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AN ISOPERIMETRIC INEQUALITY ON THE HEISENBERG GRUOP

We show that the 3-dimensional Heisenberg group behaves in a rather original way at infinity.

A Let us first explain what we mean by "behaviour at infinity". We shall test a few notions on two simple examples: Euclidean space $\mathbb{R}^n$ and hyperbolic space $\mathbb{H}^n$.

Clearly, two balls of radius R, in $\mathbb{R}^n$ and $\mathbb{H}^n$ respectively do not look very different. In fact, the exponential map is a diffeomorphism between them, which changes distances (or lengths of curves) a bounded amount. We call such a map a *quasiisometry*.

Definition: A diffeomorphism φ between Riemannian manifolds is a C-quasiisometry if, for any curve γ,

$$\frac{1}{C} \leqslant \frac{\text{length } \varphi(\gamma)}{\text{length } \gamma} \leqslant C$$

Example: the exponential map above is a C-quasiisometry with $C = \frac{\sinh R}{R} \underset{R \to +\infty}{\to} +\infty$.

However, one *cannot* map *all* of $\mathbb{R}^n$ onto $\mathbb{H}^n$ with a bounded distortion of distances, i.e. by a quasiisometry. Indeed, the volume of balls

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grows polynomially in $\mathbb{R}^n$, and exponentially in $\mathbb{H}^n$, while the type of growth is quasiisometry invariant.

One feels that $\mathbb{R}^n$ and $\mathbb{H}^n$ are even more different. Indeed, in the traditional Poincaré upper half space and disk models - which are conformal imbeddings of $\mathbb{H}^n$ in S^n - hyperbolic space is always accompanied by a "big"boundary. Whereas euclidean space may be conformally imbedded in S^n with boundary consisting of a single point.

This means that $\mathbb{R}^n$ and $\mathbb{H}^n$ are not conformally equivalent. When $n=2$, it follows from a celebrated theorem of Liouville. Since $\mathbb{H}^2$ is conformal to a disk in $\mathbb{R}^2$, a conformal mapping of $\mathbb{R}^2$ into $\mathbb{H}^2$ would be a bounded entire holomorphic function on $\mathbb{R}^2$, which does not exist.

There is a generalisation to higher dimensions, which also applies to *quasiconformal mappings.*

Definition: Let φ be a diffeomorphism between riemannian manifolds (M, g) and (N, g'). Let $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ denote the eigenvalues of $S^* g'$ with respect to g. Recall that φ is conformal iff $\lambda_1 = \ldots = \lambda_n$ (Notice that φ is a C-quasiisometry iff $\frac{1}{C} \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_n \leqslant C$) φ is said to be *C-quasiconformal* if the ratio $\frac{\lambda_n}{\lambda_1} \leqslant C$.

We have seen that volumes are helpful to deal with quasiisometries. In the case of quasiconformal mappings, one uses a kind of conformally invariant volume, called *conformal capacity,* which probably goes back to G. Polya.

Definition: In a complete riemannian manifold (M, g) a *shell* is a connected open subset S whose boundary is disconnected into two parts $\partial_0 S, \partial_1 S$. The *conformal capacity* of S is $\mathrm{cap}(S) = \inf \{\mathrm{vol}(g)/g$ is a metric pointwise conformal to g, and $d_g(\partial_0 D, \partial_1 D) \geqslant 1\}$

Remark: The normalization $d_g(\partial_0 D, \partial_1 D) \geqslant 1$ is necessary in order to obtain a non zero invariant.

Examples:

1) Cap is non zero in general: if S is bounded, and if both $\partial_0 S$ and $\partial_1 S$ have positive dimension (i.e., are not totally disconnected), then

$$\mathrm{Cap}(S) > 0 \quad . \quad \text{(see for example [19])}$$

In particular, let $\mathbb{H}^n$ be a disk in S^n. Then, for any closed $K \subset \mathbb{H}^n$ of positive dimension,

$$\mathrm{cap}(\mathbb{H}^n \backslash K) > 0 \ .$$

2) On the contrary, if $\partial_0 S$ is a point, then $\mathrm{cap}(S) = 0$. In particular, let $\mathbb{R}^n$ be the complement of a point ∞ in S^n. Then for any closed $K \subset \mathbb{R}^n$ of positive dimension, $\mathrm{cap}(\mathbb{R}^n \backslash K) = 0$.

Since φ is C-quasiconformal $\Rightarrow$ for any S, $\frac{1}{C} \leqslant \frac{\mathrm{Cap}(\varphi(S))}{\mathrm{Cap}(S)} \leqslant C$ (in fact, it is equivalent, see [19] and, more generally, [18]), we conclude that $\mathbb{R}^n$ and $\mathbb{H}^n$ are not quasiconformally equivalent.

B In 1936, L. Ahlfors discovered a link between conformal mappings and isoperimetric inequalities.

What do we mean by an isoperimetric inequality?

Classical Isoperimetric Inequality. Let us call *domain* in a manifold a bounded open set with smooth boundary. The Classical Isoperimetric Inequality (proved first by H. A. Schwarz, see [21]) states that "Among all domains in $\mathbb{R}^n$ (resp $\mathbb{H}^n$) of given volume, the domains of least boundary volume are exactly balls'

Restated, in $\mathbb{R}^n$: $\mathrm{vol}(D) \leqslant \mathrm{const}_n \ \mathrm{vol}(\partial D)^{n/n-1}$, equality for balls.

Weakened, in $\mathbb{H}^n$: $\mathrm{vol}(D) \leqslant \frac{1}{n-1} \ \mathrm{vol}(\partial D)$, no equality.

Remark: The hyperbolic inequality is much stronger than the euclidean one, for large domains.

The following theorem states that, if a complete manifold satisfies an isoperimetric inequality, which is stronger than the euclidean one, then its conformal behaviour is similar to that of hyperbolic space.

Theorem. (L. Ahlfors [2], M. Gromov [14] chap. 6)
If a complete Riemannian manifold M satisfies an isoperimetric inequality

$$\mathrm{vol}\,(D) \leqslant \text{const.}\ \mathrm{vol}\,(\partial D)^{\alpha}$$

with $\alpha < \frac{n}{n-1}$ *then, for any compact* $K \subset M$ *of positive dimension*

$$\mathrm{Cap}\,(M-K) > 0\ .$$

A sketch of the proof is given at the end of this paper.

Corollary (M. Gromov, id) *Under the same assumptions, there are no quasi-regular mappings* (roughly speaking, non smooth quasiconformal immersions with branched points, see [14]) *from* $\mathbb{R}^n$ *into* M .

C Example: the 3-dimensional Heisenberg group, with its left invariant metrics.

Definition: The *3-dimensional Heisenberg group* is the group G of triangular matrices $\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$ with 3 real entries α, β, γ .

Theorem 1: *Let* g *be a left-invariant metric on* G . *Then, for any domain* D *in* G , *one has*

$$\mathrm{vol}\,(D) \leqslant \mathrm{const}\,(g)\ \mathrm{vol}\,(\partial D)^{4/3}\ .$$

Proof: Follows from the Carnot-Caratheodory inequality, see below.

Remark: This inequality is mainly significant for large domains.

The Ahlfors Theorem implies that, conformally speaking, the Heisenberg group is "larger" than euclidean space $\mathbb{R}^3$. For example, in any quasiconformal imbedding in a complete manifold, G will have a boundary of positive dimension.

In fact, the Heisenberg group is not quasiconformal to hyperbolic space $\mathbb{H}^3$ either. This follows from L. Ahlfors'measurable Riemann mapping theorem [1], together with an idea of D. Sullivan. We will not go further into this theory. Thus the Heisenberg group appears in an intermediate position between $\mathbb{R}^3$ and $\mathbb{H}^3$. In constrast, recall that any simply connected surface is either conformal to $\mathbb{R}^2$, or to $\mathbb{H}^2$ - this follows from the uniformization theory of Riemann surfaces, see [25].

D Let us try to explain the origin of the exponent 4/3.

Let $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ be a basis of the Lie algebra. The bracket relations $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$ show that G is nilpotent with center Z .

Due to many automorphisms, left invariant metrics on G may be reduced to a normal form, depending on only one parameter.

Up to automorphism, any left-invariant metric is equivalent to one of the g_ϵ , whose matrix in basis (X, Y, Z) is $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \epsilon^2 \end{pmatrix}$. In coordinates,

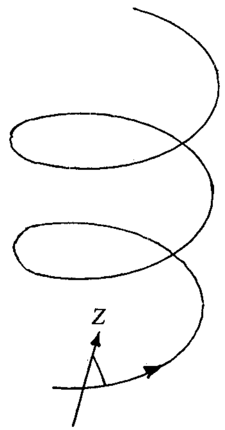
$$g_\epsilon = d\alpha^2 + d\beta^2 + \epsilon^2\,(d\gamma - \alpha\, d\beta)^2\ .$$

In fact, all g_ϵ are isometric to g_1 , up to scaling. Indeed, we define the *dilation* δ_ϵ as the automorphism whose matrix, in the basis (X, Y, Z) is $\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix}$. Then $g_\epsilon = \frac{1}{\epsilon^2}\ \delta_\epsilon^*\ g_\epsilon$.

We are interested in large domains, their volume and boundary area relative to g_1 . But, up to scaling, a ball, relative to g_1 , of large radius ϵ , is identical to the unit ball relative to g_ϵ

$$\mathrm{Ball}_{g_1}(\epsilon) = \mathrm{Ball}_{g_\epsilon}(1) .$$

Therefore, we should concentrate on the behaviour of the metrics g_ϵ, inside a fixed ball, when ϵ goes to $+\infty$.



Let π denote the projection $G \to G/Z = \mathbb{R}^2$. Then, for any ϵ , π is a *Riemannian submersion* of (G, g_ϵ) onto euclidean $\mathbb{R}^2$. This means that π induces an isometry on each *horizontal plane,* i.e. orthogonal to Z , for any ϵ . The geodesics of g_ϵ are *helices,* i.e., they make a constant angle with the center Z , and they project onto circles of $\mathbb{R}^2$.

A. Koranyi [15] has observed that the (in general) unique g_ϵ-geodesic joining two points X and Y converges, when ϵ goes to $+\infty$, towards a *horizontal* curve, that is, orthogonal to the center Z .

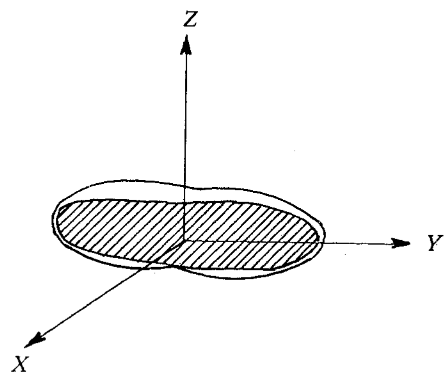
Distances also converge: $\lim_{\epsilon \to +\infty} d_\epsilon(x, y) = d_\infty(x, y)$, where the left-invariant (though non Riemannian) metric d_∞ is obtained as follows: $d_\infty(x, y)$ is the minimum of the length of horizontal curves joining x to y . The distance d_∞ is well-behaved: it defines the usual topology of G ; its balls are compact; it admits geodesics, which satisfy a differential equation - they are horizontal helices. It is called the *Carnot-Carathéodory metric*.

Remark: The fact that the family of homothetic metric spaces $(G, \frac{1}{\epsilon^2} g_1)$

"converges" generalizes to all nilpotent Lie groups - see [23] and [13] for the relevant notion of convergence. In some sense, this phenomenon characterizes nilpotent groups, see M. Gromov [13].

It appears now that this Carnot-Carathéodory metric space (G, d_∞) contains information about the asymptotic behaviour of g_1 . Let us describe this metric space.

As the dilations δ_ϵ are *homotheties* for d_∞ , small balls are images by δ_ϵ of unit ball. They are very flat in the Z direction. In particular, to cover a given open set with balls of radius ϵ , it requires as usual $\sim \frac{1}{\epsilon^2}$ balls in the horizontal plane, but also $\sim \frac{1}{\epsilon^2}$ balls in the vertical direction.



This is exactly what the following statement means: "The Hausdorff dimension of the metric space (G, d_∞) is equal to 4". There is a canonical *Hausdorff measure,* see [9],denoted by $\mathcal{H}^4$, homogeneous of degree 4 under the δ_ϵ . It is just the Haar measure $d\alpha \wedge d\beta \wedge d\gamma$. In the same way, if S is a smooth compact surface, the Hausdorff dimension of the metric space (S, d_∞) is equal to 3. There is a canonical Hausdorff measure $\mathcal{H}^3$ on surfaces, homogeneous of degree 3 under the δ_ϵ . Theorem 1 is an easy consequence of the following *"Carnot-Carathéodory isoperimetric inequality"*: for any domain D in G ,

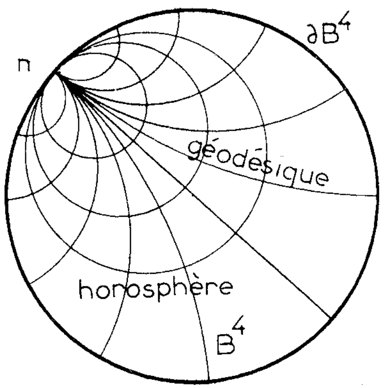
$$\mathcal{H}^4(D) \leqslant \left(\frac{12}{\pi}\right)^{1/3} \mathcal{H}^3(\partial D)^{4/3} .$$

A sketch of proof is given at the end of this paper.

E *Comments*

(*i*) The preceding considerations - construction of a limiting metric - obviously generalize to any nilpotent group with "dilations" δ_ϵ . We conjecture that the isoperimetric inequality should generalize too.

(*ii*) Our feeling is that the Carnot-Caratheodory metric deserves further attention. In some sense, it is more canonically associated to a nilpotent group than left invariant Riemannian metrics. For instance, let M be a bounded symmetric domain of rank one in $\mathbb{C}^n$, $\mathbb{H}^n$ or $\mathbb{C}a^2$. Then $M = G/K$, G Iwasawa decomposes in $G = KAN$ where N is nilpotent. Then N is the stabilizer of a point n on the boundary. It is simply transitive on the horospheres centered at n (i.e. orthogonal hypersurfaces to the family of geodesics with limit point n) . The family of metrics induced by horospheres considered as imbeddings of N is of type g_ϵ - up to normalization. Thus a Carnot-Caratheodory metric is obtained on the boundary.



This metric depends on the choice of a point n . What does not depend on n is the conformal structure it defines. This point of view is developped in [16].
This conformal structure may be expressed in terms of the Levi form, and thus generalizes to pseudoconvex domains, see D. Burns Jr and S. Shnider [6].

G. D. Mostow uses a coarse version of this Carnot Caratheodory conformal structure to prove rigidity of the compact quotiens of the above symmetric spaces, see [20], chapter 20.

(*iii*) More generally, the differential geometry of Carnot-Caratheodory metrics is still to be done. In general, a Carnot-Caratheodory metric on a manifold M consists of the data of a non integrable subbundle H (in the strong sense of [26]) of the tangent bundle TM , and of a metric on H (euclidean on each fibre). Then the length of horizontal curves is defined, and an infimum defines the distance. One should be able to associate to these data a volume element, curvature, and so on. It is unclear wether a reasonable exponential map can be constructed.

(*iv*) Consider the horizontal vectors X and Y in the Lie algebra as left-invariant vector-fields on G . Then the second order differential operator $\Delta = X^2 + Y^2$ is *hypoelliptic*, that is, for any function u . $\Delta u \in C^\infty \Rightarrow$ $\Rightarrow u \in C^\infty$. In some sense, this operator C is the prototype of all hypoelliptic operators, in the same way as the euclidean Laplacian is the prototype of all elliptic second order operators. Its symbol is well-defined on covectors: it is $X^2 + Y^2$ considered as a (positive semi definite) quadratic form on T^*G . Under Legendre transform, it corresponds to a function on TG which is quadratic on the horizontal planes and infinite elsewhere. This exactly is our Carnot-Caratheodory data. In fact in the study of hypoelliptic operators, the Carnot-Caratheodory distance plays the role of the Riemannian distance associated to the symbol of an elliptic second order operator .... For instance, in the estimation of the corresponding heat kernel, see B. Gaveau's work in [3] and [12].

(*v*) A typical situation in optimal control theory is the following.
The evolution of a system - for instance, position and speed of a rocket - is described by a curve in a phase space - a manifold M .
Assume that the acceleration - a tangent vector to M - is subject to linear constraints: for example, the rocket is propelled via a few jets. This defines a subbundle H of TM . Assume that the energy necessary to modify the motion is measured by some quadratic form on H . Then, the total energy needed to attain some state is the value of some Carnot-Caratheodory distance.

In [26], H. Sussman gives a sufficient condition for accessibility (i.e. any state may be attained, or, equivalently, the Carnot-Caratheodory distance is finite): the brackets - of any order - of vector fields tangent to H should span the ful. tangent space at each point.
The optimal way to join two states is a geodesic for the Carnot-Caratheodory distance. Thus, a key problem in optimal control amounts to describing a Carnot-Caratheodory distance, cf. [17].

F Further applications of theorem 1.

- *A combinatorial property*

Let Γ be the subgroup of matrices in G with integral entries. Then Γ is discrete, cocompact in G.

Let us say that two matrices in Γ are *neighbours* if they differ by only one unit in one of their entries. Given a finite subset A in Γ, define the *boundary* $\partial A = \{a \in A$/some neighbour of a is not in $A\}$.

Then, up to a constant, theorem 1 is equivalent to the following statement, which is purely combinatorial:
"For any finite subset A in Γ, one has

$$\text{Card}\,(A) \leqslant \text{Const.}\,(\text{Card}\,\partial A)^{4/3}\ ."$$

It follows that theorem 1 generalizes to any universal covering $(\widetilde{M}, \widetilde{g})$ of a compact (M, g) with $\pi_1(M) \cong \Gamma$. Of course, the exponent 4/3 is independent of the dimension of M (take $M = S^n \times \Gamma/G$, $n \geqslant 2$, for instance).

Remark: The link between isoperimetric inequalities on manifolds, and on discrete groups is explained in [14] chapter 6).

- *A Sobolev-type inequality*

Recall the classical Sobolev inequality in $\mathbb{R}^n$. Let D be some bounded open set, u a smooth function with support in D. Then

$$(\textstyle\int_D u^q)^{1/q} \leqslant \text{const}\,(p, q, D)\,(\int_D |\text{grad}\, u|^p)^{1/p}$$

for any $1 \leqslant p \leqslant q \leqslant +\infty$, where $\frac{1}{q} \leqslant \frac{1}{p} - \frac{1}{n}$. Such an inequality means that any two points in $\mathbb{R}^n$ are linked by some short curve, and one can estimate the variation of u by integrating $|\text{grad}\, u|$ along this curve.

In the Heisenberg group G, any two points may be joined by a *horizontal* curve (non integrability of the plane field H), so it should suffice to integrate the norm $|\text{grad}^H u|$ of the horizontal component of the gradient of u.
Indeed, it follows from the Carnot-Caratheodory inequality that

$$(\textstyle\int_D u^q)^{1/q} \leqslant \text{const}\,(p, q, D)\,(\int_D |\text{grad}^H u|^p)^{1/p}$$

for $\frac{1}{q} \leqslant \frac{1}{p} - \frac{1}{4}$.

In their studv of the horizontal laplacian Δ, G. B. Folland and E. M. Stein have proved such an inequality, with $\frac{1}{q} < \frac{1}{p} - \frac{1}{4}$. Their method is abstract, using Fourier transforms. I think that we have gained more geometric understanding of this inequality.

Remark: The link between isoperimetric inequalities and Sobolev inequalities goes back to G. Faber [8] and E. Krahn [17], see [4] for instance.

G *Proofs.*

-Proof of the Ahlfors-Gromov theorem.
First notice that, for any shell S,

$$\text{Cap}\,(S) = \inf\,\{\textstyle\int_S |\text{grad}\, u|^n / u = (S, \partial_0 S, \partial_1 S) \to ([0,1], 0, 1)\}\,.$$

For such a smooth function u, *denote by*

$$a(t) = \left(\int_{\{u=t\}} |\text{grad}\, u|^{n-1}\right)^{\frac{1}{n-1}} \quad t \in\,]0, 1[\ ,$$

and

$$A(u) = \left(\int_0^1 a(t)^{-1}\, dt\right)^{1-n}$$

Then the Hölder inequality (together with the equality case) implies that

$$\operatorname{cap}(S) = \inf \{A(u)/u = (\bar{S}, \partial_0 S, \partial_1 S) \to ([0,1], 0, 1)\}.$$

Now assume that S is the complement of a compact set K of positive volume in a complete manifold M, which satisfies an isoperimetric inequality with exponent $\alpha < \frac{n}{n-1}$.
For a smooth function $u = (M, K, \infty) \to ([0,1], 1, 0)$, *denote by*

$$Y(t) = \operatorname{vol}\{u = t\}, \quad X(t) = \operatorname{vol}\{u \geqslant t\}.$$

Then

$$X'(t) = \frac{dX}{dt} = \int_{\{u=t\}} |\operatorname{grad} u|^{-1}.$$

Again, Hölder's inequality yields

$$Y(t) \leqslant a(t)^{\frac{n-1}{n}} \quad X'(t)^{\frac{n-1}{n}}.$$

With the isoperimetric inequality $\operatorname{vol}(D) \leqslant \text{Const.} \operatorname{vol}(\partial D)^{\alpha}$, one gets

$$a(t)^{-1} \leqslant \text{const.}\ X'(t)\, X(t)^{-\frac{n}{\alpha(n-1)}}.$$

Integrating, it yields

$$A(u) \geqslant \text{const.} \left[\int_0^1 X^{-\frac{n}{\alpha(n-1)}}\, dX\right]^{1-n} = \text{const.} \operatorname{vol}(K)^{\frac{n}{\alpha}-(n-1)} > 0.$$

This is the desired property for compact K with positive volume. The general case (K of positive dimension) is obtained using the fact that semi-open functions u satisfy a uniform estimate in terms of $\int |\operatorname{grad} u|^n$, see [10] and [19].

- Proof of the Carnot Caratheodory isoperimetric inequality.

It imitates Ch. Croke's proof of an isoperimetric inequality in simply connected manifolds M of non positive sectional curvature, see [7]. Ch. Croke

uses a formula, due to S. Santalo ([24], p), which computes the volume of a domain D as an integral over the unit tangent bundle $UM_{|\partial D}$ of the boundary ∂D:

$$\operatorname{vol}(S^{n-1}) \operatorname{vol}(D) = \operatorname{vol}(UM_{|D}) = \int_{UM_{|\partial D}} \ell(x, v) \cos \langle v, \nu \rangle\, d(x, v)$$

where for $x \in \partial D$, v a unit tangent vector at x, ν is the unit normal vector of ∂D at x, $\ell(x, v)$ is the first time when the geodesic starting from x with initial speed v leaves D. This formula is a consequence of the fact that the geodesic flow on the unit tangent bundle UM preserves the natural volume. Since geodesics through x sweep all of D, the sectional curvature assumption allows him to estimate

$$\int_{U_x \partial D} \ell(x, v)\, dv \leqslant \text{const}(n) \operatorname{vol}(D)^{1/n}$$

and thus

$$\operatorname{vol}(D)^{1-\frac{1}{n}} \leqslant \text{const}(n) \operatorname{vol}(\partial D).$$

In our situation, we replace the unit tangent bundle UG - where no geodesic flow is defined - by the unit *horizontal* bundle UH. We consider special geodesics, those which are horizontal lifts of lines in $\mathbb{R}^2$. Since, through any point and horizontal vector, there is one unique such geodesic, a flow is defined on UH. It preserves a natural volume on UH - this relies essentially on the fact that the distribution H is not integrable. Santalo's formula now reads

$$2\pi\, \mathcal{H}^4(D) = \int_{\partial D} \int_{UH_x} \ell(x, v) \cos \langle v, \nu \rangle\, dv\, d\mathcal{H}^3(x).$$

The special geodesics through x only sweep a surface Σ_x, so

$$\int_{UH_x} \ell(x, v)\, dv \leqslant \mathcal{H}^3(\Sigma_x \cap D)^{1/3}.$$

Since Σ_x is a minimal surface, one has

$$\mathcal{H}^3(\Sigma_x \cap D) \leqslant \frac{1}{2}\, \mathcal{H}^3(\partial D)$$

which leads to

$$\mathcal{H}^4 (D) \leq \left(\frac{12}{\pi}\right)^{1/3} \mathcal{H}^3 (\partial D)^{4/3} .$$

For a complete proof, see [22] .

Remark: The constant $\left(\frac{12}{\pi}\right)^{1/3}$ not optimal. Nevertheless, one may conjecture the optimal constant. An extremal domain ∂D , if smooth, would have "constant mean curvature". One can define Carnot-Caratheodory mean curvature. A smooth surface has constant mean curvature h if and only if it is foliated by horizontal lifts of circles in $\mathbb{R}^2$ of common radius $\frac{1}{h}$.

Therefore, we believe that, relative to this isoperimetric problem the extremal surfaces are those obtained by rotating a geodesic joining two points in the center, around the center.

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