GROUPE DE TRAVAIL: GEOMETRIZATION OF REAL REPRESENTATIONS, after Peter Scholze

Let G be a reductive group over $\mathbb{R}$ with Lie algebra $\mathfrak{g}$ and a fixed maximal compact subgroup K. The aim of this working group is to understand the following theorem due to Scholze.

Theorem 1 ([5, Theorem III.4.1]). *After restricting to the bounded part of the Harish-Chandra center of $Z(U(\mathfrak{g}))$, there is a natural t-exact equivalence between the categories $D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}})^\wedge)$ and $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$.*

Some explanations are in order. First of all, the theorem deals with so-called "derived categories of quasi-coherent sheaves" $D_{\mathrm{qc}}(X)$ of two different "analytic stacks" X. Here, analytic stacks are understood in the sense of Clausen-Scholze ([1]). For example, $* = \mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})$ denotes the analytic ring of gaseous complex numbers, $K^{\mathrm{alg}}, G^{\mathrm{alg}}$ the "algebraic" incarnations of K, G defined by their rings of algebraic functions, $(K^{\mathrm{alg}} \subseteq G^{\mathrm{alg}})^\wedge$ the formal completion of K in G and G^{la} the "locally analytic" incarnation of G^{la} defined locally by the complex-valued real analytic functions on G^{la}. The quotients $*/(-)$ refer to classifying stacks for the respective analytic group. The category $D_{\mathrm{qc}}(*)$ can be thought of as a "derived category of topological $\mathbb{C}$-vector spaces" and hence the category $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$ can reasonably be defined as the "derived category of locally analytic representations of $G(\mathbb{R})$". On the other hand, $D_{\mathrm{qc}}(*/(K^{\mathrm{alg}} \to G^{\mathrm{alg}})^\wedge)$ is the derived category of $(\mathfrak{g}, K)$-modules (on gaseous $\mathbb{C}$-vector spaces). Thus, 1 gives a good understanding of the a priori mysterious category $D_{\mathrm{qc}}(*/G^{\mathrm{la}})$. In fact, the equivalence is realized geometrically through the correspondence

$$*/(K^{\mathrm{alg}} \to G^{\mathrm{alg}})^\wedge \xleftarrow{a} */(K^{\mathrm{la}} \to G^{\mathrm{la}})^\wedge \xrightarrow{b} */G^{\mathrm{la}}$$

via (a slight variant of) the functor $b_! a^*$. The proof of 1 involves new versions of the Riemann–Hilbert correspondence and the Bernstein–Beilinson localization. In particular, going through the proof of 1 is our excuse to learn this material and the surrounding use of analytic stacks as well.

Organization

We meet weekly on Wednesday, 10-12, Room 3L15 of LMO. The first talk is on 09/10. The subsequent talks may be divided into more talks if necessary.

Talks

Talk 1, 09/10

10-11:30 : An overview of the (classical) real Langlands correspondence (L. Clozel)

11:30-12 : Overview and distribution of talks (J. Anschuetz).

Talks 2/3: Analytic rings

- light profinite sets, light condensed sets ([1, Lecture 1, Lecture 2])
- topological spaces as light condensed sets, (qc)qs condensed sets ([3, Theorem 2.16])
- $\mathbb{Z}[\mathbb{N} \cup \infty]$ is internally projective
- analytic rings [1, Lecture 8], properties of $D(A)$
- induced analytic ring structures
- (light) solid abelian groups ([1, Lecture 5])

- analytic rings constructed by pre-analytic rings ([1, Lecture 13])
- gaseous ring structure on $\mathbb{Z}((q)), \mathbb{R}, \mathbb{C}$ ([1, Lecture 12], [1, Lecture 14])
- tensor products of analytic rings ([1, Lecture 13])

Talks 4/5: Analytic stacks

- !-topology on $\mathrm{AnRings}^{\mathrm{op}}$, !-able maps, proper morphisms, open immersions ([1, Lecture 16]) of analytic rings
- !-descent implies universal $*$- and !-descent ([1, Lecture 16])
- $\mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z}) \to \mathrm{AnSpec}(\mathbb{Z}, \mathbb{Z})$ (a proper morphism), $\mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z}[T]) \to \mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z})$ (an open immersion), and the respective !-functors ([3, Lecture 8])
- analytic stacks ([1, Lecture 19])
- examples: analytification of schemes, complex analytic spaces as analytic stacks ([2]),...

Talks 6/7: The analytic Riemann–Hilbert correspondence)

- Details on Betti stacks [5, Lectures 15, 19, 20]
- [5, Chapter II]

Talks 8/9/10: Locally analytic representations of real groups

- [5, Chapter III]

Talks 11/12: Analytic Beilinson–Bernstein

- [5, Chapter IV]

References

[1] D. Clausen, P. Scholze, *Lectures on analytic stacks*, available at `https://www.youtube.com/watch?v=YxSZ1mTIpaA&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO`

[2] D. Clausen, P. Scholze, *Lectures on complex geometry*, available at `https://people.mpim-bonn.mpg.de/scholze/Complex.pdf`

[3] P. Scholze, *Lectures on condensed mathematics*, available at `https://people.mpim-bonn.mpg.de/scholze/Condensed.pdf`

[4] P. Scholze, *Emmy-Noether lectures on "Real local Langlands as geometric Langlands on the twistor-$\mathbb{P}^1$"*, available at `https://www.youtube.com/watch?v=xqbdkxEhByo`, `https://www.youtube.com/watch?v=pB7xF4M38XY`, `https://www.youtube.com/watch?v=GlUkC0JHNSw`

[5] P. Scholze, *Geometrization of the real local Langlands correspondence*, available at `https://people.mpim-bonn.mpg.de/scholze/RealLocalLanglands.pdf`