p-Adic families of Siegel modular cuspforms

Fabrizio Andreatta, Adrian Iovita and Vincent Pilloni

December 12, 2013

Abstract

Let p be an odd prime and $g \geq 2$ an integer. We prove that a finite slope Siegel cuspidal eigenform of genus g can be p-adically deformed over the g-dimensional weight space. The proof of this theorem relies on the construction of a family of sheaves of locally analytic overconvergent modular forms.

Contents

1 Introduction

After its glorious start in 1986 with H. Hida's article [Hi1], the theory of *p-adic families of modular forms* was developed in various directions and was applied in order to prove many strong results in Arithmetic Geometry. One of its first applications was in the proof of the weight two Mazur-Tate-Teitelbaum conjecture by R. Greenberg and G. Stevens in [GS] and the proof of certain cases of the Artin conjecture by K. Buzzard, M. Dickinson, N. Shepherd-Barron, R. Taylor in [BDS-BT]. An important turn in its history was marked by R. Coleman's construction of finite slope p-adic families of elliptic modular forms ([Col1] and [Col2]) and by the construction of the eigencurve by R. Coleman and B. Mazur in [C-M]. The eigencurve is a p-adic rigid analytic curve which parametrizes overconvergent elliptic modular eigenforms of finite slope.

During the last fifteen years many authors have contributed to set up a general theory of p-adic automorphic forms on higher rank groups. Some of them used an approach based on the cohomology of arithmetic groups initiated by Hida and Stevens. Hida's idea, later on developed by Emerton ([E]) was to amalgamate (more precisely take the projective limit, or as in [E], alternatively consider the inductive limit followed by p-adic completion of) cohomology groups with trivial ($\mathbb{Z}_p$) coefficients of a chosen tower of Shimura varieties. One obtains a large $\mathbb{Q}_p$-Banach space with an action of an appropriate Galois group, of the $\mathbb{Q}_p$ (or even adelic, depending on the choice of the tower) points of the group and of a certain Hecke algebra. These data were used by H. Hida in order to produce a construction of the ordinary part of the eigenvariety for a large class of Shimura varieties.

In [E] there is also a construction of finite slope eigenvarieties but so far it could not be proved that the eigenvarieties thus constructed have the right dimension except in the cases already known: for elliptic modular forms and modular forms on Shimura curves. Nevertheless the very rich structure of the completed cohomology of towers of Shimura varieties was successfully used to prove results about the p-adic local (and global) Langlands correspondence.

Stevens' approach is different, namely he uses the cohomology of one Shimura variety (of level type $\Gamma_0(Np)$ with $(N,p)=1$) with complicated coefficients (usually certain locally analytic functions or distributions on the $\mathbb{Z}_p$-points of the group) as the space (called overconvergent modular symbols or p-adic families of modular symbols) on which the Hecke operators act. These data were used by A. Ash and G. Stevens to produce eigenvarieties for $\mathrm{GL}_g/\mathbb{Q}$ in [AS]. Recently E. Urban ([Ur]) developed this method to construct equidimensional eigenvarieties of the expected dimension for modular eigensymbols of finite slope associated to reductive groups G over $\mathbb{Q}$ such that $G(\mathbb{R})$ admits discrete series.

Finally, in constructing the eigencurve some authors (including Hida and Coleman) used a geometric approach based on Dwork's ideas and Katz's theory of p-adic modular forms and overconvergent modular forms. Namely they interpolated directly the classical modular forms seen as sections of certain automorphic line bundles on the modular curve (of level prime to p) by defining overconvergent modular forms and allowing them to have essential singularities in "small p-adic disks of very supersingular points". So far this geometric approach has only been successful, in the case of higher rank groups, in producing ordinary families ([Hi2]) or one dimensional finite slope families ([K-L]). The main theme of our work is to by-pass these restrictions. In the articles [AIS] and [Pi3] we explained new points of view on the construction of the eigencurve of Coleman and Mazur. Namely, we showed that over the open modular curve (of level prime to p), which is the complement of a disjoint union of "small disks of very supersingular points", one can interpolate the classical automorphic sheaves and even construct p-adic families of such sheaves. We showed that Coleman's p-adic families can be seen as global sections of such p-adic families of sheaves.

The present paper is a development of these ideas for Siegel varieties. We prove that a cuspidal automorphic form which occurs in the H^0 of the coherent cohomology of some automorphic vector bundle on a Siegel variety can be p-adically deformed over the g dimensional weight space. We believe that the methods used in this paper would certainly apply to any PEL Shimura variety of type A and C and maybe even to those of type D (see also [Bra]).

We now give a more precise description of our main result. Let p be an odd prime, $g \geq 2$ and $N \geq 3$ two integers. We assume that $(p, N) = 1$. Let Y_{Iw} be the Siegel variety of genus g, principal level N and Iwahori level structure at p. This is the moduli space over Spec $\mathbb{Q}$ of principally polarized abelian schemes A of dimension g, equipped with a symplectic isomorphism $A[N] \simeq (\mathbb{Z}/N\mathbb{Z})^{2g}$ and a totally isotropic flag $\mathrm{Fil}_\bullet A[p]$: $0 = \mathrm{Fil}_0 A[p] \subset \ldots \subset \mathrm{Fil}_g A[p] \subset A[p]$ where $\mathrm{Fil}_i A[p]$ has rank p^i. To any g-uple $\kappa = (k_1, \ldots, k_g) \in \mathbb{Z}^g$ satisfying $k_1 \geq k_2 \ldots \geq k_g$, one attaches an automorphic locally free sheaf ω^κ on Y_{Iw}. Its global sections $\mathrm{H}^0(Y_{\mathrm{Iw}}, \omega^\kappa)$ constitute the module of classical Siegel modular forms of weight κ. It contains the sub-module of cuspidal forms $\mathrm{H}^0_{\mathrm{cusp}}(Y_{\mathrm{Iw}}, \omega^\kappa)$ which vanish at infinity. On these modules we have an action of the unramified Hecke algebra $\mathbb{T}^{Np}$ and of the dilating Hecke algebra $\mathbb{U}_p = \mathbb{Z}[U_{p,1}, \ldots, U_{p,g}]$ at p. Let f be a cuspidal eigenform and $\Theta_f \colon \mathbb{T}^{Np} \otimes \mathbb{U}_p \to \bar{\mathbb{Q}}$ be the associated character. Since f has Iwahori level at p, $\Theta_f(U_{p,i}) \neq 0$ and f is of finite slope. We fix an embedding of $\bar{\mathbb{Q}}$ in $\mathbb{C}_p$ and denote by v the valuation on $\mathbb{C}_p$ normalized by $v(p) = 1$.

Theorem 1.1. *Let f be a weight κ cuspidal eigenform of Iwahori level at p. Then there is an affinoid neighbourhood $\mathcal{U}$ of $\kappa \in \mathcal{W} = \mathrm{Hom}_{cont}((\mathbb{Z}_p^\times)^g, \mathbb{C}_p^\times)$, a finite surjective map of rigid analytic varieties*

$$w \colon \mathcal{E}_f \to \mathcal{U}$$

and a faithful, finite $\mathscr{O}_{\mathcal{E}_f}$-module $\mathcal{M}$ which is projective as an $\mathscr{O}_{\mathcal{U}}$-module, such that:

1. *$\mathcal{E}_f$ is equidimensional of dimension g.*
2. *We have a character $\Theta \colon \mathbb{T}^{Np} \otimes \mathbb{U}_p \to \mathscr{O}_{\mathcal{E}_f}$.*
3. *$\mathcal{M}$ is an $\mathscr{O}_{\mathcal{E}_f}$-module consisting of finite-slope locally analytic cuspidal overconvergent modular forms. The modular form f is an element of $\mathcal{M} \otimes^\kappa_{\mathscr{O}_{\mathcal{U}}} \mathbb{C}_p$, where the notation means that the tensor product is taken with respect to the $\mathscr{O}_{\mathcal{U}}$-module structure on $\mathbb{C}_p$ given by the ring homomorphism $\mathscr{O}_{\mathcal{U}} \longrightarrow \mathbb{C}_p$ which is evaluation of the rigid functions on $\mathcal{U}$ at κ.*
4. *There is a point $x_f \in \mathcal{E}_f$, with $w(x_f) = \kappa$ and such that the specialization of Θ at x_f is Θ_f.*
5. *For all $\mu = (m_1, \ldots, m_g) \in \mathbb{Z}^g \cap \mathcal{U}$ satisfying $m_1 \geq m_2 \ldots \geq m_g$, $v(\Theta_f(U_{p,i})) < m_{g-i} - m_{g-i+1} + 1$ for $1 \leq i \leq g-1$ and $v(\Theta_f(U_{p,g})) < m_g - \frac{g(g+1)}{2}$ the following hold*
   - *There is an inclusion $\mathcal{M} \otimes^\mu_{\mathscr{O}_{\mathcal{U}}} \mathbb{C}_p \hookrightarrow \mathrm{H}^0_{\mathrm{cusp}}(Y_{\mathrm{Iw}}, \omega^\mu)$,*
   - *For any point y in the fiber $w^{-1}(\mu)$, the character Θ_y comes from a weight μ cuspidal Siegel eigenform on Y_{Iw}.*

The rigid space $\mathcal{E}_f$ is a neighbourhood of the point x_f in an eigenvariety $\mathcal{E}$. We actually prove the following:

Theorem 1.2. *1. There is an equidimensional eigenvariety $\mathcal{E}$ and a locally finite map to the weight space $w\colon \mathcal{E} \to \mathcal{W}$. For any $\kappa \in \mathcal{W}$, $w^{-1}(\kappa)$ is in bijection with the eigensystems of $\mathbb{T}^{Np} \otimes_{\mathbb{Z}} \mathbb{U}_p$ acting on the space of finite slope locally analytic overconvergent cuspidal modular forms of weight κ.*

2. Let f be a finite slope locally analytic overconvergent cuspidal eigenform of weight κ and x_f be the point corresponding to f in $\mathcal{E}$. If w is unramified at x_f, then there is a neighbourhood $\mathcal{E}_f$ of x_f in $\mathcal{E}$ and a family F of finite slope locally analytic overconvergent cuspidal eigenforms parametrized by $\mathcal{E}_f$ and passing through f at x_f.

We expect w to be unramified at all classical points x_f that satisfy the slope conditions of theorem 1.1, 5, but don't have any general result in this direction. See section 8.3 for a more detailed discussion.

A key step in the proof of these theorems is the construction of the spaces of analytic overconvergent modular forms of any weight $\kappa \in \mathcal{W}$. They are global sections of sheaves $\omega_w^{\dagger\kappa}$ which are defined over some strict neighbourhood of the multiplicative ordinary locus of X_{Iw}, a toroidal compactification of Y_{Iw}. These sheaves are locally in the étale topology isomorphic to the w-analytic induction, from a Borel of GL_g to the Iwahori subgroup, of the character κ. They are particular examples of sheaves over rigid spaces which we call Banach sheaves, and whose properties are studied in the appendix. We view these sheaves as possible rigid analytic analogues of quasi-coherent sheaves in algebraic geometry.

One important feature of the sheaves $\omega_w^{\dagger\kappa}$ is that they vary analytically with the weight κ. One can thus define families of analytic overconvergent modular forms parameterized by the weight and construct Banach spaces of analytic overconvergent modular forms of varying weight. We have been able to show that the module of cuspidal families is a projective module (the $\mathscr{O}_{\mathcal{U}}$-module $\mathcal{M}$ appearing in the theorem above is a direct factor defined over $\mathcal{U}$ of this module of cuspidal families). Therefore one can use Coleman's spectral theory to construct g dimensional families of cuspidal eigenforms proving theorems 1.1 and 1.2; see section 8.1.3 for a more detailed discussion. The fifth part of the theorem 1.1 is a special case of the main result of [P-S] where a classicity criterion (small slope forms are classical) for overconvergent modular forms is proved for many Shimura varieties.

As mentioned above, E. Urban has constructed an eigenvariety using the cohomology of arithmetic groups. Following Chenevier ([Che1]), one can prove that the reduced eigenvarieties constructed in [Ur] and in our paper coincide. One way to think about our theorem is that every cuspidal eigenform gives a point on an equidimensional component of the eigenvariety of dimension g. In *loc. cit.* this is proved in general when the weight is cohomological, regular and the slope is non critical. One advantage of our construction is that it provides p-adic deformations of the "physical" modular eigenforms and of their q-expansions. For the symplectic groups, these carry more information than the Hecke eigenvalues.

The paper is organized as follows. In the second section, we gather some useful and now classical results about the p-adic interpolation of the algebraic representations of the group GL_g. The idea is to replace algebraic induction from the Borel to GL_g of a character by analytic induction from the Borel to the Iwahori subgroup. This is important because the automorphic sheaf ω^κ is locally over X_{Iw} the algebraic induction of the character κ. Thus, locally for the Zariski topology over X_{Iw}, interpolating the sheaves ω^κ for varying κ is equivalent to interpolating algebraic representations of GL_g. The third and fourth sections are about canonical subgroups. We recall results of [A-M], [A-G] and [Far2]. Using canonical subgroups we construct Iwahori-like subspaces in the GL_g-torsor of trivializations of the co-normal sheaf of the universal semi-abelian scheme. They are used in section five where we produce the Banach sheaves $\omega_w^{\dagger\kappa}$. Section six is about Hecke operators. We show that they act on our spaces of analytic overconvergent modular forms and we also construct a compact operator U. In section seven we relate classical modular forms and analytic overconvergent modular forms. This section relies heavily on the main result of [P-S]. In section eight we finally construct families. We let the weight κ vary in $\mathcal{W}$ and study the variation of the spaces of overconvergent analytic modular forms.

We were able to control this variation on the cuspidal part, i.e. we showed that the specialization in any p-adic weight of a family of cuspforms is surjective onto the space of cuspidal overconvergent forms of that weight. The proof of this result is the technical heart of the paper. The main difference between the case $g \geq 2$ and $g = 1$ (see section 8.1.3 for more details) is the fact that the strict neighbourhoods $\mathcal{X}_{\mathrm{Iw}}(v)$ of width v of the multiplicative ordinary locus, in some (any) toroidal compactification of the Siegel modular variety of Iwahori level, are not affinoids. Therefore, inspired by [Hi2], we studied the descent of our families of Banach sheaves $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ from the toroidal to the minimal compactification. The key observation is that the image of the strict neighbourhood $\mathcal{X}_{\mathrm{Iw}}(v)$ in the minimal compactification is an affinoid and we managed to show the acyclicity of certain Banach sheaves on affinoids. This allows us to prove the desired results, namely that one can apply Coleman's spectral theory to the modules of p-adic families of cusp forms and obtain eigenfamilies of finite slope. Moreover that any overconvergent modular form of finite slope is the specialization of a p-adic family of finite slope, in other words that any overconvergent modular form of finite slope deforms over the weight space.

Aknowledgements We thank Michael Harris, Benoît Stroh and Jacques Tilouine for many inspiring discussions on subjects pertaining to this research. We are grateful to Gaetan Chenevier for explaining us his proof of the unramifiedness of the map from the eigenvariety for certain unitary groups to the respective weight space at certain classical non-critical points and for suggesting how the same proof would work in the Siegel case. We are also grateful to the referee for the careful reading of the paper and useful suggestions which hopefully led to its improvement. Finally, we would like to thank Peter Scholze who pointed out a mistake in the appendix of the paper.

2 Families of representations of the group GL_g

We recall some classical results about Iwahoric induction using the BGG analytic resolution of [Jo] (see also [Ur]).

2.1 Algebraic representations

Let GL_g be the linear algebraic group of dimension g realized as the group of $g \times g$ invertible matrices. Let B be the Borel subgroup of upper triangular matrices, T the maximal torus of diagonal matrices, and U the unipotent radical of B. We let B^0 and U^0 be the opposite Borel of lower triangular matrices and its unipotent radical. We denote by $X(\mathrm{T})$ the group of characters of T and by $X^+(\mathrm{T})$ its cone of dominant weights with respect to B. We identify $X(\mathrm{T})$ with $\mathbb{Z}^g$ via the map which associates to a g-uple $(k_1, \ldots, k_g) \in \mathbb{Z}^g$

the character

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_g \end{pmatrix} \mapsto t_1^{k_1} \cdots t_g^{k_g}.$$

With this identification, $X^+(\mathrm{T})$ is the cone of elements $(k_1, \ldots, k_g) \in \mathbb{Z}^g$ such that $k_1 \geq k_2 \geq \ldots \geq k_g$. Till the end of this paragraph, all group schemes are considered over $\operatorname{Spec} \mathbb{Q}_p$. For any $\kappa \in X^+(\mathrm{T})$ we set

$$V_\kappa = \{ f \colon \mathrm{GL}_g \to \mathbb{A}^1 \text{ morphism of schemes s.t. } f(gb) = \kappa(b) f(g) \ \forall (g, b) \in \mathrm{GL}_g \times \mathrm{B} \}.$$

This is a finite dimensional $\mathbb{Q}_p$-vector space. The group GL_g acts on V_κ by the formula $\big(g \cdot f\big)(x) = f(g^{-1} \cdot x)$ for any $(g, f) \in \mathrm{GL}_g \times V_\kappa$. If L is an extension of $\mathbb{Q}_p$ we set $V_{\kappa,L} = V_\kappa \otimes_{\mathbb{Q}_p} L$.

2.2 The weight space

Let $\mathcal{W}$ be the rigid analytic space over $\mathbb{Q}_p$ associated to the noetherian, complete algebra $\mathbb{Z}_p[[\mathrm{T}(\mathbb{Z}_p)]]$, where let us recall T is the split torus of diagonal matrices in GL_g. Let us fix an isomorphism $\mathrm{T} \simeq \mathbb{G}_m^g$. We obtain an isomorphism $\mathrm{T}(\mathbb{Z}_p) \tilde{\to} \mathrm{T}(\mathbb{Z}/p\mathbb{Z}) \times (1 + p\mathbb{Z}_p)^g$ which implies that we have natural isomorphisms as $\mathbb{Z}_p$-algebras

$$\mathbb{Z}_p[[\mathrm{T}(\mathbb{Z}_p)]] \tilde{\longrightarrow} \big(\mathbb{Z}_p[\mathrm{T}(\mathbb{Z}/p\mathbb{Z})]\big)[[(1 + p\mathbb{Z}_p)^g]] \tilde{\longrightarrow} \big(\mathbb{Z}_p[\mathrm{T}(\mathbb{Z}/p\mathbb{Z})]\big)[[X_1, X_2, ..., X_g]],$$

where the second isomorphism is obtained by sending $(1, 1, ..., 1 + p, 1..., 1)$ with $1 + p$ on the i-th component for $1 \leq i \leq g$, to $1 + X_i$.

It follows that the $\mathbb{C}_p$-points of $\mathcal{W}$ are described by: $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{T}(\mathbb{Z}_p), \mathbb{C}_p^\times)$ and if we denote by $\widehat{\mathrm{T}(\mathbb{Z}/p\mathbb{Z})}$ the character group of $\mathrm{T}(\mathbb{Z}/p\mathbb{Z})$, the weight space is isomorphic to a disjoint union, indexed by the elements of $\widehat{\mathrm{T}(\mathbb{Z}/p\mathbb{Z})}$, of g-dimensional open unit polydiscs.

More precisely we have the following explicit isomorphism:

$$\begin{array}{rcl} \mathcal{W} & \tilde{\to} & \widehat{\mathrm{T}(\mathbb{Z}/p\mathbb{Z})} \times \prod_{i=1}^{g} B(1, 1^-) \\ \kappa & \mapsto & \big(\kappa|_{\mathrm{T}(\mathbb{Z}/p\mathbb{Z})}, \kappa((1+p, 1, \ldots, 1)), \kappa((1, 1+p, \ldots, 1)), \ldots, \kappa((1, \ldots, 1, 1+p))\big). \end{array}$$

The inverse of the above map is defined as follows: $(\chi, s_1, \cdots, s_g) \in \widehat{\mathrm{T}(\mathbb{Z}/p\mathbb{Z})} \times \prod_{i=1}^g B(1, 1^-)$ is assigned to the character which maps $(\lambda, x_1, \cdots, x_g) \in \mathrm{T}(\mathbb{Z}/p\mathbb{Z}) \times (1 + p\mathbb{Z}_p)^g$ to

$$\chi(\lambda) \prod_{i=1}^{g} s_i^{\frac{\log(x_i)}{\log(1+p)}}.$$

The universal character

If we denote by $\mathscr{O}_{\mathcal{W}}$ the sheaf of rigid analytic functions on $\mathcal{W}$, we have a natural continuous group homomorphism, obtained as the composition

$$\kappa^{\mathrm{un}} : \mathrm{T}(\mathbb{Z}_p) \longrightarrow \big(\mathbb{Z}_p[[\mathrm{T}(\mathbb{Z}_p)]]\big)^\times \longrightarrow \mathscr{O}_{\mathcal{W}}(\mathcal{W})^\times,$$

where the first map is the tautological one. We call κ^{un} the universal character. It can alternatively be seen as a pairing $\kappa^{\mathrm{un}} : \mathcal{W}(\mathbb{C}_p) \times \mathrm{T}(\mathbb{Z}_p) \longrightarrow \mathbb{C}_p^\times$ satisfying the property: for every $t \in \mathrm{T}(\mathbb{Z}_p), \kappa \in \mathcal{W}(\mathbb{C}_p)$ we have $\kappa^{\mathrm{un}}(t)(\kappa) = \kappa(t)$. If $\mathcal{U} = \mathrm{Spm}\, A \subset \mathcal{W}$ is an admissible affinoid open, we obtain a universal character for $\mathcal{U}$, $\mathrm{T}(\mathbb{Z}_p) \to A^\times$ which is the composition of κ^{un} with the natural restriction homomorphism $\mathscr{O}_{\mathcal{W}}(\mathcal{W})^\times \to A^\times$. This character will be also denoted by κ^{un} and it may be seen as an A-valued weight, i.e. $\kappa^{\mathrm{un}} \in \mathcal{W}(A)$.

Definition 2.2.1. *Let $w \in \mathbb{Q}_{>0}$. We say that a character $\kappa \in \mathcal{W}(\mathbb{C}_p)$ is w-analytic if κ extends to an analytic map*

$$\kappa \colon \mathrm{T}(\mathbb{Z}_p)(1 + p^w \mathcal{O}_{\mathbb{C}_p})^g \to \mathbb{C}_p^\times.$$

It follows from the classical p-adic properties of the exponential and the logarithm that any character κ is w-analytic for some $w > 0$. In fact, we have the following proposition:

Proposition 2.2.2 ([Ur], lem. 3.4.6)**.** *For any quasi-compact open subset $\mathcal{U} \subset \mathcal{W}$, there exists $w_{\mathcal{U}} \in \mathbb{R}_{>0}$ such that the universal character $\kappa^{\mathrm{un}} \colon \mathcal{U} \times \mathrm{T}(\mathbb{Z}_p) \to \mathbb{C}_p^\times$, extends to an analytic function $\kappa^{\mathrm{un}} \colon \mathcal{U} \times \mathrm{T}(\mathbb{Z}_p)(1 + p^{w_{\mathcal{U}}} \mathcal{O}_{\mathbb{C}_p})^g \to \mathbb{C}_p^\times$.*

In what follows we construct an admissible affinoid covering $\cup_{w>0}\mathcal{W}(w)$ of the weight space $\mathcal{W}$ with the property that for every w the restriction of the universal character κ^{un} to $\mathcal{W}(w)$ is w-analytic.

We start by fixing $w \in]n-1, n] \cap \mathbb{Q}$ and we choose a finite extension K of $\mathbb{Q}_p$ whose ring of integers, denoted $\mathcal{O}_K$ contains an element p^w of valuation w. We set $\mathfrak{W}(w)^o = \mathrm{Spf}\, \mathcal{O}_K\langle\langle S_1, \dots, S_g \rangle\rangle$, it is a formal unit polydisc over $\mathrm{Spf}\, \mathcal{O}_K$. The formal scheme $\mathfrak{W}(w)^o$ does not depend on w, but the character it carries will depend on w.

We let $\mathfrak{T}$ be the formal torus associated to T and define the formal sub-torus $\mathfrak{T}_w$ by

$$\mathfrak{T}_w(R) = \mathrm{Ker}\ \mathrm{T}(R) \to \mathrm{T}(R/p^w R)$$

for any flat, p-adically complete $\mathcal{O}_K$-algebra R.

We let $X_1', ..., X_g'$ be the coordinates on $\mathfrak{T}_w$ (so that $1 + X_i = 1 + p^w X_i'$), and attach to $\mathfrak{W}(w)^o$ a formal universal character

$$\begin{array}{rcl} \kappa^{o\mathrm{un}} \colon \mathfrak{T}_w \times \mathfrak{W}(w)^o & \to & \widehat{\mathbb{G}}_m \\ (1 + p^w X_1', \dots, 1 + p^w X_g', S_1, \dots, S_g) & \mapsto & \displaystyle\prod_{i=1}^{g} (1 + p^w X_i')^{S_i p^{-w + \frac{2}{p-1}}} \end{array}$$

Let $\mathcal{W}(w)^o$ be the rigid analytic generic fiber of $\mathfrak{W}(w)^o$. We define $\mathcal{W}(w)$ to be the fiber product:

$$\mathcal{W} \times_{\mathrm{Hom}_{cont}\left((1+p^n\mathbb{Z}_p)^g, \mathbb{C}_p^\times\right)} \mathcal{W}(w)^o,$$

where the maps used to define the fiber product are the following: $\mathcal{W} \longrightarrow \mathrm{Hom}_{cont}\big((1 + p^n\mathbb{Z}_p)^g, \mathbb{C}_p^\times\big)$ is restriction and the map $\mathcal{W}(w)^o \longrightarrow \mathrm{Hom}_{cont}\big((1 + p^n\mathbb{Z}_p)^g, \mathbb{C}_p^\times\big)$ is given by

$$(s_1, s_2, \dots, s_g) \longrightarrow \Big((1 + p^n x_1, 1 + p^n x_2, \dots, 1 + p^n x_g) \to \prod_{i=1}^{g} (1 + p^n x_i)^{s_i p^{-w + \frac{2}{p-1}}}\Big).$$

Then, $\mathcal{W} = \cup_{w>0}\mathcal{W}(w)$ is an increasing cover by affinoids. By construction, the restriction of the universal character κ^{un} of $\mathcal{W}$ to $\mathcal{W}(w)$ is w-analytic.

2.3 Analytic representations

Let I be the Iwahori sub-group of $\mathrm{GL}_g(\mathbb{Z}_p)$ of matrices whose reduction modulo p is upper triangular. Let N^0 be the subgroup of $\mathrm{U}^0(\mathbb{Z}_p)$ of matrices which reduce to the identity modulo p. The Iwahori decomposition is an isomorphism: $\mathrm{B}(\mathbb{Z}_p) \times \mathrm{N}^0 \to \mathrm{I}$. We freely identify N^0 with $(p\mathbb{Z}_p)^{\frac{g(g-1)}{2}} \subset \mathbb{A}_{\mathrm{an}}^{\frac{g(g-1)}{2}}$, where $\mathbb{A}_{\mathrm{an}}$ denotes the rigid analytic affine line defined over $\mathbb{Q}_p$. For $\epsilon > 0$, we let N^0_ϵ be the rigid analytic space

$$\bigcup_{x \in (p\mathbb{Z}_p)^{\frac{g(g-1)}{2}}} B(x, p^{-\epsilon}) \subset \mathbb{A}_{\mathrm{an}}^{\frac{g(g-1)}{2}}.$$

Let L be an extension of $\mathbb{Q}_p$ and $\mathcal{F}(\mathrm{N}^0, L)$ the ring of L-valued functions on N^0. We say that a function $f \in \mathcal{F}(\mathrm{N}^0, L)$ is ϵ-analytic if it is the restriction to N^0 of a necessarily unique analytic function on N^0_ϵ. We denote by $\mathcal{F}^{\epsilon-\mathrm{an}}(\mathrm{N}^0, L)$ the set of ϵ-analytic functions. A function is analytic if it is 1-analytic. We simply denote by $\mathcal{F}^{\mathrm{an}}(\mathrm{N}^0, L)$ the set of analytic functions. We let $\mathcal{F}^{l-\mathrm{an}}(\mathrm{N}^0, L)$ be the set of locally analytic functions on N^0 i.e. the direct limit of the sets $\mathcal{F}^{\epsilon-\mathrm{an}}(\mathrm{N}^0, L)$ for all $\epsilon > 0$.

Let $\epsilon > 0$ and $\kappa \in \mathcal{W}(L)$ be an ϵ-analytic character. We set

$$V^{\epsilon-\mathrm{an}}_{\kappa,L} = \big\{ f \colon \mathrm{I} \to L,\ f(ib) = \kappa(b) f(i)\ \forall (i,b) \in \mathrm{I} \times \mathrm{B}(\mathbb{Z}_p),\ f|_{\mathrm{N}^0} \in \mathcal{F}^{\epsilon-\mathrm{an}}(\mathrm{N}^0, L) \big\}.$$

We define similarly $V^{\mathrm{an}}_{\kappa,L}$ and $V^{l-\mathrm{an}}_{\kappa,L}$. They are all representations of the Iwahori group I.

2.4 The BGG resolution

Let W be the Weyl group of GL_g, it acts on $X(\mathrm{T})$. We set $\mathfrak{g}$ and $\mathfrak{t}$ for the Lie algebras of GL_g and T. The choice of B determines a system of simple positive roots $\Delta \subset X(\mathrm{T})$. To any $\alpha \in \Delta$ are associated an element $H_\alpha \in \mathfrak{t}$, elements $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$ and a co-root $\alpha^\vee$. We let $s_\alpha \in W$ be the symmetry $\lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. For any $w \in W$ and $\lambda \in X(\mathrm{T})$, we set $w \bullet \lambda = w(\lambda + \rho) - \rho$, where ρ is half the sum of the positive roots. By the main result of [Jo], for all $\kappa \in X^+(\mathrm{T})$, and any field extension L of $\mathbb{Q}_p$, we have an exact sequence of I-representations:

$$0 \longrightarrow V_{\kappa,L} \xrightarrow{d_0} V^{\mathrm{an}}_{\kappa,L} \xrightarrow{d_1} \bigoplus_{\alpha \in \Delta} V^{\mathrm{an}}_{s_\alpha \bullet \kappa, L} \tag{2.4.A}$$

Let us make explicit the differentials. The map d_0 is the natural inclusion, the map d_1 is the sum of maps $\Theta_\alpha \colon V^{\mathrm{an}}_{\kappa,L} \to V^{\mathrm{an}}_{s_\alpha \bullet \kappa, L}$ whose definitions we now recall. We let I act on the space of analytic functions on I by the formula $(i \star f)(j) = f(j \cdot i)$ for any analytic function f and $i, j \in I$. By differentiating we obtain an action of $\mathfrak{g}$ and hence of the enveloping algebra $U(\mathfrak{g})$ on the space of analytic functions on I. If $f \in V^{\mathrm{an}}_{\kappa,L}$ we set $\Theta_\alpha(f) = X_{-\alpha}^{\langle \kappa, \alpha^\vee \rangle + 1} \star f$. We now show that $\Theta_\alpha(f) \in V^{\mathrm{an}}_{s_\alpha \bullet \kappa, L}$. First of all let us check that $\Theta_\alpha(f)$ is $\mathrm{U}(\mathbb{Z}_p)$-invariant. It will be enough to prove that $X_\beta \star \Theta_\alpha(f) = 0$ for all $\beta \in \Delta$. If $\beta \neq \alpha$, this follows easily for $[X_\beta, X_{-\alpha}] = 0$. If $\beta = \alpha$, we have to use the relation

$$[X_\alpha, X_{-\alpha}^{\langle \kappa, \alpha^\vee \rangle}] = (\langle \kappa, \alpha^\vee \rangle + 1) X_{-\alpha} (H_\alpha - \langle \kappa, \alpha^\vee \rangle).$$

We now have

$$\begin{aligned} X_\alpha \star \Theta_\alpha(f) &= [X_\alpha, X_{-\alpha}^{\langle \kappa, \alpha^\vee \rangle + 1}] \star f \\ &= (\langle \kappa, \alpha^\vee \rangle + 1) X_{-\alpha} (H_\alpha - \langle \kappa, \alpha^\vee \rangle) \star f \\ &= 0. \end{aligned}$$

Let us find the weight of $\Theta_\alpha(f)$. For any $t \in \mathrm{T}(\mathbb{Q}_p)$, We have

$$\begin{aligned} t \star \Theta_\alpha(f) &= \mathrm{Ad}(t)(X_{-\alpha}^{\langle\kappa,\alpha^\vee\rangle+1})t \star f \\ &= \alpha^{-\langle\kappa,\alpha^\vee\rangle-1}(t)\kappa(t)\Theta_\alpha(f). \end{aligned}$$

Since we have $\alpha^{-\langle\kappa,\alpha^\vee\rangle-1}\kappa = s_\alpha \bullet \kappa$, the map Θ_α is well defined.

2.5 A classicity criterion

For $1 \leq i \leq g-1$ we set $d_i = \begin{pmatrix} p^{-1}1_{g-i} & 0 \\ 0 & 1_i \end{pmatrix} \in \mathrm{GL}_g(\mathbb{Q}_p)$. The adjoint action of d_i on $\mathrm{GL}_g/\mathbb{Q}_p$ stabilizes the Borel subgroup B. The formula $(\delta_i \cdot f)(g) := f(d_i g d_i^{-1})$ defines an action on the space V_κ for any $\kappa \in X^+(\mathrm{T})$. We now define the action on the spaces $V_{\kappa,L}^{\epsilon-\mathrm{an}}$ for any $\kappa \in \mathcal{W}(L)$. We have a well-defined adjoint action of d_i on the group N^0. Let $f \in V_\kappa^{\epsilon-\mathrm{an}}$ and $j \in \mathrm{I}$. Let $j = n \cdot b$ be the Iwahori decomposition of j. We set $\delta_i f(j) := f(d_i n d_i^{-1} b)$. We hence get operators δ_i on $V_{\kappa,L}^{\epsilon-\mathrm{an}}$ and $V_{\kappa,L}^{l-\mathrm{an}}$. Let $z_{k,l}$ be the (k,l)-matrix coefficient on GL_g. If we use the isomorphism $V_{\kappa,L}^{\epsilon-\mathrm{an}} \to \mathcal{F}^{\epsilon-\mathrm{an}}(\mathrm{N}^0, L)$ given by the restriction of functions to N^0, then the operator δ_i is given by

$$\begin{aligned} \delta_i \colon \mathcal{F}^{\epsilon-\mathrm{an}}(\mathrm{N}^0, L) &\to \mathcal{F}^{\epsilon-\mathrm{an}}(\mathrm{N}^0, L) \\ f &\mapsto [(z_{k,l})_{k<l} \mapsto f(p^{n_{k,l}} z_{k,l})] \end{aligned}$$

where $n_{k,l} = 1$ if $k \geq g-i+1$ and $l \leq g-i$ and $n_{k,l} = 0$ otherwise. The operator δ_i is norm decreasing and the operator $\prod_i \delta_i$ on $V_\kappa^{\epsilon-\mathrm{an}}$ is completely continuous.

If $\kappa \in X(\mathrm{T})^+$ the map d_0 in the exact sequence (2.4.A) is δ_i-equivariant. Regarding the map d_1 we have the following variance formula

$$\delta_i \Theta_\alpha = \alpha(d_i)^{\langle\kappa,\alpha^\vee\rangle+1}\Theta_\alpha\delta_i.$$

Indeed for any $f \in V_{\kappa,L}^{\epsilon-\mathrm{an}}$ we have

$$\begin{aligned} \delta_i\Theta_\alpha(f) &= d_i \cdot (d_i^{-1} X_{-\alpha}^{\langle\kappa,\alpha^\vee\rangle+1} \star f) \\ &= \alpha(d_i)^{\langle\kappa,\alpha^\vee\rangle+1} d_i \cdot (X_{-\alpha}^{\langle\kappa,\alpha^\vee\rangle+1} d_i^{-1} \star f) \\ &= \alpha(d_i)^{\langle\kappa,\alpha^\vee\rangle+1}\Theta_\alpha(\delta_i f). \end{aligned}$$

Let $\underline{v} = (v_1, \ldots, v_{g-1}) \in \mathbb{R}^{g-1}$. We let $V_\kappa^{l-\mathrm{an},<\underline{v}}$ be the union of the generalized eigenspaces where δ_i acts by eigenvalues of valuation strictly smaller than v_i. We are now able to give the classicity criterion.

Proposition 2.5.1. *Let $\kappa = (k_1, \ldots, k_g) \in X^+(\mathrm{T})$. Set $v_{g-i} = k_i - k_{i+1} + 1$ for $1 \leq i \leq g-1$. Then any element $f \in V_{\kappa,L}^{l-\mathrm{an},<\underline{v}}$ is in $V_{\kappa,L}$.*

Proof One easily checks that any element $f \in V_{\kappa,L}^{l-\mathrm{an},<\underline{v}}$ is actually analytic because the operators δ_i increase the radius of analyticity. Using the exact sequence (2.4.A), we need to see that $d_1.f = 0$. Let α be the simple positive root given by the character $(t_1, \ldots, t_g) \mapsto t_i.t_{i+1}^{-1}$. Since $\delta_{g-i}\Theta_\alpha(f) = p^{k_{i+1}-k_i-1}\Theta_\alpha\delta_{g-i}(f)$ we see that $\Theta_\alpha(f)$ is a generalized eigenvector for δ_{g-i} for eigenvalues of negative valuation. But the norm of δ_{g-i} is less than 1 so $\Theta_\alpha(f)$ has to be zero. □

3 Canonical subgroups over complete discrete valuation rings

3.1 Existence of canonical subgroups

Let $p > 2$ be a prime integer and K a complete valued extension of $\mathbb{Q}_p$ for a valuation $v\colon K \to \mathbb{R} \cup \{\infty\}$ such that $v(p) = 1$. Let $\bar{K}$ be an algebraic closure of K. We denote by $\mathcal{O}_K$ the ring of elements of K having non-negative valuation and set $v\colon \mathcal{O}_K/p\mathcal{O}_K \to [0,1]$ to be the truncated valuation defined as follows: if $x \in \mathcal{O}_K/p\mathcal{O}_K$ and $\hat{x}$ is a (any) lift of x in $\mathcal{O}_K$, set $v(x) = \inf\{v(\hat{x}), 1\}$. For any $w \in v(\mathcal{O}_K)$ we set $\mathfrak{m}(w) = \{x \in K, v(x) \geq w\}$ and $\mathcal{O}_{K,w} = \mathcal{O}_K/\mathfrak{m}(w)$. If M is an $\mathcal{O}_K$-module, then M_w denotes $M \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w}$. If M is a torsion $\mathcal{O}_K$-module of finite presentation, there is an integer r such that $M \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{K,a_i}$ for real numbers $a_i \in v(\mathcal{O}_K)$. We set $\deg M = \sum_i v(a_i)$.

Let H be a group scheme over $\mathcal{O}_K$ and let ω_H denote the co-normal sheaf along the unit section of H. If H is a finite flat group scheme, ω_H is a torsion $\mathcal{O}_K$-module of finite presentation and the degree of H, denoted $\deg H$, is by definition the degree of ω_H (see [Far1] where the degree is used to define the Harder-Narasimhan filtration of finite flat group schemes).

Let G be a Barsotti-Tate group over Spec $\mathcal{O}_K$ of dimension g (for example the Barsotti-Tate group associated to an abelian scheme of dimension g). Consider the $\mathcal{O}_{K,1}$-module Lie $G[p]$. We denote by σ the Frobenius endomorphism of $\mathcal{O}_{K,1}$. The module Lie $G[p]$ is equipped with a σ-linear Frobenius endomorphism whose determinant, called the Hasse invariant of G is denoted $\mathrm{Ha}(G)$. The Hodge height of G, denoted $\mathrm{Hdg}(G)$ is the truncated valuation of $\mathrm{Ha}(G)$.

Canonical subgroups have been constructed by Abbes-Mokrane, Andreatta-Gasbarri, Tian and Fargues. In the sequel we quote mostly results of Fargues.

Theorem 3.1.1 ([Far2], thm. 6). *Let $n \in \mathbb{N}$. Assume that $\mathrm{Hdg}(G) < \frac{1}{2p^{n-1}}$ (resp. $\frac{1}{3p^{n-1}}$ if $p = 3$). Then the n-th step of the Harder-Narasimhan filtration of $G[p^n]$, denoted H_n is called the canonical subgroup of level n of G. It enjoys the following properties.*

1. $H_n(\bar{K}) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$.
2. $\deg H_n = ng - \frac{p^n-1}{p-1}\mathrm{Hdg}(G)$.
3. *For any $1 \leq k \leq n$, $H_n[p^k]$ is the canonical subgroup of level k of G.*
4. *In $G|_{\mathrm{Spec}\ \mathcal{O}_{K,1-\mathrm{Hdg}(G)}}$ we have that $H_1|_{\mathrm{Spec}\ \mathcal{O}_{K,1-\mathrm{Hdg}(G)}}$ is the Kernel of Frobenius.*
5. *For any $1 \leq k < n$, $\mathrm{Hdg}(G/H_k) = p^k\mathrm{Hdg}(G)$ and H_n/H_k is the canonical subgroup of level $n-k$ of G/H_k.*
6. *Let G^D be the dual Barsotti-Tate group of G. Denote by $H_n^\perp$ the annihilator of H_n under the natural pairing $G[p^n] \times G^D[p^n] \to \mu_{p^n}$. Then $\mathrm{Hdg}(G^D) = \mathrm{Hdg}(G)$ and $H_n^\perp$ is the canonical subgroup of level n of G^D.*

The theorem states in particular that if the Hodge height of G is small, there is a (canonical) subgroup of high degree and rank g inside $G[p]$. The converse is also true.

Proposition 3.1.2. *Let $H \hookrightarrow G[p]$ be a finite flat subgroup scheme of $G[p]$ of rank g. The following are equivalent:*

1. $\deg H > g - \frac{1}{2}$ *if $p \neq 3$ or* $\deg H > g - \frac{1}{3}$ *if $p = 3$,*
2. $\mathrm{Hdg}(G) < \frac{1}{2}$ *if $p \neq 3$ or* $\mathrm{Hdg}(G) < \frac{1}{3}$ *if $p = 3$, and H is the canonical subgroup of level 1 of G.*

Proof In view of theorem 3.1.1, we only need to show that the first point implies the second. Set $v = g - \deg H$. It is enough to prove that $v < \frac{1}{2}$ if $p \neq 3$ or $v < \frac{1}{3}$ if $p = 3$ implies that $\mathrm{Hdg}(G) \leq v$. Indeed, by theorem 3.1.1, G will admit a canonical subgroup of level 1, which is a step of the Harder-Narasimhan filtration of $G[p]$. On the other hand, proposition 15 of [Far2] shows that H is a step of the Harder-Narasimhan filtration of $G[p]$. It follows that H is the canonical subgroup of level 1 of G.

Let $\bar{H}$ and $\bar{G}[p]$ denote the restrictions of H and $G[p]$ to Spec $\mathcal{O}_{K,1}$. Note that there are canonical identifications $\omega_{\bar{G}[p]} \simeq \omega_{G[p]} \simeq \omega_G/p\omega_G$ and $\omega_H \simeq \omega_{\bar{H}}$. We use the superscripts $\_^{(p)}$ to denote base change by the Frobenius map $\sigma : \mathcal{O}_{K,1} \to \mathcal{O}_{K,1}$. We have a functorial Vershiebung morphism $V\colon \bar{H}^{(p)} \to \bar{H}$ and $V\colon \bar{G}[p]^{(p)} \to \bar{G}[p]$. Taking the induced map on the co-normal sheaves we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & \omega_{\bar{G}[p]/\bar{H}} & \longrightarrow & \omega_{\bar{G}[p]} & \xrightarrow{\phi} & \omega_{\bar{H}} & \longrightarrow 0 \\
& \downarrow{\scriptstyle V^*} & & \downarrow{\scriptstyle V^*} & & \downarrow{\scriptstyle V^*} & \\
& \omega^{(p)}_{\bar{G}[p]/\bar{H}} & \longrightarrow & \omega^{(p)}_{\bar{G}[p]} & \xrightarrow{\phi\otimes 1} & \omega^{(p)}_{\bar{H}} & \longrightarrow 0
\end{array}$$

We have an isomorphism $\mathcal{O}^g_{K,1} \simeq \omega_{G[p]}$ and $\mathrm{Ker}\phi \subset p^{1-v}\omega_{G[p]}$ since $\deg G[p]/H = v$ by [Far1], lem. 4. As a result, there is a surjective map $\omega_{\bar{H}} \to \mathcal{O}^g_{K,1-v}$. We thus obtain a surjective map $\omega^{(p)}_{\bar{H}} = \omega_{\bar{H}} \otimes_{\mathcal{O}_{K,1},\sigma} \mathcal{O}_{K,1} \to \mathcal{O}^g_{K,1-v} \otimes_{\mathcal{O}_{K,1},\sigma} \mathcal{O}_{K,1} \simeq \mathcal{O}^g_{K,1}$. The map $\phi \otimes 1\colon \omega^{(p)}_{\bar{G}[p]} \to \omega^{(p)}_{\bar{H}}$ is a surjective map between two finite $\mathcal{O}_{K,1}$-modules which are isomorphic, so it is an isomorphism. As $\mathrm{Hdg}(G)$ can also be computed as the truncated valuation of the determinant of V^* on $\omega^{(p)}_{\bar{G}[p]}$, we conclude that $\mathrm{Hdg}(G) = \deg(\omega^{(p)}_{\bar{G}[p]}/V^*\omega_{\bar{G}[p]}) = \deg(\omega^{(p)}_{\bar{H}}/V^*\omega_{\bar{H}})$. We are thus reduced to compute the map V^* at the level of the group $\bar{H}$.

After possibly extending K, we can find an increasing filtration of H_K by finite flat subgroups $\{\mathrm{Fil}_i H_K\}_{0\leq i\leq g}$ where $\mathrm{Fil}_i H_K$ has rank p^i. Taking schematic closures we obtain an increasing filtration of H by finite flat subgroups $\{\mathrm{Fil}_i H\}_{0\leq i\leq g}$ where $\mathrm{Fil}_i H$ has rank p^i. We set $\mathrm{Gr}_k H = \mathrm{Fil}_k H/\mathrm{Fil}_{k-1} H$. This is a finite flat group scheme of order p for every k. We let $\{\mathrm{Fil}_i \bar{H}\}_{0\leq i\leq g}$ be the filtration of $\bar{H}$ obtained via base change to Spec $\mathcal{O}_{K,1}$ and $\{\mathrm{Fil}_i \bar{H}^{(p)}\}_{0\leq i\leq g}$ be the filtration of $\bar{H}^{(p)}$ induced by pull-back under σ. We obtain a decreasing filtration on the differentials by setting $\mathrm{Fil}^i\omega_{\bar{H}} = \mathrm{Ker}(\omega_{\bar{H}} \to \omega_{\mathrm{Fil}_i\bar{H}})$. Taking differentials in the exact sequence $0 \to \mathrm{Fil}_{k-1}H \to \mathrm{Fil}_k H \to \mathrm{Gr}_k H \to 0$ provides an isomorphism $\mathrm{Gr}^k\omega_{\bar{H}} := \mathrm{Fil}^{k-1}\omega_{\bar{H}}/\mathrm{Fil}^k\omega_{\bar{H}} \simeq \omega_{\mathrm{Gr}_k\bar{H}}$. Similarly, we set $\mathrm{Fil}^i\omega^{(p)}_{\bar{H}} = \mathrm{Ker}(\omega^{(p)}_{\bar{H}} \to \omega^{(p)}_{\mathrm{Fil}_i\bar{H}})$ and there is a surjective map $\omega^{(p)}_{\mathrm{Gr}_k\bar{H}} \to \mathrm{Gr}^k(\omega^{(p)}_{\bar{H}})$. But as before, it is easy to see that both modules are isomorphic to $\mathcal{O}_{K,1}$ and this map is an isomorphism.

The map V^* respects these filtrations and a straightforward calculation using Oort-Tate theory shows that $\deg(\omega^{(p)}_{\mathrm{Gr}^k\bar{H}}/V^*\omega_{\mathrm{Gr}^k\bar{H}}) = 1 - \deg(\mathrm{Gr}^k H)$. Hence, $\deg(\omega^{(p)}_{\bar{H}}/V^*\omega_{\bar{H}}) \leq \sum_k \deg \mathrm{Gr}^k\omega^{(p)}_{\bar{H}}/V^*\mathrm{Gr}^k\omega_{\bar{H}} = g - \sum_k \deg(\mathrm{Gr}^k H)$. Since $\sum_k \deg(\mathrm{Gr}^k H) = \deg H$, we conclude that $\mathrm{Hdg}(G) = \deg(\omega^{(p)}_{\bar{H}}/V^*\omega_{\bar{H}}) \leq g - \deg H = v$ as claimed. □

3.2 The Hodge-Tate maps for H_n and $G[p^n]$

In this section we work under the hypothesis of theorem 3.1.1, i.e let us recall that G was a Barsotti-Tate group of dimension g such that $v := \mathrm{Hdg}(G) < \frac{1}{2p^{n-1}}$ (resp. $\frac{1}{3p^{n-1}}$

if $p = 3$) and we denoted $H_n \subset G[p^n]$ its level n canonical subgroup. We now define the Hodge-Tate map for H_n^D (viewed as a map of abelian sheaves on the $fppf$-topology):

$$\mathrm{HT}_{H_n^D}\colon H_n^D \to \omega_{H_n},$$

by sending an S-valued point $x \in H_n^D(S)$, i.e., a homomorphism of S-group schemes $x\colon H_{n,S} \to \mu_{p^n,S}$, to the pull–back $x^*(dt/t) \in \omega_{H_n}(S)$ of the invariant differential dt/t of $\mu_{p^n,S}$ (see [Me], p. 117 for a more complete discussion).

Following the conventions of section 3.1 we write $\omega_{G[p^n],w}$, resp. $\omega_{H_n,w}$ for $\omega_{G[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w}$, resp. $\omega_{H_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w}$.

Proposition 3.2.1. *1. The differential of the inclusion $H_n \hookrightarrow G[p^n]$ induces an isomorphism*

$$\omega_{G[p^n],n-v\frac{p^n-1}{p-1}} \xrightarrow{\sim} \omega_{H_n,n-v\frac{p^n-1}{p-1}}.$$

2. The linearized Hodge-Tate map

$$\mathrm{HT}_{H_n^D} \otimes 1\colon H_n^D(\bar{K}) \otimes_{\mathbb{Z}} \mathcal{O}_{\bar{K}} \to \omega_{H_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$$

has cokernel of degree $\frac{v}{p-1}$.

Proof We have an exact sequence:

$$0 \to H_n \to G[p^n] \to G[p^n]/H_n \to 0$$

which induces an exact sequence:

$$0 \to \omega_{G[p^n]/H_n} \to \omega_{G[p^n]} \to \omega_{H_n} \to 0$$

We know that $\omega_{G[p^n]} \simeq \mathcal{O}_{K,n}^g$ and that $\deg G[p^n]/H_n = \frac{p^n-1}{p-1}v$ so the first claim follows.

There is a commutative diagram:

$$\begin{array}{ccc} H_n^D(\bar{K}) \otimes_{\mathbb{Z}} \mathcal{O}_{\bar{K}} & \xrightarrow{\mathrm{HT}_{H_n^D}\otimes 1} & \omega_{H_n} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}} \\ \downarrow & & \downarrow \\ H_1^D(\bar{K}) \otimes_{\mathbb{Z}} \mathcal{O}_{\bar{K}} & \xrightarrow{\mathrm{HT}_{H_1^D}\otimes 1} & \omega_{H_1} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}} \end{array}$$

We know by the proof of theorem 4 of [Far2] that $\mathrm{HT}_{H_1^D} \otimes 1$ has cokernel of degree $\frac{v}{p-1}$. The same holds for the Hodge-Tate map $\mathrm{HT}_{H_n^D} \otimes 1$. □

Although the results of proposition 3.2.1 are all that we need for later use, we'd like to go further and analyze the Hodge-Tate map for the group $G[p^n]$:

$$\mathrm{HT}_n\colon G[p^n](\mathcal{O}_{\bar{K}}) \to \omega_{G^D[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}.$$

The following result is implicit in [Far2] (see also [A-G], sect. 13.2 when $n = 1$).

Proposition 3.2.2. *We use the notations of proposition 3.2.1. Assume that $v < \frac{p-1}{p(p^n-1)}$. Then the following natural sequence: is exact.*

$$0 \to H_n(\bar{K}) \to G[p^n](\bar{K}) \overset{\mathrm{HT}_{G[p^n]}}{\to} \omega_{G^D[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}.$$

Furthermore, the cokernel of the map $\mathrm{HT}_{G[p^n]} \otimes 1\colon G[p^n](\bar{K}) \otimes_{\mathbb{Z}} \mathcal{O}_{\bar{K}} \to \omega_{G^D[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$ is of degree $\frac{v}{p-1}$.

Proof An easy calculation using Oort-Tate theory shows that for any group scheme $H \to \operatorname{Spec} \mathcal{O}_{\bar{K}}$ of order p and degree at least $1 - \frac{1}{p}$, the Hodge-Tate map $\mathrm{HT}_H\colon H(\bar{K}) \to \omega_{H^D}$ is zero. By hypothesis we have $\deg H_n \geq ng - \frac{1}{p}$ and we can thus filter H_n by group schemes such that each graded quotient is of order p and has degree at least $1 - \frac{1}{p}$. A straightforward dévissage now proves that the map $\mathrm{HT}_{H_n}\colon H_n(\bar{K}) \to \omega_{H_n^D} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$ is zero and so the map $\mathrm{HT}_{G[p^n]}\colon H_n(\bar{K}) \to \omega_{G^D[p^n]}$ is also zero. The rest of the proposition follows from the proof of theorem 4 of [Far2] as in proposition 3.2.1. □

Applying this proposition to G^D and using the fact that $H_n^{\perp}$ is the canonical subgroup of G^D we obtain a map

$$\hat{\mathrm{HT}}\colon H_n^D(\bar{K}) \to \omega_{G[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}},$$

which is a lift of the map $\mathrm{HT}_{H_n^D}$.

Remark 3.2.3. There is a rationality issue with the map $\hat{\mathrm{HT}}$. If K' is the finite extension of K fixed by an open sub-group Γ of $\mathrm{Gal}(\bar{K}/K)$ we obtain an induced map:

$$\hat{\mathrm{HT}}\colon H_n^D(K') \to (\omega_{G[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}})^{\Gamma}.$$

There is an injection $\omega_{G[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \hookrightarrow (\omega_{G[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{\bar{K}})^{\Gamma}$ which may be strict and there is no reason for $\hat{\mathrm{HT}}(H_n^D(K'))$ to lie in $\omega_{G[p^n]} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. But if we reduce modulo $p^{n - v\frac{p^n-1}{p-1}}$, then $\hat{\mathrm{HT}}$ coincides with $\mathrm{HT}_{H_n^D}$ in $\omega_{G[p^n], n - v\frac{p^n-1}{p-1}} = \omega_{H_n, n - v\frac{p^n-1}{p-1}}$.

3.3 Canonical subgroups for semi-abelian schemes

We will need to apply the results of the last section in the setting of semi-abelian schemes. Let S be a noetherian scheme and U a dense open subset. We will use the notions of 1-motives over U and Mumford 1-motives over $U \hookrightarrow S$ as follows.

Definition 3.3.1 ([De], def. 10.1.1, [Str2], def. 1.3.1)**.** *A* 1*-motive over* U *is a complex of* fppf *abelian sheaves* $[Y \to \tilde{G}]$ *concentrated in degree* -1 *and* 0 *where*

1. $\tilde{G} \to U$ *is a semi-abelian scheme which is an extension*

$$0 \to T \to \tilde{G} \to A \to 0$$

   with T *a torus and* A *an abelian scheme,*

2. $Y \to U$ *is an isotrivial sheaf.*

A Mumford 1*-motive over* $U \hookrightarrow S$ *is the data of:*

1. *A semi-abelian scheme* $\tilde{G} \to S$ *which is an extension*

$$0 \to T \to \tilde{G} \to A \to 0$$

   with T *a torus over* S *and* A *an abelian scheme over* S*,*

2. $Y \to S$ *an isotrivial sheaf,*

3. $[Y_U \to \tilde{G}_U]$ *a* 1*-motive over* U*.*

Given $M = [Y \to \tilde{G}]$ a 1-motive over U and an integer n, we define the n-torsion of M as the H^{-1} of the cone of the multiplication by n map $M \to M$. It comes with a filtration $\mathrm{Fil}_\bullet$. The group $\tilde{G}$ is an extension $0 \to T \to \tilde{G} \to A \to 0$, and $\mathrm{Fil}_0 = T[n]$, $\mathrm{Fil}_1 = \tilde{G}[n]$, $\mathrm{Fil}_2 = M[n]$.

Assume that $S = \mathrm{Spec}\ \mathcal{O}_K$, that $U = \mathrm{Spec}\ K$. We say that a Mumford 1-motive $M = [Y \to \tilde{G}]$ over $U \hookrightarrow S$ has a canonical subgroup of level n if $\tilde{G}$ has a canonical subgroup of level n.

4 Canonical subgroups in families and applications

4.1 Families of canonical subgroups

We start by introducing some categories of $\mathcal{O}_K$-algebras. We let **Adm** be the category of admissible $\mathcal{O}_K$-algebras, by which we mean flat $\mathcal{O}_K$-algebras which are quotients of rings of restricted power series $\mathcal{O}_K\langle X_1, \dots, X_r\rangle$, for some $r \geq 0$. We let **NAdm** be the category of normal admissible $\mathcal{O}_K$-algebras.

Let R be an object of **Adm**. We have a supremum semi-norm on $R[1/p]$ denoted by $|\ |$. If R is in **NAdm** then $|\ |$ is a norm and the unit ball for this norm is precisely R.

For any object R in **Adm**, we let $R-\mathbf{Adm}$ be the category of R-algebras which are admissible as $\mathcal{O}_K$-algebras. We define similarly $R-\mathbf{NAdm}$.

If $w \in v(\mathcal{O}_K)$ we set as before $R_w = R \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w}$ and for any R-module M, M_w means $M \otimes_R R_w$.

Till the end of this section we fix an object R of **NAdm**. We set $S = \mathrm{Spec}\ R$, and S_{rig} is the rigid analytic space associated to $R[\frac{1}{p}]$. We will study the p-adic properties of certain semi-abelian schemes over S and their canonical subgroups. We make the following assumptions.

Let U be a dense open sub-scheme of S and G a semi-abelian scheme over S such that $G|_U$ is abelian. We assume that there exists $\tilde{G}$, a semi-abelian scheme over S with constant toric rank, Y an isotrivial sheaf over S and $M = [Y \to \tilde{G}]$ a Mumford 1-motive over $U \hookrightarrow S$ such that $M[p^n] \simeq G[p^n]$ over U and that $\tilde{G}[p^n] \hookrightarrow G[p^n]$. For $x \in S_{\mathrm{rig}}$ we write $\mathrm{Hdg}(x)$ for $\mathrm{Hdg}\big(\tilde{G}_x[p^\infty]\big)$.

Remark 4.1.1. The group $G[p^n]$ is not finite flat in general (unless G has constant toric rank over S) but under the hypothesis above it has a finite flat sub-group $\tilde{G}[p^n]$ which we use as a good substitute.

Remark 4.1.2. In our applications R will come from the p-adic completion of an étale affine open sub-set of the toroidal compactification of the Siegel variety. If this open sub-set does not meet the boundary, then the semi-abelian scheme G will be abelian and the situation is simple. On the other hand we can cover the boundary by étale affine open sub-sets such that G comes by approximation from a semi-abelian scheme constructed out of a 1-motive M (by Mumford's construction). In the approximation process it is possible to preserve the p^n-torsion of M as explained in [Str2], section 2.3.

Using the previous notations and assumptions on S and G we now make further assumptions on the Hodge height. Let $v < \frac{1}{2p^{n-1}}$ (resp. $v < \frac{1}{3p^{n-1}}$ if $p = 3$) such that for any $x \in S_{\mathrm{rig}}$, $\mathrm{Hdg}(x) < v$. For any point $x \in S_{\mathrm{rig}}$, G has a canonical sub-group of order n. By the properties of the Harder-Narasimhan filtration there is a finite flat sub-group $H_{n,K} \subset G|_{S_{\mathrm{rig}}}$ interpolating the canonical sub-groups of level n for all the points $x \in S_{\mathrm{rig}}$.

Proposition 4.1.3. *The canonical subgroup extends to a finite flat subgroup scheme $H_n \hookrightarrow G[p^n]$ over S.*

Proof We first assume $n = 1$. Let $Gr \to S = \mathrm{Spec}\ R$ be the proper scheme which parametrizes all finite flat subgroups of $\tilde{G}[p]$ of rank p^g over S. We have a section $s\colon S_K \to Gr_K$ given by the canonical subgroup. Let T be the schematic closure of $s(S_K)$ in Gr. The map $T \to S$ is proper. We let $H \to T$ be the universal subgroup. We first show that $T \to S$ is finite. Let k be the residue field of $\mathcal{O}_K$. It is enough to prove that for all $x \in T_k$, H_x is the kernel of the Frobenius morphism; indeed this will imply that $T \to S$ is quasi-finite, hence finite. So let $x \in T_k$ and let $x_1 \rightsquigarrow x_2 \ldots \rightsquigarrow x$ be a sequence of immediate specializations of maximal length. Clearly, $x_1 \in T_K$ since T is the closure of its generic fiber. So let x_j and x_{j+1} be such that $x_j \in T_K$ and $x_{j+1} \in T_k$. Let V be the closure of x_j in T, V' be the localization of V at x_{j+1} and V'' be the normalization of V'. Then V'' is a discrete valuation ring of mixed characteristic. So $H_{V''}$ is generically the canonical subgroup and by the general theory over discrete valuation rings (see theorem 3.1.1), H_{x_k} is the kernel of Frobenius. As a result H_x is the kernel of Frobenius as well. Now set $T = \mathrm{Spec}\ B$. By construction B is torsion free, and hence it is flat, as $\mathcal{O}_K$-module. Furthermore it is a finite R-module and $R_K = B_K$. Since R is normal, $B = R$.

By induction, we assume that the proposition is known for $n-1$ and prove it for $n \geq 2$. We define H_n by the cartesian square (where H_n/H_1 is the canonical subgroup of level $n-1$ for $\tilde{G}[p^n]/H_1$ by theorem 3.1.1):

$$\begin{array}{ccc} H_n & \longrightarrow & \tilde{G}[p^n] \\ \downarrow & & \downarrow \\ H_n/H_1 & \longrightarrow & \tilde{G}[p^n]/H_1 \end{array}$$

all vertical maps are finite flat. Since H_n/H_1 is finite flat over S, we are done. □

4.2 The Hodge-Tate map in families

In this paragraph we investigate the properties of the map of $fppf$ abelian sheaves $\mathrm{HT}_{H_n^D}\colon H_n^D \to \omega_{H_n}$. We work using the notations and assumptions of section §4.1.

Proposition 4.2.1. *Let $w \in v(\mathcal{O}_K)$ with $w < n - v\frac{p^n-1}{p-1}$. The morphism of coherent sheaves $\omega_G \to \omega_{H_n}$ induces an isomorphism $\omega_{G,w} \to \omega_{H_n,w}$.*

Proof Possibly after replacing R with an open affine formal covering, we may assume that ω_G is a free R-module. Fix an isomorphism $\omega_G \cong R^g$. Consider the surjective map $\alpha\colon R^g \cong \omega_G \to \omega_{H_n,w}$ given by the inclusion $H_n \subset G$. It suffices to show that any element $(x_1, \ldots, x_g) \in \mathrm{Ker}(\alpha)$ satisfies $x_i \in p^w R$ for every $i = 1, \ldots, g$. As R is normal, it suffices to show that for every codimension 1 prime ideal $\mathfrak{P}$ of R containing (p) we have $x_i \in p^w R_{\mathfrak{P}}$ or equivalently $x_i \in p^w \widehat{R}_{\mathfrak{P}}$. Here, $R_{\mathfrak{P}}$ is a discrete valuation ring of mixed characteristic and $\widehat{R}_{\mathfrak{P}}$ is its p-adic completion. We are then reduced to prove the claim over a complete dvr and this is the content of proposition 3.2.1. □

Proposition 4.2.2. *Assume that $H_n^D(R) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$. The cokernel of the map*

$$\mathrm{HT}_{H_n^D} \otimes 1\colon H_n^D(R) \otimes_{\mathbb{Z}} R \to \omega_{H_n}$$

is killed by $p^{\frac{v}{p-1}}$.

Proof Possibly after localization on R, we may assume that ω_G is a free R-module of rank g. We have a surjection $R^g \simeq \omega_G \to \omega_{H_n}$. If we fix a basis of $H_n^D(R)$, we also have a surjection $R^g \to H_n^D(R) \otimes_{\mathbb{Z}} R \simeq R_n^g$. In these presentations, the map $\mathrm{HT}_{H_n^D} \otimes 1$ is given by a matrix $\gamma \in \mathrm{M}_g(R)$. Let $d \in R$ be the determinant of the matrix γ . Then, d annihilates the cokernel of γ. It suffices to prove that $p^{\frac{v}{p-1}} \in dR$. As R is normal, it suffices to prove that $p^{\frac{v}{p-1}} \in dR_{\mathfrak{P}}$ for every codimension 1 prime ideal $\mathfrak{P}$ of R containing p. It follows from proposition 3.2.1 that $p^{\frac{v}{p-1}} \in dR_{\mathfrak{P},n-v\frac{p^n-1}{p-1}}$. As $\frac{v}{p-1} < n - v\frac{p^n-1}{p-1}$, we conclude that $p^{\frac{v}{p-1}} \in dR_{\mathfrak{P}}$ as wanted. □

4.3 The locally free sheaf $\mathcal{F}$

We work in the hypothesis of section §4.1, i.e. let us recall that we have fixed $R \in$ **NAdm** and a semi-abelian scheme G over $S := \mathrm{Spec}(R)$ such that the restriction of G to a dense open sub-scheme U of S is abelian. We also fix a rational number v such that $v < \frac{1}{2p^{n-1}}$ (resp. $v < \frac{1}{3p^{n-1}}$ if $p = 3$) with the property that for any $x \in S_{\mathrm{rig}}$, $\mathrm{Hdg}(x) < v$. Here $\mathrm{Hdg}(x) := \mathrm{Hdg}\big(G_x[p^\infty]\big)$. Let H_n denote the canonical subgroup of G of level n over S. From now on, we also assume that $H_n^D(R) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$. We then have the following fundamental proposition.

Proposition 4.3.1. *There is a free sub-sheaf of R-modules $\mathcal{F}$ of ω_G of rank g containing $p^{\frac{v}{p-1}}\omega_G$ which is equipped, for all $w \in]0, n - v\frac{p^n}{p-1}]$, with a map*

$$\mathrm{HT}_w \colon H_n^D(R[1/p]) \to \mathcal{F} \otimes_R R_w$$

deduced from $\mathrm{HT}_{H_n^D}$ which induces an isomorphism:

$$\mathrm{HT}_w \otimes 1 \colon H_n^D(R[1/p]) \otimes_{\mathbb{Z}} R_w \to \mathcal{F} \otimes_R R_w.$$

Proof Set $w_0 = n - v\frac{p^n-1}{p-1}$. Let $x_1, \ldots, x_g$ be a $\mathbb{Z}/p^n\mathbb{Z}$-basis of $H_n^D(R[1/p])$. Let $\tilde{\mathrm{HT}}_{H_n^D}(x_i)$ be lifts to ω_G of $\mathrm{HT}_{H_n^D}(x_i) \in \omega_{H_n}$. We set $\mathcal{F}$ to be the sub-module of ω_G generated by

$$\{\tilde{\mathrm{HT}}_{H_n^D}(x_1), \ldots, \tilde{\mathrm{HT}}_{H_n^D}(x_g)\}.$$

This module is free of rank g. Indeed, let $\sum_{i=1}^g \lambda_i \tilde{\mathrm{HT}}_{H_n^D}(x_i) = 0$ be a non-zero relation with coefficients in R. We may assume that there is an index i_0 such that $\lambda_{i_0} \notin p^{w_0 - \frac{v}{p-1}} R$. Projecting this relation in $\omega_{G,w_0} = \omega_{H_n,w_0}$ (see proposition 4.2.1) we contradict the proposition 4.2.2. By proposition 4.2.2, $p^{\frac{v}{p-1}}\omega_G \subset \mathcal{F}$ and the module $\mathcal{F}$ is independent of the choice of a particular lifts $\tilde{\mathrm{HT}}_{H_n^D}(x_i)$. Let $r \colon \omega_{H_n} \to \omega_{G,w_0}$ denote the projection. The map $\mathrm{HT}_{H_n^D} \circ r$ factors through $\mathcal{F}/\mathcal{F} \cap p^{w_0}\omega_G$. For all $w \in]0, n - v\frac{p^n}{p-1}]$, we have $\mathcal{F} \cap p^{w_0}\omega_G \subset p^w\mathcal{F}$. We can thus define HT_w as the composite of $\mathrm{HT}_{H_n^D} \circ r$ and the projection $\mathcal{F}/\mathcal{F} \cap p^{w_0}\omega_G \to \mathcal{F}/p^w\mathcal{F}$. Finally, the last claim follows because the map $\mathrm{HT}_w \otimes 1$ is a surjective map between two free modules of rank g over R_w and so has to be an isomorphism. □

Remark 4.3.2. The sheaf $\mathcal{F}$ is independent of $n \geq 1$, it is functorial in R and it coincides with the sheaf constructed using p-adic Hodge theory in [AIS], prop. 2.6. where it was denoted F_0.

Remark 4.3.3. Let Ω be an algebraic closure of $\mathrm{Frac}(R)$. Let $\bar{R}$ be the inductive limit of all finite, étale R-algebras contained in Ω and let $\widehat{\bar{R}}$ denote its p-adic completion. Assume that G is ordinary. Let $H_\infty \subset G$ be the canonical subgroup of order "∞" and $T_p(H_\infty^D)(\widehat{\bar{R}})$ be the Tate module of its dual H_∞^D. We have an isomorphism:

$$\mathrm{HT}_{H_\infty^D} \otimes 1 \colon T_p(H_\infty^D)\big(\widehat{\bar{R}}\big) \otimes_{\mathbb{Z}} \widehat{\bar{R}} \to \omega_G \otimes_R \widehat{\bar{R}}.$$

The proposition 4.3.1 is a good substitute for this isomorphism in the non ordinary case.

4.4 Functoriality in G

We assume the hypothesis of section §4.1.

Moreover we suppose that we have an isogeny $\phi\colon G \to G'$ over S, where G' is a second semi-abelian scheme over S satisfying the same assumptions as G, i.e. for all $x \in S_{\mathrm{rig}}$, $\mathrm{Hdg}(G_x), \mathrm{Hdg}(G'_x) \leq v$.

By functoriality of the Harder-Narasimhan filtration the isogeny induces a map $\phi\colon H_n \to H'_n$ where H_n and H'_n are the canonical subgroups of level n of G and G'.

We assume further that $H_n^D(R[1/p]) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$ and that ${H'_n}^D(R[1/p]) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$. We let $\mathcal{F}$ and $\mathcal{F}'$ be the sub-sheaves of ω_G and $\omega_{G'}$ constructed in proposition 4.3.1.

Proposition 4.4.1. *Let $w \in]0, n - v\frac{p^n}{p-1}]$. The isogeny ϕ gives rise to the following diagram:*

$$\begin{array}{ccc}
\omega_{G'} & \xrightarrow{\phi^*} & \omega_G \\
\uparrow & & \uparrow \\
\mathcal{F}' & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
\mathcal{F}'/p^w\mathcal{F}' & \longrightarrow & \mathcal{F}/p^w\mathcal{F} \\
{\scriptstyle \mathrm{HT}_w}\uparrow & & {\scriptstyle \mathrm{HT}_w}\uparrow \\
{H'_n}^D(R[1/p]) & \xrightarrow{\phi^D} & H_n^D(R[1/p])
\end{array}$$

Proof Set $w_0 = n - v\frac{p^n-1}{p-1}$. We check that $\phi^*(\mathcal{F}') \subset \mathcal{F}$. Let $\omega \in \omega_{G'}$ such that $\omega \mod p^{w_0}$ belongs to the R-span $\langle \mathrm{HT}_{w_0}({H'_n}^D(R[1/p])\rangle$. Then $\phi^*\omega \mod p^{w_0}$ belongs to the R-span $\langle \mathrm{HT}_{w_0}(\phi^D {H'_n}^D(R[1/p])\rangle$. The rest now follows easily. □

4.5 The main construction

In this section we work in the hypothesis of section §4.1 and we make the further assumptions that $v < \frac{1}{2p^{n-1}}$ (resp. $v < \frac{1}{3p^{n-1}}$ if $p = 3$), that $H_n^D(R[1/p]) \simeq (\mathbb{Z}/p^n\mathbb{Z})^g$ and that $w \in]0, n - v\frac{p^n}{p-1}]$.

Let $\mathcal{GR}_{\mathcal{F}} \to S$ be the Grassmannian parametrizing all flags $\mathrm{Fil}_0\mathcal{F} = 0 \subset \mathrm{Fil}_1\mathcal{F} \ldots \subset \mathrm{Fil}_g\mathcal{F} = \mathcal{F}$ of the free module $\mathcal{F}$; see [Ko, §I.1.7] for the construction. Let $\mathcal{GR}_{\mathcal{F}}^+$ be the T-torsor over $\mathcal{GR}_{\mathcal{F}}$ which parametrizes flags $\mathrm{Fil}_\bullet\mathcal{F}$ together with basis ω_i of the graded pieces $\mathrm{Gr}_i\mathcal{F}$.

We fix an isomorphism $\psi\colon (\mathbb{Z}/p^n\mathbb{Z})^g \simeq H_n^D(R[1/p])$ and call $x_1, \ldots, x_g$ the $\mathbb{Z}/p^n\mathbb{Z}$-basis of $H_n^D(R[1/p])$ corresponding to the canonical basis of $(\mathbb{Z}/p^n\mathbb{Z})^g$. Out of ψ, we obtain a flag $\mathrm{Fil}^{\psi}_{\bullet} = \{0 \subset \langle x_1\rangle \subset \langle x_1, x_2\rangle \ldots \subset \langle x_1, \ldots, x_g\rangle = H^D_n(R[1/p])\}$. We also have a basis $x_i \mod \mathrm{Fil}^{\psi}_{i-1}$ of the graded piece Gr^{ψ}_i.

Let R' be an object in $R-\mathbf{Adm}$. We say that an element $\mathrm{Fil}_{\bullet}\mathcal{F}\otimes_R R' \in \mathcal{GR}_{\mathcal{F}}(R')$ is w-compatible with ψ if $\mathrm{Fil}_{\bullet}\mathcal{F}\otimes_R R'_w = \mathrm{HT}_w(\mathrm{Fil}^{\psi}_{\bullet})\otimes_{\mathbb{Z}} R'_w$.

We say that an element $(\mathrm{Fil}_{\bullet}\mathcal{F}\otimes_R R', \{w_i\}) \in \mathcal{GR}^+_{\mathcal{F}}(R')$ is w-compatible with ψ if $\mathrm{Fil}_{\bullet}\mathcal{F}\otimes_R R'_w = \mathrm{HT}_w(\mathrm{Fil}^{\psi}_{\bullet})\otimes_{\mathbb{Z}} R'_w$ and $w_i \mod p^w\mathcal{F}\otimes_R R' + \mathrm{Fil}_{i-1}\mathcal{F}\otimes_R R' = \mathrm{HT}_w(x_i \mod \mathrm{Fil}^{\psi}_{i-1})$.

We now define functors

$$\begin{array}{rcl} \mathfrak{IW}_w : R-\mathbf{Adm} & \to & SET \\ R' & \mapsto & \{w-\text{compatible } \mathrm{Fil}_{\bullet}\mathcal{F}\otimes_R R' \in \mathcal{GR}_{\mathcal{F}}(R')\} \\ \mathfrak{IW}^+_w : R-\mathbf{Adm} & \to & SET \\ R' & \mapsto & \{w-\text{compatible } (\mathrm{Fil}_{\bullet}\mathcal{F}\otimes_R R', \{w_i\}) \in \mathcal{GR}^+_{\mathcal{F}}(R')\} \end{array}$$

These two functors are representable by affine formal schemes which can be described as follows. Let $f_1, \ldots, f_g$ be an R-basis of $\mathcal{F}$ lifting the vectors $\mathrm{HT}_w(x_1), \ldots, \mathrm{HT}_w(x_g)$.

The given basis identifies $\mathcal{GR}_{\mathcal{F}}$ with $\mathrm{GL}_g/\mathrm{B}\times S$ and $\mathfrak{IW}_w$ with the set of matrices:

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ p^w\mathfrak{B}(0,1) & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^w\mathfrak{B}(0,1) & p^w\mathfrak{B}(0,1) & \ldots & 1 \end{pmatrix} \times_{\mathrm{Spf}\ \mathcal{O}_K} \mathrm{Spf}\ R$$

where we have denoted by $\mathfrak{B}(0,1) = \mathrm{Spf}\ \mathcal{O}_K\langle X\rangle$ the formal unit ball.

Similarly, the given basis identifies $\mathcal{GR}^+_{\mathcal{F}}$ with $\mathrm{GL}_g/\mathrm{U}\times S$ and $\mathfrak{IW}^+_w$ with the set of matrices:

$$\begin{pmatrix} 1+p^w\mathfrak{B}(0,1) & 0 & \ldots & 0 \\ p^w\mathfrak{B}(0,1) & 1+p^w\mathfrak{B}(0,1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^w\mathfrak{B}(0,1) & p^w\mathfrak{B}(0,1) & \ldots & 1+p^w\mathfrak{B}(0,1) \end{pmatrix} \times_{\mathrm{Spf}\ \mathcal{O}_K} \mathrm{Spf}\ R.$$

We let $\mathfrak{T}\to \mathrm{Spf}\ \mathcal{O}_K$ be the formal completion of T along its special fiber. Let $\mathfrak{T}_w$ be the formal torus defined by

$$\mathfrak{T}_w(R') = \mathrm{Ker}\big(\mathrm{T}(R') \to \mathrm{T}(R'/p^wR')\big)$$

for any object $R' \in \mathbf{Adm}$. The formal scheme $\mathfrak{IW}^+_w$ is a torsor over $\mathfrak{IW}_w$ under $\mathfrak{T}_w$.

All these constructions are functorial in R. They do not depend on n but only on w. We denote by $\mathcal{IW}_w$ and $\mathcal{IW}^+_w$ the rigid analytic generic fibers of these formal schemes. They are admissible opens of the rigid spaces associated to $\mathcal{GR}_{\mathcal{F}}$ and $\mathcal{GR}^+_{\mathcal{F}}$ respectively.

5 The overconvergent modular sheaves

5.1 Classical Siegel modular schemes and modular forms

We fix an integer $N\geq 3$ such that $(p,N)=1$. Recall that K denotes a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K$ its ring of integers and k its residue field. The valuation v of K is normalized such that $v(p)=1$.

The Siegel variety of prime to p level. Let Y be the moduli space of principally polarized abelian schemes (A,λ) of dimension g equipped with a principal level N structure ψ_N over Spec $\mathcal{O}_K$. Let X be a toroidal compactification of Y and $G \to X$ be the semi-abelian scheme extending the universal abelian scheme (see [F-C]).

The Siegel variety of Iwahori level. Let $Y_{\mathrm{Iw}} \to$ Spec $\mathcal{O}_K$ be the moduli space parametrizing principally polarized abelian schemes (A,λ) of dimension g, equipped with a level N structure ψ_N and an Iwahori structure at p: this is the data of a full flag $\mathrm{Fil}_\bullet A[p]$ of the group $A[p]$ satisfying $\mathrm{Fil}_\bullet^\perp = \mathrm{Fil}_{2g-\bullet}$. Let X_{Iw} be a toroidal compactification of this moduli space (see [Str2]). We choose the polyhedral decompositions occurring in the constructions of X and X_{Iw} in such a way that the forgetful map $Y_{\mathrm{Iw}} \to Y$ extends to a map $X_{\mathrm{Iw}} \to X$.

The classical modular sheaves. Let ω_G be the co-normal sheaf of G along its unit section, $\mathcal{T} = \mathrm{Hom}_X(\mathscr{O}_X^g, \omega_G)$ be the space of ω_G and $\mathcal{T}^\times = \mathrm{Isom}_X(\mathscr{O}_X^g, \omega_G)$ be the GL_g-torsor of trivializations of ω_G. We define a left action $\mathrm{GL}_g \times \mathcal{T} \to \mathcal{T}$ by sending $\omega\colon \mathscr{O}_X^g \to \omega_G$ to $\omega \circ h^{-1}$ for any $h \in \mathrm{GL}_g$.

We define an automorphism $\kappa \mapsto \kappa'$ of $X(\mathrm{T})$ by sending any $\kappa = (k_1, \dots, k_g) \in X(\mathrm{T})$ to $\kappa' = (-k_g, -k_{g-1}, \dots, -k_1) \in X(\mathrm{T})$. This automorphism stabilizes the dominant cone $X^+(\mathrm{T})$. Let $\pi\colon \mathcal{T}^\times \to X$ be the projection. For any $\kappa \in X^+(\mathrm{T})$, we let ω^κ be the subsheaf of $\pi_*\mathscr{O}_{\mathcal{T}^\times}$ of κ'-equivariant functions for the action of B (with GL_g acting on the left on $\pi_*\mathscr{O}_{\mathcal{T}^\times}$ by $f(\omega) \mapsto f(\omega g)$ for any section f of $\pi_*\mathscr{O}_{\mathcal{T}^\times}$ viewed has a function over the trivializations ω, and any $g \in \mathrm{GL}_g$). The global sections $\mathrm{H}^0(X, \omega^\kappa)$ form the module of Siegel modular forms of weight κ over X.

5.2 Application of the main construction: the sheaves $\omega_w^{\dagger,\kappa}$

We denote by $\mathfrak{X}$ the formal scheme obtained by completing X along its special fiber X_k and by X_{rig} the associated rigid space. We have a Hodge height function $\mathrm{Hdg}\colon X_{\mathrm{rig}} \to [0,1]$ (see sections 3.1 and 4.1). Let $v \in [0,1]$, we set $\mathcal{X}(v) = \{x \in X^{\mathrm{rig}},\ \mathrm{Hdg}(x) \leq v\}$, this is an open subset of X_{rig}. Let $v \in v(\mathcal{O}_K)$. Consider the blow-up $\tilde{\mathfrak{X}}(v) = \mathbf{Proj}\mathscr{O}_{\mathfrak{X}}[X,Y]/\big(\mathrm{Ha}X + p^vY\big)$ of $\mathfrak{X}$ along the ideal (Ha, p^v). Let $\mathfrak{X}(v)$ be the p-adic completion of the normalization of the greatest open formal sub-scheme of $\tilde{\mathfrak{X}}(v)$ where the ideal (Ha, p^v) is generated by Ha. This is a formal model of $\mathcal{X}(v)$.

Let $n \in \mathbb{N}_{>0}$ and $v < \frac{1}{2p^{n-1}} \in v(\mathcal{O}_K)$ (rep. $v < \frac{1}{3p^{n-1}} \in v(\mathcal{O}_K)$ if $p = 3$). We have a canonical subgroup H_n of level n over $\mathcal{X}(v)$. Let $\mathcal{X}_1(p^n)(v) = \mathrm{Isom}_{\mathcal{X}(v)}((\mathbb{Z}/p^n\mathbb{Z})^g, H_n^D)$ be the finite étale cover of $\mathcal{X}(v)$ parametrizing trivializations of H_n^D. We let ψ be the universal trivialization over $\mathcal{X}_1(p^n)(v)$. Let $\mathfrak{X}_1(p^n)(v)$ be the normalization of $\mathfrak{X}(v)$ in $\mathcal{X}_1(p^n)(v)$. The group $\mathrm{GL}_g(\mathbb{Z}/p^n\mathbb{Z})$ acts on $\mathfrak{X}_1(p^n)(v)$. We let $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$ be the quotient $\mathfrak{X}_1(p^n)(v)/\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$. It is also the normalization of $\mathfrak{X}(v)$ in $\mathcal{X}_1(p^n)(v)/\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$. We also denote by $\mathfrak{X}_{\mathrm{Iw}^+}(p^n)(v)$ the quotient $\mathfrak{X}_1(p^n)(v)/\mathrm{U}(\mathbb{Z}/p^n\mathbb{Z})$.

5.2.1 Modular properties

The formal schemes $\mathfrak{X}_1(p^n)(v)$ and $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$ have nice modular interpretations away from the boundary. Let $\mathfrak{Y}_1(p^n)(v)$ and $\mathfrak{Y}_{\mathrm{Iw}}(p^n)(v)$ be the open formal sub-schemes that are the complements of the boundaries in $\mathfrak{X}_1(p^n)(v)$ and $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$ respectively.

Proposition 5.2.1.1. *For any object $R \in$ **NAdm**,*

1. *$\mathfrak{Y}_1(p^n)(v)(R)$ is the set of isomorphism classes of quadruples $(A, \lambda, \psi_N, \psi)$ where $(A \to \mathrm{Spf}\ R, \lambda)$ is a principally polarized formal abelian scheme of dimension g*

such that for all rig-point x in R, we have $\mathrm{Hdg}\big(A_x[p^\infty]\big) \leq v$; ψ_N is a principal level N structure; $\psi\colon \mathbb{Z}/p^n\mathbb{Z}^g \to H_n^D$ is a trivialization of the dual canonical subgroup of level n over $R[1/p]$.

2. *$\mathfrak{Y}_{\mathrm{Iw}}(p^n)(v)(R)$ is the set of isomorphism classes of quadruples $(A, \lambda, \psi_N, \mathrm{Fil}_\bullet)$ where $(A \to \mathrm{Spf}\ R, \lambda)$ is a principally polarized formal abelian scheme of dimension g such that for all rig-point x in R, we have $\mathrm{Hdg}\big(A_x[p^\infty]\big) \leq v$; ψ_N is a principal level N structure; $\mathrm{Fil}_\bullet$ is a full flag of locally free $\mathbb{Z}/p^n\mathbb{Z}$-modules of the dual canonical subgroup of level n over $R[1/p]$.*

Proof The proof is similar to the proof of [AIS1], lemma 3.1. □

5.2.2 The modular sheaves $\omega_w^{\dagger,\kappa}$

Let $w \in v(\mathcal{O}_K)\cap]n-1+\frac{v}{p-1}, n-v\frac{p^n}{p-1}]$. By proposition 4.3.1 there is a rank g locally free sub-sheaf $\mathcal{F}$ of $\omega_G/_{\mathfrak{X}_1(p^n)(v)}$. It is equipped with an isomorphism:

$$(\mathrm{HT}_w \circ \psi) \otimes 1\colon (\mathbb{Z}/p^n\mathbb{Z})^g \otimes_{\mathbb{Z}} \mathscr{O}_{\mathfrak{X}_1(p^n)(v)}/p^w\mathscr{O}_{\mathfrak{X}_1(p^n)(v)} \simeq \mathcal{F} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,w}.$$

Remark 5.2.2.1. The hypothesis $w \in]n-1+\frac{v}{p-1}, n-v\frac{p^n}{p-1}]$ is motivated by proposition 5.3.1. All this paragraph would make sense under the hypothesis $0 < w < n - v\frac{p^n}{p-1}$ but in this way we normalize n and our construction only depends on κ, w and v. Remark that if $w < n-1-v\frac{p^{n-1}}{p-1}$ we could use $\mathfrak{X}_1(p^{n-1})(v)$ as a base.

By section 4.5 we have a chain of formal schemes:

$$\mathfrak{IW}_w^+ \xrightarrow{\pi_1} \mathfrak{IW}_w \xrightarrow{\pi_2} \mathfrak{X}_1(p^n)(v) \xrightarrow{\pi_3} \mathfrak{X}_{\mathrm{Iw}}(p^n)(v) \xrightarrow{\pi_4} \mathfrak{X}_{\mathrm{Iw}}(p)(v).$$

We recall that $\mathfrak{IW}_w$ parametrizes flags in the locally free sheaf $\mathcal{F}$ that are w-compatible with ψ and that $\mathfrak{IW}_w^+$ parametrizes flags and bases of the graded pieces that are w-compatible with ψ.

We recall that $\mathfrak{IW}_w^+$ is a torsor over $\mathfrak{IW}_w$ under the formal torus $\mathfrak{T}_w$. We also have an action of the group $\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathfrak{X}_1(p^n)(v)$ over $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$. We let $\mathfrak{B}_w$ be the formal group defined by

$$\mathfrak{B}_w(R) = \mathrm{Ker}\big(\mathrm{B}(R) \to \mathrm{B}(R/p^wR)\big)$$

for all $R \in \mathbf{Adm}$.

There is a surjective map $\mathfrak{B}_w \to \mathfrak{T}_w$ with kernel the "unipotent radical" $\mathfrak{U}_w$. All these actions fit together in an action of $B(\mathbb{Z}_p)\mathfrak{B}_w$ on $\mathfrak{IW}_w^+$ over $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$ (the unipotent radical $\mathfrak{U}_w$ acts trivially).

The morphisms π_1, π_2, π_3 and π_4 are affine. Set $\pi = \pi_4 \circ \pi_3 \circ \pi_2 \circ \pi_1$. Let $\kappa \in \mathcal{W}(K)$ be a w-analytic character. The involution $\kappa \mapsto \kappa'$ of $X(\mathrm{T})$ extends to an involution of $\mathcal{W}$, mapping w-analytic characters to w-analytic characters. The character $\kappa'\colon \mathrm{T}(\mathbb{Z}_p) \to \mathcal{O}_K^\times$ extends to a character $\kappa'\colon \mathrm{T}(\mathbb{Z}_p)\mathfrak{T}_w \to \widehat{\mathbb{G}_m}\mathcal{O}_K^\times$ and to a character $\kappa'\colon \mathrm{B}(\mathbb{Z}_p)\mathfrak{B}_w \to \widehat{\mathbb{G}_m}$ with $\mathrm{U}(\mathbb{Z}_p)\mathfrak{U}_w$ acting trivially.

We use the notion of formal Banach sheaf given in definition A.1.1.1.

Proposition 5.2.2.2. *The sheaf $\pi_*\mathscr{O}_{\mathfrak{IW}_w^+}[\kappa']$ is a formal Banach sheaf.*

Proof Let $\kappa^{o\prime}$ be the restriction of κ' to $\mathfrak{T}_w$. Since the map π_1 is a torsor under the group $\mathfrak{T}_w$, the sheaf $(\pi_1)_*\mathscr{O}_{\mathfrak{IW}_w^+}[\kappa^{o\prime}]$ is an invertible sheaf). Since the map π_2 is affine, $(\pi_2\circ\pi_1)_*\mathscr{O}_{\mathfrak{IW}_w^+}[\kappa^{o\prime}]$ is a formal Banach sheaf. It remains to take the pushforward by the finite map $\pi_4\circ\pi_3$ and the invariants under $\mathrm{T}(\mathbb{Z}/p^n\mathbb{Z})$ for the action twisted by κ as in the paragraph preceeding proposition A.2.2.4. □

Definition 5.2.2.3. *The formal Banach sheaf of w-analytic, v-overconvergent modular forms of weight κ is*

$$\mathfrak{w}_w^{\dagger\kappa} = \pi_*\mathscr{O}_{\mathfrak{IW}_w^+}[\kappa'].$$

5.2.3 Integral overconvergent modular forms

Definition 5.2.3.1. *The space of integral w-analytic, v-overconvergent modular forms of genus g, weight κ, principal level N is*

$$\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(v)) = \mathrm{H}^0(\mathfrak{X}_{\mathrm{Iw}}(p)(v), \mathfrak{w}_w^{\dagger\kappa})$$

An element f of the module

$$\mathrm{H}^0(\mathfrak{Y}_{\mathrm{Iw}}(p)(v), \mathfrak{w}_w^{\dagger\kappa}),$$

called *weakly modular form* is a rule which associates functorially to an object R in **NAdm**, a quadruple $(A,\lambda,\psi_N,\psi)\in\mathfrak{Y}_1(p^n)(v)(R)$, a w-compatible flag $\mathrm{Fil}_\bullet\mathcal{F}_R$ and a w-compatible basis w_i of each $\mathrm{Gr}_i\mathcal{F}_R$ an element

$$f(R,A,\lambda,\psi_N,\psi,Fil_\bullet\mathcal{F}_R,\{\omega_i\})\in R$$

satisfying the functional equation:

$$f(R,A,\lambda,\psi_N,b.\psi,Fil_\bullet\mathcal{F}_R,b\cdot\{\omega_i\}) = \kappa'(b)f(R,A,\lambda,\psi_N,\psi,Fil_\bullet\mathcal{F}_R,\{w_i\})$$

for all $b\in\mathrm{B}(\mathbb{Z}_p)\mathfrak{B}_w(R)$.

5.2.4 Locally analytic overconvergent modular forms

Let $\kappa\in\mathcal{W}(K)$ and $v,w>0$. Suppose that κ is w-analytic. If there is $n\in\mathbb{N}$ satisfying $v<\frac{1}{2p^{n-1}}$ (resp. $v<\frac{1}{3p^{n-1}}$ if $p=3$) and $w\in]n-1+\frac{v}{p-1}, n-v\frac{p^n}{p-1}]$, then we have constructed a sheaf $\mathfrak{w}_w^{\dagger\kappa}$ on $\mathfrak{X}_{\mathrm{Iw}}(v)$. As a result for all κ, if we take $v>0$ sufficiently small and $w\notin\mathbb{N}$ big enough (so that κ is w-analytic), there is a unique $n\in\mathbb{N}$ satisfying the conditions above and so the modular sheaf $\mathfrak{w}_w^{\dagger\kappa}$ exists. Let κ,n,v,w satisfying all the required conditions for the existence of the sheaf $\mathfrak{w}_w^{\dagger\kappa}$. Clearly, if $v'<v$ then κ,n,v',w satisfy also the conditions and the sheaf $\mathfrak{w}_w^{\dagger\kappa}$ on $\mathfrak{X}_{\mathrm{Iw}}(v')$ is the restriction of the sheaf on $\mathfrak{X}_{\mathrm{Iw}}(v)$.

If κ is w-analytic, it is also w' analytic for any $w'>w$. Let $n'\in\mathbb{N}$ and $v>0$ such that κ,n,v,w and κ,n',v,w' satisfy the conditions, so that we have two sheaves $\mathfrak{w}_w^{\dagger\kappa}$ and $\mathfrak{w}_{w'}^{\dagger\kappa}$ over $\mathfrak{X}_{\mathrm{Iw}}(v)$.

There is a natural inclusion:

$$\mathfrak{IW}_{w'}^+ \hookrightarrow \mathfrak{IW}_w^+\times_{\mathfrak{X}_1(p^n)(v)}\mathfrak{X}_1(p^{n'})(v)$$

which follows from the fact that w'-compatibility implies w-compatibility.

This induces a natural map $\mathfrak{w}_w^{\dagger\kappa}\to\mathfrak{w}_{w'}^{\dagger\kappa}$ and thus a map $\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(v))\to\mathrm{M}_{w'}^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(v))$. We are led to the following definition.

Definition 5.2.4.1. *Let $\kappa \in \mathcal{W}$. The space of integral locally analytic overconvergent modular forms of weight κ and principal level N is the inductive limit:*

$$\mathrm{M}^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(v)) = \lim_{v\to 0, w\to\infty} \mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(v)).$$

5.3 Rigid analytic interpretation

We let $\mathcal{T}_{\mathrm{an}}$, $\mathcal{T}_{\mathrm{an}}^{\times}$, $\mathrm{GL}_{g,\mathrm{an}}$ be the rigid analytic spaces associated to the $\mathcal{O}_K$-schemes $\mathcal{T}$, $\mathcal{T}^{\times}$ and GL_g. We also let $\mathfrak{T}$, $\mathfrak{T}^{\times}$ and $\widehat{\mathrm{GL}_g}$ be the completions of $\mathcal{T}$, $\mathcal{T}^{\times}$ and GL_g along their special fibers and $\mathcal{T}_{\mathrm{rig}}$, $\mathcal{T}_{\mathrm{rig}}^{\times}$ and $\mathrm{GL}_{g,rig}$ be their rigid analytic fibers. We have actions $\mathrm{GL}_{g,\mathrm{an}} \times \mathcal{T}_{\mathrm{an}} \to \mathcal{T}_{\mathrm{an}}$, $\mathrm{GL}_{g,rig} \times \mathcal{T}_{\mathrm{rig}} \to \mathcal{T}_{\mathrm{rig}}$ and $\widehat{\mathrm{GL}_g} \times \mathfrak{T} \to \mathfrak{T}$. When the context is clear, we just write GL_g instead of $\mathrm{GL}_{g,\mathrm{an}}$, $\mathrm{GL}_{g,rig}$ or $\widehat{\mathrm{GL}_g}$. For example, we have $\mathcal{T}_{\mathrm{rig}}^{\times}/\mathrm{B} = \mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{B}$, because $\mathcal{T}^{\times}/\mathrm{B}$ is complete. Over $\mathcal{T}_{\mathrm{rig}}^{\times}/\mathrm{B}$, we have a diagram:

$$\begin{array}{ccc} \mathcal{T}_{\mathrm{rig}}^{\times}/\mathrm{U} & \longrightarrow & \mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{U} \\ \downarrow & \swarrow & \\ \mathcal{T}_{\mathrm{rig}}^{\times}/\mathrm{B} & & \end{array}$$

where $\mathcal{T}_{\mathrm{rig}}^{\times}/\mathrm{U}$ is a torsor under $\mathrm{U}_{\mathrm{rig}}/\mathrm{B}_{\mathrm{rig}} = \mathrm{T}_{\mathrm{rig}}$ and $\mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{U}$ is a torsor under $\mathrm{U}_{\mathrm{an}}/\mathrm{B}_{\mathrm{an}} = \mathrm{T}_{\mathrm{an}}$.

We let $\mathcal{IW}_w^+$ and $\mathcal{IW}_w$ be the rigid spaces associated to $\mathfrak{IW}_w^+$ and $\mathfrak{IW}_w$ respectively. We have a chain of rigid spaces:

$$\mathcal{IW}_w^+ \to \mathcal{IW}_w \to \mathcal{X}_1(p^n)(v) \to \mathcal{X}_{\mathrm{Iw}}(p)(v)$$

The natural injection $\mathcal{F} \hookrightarrow \omega_{G\mathfrak{X}_1(p^n)(v)}$ is an isomorphism on the rigid fiber. More precisely, we can cover $\mathfrak{X}_1(p^n)(v)$ by affine open formal schemes Spf R such that $\mathcal{F}$ and ω_G are free R modules of rank g. We choose a basis for $\mathcal{F}$ compatible with ψ and a basis for ω_G such that the inclusion is given by an upper triangular matrix $M \in \mathrm{M}_g(R)$ with diagonal given by $\mathrm{diag}(\beta_1, \dots, \beta_g)$ where $\beta_i \in R$ and $v(\prod_i \beta_i(x)) = \frac{1}{p-1}\mathrm{Hdg}(G_x)$ for all closed points x of Spec $R[1/p]$ (see proposition 3.2.1).

We thus obtain an open immersion:

$$\mathcal{IW}_w \hookrightarrow \mathcal{T}_{\mathrm{rig}}^{\times}/\mathrm{B}_{\mathcal{X}_1(p^n)(v)}.$$

This immersion is locally isomorphic to the inclusion:

$$M \begin{pmatrix} 1 & 0 & \cdots & 0 \\ p^w\mathrm{B}(0,1) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ p^w B(0,1) & \cdots & p^w\mathrm{B}(0,1) & 1 \end{pmatrix} \hookrightarrow (\mathrm{GL}_g)_{\mathrm{rig}}/\mathrm{B}.$$

Similarly, we have an open immersion:

$$\mathcal{IW}_w^+ \hookrightarrow \mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{B}_{\mathcal{X}_1(p^n)(v)}$$

which is locally isomorphic to the inclusion

$$M\begin{pmatrix} 1+p^w\mathrm{B}(0,1) & 0 & \cdots & 0 \\ p^w\mathrm{B}(0,1) & 1+p^w\mathrm{B}(0,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ p^wB(0,1) & \cdots & p^w\mathrm{B}(0,1) & 1+p^w\mathrm{B}(0,1) \end{pmatrix} \hookrightarrow (\mathrm{GL}_g)_{\mathrm{an}}/\mathrm{U}$$

Let $\mathcal{X}_{\mathrm{Iw}}(p^n)(v) = \mathcal{X}_1(p^n)(v)/\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$ be the rigid space associated to $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$ and let $\mathcal{X}_{\mathrm{Iw}^+}(p^n)(v) = \mathcal{X}_1(p^n)(v)/\mathrm{U}(\mathbb{Z}/p^n\mathbb{Z})$. The map $\mathcal{X}_{\mathrm{Iw}^+}(p^n)(v) \to \mathcal{X}_{\mathrm{Iw}}(p^n)(v)$ is an étale cover with group $\mathrm{T}(\mathbb{Z}/p^n\mathbb{Z})$. The action of $\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}_1(p^n)(v)$ lifts to an action on the rigid space $\mathcal{IW}_w \to \mathcal{X}_1(p^n)(v)$ since the notion of w-compatibility of the flag only depends on $\psi \mod \mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$. Taking quotients we get a rigid space $\mathcal{IW}_w^o \to \mathcal{X}_{\mathrm{Iw}}(p^n)(v)$. Similarly, the action of $\mathrm{U}(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}_1(p^n)(v)$ lifts to an action on the rigid space $\mathcal{IW}_w^+ \to \mathcal{X}_1(p^n)(v)$ since the notion of w-compatibility of the flag and the bases of the graded pieces only depends on $\psi \mod \mathrm{U}(\mathbb{Z}/p^n\mathbb{Z})$. Taking quotients we obtain a rigid space $\mathcal{IW}_w^{o+} \to \mathcal{X}_{\mathrm{Iw}^+}(p^n)(v)$. From the open immersions above we get open immersions:

$$\mathcal{IW}_w^o \hookrightarrow \mathcal{T}_{\mathrm{rig}}^\times/\mathrm{B}_{\mathcal{X}_{\mathrm{Iw}}(p^n)(v)} \text{ and } \mathcal{IW}_w^{o+} \hookrightarrow \mathcal{T}_{\mathrm{an}}^\times/\mathrm{U}_{\mathcal{X}_{\mathrm{Iw}^+}(p^n)(v)},$$

where $\mathrm{B}_{\mathcal{X}_{\mathrm{Iw}}(p^n)(v)}$ and $\mathrm{U}_{\mathcal{X}_{\mathrm{Iw}^+}(p^n)(v)}$ are the base changes of the algebraic groups B and U to $\mathcal{X}_{\mathrm{Iw}^+}(p^n)(v)$.

Proposition 5.3.1. *Since $w > n-1+\frac{v}{p-1}$, the compositions $\mathcal{IW}_w^o \hookrightarrow \mathcal{T}_{\mathrm{rig}}^\times/\mathrm{B}_{\mathcal{X}_{\mathrm{Iw}}(p^n)(v)} \to \mathcal{T}_{\mathrm{rig}}^\times/\mathrm{B}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)}$ and $\mathcal{IW}_w^{o+} \hookrightarrow \mathcal{T}_{\mathrm{an}}^\times/\mathrm{U}_{\mathcal{X}_{\mathrm{Iw}}(p^n)(v)} \to \mathcal{T}_{\mathrm{an}}^\times/\mathrm{U}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)}$ are open immersions.*

Proof We can work étale locally over $\mathcal{X}_{\mathrm{Iw}}(p)(v)$. Let S be a set of representatives in the Iwahori subgroup $\mathrm{I}(\mathbb{Z}_p) \subset \mathrm{GL}_g(\mathbb{Z}_p)$ of $\mathrm{I}(\mathbb{Z}/p^n\mathbb{Z})/\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$. Over a suitable open affinoid U of $\mathcal{X}_1(p^n)(v)$, the map $(\mathcal{IW}_w^o)|_U \to (\mathcal{T}_{\mathrm{rig}}^\times/\mathrm{B}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)})|_U$ is isomorphic to the following projection:

$$h\colon \coprod_{\gamma\in S} M\begin{pmatrix} 1 & 0 & \cdots & 0 \\ p^w\mathrm{B}(0,1) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ p^wB(0,1) & \cdots & p^w\mathrm{B}(0,1) & 1 \end{pmatrix}.\gamma \to (\mathrm{GL}_g)_{\mathrm{rig}}/\mathrm{B}$$

There is a matrix M' with integral coefficients such that $M'\cdot M = p^{\frac{v}{p-1}}\mathrm{Id}_g$. It is trivial to check that $M'\circ h$ is injective and so h is injective. The proof of the second part of the proposition is similar. $\square$

We have a diagram:

$$\begin{array}{ccccc} \mathcal{T}_{\mathrm{an}}^\times/\mathrm{U}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)} & \xleftarrow{i_1} & \mathcal{IW}_w^{o+} & \xrightarrow{f_1} & \mathcal{X}_{\mathrm{Iw}^+}(p^n)(v) \\ \downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow \\ \mathcal{T}_{\mathrm{rig}}^\times/\mathrm{B}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)} & \xleftarrow{i_2} & \mathcal{IW}_w^{o} & \xrightarrow{f_2} & \mathcal{X}_{\mathrm{Iw}}(p^n)(v) \\ & \searrow & \downarrow{\scriptstyle g_2} & \swarrow & \\ & & \mathcal{X}_{\mathrm{Iw}}(p)(v) & & \end{array}$$

The maps i_1 and i_2 are open immersions, the maps f_1 and f_2 have geometrically connected fibers. The torus $\mathrm{T}(\mathbb{Z}_p)$ acts on $\mathcal{IW}_w^{o+}$ over $\mathcal{IW}_w^{o}$. This action is compatible with the maps f_1 and f_2 and the action of $\mathrm{T}(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}_{\mathrm{Iw}^+}(p^n)(v)$ over $\mathcal{X}_{\mathrm{Iw}}(p^n)(v)$. It is also compatible with the maps i_1 and i_2 and the action of T_{an} on $\mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{U}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)}$ over $\mathcal{T}_{\mathrm{rig}}^{\times}/\mathrm{B}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)}$.

Set $g = g_2 \circ g_1$. Let κ be a w-analytic character. Then $\omega_w^{\dagger\kappa} = g_* \mathscr{O}_{\mathcal{IW}_w^{o+}}[\kappa']$ is the projective Banach sheaf of w-analytic, v-overconvergent weight κ modular forms over $\mathcal{X}_{\mathrm{Iw}}(p)(v)$. It is the Banach sheaf associated to the formal Banach sheaf $\mathfrak{w}_w^{\dagger\kappa}$ by proposition A.2.2.4.

Remark 5.3.2. Let T_w be the rigid-analytic torus which is the rigid analytic fiber of $\mathfrak{T}_w$. For example, we have $\mathrm{T}_w(\mathbb{C}_p) = (1+p^w\mathcal{O}_{\mathbb{C}_p})^g$. The rigid space $\mathcal{IW}_w^{o+}$ is a $\mathrm{T}(\mathbb{Z}_p)\mathrm{T}_w$-torsor over $\mathcal{IW}_w^{o}$. By lemma 2.1 of [Pi3], it makes no difference in the definition of $\omega_w^{\dagger\kappa}$ to take κ'-equivariant functions for the action of $\mathrm{T}(\mathbb{Z}_p)$ or of the bigger group $\mathrm{T}(\mathbb{Z}_p)\mathrm{T}_w$.

Definition 5.3.3. *Let $\kappa \in \mathcal{W}$. The space of w-analytic, v-overconvergent modular forms of weight κ is:*

$$\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v)) = \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger\kappa}).$$

The space of locally analytic overconvergent modular forms of weight κ is:

$$\mathrm{M}^{\dagger\kappa}(X_{\mathrm{Iw}}(p)) = \mathrm{colim}_{v\to 0, w\to\infty}\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v)).$$

The space $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$ is a Banach space, for the norm induced by the supremum norm on $\mathcal{IW}_w^{o+}$. Its unit ball is the space $\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(v))$ of integral forms.

Proposition 5.3.4. *If $\kappa \in X_+(\mathrm{T})$, then there is a canonical restriction map:*

$$\omega^{\kappa}|_{\mathcal{X}_{\mathrm{Iw}}(p)(v)} \hookrightarrow \omega_w^{\dagger\kappa}$$

induced by the open immersion: $\mathcal{IW}_w^{o+} \hookrightarrow \mathcal{T}_{\mathrm{an}}/\mathrm{U}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)}$.

This map is locally for the étale topology isomorphic to the inclusion

$$V_{\kappa'} \hookrightarrow V_{\kappa'}^{w-\mathrm{an}}$$

of the algebraic induction into the analytic induction.

Corollary 5.3.5. *For any $\kappa \in X_+(\mathrm{T})$, we have an inclusion:*

$$\mathrm{H}^0(X_{\mathrm{Iw}}, \omega^{\kappa}) \hookrightarrow \mathrm{M}_w^{\dagger,\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$$

from the space of classical forms of weight κ into the space of w-analytic, v-overconvergent modular forms of weight κ.

5.4 Overconvergent and p-adic modular forms

We compare the notion p-adic modular forms introduced by Katz and used by Hida ([Hi2]) to construct ordinary eigenvarieties to the notion of overconvergent locally analytic modular forms. Let $\mathfrak{X}_1(p^\infty)(0)$ be the projective limit of the formal schemes $\mathfrak{X}_1(p^n)(0)$. It is a pro-étale cover of $\mathfrak{X}_{\mathrm{Iw}}(p)(0)$ with group the Iwahori subgroup of $\mathrm{GL}_g(\mathbb{Z}_p)$, denoted by I. In particular we have an action of $\mathrm{B}(\mathbb{Z}_p)$ on the space $\mathrm{H}^0(\mathfrak{X}_1(p^\infty)(0), \mathscr{O}_{\mathfrak{X}_1(p^\infty)(0)})$. Any character $\kappa \in \mathcal{W}$ can be seen as a character of $\mathrm{B}(\mathbb{Z}_p)$, trivial on the unipotent radical.

Definition 5.4.1. *Let $\kappa \in \mathcal{W}(K)$ be an $\mathcal{O}_K$-valued character of the group $\mathrm{T}(\mathbb{Z}_p)$. The space of p-adic modular forms of weight κ is:*

$$\mathrm{M}^{\infty\kappa} := \mathrm{H}^0(\mathfrak{X}_1(p^\infty)(0), \mathscr{O}_{\mathfrak{X}_1(p^\infty)(0)})[\kappa']$$

Over $\mathfrak{X}_1(p^\infty)(0)$, we have a universal trivialization $\psi\colon \mathbb{Z}_p^g \simeq T_p\big(G^D[p^\infty]\big)^{\mathrm{et}}$ of the p-adic étale Tate module of $G^D[p^\infty]$ and a comparison theorem:

$$\mathrm{HT}_{G^D_\infty} : \mathbb{Z}_p^g \otimes_{\mathbb{Z}} \mathscr{O}_{\mathfrak{X}_1(p^\infty)(0)} \xrightarrow{\sim} \omega_G \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{\mathfrak{X}_1(p^\infty)(0)}.$$

As a result, for all $w \in]n-1, n]$, we have a diagram:

$$\begin{array}{ccc} \mathfrak{X}_1(p^\infty)(0) & \xrightarrow{i} & \mathfrak{IW}_w^+ \\ \downarrow & \swarrow & \\ \mathfrak{X}_1(p^n)(0) & & \end{array}$$

Let $\mathfrak{U}_n(\mathbb{Z}_p)$ be the sub-group of $\mathrm{U}(\mathbb{Z}_p)$ of matrices which are trivial modulo $p^n\mathbb{Z}_p$. The map i factorizes through an immersion $\mathfrak{X}_1(p^\infty)(0)/\mathfrak{U}_n(\mathbb{Z}_p) \hookrightarrow \mathfrak{IW}_w^+$ which is equivariant under the action of $\mathrm{B}(\mathbb{Z}_p)$. This provides a map:

$$\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{Iw}(p)(0)) \to \mathrm{M}^{\infty\kappa}.$$

Remark 5.4.2. A space analogue to $\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(0))$ appears in the work [S-U] of Skinner-Urban for GSp_4 . The space of semi-ordinary modular forms is a direct factor of $\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(0))$ cut out by a projector.

Proposition 5.4.3. *There is a natural injective map:*

$$\mathrm{M}^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)) \hookrightarrow \mathrm{M}^{\infty\kappa}.$$

As a result, locally analytic overconvergent modular forms are p-adic modular forms.

Proof This map is obtained as the limit of the maps $\mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(v)) \to \mathrm{M}_w^{\dagger\kappa}(\mathfrak{X}_{\mathrm{Iw}}(p)(0)) \to \mathrm{M}^{\infty\kappa}$. All spaces are torsion free $\mathcal{O}_K$-module so the injectivity can be checked after inverting p. The injectivity of $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v)) \to \mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(0))$ for v small enough follows from the the surjectivity of the map on the connected components $\Pi_0(\mathcal{X}_{\mathrm{Iw}}(p)(0)) \to \Pi_0(\mathcal{X}_{\mathrm{Iw}}(p)(v))$. The injectivity of the map $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(0)) \to \mathrm{M}^{\infty\kappa}[1/p]$ can be checked locally over $\mathcal{X}_{\mathrm{Iw}}(p)(0)$. This boils down to the injectivity of the restriction map: $V_{\kappa'}^{w-\mathrm{an}} \to \mathcal{F}^0(\mathrm{I})$ where $\mathcal{F}^0(\mathrm{I})$ is the space of continuous, $\mathcal{O}_K$-valued functions on I. □

5.5 Independence of the compactification

We will see that our modules of overconvergent modular forms are in fact independent of the compactifications.

If S is a rigid space, we say that a function f on S is bounded if the supremum norm $\sup_{x\in S} |f(x)|$ is finite. If S is quasi-compact, then this property is automatically satisfied (see [Bos], p. 23). We now recall the following result:

Theorem 5.5.1 ([Lu], thm 1.6.1)**.** *Let S be a smooth, quasi-compact rigid space and Z a co-dimension ≥ 1 Zariski-closed subspace. Then any bounded function on $S\backslash Z$ extends uniquely to S.*

Let $Y_{\mathrm{Iw}}^{\mathrm{an}}$ be the analytification of Y_{Iw} (see for example [Bos], sect. 1.13). Set $\mathcal{X}_{\mathrm{Iw}}(p)(v) \cap Y_{\mathrm{Iw}}^{\mathrm{an}} = \mathcal{Y}_{\mathrm{Iw}}(v)$. The space $\mathcal{Y}_{\mathrm{Iw}}(v)$ does not depend on the compactification. We say that $f \in \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v), \omega_w^{\dagger\kappa})$ is bounded if it is bounded when considered as a function on the rigid space $\mathcal{IW}_w^{o+} \times_{\mathcal{X}_{\mathrm{Iw}}(v)} \mathcal{Y}_{\mathrm{Iw}}(v)$.

Proposition 5.5.2. *The module of w-analytic and v-overconvergent modular forms is exactly the submodule of bounded sections of* $\mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v), \omega_w^{\dagger\kappa})$*. In particular,* $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$ *is independent on the choice of the toroidal compactification.*

Proof The map $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v)) \to \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v), \omega_w^{\dagger\kappa})$ is clearly injective. Let

$$f \in \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v), \omega_w^{\dagger\kappa})$$

be a bounded section. This is a bounded function on $\mathcal{IW}_w^{o+} \times_{\mathcal{X}_{\mathrm{Iw}}(v)} \mathcal{Y}_{\mathrm{Iw}}(v)$, homogeneous for the action of the torus $\mathrm{T}(\mathbb{Z}_p)$. By theorem 5.5.1, it extends to a function on $\mathcal{IW}_w^{o+}$, which is easily seen to be homogeneous of the same weight. □

Remark 5.5.3. The module $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$ could thus have been defined without reference to any compactification. Nevertheless, compactifications will turn out to be quite useful in the last sections allowing us to prove properties of these modules.

5.6 Dilations

For our study of Hecke operators it is useful to define slight generalizations of the spaces $\mathcal{IW}_w^{o+}$. This section is technical and may be skipped at the first reading. Let $n \in \mathbb{N}, v < \frac{1}{2p^{n-1}}$ (resp. $v < \frac{1}{3p^{n-1}}$ if $p = 3$) and let $\underline{w} = (w_{i,j})_{1 \leq j \leq i \leq g} \in]\frac{v}{p-1}, n - v\frac{p^n}{p-1}]^{\frac{g(g+1)}{2}}$ satisfying $w_{i+1,j} \geq w_{i,j}$ and $w_{i,j-1} \geq w_{i,j}$. We will call $(w_{i,j})$ a dilation parameter. Let $\mathcal{IW}_{\underline{w}}^{o+}$ be the open subset of $\mathcal{T}_{\mathrm{an}}/\mathrm{U}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)}$ such that for any finite extension L of K, an element in $\mathcal{IW}_{\underline{w}}^{o+}(L)$ is the data of:

- — an $\mathcal{O}_L$-point of $\mathcal{X}_{\mathrm{Iw}}(p)(v)$, coming from a semi-abelian scheme $G \to \mathcal{O}_L$, with H_n its canonical subgroup of level n, and a flag $\mathrm{Fil}_\bullet H_n$,
- — a flag of differential forms $\mathrm{Fil}_\bullet \mathcal{F} \in \mathcal{T}_{\mathrm{rig}}/\mathrm{B}(\mathcal{O}_L)$ and for all $1 \leq j \leq g$, an element $\omega_j \in \mathrm{Gr}_j \mathcal{F}$ such that there is a trivialization $\psi\colon H_n^D(\bar{K}) \simeq \mathbb{Z}/p^n\mathbb{Z}^g$ where ψ is compatible with $\mathrm{Fil}_\bullet H_1$, and the following holds:
  Denote by $e_1, \ldots, e_g$ the canonical basis of $\mathbb{Z}/p^n\mathbb{Z}^g$, set $w_0 = n - v\frac{p^n}{p-1}$ and by abuse of notation set HT_{w_0} for the map $\mathrm{HT}_{w_0} \circ \psi$; then for all $1 \leq i \leq g$, we have:

  $$\omega_i \mod \mathrm{Fil}_{i-1}\mathcal{F} + p^{w_0}\mathcal{F} = \sum_{j \geq i} a_{j,i} \mathrm{HT}_{w_0}(e_j)$$

  where $a_{j,i} \in \mathcal{O}_L$ and $v(a_{j,i}) \geq w_{j,i}$ if $j > i$ and $v(a_{i,i} - 1) \geq w_{i,i}$.

When $w_{i,j} = w$ for all $1 \leq j \leq i \leq g$ and there exists $n \in \mathbb{N}$ such that $w \in]n - 1 + \frac{v}{p-1}, n - v\frac{p^n}{p-1}]$, we have $\mathcal{IW}_{\underline{w}}^{o+} = \mathcal{IW}_w^{o+}$. The spaces $\mathcal{IW}_{\underline{w}}^{o+}$ are dilations of the space $\mathcal{IW}_{w_0}^{o+}$, in the sense that we relax the w_0-compatibility with ψ and impose a weaker condition. The rigid space $\mathcal{IW}_{\underline{w}}^{o+}$ is locally for the étale topology over $\mathcal{X}_{\mathrm{Iw}}(p)(v)$ isomorphic to

$$\begin{pmatrix} 1+p^{w_{1,1}}\mathrm{B}(0,1) & 0 & \cdots & 0 \\ p^{w_{2,1}}\mathrm{B}(0,1) & 1+p^{w_{2,2}}\mathrm{B}(0,1) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ p^{w_{g,1}}\mathrm{B}(0,1) & \cdots & p^{w_{g,g-1}}\mathrm{B}(0,1) & 1+p^{w_{g,g}}\mathrm{B}(0,1) \end{pmatrix}.\mathrm{I}/\mathrm{U}(\mathbb{Z}_p).$$

If κ is a $\inf_i\{w_{i,i}\}$-analytic character, we can define Banach sheaves $\omega_{\underline{w}}^{\dagger\kappa}$ and the space $\mathrm{M}_{\underline{w}}^{\dagger\kappa}\big(\mathcal{X}_{\mathrm{Iw}}(p)(v)\big)$ of $\underline{w}$-analytic, v-overconvergent modular forms as in section 5.3.

Remark 5.6.1. If $\underline{w}$ and $\underline{w'}$ are two dilation parameters satisfying $w_{i,j}=w'_{i,j}$ as soon as $i\neq j$, and if κ is a $\inf_{i,j}\{w_{i,i},w'_{j,j}\}$ analytic character then the sheaves $\omega_{\underline{w}}^{\dagger\kappa}$ and $\omega_{\underline{w'}}^{\dagger\kappa}$ are canonically isomorphic.

6 Hecke operators

In this section we define an action of the Hecke operators on the space of overconvergent modular forms and we single out one of these operators which is compact.

6.1 Hecke operators outside p

Let q be a prime integer with $(q,p)=1$ and let $\gamma\in\mathrm{GSp}_{2g}(\mathbb{Q}_q)\cap\mathrm{M}_{2g}(\mathbb{Z}_q)$. Let $C_\gamma\subset Y_{\mathrm{Iw}}\times Y_{\mathrm{Iw}}\times\mathrm{Spec}\,K$ be the moduli space over K classifying pairs (A,A') of principally polarized abelian schemes of dimension g, equipped with level N structures (ψ_N,ψ'_N), flags $\mathrm{Fil}_\bullet A[p]$ and $\mathrm{Fil}_\bullet A'[p]$ of $A[p]$ and $A'[p]$, and an isogeny $\pi\colon A\to A'$ of type γ, compatible with the level structures, the flags and the polarizations (see [F-C], chapter 7). We have two finite étale projections $p_1,\,p_2\colon C_\gamma\to Y_{\mathrm{Iw},K}$. They extend to projections on the analytifications $p_1,\,p_2\colon C_\gamma^{\mathrm{an}}\to Y_{\mathrm{Iw},K}^{\mathrm{an}}$. There is an issue with the boundary: in general it is not possible to find toroidal compactifications for C_γ and Y_{Iw} in such a way that the projections p_1 and p_2 extend to finite morphisms. Moreover if one varies γ it is not possible to find toroidal compactifications of Y_{Iw} and the C_γ's such that all projections extend to the compactifications. Therefore we will define Hecke operators on $\mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v),\omega_w^{\dagger\kappa})$ and show that these Hecke operators map bounded functions to bounded functions thus defining an action on $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$ (see section §5.5).

The isogeny π induces a map $\pi^*\colon\omega_A\to\omega_{A'}$, hence a map $\pi^*\colon p_2^*\mathcal{T}_{\mathrm{an}}^\times/\mathrm{U}\to p_1^*\mathcal{T}_{\mathrm{an}}^\times/\mathrm{U}$ which is an isomorphism. Let $n\in\mathbb{N}$, $v<\frac{1}{2p^{n-1}}$ (resp $<\frac{1}{3p^{n-1}}$ if $p=3$. Set $C_\gamma^{\mathrm{an}}(v)=C_\gamma^{\mathrm{an}}\times_{p_1}\mathcal{Y}_{\mathrm{Iw}}(v)=C_\gamma^{\mathrm{an}}\times_{p_2}\mathcal{Y}_{\mathrm{Iw}}(v)$. The last equality follows from the facts that level prime-to-p isogenies are étale in characteristic p and that the Hodge height is preserved under étale isogenies. Let $w\in]n-1+\frac{v}{p-1},n-v\frac{p^n}{p-1}]$.

Lemma 6.1.1. *The map π^* induces an isomorphism*

$$\pi^*\colon p_2^*\mathcal{IW}_w^{o+}|_{\mathcal{Y}_{\mathrm{Iw}}(v)}\simeq p_1^*\mathcal{IW}_w^{o+}|_{\mathcal{Y}_{\mathrm{Iw}}(v)}.$$

Proof. We can check the equality at the level of points. Let L be a finite extension of K, and A, A' be two semi-abelian schemes over $\mathrm{Spec}\,\mathcal{O}_K$ with canonical subgroup of level n, say H_n and H'_n. Enlarging L, we may assume that $H_n(L)\simeq H'_n(L)\simeq\mathbb{Z}/p^n\mathbb{Z}^g$. Let

$\pi\colon A \to A'$ be an isogeny of type γ. It induces a group isomorphism $H_n(L)\tilde{\to}H'_n(L)$. We have a commutative diagram (see section 4.4):

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{\pi^*} & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{F}'/p^w & \longrightarrow & \mathcal{F}/p^w \\ {\scriptstyle \mathrm{HT}_w}\uparrow & & {\scriptstyle \mathrm{HT}_w}\uparrow \\ (H'_n)^D(L) & \xrightarrow{\pi^D} & H_n^D(L) \end{array}$$

Since the bottom line is an isomorphism and the maps $\mathrm{HT}_w \otimes 1$ are isomorphisms it follows that π^* is an isomorphism.

□

We let π^{*-1} be the inverse of the isomorphism given by the proposition. We can now define the Hecke operator T_γ as the composition:

$$T_\gamma\colon \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v),\omega_w^{\dagger\kappa}) \overset{p_2^*}{\to} \mathrm{H}^0(C_\gamma(v),p_2^*\omega_w^{\dagger\kappa}) \overset{\pi^{*-1}}{\to} \mathrm{H}^0(C_\gamma(v),p_1^*\omega_w^{\dagger\kappa}) \overset{\mathrm{Tr}p_1}{\to} \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v),\omega_w^{\dagger\kappa}).$$

6.2 Hecke operators at p

We now define an action of the dilating Hecke algebra at p. For $i = 1,\ldots,g$, let C_i be the moduli scheme over K parametrizing principally polarized abelian schemes A, a level N structure ψ_N, an self-dual flag $\mathrm{Fil}_\bullet A[p]$ of subgroups of $A[p]$ and a lagrangian sub-group $L \subset A[p^2]$ if $i = 1,\ldots,g-1$ or $L \subset A[p]$ if $i = g$, such that $L[p] \oplus \mathrm{Fil}_i A[p] = A[p]$. There are two projections p_1, $p_2\colon C_i \to Y_{\mathrm{Iw},K}$. The first projection is defined by forgetting L. The second projection is defined by mapping $(A,\psi_N,\mathrm{Fil}_\bullet A[p])$ to $(A/L,\psi'_N,\mathrm{Fil}_\bullet A/L[p])$ where ψ'_N is the image of the level N structure and $\mathrm{Fil}_\bullet A/L[p]$ is defined as follows:

— For $j = 1,\ldots,i$, $\mathrm{Fil}_j A/L[p]$ is simply the image of $\mathrm{Fil}_j A[p]$ in A/L,

— For $j = i+1,\ldots,g$, $\mathrm{Fil}_j A/L[p]$ is the image in A/L of $\mathrm{Fil}_j A[p]+p^{-1}(\mathrm{Fil}_j A[p]\cap pL)$.

As before we consider the analytifications p_1, $p_2\colon C_i^{\mathrm{an}} \to Y_{\mathrm{Iw}}^{\mathrm{an}}$.

6.2.1 The operator $U_{p,g}$

We start by recalling the following result:

Proposition 6.2.1.1 ([Far2], prop. 17)**.** *Let G be a semi-abelian scheme of dimension g over $\mathcal{O}_K$, generically abelian. Assume that $\mathrm{Hdg}(G) < \frac{p-2}{2p-2}$. Let H_1 be the canonical sub-group of level 1 of G and let L be a sub-group of $G_K[p]$ such that $H_1 \oplus L = G_K[p]$. Then $\mathrm{Hdg}(G/L) = \frac{1}{p}\mathrm{Hdg}(G)$, and $G[p]/L$ is the canonical sub-group of level 1 of G/L.*

Let $\mathcal{C}_g(v) = C_g^{\mathrm{an}} \times_{p_1,Y_{\mathrm{Iw}}^{\mathrm{an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$. If $v < \frac{p-2}{2p-2}$, by the previous proposition, we have a diagram:

$$\begin{array}{ccccc} & & \mathcal{C}_g(v) & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & & & \mathcal{Y}_{\mathrm{Iw}}(\frac{v}{p}) \end{array}$$

Let $\pi\colon A \to A'$ be the universal isogeny over $\mathcal{C}_g(v)$. It induces a map $\pi^*\colon \omega_{A'} \to \omega_A$ and a map $\pi^*\colon p_2^*\mathcal{T}_{\mathrm{an}}^\times/\mathrm{U} \to p_1^*\mathcal{T}_{\mathrm{an}}^\times/\mathrm{U}$. This map is an isomorphism. Let $n \in \mathbb{N}$, and $v < \inf\{\frac{1}{4p^{n-1}}\}$. Let $w \in]n-1+v\frac{1}{p-1}, n - v\frac{p^n}{p-1}]$. We have the lemma whose proof is identical to the proof of lemma 6.1.1:

Lemma 6.2.1.2. *The map π^* induces an isomorphism*

$$\pi^*\colon p_2^*\mathcal{IW}_w^{o+}|_{\mathcal{Y}_{\mathrm{Iw}}(v)} \simeq p_1^*\mathcal{IW}_w^{o+}|_{\mathcal{Y}_{\mathrm{Iw}}(v)}.$$

We let π^{*-1} be the inverse of this map. Let κ be a w-analytic character. We now define the Hecke operator $U_{p,g}$ as the composition:

$$\mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(\frac{v}{p}), \omega_w^{\dagger\kappa}) \stackrel{p_2^*}{\to} \mathrm{H}^0(\mathcal{C}_g(v), p_2^*\omega_w^{\dagger\kappa}) \stackrel{\pi^{*-1}}{\longrightarrow} \mathrm{H}^0(\mathcal{C}_g(v), p_1^*\omega_w^{\dagger\kappa}) \stackrel{p^{\frac{-g(g+1)}{2}}\mathrm{Tr}p_1}{\longrightarrow} \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v), \omega_w^{\dagger\kappa}).$$

The operator $U_{p,g}$ hence improves the radius of overconvergence. Remark also that we normalize the trace of the map p_1 by a factor $p^{-\frac{g(g+1)}{2}}$ which is an inseparability degree (see [Pi2], sect. A.1). By a slight abuse of notation we also denote by $U_{p,g}$ the endomorphism of $\mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v), \omega_w^{\dagger\kappa})$ defined as the composition of the operator we just defined with the restriction map $\mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v), \omega_w^{\dagger\kappa}) \to \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(\frac{v}{p}), \omega_w^{\dagger\kappa})$.

6.2.2 The operators $U_{p,i}$, $i = 1, \ldots, g-1$

Let F be a finite extension of K and $(A, \psi_N, \mathrm{Fil}_\bullet A[p], L)$ be an F-point of C_i. Let $(A' = A/L, \psi'_N, \mathrm{Fil}_\bullet A'[p])$ be the image by p_2 of $(A, \psi_N, \mathrm{Fil}_\bullet A[p], L)$. Set $\pi\colon A \to A/L$ for the isogeny.

Proposition 6.2.2.1. *If $\mathrm{Hdg}\big(A[p^\infty]\big) < \frac{p-2}{2p^2-p}$ and $\mathrm{Fil}_g A[p]$ is the canonical sub-group of level 1, then $\mathrm{Hdg}\big(A[p^\infty]/L\big) \leq \mathrm{Hdg}\big(A[p^\infty]\big)$ and $\mathrm{Fil}_g A'[p]$ is the canonical sub-group of level 1 of A'.*

Proof. We assume that $\mathrm{Hdg}(A[p^\infty]) \leq \frac{p-2}{2p^2-p}$ and we are reduced by proposition 3.1.2 to show that $\mathrm{Fil}_g A'[p]$ has degree greater or equal to $g - \mathrm{Hdg}(A[p^\infty])$. Let H_2 be the canonical subgroup of level 2 in A and $x_1, \ldots, x_g$ be a basis of $H_2(\bar{K})$ as a $\mathbb{Z}/p^2\mathbb{Z}$-module. We complete it to a basis $x_1, \ldots, x_{2g}$ of $A[p^2](\bar{K})$. We can assume that $\mathrm{Fil}_\bullet A[p]$ is given by $0 \subset \langle px_1\rangle \subset \ldots \subset \langle px_1, \ldots, px_g\rangle = H_1$ and that L is given by $\langle px_{i+1}, \ldots, px_{2g-i}, x_{2g-i+1}, \ldots, x_{2g}\rangle$. Set $\tilde{H}_1 = \langle px_1, \ldots, px_i, x_{i+1}, \ldots, x_g\rangle$. With these notations $\mathrm{Fil}_g A'[p] = \tilde{H}_1/L$. We will show that $\deg\ \tilde{H}_1/L \geq g - \mathrm{Hdg}(A[p^\infty])$. We have a generic isomorphism

$$H_2/H_1 \stackrel{\mathrm{diag}(p1_i, 1_{g-i})}{\to} \tilde{H}_1/\langle px_{i+1}, \ldots, px_g\rangle,$$

which implies that $\deg \tilde{H}_1/\langle px_{i+1}, \ldots, px_g\rangle \geq g - p\mathrm{Hdg}(A[p^\infty])$ (by [Far1], coro. 3 on p. 10). By proposition 3.1.2,

$$\mathrm{Hdg}(A[p^\infty]/\langle px_{i+1}, \ldots, px_g\rangle) \leq p\mathrm{Hdg}(A[p^\infty])$$

and $\tilde{H}_1/\langle px_{i+1}, \ldots, px_g\rangle$ is the canonical sub-group of level 1 of $A/\langle px_{i+1}, \ldots, px_g\rangle$. At the level of the generic fiber, we have

$$A/\langle px_{i+1}, \ldots, px_g\rangle[p] = \langle px_{g+1}, \ldots, px_{2g}\rangle \oplus \tilde{H}_1/\langle px_{i+1}, \ldots, px_g\rangle.$$

By proposition 6.2.1.1 we obtain $\deg \tilde{H}_1/\langle px_{i+1}, \ldots, px_{2g}\rangle \geq g - \mathrm{Hdg}(A[p^\infty])$. We conclude, since the map $\tilde{H}_1/\langle px_{i+1}, \ldots, px_{2g}\rangle \to \tilde{H}_1/L$ is a generic isomorphism. $\square$

We set $\mathcal{C}_i(v) = C_i^{\mathrm{an}} \times_{p_1, Y_{\mathrm{Iw}}^{\mathrm{an}}} \mathcal{Y}_{\mathrm{Iw}}(v)$. If $v \leq \frac{p-2}{2p^2-p}$, we have a diagram:

$$\begin{array}{ccc} & \mathcal{C}_i(v) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathcal{Y}_{\mathrm{Iw}}(v) & & \mathcal{Y}_{\mathrm{Iw}}(v) \end{array}$$

Let $\pi\colon A \to A/L$ be the universal isogeny over $\mathcal{C}_i(v)$. We have a map $\pi^*\colon \omega_{A/L} \to \omega_A$. It induces a map $\tilde{\pi}^*\colon p_2^*\mathcal{T}_{\mathrm{an}}^{\times} \to p_1^*\mathcal{T}_{\mathrm{an}}^{\times}$ which sends a basis $\omega'_1, \ldots, \omega'_g$ of $\omega_{A/L}$ to $p^{-1}\pi^*\omega'_1, \ldots, p^{-1}\pi^*\omega_{g-i}, \pi^*\omega'_{g-i+1}, \ldots, \pi^*\omega'_g$. This map is an isomorphism, we call $\tilde{\pi}^{*-1}$ its inverse, and denote by the same symbol the quotient map $\tilde{\pi}^{*-1}\colon p_1^*\mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{U} \to p_2^*\mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{U}$.

Let $n \in \mathbb{N}$, $v < \inf\{\frac{1}{3p^{n-1}}, \frac{p-2}{2p^2-p}\}$ and $\underline{w} = (w_{i,j})_{1 \leq j \leq k \leq g}$ be a dilation parameter such that $w_{k,j} \in]0, n-2-v\frac{p^n}{p-1}]$.

Proposition 6.2.2.2. *We have* $\tilde{\pi}^{*-1}p_1^*\mathcal{IW}_{\underline{w}}^{o+} \subset p_2^*\mathcal{IW}_{\underline{w'}}^{o+}$ *where*

$$\begin{aligned} w'_{k,j} &= w_{k,j} \text{ if } j \leq k \leq i, \\ w'_{k,j} &= 1 + w_{k,j} \text{ if } j \leq i \text{ and } k \geq i+1, \\ w'_{k,j} &= w_{k,j} \text{ if } j \geq i+1. \end{aligned}$$

Proof. Let $(A, \mathrm{Fil}_\bullet A[p], \psi_N, L)$ be an F-point of $\mathcal{C}_i(v)$. We set $A' = A/L$ and assume that F is large enough to trivialize the group schemes H_n, H_n^D, H'_n and ${H'_n}^D$. There are $\mathbb{Z}/p^n\mathbb{Z}$-basis $e_1, \ldots, e_g$ for $H_n(F)$ and $e'_1, \ldots, e'_g$ for $H'_n(F)$ such that the flags on $H_1(F)$ and $H'_1(F)$ are given by $\mathrm{Fil}_j =< p^{n-1}e_g, p^{n-1}e_{g-1}, \ldots, p^{n-1}e_{g-j+1} >$ and $\mathrm{Fil}'_j =< p^{n-1}e'_g, p^{n-1}e'_{g-1}, \ldots, p^{n-1}e'_{g-j+1} >$ and the isogeny π induces a map $H_n(F) \to H'_n(F)$ given by $\mathrm{diag}(p\mathrm{Id}_{g-i}, \mathrm{Id}_i)$ in the basis. Let $x_1, \ldots, x_g$ and $x'_1, \ldots, x'_g$ be the dual basis of $H_n^D(F)$ and ${H'_n}^D(F)$ (for a choice of a primitive p^n-th root of unity). The flags on $H_1^D(F)$ and ${H'_1}^D(F)$ are given by $\mathrm{Fil}_j =< x_1, x_2, \ldots, x_j >$ and $\mathrm{Fil}'_j =< x'_1, x'_2, \ldots, x'_j >$ and the map $\pi^D\colon {H'_n}^D(F) \to H_n^D(F)$ is given by $\mathrm{diag}(p\mathrm{Id}_{g-i}, \mathrm{Id}_i)$. Set $w_0 = n - v\frac{p^n}{p-1}$. We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{\pi^*} & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{F}'/p^{w_0} & \longrightarrow & \mathcal{F}/p^{w_0} \\ {\scriptstyle \mathrm{HT}_{w_0}}\uparrow & & {\scriptstyle \mathrm{HT}_{w_0}}\uparrow \\ (H'_n)^D(L) & \xrightarrow{\pi^D} & H_n^D(L) \end{array}$$

Let $(\mathrm{Fil}_\bullet\mathcal{F}', \{\omega'_i \in \mathrm{Gr}_i\mathcal{F}'\})$ be an element of $p_2^*\mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{U}$ over A'. We assume that $\tilde{\pi}^*(\mathrm{Fil}_\bullet\mathcal{F}', \{\omega'_i \in \mathrm{Gr}_i\mathcal{F}'\}) = (\mathrm{Fil}_\bullet\mathcal{F}, \{\omega_i \in \mathrm{Gr}_i\mathcal{F}\}) \in p_1^*\mathcal{IW}_{\underline{w}}^{o+}$.

This means that:

$$\begin{aligned} p^{-1}\pi^*\omega'_j &= \sum_{k=j}^{g} a_{k,j}\mathrm{HT}_{w_0}(x_k) \quad \mathrm{mod}\ p^{w_0}\mathcal{F} + \mathrm{Fil}_{j-1}\mathcal{F} \text{ for } 1 \leq j \leq g-i, \\ \pi^*\omega'_j &= \sum_{k=j}^{g} a_{k,j}\mathrm{HT}_{w_0}(x_k) \quad \mathrm{mod}\ p^{w_0}\mathcal{F} + \mathrm{Fil}_{j-1}\mathcal{F} \text{ for } g-i+1 \leq j \leq g. \end{aligned}$$

where $(a_{k,j})_{1\leq j\leq k\leq g}\in \mathcal{O}_L^{\frac{g(g+1)}{2}}$ satisfy $v(a_{k,k}-1)\geq w_{k,k}$ and $v(a_{k,j})\geq w_{k,j}$ for $k>j$. We obtain:

$$\pi^*\omega'_j \;=\; \sum_{k=j}^{g-i} a_{k,j}\mathrm{HT}_{w_0}(\pi^D x'_k)+\sum_{k=g-i+1}^{g} p a_{k,j}\mathrm{HT}_{w_0}(\pi^D x'_k) \mod p^{w_0}\mathcal{F}+\mathrm{Fil}_{j-1}\mathcal{F}$$

for $1\leq j\leq g-i$,

$$\pi^*\omega'_j \;=\; \sum_{k=j}^{g} a_{k,j}\mathrm{HT}_{w_0}(\pi^D x'_k) \mod p^{w_0}\mathcal{F}+\mathrm{Fil}_{j-1}\mathcal{F} \text{ for } g-i+1\leq j\leq g.$$

Since $p\mathcal{F}\subset \pi^*\mathcal{F}'$, we now get that:

$$\omega'_j \;=\; \sum_{k=j}^{g-i} a_{k,j}\mathrm{HT}_{w_0}(x'_k)+\sum_{k=g-i+1}^{g} p a_{k,j}\mathrm{HT}_{w_0}(x'_k) \mod p^{w_0-1}\mathcal{F}'+\mathrm{Fil}_{j-1}\mathcal{F}'$$

for $1\leq j\leq g-i$,

$$\omega'_j \;=\; \sum_{k=j}^{g} a_{k,j}\mathrm{HT}_{w_0}(x'_k) \mod p^{w_0-1}\mathcal{F}'+\mathrm{Fil}_{j-1}\mathcal{F}' \text{ for } g-i+1\leq j\leq g.$$

□

Let $\underline{w}$ and $\underline{w}'$ be as in the proposition. Let κ be a $\inf_j\{w_{j,j}\}$-analytic character. We now define the Hecke operator $U_{p,i}$ as:

$$U_{p,i}\colon \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v),\omega^{\dagger\kappa}_{\underline{w}'}) \overset{p_2^*}{\to} \mathrm{H}^0(C_\gamma(v),p_2^*\omega^{\dagger\kappa}_{\underline{w}'}) \overset{\tilde{\pi}^{-1*}}{\to} \mathrm{H}^0(C_\gamma(v),p_1^*\omega^{\dagger\kappa}_{\underline{w}}) \overset{p^{-i(g+1)}\mathrm{Tr}p_1}{\to} \mathrm{H}^0(\mathcal{Y}_{\mathrm{Iw}}(v),\omega^{\dagger\kappa}_{\underline{w}}).$$

This Hecke operator improves analyticity. Note the normalization of the trace map by a factor $p^{-(g+1)i}$ which is an inseparability degree. We also denote by $U_{p,i}$ the endomorphism of $\mathrm{M}^{\dagger\kappa}_{\underline{w}}\big(\mathcal{X}_{\mathrm{Iw}}(p)(v)\big)$ obtained by composing the above operator with the restriction

$$\mathrm{M}^{\dagger\kappa}_{\underline{w}}\big(\mathcal{X}_{\mathrm{Iw}}(p)(v)\big)\to \mathrm{M}^{\dagger\kappa}_{\underline{w}'}\big(\mathcal{X}_{\mathrm{Iw}}(p)(v)\big).$$

6.2.3 The relationship between $U_{p,i}$ and δ_i

In this paragraph we establish the relationship between the operators $U_{p,i}$ and the operators δ_i of section §2.5. Let $1\leq i\leq g-1$ and consider the correspondence p_1, $p_2\colon \mathcal{C}_i(v)\to\mathcal{X}_{\mathrm{Iw}}(v)$.

Proposition 6.2.3.1. *Let L be a finite extension of K, $x,y\in\mathcal{Y}_{\mathrm{Iw}}(v)(L)$ such that $y\in p_2(p_1^{-1})\{x\}$. Let $w>0$ and κ be a w-analytic character. There exists a commutative diagram where the vertical maps are isomorphisms:*

$$\begin{array}{ccc} (\omega^{\dagger\kappa}_w)_y & \xrightarrow{\tilde{\pi}^{*-1}} & (\omega^{\dagger\kappa}_w)_x \\ \uparrow & & \uparrow \\ V^{w-\mathrm{an}}_{\kappa',L} & \xrightarrow{\delta_i} & V^{w-\mathrm{an}}_{\kappa',L} \end{array}$$

Proof. This follows from the definition (see also lemma 5.1 of [Pi2]). □

6.2.4 A compact operator

Let $n \in \mathbb{N}$, $v < \inf\{\frac{1}{3p^{n-1}}, \frac{p-2}{2p^2-p}\}$, $w \in]\frac{v}{p-1}, n-g-1-v\frac{p^n}{p-1}]$ and κ a w-analytic character.

The composite operator $\prod_{i=1}^g U_{p,i}$ induces a map from $\mathrm{M}^{\dagger\kappa}_{\underline{w}'}(\mathcal{X}_{\mathrm{Iw}}(\frac{v}{p})) \to \mathrm{M}^{\dagger\kappa}_{w}(\mathcal{X}_{\mathrm{Iw}}(v))$ where $\underline{w}' = (w'_{i,j})$ is defined by:

$$w'_{i,j} = i - j + w.$$

The natural restriction map $res\colon \mathrm{M}^{\dagger\kappa}_{w}(\mathcal{X}_{\mathrm{Iw}}(v)) \to \mathrm{M}^{\dagger\kappa}_{\underline{w}'}(\mathcal{X}_{\mathrm{Iw}}(\frac{v}{p}))$ is compact. We let $U = \prod_{i=1}^g U_{p,i} \circ res$. This is a compact endomorphism of $\mathrm{M}^{\dagger\kappa}_{w}(\mathcal{X}_{\mathrm{Iw}}(v))$.

6.3 Summary

For all $q \nmid pN$, let $\mathbb{T}_q$ be the spherical Hecke algebra

$$\mathbb{Z}[\mathrm{GSp}_{2g}(\mathbb{Q}_q)/\mathrm{GSp}_{2g}(\mathbb{Z}_q)].$$

Let $\mathbb{T}^{Np}$ be the restricted tensor product of the algebras $\mathbb{T}_q$. We have defined an action of $\mathbb{T}^{Np}$ on the Frechet space $\mathrm{M}^{\dagger\kappa}(X_{\mathrm{Iw}})$. Consider the dilating Hecke algebra, $\mathbb{U}_p$, defined as the polynomial algebra over $\mathbb{Z}$ with indeterminates $X_1, \ldots, X_g$. We have also defined an action of $\mathbb{U}_p$, sending X_i to $U_{p,i}$. We proved that the operator $U = \prod_i U_{p,i}$ is compact. Let us denote by $\mathbb{T}^{\dagger\kappa}$ the image of $\mathbb{T}^{Np} \otimes_{\mathbb{Z}} \mathbb{U}_p$ in $\mathrm{End}\big(\mathrm{M}^{\dagger\kappa}(X_{\mathrm{Iw}})\big)$ and call it the overconvergent Hecke algebra of weight κ.

7 Classicity

7.1 Statement of the main result

Let $\kappa = (k_1, \ldots, k_g) \in X^+(\mathrm{T})$. We have a series of natural restriction maps:

$$\mathrm{H}^0(X_{\mathrm{Iw}}, \omega^\kappa) \overset{r_1}{\to} \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega^\kappa) \overset{r_2}{\to} \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega^{\dagger\kappa}_w)$$

and we establish a criterion for an element in $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega^{\dagger\kappa}_w)$ to be in the image of $r_2 \circ r_1$. Let $\underline{a} = (a_1, \ldots, a_g) \in \mathbb{R}^g_{\geq 0}$. We set $\mathrm{M}^{\dagger\kappa}_w(\mathcal{X}_{\mathrm{Iw}}(p)(v))^{<\underline{a}}$ for the union of the generalized eigenspaces where $U_{p,i}$ has slope $< a_i$ for $1 \leq i \leq g$.

Theorem 7.1.1. *Let $\underline{a} = (a_1, \ldots, a_g)$ with $a_i = k_{g-i} - k_{(g-i)+1} + 1$ when $1 \leq i \leq g-1$ and $a_g = k_g - \frac{g(g+1)}{2}$. Then we have:*

$$\mathrm{M}^{\dagger\kappa}_w(\mathcal{X}_{\mathrm{Iw}}(v))^{<\underline{a}} \subset \mathrm{H}^0(X_{\mathrm{Iw}}, \omega^\kappa).$$

The proof of this theorem is split in two parts. We first show that $\mathrm{M}^{\dagger\kappa}_w(\mathcal{X}_{\mathrm{Iw}}(p)(v))^{<\underline{a}} \subset \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega^\kappa)$. This is a classicity statement at the level of sheaves and it is easily deduced from the results of section 2, see proposition 7.2.1.

We conclude by applying the main theorem of [P-S] as follows. Since $U_{p,g}$ is a compact operator on $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega^\kappa)$, for all $a_g \in \mathbb{R}_{\geq 0}$ we can define $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega^\kappa)^{<a_g}$ which is the sum of generalized eigenspaces for $U_{p,g}$ with eigenvalues of slope less that a_g.

Theorem 7.1.2 ([P-S])**.** *Let $a_g = k_g - \frac{g(g+1)}{2}$. Then $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega^\kappa)^{<a_g}$ is a space of classical forms.*

7.2 Relative BGG resolution

We now take $w \in]\frac{v}{p-1}, 1 - v\frac{p}{p-1}]$. We remark that for such a w, the fibers of the morphism $\pi\colon \mathcal{IW}_w^o \to \mathcal{X}_{\mathrm{Iw}}(p)(v)$ are connected. Consider the cartesian diagram:

$$\begin{array}{ccc} \mathcal{IW}_w^{o+} \times \mathcal{T}_{\mathrm{an}}^{\times} & \longrightarrow & \mathcal{T}_{\mathrm{an}}^{\times} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ \mathcal{IW}_w^{o+} & \longrightarrow & \mathcal{T}_{\mathrm{an}}^{\times}/\mathrm{U} \\ & \searrow^{\pi_2} & \downarrow \\ & & \mathcal{X}_{\mathrm{Iw}}(p)(v) \end{array}$$

We have an action of the Iwahori sub-group I on $\mathcal{IW}_w^{o+} \times \mathcal{T}_{\mathrm{an}}^{\times}$ and by differentiating we obtain an action of the enveloping algebra $U(\mathfrak{g})$ on

$$(\pi_2 \circ \pi_1)_* \mathscr{O}_{\mathcal{IW}_w^{o+} \times \mathcal{T}_{\mathrm{an}}^{\times}}$$

denoted $\star$. We have already defined an inclusion $d_0\colon \omega^{\kappa} \to \omega_w^{\dagger\kappa}$. For all $\alpha \in \Delta$ we now define a map $\Theta_\alpha\colon \omega_w^{\dagger\kappa} \to \omega_w^{\dagger s_\alpha \bullet \kappa}$. We first define an endomorphism of

$$(\pi_2 \circ \pi_1)_* \mathscr{O}_{\mathcal{IW}_w^{o+} \times \mathcal{T}_{\mathrm{an}}^{\times}}$$

by sending a section f to $X_{-\alpha}^{<\kappa,\alpha^\vee>+1} \star f$. It follows from section 2.4 that this map restricted to $\omega_w^{\dagger\kappa}$ produces the expected map Θ_α. We then set $d_1\colon \oplus\, \Theta_\alpha\colon \omega_w^{\dagger\kappa} \to \bigoplus_{\alpha\in\Delta} \omega_w^{\dagger s_\alpha \bullet \kappa}$. We have the following relative BGG resolution, which is a relative version of the theory recalled in section 2.4:

Proposition 7.2.1. *There is an exact sequence of sheaves over $\mathcal{X}_{\mathrm{Iw}}(p)(v)$:*

$$0 \to \omega^{\kappa} \xrightarrow{d_0} \omega_w^{\dagger\kappa} \xrightarrow{d_1} \bigoplus_{\alpha\in\Delta} \omega_w^{\dagger s_\alpha \bullet \kappa}$$

Proof Tensoring-completing the exact sequence 2.4.A (or more precisely its w-analytic version, see the remark below) by $\mathscr{O}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)}$ we get a sequence:

$$0 \longrightarrow V_{\kappa'} \hat{\otimes} \mathscr{O}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)} \xrightarrow{d_0 \otimes 1} V_{\kappa'}^{w-\mathrm{an}} \hat{\otimes} \mathscr{O}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)} \xrightarrow{d_1 \otimes 1} \bigoplus_{\alpha\in\Delta} V_{s_\alpha \bullet \kappa'}^{w-\mathrm{an}} \hat{\otimes} \mathscr{O}_{\mathcal{X}_{\mathrm{Iw}}(p)(v)} \qquad (7.2.A)$$

Note that the image of d_1 is closed in $\bigoplus_{\alpha\in\Delta} V_{s_\alpha \bullet \kappa'}^{\mathrm{an}}$ and is a direct factor of an orthonormalizable Banach module by the main theorem of [Jo]. It follows that there exists an isomorphism of Banach modules $V_{\kappa'}^{w-\mathrm{an}} = \mathrm{Im}(d_1) \oplus V_{\kappa'}$ splitting the sequence $0 \to V_{\kappa'} \to V_{\kappa'}^{w-\mathrm{an}} \to \mathrm{Im}(d_1) \to 0$. As a result, the sequence 7.2.A is exact. By proposition 5.3.4, this exact sequence is locally for the étale topology isomorphic to the sequence of the proposition which is exact. $\square$

Remark 7.2.2. The definition of the map Θ_α does not require the condition $w \in]\frac{v}{p-1}, 1 - v\frac{p}{p-1}]$ but it is needed for the exactness of the sequence.

The maps Θ_α do not commute with the action of the Hecke operators $U_{p,i}$, for $i = 1, \ldots, g-1$. Precisely, we have the following result for $v \leq \frac{p-2}{2p^2-p}$.

Proposition 7.2.3. *For $1 \leq i \leq g-1$ we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger\kappa}) & \xrightarrow{\Theta_\alpha} & \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger s_\alpha \bullet \kappa}) \\ \downarrow U_{p,i} & & \downarrow \alpha(d_{g-i})^{<\kappa,\alpha^\vee>+1} U_{p,i} \\ \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger\kappa}) & \xrightarrow{\Theta_\alpha} & \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger s_\alpha \bullet \kappa}) \end{array}$$

Proof. Let $f \in \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger\kappa})$. We need to check that

$$\Theta_\alpha U_{p,i} f = \alpha(d_{g-i})^{<\kappa,\alpha^\vee>+1} U_{p,i} \Theta_\alpha f.$$

We apply proposition 6.2.3.1 to reduce to the results of section 2.5. □

7.3 Classicity at the level of the sheaves

We now assume only that v is small enough for the operators $U_{p,i}$ to be defined. We make no particular assumption on w.

Proposition 7.3.1. *The submodule of $\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$ on which $U_{p,i}$ acts with slope strictly less than $k_{g-i} - k_{g-i+1} + 1$ for $1 \leq i \leq g-1$ and $U_{p,g}$ acts with finite slope is contained in $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega^\kappa)$.*

Proof. Let $f \in \mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$. For simplicity let us assume that f is an eigenvector for all operators $U_{p,i}$ with corresponding eigenvalue a_i. The operator $\prod_{i=1}^{g-1} U_{p,i}$ increases analyticity. Since we have $f = \prod_{i=1}^{g-1} a_{p,i}^{-1} \prod_{i=1}^{g-1} U_{p,i} f$ we can assume that $w \in]\frac{v}{p-1}, 1 - v\frac{p}{p-1}]$. We endow the space $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger\kappa})$ and $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger s_\alpha \bullet \kappa})$ for all simple positive root α with the supremum norm over the ordinary locus. This is indeed a norm by the analytic continuation principle (but of course $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger\kappa})$ may not be complete for this norm). For this choice, the $U_{p,i}$ operators have norm less or equal to 1. By the relative BGG exact sequence of proposition 7.2.1 it is enough to prove that $\Theta_\alpha f$ is 0. Let α be the character $(t_1, \ldots, t_g) \mapsto t_i.t_{i+1}^{-1}$. Since $U_{p,g-i}\Theta_\alpha(f) = p^{k_{i+1}-k_i-1}\Theta_\alpha U_{p,g-i}(f)$, we see that $\Theta_\alpha(f)$ is an eigenvector for $U_{p,g-i}$ for an eigenvalue of negative valuation. Since the norm of $U_{p,g-i}$ is less than 1, $\Theta_\alpha(f)$ has to be zero. □

8 Families

Recall that the weight space $\mathcal{W} = \mathrm{Hom}(\mathrm{T}(\mathbb{Z}_p), \mathbb{C}_p^\times)$ was defined in section 2.2. For any affinoid open subset $\mathcal{U}$ of $\mathcal{W}$, by proposition 2.2.2, there exists $w_U > 0$ such that the universal character $\kappa^{\mathrm{un}} \colon \mathrm{T}(\mathbb{Z}_p) \times \mathcal{W} \to \mathbb{C}_p^\times$ restricted to $\mathcal{U}$ extends to an analytic character $\kappa^{\mathrm{un}} \colon \mathrm{T}(\mathbb{Z}_p)(1 + p^{w_\mathcal{U}}\mathcal{O}_{\mathbb{C}_p}) \times \mathcal{U} \to \mathbb{C}_p^\times$.

8.1 Families of overconvergent modular forms

8.1.1 The universal sheaves $\omega_w^{\dagger\kappa^{\mathrm{un}}}$

Let $n \in \mathbb{N}, v \leq \frac{1}{2p^{n-1}}$ (resp. $\frac{1}{3p^{n-1}}$ if $p = 3$) and $w \in]n - 1 + \frac{v}{p-1}, n - v\frac{p^n}{p-1}]$ satisfying $w \geq w_\mathcal{U}$. We deduce immediately from proposition 2.2.2 that the construction given in section 5 works in families:

Proposition 8.1.1.1. *There exists a sheaf* $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ *on* $\mathcal{X}_{\mathrm{Iw}}(p)(v)\times\mathcal{U}$ *such that for any weight* $\kappa\in\mathcal{U}$*, the fiber of* $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ *over* $\mathcal{X}_{\mathrm{Iw}}(p)(v)\times\{\kappa\}$ *is* $\omega_w^{\dagger\kappa}$*.*

Proof. We consider the projection $\pi\times 1\colon \mathcal{IW}_w^{o+}\times\mathcal{U}\to\mathcal{X}_{\mathrm{Iw}}(p)(v)\times\mathcal{U}$. We take $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ to be the sub-sheaf of $(\pi\times 1)_*\mathscr{O}_{\mathcal{IW}_w^{o+}\times\mathcal{U}}$ of $(\kappa^{\mathrm{un}})'$-invariant sections for the action of $\mathrm{T}(\mathbb{Z}_p)$. □

Let A be the ring of rigid analytic functions on $\mathcal{U}$. Let $M_{v,w}$ be the A-Banach module $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{U},\omega_w^{\dagger\kappa^{\mathrm{un}}})$. Passing to the limit on v and w we get the A-Frechet space $M^\dagger=\lim_{v\to 0,w\to\infty} M_{v,w}$.

It is clear that the geometric definition of Hecke operators given in section 6 works in families. We thus have an action of the Hecke algebra of level prime to Np, $\mathbb{T}^{Np}$ on the space $M_{v,w}$. We also have an action of $\mathbb{U}_p$, the dilating Hecke algebra at p, on $M_{v,w}$ for v small enough.

Let D be the boundary in $\mathcal{X}_{\mathrm{Iw}}(v)$. We let $\omega_w^{\dagger\kappa^{\mathrm{un}}}(-D)$ be the cuspidal sub-sheaf of $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ of sections vanishing along D. Let $M_{v,w,\mathrm{cusp}}$ be the A-Banach module $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v)\times\mathcal{U},\omega_w^{\dagger\kappa}(-D))$ and $M^\dagger_{\mathrm{cusp}}=\lim_{v\to 0,w\to\infty} M_{v,w,\mathrm{cusp}}$. All these modules are stable under the action of the Hecke algebra. We wish to construct an eigenvariety out of this data.

8.1.2 Review of Coleman's Spectral theory

A convenient reference for the material in this section is [Buz]. The data we are given is:

- A reduced, equidimensional affinoid Spm A (e.g. $\mathcal{U}=\mathrm{Spm}\ A$ is an admissible affinoid open of the weight space $\mathcal{W}$),
- A Banach A-module M (e.g. the A- module of p-adic families of modular forms $M_{v,w}$ defined above for suitable v,w),
- A commutative endomorphism algebra $\mathbb{T}$ of M over A (e.g. the Hecke algebra),
- A compact operator $U\in\mathbb{T}$ (e.g. the operator $\prod_i U_{p,i}$).

Definition 8.1.2.1.
1. *Let I be a set. Let $C(I)$ be the A-module of functions $\{f\colon I\to A,\ \lim_{i\to\infty} f(i)=0\}$, where the limit is with respect to the filter of complements of finite subsets of I. The module $C(I)$ is equipped with its supremum norm.*
2. *A Banach A-module M is orthonormalizable if there is a set I such that $M\simeq C(I)$.*
3. *A Banach-A module M is projective if there is a set I and a Banach A-module M' such that $M\oplus M'=C(I)$.*

The following lemma follows easily from the universal property of projective Banach modules given in [Buz], p. 18.

Lemma 8.1.2.2. *Let*

$$0\to M\to M_1\to\cdots\to M_n\to 0$$

be an exact sequence of Banach A-modules, where the differentials are continuous and for all $1\le i\le n$, M_i is projective. Then M is projective.

We suppose now that M is projective. Since U is a compact operator, the following power series $P(T):=\det(1-TU|M)\in A[\![T]\!]$ exists. It is known that $P(T)=1+\sum_{n\geq 1}c_nT^n$ where $c_n\in A$ and $|c_n|r^n\to 0$ when $n\to\infty$ for all positive $r\in\mathbb{R}$. As a result, $P(T)$ is a rigid analytic function on Spm $A\times\mathbb{A}^1_{\mathrm{an}}$.

Definition 8.1.2.3. *The spectral variety* $\mathcal{Z}$ *is the closed rigid sub-space of* Spm $A \times \mathbb{A}^1_{\mathrm{an}}$ *defined by the equation* $P(T) = 0$.

A pair $(x, \lambda) \in \mathrm{Spm}\ A \times \mathbb{A}^1_{\mathrm{an}}$ is in $\mathcal{Z}$ if and only if there is an element $m \in M \otimes_A \overline{k(x)}$ such that $U \cdot m = \lambda^{-1} m$.

Proposition 8.1.2.4 ([Buz], thm. 4.6). *The map* $p_1 \colon \mathcal{Z} \to \mathrm{Spm}\ A$ *is locally finite flat. More precisely, there is an admissible cover of* $\mathcal{Z}$ *by open affinoids* $\{\mathcal{U}_i\}_{i \in I}$ *with the property that the map* $\mathcal{U}_i \to p_1(\mathcal{U}_i)$ *is finite flat.*

Let $i \in I$ and let B be the ring of functions on $p_1(\mathcal{U}_i)$. To $\mathcal{U}_i$ is associated a factorization $P(T) = Q(T)R(T)$ of P over $B[\![T]\!]$ where $Q(T)$ is a polynomial and $R(T)$ is a power series co-prime to $Q(T)$. Moreover, $\mathcal{U}_i$ is defined by the equation $Q(T) = 0$ in $p_1(\mathcal{U}_i) \times \mathbb{A}^1_{\mathrm{an}}$. To $\mathcal{U}_i$, one can associate a direct factor $\mathcal{M}(\mathcal{U}_i)$ of M. This is the generalized eigenspace of $M \otimes_A B$ for the eigenvalues of U occurring in $Q(T)$. The rule $\mathcal{U}_i \mapsto \mathcal{M}(\mathcal{U}_i)$ gives a coherent sheaf $\mathcal{M}$ of $\mathscr{O}_{\mathcal{Z}}$-modules which can be viewed as the universal generalized eigenspace.

Definition 8.1.2.5. *The eigenvariety* $\mathcal{E}$ *is the affine rigid space over* $\mathcal{Z}$ *associated to the coherent* $\mathscr{O}_{\mathcal{Z}}$*-algebra generated by the image of* $\mathbb{T}$ *in* $\mathrm{End}_{\mathscr{O}_{\mathcal{Z}}} \mathcal{M}$.

The map $w \colon \mathcal{E} \to \mathrm{Spm}\ A$ is locally finite and $\mathcal{E}$ is equidimensional. For each $x \in \mathrm{Spm}\ A$, the geometric points of $w^{-1}(x)$ are in bijection with the set of eigenvalues of $\mathbb{T}$ acting on $M \otimes_A \overline{k(x)}$, which are of finite slope for U (i.e the eigenvalue of U is non-zero).

The space $\mathcal{E}$ parametrizes eigenvalues. One may sometimes ask for a family of eigenforms. We have the following:

Proposition 8.1.2.6. *Let* $x \in \mathcal{E}$ *and* $f \in \mathcal{M} \otimes_A k(x)$ *be an eigenform corresponding to* x. *Assume that* w *is unramified at* x. *Then there is a family of eigenforms* F *passing through* f. *More precisely, there exist* Spm B, *an admissible open affinoid of* $\mathcal{E}$ *containing* x *and* $F \in M \otimes_A B$ *such that:*

— $\forall \phi \in \mathbb{T}, \phi \cdot F = F \otimes \phi$,

— *the image of* F *in* $M \otimes_A k(x)$ *is* f.

Proof Let Spm B be an admissible open affinoid of $\mathcal{E}$ containing x such that $w \colon \mathrm{Spm}\ B \to w(\mathrm{Spm}\ B)$ is finite unramified. Let C be the ring of rigid analytic functions of $w(\mathrm{Spm}\ B)$. Let e be the projector in $B \otimes_C B$ corresponding to the diagonal. The projective B-module $e\mathcal{M}(B) \otimes_C B$ is the sub-module of $\mathcal{M}(B) \otimes_C B$ of elements m satisfying $b.m = m \otimes b$ for all $b \in B$. We have a reduction map:

$$\mathcal{M}(B) \otimes_C B \to \mathcal{M}(B) \otimes_A k(x)$$

and f is in the image of $e\mathcal{M}(B) \otimes_C B$. Any element F of $e\mathcal{M}(B) \otimes_C B$, mapping to f is a family of eigenforms passing through f. □

8.1.3 Properties of the module $M_{v,w,\mathrm{cusp}}$

In proposition 8.2.3.3 we will prove the following structure result about the module $M_{v,w,\mathrm{cusp}}$:

Proposition 8.1.3.1. *a) The Banach A-module $M_{v,w,\mathrm{cusp}}$ is projective.*

b) For any $\kappa \in \mathcal{U}$, the specialization map

$$M_{v,w,\mathrm{cusp}} \to \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(p)(v), \omega_w^{\dagger\kappa}(-D))$$

is surjective.

Granted this proposition, one can apply Coleman's spectral theory as described in section §8.1.2 to construct an equidimensional eigenvariety over the weight space. Thanks to theorem 7.1.1 we also get precise information about the points of this eigenvariety. This is enough to prove theorems 1.1 and 1.2 of the introduction.

The rest of this chapter will be devoted to the proof of proposition 8.2.3.3. Let us point out two main differences between the case $g = 1$ treated in [AIS] and [Pi3] and the case $g \geq 2$ treated in the present article. First of all, the ordinary locus in modular curves over a p-adic field is an affinoid, whereas it is not an affinoid in the toroidal compactification of Siegel modular varieties of genus $g \geq 2$. Secondly, for modular curves the classical modular sheaves are interpolated by coherent sheaves, whereas for $g \geq 2$, the sheaves $\omega_w^{\dagger\kappa}$ are only Banach sheaves.

In the modular curve case, because of the two reasons mentioned above, it is easy to see by a cohomological argument that the proposition holds even in the non-cuspidal case (see [Pi3], cor. 5.1). We believe that the cuspidality assumption is necessary when $g \geq 2$.

Because the proof of proposition 8.2.3.3 is quite involved we will first explain the strategy of the proof (for technical reasons the actual proof of the proposition follows a slightly different line of arguments then the one sketched below, but the ideas are presented faithfully). Let $X_{\mathrm{Iw}}(p)^\star$ be the minimal compactification of $Y_{\mathrm{Iw}}(p)$. Let $X_{\mathrm{Iw}}(p)^\star_{\mathrm{rig}}$ be the rigid analytic fiber of $X_{\mathrm{Iw}}(p)^\star$ and $\xi\colon X_{\mathrm{Iw}}(p)_{\mathrm{rig}} \to X_{\mathrm{Iw}}(p)^\star_{\mathrm{rig}}$ be the projection. We define $\mathcal{X}_{\mathrm{Iw}}(p)^\star(v)$ as the image of $\mathcal{X}_{\mathrm{Iw}}(p)(v)$ in $X_{\mathrm{Iw}}(p)^\star_{\mathrm{rig}}$. If $v \in \mathbb{Q}$, this is an affinoid. We have a Banach sheaf $\omega_w^{\dagger\kappa^{\mathrm{un}}}$ on $\mathcal{X}_{\mathrm{Iw}}(p)(v) \times \mathcal{U}$. We will show that there exists an affinoid covering $\mathfrak{U} = (\mathcal{X}_i)_{i\in I}$ of $\mathcal{X}^\star_{\mathrm{Iw}}(v)$ such that (1) for every multi indexes $\underline{i} \in I^t$ for $1 \leq t \leq \sharp I$ the Banach A-module $\mathrm{H}^0\big((\xi\times 1)^{-1}\big(\mathcal{X}_{\underline{i}} \times \mathcal{U}\big), \omega_w^{\dagger\kappa^{\mathrm{un}}}(-D)\big)$ is projective, (2) the Chech sequence

$$0 \to M \to \oplus_i \mathrm{H}^0\big((\xi \times 1)^{-1}\big(\mathcal{X}_i \times \mathcal{U}\big), \omega_w^{\dagger\kappa^{\mathrm{un}}}(-D)\big) \to \cdots$$

is exact. This implies that M is a projective A-module. In other words, the first part of the proposition can be checked working locally on the minimal compactification. Of course, locally on the toroidal compactification the projectivity holds. On the other hand the acyclicity of the Chech complex relies on the fact that $X_{\mathrm{Iw}}(p)^\star_{\mathrm{rig}}$ is affinoid. To use both facts we are reduced to study the sheaf $\omega_w^{\dagger\kappa^{\mathrm{un}}}(-D)$ along the boundary and we conclude by some explicit computations. The second point of the proposition follows by similar arguments.

If the sheaf $(\xi \times 1)_*\omega_w^{\dagger\kappa^{\mathrm{un}}}(-D)$ on $\mathcal{X}_{\mathrm{Iw}}(p)^\star(v) \times \mathcal{U}$ were a Banach sheaf and we had an acyclicity result à la Kiehl for Banach sheaves relatively to affinoid coverings of $\mathcal{X}_{\mathrm{Iw}}(p)^\star(v)$, the existence of an affinoid covering satisfying properties (1) and (2) would be immediately true. Our approach to prove this property in this special situation is via formal models as discussed in section A.1.

8.1.4 An integral family

We consider the map defined in section 5.2.2:

$$\zeta = \pi_1 \circ \pi_2 \circ \pi_3 \colon \mathfrak{IW}_w^+ \to \mathfrak{X}_1(p^n)(v).$$

There is an action of the torus $\mathfrak{T}_w$ on $\mathfrak{IW}_w^+$ over $\mathfrak{X}_1(p^n)(v)$. For all $\kappa^o \in \mathcal{W}(w)^o(K)$ we set $\tilde{\mathfrak{w}}_w^{\dagger\kappa^o} = \zeta_* \mathscr{O}_{\mathfrak{IW}_w^+}[\kappa^{o\prime}]$.

Let $\kappa \in \mathcal{W}(w)$ mapping to κ^o. Let $\pi_4 \colon \mathfrak{X}_1(p^n)(v) \to \mathfrak{X}_{\mathrm{Iw}}(p)(v)$ be the finite projection. One recovers $\mathfrak{w}_w^{\dagger\kappa}$ by taking the direct image $\pi_{4,*}\tilde{\mathfrak{w}}_w^{\dagger\kappa^o}$ and the κ'-equivariant sections for the action of $\mathrm{B}(\mathbb{Z}_p)\mathfrak{B}_w$ or equivalently the invariants under the action $\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$ of the sheaf $\pi_{4,*}\tilde{\mathfrak{w}}_w^{\dagger\kappa^o}(-\kappa')$ with twisted action by $-\kappa'$ (after this twist, the action of $\mathrm{B}(\mathbb{Z}_p)\mathfrak{B}_w$ factors through its quotient $\mathrm{B}(\mathbb{Z}/p\mathbb{Z})$). The group $\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$ is of order divisible by p and has higher cohomology on $\mathbb{Z}_p$-modules. For this reason, we will implement the strategy of section 8.1.3 at the level of $\mathfrak{X}_1(p^n)$ for a while, and at the very end invert p and take into account the action of $\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$; see proposition 8.2.3.3.

The sheaves $\tilde{\mathfrak{w}}_w^{\dagger\kappa^o}$ can be interpolated. Consider the projection

$$\zeta \times 1 \colon \mathfrak{IW}_w^+ \times \mathfrak{W}(w)^o \to \mathfrak{X}_1(p^n)(v) \times \mathfrak{W}(w)^o$$

and the family of formal Banach sheaves

$$\tilde{\mathfrak{w}}_w^{\dagger\kappa^{\mathrm{oun}}} = \big(\zeta \times 1\big)_* \mathscr{O}_{\mathfrak{IW}_w^+ \times \mathfrak{W}(w)^o}[\kappa^{\mathrm{oun}\prime}].$$

8.1.5 Description of the sections

We denote by $\mathrm{Spf} R$ an open affine sub-formal scheme of $\mathfrak{X}_1(p^n)$. We let $\psi\colon (\mathbb{Z}/p^n\mathbb{Z})^g \to H_n^D(R[1/p])$ be the pull back of the universal trivialization. We have an isomorphism:

$$(\mathrm{HT}_w \circ \psi) \otimes 1 \colon \mathbb{Z}/p^n\mathbb{Z}^g \otimes_{\mathbb{Z}} R/p^wR \to \mathcal{F}/p^w\mathcal{F}.$$

We denote by $e_1, \ldots, e_g$ the canonical basis of $(\mathbb{Z}/p^n\mathbb{Z})^g$. Let $f_1, \ldots, f_g$ be a basis of $\mathcal{F}$ lifting the vectors $\mathrm{HT}_w \circ \psi(e_1), \ldots, \mathrm{HT}_w \circ \psi(e_g)$. With these choices, $\mathfrak{IW}_w^+|_{\mathrm{Spf} R}$ is identified with the set of matrices:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ p^w\mathfrak{B}(0,1) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p^w\mathfrak{B}(0,1) & p^w\mathfrak{B}(0,1) & \dots & 1 \end{pmatrix} \times \begin{pmatrix} 1+p^w\mathfrak{B}(0,1) \\ 1+p^w\mathfrak{B}(0,1) \\ \vdots \\ 1+p^w\mathfrak{B}(0,1) \end{pmatrix} \times_{\mathrm{Spf}\ \mathcal{O}_K} \mathrm{Spf}\ R$$

where the first $g \times g$ matrix parametrizes the position of the flag and the second column vector the basis of the graded pieces.

For $1 \leq j < i \leq g$, we let $X_{i,j}$ be the coordinate of the ball on the i-th line and j-th column in the $g \times g$ matrix and we let $X_1, \ldots, X_g$ be the coordinates on the column vector.

A function f on $\mathfrak{IW}_w^+|_{\mathrm{Spf}\ R}$ is a power series:

$$f(X_{i,j}, X_k) \in R\langle\langle X_{i,j}, X_k, 1 \leq j < i \leq g, 1 \leq k \leq g\rangle\rangle.$$

Let $\kappa^o \in \mathfrak{W}(w)^o$. Then $f \in \tilde{\mathfrak{w}}_w^{\dagger\kappa^o}(R)$ if and only if:

$$f(X_{i,j}, \lambda.X_k) = \kappa^{o\prime}(\lambda) f(X_{i,j}, X_k)\ \forall \lambda \in \mathfrak{T}_w(R').$$

We deduce the lemma:

Lemma 8.1.5.1. *A section $f \in \tilde{\mathfrak{w}}_w^{\dagger\kappa^o}(R)$ has a unique decomposition:*

$$f(X_{i,j}, X_k) = g(X_{i,j})\kappa^{o'}(1 + p^w X_1, \dots, 1 + p^w X_g),$$

where $g(X_{i,j}) \in R\langle\langle X_{i,j}, 1 \leq i < j \leq g\rangle\rangle$. This decomposition sets a bijection:

$$\tilde{\mathfrak{w}}_w^{\dagger\kappa^o}(R) \simeq R\langle\langle X_{i,j}, 1 \leq i < j \leq g\rangle\rangle.$$

Similarly:

Lemma 8.1.5.2. *A section $f \in \tilde{\mathfrak{w}}_w^{\dagger\kappa^{\mathrm{oun}}}(R\hat{\otimes}\mathcal{O}_K\langle\langle S_1, \dots, S_g\rangle\rangle)$ has a unique decomposition:*

$$f(X_{i,j}, X_k) = g(X_{i,j})\kappa^{\mathrm{oun}\prime}(1 + p^w X_1, \dots, 1 + p^w X_g),$$

where $g(X_{i,j}) \in R\langle\langle S_1, \dots, S_g, X_{i,j}\rangle\rangle$. This decomposition sets a bijection:

$$\tilde{\mathfrak{w}}_w^{\dagger\kappa^{\mathrm{oun}}}(R) \simeq R\langle\langle S_1, \dots, S_g, X_{i,j}\rangle\rangle.$$

Lemma 8.1.5.3. *Let $\varpi \in \mathcal{O}_K$ be the uniformizing element. We have*

$$\kappa^{\mathrm{oun}}\big((1 + p^w X_i)_{1\leq i\leq g}\big) \in 1 + \varpi\mathcal{O}_K\langle\langle S_1, \dots, S_g, X_1, \dots, X_g\rangle\rangle^\times$$

Proof We have

$$(1 + p^w X_i)^{S_i p^{-w+\frac{2}{p-1}}} = \sum_{k\geq 0} \frac{S_i p^{-w+\frac{2}{p-1}}(S_i p^{-w+\frac{2}{p-1}} - 1)\cdots(S_i p^{-w+\frac{2}{p-1}} - k + 1)}{k!}(p^w X_i)^k.$$

The constant term of this series is 1. Recall that for any integer $k \geq 1$, $v(k!) \leq \frac{k-1}{p-1}$. As a result the k-th coefficient of the series, for $k > 0$ has valuation at least $kw - kw + \frac{2k}{p-1} - \frac{k-1}{p-1} > 0$. □

For all $m \in \mathbb{N}$, we let $X_1(p^n)(v)_m$, $\mathcal{W}(w)^o_m$ be the schemes over $\mathcal{O}_K/\varpi^m\mathcal{O}_K$ obtained by reduction modulo ϖ^m from $\mathfrak{X}_1(p^n)(v)$ and $\mathfrak{W}(w)^o$. We let $\tilde{\mathfrak{w}}_{w,m}^{\dagger\kappa^{\mathrm{oun}}}$ (resp. $\tilde{\mathfrak{w}}_{w,m}^{\dagger\kappa^o}$), be the quasi-coherent sheaf over $X_1(p^n)(v)_m \times \mathcal{W}(w)^o_m$ (resp. $X_1(p^n)(v)_m$) obtained by pull back.

Corollary 8.1.5.4. *The quasi-coherent family of sheaves $\tilde{\mathfrak{w}}_{w,1}^{\dagger\kappa^{\mathrm{oun}}}$ over $X_1(p^n)(v)_1 \times \mathcal{W}(w)^o_1$ is constant: the sheaf $\tilde{\mathfrak{w}}_{w,1}^{\dagger\kappa^{\mathrm{oun}}}$ is the inverse image on $X_1(p^n)(v)_1 \times \mathcal{W}(w)^o_1$ of a sheaf defined on $X_1(p^n)(v)_1$.*

Proof In view of lemmas 8.1.5.1, 8.1.5.2 and 8.1.5.3, the sheaf $\tilde{\mathfrak{w}}_{w,1}^{\dagger\kappa^{\mathrm{oun}}}$ equals the pull back of the sheaf $\omega_{w,1}^{\dagger\kappa^o}$ for any $\kappa^o \in \mathcal{W}(w)^o(K)$. □

8.1.6 Dévissage of the sheaves

We have just given a description of the local sections of $\tilde{\mathfrak{w}}_w^{\dagger\kappa^o}$. This description depends on the choice of a basis $f_1, \dots, f_g$ of $\mathcal{F}$ lifting the universal basis $e_1, \dots, e_g$ of $H_1^D(R'[1/p])$. We'd now like to investigate the dependence on the choice of the basis $f_1, \dots, f_g$.

Let $(f'_1, \dots, f'_g)$ be an other compatible choice of basis for $\mathcal{F}$ with $P = \mathrm{Id}_g + p^w M \in \mathrm{GL}_g(R')$ be the base change matrix from $f_1, \dots, f_g$ to $f'_1, \dots, f'_g$. This second trivialization of $\mathcal{F}$ determines new coordinates $X'_{i,j}, X'_k$ on $\mathfrak{IW}_w^+|_{\mathrm{Spf}\ R}$.

Lemma 8.1.6.1. *We have the congruences:*

$$\begin{aligned} X'_{i,j} &= X_{i,j} + m_{i,j} \mod p^w R\langle\langle X_{s,t}, X_u\rangle\rangle \\ X'_k &= X_k + m_k \mod p^w R\langle\langle X_{s,t}, X_u\rangle\rangle \end{aligned}$$

for all $1 \leq j < i \leq g$ and all $1 \leq k \leq g$ where $m_{i,j}$ is the coefficient of M on the i-th line and j-th column and m_k the coefficient on the k-th diagonal entry.

Proof Let $\underline{X}$ and $\underline{X}'$ be the lower triangular matrices with $X_{i,j}$ and $X'_{i,j}$ on the i-th line and j-th column and X_k, X'_k on the k-th diagonal entry. We have

$$(\mathrm{Id}_g + p^w M)(\mathrm{Id}_g + p^w \underline{X}) = \mathrm{Id}_g + p^w(M + \underline{X}) + O(p^{2w}).$$

There is a unique upper triangular matrix N with 0 on the diagonal such that

$$(\mathrm{Id}_g + p^w M)(\mathrm{Id}_g + p^w \underline{X})(\mathrm{Id}_g + p^w N) = \mathrm{Id}_g + p^w \underline{X}'.$$

We have

$$(\mathrm{Id}_g + p^w M)(\mathrm{Id}_g + p^w \underline{X})(\mathrm{Id}_g + p^w N) = \mathrm{Id}_g + p^w(M + \underline{X} + N) + O(p^{2w}).$$

We deduce that $N = (-m_{i,j})_{1\leq i<j\leq g} + O(p^w)$ and that $M + \underline{X} + N = \underline{X}' \mod p^w$. □

Corollary 8.1.6.2. *Let $\kappa^o \in \mathcal{W}(w)^o(K)$. The quasi-coherent sheaf $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,1}$ is an inductive limit of coherent sheaves which are extensions of the trivial sheaf.*

Proof Covering $\mathfrak{X}_1(p^n)(v)$ by affine open formal sub-schemes Spf R and choosing a basis $(f_1, \dots, f_g)$ of $\mathcal{F}$ compatible with ψ, we can expand the sections of $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,1}|_{\mathrm{Spf}\ R}$ as polynomials in the variables $(X_{i,j})_{1\leq j<i\leq g}$. By lemma 8.1.6.1, the total degree of a section is independent of the choice of the basis, so we can write $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,1}$ as the inductive limit as $r \in \mathbb{N}$ grows of the sub-sheaves $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,1}|^{\leq r}$ of sections of degree bounded by r. In $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,1}|^{\leq r}$, we can consider for all $1 \leq k, l \leq g$, the sub-sheaf $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,1}|^{\leq r,k,l}$ locally generated by the polynomials of degree less than r in the variables $X_{i,j}$ for $i \geq k$ and $j \leq l$. This sub-sheaf is well defined by lemma 8.1.6.1. The sheaves

$$\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,1}|^{\leq r,k,l} \mod \tilde{\omega}^{\dagger\kappa^o}_{w,1}|^{\leq r,k-1,l} + \tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,1}|^{\leq r,k,l-1}$$

are isomorphic to the trivial sheaf. □

In general, one can always write $\tilde{\mathfrak{w}}^{\dagger\kappa}_{w,m}$ as an inductive limit of coherent sheaves in a reasonable way as follows. Let $\cup_i \mathfrak{U}_i$ be a finite Zariski cover of $\mathfrak{X}_1(p^n)$ by affine formal sub-schemes $\mathfrak{U}_i =$ Spf R_i such that over each $\mathfrak{U}_i$ we have the description of the sections of $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w}(R_i)$ as in lemma 8.1.5.1. We let $R_{i,m}$ be the reduction modulo ϖ^m of R_i and $U_{i,m} =$ Spec $R_{i,m}$. We have that $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m}(U_{i,m}) \simeq R_{i,m}[X_{i,j}, 1 \leq i < j \leq g]$. We let $\tilde{\mathfrak{w}}^{\dagger\kappa^o \leq r}_{w,i,m}$ be the coherent sheaf over $U_{i,m}$ associated to the sub-module of $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m}(U_{i,m})$ of polynomials of degree bounded by r. We also let $\tilde{\mathfrak{w}}^{\dagger\kappa^o \leq r}_{w,m}$ be the sub-sheaf of $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m}$ defined as the kernel of

$$\prod_i \tilde{\mathfrak{w}}^{\dagger\kappa^o \leq r}_{w,i,m} \longrightarrow \prod_{i,j} \tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m}|_{U_{i,m}\cap U_{j,m}}.$$

It is a quasi-coherent sheaf as it is the kernel of a morphism of quasi-coherent sheaves. Furthermore, for every i the natural map $\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,m}|_{U_{i,m}} \to \tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,i,m}$ is injective. As $\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,i,m}$ is a coherent sheaf over $U_{i,m}$ for every i, we deduce that $\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,m}|_{U_{i,m}}$ is a coherent sheaf as well. Furthermore, as $\mathrm{colim}_r\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,j,m} = \tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m}|_{U_{j,m}}$ for every j, we conclude that $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m} = \mathrm{colim}_r\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,m}$. Of course, the sheaves $\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,m}$ depend on the choice of the cover $\cup_i\mathfrak{U}_i$ and on the choice of a basis of $\mathcal{F}|_{\mathfrak{U}_i}$ if $m \geq 2$. If we fix two choices of cover and basis, we get two inductive limits:

$$\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m} = \mathrm{colim}_r\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r,1}_{w,m} = \mathrm{colim}_r\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r,2}_{w,m}.$$

It is easy to check that in any case, for all r, there is $r' \leq r''$ such that

$$\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r',2}_{w,m} \hookrightarrow \tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r,1}_{w,m} \hookrightarrow \tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r'',2}_{w,m}.$$

The preceding discussion still makes sense for $\tilde{\mathfrak{w}}^{\dagger\kappa^{o\mathrm{un}}}_{w,m}$.

8.2 The base change theorem

8.2.1 The boundary of the compactification

Let $V = \oplus_{i=1}^{2g}\mathbb{Z}e_i$ be a free $\mathbb{Z}$-module of rank $2g$ equipped with the symplectic form of matrix $J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$. For all totally isotropic direct factor V' we consider $C(V/V'^{\perp})$ the cone of symmetric semi-definite bilinear forms on $V/V'^{\perp} \otimes \mathbb{R}$ with rational radical. If $V' \subset V''$ we have an inclusion $C(V/V''^{\perp}) \subset C(V/V'^{\perp})$. We let $\mathfrak{C}$ be the set of all totally isotropic direct factors $V' \subset V$ and $\mathcal{C}$ be the quotient of the disjoint union:

$$\coprod_{V'\in\mathfrak{C}} C(V/V'^{\perp})$$

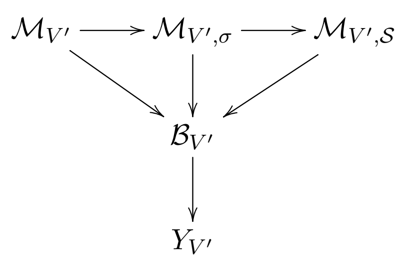
by the equivalence relation induced by the inclusions $C(V/V''^{\perp}) \subset C(V/V'^{\perp})$.

The given basis of V gives a "principal level N structure": $\psi_N\colon \mathbb{Z}/N\mathbb{Z}^{2g} \simeq V/NV$. The vectors $e_1, \ldots, e_g$ give a "Siegel principal level p^n structure":

$$\psi\colon (\mathbb{Z}/p^n\mathbb{Z})^g \hookrightarrow V/p^nV.$$

Let Γ be the congruence sub-group of $\mathrm{G}(\mathbb{Z})$ stabilizing ψ_N and $\Gamma_1(p^n)$ be the congruence subgroup stabilizing ψ and ψ_N. Let $\mathcal{S}$ be a rational polyhedral decomposition of $\mathcal{C}$ which is Γ-admissible (see [F-C], section IV. 2).

We now recall some facts about the toroidal compactifications following the presentation adopted in [Str2] (see for example sections 1.4.3, 2.1 and 2.2). For any $V' \in \mathfrak{C}$ and $\sigma \in \mathcal{S}$ lying in the interior of $C(V/V'^{\perp})$ there is a diagram:



Where:

— $Y_{V'}$ is the moduli space of principally polarized abelian schemes of dimension $g-r$, with $r = \mathrm{rk}_{\mathbb{Z}} V'$, with principal level N structure;
— Let $A_{V'}$ is the universal abelian scheme over $Y_{V'}$. The scheme $\mathcal{B}_{V'} \to Y_{V'}$ is an abelian scheme. Moreover, there is an isogeny $i\colon \mathcal{B}_{V'} \to A^r_{V'}$ of degree a power of N. Over $\mathcal{B}_{V'}$, there is a universal semi-abelian scheme

$$0 \to \mathrm{T}_{V'} \to \tilde{G}_{V'} \to A_{V'} \to 0$$

where $\mathrm{T}_{V'}$ is the torus $V' \otimes_{\mathbb{Z}} \mathbb{G}_m$;
— $\mathcal{M}_{V'}$ is a moduli space of principally polarized 1-motives with principal level N structure, the map $\mathcal{M}_{V'} \to \mathcal{B}_{V'}$ is a torsor under a torus with character group $S_{V'}$, isogenous to $\mathrm{Hom}(\mathrm{Sym}^2 V/V'^{\perp}, \mathbb{G}_m)$;
— $\mathcal{M}_{V'} \to \mathcal{M}_{V',\sigma}$ is an affine toroidal embedding attached to the cone $\sigma \in C(V/V'^{\perp})$ and the $\mathbb{Z}$-module $S_{V'}$;
— $\mathcal{M}_{V'} \to \mathcal{M}_{V',\mathcal{S}}$ is a toroidal embedding, locally of finite type associated to the polyhedral decomposition $\mathcal{S}$. The scheme $\mathcal{M}_{V',\sigma}$ is open affine in $\mathcal{M}_{V',\mathcal{S}}$.

We shall denote by Z_σ the closed stratum in $\mathcal{M}_{V',\sigma}$ and by $Z_{V'}$ the closed stratum in $\mathcal{M}_{V',\mathcal{S}}$. We let $\Gamma_{V'}$ be the stabilizer of V' in Γ, it acts on $C(V/V'^{\perp})$ and on the toroidal embedding $\mathcal{M}_{V',\mathcal{S}}$. We assume that $\{\mathcal{S}\}$ is a smooth and projective admissible polyhedral decomposition. The existence is guaranteed by the discussion in [F-C, §V.5]. Let Y be the moduli space classifying principally polarized abelian varieties over $\mathcal{O}_K$ with principal level N structure. Let $Y \subset X$ be the toroidal compactification associated to $\mathcal{S}$. The hypothesis that $\mathcal{S}$ is projective guarantees that X is a projective scheme and not simply an algebraic space.

Theorem 8.2.1.1 ([F-C])**.** *1. The toroidal compactification X carries a stratification indexed by $\mathfrak{C}/\Gamma$. For all $V' \in \mathfrak{C}$ the completion of X along the V'-stratum is isomorphic to the space $\widehat{\mathcal{M}}_{V',\mathcal{S}}/\Gamma_{V'}$ where $\widehat{\mathcal{M}}_{V',\mathcal{S}}$ is the completion of $\mathcal{M}_{V',\mathcal{S}}$ along the strata $Z_{V'}$.*

2. The toroidal compactification X carries a finer stratification indexed by $\mathcal{S}/\Gamma$. Let $\sigma \in \mathcal{S}$. The corresponding stratum in X is Z_σ. Let Z be an affine open subset of Z_σ. Then the henselization of X along Z is isomorphic to the henselization of $\mathcal{M}_{V,\sigma}$ along Z.

The Hasse invariant of the semi-abelian scheme on the special fiber of $\widehat{\mathcal{M}}_{V',\mathcal{S}}$ is the Hasse invariant of the abelian part of the semi-abelian scheme. We can thus identify Ha with the Hasse invariant defined over the special fiber of $Y_{V'}$.

Recall from section 5.2 that we have defined a formal scheme $\mathfrak{X}(v)$ with a morphism $\mathfrak{X}(v) \to \mathfrak{X}$ to the formal completion $\mathfrak{X}$ of X along its special fiber. We can now describe the boundary of $\mathfrak{X}(v)$, namely the complement of the inverse image $\mathfrak{Y}(v)$ of the formal completion $\mathfrak{Y} \subset \mathfrak{X}$ of Y. We will need some notations:
— $\mathfrak{Y}_{V'}$ is the completion of $Y_{V'}$ along its special fiber;
— $\mathfrak{Y}_{V'}(v)$ is largest formal open subset of the formal admissible blow-up of $\mathfrak{Y}_{V'}$ along the ideal (p^v, Ha) where the ideal (p^v, Ha) is generated by Ha (see 5.2);
— $\mathfrak{B}_{V'}$ is the completion of $\mathcal{B}_{V'}$ along its special fiber and $\mathfrak{B}_{V'}(v)$ is the fiber product $\mathfrak{B}_{V'} \times_{\mathfrak{Y}_{V'}} \mathfrak{Y}_{V'}(v)$. We define similarly $\mathfrak{M}_{V'}(v)$, $\mathfrak{M}_{V',\sigma}(v)$ and $\mathfrak{M}_{V',\mathcal{S}}(v)$, $\mathfrak{Z}_\sigma(v)$, $\mathfrak{Z}_{V'}(v)$.

Proposition 8.2.1.2. *The formal scheme $\mathfrak{X}(v)$ has a fine stratification indexed by $\mathcal{S}/\Gamma$ over a coarse stratification indexed by $\mathfrak{C}/\Gamma$. For all $\sigma \in \mathcal{S}$ the corresponding strata is $\mathfrak{Z}_{V',\sigma}(v)$. For any open affine sub-scheme $\mathfrak{Z}$ of $\mathfrak{Z}_{V',\sigma}(v)$ the henselization of $\mathfrak{X}(v)$ along $\mathfrak{Z}$ is isomorphic to the henselization of $\mathfrak{M}_{V',\sigma}(v)$ along $\mathfrak{Z}$. For all $V' \in \mathfrak{C}'$ the completion of $\mathfrak{X}(v)$ along the V'-strata of $\mathfrak{X}(v)$ is isomorphic to the completion $\widehat{\mathfrak{M}}_{V',\mathcal{S}}(v)$ of $\mathfrak{M}_{V',\mathcal{S}}(v)$ along $\mathfrak{Z}_{V'}(v)$.*

Proof As admissible blow-ups commute with flat base change, this follows easily from theorem 8.2.1.1. □

In section 5.2 we have introduced a covering $\mathfrak{X}_1(p^n)(v) \to \mathfrak{X}(v)$. We now describe the boundary of $\mathfrak{X}_1(p^n)(v)$, i.e., the complement of the inverse image of $\mathcal{Y}(v) \subset \mathfrak{X}(v)$. Let $\mathfrak{C}'$ be the subset of $\mathfrak{C}$ of totally isotropic spaces satisfying $\psi((\mathbb{Z}/p^n\mathbb{Z})^g) \subset {V'}^{\perp}$. We let $\mathcal{C}'$ be the quotient of the disjoint union:

$$\coprod_{V' \in \mathfrak{C}'} C(V/{V'}^{\perp})$$

by the equivalence relation induced by the inclusions $C(V/{V''}^{\perp}) \subset C(V/{V'}^{\perp})$. Clearly, $\Gamma_1(p^n)$ acts on $\mathfrak{C}'$ and the polyhedral decomposition $\mathcal{S}$ induces a polyhedral decomposition $\mathcal{S}'$ of $\mathcal{C}'$ which is $\Gamma_1(p^n)$-admissible.

Let $V' \in \mathfrak{C}'$ of rank r. We have an exact sequence

$$0 \to V'/p^nV' \to {V'}^{\perp}/p^n{V'}^{\perp} \to {V'}^{\perp}/V' + p^n{V'}^{\perp} \to 0.$$

The image of $\psi(\mathbb{Z}/p^n\mathbb{Z}^g)$ in ${V'}^{\perp}/V' + p^n{V'}^{\perp}$ is a totally isotropic subspace of rank p^{g-r}, that we denote by W. The map ψ also provides a section $s\colon W \hookrightarrow {V'}^{\perp}/p^n{V'}^{\perp}$. By duality, ψ provides an isomorphism that we again denote by ψ:

$$\psi\colon (\mathbb{Z}/p^n\mathbb{Z})^g \simeq W^{\vee} \oplus V/(p^nV + {V'}^{\perp})$$

To describe the local charts, we need some notations.

— $\mathcal{Y}_{V'}(v)$ is the rigid fibre of $\mathfrak{Y}_{V'}(v)$.

— We let $H_{n,V'}$ be the canonical subgroup of the universal abelian scheme $\mathfrak{A}_{V'}$ over $\mathfrak{Y}_{V'}(v)$ and we denote by $\mathcal{Y}_1(p^n)_{V'}(v)$ the torsor $\mathrm{Isom}_{\mathcal{Y}_{V'}(v)}(W^{\vee}, H^D_{n,V'})$. We let $\psi_{V'}$ be the universal trivialisation.

— $\mathfrak{Y}_1(p^n)_{V'}(v)$ is the normalization of $\mathfrak{Y}_{V'}(v)$ in $\mathcal{Y}_1(p^n)_{V'}(v)$.

— Recall that there is an isogeny $i\colon \mathfrak{B}(v) \to \mathfrak{A}^r_{V'}$ of degree a power of N. We let $i_{\mathrm{can}}\colon \mathfrak{A}_{V'} \to \mathfrak{A}_{V'}/H_{n,V'}$ be the canonical projection. We set

$$\mathfrak{B}_1(p^n)_{V'} = \mathfrak{B}(v) \times_{i,\mathfrak{A}^r_{V'},i^D_{\mathrm{can}}} (\mathfrak{A}_{V'}/H_{n,V'})^r.$$

The abelian scheme $\mathfrak{B}_1(p^n)_{V'} \to \mathfrak{Y}_1(p^n)_{V'}$ carries a universal diagram:

$$\begin{array}{ccc} {V'}^{\vee} & \longrightarrow & \mathfrak{A}'_V \\ \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle i^D_{\mathrm{can}}} \\ {V'}^{\vee} & \longrightarrow & \mathfrak{A}_{V'}/H_{n,V'} \end{array}$$

which is equivalent to the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{T}_{V'} & \longrightarrow & \tilde{\mathfrak{G}}_{V'} & \longrightarrow & \mathfrak{A}_{V'} & \longrightarrow & 0 \\ & & \downarrow \mathrm{Id} & & \downarrow j & & \downarrow i_{\mathrm{can}} & & \\ 0 & \longrightarrow & \mathfrak{T}_{V'} & \longrightarrow & \tilde{\mathfrak{G}}'_{V'} & \longrightarrow & \mathfrak{A}_{V'}/H_{n,V'} & \longrightarrow & 0 \end{array}$$

The group $\mathrm{Ker} j$ is a lift to $\tilde{\mathfrak{G}}_{V'}[p^n]$ of $H_{n,V'}$.
Over the rigid fiber, we thus have $H_n = \mathrm{T}_{V'}[p^n] \oplus \mathrm{Ker} j$ and $H_n^D = V/(p^nV + V'^{\perp}) \oplus \mathrm{Ker} j^D$. The map $\psi_{V'}$ provides an isomorphism $W^\vee \simeq Kerj^D$. The map ψ now provides an isomorphism $\psi\colon \mathbb{Z}/p^n\mathbb{Z}^g \simeq H_n^D$.

— We define $\mathfrak{M}_1(p^n)_{V'}(v)$, $\mathfrak{M}_1(p^n)_{V',\sigma}(v)$ and $\mathfrak{M}_1(p^n)_{V',\mathcal{S}'}(v)$, $\mathfrak{Z}_1(p^n)_\sigma(v)$, $\mathfrak{Z}_1(p^n)_{V'}(v)$ by fiber product of $\mathfrak{M}_{V'}(v)$, $\mathfrak{M}_{V',\sigma}(v)$, $\mathfrak{M}_{V',\mathcal{S}'}(v)$, $\mathfrak{Z}_\sigma(v)$, $\mathfrak{Z}_{V'}(v)$ with $\mathfrak{B}_1(p^n)_{V'}$ over $\mathfrak{B}_{V'}$.

Proposition 8.2.1.3. *The formal scheme $\mathfrak{X}_1(p^n)(v)$ has a fine stratification indexed by $\mathcal{S}'/\Gamma_1(p^n)$ over a coarse stratification indexed by $\mathfrak{C}'/\Gamma_1(p^n)$. For all $\sigma \in \mathcal{S}'$ the corresponding strata is $\mathfrak{Z}_1(p^n)_{V',\sigma}(v)$. For any open affine sub-scheme $\mathfrak{Z}$ of $\mathfrak{Z}_1(p^n)_{V',\sigma}(v)$ the henselization of $\mathfrak{X}_1(p^n)(v)$ along $\mathfrak{Z}$ is isomorphic to the henselization of $\mathfrak{M}_1(p^n)_{V',\sigma}(v)$ along $\mathfrak{Z}$. For all $V' \in \mathfrak{C}'$ the completion of $\mathfrak{X}_1(p^n)(v)$ along the V'-strata of $\mathfrak{X}_1(p^n)(v)$ is isomorphic to the completion $\widehat{\mathfrak{M}_1(p^n)}_{V',\mathcal{S}'}(v)$ of $\mathfrak{M}_1(p^n)_{V',\mathcal{S}'}(v)$ along $\mathfrak{Z}_1(p^n)_{V'}(v)$.*

Proof Over the rigid fiber, this is a variant of theorem 8.2.1.1. In particular we know that the rigid fiber of the local charts of level $\Gamma_1(p^n)$ are correctly described. It is now easy to check that our formal local charts of level $\Gamma_1(p^n)$ are normal, and as a result, they are the normalization of the formal local charts of level Γ. Since normalization commutes with étale localization, we conclude. □

Remark 8.2.1.4. The process of obtaining toroidal compactifications by normalization is studied in [F-C] p. 128, in a different situation.

8.2.2 Projection to the minimal compactification

There is a projective scheme $X^\star$ called minimal compactification and a proper morphism $\xi\colon X \to X^\star$. Let us recall some properties of the minimal compactification:

Theorem 8.2.2.1 ([F-C], Thm. V.2.7)**.** *The minimal compactification $X^\star$ is stratified by $\mathfrak{C}/\Gamma$ and the morphism ξ is compatible with the stratification.*

— for any $V' \in \mathfrak{C}$ the V'-strata is $Y_{V'}$,

— For a geometric point $\bar{x}$ of the V' strata, we have

$$\widehat{\mathscr{O}_{X^\star,\bar{x}}} = \Big(\prod_{\lambda \in S_{V'} \cap C(V/V'^{\perp})^\vee} \mathrm{H}^0\big(\widehat{\mathcal{B}_{V',\bar{x}}}, \mathcal{L}(\lambda)\big)\Big)^{\Gamma_{V'}}$$

where

- *$\widehat{\mathscr{O}_{X^\star,\bar{x}}}$ is the completion of the strict henselization of $\mathscr{O}_{X^\star}$ along $\bar{x}$,*
- *$\widehat{\mathcal{B}_{V',\bar{x}}}$ is the formal completion of $\mathcal{B}_{V'}$ along its fiber over $\bar{x}$,*
- *$\mathcal{L}(\lambda)$ is the invertible sheaf over $\mathcal{B}_{V'}$ of λ-homogeneous functions on $\mathcal{M}_{V'}$.*

The Hasse invariant Ha descends to a function on the special fiber of $X^\star$. We let $\mathfrak{X}^\star$ be the completion of $\mathfrak{X}$ along its special fiber. We denote by $\mathfrak{X}^\star(v)$ the p-adic completion of the normalization of the greatest open sub-scheme of the blow-up of $\mathfrak{X}^\star$ along the ideal (p^v, Ha), on which this ideal is generated by Ha.

Proposition 8.2.2.2. *For all $V' \in \mathfrak{C}'$ the V'-stratum of $\mathfrak{X}^\star(v)$ is $\mathfrak{Y}_{V'}(v)$.*

Proof This follows from the fact that (Ha, p^v) is a regular sequence in $Y_{V'}$ and in $X^\star$. This implies that the blow-up along (Ha, p^v) is in both cases a closed sub-scheme of a relative 1-dimensional projective space with equation $T\mathrm{Ha} - Sp^v$ (where T, S are homogeneous coordinates). □

We have a diagram:

$$\begin{array}{ccc} \mathfrak{X}_1(p^n)(v) & \xrightarrow{\pi_4} & \mathfrak{X}(v) \\ & \searrow^{\eta} & \downarrow{\scriptstyle \xi} \\ & & \mathfrak{X}^\star(v) \end{array}$$

We let $X_1(p^n)(v)_m$, $X^\star(v)_m$, $\mathcal{M}_1(p^n)_{V',\sigma}(v)_m$, $\mathcal{B}_1(p^n)_{V'}(v)_m$, ... be the schemes obtained by reduction modulo ϖ^m from $\mathfrak{X}_1(p^n)(v)$, $\mathfrak{M}_1(p^n)_{V',\sigma}(v)$, $\mathfrak{B}_1(p^n)_{V'}(v)$, ... We will also consider the projection $\eta \times 1\colon \mathfrak{X}_1(p^n)(v) \times \mathfrak{W}(w)^o \to \mathfrak{X}^\star(v) \times \mathfrak{W}(w)^o$. Finally, we use D to denote the boundary in $\mathfrak{X}_1(p^n)$, $\mathfrak{X}_1(p^n)(v)$, ...

Theorem 8.2.2.3. *Consider the following diagram for l, $m \in \mathbb{N}$ and $m \geq l$:*

$$\begin{array}{ccc} X_1(p^n)(v)_l & \xrightarrow{i} & X_1(p^n)(v)_m \\ \downarrow{\scriptstyle \eta_l} & & \downarrow{\scriptstyle \eta_m} \\ X^\star(v)_l & \xrightarrow{i'} & X^\star(v)_m \end{array}$$

Then we have the base change property:

$$i'^*(\eta_m)_*\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m}(-D) = \eta_{l,*}\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,l}(-D).$$

In particular, $\big(\eta_*\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w}(-D)\big)$ is a formal Banach sheaf over $\mathfrak{X}^\star(v)$. Similarly, also $(\eta \times 1)_*\big(\tilde{\mathfrak{w}}^{\dagger\kappa^{oun}}_{w}(-D)\big)$ is a formal Banach sheaf over $\mathfrak{X}^\star(v) \times \mathfrak{W}(w)^o$.

Proof The property is local for the $fppf$-topology on $X^\star(v)_m$. Let $\bar{x} \in X^\star(v)_m$ be a geometric point. We can write $\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m}$ as an inductive limit of coherent sheaves $\mathrm{colim}\tilde{\mathfrak{w}}^{\dagger\kappa^o \leq r}_{w,m}$ by the discussion at the end of section 8.1.6. By the theorem on formal functions [EGAIII, §4], and because direct images commute with inductive limits, we have that

$$\eta_{m,*}\tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w,m}(-D)\big(\widehat{X^\star(v)_{m,\bar{x}}}\big) = \mathrm{colim}_r \mathrm{H}^0\big(X_1\widehat{(p^n)(v)}_{m,\bar{x}}, \tilde{\mathfrak{w}}^{\dagger\kappa^o \leq r}_{w,m}(-D)\big)$$

where $\widehat{X^\star(v)_{m,\bar{x}}}$ is the completion of the strict henselization of $X^\star(v)_m$ at $\bar{x}$ and $X_1\widehat{(p^n)(v)}_{m,\bar{x}}$ is the completion of $X_1(p^n)(v)_m$ along $\eta_m^{-1}(\bar{x})$. This completion is isomorphic to a finite disjoint union of spaces $\mathcal{M}_1\widehat{(p^n)_{V',\mathcal{S}'}(v)}_{m,\bar{y}}/\Gamma_1(p^n)_{V'}$ where $\bar{y}$ is a geometric point in

$Y_1(p^n)_{V'}(v)_m$. This space fits in the following diagram:

$$\begin{array}{ccccc} \widehat{\mathcal{M}_1(p^n)_{V',\mathcal{S}'}}(v)_{m,\bar{y}} & \xrightarrow{h_2} & \widehat{\mathcal{M}_1(p^n)_{V',\mathcal{S}'}}(v)_{m,\bar{y}}/\Gamma_1(p^n)_{V'} & \longrightarrow & \widehat{X_1(p^n)(v)}_{m,\bar{x}} \\ \downarrow h_1 & & \downarrow h_3 & & \\ \widehat{\mathcal{B}_1(p^n)_{V'}}(v)_{m,\bar{y}} & \xrightarrow{h_4} & \widehat{Y_1(p^n)_{V'}(v)}_{m,\bar{y}} & & \end{array}$$

where $\widehat{\mathcal{B}_1(p^n)_{V'}}(v)_{m,\bar{y}}$, $\widehat{\mathcal{M}_1(p^n)_{V',\mathcal{S}}}(v)_{m,\bar{y}}$ are the formal completions of $\mathcal{B}_1(p^n)_{V'}(v)_m$ and $\mathcal{M}_1(p^n)_{V',\mathcal{S}}(v)_m$ over their fibers at $\bar{y}$. We are thus reduced to prove the following:

Claim: the formation of the module

$$\mathrm{colim}_r \mathrm{H}^0\big(\widehat{\mathcal{M}_1(p^n)_{V',\mathcal{S}'}}(v)_{m,\bar{y}}/\Gamma_1(p^n)_{V'}, \tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,m}(-D)\big)$$

commutes with reduction modulo ϖ^l for $l \leq m$.

We provide two proofs.

First Proof: We identify the module in the claim with

$$\mathrm{H}^0\Big(\Gamma_1(p^n)_{V'}, \mathrm{colim}_r \mathrm{H}^0\big(\widehat{\mathcal{M}_1(p^n)_{V',\mathcal{S}'}}(v)_{m,\bar{y}}, h_2^*\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,m}(-D)\big)\Big).$$

We remark that we have a formal semi-abelian scheme $\tilde{\mathfrak{G}}_{V'}$ over $\widehat{\mathcal{B}_1(p^n)_{V'}}(v)_{m,\bar{y}}$, extension of the universal $g-r$-dimensional formal abelian scheme $\mathfrak{A}_{V'}$ by the r-dimensional formal torus $\widehat{T}_{V'} := V' \otimes \widehat{\mathbb{G}}_m$. It admits a canonical subgroup scheme $H_n \subset \tilde{\mathfrak{G}}_{V'}[p^n]$ extension of the canonical subgroup $H_{n,V'} \subset \mathfrak{A}_{V'}[p^n]$ by $\widehat{T}_{V'}[p^n]$ and, as explained in propositions 4.2.1 and 4.2.2, the tautological principal p^n-level structure and the Hodge-Tate morphism for H_n provide a morphism $\mathrm{HT}\colon (\mathbb{Z}/p^n\mathbb{Z})^g \to \omega_{\tilde{\mathfrak{G}}_{V'}}/p^w$. Thus, proceeding as in proposition 4.3.1 we obtain a sheaf $\mathcal{F} \subset \omega_{\tilde{\mathfrak{G}}_{V'}}$ and an isomorphism $\mathrm{HT}_w\colon (\mathbb{Z}/p^n\mathbb{Z})^g \otimes \mathscr{O}_{\widehat{\mathcal{B}_1(p^n)_{V'}}(v)_{m,\bar{y}}} \longrightarrow \mathcal{F}_w$. The Levi quotient of $\Gamma_1(p^n)_{V'}$ is a subgroup of $\mathrm{GL}_g(V') \times \mathrm{GSp}(V'^{\perp}/V')$. We let $\Gamma'_1(p^n)_{V'}$ be the projection of $\Gamma_1(p^n)_{V'}$ onto its $\mathrm{GL}_g(V')$ factor. As $\widehat{\mathcal{B}_1(p^n)_{V'}}(v)_{m,\bar{y}}$ classifies extensions by $\mathfrak{A}_{V'}$ and $\widehat{T}_{V'}$ with a level N-structure, the group $\Gamma'_1(p^n)_{V'}$ acts on $\widehat{T}_{V'}$, on $\widehat{\mathcal{B}_1(p^n)_{V'}}(v)_{m,\bar{y}}$ and on $\mathcal{F}$ and we get an induced action of the group $\Gamma_1(p^n)_{V'}$ through the natural morphism $\Gamma_1(p^n)_{V'} \to \Gamma'_1(p^n)_{V'}$. We thus obtain an action of $\Gamma_1(p^n)_{V'}$ on $\mathcal{F}$ so that HT_w is $\Gamma_1(p^n)_{V'}$-equivariant. The functoriality of $\mathcal{F}$ and HT_w implies that their base change via h_1 coincide with the base change via h_2 of the sheaf $\mathcal{F}$ and the map HT_w for the universal degenerating semi-abelian scheme over $\widehat{X_1(p^n)(v)}_{m,\bar{x}}$. As in §4.5 we get an affine formal scheme $\mathfrak{IW}^+_{w,m} \longrightarrow \widehat{\mathcal{B}_1(p^n)_{V'}}(v)_{m,\bar{y}}$, with an equivariant action of $\Gamma_1(p^n)_{V'}$, such that its base change via h_1 is the reduction modulo ϖ^m of the base change via h_2 of the formal scheme $\mathfrak{IW}^+_w$ naturally defined over $\widehat{X_1(p^n)(v)}_{m,\bar{x}}$. Taking κ' invariant functions on $\mathfrak{IW}^+_{w,m}$ as in definition 5.2.2.3 we introduce a quasi-coherent sheaf $\underline{\tilde{\mathfrak{w}}}^{\dagger\kappa^o}_{w,m}$ over $\widehat{\mathcal{B}_1(p^n)_{V'}}(v)_{m,\bar{y}}$, with an equivariant action of $\Gamma'_1(p^n)_{V'}$ and hence of $\Gamma_1(p^n)_{V'}$. As explained in section 8.1.6 we can write $\underline{\tilde{\mathfrak{w}}}^{\dagger\kappa^o}_{w,m}$ as an inductive limit of coherent sheaves $\mathrm{colim}_r\underline{\tilde{\mathfrak{w}}}^{\dagger\kappa^o\leq r}_{w,m}$. Then, each $\mathrm{colim}_r h_1^*\underline{\tilde{\mathfrak{w}}}^{\dagger\kappa^o\leq r}_{w,m}$ is naturally a $\Gamma_1(p^n)_{V'}$-equivariant sheaf through the diagonal action of $\Gamma_1(p^n)_{V'}$ on $\mathrm{colim}_r\underline{\tilde{\mathfrak{w}}}^{\dagger\kappa^o\leq r}_{w,m}$ and on $\mathscr{O}_{\widehat{\mathcal{M}_1(p^n)_{V',\mathcal{S}}}(v)_{m,\bar{y}}}$. Due to the definition of $\tilde{\mathfrak{w}}^{\dagger\kappa^o\leq r}_{w,m}$ in section 8.1.6 it follows that we have a $\Gamma_1(p^n)_{V'}$-equivariant isomorphism of quasi-coherent sheaves over $\widehat{\mathcal{M}_1(p^n)_{V',\mathcal{S}'}}(v)_{m,\bar{y}}$:

$$\mathrm{colim}_r h_2^*\left(\tilde{\mathfrak{w}}_{w,m}^{\dagger\kappa^o\leq r}\right) = \mathrm{colim}_r h_1^*\left(\underline{\tilde{\mathfrak{w}}}_{w,m}^{\dagger\kappa^o\leq r}\right).$$

By the projection formula, we have that

$$\mathrm{colim}_r \mathrm{H}^0\big(\widehat{\mathcal{M}_1(p^n)_{V'}}_{,\mathcal{S}'}(v)_{m,\bar{y}}/\Gamma_1(p^n)_{V'}, \tilde{\mathfrak{w}}_{w,m}^{\dagger\kappa^o\leq r}(-D)\big)$$

$$\begin{aligned} &= \mathrm{H}^0\Big(\Gamma_1(p^n)_{V'}, \mathrm{colim}_r \mathrm{H}^0\big(\widehat{\mathcal{M}_1(p^n)_{V'}}_{,\mathcal{S}'}(v)_{m,\bar{y}}, h_1^*\underline{\tilde{\mathfrak{w}}}_{w,m}^{\dagger\kappa^o\leq r}(-D)\big)\Big) \\ &= \Big(\mathrm{colim}_r \prod_{\lambda\in S'_V\cap C(V/V'^{\perp})^\vee,\lambda>0} \mathrm{H}^0\big(\mathcal{B}_1(\widehat{p^n)_{V'}(v)}_{m,\bar{y}}, \mathcal{L}(\lambda)\otimes\underline{\tilde{\mathfrak{w}}}_{w,m}^{\dagger\kappa^o\leq r}\big)\Big)^{\Gamma_1(p^n)_{V'}}. \end{aligned}$$

The action of $\Gamma_1(p^n)_{V'}$ on $S_{V'}$ and on the product above factors via $\Gamma'_1(p^n)_{V'}$. Furthermore, $\Gamma'_1(p^n)_{V'}$ acts freely on the elements $\lambda \in S'_V \cap C(V/V'^{\perp})^\vee$ which are definite positive (indeed, the stabilizer of an element would be a compact group, hence finite, but $\Gamma'_1(p^n)_{V'}$ has no finite subgroups because of the principal level N structure). Let S_0 be a set of representative of the orbits. We then have

$$\Big(\mathrm{colim}_r \prod_{\lambda\in S'_V\cap C(V/V'^{\perp})^\vee,\lambda>0} \mathrm{H}^0\big(\mathcal{B}_1(\widehat{p^n)_{V'}(v)}_{m,\bar{y}}, \mathcal{L}(\lambda)\otimes\underline{\tilde{\mathfrak{w}}}_{w,m}^{\dagger\kappa^o\leq r}\big)\Big)^{\Gamma_1(p^n)_{V'}}$$

$$= \mathrm{colim}_r \prod_{\lambda\in S_0} \mathrm{H}^0\big(\mathcal{B}_1(\widehat{p^n)_{V'}(v)}_{m,\bar{y}}, \mathcal{L}(\lambda)\otimes\underline{\tilde{\mathfrak{w}}}_{w,m}^{\dagger\kappa^o\leq r}\big).$$

So it remains to see that the formation of $\mathrm{H}^0\big(\mathcal{B}_1(\widehat{p^n)_{V'}(v)}_{m,\bar{y}}, \mathcal{L}(\lambda)\otimes\underline{\tilde{\mathfrak{w}}}_{w,m}^{\dagger\kappa^o\leq r}\big)$ commutes with reduction modulo ϖ^l for $l\leq m$. We have an exact sequence of sheaves over $\mathcal{B}_1(\widehat{p^n)_{V'}(v)}_{m,\bar{y}}$:

$$0\to\mathcal{L}(\lambda)\otimes\underline{\tilde{\mathfrak{w}}}_{w,m-l}^{\dagger\kappa^o\leq r}\xrightarrow{\varpi^l}\mathcal{L}(\lambda)\otimes\underline{\tilde{\mathfrak{w}}}_{w,m}^{\dagger\kappa^o\leq r}\to\mathcal{L}(\lambda)\otimes\underline{\tilde{\mathfrak{w}}}_{w,l}^{\dagger\kappa^o\leq r}\to 0$$

By induction, we may assume that $l = m-1$. It is enough to show that

$$\mathrm{H}^1\big(\mathcal{B}_1(\widehat{p^n)_{V'}(v)}_{1,\bar{y}}, \mathcal{L}(\lambda)\otimes\underline{\tilde{\mathfrak{w}}}_{w,1}^{\dagger\kappa^o\leq r}\big)\cong 0$$

Note that $\mathcal{L}(\lambda)$, for $\lambda\in S_0$, is a very ample sheaf on the abelian scheme $\mathfrak{B}_1(p^n)_{V'}(v)$, due to the principal level N-structure with $N\geq 3$; see the proof of [F-C, Thm. V.5.8]. The vanishing of the cohomology follows by the vanishing theorem of [Mu, §III.16], the theorem of formal functions [EGAIII, §4] and the fact that $\underline{\tilde{\mathfrak{w}}}_{w,1}^{\dagger\kappa^o\leq r}$ is an iterated extension of the trivial sheaf as seen by corollary 8.1.6.2 and its proof.

Second Proof: In order to prove the claim, it suffices to consider the case $l = m-1$. From the local description of the sheaves $\tilde{\mathfrak{w}}_{w,m}^{\dagger\kappa^o\leq r}(-D)$, see section 8.1.6, it follows that the kernel of $\mathrm{colim}_r\tilde{\mathfrak{w}}_{w,m}^{\dagger\kappa^o\leq r}\to\mathrm{colim}_r\tilde{\mathfrak{w}}_{w,m-1}^{\dagger\kappa^o\leq r}$ is isomorphic to $\tilde{\mathfrak{w}}_{w,1}^{\dagger\kappa^o}$. The latter is an inductive limit of coherent sheaves which are extensions of the trivial sheaf $\mathscr{O}_{X_1(p^n)(v)_1}$ by corollary 8.1.6.2. To prove the claim it then suffices to show that $R^1\eta_*\mathscr{O}_{X_1(p^n)(v)_1}(-D) = 0$ and this follows from proposition 8.2.2.4 below.

The proof of the second part of the proposition goes exactly along the same lines since the family of sheaves $\tilde{\mathfrak{w}}_{w,1}^{\dagger\kappa^{oun}}$ is trivial over the weight space by corollary 8.1.5.4.

□

Recall that $\mathcal{X}_1(p^n)(v)$ is defined by the choice of a smooth and projective admissible polyhedral decomposition in the sense of [F-C, Def. V.5.1]. In particular for every V' as above there exists a $\Gamma_1(p^n)_{V'}$-admissible *polarization function* $h\colon C_{V'} \to \mathbb{R}$, i.e. a function satisfying:

(i) $h(x) > 0$ if $x \neq 0$ and $h(tx) = th(x)$ for all $t \in \mathbb{R}_{\geq 0}$ and every $x \in C_{V'}$;

(ii) h is upper convex namely $h\big(tx + (1-t)y\big) \geq th(x) + (1-t)h(y)$ for every x and $y \in C_{V'}$ and every $0 \leq t \leq 1$;

(iii) h is $\mathcal{S}$-linear, i.e., h is linear on each $\Sigma \in \mathcal{S}$;

(iv) h is strictly upper convex for $\mathcal{S}$, i.e, $\mathcal{S}$ is the coarsest among the fans $\mathcal{S}'$ of $C_{V'}$ for which h is $\mathcal{S}'$-linear. Equivalently, the closure of the top dimensional cones of $\mathcal{S}$ are exactly the maximal polyhedral cones of $C_{V'}$ on which h is linear.

(v) h is $\mathbb{Z}$-valued on the set of N times the subset of $C_{V'}$ consisting of symmetric semi-definite bilinear and integral valued forms on $V/V'^{\perp}$.

Consider the morphism $\eta\colon \mathcal{X}_1(p^n)(v) \to \mathcal{X}^\star(v)$ from a toroidal to the minimal compactification. Then,

Proposition 8.2.2.4. *We have* $R^q\eta_*\mathscr{O}_{\mathcal{X}_1(p^n)(v)}(-D) = 0$ *for every* $q \geq 1$.

Proof We use the notations of the proof of theorem 8.2.2.3. We write $\widehat{Z}_{V'} := \mathcal{M}_1\widehat{(p^n)_{V'}},_{\mathcal{S}'}(v)_{1,\bar{y}}$ and $\widetilde{Z}_{V'} := \widehat{Z}_{V'}/\Gamma_1(p^n)_{V'}$ to simplify the notation. By the theorem of formal functions [EGAIII, §4] it suffices to prove that $\mathrm{H}^q\big(\widetilde{Z}_{V'}, \mathscr{O}_{\widehat{Z}_{V'}}(-D)\big) = 0$ for every $q \geq 1$.

We recall the construction of $\widehat{Z}_{V'}$. We have fixed a smooth $\Gamma_1(p^n)_{V'}$-admissible polyhedral decomposition $\mathcal{S}$ of the cone $C_{V'} := C(V/V'^{\perp})$ of symmetric semi-definite bilinear forms on $V/V'^{\perp} \otimes \mathbb{R}$ with rational radical. Every $\Sigma \in \mathcal{S}$ defines an affine relative torus embedding Z_Σ over the abelian scheme $\mathcal{B}_1(p^n)_{V'}(v)_{1,\bar{y}}$ which we view over the spectrum of the local ring underlying $Y_1\widehat{(p^n)_{V'}(v)}_{1,\bar{y}}$. The Z_Σ's glue to define a relative torus embedding $Z_{V'}$ stable for the action of $\Gamma_1(p^n)_{V'}$. For every Σ we let $W_\Sigma := \sum_{\rho\in\Sigma(1)^o} D_\rho$ be the relative Cartier divisor defined by the set $\Sigma(1)^o$ of 1-dimensional faces of Σ contained in the interior $C_{V'}^o$ of the cone $C_{V'}$. Put $W_{V'} := \cup_\Sigma W_\Sigma$. Write $\widehat{Z}_\Sigma$ (resp. $\widehat{Z}_{V'}$) for the formal scheme given by the completion of Z_Σ (resp. $Z_{V'}$) with respect to the ideal $\mathscr{O}_{Z_\Sigma}\big(-W_\Sigma\big)$ (resp. $\mathscr{O}_{Z_{V'}}\big(-W_{V'}\big)$). Then, $\widehat{Z}_{V'} = \cup_\Sigma \widehat{Z}_\Sigma$.

Fix a $\Gamma_1(p^n)_{V'}$-admissible polarization function $h\colon C_{V'} \to \mathbb{R}_{\geq 0}$. As in [F-C, Def. V.5.6] we define

$$D'_{\Sigma,h} := -\sum_{\rho\in\Sigma(1)} a_\rho D_\rho, \qquad D'_h := \cup_\Sigma D_{\Sigma,h},$$

where the sum is over the set $\Sigma(1)$ of all 1-dimensional faces of Σ (not simply over the set $\Sigma(1)^o$ as in the definition of W_Σ) . More explicitly, for every $\Sigma \in \mathcal{S}$ and every $\rho \in \Sigma(1)$ there exists a unique primitive integral element $n(\rho) \in \Sigma$ such that $\rho = \mathbb{R}_{\geq 0} n_\rho$. We then set $a_\rho := h\big(n(\rho)\big)$. As $h(x) \neq 0$ if $x \neq 0$ and $h(x) \in \mathbb{Z}$ for integral elements $x \in C_{V'}$, we deduce that a_ρ is a positive integer for every Σ and every $\rho \in \Sigma(1)$. Moreover, the Cartier divisor D'_h of $Z_{V'}$ is $\Gamma_1(p^n)_{V'}$-invariant.

Take a positive integer s. As the set of integers $\{a_\rho\}$ is $\Gamma_1(p^n)_{V'}$-invariant, it is finite. Thus there exists $\ell \in \mathbb{Z}$ such that $0 < sa_\rho < \ell$ for every ρ. Recall that given a $\mathbb{Q}$-divisor $E := \sum_\rho e_\rho D_\rho$, with $e_\rho \in \mathbb{Q}$, one defines the "round down" Cartier divisor

$\lfloor E \rfloor := \sum_\rho \lfloor e_\rho \rfloor D_\rho$ by setting $\lfloor e_\rho \rfloor$ to be the smallest integer $\leq e_\rho$. In particular, we compute

$$\lfloor \ell^{-1} s D'_h \rfloor = \sum_{\rho \in \Sigma(1)} -D_\rho := -D,$$

where D defines the boundary of Z_Σ. Multiplication by ℓ on the cone $C_{V'}$ preserves the polyhedral decomposition $\mathcal{S}$ and for every $\Sigma \in \mathcal{S}$ defines a finite and flat morphism $\Phi_{\ell,\Sigma}\colon Z_\Sigma \to Z_\Sigma$ over $\mathcal{B}_1(p^n)_{V'}(v)_{1,\bar{y}}$. As $\Phi_{\ell,\Sigma}^*(W_\Sigma) = \ell W_\Sigma$, the morphism $\Phi_{\ell,\Sigma}$ induces a morphism on the completions with respect to the ideal $\mathcal{O}_{Z_\Sigma}(-W_\Sigma)$ and we get finite and flat morphisms $\widehat{\Phi}_{\ell,\Sigma}\colon \widehat{Z}_\Sigma \longrightarrow \widehat{Z}_\Sigma$. They glue to provide a finite flat, $\Gamma_1(p^n)_{V'}$-equivariant morphism of formal schemes $\widehat{\Phi}_\ell\colon \widehat{Z}_{V'} \to \widehat{Z}_{V'}$ over $\mathcal{B}_1\widehat{(p^n)_{V'}(v)}_{1,\bar{y}}$. After passing to the quotients by $\Gamma_1(p^n)_{V'}$ we get a finite and flat morphism of formal schemes

$$\widetilde{\Phi}_\ell\colon \widetilde{Z}_{V'} \to \widetilde{Z}_{V'}.$$

As $sD'_h - \Phi_\ell^*(-D) = \sum_\rho (\ell - s a_\rho) D_\rho$ is an effective Cartier divisor, we have by adjunction natural inclusions of invertible sheaves

$$\iota_\ell\colon \mathscr{O}_{Z_{V'}}(-D) \to \Phi_{\ell,*}\big(\mathscr{O}_{Z_{V'}}(sD'_h)\big), \qquad \widehat{\iota}_\ell\colon \mathscr{O}_{\widehat{Z}_{V'}}(-D) \to \widehat{\Phi}_{\ell,*}\big(\mathscr{O}_{\widehat{Z}_{V'}}(sD'_h)\big).$$

In [C-S, Lemmas 9.2.6 and 9.3.4] a canonical splitting of ι_ℓ as $\mathscr{O}_{Z_{V'}}$-modules is constructed in terms of the cone $C_{V'}$ and the integers $\{a_\rho | \rho \in \Sigma(1)\}$. In particular it is $\Gamma_1(p^n)_{V'}$-equivariant and it defines a $\Gamma_1(p^n)_{V'}$-equivariant splitting of $\widehat{\iota}_\ell$ as $\mathscr{O}_{\widehat{Z}_{V'}}$-modules after passing to completions. Taking the quotient under $\Gamma_1(p^n)_{V'}$ we get a split injective map $\widetilde{\iota}_\ell\colon \mathscr{O}_{\widetilde{Z}_{V'}}(-D) \to \widetilde{\Phi}_{\ell,*}\mathscr{O}_{\widetilde{Z}_{V'}}(sD'_h)$ of $\mathscr{O}_{\widetilde{Z}_{V'}}$-modules. Taking cohomology for every $q \in \mathbb{N}$ we get a split injective map

$$\mathrm{H}^q\big(\widetilde{Z}_{V'}, \mathscr{O}_{\widetilde{Z}_{V'}}(-D)\big) \to \mathrm{H}^q\big(\widetilde{Z}_{V'}, \widetilde{\Phi}_{\ell,*}\big(\mathscr{O}_{\widetilde{Z}_{V'}}(sD'_h)\big) \cong \mathrm{H}^q\big(\widetilde{Z}_{V'}, \mathscr{O}_{\widetilde{Z}_{V'}}(sD'_h)\big).$$

If we show that there exists $s \in \mathbb{N}$ such that $\mathrm{H}^q\big(\widetilde{Z}_{V'}, \mathscr{O}_{\widetilde{Z}_{V'}}(sD'_h)\big) = 0$ for every $q \geq 1$, we are done. This follows if we prove that there exists s such that $\mathscr{O}_{\widetilde{Z}_{V'}}(sD'_h)$ is a very ample invertible sheaf.

It follows from [F-C, Thm. V.5.8] that the map $\eta\colon \mathcal{X}_1(p^n)(v) \to \mathcal{X}^\star(v)$ is the normalization of the blow-up of $\mathcal{X}^\star(v)$ defined by a sheaf of ideals $\mathcal{J}$ such that $\eta^*(\mathcal{J})$ restricted to $\widetilde{Z}_{V'}$ is $\mathscr{O}_{\widetilde{Z}_{V'}}(dD'_h)$ for a suitable d. In particular, as η is the composite of a finite map and a blow-up, the sheaf $\eta^*(\mathcal{J})$ is ample relatively to η. We conclude that $\mathscr{O}_{\widetilde{Z}_{V'}}(dD'_h)$ is an ample sheaf on $\widetilde{Z}_{V'}$. In particular, there exists a large enough multiple s of d so that $\mathscr{O}_{\widetilde{Z}_{V'}}(sD'_h)$ is very ample as claimed.

Alternatively, to prove that $\mathscr{O}_{\widetilde{Z}_{V'}}(D'_h)$ is ample, it suffices to prove that its restriction to the boundary $\partial \widetilde{Z}_{V'}$ is ample. As $\partial \widetilde{Z}_{V'} = W_{V'}/\Gamma_1(p^n)_{V'}$ is proper over $\widehat{Y_1(p^n)}_{V'}(v)_{m,\bar{y}}$ it suffices to prove ampleness after passing to the residue field $k(\bar{y})$ of $\widehat{Y_1(p^n)}_{V'}(v)_{m,\bar{y}}$. It then follows from the Nakai-Moishezon criterion for ampleness [Kl] that it suffices to prove that the restriction of $\mathscr{O}_{\widehat{Z}_{V'}}(D'_h)$ to the fiber of the boundary $W_{V'}$ of $\widehat{Z}_{V'}$ over $\bar{y}$ is ample in the sense that the global sections of $\mathscr{O}_{\widehat{Z}_{V'} \otimes k(\bar{y})}(dD'_h)$ for $d \geq 1$ form a basis of the topology of $W_{V'} \otimes k(\bar{y})$ (see the footnote to [F-C, Def. 2.1, Appendix]). This follows if we prove the stronger statement that $\mathscr{O}_{Z_{V'}}(dD'_h)$ is very ample for every $d \geq 1$, i.e., that the elements of $\mathrm{H}^0\big(Z_{V'}, \mathscr{O}_{Z_{V'}}(dD'_h)\big)$ form a basis of the Zariski topology of $Z_{V'}$. If $f\colon Z_{V'} \to \mathcal{B}_1(p^n)_{V'}(v)_{1,\bar{y}}$ is the structural morphism, then $f_*\big(\mathscr{O}_{Z_{V'}}(dD'_h)\big) = \oplus_{\lambda \geq dh}\mathcal{L}(\lambda)$, where the sum is taken over all integral elements $\lambda \in S_{V'} \cap C(V/V'^{\perp})^\vee$ such that for every $\Sigma \in \mathcal{S}$ and every $\rho \in \Sigma$ we have $\lambda(\rho) \geq dh(\rho)$, see [Dm, §IV.4]. In particular,

$$\mathrm{H}^0\big(Z_{V'}, \mathscr{O}_{Z_{V'}}(dD'_h)\big) = \oplus_{\lambda \geq dh}\mathrm{H}^0\big(\mathcal{B}_1(p^n)_{V'}(v)_{1,\bar{y}}, \mathcal{L}(\lambda)\big).$$

Assuming conditions (i), (iii) and (v) in the definition of polarization function given above then condition [Dm, Cor. IV.4.1(iii)] is equivalent to conditions (ii) and (iv) above. As dh is also a polarization function, we conclude from [Dm, Cor. IV.4.1(i) and Pf. Thm. IV.4.2] that the morphism $Z_{V'} \to \mathbf{Proj}\left(\oplus_s f_*\big(\mathscr{O}_{Z_{V'}}(dD'_h)\big)^{\otimes^s}\right)$ of schemes over $\mathcal{B}_1(p^n)_{V'}(v)_{1,\bar{y}}$ defined by $f_*\big(\mathscr{O}_{Z_{V'}}(dD'_h)\big)$ is a closed immersion. The sheaf $\mathcal{L}(\lambda)$ is very ample on the abelian scheme $\mathcal{B}_1(p^n)_{V'}(v)_{1,\bar{y}}$ for every integral, non-zero element $\lambda \in S_{V'} \cap C(V/V'^{\perp})^\vee$ due to the principal level N-structure with $N \geq 3$; see the proof of [F-C, Thm. V.5.8]. As the condition $\lambda \geq dh$ implies $\lambda > 0$, the very ampleness of $\mathscr{O}_{Z_{V'}}(dD'_h)$ follows.

□

8.2.3 Applications of the base change theorem: the proof of proposition 8.2.3.3

We let $\mathfrak{U} = (\mathfrak{V}_i)_{1\leq i\leq r}$ be an affine covering of $\mathfrak{X}^\star(v)$. We let $\underline{i} = (i_1, i_2, \ldots, i_{n'})$ be a multi index with $1 \leq i_1 < \ldots < i_{n'} \leq r$. We let $\mathfrak{V}_{\underline{i}}$ be the intersection of $\mathfrak{V}_{i_1}$, $\mathfrak{V}_{i_2}$, …, $\mathfrak{V}_{i_{n'}}$. This is again an affine formal scheme. We denote by $V_{\underline{i},m}$ the scheme obtained by reduction modulo ϖ^m. We let $M_{\underline{i},m} = \mathrm{H}^0(V_{\underline{i},m} \times \mathcal{W}(w)^o_m, (\eta\times 1)_*\tilde{\mathfrak{w}}^{\dagger\kappa^{oun}}_{w,m}(-D))$ and $M_{\underline{i},\infty} = \mathrm{H}^0(\mathfrak{V}_{\underline{i}} \times \mathfrak{W}(w)^o, (\eta\times 1)_*\tilde{\mathfrak{w}}^{\dagger\kappa^{oun}}_{w}(-D)) = \lim_m M_{\underline{i},m}$. Finally let A be the algebra of $\mathfrak{W}(w)^o$.

Corollary 8.2.3.1. *The module $M_{\underline{i},\infty}$ is isomorphic to the p-adic completion of a free A-module.*

Proof The module $M_{\underline{i},\infty}$ is p-torsion free and the reduction map $M_{\underline{i},\infty} \to M_{\underline{i},1}$ is surjective. The $A/\varpi A$-module $M_{\underline{i},1}$ is free by corollary 8.1.5.4. Fix a basis $(\bar{e}_i)_{i\in I}$ for this module. We lift the vectors $\bar{e}_i$ to vectors e_i in $M_{\underline{i},\infty}$. We let $\widehat{A^I}$ be the p-adic completion of the module A^I. Now consider the map $\widehat{A^I} \to M_{\underline{i},\infty}$ which sends $(a_i)_{i\in I} \in \widehat{A^I}$ to $\sum_i a_ie_i$. We claim that this map is an isomorphism. It is surjective by the topological Nakayama lemma. It is injective, for if $\sum_i a_ie_i$ is 0 and $(a_i)_{i\in I} \neq 0$, there is $n \in \mathbb{N}$, $(a'_i)_{i\in I} \in \widehat{A^I}$ such that $\varpi^n a'_i = a_i$ and an index i_0 such that $a'_{i_0} \notin \varpi A$. Since $M_{\underline{i},\infty}$ is ϖ-torsion free we have $\sum_i a'_ie_i = 0$ and reducing this relation modulo ϖ we get a contradiction. □

We set $M = \mathrm{H}^0\left(\mathfrak{X}_1(p^n)(v) \times \mathfrak{W}(w)^o, \tilde{\mathfrak{w}}^{\dagger\kappa^{oun}}_{w}(-D)\right)[p^{-1}]$, and $M_{\underline{i}} = M_{\underline{i},\infty}[p^{-1}]$.

Corollary 8.2.3.2. *The module M is a projective Banach-$A[\frac{1}{p}]$-module. For any $\kappa \in \mathcal{W}(w)^o$, the specialization map $M \to \mathrm{H}^0\big(\mathfrak{X}_1(p^n)(v), \tilde{\mathfrak{w}}^{\dagger\kappa^o}_{w}(-D)\big)[p^{-1}]$ is surjective.*

Proof Notice that $\tilde{\mathfrak{w}}^{\dagger\kappa^{oun}}_{w}(-D)$ is a small formal Banach sheaf thanks to corollary 8.1.6.2. It follows from proposition 8.2.2.4 and proposition A.1.3.1 that also $(\eta\times 1)_*\tilde{\mathfrak{w}}^{\dagger\kappa^{oun}}_{w}(-D)$ is a small formal Banach sheaf. Since $\mathcal{X}^\star(v) \times \mathcal{W}^o$ is affinoid and $\mathfrak{X}^\star(v)$ is a normal integral formal scheme by construction, theorem A.1.2.2 implies that the Chech complex associated the covering $\mathfrak{U}$ of $\mathfrak{X}^\star(v)$ provides a resolution of the module M by the projective A-modules $M_{\underline{i}}$ and as a result M is projective.

We now prove the surjectivity of the specialization map. Let P_{κ^o} be the maximal ideal of κ^o in $A[p^{-1}]$. We consider the Koszul resolution of $A[p^{-1}]/P_{\kappa^o}$:

$$\mathrm{Ko}(\kappa^o)\colon 0 \to A[p^{-1}] \to A[p^{-1}]^g \cdots \to A[p^{-1}]^g \to A[p^{-1}] \to A[p^{-1}]/P_{\kappa^o} \to 0.$$

For any multi-index $\underline{i}$ the tensor product $\mathrm{Ko}(\kappa^o)\otimes(\eta\times 1)_*\tilde{\mathfrak{w}}_w^{\dagger\kappa^{o\mathrm{un}}}(-D)\big(\mathfrak{V}_{\underline{i}}\big)$ is a resolution of $\eta_*\tilde{\mathfrak{w}}_w^{\dagger\kappa^o}(-D)\big(\mathfrak{V}_{\underline{i}}\big)[p^{-1}]$ by A-modules which are isomorphic to direct sums of the A-modules $(\eta\times 1)_*\tilde{\mathfrak{w}}_w^{\dagger\kappa^{o\mathrm{un}}}(-D)\big(\mathfrak{V}_{\underline{i}}\big)[p^{-1}]$.

We consider the following double complex, obtained by taking the Chech complex $\mathrm{Ko}(\kappa)\otimes(\eta\times 1)_*\tilde{\mathfrak{w}}_w^{\dagger\kappa^{o\mathrm{un}}}(-D)\big(\mathfrak{V}_{\underline{i}}\big)$ attached to the covering $\mathfrak{U}=(\mathfrak{V}_{\underline{i}})$ (we think of $\mathrm{Ko}(\kappa)\otimes(\eta\times 1)_*\tilde{\mathfrak{w}}_w^{\dagger\kappa^{o\mathrm{un}}}(-D)\big(\mathfrak{V}_{\underline{i}}\big)$ as a vertical complex for fixed $\underline{i}$):

$$0 \to \mathrm{Ko}(\kappa^o)\otimes(\eta\times 1)_*\tilde{\mathfrak{w}}_w^{\dagger\kappa^{o\mathrm{un}}}(-D)\big(\mathfrak{X}^\star(v)\big) \longrightarrow \oplus_{\underline{i}}\mathrm{Ko}(\kappa^o)\otimes(\eta\times 1)_*\tilde{\mathfrak{w}}_w^{\dagger\kappa^{o\mathrm{un}}}(-D)\big(\mathfrak{V}_{\underline{i}}\big) \to \cdots.$$

For any multi-index $\underline{i}$ the complex $\mathrm{Ko}(\kappa^o)\otimes(\eta\times 1)_*\tilde{\mathfrak{w}}_w^{\dagger\kappa^{o\mathrm{un}}}(-D))\big(\mathfrak{V}_{\underline{i}}\big)$ is exact as remarked above. All the rows of the double complex are exact by the acyclicity theorem A.1.2.2. It follows that the first column is also exact proving the claim on specialization. □

We now prove proposition 8.2.3.3. Let B be the algebra of rigid analytic functions on $\mathcal{W}(w)$.

Proposition 8.2.3.3. *a) The module* $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{W}(w),\omega_w^{\dagger\kappa^{\mathrm{un}}}(-D))$ *is a projective Banach-B-module.*

and

b) For every $\kappa\in\mathcal{W}(w)$ *the specialization map*

$$\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{W}(w),\omega_w^{\dagger\kappa^{\mathrm{un}}}(-D)) \to \mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(v),\omega_w^{\dagger\kappa}(-D))$$

is surjective.

Proof We use the notations of the corollary 8.2.3.2. By definition,

$$\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{W}(w),\omega_w^{\dagger\kappa^{\mathrm{un}}}(-D)) = \big(M\otimes_{A[\frac{1}{p}]} B(-\kappa^{\mathrm{un}\prime})\big)^{\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})}$$

Here, $B(-\kappa^{\mathrm{un}\prime})$ is the free B-module with action of $B(\mathbb{Z}/p^n\mathbb{Z})\mathcal{B}_w$ via the character $-\kappa^{\mathrm{un}\prime}$. Then, $M\otimes_A B(-\kappa^{\mathrm{un}\prime})$ is viewed as a $B(\mathbb{Z}/p^n\mathbb{Z})\mathcal{B}_w$-module with diagonal action. This action factors through the group $\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$ and the invariants are precisely $\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(v)\times\mathcal{W}(w),\omega_w^{\dagger\kappa^{\mathrm{un}}}(-D))$. Thus, this module is a direct factor in a projective B-module by corollary 8.2.3.2 so it is projective. Now, let $\kappa\in\mathcal{W}(w)$. We let κ^o be its image in $\mathcal{W}(w)^o$. Let m_κ be the maximal ideal of κ in B. Set $M_{\kappa^o}=\mathrm{H}^0\big(\mathfrak{X}_1(p^n)(v),\tilde{\mathfrak{w}}_w^{\dagger\kappa^o}(-D)\big)[p^{-1}]$. The specialization map $M\to M_{\kappa^o}$ is surjective thanks to corollary 8.2.3.2. The map $M\otimes_A B(-\kappa^{\mathrm{un}})\to M_{\kappa^o}\otimes_A B/m_\kappa(-\kappa)$ is surjective and the map

$$\big(M\otimes_A B(-\kappa^{\mathrm{un}\prime})\big)^{\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})} \to \big(M_{\kappa^o}\otimes_A B/m_\kappa(-\kappa')\big)^{\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})}$$

is still surjective since $\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})$ has no higher cohomology on $\mathbb{Q}_p$-modules. As

$$\mathrm{H}^0(\mathcal{X}_{\mathrm{Iw}}(v),\omega_w^{\dagger\kappa}(-D)) = \big(M_{\kappa^o}\otimes_A B/m_\kappa(-\kappa')\big)^{\mathrm{B}(\mathbb{Z}/p^n\mathbb{Z})},$$

the claim concerning the specialization follows. □

8.3 Properties of the morphism from the eigenvariety to the weight space

We end with some comments concerning the unramifiedness hypothesis in theorem 1.2. We would like to rise the following question:

Open problem 1. *Let $x_f \in \mathcal{E}$ be a classical point. Is the map $w\colon \mathcal{E} \to \mathcal{W}$ unramified at x_f ?*

When $g = 1$, the tame level is trivial, f is of weight k and $v(U_p(x_f)) \neq \frac{k-2}{2}$, Coleman and Mazur have proved that the answer is positive (see [C-M], coro. 7.6.3). Coleman and Mazur's approach is purely Hecke theoretic. It relies on the duality between the Hecke algebra and the cuspidal modular forms provided by the first Fourier coefficient in the q-expansion. This duality does not exist when $g \geq 2$. When $g = 1$, M. Kisin ([Ki], thm. 11.10) proved that the answer is positive in many cases using Galois deformation theoretic methods. Moreover G. Chenevier studied this problem for certain unitary groups in [Che2] and also obtained a positive answer in many cases. His method uses multiplicity results for automorphic forms on unitary groups and some properties of the Galois representations attached to these automorphic forms. As he suggested to us, his results should hold in our (Siegel modular forms) case if we knew certain facts about the automorphic forms for GSp_{2g}. To conclude this paper, we state a result for level 1 forms and $g = 2$ in this spirit.

Remark 8.3.1. In the paper at hand, we have so far worked using an auxiliary tame level $N \geq 3$ structure in order to have the representability of the Siegel variety. It is easy to define the level 1 eigenvariety as a closed subvariety of the level N eigenvariety as follows.

We freely use the notations of section 8.1.1 and fix an integer $N \geq 3$. Let $M_{v,w,cusp}$ be the A-Banach module of v-overconvergent, w-analytic cuspidal modular forms of tame level N. This module carries an action of the finite group $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, which commutes with all Hecke operators of level prime to Np. Let $N_{v,w,cusp}$ be the direct factor of $M_{v,w,cusp}$ fixed by $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. We define $N_{n,w,cusp}$ to be the A-Banach module of v-overconvergent, w-analytic cuspidal modular forms of tame level 1. We remark that taking invariants by this finite group commutes with any base change on A (the weight space), so there is no ambiguity in the definition and moreover $N_{v,w,cusp}$ is independent on the choice of N. One can apply the recipe of section 8.1.2 to $N_{v,w,cusp}$ in order to obtain the tame level 1 spectral variety and the tame level 1 eigenvariety. Finally, let us remark that the classicity theorem holds for tame level 1 forms as well. Indeed, let f be a classical form of level N whose restriction to the space of overconvergent, locally analytic modular forms has tame level 1 (or equivalently, is invariant under $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$). Then f has tame level 1.

Proposition 8.3.2. *Let $\mathcal{E}$ be the tame level* 1 *eigenvariety for* GSp_4. *Let $x_f \in \mathcal{E}$ be a classical point of weight $(k_1, k_2) \in \mathbb{Z}^2$ which satisfies the slope condition*

$$v(\Theta_f(U_{p,1})) < k_2 - k_1 + 1 \text{ and } v(\Theta_f(U_{p,2})) < k_2 - 3.$$

Let $\pi(f)$ be an irreducible constituant of the automorphic representation generated by f. Assume that $\pi(f)$ is tempered and that $\pi(f)_p$ is an unramified principal series with distinct Satake parameters. Then w is étale at x_f.

Proof We recall that the eigenvariety and the map w can be described as follows using affinoid neighborhoods of x_f and $w(x_f)$.

— Spm A is an admissible affinoid open of $\mathcal{W}$ containing $w(x_f)$,

— M is a free A-module of finite rank. It consists of locally analytic overconvergent finite slope modular forms and $f \in M \otimes_A k(w(x_f))$.
— C is a finite A-algebra (a quotient of the Hecke algebra), which acts faithfully on M and Spm C is an admissible affinoid open neighborhood of x_f in $\mathcal{E}$.

Because local rings in rigid geometry are henselian, we may also assume that x_f is the only point of Spm C over $w(x_f)$. Let Π_f be the global L-packet associated to f. By [Ar], thm. on p. 76, all $\pi \in \Pi_f$ are tempered and occur with multiplicity one. Now, $M \otimes_A k(w(x_f))$ is included in the space of classical cuspidal forms of weight (k_1, k_2) and Iwahori level at p by the classicity theorem. Moreover, it consists of the generalized eigenvectors for Θ_f. We claim that M has dimension 1. This implies that $C = A$ and the conclusion of the proposition follows. Let $K = \prod_{\ell \neq p} \mathrm{GSp}_4(\mathbb{Z}_\ell) \times I$ with I the Iwahori subgroup of $\mathrm{GSp}_4(\mathbb{Z}_p)$. Let $\pi \in \Pi_f$ such that $\pi^K \neq 0$. For such a π, π_ℓ is an unramified principal series for all $\ell \neq p$ and must be equal to $\pi(f)_\ell$. At p, π_p is tempered, has Iwahori fixed vectors and π_p^I contains an eigenvector for $U_{p,1}$ and $U_{p,2}$ with eigenvalues $\Theta_f(U_{p,1})$ and $\Theta_f(U_{p,2})$. An examination of the tables 1, 2 and 3 of [Sch] tells us that if π_p had no spherical vectors this would contradict the temperdness of $\pi(f)_p$. It follows that $\pi_p = \pi(f)_p$. Finally, π_∞ is a holomorphic discrete series. Thus $\pi = \pi(f)$ is uniquely determined. It follows that $M \otimes_A k(w(x))$ can be identified with a subspace of $\pi(f)_p^I$. The vector space $\pi(f)_p^I$ has dimension 8 and by assumption, the operators $U_{p,1}$ and $U_{p,2}$ act semi-simply on it with distinct systems of eigenvalues. The character Θ_f encodes the choice of one system of eigenvalues (the p-stabilisation of f) and thus, the generalized eigenspace for Θ_f in $\pi(f)_p^I$ is one dimensional. In other words, $M \otimes_A k(w(x))$ is generated by f.

□

A Banach and formal Banach sheaves

A.1 Formal Banach sheaves and their properties

A.1.1 Definition

Let $\mathfrak{X}$ be a flat formal scheme of finite type over Spf $\mathcal{O}_K$. Let ϖ be a uniformizing element in $\mathcal{O}_K$. We denote by X_n the scheme over Spec $\mathcal{O}_K/\varpi^n\mathcal{O}_K$ deduced from $\mathfrak{X}$ by reduction modulo ϖ^n.

Definition A.1.1.1. *A formal Banach sheaf on $\mathfrak{X}$ is a family of quasi-coherent sheaves $\mathfrak{F} = (\mathfrak{F}_n)_{n\in\mathbb{N}}$ where:*

1. *$\mathfrak{F}_n$ is a sheaf on X_n, flat over $\mathcal{O}_K/\varpi^n$,*
2. *for all $n \geq m$, if $i\colon X_m \hookrightarrow X_n$ is the closed immersion, we have $i^*\mathfrak{F}_n = \mathfrak{F}_m$.*

For $\mathfrak{U} \hookrightarrow \mathfrak{X}$ is an open formal sub-scheme we set $\mathfrak{F}(\mathfrak{U}) := \lim_n \mathfrak{F}_n(\mathfrak{U})$. It is the sheaf on $\mathfrak{X}$ defined by the inverse limit $\lim_n \mathfrak{F}_n$.

If $f\colon \mathfrak{X}' \to \mathfrak{X}$ is a morphism of formal schemes and if $\mathfrak{F}$ is a formal Banach sheaf over $\mathfrak{X}$, then $f^*\mathfrak{F} := (f^*\mathfrak{F}_n)_n$ is readily verified to be a formal Banach sheaf.

We say that a formal Banach sheaf $\mathfrak{F} = (\mathfrak{F}_n)_{n\in\mathbb{N}}$ on $\mathfrak{X}$ is a sheaf of flat $\mathscr{O}_{\mathfrak{X}}$-modules if $\mathfrak{F}_n$ is a flat $\mathscr{O}_{X_n}$-module for every $n \in \mathbb{N}$.

A.1.2 An acyclicity criterion for formal Banach sheaves

Let $\mathfrak{X}$ be a flat formal scheme, locally of finite type over Spf $\mathcal{O}_K$. We say that $\mathfrak{X}$ is normal (respectively integral) if there exists a covering by open affine formal sub-schemes $\mathfrak{U}_i = \mathrm{Spf} A_i$ such that A_i is a normal ring (respectively an integral domain). We say that $\mathfrak{X}$ is quasi-projective if there exists an immersion (namely an isomorphism onto an open formal sub-scheme of a closed formal sub-scheme) of ϖ-adic formal schemes from $\mathfrak{X}$ into the formal scheme associated to $\mathbb{P}^n_{\mathcal{O}_K}$.

We now introduce a finiteness condition:

Definition A.1.2.1. *We say that a formal Banach sheaf $\mathfrak{F}$ is* **small** *if there exists a coherent sheaf $\mathscr{G}$ on X_1 such that:*

a) $\mathfrak{F}_1$ can be written as the direct limit of coherent sheaves $\lim_{j\in\mathbb{N}} \mathfrak{F}_{1,j}$, and

b) $\mathfrak{F}_{1,0}$ and for every $j \in \mathbb{N}$ the quotient $\mathfrak{F}_{1,j+1}/\mathfrak{F}_{1,j}$ are direct summands of $\mathscr{G}$.

The following acyclicity result justifies the definition of a small formal Banach sheaf.

Theorem A.1.2.2. *Let $\mathfrak{X}$ be an integral, normal, quasi-projective formal scheme over* Spf $\mathcal{O}_K$ *such that the rigid analytic generic fiber $\mathcal{X}$ of $\mathfrak{X}$ is an affinoid. Let $\mathfrak{F}$ be a small formal Banach sheaf on $\mathfrak{X}$. Let $\mathfrak{U} = \{\mathfrak{U}_i\}_{i\in I}$ be a finite, open, affine covering of $\mathfrak{X}$. Then the augmented Chech complex tensored with K:*

$$\mathrm{C}^\bullet(\mathfrak{F})[1/p]: \quad 0 \to \mathrm{H}^0(\mathfrak{X}, \mathfrak{F})[1/p] \to \oplus_i \mathrm{H}^0(\mathfrak{U}_i, \mathfrak{F})[1/p] \to \cdots$$

is exact.

Proof. (1) Assume first that there exists a projective morphism of formal schemes $\gamma\colon \mathfrak{X} \to \mathfrak{Z}$ with $\mathfrak{Z} = \mathrm{Spf} A$ affine formal scheme, where A is a flat and topologically of finite type $\mathcal{O}_K$-algebra.

Let $\mathfrak{L}$ be an ample invertible sheaf on $\mathfrak{X}$ relatively to $\mathfrak{Z}$ such that $\mathrm{H}^i\big(X_1, \mathscr{G} \otimes \mathfrak{L}_1\big) = 0$ for all $i > 0$. It follows that $\mathrm{H}^i\big(X_1, \mathfrak{F}_{1,j} \otimes \mathfrak{L}_1\big) = 0$ for all $i > 0$ and $j \geq 0$, hence, $\mathrm{H}^i\big(X_1, \mathfrak{F}_1 \otimes \mathfrak{L}_1\big) = 0$ for all $i > 0$. As the cohomology groups $\mathrm{H}^i\big(X_1, \mathfrak{F}_1 \otimes \mathfrak{L}\big)$ are computed by the Chech complex:

$$\mathrm{C}^\bullet\big(\mathfrak{F}_1 \otimes \mathfrak{L}_1\big): \quad 0 \to \mathrm{H}^0\big(X_1, \mathfrak{F}_1 \otimes \mathfrak{L}_1\big) \to \oplus_i \mathrm{H}^0\big(U_{i,1}, \mathfrak{F}_1 \otimes \mathfrak{L}_1\big) \to \cdots,$$

it follows that $\mathrm{C}^\bullet\big(\mathfrak{F}_1 \otimes \mathfrak{L}_1\big)$ is exact. Moreover we have $\mathrm{C}^\bullet\big(\mathfrak{F}_1 \otimes \mathfrak{L}_1\big) \cong \mathrm{C}^\bullet\big(\mathscr{F} \otimes \mathfrak{L}\big) \otimes_{\mathcal{O}_K} \mathcal{O}_K/\varpi\mathcal{O}_K$ and as $\mathrm{C}^\bullet\big(\mathfrak{F} \otimes \mathfrak{L}\big)$ is a complex of flat and ϖ-adically complete and separated $\mathcal{O}_K$-modules, it follows that it is exact.

Let $\mathcal{L}$ be the invertible sheaf on $\mathcal{X}$ associated to $\mathfrak{L}$. We denote $\mathrm{H}^0(\mathcal{X}, \mathcal{L}) = L$ and $\mathrm{H}^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = B$. As $\mathcal{X}$ is an affinoid and $\mathcal{L}$ is coherent, $\mathcal{L}$ is the sheaf associated to the B-module L. It follows that L is a projective, rank one B-module. Moreover since X_1 is quasi-projective it is separated as a scheme and therefore all the opens $\mathfrak{U}_{i_1,i_2,\cdots,i_k}$ appearing in the Chech complex are affine. It follows that we have a natural isomorphism of B-modules:

$$\mathrm{C}^\bullet(\mathfrak{F} \otimes \mathfrak{L})[1/p] \cong \mathrm{C}^\bullet(\mathfrak{F})[1/p] \otimes_B L$$

which implies the claim.

(2) We now show that, under the assumptions of the proposition, there exists a projective morphism of formal schemes $\gamma\colon \mathfrak{X} \to \mathfrak{Z}$ as claimed. Since $\mathcal{X}$ is assumed to be

affinoid, there exits an affine formal scheme $\mathfrak{Z} = \mathrm{Spf} A$ with A flat and topologically of finite type as $\mathcal{O}_K$-algebra such that $\mathcal{X}$ is the associated rigid analytic fiber. In particular, using Raynaud's description of quasi-compact and quasi-separated rigid varieties as the category of formal schemes localized with respect to admissible blow-ups, we deduce that there exits a formal scheme $\mathfrak{Y}$ and admissible blow-ups $f\colon \mathfrak{Y} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{X}$. In particular the rigid analytic fibers of f and g are isomorphisms.

Let $\mathfrak{U} \subset \mathfrak{X}$ be an open affine formal scheme with $\mathfrak{U} = \mathrm{Spf} C$ and C an integral normal domain. Since g is an admissible blow-up, its restriction $g^{-1}(\mathfrak{U}) \to \mathfrak{U}$ is the map of formal schemes associated to an algebraic blow-up of schemes $\tilde{g}\colon Y_{\mathfrak{U}} \to U = \mathrm{Spec}(C)$. As C is normal and $\tilde{g}$ is birational, then $\tilde{g}_*\big(\mathcal{O}_{Y_{\mathfrak{U}}}\big) = \mathcal{O}_U$ by Zariski's Main Theorem. Furthermore $\tilde{g}$ is surjective as its image is closed and dense since U is irreducible. This implies that g is surjective as a map of topological spaces and that $g_*\big(\mathcal{O}_{\mathfrak{Y}}\big) = \mathcal{O}_{\mathfrak{X}}$. We conclude that the map f factors via g, i.e., that there exists a morphism $h\colon \mathfrak{X} \to \mathfrak{Z}$ such that $f = h \circ g$; it is defined on each formal affine subscheme $\mathfrak{U} = \mathrm{Spf} C$ of $\mathfrak{X}$ by the map of $\mathcal{O}_K$-algebras

$$A \to C = \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \to g_*\big(\mathcal{O}_{\mathfrak{Y}}\big)(\mathfrak{U}) = \mathcal{O}_{\mathfrak{Y}}\big(g^{-1}(\mathfrak{U})\big)$$

defined by the map of ringed spaces f.

The map of schemes $h_1\colon X_1 \to Z_1 := \mathrm{Spec}(A/\varpi A)$, defined by h modulo ϖ, is separated and of finite type as X_1 is quasi-projective over k. It is also universally closed as f is projective, and hence universally closed and g is surjective. We conclude that h_1 is proper. Since $\mathfrak{X}$ is quasi-projective over $\mathcal{O}_K$, then h factors via an immersion into the formal scheme associated to $\mathbb{P}^n_A$ and by the properness of h_1 it is a closed immersion. Thus h is projective.

□

A.1.3 Direct images of formal Banach sheaves

Let $\phi\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism between two flat formal schemes, locally of finite type over Spf $\mathcal{O}_K$. As before we denote by X_n and Y_n the schemes obtained by reduction modulo ϖ^n and by $\phi_n\colon X_n \to Y_n$ the induced map.

Let $\mathfrak{F} = (\mathfrak{F}_n)_{n\in\mathbb{N}}$ be a small formal Banach sheaf on $\mathfrak{X}$. In particular there exists a coherent sheaf $\mathscr{G}$ on X_1 such that $\mathfrak{F}_1$ is the direct limit of coherent sheaves $\lim_{j\in\mathbb{N}} \mathfrak{F}_{1,j}$ and $\mathfrak{F}_{1,0}$ and for every $j \in \mathbb{N}$ the quotient $\mathfrak{F}_{1,j+1}/\mathfrak{F}_{1,j}$ are direct summands of $\mathscr{G}$.

For all $n \geq m$, we have a cartesian diagram:

$$\begin{array}{ccc} X_n & \xrightarrow{\;i\;} & X_m \\ \big\downarrow{\scriptstyle \phi_n} & & \big\downarrow{\scriptstyle \phi_m} \\ Y_n & \xrightarrow{\;j\;} & Y_m \end{array}$$

Proposition A.1.3.1. *Assume that for all $n \geq m$ we have the base change property:*

$$j^*\big(\phi_{m,*}\mathfrak{F}_m\big) \simeq \phi_{n,*}\mathfrak{F}_n$$

and that

$$\mathrm{R}^i\phi_{1,*}\mathscr{G} = 0 \,\forall\, i \geq 1.$$

Then $\phi_*\mathscr{F}$ is a small formal Banach sheaf.

Proof Indeed $\phi_*\mathfrak{F} = \big(\phi_{n,*}\mathfrak{F}_n\big)_{n\in\mathbb{N}}$ is a formal Banach sheaf. Furthermore $\phi_{1,*}\mathfrak{F}_1 = \lim_n \phi_{1,*}\mathfrak{F}_{1,n}$ and moreover $\phi_{1,*}\mathfrak{F}_{1,n}$ and $\phi_{1,*}\mathscr{G}$ are coherent for all n. By induction on i one proves that $\mathrm{R}^i\phi_{1,*}\mathfrak{F}_{1,n} = 0$ for every n and every $i \geq 1$. This implies that $\phi_{1,*}\mathfrak{F}_{1,n+1}/\phi_{1,*}\mathfrak{F}_{1,n} \cong \phi_{1,*}\big(\mathfrak{F}_{1,n+1}/\mathfrak{F}_{1,n}\big)$ and the latter is a direct summand in $\phi_{1,*}\mathscr{G}$. The claim follows. □

A.2 Banach sheaves

A.2.1 Banach modules

Let A be a K-affinoid algebra equipped with a norm $|\_|$ and let M be an A-module. We say that M is a normed A module if there is a norm function $|\_|\colon M \to \mathbb{R}_{\geq 0}$ such that:

1. $|m| = 0$, for some $m \in M$, implies that $m = 0$,
2. $|m+n| \leq \sup\{|m|, |n|\}$ for every m and $n \in M$,
3. $|am| \leq |a||m|$, for every $a \in A$ and every $m \in M$.

If $|\_|$ satisfies only conditions (2) and (3), we call it a semi-norm. We say that M is a Banach A-module if M is a complete normed A-module. It may be useful to recall the open mapping theorem:

Theorem A.2.1.1 ([Bou], Chap. I, sect. 3.3, thm. 1)**.** *A surjective continuous map $\phi\colon M \to N$ between Banach A-modules is open.*

If $(M, |\_|)$ is a Banach A-module then any other norm $|\_|'$ on M inducing the same topology on M is equivalent to $|\_|$. For this reason, from now on, we will not consider that our Banach modules are equipped with a specific norm.

If M is an A-module, and if A_0 is an open and bounded sub-ring of A and M_0 is p-adically complete sub-A_0-module of M such that $M_0[1/p] = M$, then M is naturally an A-Banach module, with unit ball M_0.

If M and M' are two A-Banach modules, we define an A-Banach module $M\hat{\otimes}_A M'$ as follows. Let $|\_|$ and $|\_|'$ be norms on M and M'. Denote by $M\hat{\otimes}_A M'$ the separation and completion of the semi-normed A-module $M \otimes_A M'$, where the semi-norm of an element x is the infimum over all the expressions $x = \sum_i m_i \otimes m'_i$ of the supremum $\sup_i |m_i||m'_i|'$.

A Banach A-module is called projective if it is a direct factor in an orthonormalizable Banach A-module.

We now make the following definition:

Definition A.2.1.2. *Let $\mathcal{X}$ be a rigid space and $\mathscr{F}$ be a sheaf of $\mathscr{O}_{\mathcal{X}}$-modules on $\mathcal{X}$. We say that $\mathscr{F}$ is a Banach sheaf if:*

- *I. for every affinoid open sub-set $\mathcal{U}$ of $\mathcal{X}$, the $\mathscr{O}_{\mathcal{X}}(\mathcal{U})$-module $\mathscr{F}(\mathcal{U})$ is a Banach module,*
- *II. the restriction maps are continuous,*
- *III. there exists an admissible affinoid covering $\mathfrak{U} = \{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{X}$ such that for every $i \in I$ and for every affinoid $\mathcal{V} \subset \mathcal{U}_i$, the map induced by restriction:*

$$\mathscr{O}_{\mathcal{X}}(\mathcal{V})\hat{\otimes}_{\mathscr{O}_{\mathcal{X}}(\mathcal{U}_i)}\mathscr{F}(\mathcal{U}_i) \to \mathscr{F}(\mathcal{V})$$

is an isomorphism of $\mathscr{O}_{\mathcal{X}}(\mathcal{V})$-Banach modules.

If the admissible affinoid covering in (III.) can be chosen in such a way that $\mathscr{F}(\mathcal{U}_i)$ is a projective Banach $\mathscr{O}_{\mathcal{X}}(\mathcal{U}_i)$-module for all i, then $\mathscr{F}$ is called a projective Banach sheaf.

A.2.2 The rigid analytic generic fiber of a formal Banach sheaf

Let $\mathfrak{X}$ be a flat formal scheme locally of finite type over Spf $\mathcal{O}_K$. Let $\mathcal{X}$ be its rigid analytic fiber. Let $\mathfrak{F} = \big(\mathfrak{F}_n\big)_{n\in\mathbb{N}}$ be a formal Banach sheaf over $\mathfrak{X}$. We associate to $\mathfrak{F}$ a sheaf $\mathscr{F}$ on $\mathfrak{X}$, valued in the category of K-vector spaces, by setting

$$\mathfrak{U} \mapsto \mathscr{F}(\mathfrak{U}) := \mathfrak{F}(\mathfrak{U}) \otimes_{\mathcal{O}_K} K.$$

For every open subset $\mathfrak{U} \hookrightarrow \mathfrak{X}$, with rigid fiber $\mathcal{U}$, the $\mathscr{O}_{\mathcal{X}}(\mathcal{U})$-module $\mathscr{F}(\mathfrak{U})$ is a Banach module for the norm for which $\mathfrak{F}(\mathfrak{U})$ is the unit ball of $\mathscr{F}(\mathfrak{U})$.

We recall that if $\mathfrak{F}$ were a coherent sheaf, then $\mathscr{F}$ would extend uniquely to a sheaf on $\mathcal{X}$. The main goal of this section is to prove a similar result for the class of flat formal Banach sheaves. We start with the following technical lemma.

Lemma A.2.2.1. *Let $h\colon \mathfrak{X}' \to \mathfrak{X}$ be an admissible blow-up of $\mathfrak{X}$ and let $\mathfrak{F}$ be a flat formal Banach sheaf. Then $h^*\mathfrak{F} := \big(h^*\mathfrak{F}_n\big)_{n\in\mathbb{N}}$ is a flat formal Banach sheaf on $\mathfrak{X}'$.*

Moreover the adjunction maps $\mathfrak{F}_n \to h_*h^*\mathfrak{F}_n := \mathfrak{F}'_n$ give rise to a map $\mathfrak{F} \to \lim_n \mathfrak{F}'_n$ whose kernel and cokernel are annihilated by a power of ϖ.

Proof. The fact that $h^*\mathfrak{F}$ is a flat formal Banach sheaf follows directly from the flatness of $\mathfrak{F}$. For the other statements we can work locally on $\mathfrak{X}$, therefore we may assume that $\mathfrak{X} = \mathrm{Spf}\ R$ is affine. Put $R_n := R/\varpi^n R$. Let $F_n = \mathfrak{F}_n(\mathfrak{X})$ so that $\mathfrak{F}(\mathfrak{X}) = F = \lim_n F_n$. By assumption F_n is a flat R_n-module for every $n \in \mathbb{N}$. Let $R' = \mathrm{H}^0(\mathfrak{X}', \mathscr{O}_{\mathfrak{X}'})$ and $R'_n = \mathrm{H}^0(\mathfrak{X}'_n, \mathscr{O}_{\mathfrak{X}'_n})$. Thus, $R' = \lim_n R'_n$. Since h is an admissible blow-up, it is the map of formal schemes defined by an algebraic blow-up $\tilde{h}\colon X' \to \mathrm{Spec}\ R$ concentrated on the special fiber. In particular, R' is a finite R-module as $\tilde{h}$ is a projective morphism and the map $\alpha\colon R \to R'$ is injective with cokernel killed by ϖ^N for some integer N as $\tilde{h}$ is an isomorphism after inverting ϖ. Hence multiplication by ϖ^N on R' factors via an R-linear morphism $\beta\colon R' \to R$. Since R and R' are flat $\mathcal{O}_K$-modules, the composite maps $\beta \circ \alpha$ and $\alpha \circ \beta$ are multiplication by ϖ^N on R (resp. on R'). As $\mathfrak{X}'$ is flat over Spf $\mathcal{O}_K$, the map $R'/\varpi^n R' \to R'_n$ is injective with cokernel contained in $\big(R^1 h_* \mathscr{O}_{\mathfrak{X}'}\big)(\mathfrak{X})$. The latter is a finite, ϖ-torsion R-module so that it is annihilated by ϖ^M for some M. We deduce that the cokernel of $R'/\varpi^n R' \to R'_n$ is annihilated by ϖ^M. Reducing α and β modulo ϖ^n we get maps $\alpha_n\colon R_n \to R'_n$ and $\gamma_n := \beta_n \cdot \varpi^M\colon R'_n \to R_n$, compatible for varying n, and such that $\gamma_n \circ \alpha_n$ and $\alpha_n \circ \gamma_n$ are multiplication by ϖ^{N+M}.

Write $\mathfrak{F}'_n := h_*h^*\mathfrak{F}_n$. We claim that $\mathrm{H}^0(\mathfrak{X}, \mathfrak{F}'_n) = F_n \otimes_{R_n} R'_n$ (projection formula). Let us remark that it is enough to prove that $\mathrm{H}^0\big(\mathfrak{X}', h^*(\mathfrak{F}_n)\big) = F_n \otimes_{R_n} R'_n$, therefore let $\{\mathfrak{U}_i = \mathrm{Spf}\ A_i\}_i$ be a finite covering of $\mathfrak{X}'$ by open affine formal sub-schemes (the topological space of $\mathfrak{X}'$ is quasi-compact). As h is separated, the intersections of $\mathfrak{U}_i$ and $\mathfrak{U}_j$ are still affine formal schemes Spf B_{ij}. Let $A_{i,n}$ and $B_{i,j,n}$ be the reductions modulo ϖ^n of these rings. We tensor the exact sequence $0 \to R'_n \to \oplus_i A_{i,n} \to \oplus_{ij} B_{i,j,n}$ by F_n over R_n and we use on the one hand the flatness of F_n as R_n-module and on the other hand the fact that we have natural isomorphisms $\mathrm{H}^0\big(\mathfrak{U}_i, h^*(\mathfrak{F}_n)\big) \cong A_{i,n} \otimes_{R_n} F_n$ and $\mathrm{H}^0\big(\mathfrak{U}_{i,j}, h^*(\mathfrak{F}_n)\big) \cong B_{i,j,n} \otimes_{R_n} F_n$. The claim follows.

We can now show that the map $a\colon F \to \lim_n F_n \otimes_{R_n} R'_n$ has kernel and cokernel killed by ϖ^{N+M}. Using the maps $\lim_n 1 \otimes \alpha_n$ we get the adjunction map $a\colon F = \lim_n F_n \to \lim_n\big(F_n \otimes_{R_n} R'_n\big) := F'$ and using the maps $\lim_n 1 \otimes \gamma_n$ we get a map $b\colon F' \to F$ such that $a \circ b$ and $b \circ a$ are multiplication by ϖ^{N+M}. Thus a has kernel and cokernel annihilated by ϖ^{N+M} as wanted.

□

Assume that $\mathfrak{F}$ is a flat formal Banach sheaf on $\mathfrak{X}$. Let $\mathcal{U}$ be a quasi-compact open subset of $\mathcal{X}$. By Raynaud's description of quasi-compact and quasi-separated rigid varieties as the category of formal schemes localized with respect to admissible blow-ups, there exists an admissible blow up $h\colon \mathfrak{X}' \to \mathfrak{X}$ such that $\mathcal{U}$ is the rigid analytic fiber of an open formal sub-scheme $\mathfrak{U}'$ of $\mathfrak{X}'$. We define

$$\mathscr{F}(\mathcal{U}) := h^*\mathfrak{F}(\mathfrak{U}') \otimes_{\mathcal{O}_K} K.$$

Lemma A.2.2.2. *$\mathscr{F}(\mathcal{U})$ described above is independent of the admissible blow-up $h\colon \mathfrak{X}' \to \mathfrak{X}$ used to define it.*

Proof. If $g\colon \mathfrak{X}'' \to \mathfrak{X}'$ is an admissible blow-up, it follows from lemma A.2.2.1 that

$$h^*\mathfrak{F}(\mathfrak{U}') \otimes_{\mathcal{O}_K} K = g_* g^* h^*\mathfrak{F}(\mathfrak{U}') \otimes_{\mathcal{O}_K} K = (h \circ g)^*\mathfrak{F}\big(g^{-1}(\mathfrak{U}')\big) \otimes_{\mathcal{O}_K} K.$$

Let $\mathcal{U} \subset \mathcal{X}$ be a quasi-compact open subspace. Using Raynaud's theory any two admissible blow-ups of $\mathfrak{X}$, such that $\mathcal{U}$ is the rigid fiber of an open formal sub-scheme in both, are dominated by a third one. We deduce that the definition of $\mathscr{F}(\mathcal{U})$ is independent of the choice of the blow-up h. □

Let now $\mathcal{V} \subset \mathcal{U}$ be an inclusion of quasi-compact open sub-spaces of $\mathcal{X}$, then there exists an admissible blow up $h\colon \mathfrak{X}' \to \mathfrak{X}$ such that the inclusion $\mathcal{V} \subset \mathcal{U}$ is the rigid analytic fiber of an inclusion of open formal sub-schemes $\mathfrak{V}' \subset \mathfrak{U}'$ of $\mathfrak{X}'$. Using Lemma A.2.2.2 we define a restriction map $\mathscr{F}(\mathcal{U}) \to \mathscr{F}(\mathcal{V})$ using the restriction map $h^*\mathfrak{F}(\mathfrak{U}') \to h^*\mathfrak{F}(\mathfrak{V}')$. It is immediate that $\mathscr{F}$ thus defined is a presheaf on $\mathcal{X}$.

Proposition A.2.2.3. *The definition above attaches functorially to every flat formal Banach sheaf $\mathfrak{F}$ on $\mathfrak{X}$ a Banach sheaf $\mathscr{F}$, called the rigid analytic generic fiber of $\mathfrak{F}$. Moreover condition (III) in Definition A.2.1.2 holds for every admissible affinoid covering of $\mathcal{X}$ defined upon taking the rigid analytic fiber of a covering of $\mathfrak{X}$ by open affine formal subschemes.*

Proof. Using Raynaud's theory every admissible (finite) covering $\{\mathcal{U}_i\}_i$ of $\mathcal{U}$ by quasi-compact open subspaces of $\mathcal{X}$ can be realized as follows: there is a formal blow-up $h\colon \mathfrak{X}' \to \mathfrak{X}$, a formal open sub-scheme $\mathfrak{U}'$ of $\mathfrak{X}'$ and formal open covering $\{\mathfrak{U}'_i\}_i$ of $\mathfrak{U}'$ such that the rigid analytic generic fibers of $\mathfrak{U}'$, $\mathfrak{U}'_i$ are respectively $\mathcal{U}$, $\mathcal{U}_i$, for all i. The sheaf property for $h^*\mathfrak{F}$ with respect to the covering $\{\mathfrak{U}'_i\}_i$ implies that $\mathscr{F}$ satisfies the sheaf property for the cover $\{\mathcal{U}_i\}_i$ of $\mathcal{U}$.

Let now $\{\mathfrak{T}_i\}_i$ be a covering of $\mathfrak{X}$ by formal open affine sub-schemes with rigid analytic generic fibres $\{\mathcal{T}_i\}_i$. Let $\mathcal{V}$ be an open affinoid of $\mathcal{T}_i$. Let Spf S be an admissible affine formal model of $\mathcal{V}$. There exists an admissible blow-up $h\colon \mathfrak{Y}_i \to \mathfrak{T}_i = \mathrm{Spf}\ R$ and an open formal sub-scheme $\mathfrak{V} \subset \mathfrak{Y}_i$ such that $\mathcal{V}$ is its rigid analytic generic fiber. Moreover, using Raynaud's theory, we can assume that $\mathfrak{V}$ is an admissible blow up of Spf S. Then $\mathscr{F}(\mathcal{V}) = h^*\mathfrak{F}(\mathfrak{V}) \otimes_{\mathcal{O}_K} K$ by definition. Set V_n the scheme obtained from $\mathfrak{V}$ by reduction modulo ϖ^n, $R_n := R/\varpi^n R$, $F_n := \mathfrak{F}_n(\mathfrak{T}_i)$, $R'_n := \mathrm{H}^0\big(V_n, \mathscr{O}_{V_n}\big)$, $R' := \lim_n R'_n$ and $R''_n := R'/\varpi^n R' \hookrightarrow R'_n$. As in the proof of the projection formula in lemma A.2.2.1 it follows from the flatness of F_n as R_n-module that $h^*(\mathfrak{F}_n)(\mathfrak{V}) = F_n \otimes_{R_n} R'_n$ so that passing to the limits over n and inverting p we get that

$$(\lim_n F_n \otimes_{R_n} R'_n) \otimes_{\mathcal{O}_K} K = \mathscr{F}(\mathcal{V}).$$

Note that $\mathrm{H}^1(\mathfrak{V}, \mathscr{O}_{\mathfrak{V}})$ is a finite type torsion S-module, so it is annihilated by ϖ^N for some integer N. Set $F := \mathfrak{F}(\mathfrak{T}_i)$. By the very definition of the completed tensor product, to check that $\mathscr{F}(\mathcal{V}) = \mathscr{F}(\mathcal{T}_i)\hat{\otimes}_{\mathscr{O}_{\mathcal{X}}(\mathcal{T}_i)}\mathscr{O}_{\mathcal{X}}(\mathcal{V})$, it suffices to check that the map:

$$s\colon \lim_n F \otimes_{R_n} R''_n \to \lim_n F_n \otimes_{R_n} R'_n$$

is injective with cokernel killed by ϖ^N. Since R''_n injects in R'_n and F_n is flat over R_n then $F_n \otimes_{R_n} R''_n \to F_n \otimes_{R_n} R'_n$ is injective and passing to the limit, we get the injectivity of the map s. On the other hand $\varpi^N R'_n \subset R''_n$ so that there exists a map $s' : \lim_n F_n \otimes_{R_n} R'_n \to \lim_n F \otimes_{R_n} R''_n$ with the property that $s \circ s'$ is multiplication by ϖ^N. This implies the claim. □

Example 1. Let A be an affinoid algebra and $\mathcal{X} = \mathrm{Spm}\ A$. Let M be an A-Banach module. We can define a presheaf $\mathcal{M}$ on the category of affinoid open subsets of $\mathcal{X}$ by $\mathcal{M}(\mathcal{U}) = M\hat{\otimes}_A A_{\mathcal{U}}$, where $\mathcal{U} = \mathrm{Spm}\ A_U$ is an open affinoid subset of $\mathcal{X}$. Recall we say that a Banach A-module M is projective if it is a direct factor of an orthonormalizable Banach module. Assume that $A = A_0[1/p]$ with A_0 ϖ-adically complete and separated, flat and topologically of finite type as $\mathcal{O}_K$-algebra. If M is projective, then M admits an open and bounded sub-A_0-module M_0 such that for all n, M_0/ϖ^n is A_0/ϖ^n-flat. The proposition A.2.2.3 shows that $\mathcal{M}$ is a sheaf in that case.

As mentioned above, our main interest in this article is to study overconvergent modular sheaves. These are formal Banach sheaves but they are not necessarily flat. Therefore we need a slight generalization of the above result as follows. Let $g\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite map of admissible formal schemes, we denote by $f\colon \mathcal{X} \to \mathcal{Y}$ the associated morphism of rigid spaces. Let $\mathfrak{F}$ be a flat formal Banach sheaf on $\mathfrak{X}$ with rigid analytic fiber $\mathscr{F}$, a Banach sheaf on $\mathcal{X}$. We denote by G a finite group acting on $\mathfrak{X}$ over $\mathfrak{Y}$ and we suppose that the action lifts compatibly to $\mathfrak{F}$.

Remark that $g_*\mathfrak{F} := (g_*\mathfrak{F}_n)_n$ is a formal Banach sheaf as g_* is an exact functor. We now define the sheaf of invariants $\big(g_*\mathfrak{F}\big)^G := (\big(g_*\mathfrak{F}\big)^G_n)_{n\in\mathbb{N}}$ as follows. For every $n \in \mathbb{N}$ we let $\big(g_*\mathfrak{F}\big)^G_n$ be the sheaf associated to the presheaf $\mathfrak{H}_n$ whose values over an open subset $\mathfrak{U}$ of $\mathfrak{Y}$ is $\big(g_*\mathfrak{F}(\mathfrak{U})\big)^G \otimes_{\mathcal{O}_K} \mathcal{O}_K/\varpi^n$. As the presheaves $\mathfrak{H}_n$, for varying n, satisfy the properties of definition A.1.1.1, we conclude that $\big(g_*\mathfrak{F}\big)^G$ is a formal Banach sheaf.

Proposition A.2.2.4. *The sheaf $\big(f_*\mathscr{F}\big)^G$ is a Banach sheaf over $\mathcal{Y}$. Moreover, $\big(f_*\mathscr{F}\big)^G$ is related to $\big(g_*\mathfrak{F}\big)^G$ by the property that for every admissible blow-up $h\colon \mathfrak{Y}' \to \mathfrak{Y}$ and every open formal sub-scheme $\mathfrak{V} \subset \mathfrak{Y}'$ with rigid analytic fiber $\mathcal{V} \subset \mathcal{Y}$ we have*

$$\big(f_*\mathscr{F}\big)^G(\mathcal{V}) = h^*\big(g_*\mathfrak{F}\big)^G(\mathfrak{V}) \otimes_{\mathcal{O}_K} K.$$

Proof. We claim that the sheaf $\mathscr{G} := f_*\mathscr{F}$ is a Banach sheaf. Condition (I) and (II) in definition A.2.1.2 are easily verified. For condition (III) we remark that thanks to proposition A.2.2.3 we can take an admissible affinoid covering of $\mathcal{X}$ defined by the inverse image under f of the rigid analytic fiber of an affine covering of $\mathfrak{Y}$, as g is finite.

For any $\mathcal{O}_K[G]$-module M there is a trace map $\mathrm{tr}\colon M \to M^G$ which maps $m \in M$ to $\sum_{g\in G} g \cdot m$. If $i\colon M^G \to M$ is the inclusion, then $\mathrm{tr} \circ i$ is the multiplication by the order a of G. As a is invertible in K, the map $e := (1/a)i \circ \mathrm{tr}$ for the sheaf $\mathscr{G}$ is an idempotent such that $\mathrm{Im}(e) = \big(\mathscr{G}\big)^G$. This implies that $\big(\mathscr{G}\big)^G$ is a Banach sheaf, being a direct summand of $\mathscr{G}$.

Next, as G is a finite group, for any $\mathcal{O}_K[G]$-module M and for every integer $n \geq 0$ we denote by $M_n := M/\varpi^n M$. If M is flat as $\mathcal{O}_K$-module, by taking the G-invariants of the exact sequence

$$0 \to M \stackrel{\varpi^n}{\to} M \to M_n \to 0$$

we obtain an injective map $M^G/\varpi^n M^G \hookrightarrow M_n^G$ whose cokernel is a sub-group of $\mathrm{H}^1\big(G, M\big)$ and therefore it is annihilated by the order a of G.

In particular, using the notation above, we have a natural injective map $\mathfrak{H}_n \to \big(g_*\mathfrak{F}_n\big)^G$ with cokernel annihilated by a. Thus $a \cdot \mathrm{tr}$ (where tr is the trace for the G-action) defines a map $g_*\mathfrak{F}_n \to \mathfrak{H}_n$. This induces, after sheafification, an injective map $\alpha_n \colon \big(g_*\mathfrak{F}\big)_n^G \to \big(g_*\mathfrak{F}_n\big)$ and a map $\beta_n \colon \big(g_*\mathfrak{F}_n\big) \to \big(g_*\mathfrak{F}\big)_n^G$ such that $\beta_n \circ \alpha_n$ is multiplication by a^2 and $\alpha_n \circ \beta_n = a \cdot \mathrm{tr}$. Using $h^*(\alpha_n)$ and $h^*(\beta_n)$ we obtain maps

$$s_n \colon h^*\big(g_*\mathfrak{F}\big)_n^G \to h^* g_*\mathfrak{F}_n, \qquad t_n \colon h^* g_*\mathfrak{F}_n \to h^*\big(g_*\mathfrak{F}\big)_n^G$$

such that $t_n \circ s_n$ is multiplication by a^2 and $s_n \circ t_n = a \cdot \mathrm{tr}$. Passing to the inverse limits sheaves we obtain maps s and t such that $t \circ s$ is multiplication by a^2 and $s \circ t = a \cdot \mathrm{tr}$. We conclude that the map s defines an isomorphism of sheaves on $\mathfrak{Y}$:

$$\lim_n h^*\big(g_*\mathfrak{F}\big)_n^G \otimes_{\mathcal{O}_K} K \to \big(\lim_n h^* g_*\mathfrak{F}_n\big)^G \otimes_{\mathcal{O}_K} K.$$

Thus to conclude the proof of the claim we are left to show that for every open formal sub-scheme $\mathfrak{V} \subset \mathfrak{Y}'$ with rigid analytic fiber $\mathcal{V} \subset \mathcal{Y}$ we have

$$\mathscr{G}(\mathcal{V}) = \mathfrak{G}'(\mathfrak{V}) \otimes_{\mathcal{O}_K} K,$$

where $\mathfrak{G} := g_*\mathfrak{F}$ and $\mathfrak{G}' := h^*\mathfrak{G}$.

Let $h \times g \colon \mathfrak{X}' \to \mathfrak{X}$ be the projective map obtained by base-change. It is dominated via a map $u \colon \mathfrak{X}'' \to \mathfrak{X}'$ by an admissible blow-up $t \colon \mathfrak{X}'' \to \mathfrak{X}$. Let $\mathfrak{F}' := (h \times g)^*\mathfrak{F}$. Arguing as in lemma A.2.2.1, using that u is projective, one shows that the adjunction $\mathfrak{F}' \to u_* u^* \mathfrak{F}'$ has kernel and cokernel killed by a power of ϖ. If $g' \colon \mathfrak{X}' \to \mathfrak{Y}'$ is the induced finite map, then by construction we have: $\mathscr{G}(\mathcal{V}) = g'_* u_* u^*(\mathfrak{F}')(\mathfrak{V}') \otimes K$. But we have just proven that this K-module coincides with $g'_*\mathfrak{F}'\big(\mathfrak{V}'\big) \otimes_{\mathcal{O}_K} K$. As g is finite, we have $g'_*\mathfrak{F}' = \mathfrak{G}'$. This proves the displayed equality and concludes the proof. □

B List of symbols

B standard Borel in GL_g, §2.1
$U \subset B$ unipotent radical, §2.1
$T \subset B$ standard torus, §2.1
B^0 Borel opposite to B, §2.1
$U^0 \subset B^0$ unipotent radical, §2.1
I Iwahori subgroup of $\mathrm{GL}_g(\mathbb{Z}_p)$, §2.3
$\mathfrak{T}$, $\mathfrak{T}_w$, formal torus, §4.5
$\mathfrak{B}_w$, $\mathfrak{U}_w$, formal groups, §5.2.2
$\mathcal{W}$ weight space, §2.2
$\mathcal{W}(w)$, $\mathcal{W}(w)^o$, §2.2
κ^{un}, universal character, proposition 2.2.2
$\kappa \mapsto \kappa'$, involution on weights, §5.1

Y moduli space of principally polarized abelian schemes (A, λ) of dimension g equipped with a principal level N, §5.1

$Y \subset X$ toroidal compactification, §5.1

$Y \subset X^{\star}$, minimal compactification, §8.2.2

$\mathfrak{X}$, formal scheme associated to X, §5.2

Y_{Iw} moduli space with principal level N structure and Iwahori structure at p, §5.1

$Y_{\mathrm{Iw}} \subset X_{\mathrm{Iw}}$ toroidal compactification, §5.1

$\mathfrak{X}_1(p^n)(v)$, $\mathfrak{X}_{\mathrm{Iw}^+}(p^n)(v)$, $\mathfrak{X}_{\mathrm{Iw}}(p^n)(v)$, $\mathfrak{X}(v)$ formal schemes, §5.2

$\mathcal{X}(v)$ rigid space, neighbourhood of ordinary locus of width v, §5.2

$\mathcal{X}_1(p^n)(v)$, $\mathcal{X}_{\mathrm{Iw}^+}(p^n)(v)$, $\mathcal{X}_{\mathrm{Iw}}(p^n)(v)$, rigid spaces, §5.3

$\mathcal{F}$, proposition 4.3.1

$\mathfrak{IW}_w$, Grassmannian of w-compatible flags in $\mathcal{F}$, §4.5

$\mathfrak{IW}_w^+$, Grassmannian of w-compatible flags in $\mathcal{F}$ and bases elements of the graded pieces, §4.5

$\mathcal{IW}_w^+$, rigid space over $\mathcal{X}_1(p^n)(v)$ associated to $\mathfrak{IW}_w^+$, §5.3

$\mathcal{IW}_w$, rigid space over $\mathcal{X}_1(p^n)(v)$ associated to $\mathfrak{IW}_w$, §5.3

$\mathcal{IW}_w^{o+}$, descent of $\mathcal{IW}_w^+$ to $\mathcal{X}_{\mathrm{Iw}^+}(p^n)(v)$, §5.3

$\mathcal{IW}_w^{o}$, descent of $\mathcal{IW}_w$ to $\mathcal{X}_{\mathrm{Iw}}(p^n)(v)$, §5.3

$\mathcal{IW}_{\underline{w}}^{o+}$, rigid space with dilations parameters, §5.6

$\mathfrak{w}_w^{\dagger\kappa}$ the formal Banach sheaf of w-analytic, v-overconvergent modular forms of weight κ, definition 5.2.2.3

$\omega_w^{\dagger\kappa}$ Banach sheaf of w-analytic, v-overconvergent weight κ modular forms, §5.3

$\mathrm{M}_w^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$, $\mathrm{M}^{\dagger\kappa}(X_{\mathrm{Iw}}(p))$ space of overconvergent modular forms of weight κ, definition 5.3.3

$\omega_w^{\dagger\kappa^{\mathrm{un}}}$, $M_{v,w}$, $M^{\dagger}$ families of overconvergent modular forms, §8.1.1

$\tilde{\mathfrak{w}}_w^{\dagger\kappa^{o\mathrm{un}}}$, family of integral overconvergent modular forms, §8.1.4

$\omega_{\underline{w}}^{\dagger\kappa}$, $\mathrm{M}_{\underline{w}}^{\dagger\kappa}(\mathcal{X}_{\mathrm{Iw}}(p)(v))$, variants with dilations parameters, §5.6

$U_{p,g}$, U operator, §6.2.1

$U_{p,i}$, U operator, §6.2.2

References

[A-M] A. Abbes, F. Mokrane, *Sous-groupes canoniques et cycles évanescents p-adic pour les variétés abéliennes*, Publ. Math. IHES **99** (2004), 117-162.

[A-G] F. Andreatta, C. Gasbarri, *The canonical subgroup for families of abelian varieties*, Compositio Math. **143** (2007), 566-602.

[AIS] F. Andreatta, A. Iovita, G. Stevens, *On overconvergent modular sheaves for* $\mathrm{GL}_{2/F}$, preprint (2010).

[AIS1] F. Andreatta, A. Iovita, G. Stevens, *On overconvergent modular forms*, preprint (2010).

[Ar] J. Arthur, *Automorphic forms for* GSp_4, Contributions to Automorphic Forms, Geometry, and Number Theory: A Volume in Honor of Joseph Shalika. Johns Hopkins Univ. Press, Baltimore, (2004), 65-81.

[AS] A. Ash, G. Stevens, *p-Adic families of modular symbols for* GL_g, preprint (2009).

[Bou] N. Bourbaki, *Espaces vectoriels topologiques*, Chapter 1 to 5. Masson, Paris, new edition, (1981).

[Bos] S. Bosch, *Lectures on formal and rigid geometry*, available at `http://www.math.uni-muenster.de/sfb/about/publ/heft378.pdf`.

[Bra] R. Brasca, *p-Adic modular forms of non-integral weight over Shimura curves*, Compositio Math., Volume 149, (2013), 32-62.

[Buz] K. Buzzard, *Eigenvarieties*, Proceedings of the LMS Durham conference on *L-functions and arithmetic geometry*, 59-120, London Math. Soc. Lecture Note Ser. **320** (2007), Cambridge Univ. Press, Cambridge.

[BDS-BT] K. Buzzard, M. Dickinson, N. Shepherd-Barron, R. Taylor, *On icosahedral Artin representations*, Duke Math. J. **109** (2001), no 2, 283-318.

[Che1] G. Chenevier, *Une correspondance de Jacquet-Langlands p-adique*, Duke Math. J. **126** (2005), 161-194.

[Che2] G. Chenevier, *On the infinite fern of Galois representations of unitary type* Annales Scientifiques de l'E.N.S. 44, 963-1019 (2011) .

[Col1] R. Coleman, *Classical and overconvergent modular forms*, Invent. Math. **124** (1996), 215-241.

[Col2] R. Coleman, *p-adic Banach spaces and families of modular forms*, Invent. Math. **127** (1997), 417-479.

[C-M] R. Coleman, B. Mazur, *The eigencurve*, In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, London Math. Soc. Lecture Note Ser. **254** (1998), 1-113. Cambridge Univ. Press, Cambridge.

[C-S] D. A. Cox, J. B. Little, H. K. Schenck, *Toric varieties.* Graduate Studies in Mathematics **124** (2011), AMS, Providence.

[De] P. Deligne, *Théorie de Hodge. III*, Publ. Math. IHES, **44**, (1974), 5-77.

[Dm] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona.* Ann. Sci. École Norm. Sup. **3** (1970), 507-588.

[E] M. Emerton, *On the interpolation of systems of eigenvalues attached to automorphic Hecke aigenforms*, Invent. Math. **164** (2006), 1-84.

[F-C] G. Faltings, C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzbgebiete (3), vol. **22**, Springer-Verlag, Berlin, (1990), with an appendix by D. Mumford.

[Far1] L. Fargues, *La Filtration de Harder-Narasimhan des schémas en groupes finis et plats*, J. Reine Angew. Math. **645** (2010), 1-39.

[Far2] L. Fargues, *La filtration canonique des points de torsion des groupes p-divisibles*, avec la collaboration de Yichao Tian, Ann. Sci. École Norm. Sup. **44** (2011), 905-961.

[GS] R. Greenberg, G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math. **111** (1993), 407-447.

[EGAIII] A. Grothendieck, *Eléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I.* Publ. Math. IHES **11** (1961), 167 pp.

[Hi1] H. Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup. **19** (1986), 231-273.

[Hi2] H. Hida, *Control theorems for coherent sheaves on Shimura varieties of PEL type*, Jour. Inst. Math. Jussieu **1** (2002), 1-76.

[Jo] O. Jones, *An analogue of the BGG resolution for locally analytic principal series*, Journal of Number Theory **131** (2011), 1616-1640.

[Ki] M. Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math, **153** (2003), no. 2, 373-454.

[K-L] M. Kisin, K.F. Lai, *Overconvergent Hilbert modular forms*, Amer. J. Math. **127** (2005), no. 4, 735-783.

[Kl] S. Kleiman, *A note on the Nakai-Moisezon test for ampleness of a divisor.* Amer. J. Math. **87** (1965), 221-226.

[Ko] J. Kollár, *Rational curves on algebraic varieties.* Ergebnisse der Mathematik und ihrer Grenzbgebiete (3), vol. **32**, Springer-Verlag, Berlin, (1996).

[Lu] W. Lütkebohmert, *Der Satz von Remmert-Stein in der nichtarchimedischen Funktionentheorie*, Math. Z. **139** (1974), 69-84.

[Me] W. Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, LNM vol **264**, Springer-Verlag, Berlin and New-York, 1972.

[Mu] D. Mumford, *Abelian Varieties*, TIFR Studies in Mathematics, vol. **5**, Oxford Univeristy Press, Oxford, 1970, with appendices by C. P. Ramanujam and Y. Manin.

[O-T] F. Oort, J. Tate, *Group schemes of prime order*, Ann. Sci. École Norm. Sup. **3** (1970), 1-21.

[Pi1] V. Pilloni, *Modularité, formes de Siegel et surfaces abéliennes*, J. reine angew. Math **666** (2012), 35-82.

[Pi2] V. Pilloni, *Sur la théorie de Hida pour* GSp_{2g}, to appear in Bull. Soc. Math. France.

[Pi3] V. Pilloni, *Formes modulaires surconvergentes*, to appear in Ann. Inst. Fourier (Grenoble).

[P-S] V. Pilloni, B. Stroh, *Surconvergence et classicité : le cas décomposé*, preprint.

[S-U] C. Skinner, E. Urban, *Sur les déformations p-adiques de certaines représentations automorphes*, Jour. Inst. Math. Jussieu **5** (2006), 629-698.

[Sch] R. Schmidt, *Some remarks on local new forms for* GSp_4 *and Siegel modular forms of degree* 2 *with square free level*, J. Math. Soc. Japan, **57** (2005), 259-293.

[Str1] B. Stroh, *Compactification minimale et mauvaise réduction*, Ann. Inst. Fourier (Grenoble) **60** (2010), 1035-1055.

[Str2] B. Stroh, *Compactifications de variétés de Siegel aux places de mauvaise réduction*, Bull. Soc. Math. France **138** (2010), 259-315.

[Ta] J. Tate, *Rigid analytic spaces.* Invent. Math. **12** (1971), 257-289.

[Ti] Y. Tian, *Canonical subgroups of Barsotti-Tate groups*, Ann. of Math. (2) **172** (2010), 955-988.

[Ur] E. Urban, *Eigenvarieties*, Ann. of Math. (2) **174** (2011), 1685-1784.