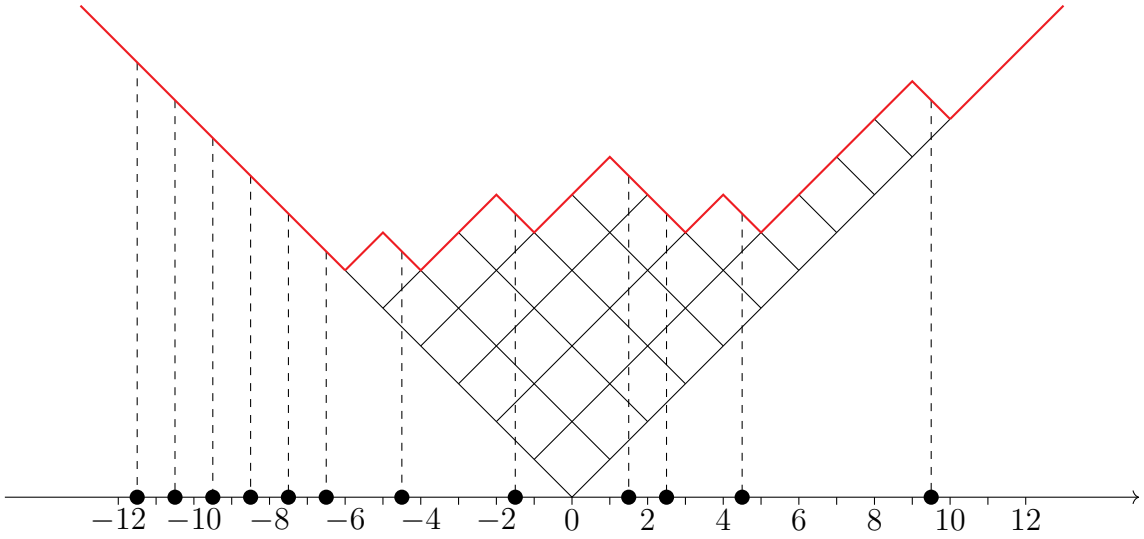

FIGURE 10. The point process M_λ on $\mathfrak{X} = \mathbb{Z} + \frac{1}{2}$ associated to the integer partition $\lambda = (10, 6, 5, 5, 3, 1)$.

The union of the two sets $M_\lambda \cap \mathbb{Z}'_+$ and $\mathbb{Z}'_- \setminus (M_\lambda \cap \mathbb{Z}'_-)$ is called the set of *Frobenius coordinates* of λ; we shall denote it F_λ. For instance, if $\lambda = (10, 6, 5, 5, 3, 1)$, then

$$F_\lambda = \left(-\frac{11}{2}, -\frac{7}{2}, -\frac{5}{2}, -\frac{1}{2}; \frac{3}{2}, \frac{5}{2}, \frac{9}{2}, \frac{19}{2}\right).$$

The sum of the absolute values of the Frobenius coordinates in F_λ is the size of the partition λ; indeed, these coordinates are the signed lengths of the rows and columns of λ measured from its diagonal. It turns out that given a random integer partition λ, the random point process M_λ is determinantal with a Hermitian kernel if and only if F_λ is determinantal with a J-Hermitian

kernel, J being the scalar product on $\ell^2(\mathbb{Z}') = \ell^2(\mathbb{Z}'_+) \oplus \ell^2(\mathbb{Z}'_-)$ associated to the infinite diagonal matrix $\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. This is related to some general properties of the determinantal point processes on a discrete space $\mathfrak{X}$, which we now explain. Fix a countable set $\mathfrak{X}$, endowed with the counting measure and with the discrete topology. Given a simple random point process F on $\mathfrak{X}$ and a finite subset $X \subset \mathfrak{X}$, we set

$$\pi(X) = \mathbb{P}\left[F = \sum_{x \in X} \delta_x\right]$$
$$\rho(X) = \sum_{X \subset Y \subset \mathfrak{X}} \pi(Y).$$

Note that if $X = \{x_1, \dots, x_n\}$, then

$$\rho(X) = \mathbb{E}\big[F^{\downarrow n}(\{x_1\} \times \{x_2\} \times \cdots \times \{x_n\})\big] = \rho_n(x_1, x_2, \dots, x_n)$$

is simply the n-th correlation function with respect to the counting measure.

Proposition 3.4 (Fermionic point processes on a discrete space)**.** *Consider a trace class non-negative Hermitian operator $\mathscr{L}$ on $\ell^2(\mathfrak{X})$, $\mathfrak{X}$ being a countable set. We assume that all the restricted operators $\mathscr{L}_X$ with X finite subset of $\mathfrak{X}$ have a non-negative determinant.*

(1) The formula

$$\pi(X) = \frac{\det \mathscr{L}_X}{\det(I + \mathscr{L})}$$

defines a probability measure on the set $\mathfrak{P}_{<\infty}(\mathfrak{X})$ of finite subsets of $\mathfrak{X}$.

(2) The corresponding finite simple random point process F on $\mathfrak{X}$ is determinantal, with kernel

$$\mathscr{K} = \mathscr{L}\,(I + \mathscr{L})^{-1}.$$

A proof of this result is given in [DV88, Section 5.4]; see also the discussion of [Ols98, Section 1]. Suppose now that one can split $\mathfrak{X}$ in two parts $\mathfrak{X}_+$ and $\mathfrak{X}_-$, and that the operator $\mathscr{L}$:

- is Hermitian with respect to the scalar product defined by the block-diagonal matrix

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

- has a block decomposition of the form

$$\mathscr{L} = \begin{pmatrix} 0 & \mathscr{D}^* \\ -\mathscr{D} & 0 \end{pmatrix}.$$

Then, the corresponding determinantal point process satisfies $\mathrm{card}(F \cap \mathfrak{X}_+) = \mathrm{card}(F \cap \mathfrak{X}_-)$ almost surely; see [Ols98, Proposition 1.7]. Moreover, the correlation kernel $\mathscr{K}$ writes then as

$$\mathscr{K} = \begin{pmatrix} (I + \mathscr{D}^*\mathscr{D})^{-1}\,\mathscr{D}^*\mathscr{D} & (I + \mathscr{D}^*\mathscr{D})^{-1}\,\mathscr{D}^* \\ -\mathscr{D}\,(I + \mathscr{D}^*\mathscr{D})^{-1} & \mathscr{D}\mathscr{D}^*\,(I + \mathscr{D}\mathscr{D}^*)^{-1} \end{pmatrix},$$

see [Ols98, Proposition 1.8 and Corollary 1.9].

Proposition 3.5 (Kerov's complementation principle)**.** *Let F be a finite determinantal point process on the discrete space $\mathfrak{X} = \mathfrak{X}_+ \sqcup \mathfrak{X}_-$, with a J-Hermitian kernel $\mathscr{K}_F$ whose block-diagonal matrix is*

$$\mathscr{K}^F = \begin{pmatrix} \mathscr{K}_{++} & \mathscr{K}_{+-} \\ \mathscr{K}_{-+} & \mathscr{K}_{--} \end{pmatrix}.$$

We define another random point process M by $M = F \,\Delta\, \mathfrak{X}_-$, where Δ is the symmetric difference of subsets of $\mathfrak{X}$. Then, M is also a determinantal point process on $\mathfrak{X}$, with a Hermitian kernel

$$\mathscr{K}^M = \begin{pmatrix} \mathscr{K}_{++} & \mathscr{K}_{+-} \\ -\mathscr{K}_{-+} & I - \mathscr{K}_{--} \end{pmatrix}.$$

Proof. We follow [BOO00, Proposition A.8]. Given $X \subset \mathfrak{X}$, we shall denote $X_+ = X \cap \mathfrak{X}_+$ and $X_- = X \cap \mathfrak{X}_-$. We compute by inclusion-exclusion

$$\begin{aligned}
\rho^M(X) &= \sum_{X \subset Y \subset \mathfrak{X}} \pi^M(Y) = \sum_{X \subset Y \subset \mathfrak{X}} \pi^F(Y \,\Delta\, \mathfrak{X}_-) \\
&= \sum_{X \subset Y \subset \mathfrak{X}} \mathbb{P}[F_+ = Y_+ \text{ and } F_- = \mathfrak{X}_- \setminus Y_-] \\
&= \sum_{Z_- \subset \mathfrak{X}_- \setminus X_-} \mathbb{P}[X_+ \subset F_+ \text{ and } Z_- = F_-] \\
&= \sum_{Z_- \subset \mathfrak{X}_- \setminus X_-} (-1)^{|Z_-|}\, \mathbb{P}[X_+ \subset F_+ \text{ and } Z_- \subset F_-] \\
&= \sum_{Z_- \subset \mathfrak{X}_- \setminus X_-} (-1)^{|Z_-|}\, \rho^F(X_+ \sqcup Z_-),
\end{aligned}$$

this identity being true for any X such that $\mathfrak{X}_- \setminus X_-$ is finite. As can be seen by expanding by multilinearity, this is precisely the determinant of $\mathscr{K}^M_X$, since $\mathscr{K}^M = \left(\begin{smallmatrix} \mathscr{K}_{++} & \mathscr{K}_{+-} \\ -\mathscr{K}_{-+} & -\mathscr{K}_{--} \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$. □

If $\mathscr{K}^F$ is associated to $\mathscr{L}^F = \left(\begin{smallmatrix} 0 & \mathscr{D}^* \\ -\mathscr{D} & 0 \end{smallmatrix}\right)$, then $\mathscr{K}^M$ writes as

$$\mathscr{K}^M = \begin{pmatrix} (I + \mathscr{D}^*\mathscr{D})^{-1}\,\mathscr{D}^*\mathscr{D} & (I + \mathscr{D}^*\mathscr{D})^{-1}\,\mathscr{D}^* \\ \mathscr{D}\,(I + \mathscr{D}^*\mathscr{D})^{-1} & (I + \mathscr{D}\mathscr{D}^*)^{-1} \end{pmatrix}.$$

This theory applies readily to the case where $F = F_\lambda$ and $M = M_\lambda$, the set $\mathfrak{X}$ being $\mathbb{Z}'$; we shall see in a moment what form the kernel $\mathscr{K}^M$ takes when λ is chosen according to a Schur measure.

In the sequel, we shall also look at *scaled* Young diagrams, and prove a law of large numbers which is the pendant of Theorem 2.4 for Plancherel measures; see Theorem 3.6 below. The scaling is performed as follows. Suppose that λ is an integer partition with size $N \geq 1$. The upper boundary of the diagram of λ drawn with the Russian convention is a piecewise affine function $\omega_\lambda : \mathbb{R} \to \mathbb{R}_+$ such that:

(1) $\omega_\lambda(s) \geq |s|$ for any $s \in \mathbb{R}$;

(2) $\omega_\lambda(s) = |s|$ for $|s|$ large enough;

(3) $\int_{\mathbb{R}} \frac{\omega_\lambda(s) - |s|}{2}\, ds = N$.

We set $\overline{\omega}_\lambda(s) = \omega_\lambda(\sqrt{N}s)/\sqrt{N}$; this amounts to rescaling the Young diagram of λ in both directions by a factor $|\lambda|^{-\frac{1}{2}}$. Therefore, $\overline{\omega}_\lambda$ satisfies the two first properties above, and

$$\int_{\mathbb{R}} \frac{\overline{\omega}_\lambda(s) - |s|}{2}\, ds = 1.$$

We introduce the space of continuous Young diagrams $\mathscr{Y}$:

$$\mathscr{Y} = \{\omega : \mathbb{R} \to \mathbb{R}_+ \mid \omega \text{ is Lipschitz with constant 1, and } \omega(s) = |s| \text{ for } |s| \text{ large enough}\}.$$

The maps $\lambda \mapsto \omega_\lambda$ and $\lambda \mapsto \overline{\omega}_\lambda$ allow one to embed $\mathfrak{Y}$ into $\mathscr{Y}$. We endow $\mathscr{Y}$ with the topology of uniform convergence on $\mathbb{R}$.

Theorem 3.6 (Logan–Shepp, Kerov–Vershik). *Let λ_N be a random integer partition chosen under the Plancherel measure $\mathbb{P}_N$. As N goes to infinity, $\overline{\omega}_{\lambda_N}$ converges in probability towards the continuous Young diagram*

$$\Omega(s) = \begin{cases} \frac{2}{\pi}\left(s \arcsin \frac{s}{2} + \sqrt{4 - s^2}\right) & \text{if } |s| < 2, \\ |s| & \text{if } |s| \geq 2. \end{cases}$$

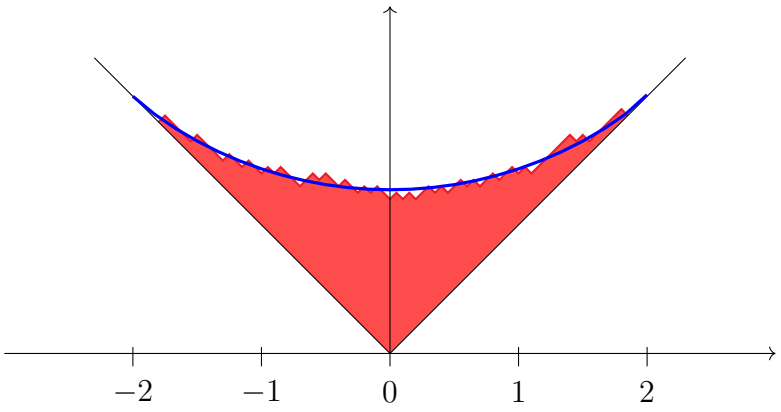

FIGURE 11. A random integer partition of size $N = 400$ under the Plancherel measure.

This result is due separately to Logan and Shepp [LS77] and to Kerov and Vershik [KV77]. We refer to Figure 11 for a drawing of a scaled random Young diagram chosen according to $\mathbb{P}_{N=400}$ (in red), and of the limit shape Ω (in blue). There is a deep connection between the Logan–Shepp–Kerov–Vershik curve Ω and the Wigner semicircle law, relying on the so-called *Markov–Krein correspondence of measures*. Denote ν the semicircle law, and μ the signed measure given by

$$\mu(ds) = -\left(\frac{\Omega(s) - |s|}{2}\right)' ds,$$

which has a density between -1 and 1 (this definition makes sense for any continuous Young diagram). The *Cauchy transform* of ν is

$$\int_{\mathbb{R}} \frac{1}{z - s}\, \nu(ds) = \frac{2}{z + \sqrt{z^2 - 4}}.$$

This function is well defined on the complex upper-half plane, and at infinity, it expands as $\frac{1+o(1)}{z}$. The numerator of this expansion is in fact the exponential of the Cauchy transform of μ:

$$\int_{\mathbb{R}} \frac{1}{z - s}\, \nu(ds) = \frac{1}{z} \exp\left(\int_{\mathbb{R}} \frac{1}{z - s}\, \mu(ds)\right).$$

We refer to [Mél17, Section 7.4] for details on this correspondence.

3.3. **Correlation kernel of a Schur measure.** In order to prove that the Schur measures yield determinantal point processes, Okounkov used an elementary but extremely powerful representation of the Schur functions by operators on the *infinite wedge space* [Oko01a]. The space $\bigwedge^\infty$ is the Hilbert space spanned by the orthonormal basis

$$v_M = x_1 \wedge x_2 \wedge \cdots \wedge x_n \wedge \cdots,$$

where $M = \{x_1 > x_2 > \cdots > x_n > \cdots\}$ is an infinite decreasing sequence in $\mathbb{Z}'$ which contains all the sufficiently large negative half-integers. Among these vectors, the *vacuum state* is

$$v_\emptyset = -\frac{1}{2} \wedge -\frac{3}{2} \wedge -\frac{5}{2} \wedge \cdots;$$

it corresponds to the empty integer partition, and more generally, we shall denote $v_\lambda = v_{M_\lambda}$. The half-integers $x \in \mathbb{Z}'$ act on $\bigwedge^\infty$ by the *free fermion operators* ψ_x and ψ_x^*:

$$\psi_x(v_M) = \begin{cases} x \wedge v_M & \text{if } x \notin M \\ 0 & \text{if } x \in M \end{cases} \quad ; \quad \psi_x^*(v_M) = \begin{cases} \varepsilon_{x,M}\, v_{M\setminus\{x\}} & \text{if } x \in M \\ 0 & \text{if } x \notin M \end{cases}$$

with the usual rules of anti-commutation for the $\wedge$ symbols (in order to replace x inside a decreasing sequence), and where $-\varepsilon_{x,M}$ is the parity of the position of x in the decreasing sequence M. We have the anti-commutation formula $\psi_x\psi_x^* + \psi_x^*\psi_x = \mathrm{id}_{\bigwedge^\infty}$. The infinite wedge space is also equipped with the *charge and energy operators*:

$$C = \sum_{x>0} \psi_x\psi_x^* - \sum_{x<0} \psi_x^*\psi_x;$$
$$H = \sum_{x>0} x\, \psi_x\psi_x^* - \sum_{x<0} x\, \psi_x^*\psi_x.$$

Note that $C(v_\lambda) = 0$ and $H(v_\lambda) = |\lambda|\, v_\lambda$ for any integer partition λ; the kernel of C is exactly the span of the vectors v_λ with $\lambda \in \mathfrak{Y}$. Consider an infinite sequence of formal parameters $t = (t_1, t_2, \ldots)$, and the operator

$$\Gamma(t) = \exp\left(\sum_{k=1}^\infty t_k\, \alpha_k\right), \quad \text{with } \alpha_k = \sum_{l\in\mathbb{Z}'} \psi_{l+k}\psi_l^*.$$

Proposition 3.7 (Schur functions and the infinite wedge space)**.** *Denote X the specialisation of the algebra* Sym *for which t is the sequence of* Miwa parameters*: $p_k(X) = k\, t_k$ for any $k \geq 1$. Then, the action of the operator $\Gamma(t)$ on* $\ker C$ *is given by*

$$\langle \Gamma(t)(v_\mu) \mid v_\lambda \rangle = \begin{cases} s_{\lambda\setminus\mu}(X) & \text{if } \mu \subset \lambda, \\ 0 & \text{otherwise}, \end{cases}$$

where $s_{\lambda\setminus\mu}$ is the skew Schur function associated to the pair of partitions $(\mu \subset \lambda)$. In particular,

$$\Gamma(t)(v_\emptyset) = \sum_{\lambda\in\mathfrak{Y}} s_\lambda(X)\, v_\lambda.$$

We refer to [Kac90, Chapter 14] and [Oko01a, Appendix A] for a proof of this formula for the operators $\Gamma(t)$, and to [Mél17, Definition 2.21] for details on skew Schur functions (we shall not use them in the sequel). Consider now two positive specialisations X and Y of the algebra Sym, and the associated sequences of Miwa parameters t_X and t_Y. The correlation function ρ^M of the random point process M_λ with $\lambda \sim \mathbb{P}_{X,Y}$ is given by:

$$\begin{aligned} \rho^M(A) = \sum_{A\subset B} \pi^M(A) &= \exp\left(-\sum_{k=1}^\infty k\, t_{X,k}\, t_{Y,k}\right) \sum_{A\subset M_\lambda} s_\lambda(X)\, s_\lambda(Y) \\ &= \exp\left(-\sum_{k=1}^\infty k\, t_{X,k}\, t_{Y,k}\right) \left\langle \Gamma(t_X)\,(v_\emptyset) \,\middle|\, \left(\prod_{a\in A} \psi_a\psi_a^*\right) \Gamma(t_Y)(v_\emptyset) \right\rangle \\ &= \exp\left(-\sum_{k=1}^\infty k\, t_{X,k}\, t_{Y,k}\right) \left\langle v_\emptyset \,\middle|\, \Gamma(t_X)^* \left(\prod_{a\in A} \psi_a\psi_a^*\right) \Gamma(t_Y)\,(v_\emptyset) \right\rangle. \end{aligned}$$

Besides, we have $\Gamma(t)^*(v_\emptyset) = v_\emptyset$, and $\Gamma(t_X)^*\,\Gamma(t_Y) = \exp(\sum_{k=1}^\infty k\, t_{X,k}\, t_{Y,k})\,\Gamma(t_Y)\,\Gamma(t_X)^*$, so

$$\rho^M(A) = \left\langle v_\emptyset \,\middle|\, \Gamma(t_X)^*\,\Gamma(-t_Y) \left(\prod_{a\in A} \psi_a\psi_a^*\right) \Gamma(t_Y)\,\Gamma(-t_X)^*\,(v_\emptyset) \right\rangle = \left\langle v_\emptyset \,\middle|\, \left(\prod_{a\in A} \Psi_a\Psi_a'\right)(v_\emptyset) \right\rangle,$$

where $\Psi_a = \Gamma(t_X)^* \, \Gamma(-t_Y) \, \psi_a \, \Gamma(t_Y) \, \Gamma(-t_X)^*$ and $\Psi'_a = \Gamma(t_X)^* \, \Gamma(-t_Y) \, \psi_a^* \, \Gamma(t_Y) \, \Gamma(-t_X)^*$ (beware that Ψ'_a is not the same as the adjoint operator Ψ_a^*). By using an analogue of the Wick principle, one can rewrite the last formula in a determinantal form, see [Oko01a, Theorem 1].

Theorem 3.8 (Okounkov). *Fix two positive specialisations X and Y of the algebra* Sym. *We set*

$$K^M_{X,Y}(x,y) = \left\langle v_\emptyset \,\middle|\, \Psi_x \Psi'_y(v_\emptyset) \right\rangle,$$

where $\Psi_a = \Gamma(t_X)^* \, \Gamma(-t_Y) \, \psi_a \, \Gamma(t_Y) \, \Gamma(-t_X)^*$ and $\Psi'_a = \Gamma(t_X)^* \, \Gamma(-t_Y) \, \psi_a^* \, \Gamma(t_Y) \, \Gamma(-t_X)^*$. The Schur measure $\mathbb{P}_{X,Y}$ defines a determinantal point process M_λ on $\mathbb{Z}'$ with correlation functions

$$\rho^M(A) = \det\left(K^M_{X,Y}(x,y)\right)_{x,y\in A}$$

for any $A \subset \mathbb{Z}'$ finite subset.

One can write down a very simple formula for the double generating of the kernel $K^M_{X,Y}$. Consider two formal variables z and w, and the associated formal series

$$\mathscr{K}^M_{X,Y}(z,w) = \sum_{x,y\in\mathbb{Z}'} z^x \, w^{-y} \, K^M_{X,Y}(x,y).$$

If $\psi(z) = \sum_{x\in\mathbb{Z}'} z^x \, \psi_x$ and $\psi^*(w) = \sum_{y\in\mathbb{Z}'} w^{-y} \, \psi_y^*$, we have

$$\begin{aligned}\mathscr{K}^M_{X,Y}(z,w) &= \langle v_\emptyset \,|\, \Gamma(t_X)^* \, \Gamma(-t_Y) \, \psi(z) \, \psi^*(w) \, \Gamma(t_Y) \, \Gamma(-t_X)^* \, (v_\emptyset)\rangle \\ &= \frac{J_{X,Y}(z)}{J_{X,Y}(w)} \, \langle v_\emptyset \,|\, \psi(z) \, \psi^*(w) \, (v_\emptyset)\rangle,\end{aligned}$$

where

$$J_{X,Y}(z) = \exp\left(\sum_{k=1}^\infty t_{X,k} \, z^k - \sum_{k=1}^\infty t_{Y,k} \, z^{-k}\right) = \frac{H(X,z)}{H(Y,z^{-1})}$$

is the Bessel function associated to the two specialisations X and Y, which comes from the generating series of homogeneous symmetric functions $H(X,z) = \sum_{n=0}^\infty h_n(X) \, z^n$. Finally, we have trivially

$$\langle v_\emptyset \,|\, \psi(z) \, \psi^*(w) \, (v_\emptyset)\rangle = \sum_{x\in\mathbb{Z}'_-} \left(\frac{z}{w}\right)^x = \frac{\sqrt{zw}}{z-w},$$

so we conclude:

Corollary 3.9 (Schur measures and generalised Bessel functions). *The generating series of the kernel $K^M_{X,Y}$ of the determinantal point process defined by a Schur measure with parameters X and Y is*

$$\mathscr{K}^M_{X,Y}(z,w) = \frac{\sqrt{zw}}{z-w} \, \frac{J_{X,Y}(z)}{J_{X,Y}(w)},$$

where $J_{X,Y}(z) = \frac{H(X,z)}{H(Y,z^{-1})}$.

One can recover the coefficients $K^M_{X,Y}(x,y)$ of the kernel of the determinantal point process either by using a double contour integral, or by expanding the generalised Bessel functions $J_{X,Y}$ in Laurent series. Hence, if $J_{X,Y}(z) = \sum_{n\in\mathbb{Z}} J_{X,Y,n} \, z^n$, one obtains

$$K^M_{X,Y}(x,y) = \sum_{k\in\mathbb{Z}'_+} J_{X,Y,x+k} \, J_{-X,-Y,-y-k},$$

where $-X$ and $-Y$ are the specialisations of Sym with opposite Miwa parameters in comparison to X and Y.

Example 3.10. Consider the Poissonised Plancherel measure

$$\mathbb{P}_{\mathcal{P}(\theta)} = \sum_{N=0}^{\infty} \frac{\mathrm{e}^{-\theta}\,\theta^N}{N!}\,\mathbb{P}_N;$$

it is the Schur measure associated to the specialisations $X = Y = \sqrt{\theta}E$. In the sequel, we use an index $\{\theta\}$ for all the functions which describe the correlations of the determinantal point process $M_{\{\theta\}}$. We have $H_{\{\theta\}}(X, z) = \exp(\sqrt{\theta}z)$, so $J_{\{\theta\}}(z) = \exp(\sqrt{\theta}(z - z^{-1}))$. Hence,

$$\mathscr{K}_{\{\theta\}}(z, w) = \frac{\sqrt{zw}}{z - w}\,\exp\left(\sqrt{\theta}((z - z^{-1}) - (w - w^{-1}))\right).$$

We recover the value of $K_{\{\theta\}}(x, y)$ by taking a double contour integral along two arbitrary non-crossing paths circling around 0:

$$K_{\{\theta\}}(x, y) = \frac{1}{(2\mathrm{i}\pi)^2}\oint\!\!\oint \frac{1}{(z - w)\sqrt{zw}}\,\exp\left(\sqrt{\theta}(z - z^{-1}) - (w - w^{-1})\right) z^{-x}w^y\,dz\,dw.$$

The saddle point analysis of this formula will provide us with asymptotic results for the Poissonised Plancherel measures. Note that a similar approach can be used in order to study *any* Schur measure. The kernel $K_{\{\theta\}}(x, y)$ is called the *discrete Bessel kernel* (with parameter θ), the terminology coming from the identity

$$K_{\{\theta\}}(x, y) = \sqrt{\theta}\,\frac{J_{x-\frac{1}{2}}(2\sqrt{\theta})\,J_{y+\frac{1}{2}}(2\sqrt{\theta}) - J_{x+\frac{1}{2}}(2\sqrt{\theta})\,J_{y-\frac{1}{2}}(2\sqrt{\theta})}{x - y},$$

where

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m + \alpha + 1)}\left(\frac{z}{2}\right)^{2m+\alpha}$$

is the standard Bessel function of the first kind (see Equation (23) in [Oko01b]). The origin of this second formula is the following. The function $J_{\{\theta\}}(z)$ is here equal to $\sum_{n\in\mathbb{Z}} J_n(2\sqrt{\theta})\,z^n$. Therefore, the general formula for the coefficients $K_{\{\theta\}}(x, y)$ yields

$$K_{\{\theta\}}(x, y) = \sum_{k\in\mathbb{Z}'_+} J_{x+k}\left(2\sqrt{\theta}\right)\,J_{-y-k}\left(-2\sqrt{\theta}\right).$$

This can be rewritten as a Christoffel–Darboux type ratio by doing an integration by parts in the integral formula for $K_{\{\theta\}}(x, y)$. On the other hand, a similar expression for the kernel of the random point process $F_{\{\theta\}}$ of Frobenius coordinates can be derived by using the complementation principle from Proposition 3.5; see Theorem 1 in [BOO00].

3.4. **Asymptotics of the Plancherel measures.** By saddle point analysis of the discrete Bessel kernel, one can compute the asymptotics of a large random integer partition chosen according to a Poissonised Plancherel measure $\mathbb{P}_{\mathcal{P}(\theta)}$; a *de-Poissonisation procedure* yields then the corresponding results for the Plancherel measures $\mathbb{P}_N$. We start from with the bulk asymptotics, and we fix a parameter $x_0 \in (-2, 2)$. As in Section 2, we renormalise the random point process $M_{\{\theta\}}$, and we set

$$M_{\{\theta\}}^{\mathrm{local},x_0}(B) = M_{\{\theta\}}\left(\lfloor x_0\sqrt{\theta}\rfloor + B\right).$$

In opposition to the case of eigenvalues of random matrices, $M_{\{\theta\}}^{\mathrm{local},x_0}$ is a random point process on $\mathbb{Z}'$ (instead of $\mathbb{R}$). The limit in the sense of Proposition 2.9 will thus be a random point process on $\mathbb{Z}'$.

Theorem 3.11 (Borodin–Okounkov–Olshanski). *For $x_0 \in (-2,2)$, we denote $\phi_0 = \arccos(\frac{x_0}{2}) \in (0,\pi)$. As θ goes to infinity, $M_{\{\theta\}}^{\mathrm{local},x_0}$ converges towards the determinantal point process with* discrete sine kernel

$$K^{\mathrm{dsine},x_0}(x,y) = \frac{\sin\phi_0(x-y)}{\pi(x-y)},$$

the reference measure being the counting measure on $\mathbb{Z}'$.

The proof of this result relies on a two-dimensional saddle point analysis, which is a common procedure when studying random particle processes; we are indebted to [Oko01b, Section 3] for the details of the computation of the limit. Since the limiting kernel is translation invariant, the theorem is equivalent to the following result: if x_θ, y_θ are half-integers such that

$$\frac{x_\theta}{\sqrt{\theta}} \to_{\theta\to\infty} x_0 \quad ; \quad \frac{y_\theta}{\sqrt{\theta}} \to_{\theta\to\infty} x_0 \quad ; \quad x_\theta - y_\theta = x - y,$$

then $K_{\{\theta\}}(x_\theta, y_\theta) \to_{\theta\to\infty} K^{\mathrm{dsine},x_0}(x,y)$. The discrete Bessel kernel writes then as

$$K_{\{\theta\}}(x_\theta,y_\theta) = \frac{1}{(2\mathrm{i}\pi)^2} \oint\!\!\oint \frac{1}{(z-w)\sqrt{zw}} \exp\left(\sqrt{\theta}\,F\left(z,\frac{x_\theta}{\sqrt{\theta}}\right) - \sqrt{\theta}\,F\left(w,\frac{y_\theta}{\sqrt{\theta}}\right)\right) dz\,dw,$$

where $F(z,t) = z - z^{-1} - t\log z$, and where the double contour integral is taken for instance over two circles with radii $|z| > |w|$. We start the saddle point analysis by determining the critical points of $F(\cdot, x_0)$. They are given by

$$1 + \frac{1}{z^2} = \frac{x_0}{z} \qquad ; \qquad z = \frac{x_0 \pm \mathrm{i}\sqrt{4-(x_0)^2}}{2} = \mathrm{e}^{\pm\mathrm{i}\phi_0}.$$

Moreover, on the circle with radii 1, $\mathrm{Re}(F(z,x_0)) = 0$, and the gradient of $\mathrm{Re}(F(z,x_0))$ is given by

$$\nabla(\mathrm{Re}\,F(\cdot,x_0))(\mathrm{e}^{\mathrm{i}\theta}) = 2\,(\cos\theta - \cos\phi_0)\,u_\theta,$$

where u_θ is the unit vector $\mathrm{e}^{\mathrm{i}\theta}$ (perpendicular to the unit circle at $z = \mathrm{e}^{\mathrm{i}\theta}$).

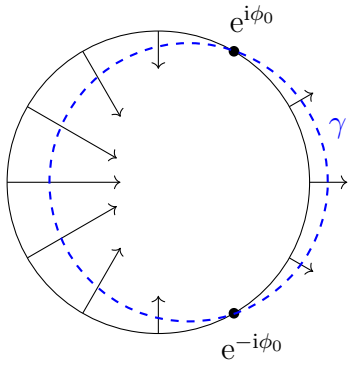

FIGURE 12. Gradient of the function $\mathrm{Re}\,F(\cdot,x_0)$ on the unit circle, and deformation of the unit circle for the steepest descent method.

Consequently, if we have a path γ which is a deformation of the unit circle crossing it at the two points $\mathrm{e}^{\pm\mathrm{i}\phi_0}$ according to Figure 12, then the steepest descent method applies to

$$\oint_\gamma \mathrm{e}^{-\sqrt{\theta}\,F(w,x_0)}\,dw,$$

and this integral goes to 0 as θ goes to infinity. We now go back to the integral form of $K_{\{\theta\}}(x_\theta,y_\theta)$, and we start with the two following contours: $|z| = 1$ (path γ_z) and $|w| = 1-\varepsilon$ (path γ_w). When

we deform γ_w into γ, we obtain an integral which is asymptotically small, but because of the ratio $\frac{1}{z-w}$, we pick up a residue each time $z = w$. This residue is equal to

$$\frac{1}{(2\mathrm{i}\pi)\, z^{x-y+1}},$$

and we have to integrate it over the part of the path γ_z which is crossed by γ during the deformation $\gamma_w \to \gamma$ (the part in red in Figure 13).

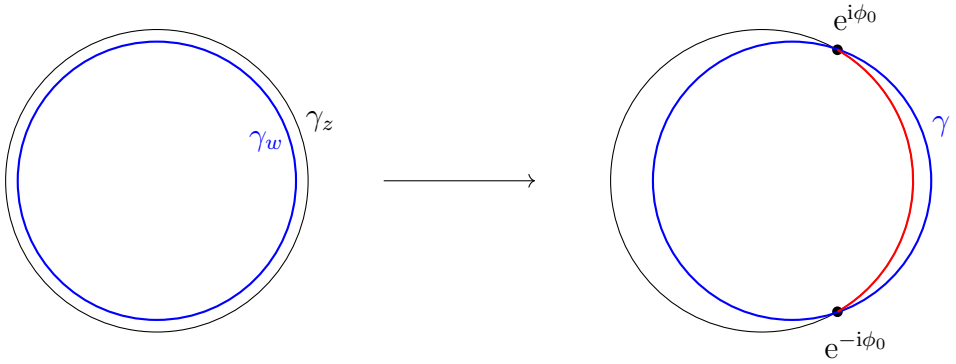

FIGURE 13. Deformation of γ_w into γ, which picks up a residue $\frac{1}{(2\mathrm{i}\pi)\, z^{x-y+1}}$ along the red curve.

So, we conclude that

$$\lim_{\theta\to\infty} K_{\{\theta\}}(x_\theta, y_\theta) = \frac{1}{2\mathrm{i}\pi} \oint_{\gamma_{\mathrm{red}}:\mathrm{e}^{-\mathrm{i}\phi_0} \to \mathrm{e}^{\mathrm{i}\phi_0}} \frac{1}{z^{x-y+1}}\, dz = \frac{\sin\phi_0 (x-y)}{\pi(x-y)}.$$

An analysis analogue to the one of Proposition 2.7 shows on the other hand that if $|x_0| > 2$, then the same kernel converges exponentially fast towards 0. The discrete sine kernel from Theorem 3.11 is related to the Logan–Shepp–Kerov–Vershik curve from Theorem 3.6 by the following computation. By taking $x_\theta = y_\theta$, we observe that the expected number of descent coordinates of the random partition $\lambda \sim \mathbb{P}_{\mathcal{P}(\theta)}$ in an interval of size L around $x_0\sqrt{\theta}$ is asymptotically equivalent

$$L\,\frac{\phi_0}{\pi} = \frac{L}{\pi}\,\arccos\left(\frac{x_0}{2}\right)$$

for $|x_0| < 2$. In terms of the random scaled diagram $\overline{\omega}_\lambda$, this means that

$$(\overline{\omega}_\lambda)'(s) \simeq_{\theta\to\infty} \frac{2}{\pi}\left(\frac{\pi}{2} - \arccos\left(\frac{s}{2}\right)\right) = \frac{2}{\pi}\,\arcsin\left(\frac{s}{2}\right),$$

which is exactly the derivative $\Omega'(s)$; see [BOO00, Remark 1.7] for more details.

Remark 3.12. Theorem 3.11 can be restated as follows. Let $X_N = (x_{N,1}, \ldots, x_{N,n})$ be a *regular sequence* of half-integers: each ratio $\frac{x_{N,i}}{\sqrt{N}}$ and each difference $x_{N,i} - x_{N,j}$ has a finite or infinite limit. If the differences have finite limits d_{ij} and the ratios all converge to the same quantity x_0, then

$$\rho_{\{N\},n}(x_{N,1}, \ldots, x_{N,n}) \to_{N\to\infty} \det\left(\frac{\sin(\phi_0\, d_{ij})}{\pi\, d_{ij}}\right)_{1\le i,j\le n},$$

where $\rho_{\{N\},n}$ is the n-th correlation function of the Poissonised Plancherel point process $M_{\{N\}}$. More generally, if the ratios have different limits, then one has asymptotic independence of the corresponding parts of the point process [BOO00, Theorem 3]. Hence, if $X_N = X'_N \sqcup X''_N$ with $\lim_{N\to\infty} d(X'_N, X''_N) = +\infty$ and $|X'_N| = n_1$, $|X''_N| = n_2$, then

$$\rho_{\{N\},n}(X_N) \simeq_{N\to\infty} \rho_{\{N\},n_1}(X'_N)\, \rho_{\{N\},n_2}(X''_N);$$

indeed, the saddle point analysis yields $\lim_{N\to\infty} K_{\{N\}}(x_{N,i}, x_{N,j}) = 0$ if $d_{ij} = +\infty$.

To prove that the non-Poissonised random partitions under the Plancherel measures $\mathbb{P}_N$ satisfy the same limit theorem in the bulk as their Poissonised counterparts (and therefore, the law of large numbers 3.6 for the scaled random diagrams), we proceed to a de-Poissonisation of the correlation functions. More precisely, given a regular sequence X_N of size n, we have

$$\rho_{\{\theta\},n}(X_N) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\theta}\,\theta^k}{k!}\,\rho_{k,n}(X_N),$$

and this formula makes sense even if θ is a complex number. We then recover $\rho_{N,n}(X_N)$ by a contour integration:

$$\rho_{N,n}(X_N) = \frac{N!}{2\mathrm{i}\pi}\oint \rho_{\{z\},n}(X_N)\,\frac{\mathrm{e}^z dz}{z^{N+1}}.$$

We take for contour the circle $|z| = N$, on which the function $z - N\log z$ admits a critical point at $z = N$. A sufficient condition in order to have $\rho_{N,n}(X_N) \simeq \rho_{\{N\},n}(X_N)$ is given by the following:

Lemma 3.13. *Let $(f_{N,k})_{N,k\in\mathbb{N}}$ be a bounded family of real numbers, and $f_N(z) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-z}z^k}{k!}\,f_{N,k}$. We assume that:*

- $\max_{|z|=N}(\mathrm{Re}\log f_N(z)) = O(\sqrt{N})$.
- *there exists a constant f_∞ such that we have the following estimate:*

$$\max_{|z-N|\le N^{1-\varepsilon}} \frac{|f_N(z) - f_\infty|}{\mathrm{e}^{\frac{\gamma|z-N|}{\sqrt{N}}}} = o(1)$$

for some $\varepsilon \in (0,1)$ and $\gamma > 0$.

Then, $\lim_{k\to\infty} f_{N,N} = f_\infty$.

We refer to [BOO00, Lemma 3.1] for a proof of this lemma, which relies on classical arguments of asymptotic analysis of integrals. It can be applied to the family $f_{N,k} = \rho_{\{k\},n}(X_N)$, with f_∞ given by the determinant of the discrete sine kernel; the details of this computation are not really interesting.

Let us now look at the edge of the random point process, and consider

$$M^{\mathrm{edge}}_{\{\theta\}}(B) = M_{\{\theta\}}\left(2\sqrt{\theta} + \theta^{1/6}\,B\right),$$

which is a random point process on $\mathbb{R}$. The kernel of this rescaled determinantal point process is

$$K^{\mathrm{edge}}_{\{\theta\}}(x,y) = I(x,\theta)I(y,\theta)\;\theta^{1/3}\;\frac{J_{r_\theta+x\theta^{1/6}-\frac{1}{2}}(r_\theta)\,J_{r_\theta+y\theta^{1/6}+\frac{1}{2}}(r_\theta) - J_{r_\theta+x\theta^{1/6}+\frac{1}{2}}(r_\theta)\,J_{r_\theta+y\theta^{1/6}-\frac{1}{2}}(r_\theta)}{x-y},$$

where $r_\theta = 2\sqrt{\theta}$ and $I(x,\theta)$ is the indicator function of the condition $2\sqrt{\theta} + x\theta^{1/6} \in \mathbb{Z}'$. By using as before the integral representation

$$J_n(2t) = \frac{1}{2\mathrm{i}\pi}\oint \mathrm{e}^{t(z-z^{-1})}\,\frac{dz}{z^{n+1}}$$

and by performing a saddle point analysis with a critical point of order 2 as in the second part of Section 2.4, one obtains the following limit formula:

$$\lim_{\theta\to\infty} \sqrt{\theta}\;\frac{J_{r_\theta+x\theta^{1/6}-\frac{1}{2}}(r_\theta)\,J_{r_\theta+y\theta^{1/6}+\frac{1}{2}}(r_\theta) - J_{r_\theta+x\theta^{1/6}+\frac{1}{2}}(r_\theta)\,J_{r_\theta+y\theta^{1/6}-\frac{1}{2}}(r_\theta)}{x-y} = K^{\mathrm{Airy}}(x,y).$$

It follows that for any family $(x_1,\dots,x_n)$,

$$\rho^{\mathrm{edge}}_{\{\theta\},n}(x_1,\dots,x_n) \simeq_{\theta\to\infty} \left(\prod_{i=1}^{n} I(x_i,\theta)\,\theta^{1/6}\right)\;\det\left(K^{\mathrm{Airy}}(x_i,x_j)\right)_{1\le i,j\le n}.$$

In the weak sense, $I(x,\theta)\,\theta^{1/6}$ converges to the Lebesgue measure dx, so one is led to the following result, which is the analogue for random partitions of Theorem 2.11:

Theorem 3.14 (Borodin–Okounkov–Olshanski)**.** *As θ goes to infinity, $M^{\mathrm{edge}}_{\{\theta\}}$ converges towards the Airy point process.*

Again, a de-Poissonisation procedure yields the same result for

$$M_N^{\mathrm{edge}}(B) = M_N\left(2\sqrt{N} + N^{1/6}B\right);$$

the details of these computations are given in [BOO00, Section 4]. As a corollary, one obtains the following answer to Ulam's problem of the longest increasing subsequence in a uniform random permutation; the result appeared first in [BDJ99], and was reproved by different means in [Oko00; Joh01].

Corollary 3.15 (Baik–Deift–Johansson)**.** *Let λ be a random partition chosen under the Plancherel measure $\mathbb{P}_N$. As N goes to infinity, the rescaled first part λ_1, which has the law of a longest increasing subsequence or a random permutation in $\mathfrak{S}(N)$, satisfies the following limit theorem:*

$$\frac{\lambda_1 - 2\sqrt{N}}{N^{1/6}} \rightharpoonup_{N\to\infty} \mathrm{TW},$$

where TW *is the Tracy–Widom distribution from Theorem 2.14.*

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