A list of open problems in Differential Geometry

Good open problems play an indispensable role in the development of differential geometry. All speakers and participants of the conference Modern Trends in Differential Geometry (São Paulo, July 2018) were invited to include open problems. We thank the contributors and the organizers. We think that this collection reflects many facets of geometry, its deep roots, and its profusion of fruit.

Frank Morgan, Pierre Pansu

1 Can you hear an orbifold singularity?

proposed by *Ian Adelstein, Yale.*

Mark Kac in the American Mathematical Monthly in 1966 famously asked if one can hear the shape of a drum, by which he meant whether the Laplace spectrum of a manifold determines its isometry class. John Milnor quickly answered this question in the negative, by producing a pair of isospectral yet non-isometric 16 dimensional tori. Many subsequent mathematicians have wondered which geometric properties are audible, i.e. determined by the spectrum of the Laplacian.

An important open question is

Question 1 *Can one hear the presence of an orbifold singularity, i.e. whether or not there exists a pair of isospectral orbifolds, one of which has singular points whereas the other does not and is therefore a manifold.*

The paper [2] discusses this question.

Recent work has shown that non-orbifold singularities are inaudible to the G-invariant spectrum. In [1] the authors construct a pair of orbit spaces with equivalent G-invariant spectra, yet where one space is isometric to an orbifold whereas the other admits a non-orbifold singularity.

References

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2 Riemannian manifolds with curvature bounds

proposed by *Lashi Bandara, Universität Potsdam, Germany.*

Can a complete metric with Ricci curvature bounded below be L^∞ approximated by one with two-sided Ricci curvature bounds?

Here is a precise statement.

Question 2 *For every $\ell, k > 0$, there exist $C, L, K > 0$ with the following effect. Let (M, g) be a complete Riemannian manifold with injectivity radius $inj(M, g) \geq \ell$ and Ricci curvature $Ric(g) \geq k$. Then there exists a metric h on M with $inj(M, h) \geq L$ and $|Ric(h)| \leq K$ such that*

$$\frac{1}{C}\, g \leq h \leq C\, g.$$

Let me motivate this problem. During my thesis, my supervisor Alan McIntosh and I studied a problem called the Kato square root problem on manifolds. In the Euclidean context, this was a 40 year outstanding problem, which Alan was involved in for that entire period of time. We considered this question in a geometric setting. In our work, we prove that this problem can be solved for metrics that have Ricci bounded above and below, as well as injectivity radius, [1]. In a later paper, [2], I showed that the solution is stable under L^∞ perturbations as expressed in the question.

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3 Reducibility of the holonomy of flat manifolds

proposed by *Renato Bettiol, City University of New York (Lehman College).*

Question 3 *Give an alternative, geometric proof that the holonomy representation of a closed flat manifold is reducible.*

By the Bieberbach Theorem, a closed flat n-dimensional manifold is finitely covered by a flat n-torus. The group of deck transformations is identified with the manifold's holonomy group H, which is a finite subgroup of $O(n)$. In a very algebraic paper, [3], Hiss and Szczepanski proved that the (orthogonal) action of H in $\mathbb{R}^n$ is always reducible, i.e., $\mathbb{R}^n$ admits a decomposition as the orthogonal direct sum of nontrivial H-invariant subspaces. Although this has important geometric consequences (e.g., it implies that all closed flat manifolds admit non-homothetic flat deformations, see [1]), the proof does not seem to provide much more geometric insight and is completely algebraic (even relies on the classification of finite simple groups). It is reasonable to expect this fact should have a purely geometric proof, and that this could shed some light on important questions about flat manifolds.

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4 Biorthogonal curvature

proposed by *Renato Bettiol, City University of New York (Lehman College).*

Question 4 *Does $S^2 \times T^2$ admit a Riemannian metric with positive biorthogonal curvature?*

The biorthogonal curvature of a 4-manifold is defined, for each tangent plane σ, as the average of the sectional curvature of σ and $\sigma^\perp$. Positive biorthogonal curvature is an intermediate condition between positive sectional curvature and positive scalar curvature. Clearly, $S^2 \times T^2$ admits metrics with nonnegative biorthogonal curvature, but it is unknown if they can be deformed into metrics with (strictly) positive biorthogonal curvature. Settling this matter would be a crucial step in classifying non-simply connected closed 4-manifolds with positive biorthogonal curvature (the simply-connected classification, up to homeomorphisms, was obtained in [1]).

References

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5 Branch points of area-minimizing surfaces

proposed by *Camillo De Lellis, Institute for Advanced Study, Princeton.*

Σ is a smooth, complete $(m+n)$-dimensional Riemannian manifold, $\Gamma \subset \Sigma$ a smooth $(m-1)$-dimensional oriented submanifold and T an area minimizing integer rectifiable current which spans Γ. We are interested in the boundary regularity of T, more precisely on the size of $\mathrm{Reg}_b(T) \subset \Gamma$, the (relatively open) set of boundary regular points, and its complement $\mathrm{Sing}_b(T) = \Gamma \setminus \mathrm{Reg}_b(T)$. The question is pertinent only in codimension $n \geq 2$, because the celebrated work of Hardt and Simon [5] shows that $\mathrm{Sing}_b(T) = \emptyset$ when $n = 1$.

In [4] we show that $\mathrm{Reg}_b(T)$ is dense in Γ. We also give an example of a 2-dimensional T with a sequence $\{P_k\} \subset \mathrm{Sing}_b(T)$ of crossing singularities accumulating towards a branching singularity $P \in \mathrm{Sing}_b(T)$. This in sharp contrast with the interior regularity theory, because we know that interior singular points of area-minimizing 2-dimensional T are isolated (cf. [1, 2, 3]).

Question 5 *1. Does* $\mathrm{Sing}_b(T)$ *have zero* $(m-1)$*-dimensional Hausdorff measure?*

2. If yes, does $\mathrm{Sing}_b(T)$ *have (Hausdorff) dimension at most* $m-2$*?*

3. A related question is: can we find an example as in [4] where the sequence of accumulating singularities $\{P_k\} \subset \mathrm{Sing}_b(T)$ *are of branching type? If yes, then one could imagine a Cantor type construction giving a negative answer to 2.*

4. White in [6] conjectures that if a 2*-dimensional area-minimizing currents spans a real analytic closed curve, then it has finitely many (interior and boundary) singularities and hence finite topological type.*

5. The example in [4] is topologically a disk. Is it possible to give an example of a smooth closed curve $\Gamma \subset \mathbb{R}^{2+n}$ *which bounds an area-minimizing* 2*-dimensional current with infinite topology?*

References

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6 Manifolds modelled on flag manifolds

proposed by *Elisha Falbel, Sorbonne Université, Paris.*

Flag manifolds are homogeneous manifolds obtained as quotients of a complex group by a Borel subgroup. One of the simplest examples is the complex full flag manifold of complex dimension three $F_{12} = SL(3,\mathbb{C})/B$, where B is the subgroup of upper triangular matrices. Each of the real forms $SL(3,\mathbb{R})$ and $SU(2,1)$ have a unique compact orbit which is a model for *flat path geometry* structures and *spherical CR structures* respectively. Both orbits have real dimension three. Here is natural question, certainly very difficult as a variant of uniformization problems, open even in dimension three:

Question 6 *Which manifolds can be modeled on an orbit of a real form in a space of flags?*

Back to real dimension three, one can define flag structures on a real 3-manifold by a choice of two complex lines in its complexified tangent space. A very special case appears when the manifold admits a totally real immersion in F_{12}. The complex lines arises when one considers natural foliations in F_{12} intersected with the real immersion. Because it is so special, this flag structure can be thought as a flat structure even though it is not locally equivalent to one of the real orbits. This situation leads to the following

Question 7 *What is the homotopy classification of totally real immersions of real* 3*-manifolds in the complex full flag manifold* F_{12}*?*

Gromov's H-principle applies here, so the problem should belong to homotopy theory.

References

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7 Area minimizing projective spaces in the projective space with the Berger metric

proposed by *Olga Gil-Medrano, University of Valencia, Spain..*

A classical result due to M. Berger [1] and A. T. Fomenko [4] is that for the real projective n-dimensional space with the standard metric, for all $0 < k < n$, the projective spaces obtained by projection of the k-dimensional great spheres (that are totally geodesic submanifolds) are the only k-dimensional volume minimizing in their homology class.

A Berger sphere is the Riemannian manifold (S^{2n+1}, g_μ) where the metric g_μ for $\mu > 0$ is defined as follows: we denote by J the complex structure of $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$ and by H the Hopf vector field JN, where $N(p) = p$ is the outwards unit normal of S^{2n+1}, then

$$g_\mu(H,H) = \mu g(H,H) = \mu, \quad (g_\mu)|_{H^\perp} = g|_{H^\perp}, \quad \text{and} \quad g_\mu(H, H^\perp) = 0,$$

where $\perp$ means orthogonal with respect to the standard metric g. By passing to the quotient we can construct the corresponding Berger metrics on the projective spaces $(\mathbb{R}P^{2n+1}, g_\mu)$

In [2] it is shown that for $(\mathbb{R}P^3, g_\mu)$ the projections of the equatorial spheres are not totally geodesic surfaces, except for the round sphere $\mu = 1$, but they are always minimal surfaces and in [5] it is shown that in fact *area minimizing projective planes in* $(\mathbb{R}P^3, g_\mu)$ *are exactly the equatorial projective planes.*

The key point for the proof is the existence of area minimizing projective planes in 3-manifolds stablished in [3].

Question 8 *For $2n+1 > 3$ and $0 < k < 2n+1$, are projective subspaces obtained by projection of the k-dimensional equatorial spheres minimal submanifolds of the Berger projective space $(\mathbb{R}P^{2n+1}, g_\mu)$? Are they the only k-dimensional volume minimizing cycle in their homology classes?*

Remark 1 *In [2] it has been shown that Berger projective space of dimension 3 can be realized as the manifold (T^rS^2, g^S) of vectors of norm r on the unit sphere; more precisely (T^rS^2, g^S) is isometric to $(\mathbb{R}P^3, 4g_{r^2})$. The isometry preserves the S^1-bundle structure of both manifolds, namely the one determined by the fibers of the Hopf fibration in the projective space and the natural one in the tangent space.This isometry is specific of dimension 3.*

References

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8 Bi-invariant metrics and multiplicity of conjugate points

proposed by *Claudio Gorodski, University of São Paulo, Brazil.*

Let G be a compact connected Lie group equipped with a bi-invariant Riemannian metric. Then G is a symmetric space, namely, the Riemann curvature tensor R is parallel. It follows that the Jacobi equation along a geodesic γ, given by

$$-J'' + R(X, J)X = 0,$$

where $X = \gamma'$, has constant coefficients in an orthonormal parallel frame along γ. More explicitly, the Jacobi operator is

$$R(X, \cdot)X = \frac{1}{4}\mathrm{ad}_X^2.$$

Here ad_X is a skew-symmetric operator, so the non-zero eigenvalues of the Jacobi operator occur in pairs and hence all conjugate points are of even order; equivalently, the index of any geodesic segment is even (Bott [1]).

We ask whether this property characterizes bi-invariant metrics on Lie groups among the left-invariant ones (of course, it can be asked among all Riemannian metrics, but we would like to consider the algebraic structure of Lie groups).

Question 9 *Assume that a left-invariant Riemannian metric is given on a compact connected Lie group G such that the index of any geodesic segment is even. Must the metric be bi-invariant?*

It is not difficult to see that for a given Riemannian metric on G, the index of any geodesic segment is even if and only if, for any two non-conjugate points p, $q \in G$, the energy functional

$$E(\gamma) = \frac{1}{2}\int |\dot{\gamma}|^2 \, dt$$

defined on the Hilbert manifold of H^1-curves joining p and q is a perfect Morse function. Complete Riemannian metrics with the latter property were called *pointwise taut* by Terng and Thorbergsson in [2].

References

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9 Toral manifolds and positive scalar curvature

proposed by *Bernhard Hanke, Augsburg.*

The classification of closed manifolds admitting Riemannian metrics of positive scalar curvature has been a major research topic for decades. While for simply connected manifolds of dimension at least 5 a complete classification has been achieved by Gromov-Lawson [1] and Stolz [6], the picture remains largely unclear even for finite fundamental groups.

Question 10 ([5], Conjecture 1.2) *Let M be a connected closed manifold with finite fundamental group of odd order. Assume that the universal cover of M admits a metric of positive scalar curvature. Does M admit a metric of positive scalar curvature?*

Remark 2 *Question 10 has a positive answer for* $\dim M \leq 2$*, and for* $\dim M = 3$ *it follows from the geometrisation theorem. For* $\dim M = 4$*, counterexamples were constructed in [4]. For* $\dim M \geq 5$ *Question 10 is open.*

Definition 3 *Let M^d be a connected closed d-manifold, and let p be a prime. The manifold M is called* p-toral*, if there exist cohomology classes* $c_1, \ldots, c_d \in H^1(M; \mathbb{F}_p)$ *with* $c_1 \cup \cdots \cup c_d \neq 0 \in H^d(M; \mathbb{F}_p)$*. Otherwise M is called* p-atoral.

Remark 4 *(i) The d-torus $T^d = (S^1 \times \cdots \times S^1)^d$ is p-toral for all p, and so are all closed manifolds which are oriented bordant to T^d over the classifying space $B(\mathbb{Z}/p)^d$.*

(ii) The real projective spaces $\mathbb{R}P^{2m+1}$ are 2-toral for all m.

(iii) If p is odd and M is p-toral, then M is orientable.

(iv) If p is odd and $\dim M > \operatorname{rank}(H^1(B\pi_1(M); \mathbb{F}_p))$*, then M is p-atoral. This follows from the fact that the classifying map $M \to B\pi_1(M)$ induces an isomorphism in $H^1(-; \mathbb{F}_p)$ and for odd p each element in $H^1(M; \mathbb{F}_p)$ has square 0.*

(v) In [3] we give a positive answer to Question 10, if $\dim M \geq 5$, $\pi_1(M)$ *is an elementary abelian p-group for some odd p and M is p-atoral.*

Question 11 *Let M be a connected closed manifold admitting a metric of positive scalar curvature. Does this imply that M is p-atoral for all odd p?*

The answer to Question 11 is positive if $\dim M \leq 2$. For $\dim M = 3$ we recall [2, Theorem 8.1] that the prime decomposition of M (which we can assume to be orientable by Remark 4 (iii)) only contains copies of $S^1 \times S^2$ and manifolds with universal covers homotopy equivalent to S^3. This implies that for odd p all prime summands of M, and hence M itself, are p-atoral.

Let p be odd and let M^d be a p-toral non-spin manifold with finite fundamental group of odd order. Such manifolds can be constructed along Remark 4 (i) for all $d \geq 4$. The universal cover of M is a closed simply connected non-spin manifold, and therefore it admits a metric of positive scalar curvature in case $d \geq 5$ by [1, Corollary C]. Together with Remark 2, we conclude that if Question 11 has a positive answer, then Question 10 has a negative answer for $\dim M \geq 4$.

References

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10 Coarse embeddings

proposed by *Dominique Hulin and Pierre Pansu, Paris-Sud (Paris-Saclay).*

A map between metric spaces is a coarse embedding if there exist functions ω and Ω, where ω tends to $+\infty$ with its argument, such that for every $D > 0$, pairs of points at distance D are mapped to pairs at distance between $\omega(D)$ and $\Omega(D)$. This large class arose in convex geometry in the 1960's, and later in geometric group theory, where coarse embeddability of a fundamental group into Hilbert space turns out to make it rather tame from the topological viewpoint. Therefore the wider problem of coarse embeddability of certain classes of metric spaces into certain classes of Banach spaces has received a lot of attention, see references in [3].

For nonlinear targets, large scale analogues of topological dimension have been studied. They are nondecreasing under coarse embeddings, see [1]. They take integer values. Only recently did noninteger valued invariants arise, like variants of the conformal dimension of the ideal boundary of a hyperbolic metric space, see [5, 4], or sharp metric cotype, see [3].

Question 12 *Find more numerical invariants of metric spaces that are nondecreasing under coarse embeddings.*

A remarkable fact about coarse embeddings is that they can sometimes be made harmonic, see [2].

Question 13 *Find applications of the harmonic map approximation of coarse embeddings.*

References

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11 Ricci pinching on solvable Lie groups

proposed by *Jorge Lauret, Córdoba, Argentina.*

Question 14 *For solvable Lie groups G, show that solvsolitons are the only local maxima of the Ricci pinching functional $g \mapsto F(g) = \frac{Scal(g)^2}{|Ric(g)|^2}$ on left-invariant Riemannian metrics on G.*

A solvsoliton is an invariant Riemannian metric g such that $Ric(g) = cI + D$ where D is a derivation of the Lie algebra $\mathfrak{g}$.

References

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12 Classification problems and Poisson structures

proposed by *Rui Loja Fernandes, University of Illinois.*

R. Bryant (see the Appendix in [1]) observed that certain classification problems in geometry, which may be called of "finite type", can be formulated as the problem of integrating a Lie algebroid $A \to X$ to a Lie groupoid $\mathcal{G} \rightrightarrows X$ (Lie's III Theorem). Examples of such classification problems include:

(i) Metrics of constant sectional curvature.

(ii) Metrics of Hessian type [2].

(iii) Bochner-Kähler metrics [1].

(iv) Symplectic connections with special holonomy [3, 4].

The Lie groupoid integrating the corresponding Lie algebroid gives a presentation of the stack of the moduli problem in question (see, e.g., [5])

On the other hand, given a Poisson manifold (M, π), it is well known that its cotangent bundle $A = T^*M$ has a natural Lie algebroid structure. If this Lie algebroid is integrable, then the source 1-connected integration $\mathcal{G} \rightrightarrows X$ is a symplectic groupoid: the space of arrows carries a multiplicative symplectic form $\omega \in \Omega^2(\mathcal{G})$.

It turns out that in all the problems above (and in others of finite type), the Lie algebroid A relevant for the classification problem arises as the cotangent Lie algebroid of some Poisson manifold. However, in some of these problems, there is no natural symplectic or Poisson structure present, and when there is one, it does not seem to arise from the symplectic structure ω on the Lie groupoid integrating the Lie algebroid A. Still, the corresponding moduli space has a "symplectic nature".

Question 15 *Explain the existence and the role of the symplectic nature of the groupoid/algebroid and its relevance for the geometry of the moduli spaces of geometric structures of finite type.*

References

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13 Morse index of embedded minimal surfaces

proposed by *Davi Maximo, University of Pennsylvania.*

Question 16
- *Do there exist embedded minimal surfaces with finite genus and Morse index* 4 *?*
- *More focussed: in the* 1*-parameter deformation of Costa's surface, does the index stay constant?*

Complete embedded minimal surfaces of index 3 have been recently ruled out in [1].

References

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14 On the Hodge spectra of lens spaces

proposed by *Roberto Miatello, Córdoba, Argentina.*

Question 17 *- Construct congruence lattices which are* $norm_1$ *and* $norm_1*$*-isospectral in all dimensions (see [1]).*

- Are there families of p*-isospectral lens spaces for all* p*, with more than two elements?*

- Give a procedure to go from p*-isospectral lens spaces (for all* p*) of dimension* m *to dimension* $m+1$*.*

- Establish connections with toric geometry (see [2]).

References

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15 Isoperimetric Problem in $\mathbb{C}P^2$

proposed by *Frank Morgan, Williams College, Williamstown.*

Question 18 *Prove that geodesic spheres provide the least-perimeter way to enclose prescribed volume in* CP^2*.*

There is enough symmetry to deduce that they have constant mean curvature but not enough to prove them optimal by symmetry arguments.

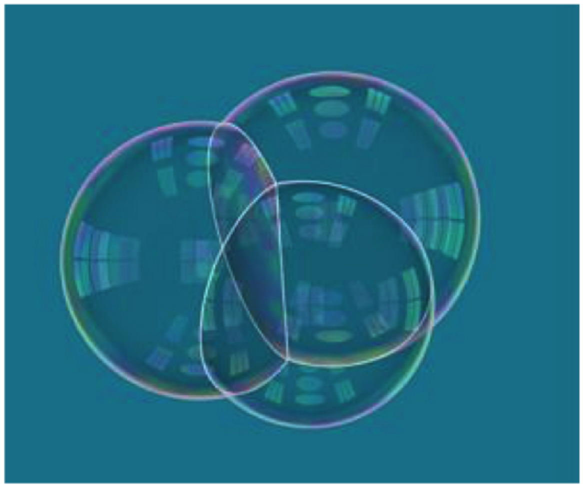

Figure 1: The triple bubble
Image by John M. Sullivan
http://torus.math.uiuc.edu/jms/Images/

References

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16 Triple Bubble in $\mathbb{R}^3$

proposed by *Frank Morgan, Williams College, Williamstown.*

Question 19 *Prove that the pictured standard triple soap bubble is the least-perimeter way to enclose and separate three given volumes in* $\mathbb{R}^3$.

The corresponding Double Bubble Theorem was proved in $\mathbb{R}^3$ by Hutchings, Morgan, Ritoré and Ros in 2002 and in $\mathbb{R}^n$ by Ben Reichardt in 2008. The double and triple bubble theorems in $\mathbb{R}^n$ with Gaussian density were announced in 2018 by Milman and Neeman.

References

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17 Homogeneous Riemannian manifolds with non-trivial nullity

proposed by *Carlos Olmos, Córdoba, Argentina.*

Question 20 *1. If the normal holonomy group of an irreducible and full homogeneous submanifold M^n of the sphere with $n \geq 2$ does not act transitively, then M is the orbit of an s-representation.*

2. The index of an irreducible symmetric space which is different from $G_2/SO(4)$ (or its symmetric dual) coincides with its reflective index,[1].

3. Open questions about homogeneous Riemannian manifolds with non-trivial nullity.

- *Are there examples which are not topologically trivial?*
- *Are there examples $M = G/H$, with G non-solvable?*
- *Are there Kähler examples?*
- *Are there examples in any dimension $d \geq 5$?*

References

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18 Constant mean curvature in homogeneous 3-manifolds

proposed by *Joaquín Pérez, University of Granada, Spain.*

Here are questions on constant mean curvature (CMC) spheres which arose from the study of the isoperimetric problem in 3-dimensional, simply-connected, noncompact homogeneous manifolds X diffeomorphic to $\mathbb{R}^3$.

Question 21

- *Do CMC spheres about a point x in such a space form a foliation of $X - \{x\}$?*
- *Could this be a way of proving embeddedness of CMC spheres in general?*

It is well known [1] that CMC spheres in X exist exactly for values of the mean curvature H lying in an interval of the form $(H(X), \infty)$ where $H(X)$ is called the critical mean curvature of X. Furthermore, spheres of CMC H are unique up to congruencies, when they exist. This uniqueness leads to a well-defined notion of center of symmetry for CMC spheres. Finally, it is also known that if we take any point x in X, the class of CMC spheres with center x is a real analytic 1-parameter manifold. It is not known (but expected) if CMC spheres are embedded in general (they are known to be Alexandrov embedded).

I include two classical problems which I partly surveyed in my talk:

Question 22

- *Calabi-Yau problem. For an embedded minimal surface in $\mathbb{R}^3$, does complete imply proper ?*
- *Hoffman-Meeks conjecture. For a complete embedded minimal surface of genus g and k ends and finite total curvature, $k \leq g + 2$.*

References

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19 Spherical submetries

proposed by *Marco Radeschi, Notre Dame.*

A submetry is a map with equidistant fibers. In [1], we show that every submetry from the round sphere is defined by an algebra of polynomials which is both maximal and Laplacian (i.e. stable under the Laplacian). The converse is true: every maximal Laplacian algebra of polynomials defines a spherical submetry.

Question 23 *Is every Laplacian algebra of polynomials maximal?*

When the submetry is given by a global quotient $S^{n-1} \to S^{n-1}/G$ for some representation $\rho : G \subset O(n)$, the Laplacian algebra is simply the algebra of invariant polynomials. In this context, the question would solve the following problem in Classical Invariant Theory: can one compute the invariant polynomials of $\rho \oplus \rho : G \to O(2n)$ in terms of the invariant polynomials of polynomials of ρ? If Question 23 was true then, letting $x_1, \ldots, x_n, y_1, \ldots, y_n$ be the coordinates of $\mathbb{R}^n \oplus \mathbb{R}^n$, it would follow that the algebra of invariants of $\rho \oplus \rho$ is the Laplacian algebra generated by the invariant polynomials of ρ and $\delta(x_1, \ldots, y_n) = \sum_i x_i y_i$. In [2], we showed that the answer to the question above is yes, when the Laplacian algebra is generated by polynomials of degree 2. Even to prove this rather basic case, though, we had to generalize Weyl's First Fundamental Theorem of Invariant theory for $O(n)$, to the more general setup of submetries.

References

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20 Minimax minimal surfaces

proposed by *Tristan Rivière, ETH Zürich.*

We consider the space of Sobolev $W^{3,2}$-immersions of a given closed surface Σ^2 into a closed riemannian manifold N^n of dimension n greater than or equal to 3. This space is denoted by $W^{3,2}_{imm}(\Sigma^2, N^n)$. An admissible family in this space is a subset of the power set $\mathcal{P}(W^{3,2}_{imm}(\Sigma^2, N^n))$ which is invariant under the action of any homeomorphism of $W^{3,2}_{imm}(\Sigma^2, N^n)$ isotopic to the identity. Let $\mathcal{A}$ be an admissible family with non zero minimax value (i.e. non zero width)

$$\beta_{\mathcal{A}} := \inf_{A \in \mathcal{A}} \sup_{\Phi \in A} \mathrm{Area}(\Phi) > 0.$$

We prove that there exists a smooth surface S satisfying

$$genus(S) \leq genus(\Sigma),$$

and a smooth, possibly branched, minimal immersion $\Phi_{\mathcal{A}}$ of S into N^n such that

$$\beta_{\mathcal{A}} = \text{Area}(\Phi_{\mathcal{A}}).$$

The admissible family is called "homotopic of dimension d" if it is given by continuous mappings from the d-dimensional sphere S^d into $W^{3,2}_{imm}(\Sigma^2, N^n)$. We prove that for such a family the Morse index of the minimal branched immersion is bounded by d :

$$\text{Index}(\Phi_{\mathcal{A}}) \leq d.$$

Question 24 *Prove the lower bound*

$$d \leq \textit{Index}(\Phi_{\mathcal{A}}) + \textit{Null}(\Phi_{\mathcal{A}}),$$

where Null$(\Phi_{\mathcal{A}})$ *is the nullity of* $\Phi_{\mathcal{A}}$ *that is the dimension of the space of its Jacobi fields.*

Question 25 *Prove that there exists infinitely many distinct minimal branched 2-dimensional immersions in* N^n*.*

References

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21 Gromov-Hausdorff convergence of Kähler Ricci flow

proposed by *Gang Tian, BICMR, Peking University.*

Let X be a compact Kähler n-manifold with $K_X \geq 0$ and Kodaira dimension $\kappa = n$. Then the normalized Ricci flow converges to a Kähler-Einstein current, smoothly on the ample locus (where $K_X > 0$). J. Song proved that the diameter stays bounded. Therefore the completion of the smooth part is a metric on the canonical model of X. This does not suffice to imply Gromov-Hausdorff convergence, which has been proved only in dimension ≤ 3, [1]. This can be thought of as an analogue of Perelman's convergence of the Ricci flow with surgery to a union of flat or hyperbolic pieces.

When Kodaira dimension satisfies $0 < \kappa < n$, collapsing may occur. A typical example is the product $\Sigma \times T$ where Σ is a higher genus curve and T a torus. The normalized Ricci flow collapses the torus factor and the Gromov-Hausdorff limit is a constant curvature metric on Σ. More generally, one expects the fibers of a map $X \to X_{can}$ to the canonical model to collapse.

Question 26 *Does the normalized Ricci flow converge in Gromov-Hausdorff sense to a generalized Kähler-Einstein space?*

In [2], a relative volume estimate is used to control the size of fibers, and this allows to handle the case where $\kappa = 1$.

References

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[2] G. Tian, Zh. Zhang *Relative volume comparison of Ricci Flow and its applications*, ArXiv: 1802.09506

22 Totally geodesic submanifolds and positive curvature

proposed by *Wolfgang Ziller, University of Pennsylvania.*

Frankel's theorem, [2], says that in a complete connected Riemannian n-manifold of positive sectional curvature, two closed totally geodesic submanifolds of dimension n_1 and n_2 must intersect provided $n_1 + n_2 \geq n$.

Question 27 *Does Frankel's theorem hold for symmetric Finsler metrics?*

We proved that the theorem holds in dimension two, but fails for non-symmetric Finsler metrics on S^2, [1].

References

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23 Closed geodesics

proposed by *Wolfgang Ziller, University of Pennsylvania, and Miguel Angel Javaloyes Victoria, Murcia.*

It was recently proved that for a bumpy Finsler metric on S^2 (i.e., the energy function on the free loop space is Morse-Bott) there are either two or infinitely many closed geodesics, [1].

Question 28 *It is true without the bumpy assumption?*

References

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