THE STACK OF VECTOR BUNDLES ON A CURVE

Contents

1. Lecture 0 : Unramified Class field theory

1.1. **Galois groups.** Let K be a field. We let K^{sep} be a separable closure of K. We let $G_K = \mathrm{Aut}_K(K^{sep})$ be the absolute Galois group of K.

Let $L \subset K^{sep}$ be a finite extension of K. We say that L is Galois if for all $\sigma \in G_K$, $\sigma(L) = L$. The Galois group of L over K is $Gal(L/K) = \mathrm{Aut}_K(L)$.

Proposition 1.1. *Let L/K be a Galois extension.*

(1) *The natural map $G_K \to Gal(L/K)$ is surjective.*
(2) *The group $Gal(L/K)$ has cardinality* $\dim_K L$.

Any finite extension $L \subset K^{sep}$ is contained in a Galois extension. Therefore, $G_K = \lim_{L/K,\text{finite galois}} Gal(L/K)$.

We equip G_K with a topology by declaring that an open basis of neighborhoods of 1 is given by the $G_L = Gal(K^{sep}/L)$ for L/K a finite extension. Then G_K is a profinite group. Moreover the Galois correspondence is :

Theorem 1.1.

$$\begin{array}{rcl} \{\textit{Open subgroups of } G_K\} & \leftrightarrow & \{\textit{Finite separable field extensions of } K\} \\ H & \mapsto & (K^{sep})^H \\ G_L & \leftarrow\!\shortmid & L \end{array}$$

Example 1. Let $q = p^r$ and let $K = \mathbb{F}_q$ be the finite field with q elements. Let $\overline{\mathbb{F}_q}$ be an algebraic closure of $\mathbb{F}_q$. For all $n \geq 0$, there is a unique extension of $\mathbb{F}_q$, $\mathbb{F}_{q^n} \subset \overline{\mathbb{F}_q}$ of degree n. Its Galois group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and a generator is given by the Frobenius $Frob_q : x \mapsto x^q$. Therefore $G_{\mathbb{F}_q} \simeq \hat{\mathbb{Z}}$ and $Frob_q$ is a topological generator.

Example 2. Let $K = \mathbb{R}$ be the field of real numbers. We have $\overline{\mathbb{R}} = \mathbb{C}$ and $G_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$, the generator is given by the complex conjugation $c : z \mapsto \overline{z}$.

1.2. Discrete valuation rings.

1.2.1. *Valuations.* Let K be a field. A discrete valuation v (of rank 1) on K is a surjective function : $v : K^\times \to \mathbb{Z}$ which satisfies :

(1) $v(xy) = v(x) + v(y)$,
(2) $v(x + y) \geq \inf\{v(x), v(y)\}$.

One extends v to K by setting $v(0) = +\infty$.

Example 3. The trivial valuation on a field K is defined by $v(x) = 1$ for all $x \in K^\times$.

Example 4. Let p be a prime number. For all $x \in \mathbb{Q}^\times$, write $x = x'p^n$ where p does not appear in the prime decomposition of x', and set $v_p(x) = n$. This is the p-adic valuation on $\mathbb{Q}$.

Example 5. Let k be a field. Let $k(T)$ be the field of rational functions over k. Let P be an irreducible polynomial. For any $x \in k(T)^\times$, write $x = x'P^n$ where P does not appear in the decomposition of x in product of prime ideals and let $v_P(x) = n$. This is the P-adic valuation on $k(T)$. Let $deg : k(T) \to \mathbb{Z} \cup \{\infty\}$ be the degree map. Then $-deg$ is a valuation.

Theorem 1.2 (Ostrowski)**.** *The only non-trivial valuations on $\mathbb{Q}$ are (up to equivalence) the p-adic valuations v_p for prime numbers p.*

Proof. [Cas67], section. 3, p. 45. □

Theorem 1.3. *The only non-trivial valuation on $k(T)$ which are trivial on k are the v_P for P an irreducible polynomial and $-deg$.*

1.2.2. *Valuation ring.* We let $A = \{x \in K, v(x) \geq 0\}$. This is the ring of the valuation v. It is easy to check that A is a discrete valuation ring, namely a principal domain which has a unique non-zero prime ideal. Conversely, A determines the valuation v. Indeed, we have a group isomorphism $K^\times/A^\times \simeq \mathbb{Z}$ which sends a generator π of the maximal ideal of A to 1 and we recover v as the composite $K^\times \to K^\times/A^\times \simeq \mathbb{Z}$.

We let $|.|_v = e^{-v(.)}$ be the associated norm. It is called non-archimedean because $|x+y|_v \leq \sup\{|x|_v, |y|_v\}$.

1.2.3. *Completion.* If A is a discrete valuation ring, we can consider its completion $\hat{A}$ with respect to the norm $|.|_v$. Concretely, $\hat{A} = \lim_n A/\mathfrak{p}^n$.

1.3. **Dedekind rings.**

1.3.1. *Definition.*

Definition 1.1. *A Dedekind ring is a noetherian domain which is integrally closed of dimension one.*

Proposition 1.2. *A noetherian domain is a Dedekind ring if and only if, for all maximal ideal $\mathfrak{p}$ of A, the localization $A_{(\mathfrak{p})}$ is a discrete valuation ring.*

Proof. See [Ser68], proposition 4 on p. 22. □

For any maximal ideal $\mathfrak{p}$ of A, we denote by $v_\mathfrak{p}$ the corresponding $\mathfrak{p}$-adic valuation. We will also denote by $A_\mathfrak{p} = \hat{A_{(\mathfrak{p})}}$ the completion of A for the $\mathfrak{p}$-adic topology.

1.3.2. *Fractional ideals.* A fractional ideal of a Dedekind ring A is a non-zero finitely generated submodule of $K = Frac(A)$. The set of fractional ideals is a monoid under multiplication, with neutral element A itself.

Proposition 1.3. *The fractional ideals of a Dedekind ring form a group. Any fractional ideal $\mathfrak{a}$ has a unique expression*

$$\mathfrak{a} = \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}}$$

where almost all the $n_\mathfrak{p}$ are zero.

Proof. See [Ser68], corollaire and proposition 7 on p. 24. □

1.3.3. *Extension of Dedekind rings.* Let A be a Dedekind ring with fraction field K. Let L be a finite extension of K. Let B be the integral closure of A in K.

Theorem 1.4. *If either A is a finite type algebra over a field, or L is a separable extension of K, B is a finite A-algebra and a Dedekind ring.*

Proof. See [Ser68], part I, chap. 4. □

We assume that the assumptions of the theorem hold. There is a (surjective) map Spec $B \to$ Spec A. We say that a prime ideal $\mathfrak{P}$ in B divides a prime ideal $\mathfrak{p}$ and write $\mathfrak{P} \mid \mathfrak{p}$ if $\mathfrak{P}$ is mapped to $\mathfrak{p}$.

If $\mathfrak{p}$ is a maximal ideal of A, we have $\mathfrak{p} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e_\mathfrak{P}}$. The integer $e_\mathfrak{P}$ is called the ramification index at $\mathfrak{P}$. The residual degree at $\mathfrak{P}$ is the degree of the finite extension $A/\mathfrak{p} \to B/\mathfrak{P}$ and is denoted by $f_\mathfrak{P}$.

Proposition 1.4. *We have the formula* $\sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}} f_{\mathfrak{P}} = \dim_K L$.

Proof. $B \otimes A_{(\mathfrak{p})}$ is a finite free $A_{(\mathfrak{p})}$-module of finite rank $\dim_K L$. By reduction modulo $\mathfrak{p}$ we find that $B/\mathfrak{p} \to \prod B/\mathfrak{P}^{e_{\mathfrak{P}}}$ is an isomorphism. The formula is obtained by comparing the dimensions as $A/\mathfrak{p}$-modules on both sides. □

Definition 1.2. *We say that* B *is unramifed over* A *at* $\mathfrak{P}$ *if* $e_{\mathfrak{P}} = 1$.

1.3.4. *Ramification.* Let $K \subset L$ be a finite separable extension of fields. We have a non-degenerate bilinear trace map $\mathrm{Tr} : L \times L \to K$. Let $A \subset K$ be a Dedekind ring with fraction field K. Let B be the integral closure of A in L. We assume that the assumptions of theorem 1.4 hold.

We can define $\mathfrak{D}^{-1}_{B/A} = \{x \in L, \mathrm{Tr}(xB) \subseteq A\}$. This is a fractional ideal of B and its inverse $\mathfrak{D}_{B/A}$ is an ideal called the different of B with respect to A.

Proposition 1.5. *The set of ramified prime of* B *over* A *is exactly the set of primes which divide the different* $\mathfrak{D}_{B/A}$. *In particular this is a finite set.*

Proof. See [Ser68], thm 1 on page 62. □

1.3.5. *Unramified extensions in complete discrete valuation rings.* Let $\mathcal{O}_K$ be a complete discrete valuation ring. Let K be its field of fraction. For any finite separable extension L of K, we let $\mathcal{O}_L$ be the inegral closure of $\mathcal{O}_K$ in L.

Lemma 1.1. *The ring* $\mathcal{O}_L$ *is a complete discrete valuation ring.*

Proof. We know that $\mathcal{O}_L$ is a Dedekind ring and has finitely many maximal ideals. Each of these ideals induce a topology on L which extends the topology of K. Since K is complete, this topology is unique (this is the product topology on K^n identified with L. Therefore there is a unique maximal prime in $\mathcal{O}_L$. □

Let K^{sep} be a separable closure of K. This is a valued field (in general not complete). Let $\mathfrak{m}_{\mathcal{O}_K^{sep}}$ be the maximal ideal of $\mathcal{O}_{K^{sep}}$. Let $k^{sep} = \mathcal{O}_{K^{sep}}/\mathfrak{m}_{\mathcal{O}_K^{sep}}$.

Theorem 1.5. k^{sep} *is a separable closure of* k *and there is an equivalence of category :*

$$\begin{array}{rcl} \{\textit{Unramified finite extensions } L \subset K^{sep}\} & \to & \{\textit{finite extensions } \ell \subset k^{sep}\} \\ L & \mapsto & \mathcal{O}_L/\mathfrak{m}_{\mathcal{O}_L} \end{array}$$

Proof. [Fr7], p. 26. □

Assume that L/K is Galois. Let $Gal(L/K)$ be the Galois group. We have a surjective map $Gal(L/K) \to Gal(l/k)$ whose kernel is denoted by $I_{L/K}$ and is called the inertia. Passing to the limit over L we have an exact sequence :

$$1 \to I_K \to G_K \to G_k \to 1.$$

1.4. **Global fields.**

1.4.1. *Definition.* A global field K is either a number field or a function field of one variable over a finite field $\mathbb{F}_q$.

(1) K is a number field. That is K is a field of characteristic 0 and is a finite extension of $\mathbb{Q}$.
(2) K is a function field of one variable over a finite field $\mathbb{F}_q$. That is, K is a field of characteristic p and is a finite type extension of transcendance degree 1 over $\mathbb{F}_p$. Moreover $\mathbb{F}_q$ is the integral closure of $\mathbb{F}_p$ in K. The simplest example of such field is $\mathbb{F}_q(T)$.

If K is a number field, K is finite over $\mathbb{Q}$ and we let $\mathcal{O}_K$ be the ring of integer. If K is a function field, we can choose $T \in K$ which is not algebraic over $\mathbb{F}_p$ and K is finite over $\mathbb{F}_q(T)$. But T is not unique.

1.4.2. *Places.* A place v of K is an equivalence class of non-trivial rank one norm :

$$|.|_v : K \to \mathbb{R}_{\geq 0}.$$

There is the following description of the places of K.

Proposition 1.6. *If K is a number field, the places of K are the non-archimedean norms $|.|_{\mathfrak{P}}$ attached to the maximal ideals $\mathfrak{P} \in \operatorname{Spec} \mathcal{O}_K$ and the archimedean norms $|.|_\sigma$ for embeddings $\sigma : K \to \mathbb{C}$.*

Proof. [Cas67], p. 45. □

Remark 1.1. Two conjugate embeddings σ and $\bar{\sigma}$ give the same archimedean norm.

Proposition 1.7. *If K is a function field, there exists a unique non-singular complete curve X with function field K and the places of K are the valuations attached to the closed points of the curve X.*

Proof. See lecture II. □

Remark 1.2. If we consider $\mathbb{F}_p(T)$, the associated curve is $\mathbb{P}^1_{\mathbb{F}_p} = \mathbb{A}^1_{\mathbb{F}_p} \cup \{\infty\}$. The closed points of $\mathbb{A}^1$ are the irreducible monic polynomials $P \in \mathbb{F}_p[T]$ with corresponding norms $|.|_P$, and ∞ corresponds to the valuation $-deg$.

In all cases, we let X (or ${}_KX$ if the context is unclear) be the set of places of K. In the number field case, we have $X = X_{fin} \cup X_\infty$ where $X_{fin} = \operatorname{Specmax} \mathcal{O}_K$ is the set of finite places and $X_\infty = \{\sigma : K \to \mathbb{C}\}/\{\text{complex conjugation}\}$ is the set of infinite places.

1.5. **From global to local fields.** If v is a place of K, we let K_v be the completion of K with respect to $|.|_v$. If v is not achimedean, we let $\mathcal{O}_v$ or $\mathcal{O}_{K_v}$ the ring of elements $x \in K_v$ with $v(x) \geq 0$. If v is archimedean, then $K_v = \mathbb{R}$ or $\mathbb{C}$.

Let L/K be a finite field extension of K. Let w be a place of L. Then w restricts to a place v of K and we say $w \mid v$. Therefore, we have a map ${}_LX \to {}_KX$.

We have the following "localization" formula :

Proposition 1.8. *The canonical map $L \otimes_K K_v \to \prod_{w|v} L_w$ is an isomorphism.*

Definition 1.3. *We say that the extension L/K is unramified at a finite place v if all the extensions L_w/K_v are unramified.*

Proposition 1.9. *A finite extension L/K is ramified at only finitely many places of K.*

1.6. **Decomposition group.** Let L/K be a finite Galois extension. Let $f : {}_LX \to {}_KX$. The group $Gal(L/K)$ acts on ${}_LX$, trivially on ${}_KX$.

Proposition 1.10. *For any $v \in {}_KX$, the action of $Gal(L/K)$ is transitive on $f^{-1}(v)$.*

Proof. See [Tat67], prop. 1.2. □

Let $w \in f^{-1}(v)$ and let $D_v = \{\sigma \in Gal(L/K), \sigma w = w\}$.

Proposition 1.11. *The map $D_v \to Gal(L_w/K_v)$ is an isomorphism.*

Proof. See [Tat67], prop. 1.2. □

The group D_v is independant of w and called the decomposition group at v. Its embedding in $Gal(L/K)$ depends on w, but its conjugacy class is independent of w.

1.7. **Frobenius substitution.** If we assume that L/K is unramified at a finite place v, then we have a canonical element $Frob_v \in D_v$, and therefore a conjugacy class $Frob_v \in Gal(L/K)$.

1.8. **The Artin reciprocity map.** We now assume that L/K is abelian. This implies that the conjugacy action of $Gal(L/K)$ on itself is trivial. Let Σ be the set of finite places where L/K is ramified.

Let I^Σ be the free abelian group generated by finite places not in Σ.

We define a map :

$$\begin{aligned} \mathrm{rec}_{L/K} : I^\Sigma &\to Gal(L/K) \\ v &\mapsto Frob_v \end{aligned}$$

Theorem 1.6 (crude reciprocity law)**.** *The map* $\mathrm{rec}_{L/K}$ *is onto and there exists* $\epsilon > 0$ *such that for all* $a \in K^\times$ *which satisfy :*

(1) $|a-1|_v < \epsilon$ *for all* $v \in \Sigma$,

(2) $\sigma(a) > 0$ *for all* $\sigma : K \to \mathbb{R}$ *in the number field case,*

we have $\mathrm{rec}_{L/K}(a) = 1$.

Remark 1.3. By $\mathrm{rec}_{L/K}(a)$ we mean $\mathrm{rec}_{L/K}(\sum_{v \notin \Sigma} v(a).v)$. This is a very hard result, you can consult [Tat67].

In this course, we will be interested in everywhere unramified extensions of K. Let H/K be the maximal abelian everywhere unramified extension of K (also called the Hilbert class field of K).

In the number field case, we have a map $I^{X_\infty} \to Gal(H/K)$. We remark that I^{X_∞} is the group of fractional ideals over Spec $\mathcal{O}_K$. Let

$$Cl^+(\mathcal{O}_K) = I^{X_\infty}/\{a \in K^\times,\ \forall\ \sigma : K \to \mathbb{R},\ \sigma(a) > 0\}$$

be the strict class group.

In the function field case we have a map $I^\emptyset \to Gal(H/K)$. We remark that $I^\emptyset$ is the group of divisors on the curve X corresponding to K. Let $Pic(X) = Div(X)/\mathrm{div}(K^\times)$ be the Picard group.

Theorem 1.7. *In the number field case, the map* $Cl^+(\mathcal{O}_K) \to Gal(H/K)$ *is an isomorphism. In the function field case, the map* $Pic(X) \to Gal(H/K)$ *is injective with dense image.*

One of the main goal of these lectures is to give Deligne's geometric proof of this theorem in the function field case. We can further geometrize the statement by interpreting $Gal(H/K)$ as $\pi_1(X)^{ab}$. Therefore the theorem reads as an injection with dense image :

$$Pic(X) \to \pi_1(X)^{ab}.$$

One can actually refine the statement. We have a degree map $Pic(X) \to \mathbb{Z}$. We also have a natural map $\pi_1(X) \to \pi_1(\mathrm{Spec}\ \mathbb{F}_q) = \hat{\mathbb{Z}}$. Let us define the Weil group of X, $W(X)$ as the preimage of $\mathbb{Z}$ in $\pi_1(X)$. Then the refined statement is that we have a commutative diagram:

$$\begin{tikzcd} Pic(X) \arrow[r] \arrow[d] & \pi_1(X)^{ab} \arrow[d] \\ \mathbb{Z} \arrow[r] & \hat{\mathbb{Z}} \end{tikzcd}$$

which induces an isomorphism between $Pic(X)$ and $W(X)^{ab}$.

1.9. **Adèles and idèles.**

1.9.1. *Adèles.* In this course we will meet at several points the ring $\mathbb{A}_K$ of adèles of a global field K. By definition, $\mathbb{A}_K$ is the subring of $\prod_{v\in\ _KX} K_v$ of elements $(x_v)_v$ such that $x_v \in \mathcal{O}_{K_v}$ for almost all v (all except finitely many ones). We equip $\mathbb{A}_K$ with a ring topology by declaring that a basis of opens of 0 are given by opens $\prod_{v\in_K X} U_v$ where for all v, U_v is an open neighborhood of 0 in K_v, and for almost all v, $U_v = \mathcal{O}_{K_v}$. The diagonal embedding $K \to \prod_v K_v$ factorizes through $\mathbb{A}_K$.

1.9.2. *Idèles.* The group of idèles is $\mathbb{A}_K^\times$ and it carries the subset topology given by the inclusion $\mathbb{A}_K^\times \to \mathbb{A}_K \times \mathbb{A}_K$, $x \mapsto (x, x^{-1})$.

Class field theory is best formulated using idèles (see [Tat67], section 5). Let us simply remark the following :

Proposition 1.12. *In the number field case, there is a natural isomorphism :*

$$K^\times\backslash\mathbb{A}_K^\times/(\prod_{v\in X_{fin}} \mathcal{O}_v^\times \prod_{v\in X_\infty} K_v^{\times,\circ}) \to Cl^+(\mathcal{O}_K).$$

In the function field case there is a natural isomorphism :

$$K^\times\backslash\mathbb{A}_K^\times/\prod_{v\in X} \mathcal{O}_v^\times \to Pic(X).$$

In the above formula $K_v^{\times,\circ}$ is the component of the identity in $K_v^\times$.

2. Lecture I : Schemes

2.1. **Affine schemes.** [Reference : [Har77], II,2; [Sta13], Chapter 01H8]

Let A be a commutative ring. We define $\mathrm{Spec}\ A = \{\text{prime ideals of } A\}$. We equip $\mathrm{Spec}\ A$ with the Zariski topology. A basis of open are the $\{D(f)\}_{f\in A}$ where $D(f) = \mathrm{Spec}\ A[1/f] \hookrightarrow \mathrm{Spec}\ A$.

We construct a sheaf of rings $\mathscr{O}_{\mathrm{Spec}\ A}$ on the topological space $\mathrm{Spec}\ A$ by putting $\mathscr{O}_{\mathrm{Spec}\ A}(D[f]) = A[1/f]$. That this defines a sheaf follows from the following proposition.

Proposition 2.1. *Let $f_1, \cdots, f_n \in A$ be such that $(f_1, \cdots, f_n) = A$. Then the following sequence is exact :*

$$0 \to A \to \prod_i A[1/f_i] \to \prod_{i,j} A[1/f_if_j]$$

where the first map is the diagonal map $a \mapsto (a)_i$ and the second map if $(f_i) \mapsto (f_{i,j})$ where $f_{i,j} = f_i - f_j$.

The pair $(\mathrm{Spec}\ A, \mathscr{O}_{\mathrm{Spec}\ A})$ is an affine scheme. Any ring morphism $f : A \to B$ induces a map of topological spaces $f : \mathrm{Spec}\ B \to \mathrm{Spec}\ A$ and a map of sheaves $\mathscr{O}_{\mathrm{Spec}\ A} \to f_\star\mathscr{O}_{\mathrm{Spec}\ B}$.

2.2. **Schemes.**

Definition 2.1. *A locally ringed space $(X, \mathscr{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ over X with the property that for all $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. A map $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of locally ringed spaces is a map $f : X \to Y$ of topological spaces together with a map of sheaves of rings :*

$$f^\star\mathscr{O}_Y \to \mathscr{O}_X$$

such that for all $x \in X$, the map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local ring map.

Definition 2.2. *A scheme is a locally ringed space $(X, \mathscr{O}_X)$ which is locally isomorphic to an affine scheme.*

Schemes are therefore a full subcategory of the category of locally ringed spaces. Inside the category of schemes, we have the full subcategory of affine schemes.

Proposition 2.2. *The category of affine schemes is equivalent to the opposite category of rings via the quasi-inverse functors $(X, \mathscr{O}_X) \to \mathrm{H}^0(X, \mathscr{O}_X)$ and $A \to (\mathrm{Spec}\ A, \mathscr{O}_{\mathrm{Spec}\ A})$, which are respectively left and right adjoints of the other.*

Remark 2.1. This proposition explains why we insist on working with locally ringed spaces and not just ringed spaces. Let k be a field and let Spec $k[[T]]$ be the affine scheme. This has a special point s and a generic point η. Consider the map Spec $k((T)) \to$ Spec $k[[T]]$ obtained by sending $(0) =$ Spec $k((T))$ to s. This induces a map of ringed spaces, but not of locally ringed spaces. The point is that the map $k[[T]] = \mathscr{O}_{\mathrm{Spec}\ k[[T]],s} \to k((T)) = \mathscr{O}_{\mathrm{Spec}\ k((T)),0}$ is not a local map. The good map Spec $k((T)) \to$ Spec $k[[T]]$ is the one induced by applying Spec to the map $k[[T]] \to k((T))$ and it sends (0) to η.

One often fixes a base scheme S and consider the category of S-schemes Sch/S. This is the category whose objects are given by a scheme X together with a "structural" morphism $X \to S$. Maps $X \to Y$ between two objects of Sch/S is a map of schemes which respects the structural morphisms.

Remark 2.2. $Sch = Sch/\mathbb{Z}$.

One is often led to impose finiteness conditions. Here is a brutal list of the most common finiteness conditions:

Finiteness conditions on a scheme :

(1) A scheme is quasi-compact if its underlying topological space is quasi compact.
(2) Quasi separated if the intersection of two quasi-compact subsets is quasi-compact.
(3) Locally noetherian : each point as an open affine neighborhood Spec R with R noetherian.
(4) Noetherian : quasi compact and locally noetherian.

Fineteness conditions on a morphism $f : X \to S$.

(1) quasi-compact : for any quasi compact open $U \hookrightarrow S$, $f^{-1}(U)$ is quasi-compact.
(2) quasi-separated : the diagonal $X \to X \times_S X$ is quasi-compact.
(3) separated : the diagonal is a closed immersion.
(4) locally of finite type : for every point $x \in X$ there are open affine $x \in \mathrm{Spec}\ R \hookrightarrow X$ and Spec $A \hookrightarrow S$ with $f(\mathrm{Spec}\ R) \subseteq \mathrm{Spec}\ A$ and R is a finite type A-algebra.
(5) locally of finite presentation : same as before with R a finite presentation A-algebra.
(6) finite type : locally of finite type + quasi-compact.
(7) finite presentation : locally of finite presentation + quasi-compact + quasi-separated.

We collect useful facts about some of these finiteness conditions.

Proposition 2.3. *All the properties of morphisms above are stable by composition, base change, product, and are local on the target.*

Proposition 2.4. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally of finite presentation (resp. locally of finite type)*
(2) *For any open affines $U \subset X$ and $V \subset S$ such that $f(U) \subset V$, the induced ring map $\mathscr{O}_S(V) \to \mathscr{O}_X(V)$ is of finite presentation (resp. of finite type).*

Proof. See [Sta13], tags 01TQ and 01T2 □

Proposition 2.5. *If X is a noetherian scheme, then for any open affine $U \subset X$, $\mathscr{O}_X(U)$ is a noetherian ring.*

2.3. **Sheaves.** In the case of a ring A, we have the abelian category $\mathrm{Mod}(A)$ of A-modules and its full subcategory $Mod_f(A)$ of finite type A-modules. The category $Mod_f(A)$ is abelian if A is Noetherian. To $M \in \mathrm{Mod}(A)$, we can associate a sheaf of $\mathscr{O}_{\mathrm{Spec}\ A}$-modules over Spec A, denoted by $\tilde{M}$ and defined by the rule that $\tilde{M}(D(f)) = M \otimes_A A[1/f]$. That this defines a sheaf follows from:

Proposition 2.6. *Let $f_1, \cdots, f_n \in A$ be such that $(f_1, \cdots, f_n) = A$. Then the following sequence is exact :*

$$0 \to M \to \prod_i M[1/f_i] \to \prod_{i,j} M[1/f_i f_j]$$

where the first map is the diagonal map $m \mapsto (m)_i$ and the second map if $(m_i) \mapsto (m_{i,j})$ where $m_{i,j} = m_i - m_j$.

Definition 2.3. *Let X be a scheme and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. The sheaf $\mathscr{F}$ is quasi-coherent if there is a covering $X = \cup \mathrm{Spec}\ A_i$ and A_i-modules M_i such that $\mathscr{F}|_{\mathrm{Spec}\ A_i} = \tilde{M}_i$. The sheaf is called coherent if theres is a covering as before such that the modules M_i are finite A_i-modules.*

We denote by $Qcoh(X)$ the category of quasi-coherent sheaves on a scheme X and $Coh(X)$ the category of coherent sheaves on X. This category $QCoh(X)$ is abelian. The category $Coh(X)$ is also abelian if X is locally Noetherian.

Remark 2.3. One finds in the literature several definitions of coherent sheaves on general schemes, which all agree in the locally Noetherian case. We have chosen the simplest one.

Proposition 2.7. *Let* Spec A *be an affine scheme. The category $QCoh(\mathrm{Spec}\ A)$ is equivalent to the category $Mod(A)$, and the category $Coh(\mathrm{Spec}\ A)$ is equivalent to the category $Mod_f(A)$ of finite A-modules via the quasi-inverse functors : $\mathscr{F} \to \mathrm{H}^0(\mathrm{Spec}\ A, \mathscr{F})$ and $M \to \tilde{M}$.*

2.4. **Functor of points.** To any scheme X we attach a functor of points :

$$\begin{array}{rcl} X(-) : Sch^{opp} & \to & SETS \\ T & \mapsto & X(T) \end{array}$$

Lemma 2.1 (Yoneda)**.** *The functor $Sch \to Func(Sch^{opp}, SETS)$ is fully faithful.*

Definition 2.4. *A functor $F : Sch^{opp} \to SETS$ is representable if it is in the essential image of the Yoneda functor.*

Proposition 2.8. *The functor $Sch \to Func(Ring, SETS) \to Func(Aff^{op}, SETS)$ that sends a scheme to the restriction of its functor of points to affine schemes is fully faithful.*

Remark 2.4. While the Yoneda lemma is purely formal and category-theoretical, the above result is not.

2.5. **Fibre products.** [Reference, [Har77], II, thm. 3.3] Let X, Y, S be schemes and $f : X \to S$, $g : Y \to S$ be maps. Then there is a scheme $X \times_S Y$ called the fibre product of X and Y over S. It fits in a commutative diagram :

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

and satisfies the following universal property:

$$\mathrm{Hom}(-, X \times_S Y) = \mathrm{Hom}(-, X) \times_{\mathrm{Hom}(-,S)} \mathrm{Hom}(-, Y).$$

In the affine case $X = \text{Spec } A, Y = \text{Spec } B, S = \text{Spec } R$ then $X \times_S Y = \text{Spec } (A \otimes_R B)$ which is in particular affine. The general case is obtained by gluing.

2.6. **Sites.** [Reference : [Sta13], Section 00VG]

Definition 2.5. *Let $\mathcal{C}$ be a category. A family of morphisms with fixed target $\{\phi_i : U_i \to U\}_{i \in I}$ is the data of an object U of $\mathcal{C}$, a set I and for all $i \in I$, a morphism $\phi_i : U_i \to U$.*

Definition 2.6. *A site is a category $\mathcal{C}$ and a collection $Cov(\mathcal{C})$ of families of morphisms with fixed target (called coverings) satisfying the following axioms :*

(1) *An isomorphism $\phi : V \to U$ is a covering,*
(2) *If $\{\phi_i : U_i \to U\}_I$ is a covering, and $\{\phi_{i,j} : U_{i,j} \to U_i\}_j$ is a covering then $\{\phi_i \circ \phi_{i,j} : U_{i,j} \to U\}_{i,j}$ is a covering.*
(3) *If $\{Ui \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then $\forall i$ the fiber product $U_i \times_U V$ exists in $\mathcal{C}$, and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.*

Definition 2.7. *A presheaf F on a site $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to SET$. A presheaf F is a sheaf if for any covering $\{\phi_i : U_i \to U\}_{i \in I}$, the diagram:*

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact. If the morphism $F(U) \to \prod_i F(U_i)$ is simply injective, the presheaf is said to be separated. A morphism of presheaves is simply a natural transformation of functors. Define $\text{Sh}(\mathcal{C})$ *to be the full subcategory of $Func(\mathcal{C}^{op}, SET)$ whose objects are sheaves.*

Before giving some examples of sites in the theory of schemes we mention a few examples:

Example 6.
(1) Let X be a topological space. Let $Op(X)$ be the category of open subsets of X, ordered by inclusion. Coverings are jointly surjective maps. A sheaf on $Op(X)$ is a sheaf in the usual sense, *ie* a topological sheaf.
(2) Let $SETS$ be the category of sets. We turn it into a site by declaring that the coverings are the jointly surjective maps.
(3) Let Top be the category of topological spaces. Coverings are open coverings.
(4) Let $CompTop$ be the category of compact Hausdorff topological spaces. Coverings are finite collections of maps, jointly surjective. A sheaf on $CompTop$ for this topology is called a "condensed set"

3. Lecture II : Topologies on schemes

[Reference [Sta13], Tag 020K]

3.1. **The Zariski topology.**

Definition 3.1. *Let X be a scheme. Let X_{Zar} be the Zariski site with underlying category the open subsets of X, and the coverings are the families $\{\phi_i : U_i \to U\}_{i \in I}$ such that $U = \cup \phi_i(U_i)$.*

Definition 3.2. *The (big) Zariski site has underlying category the category Sch, and a covering $\{\phi_i : U_i \to U\}_{i \in I}$ is the data of open immersions $U_i \to U$ such that $U = \cup \phi_i(U_i)$.*

3.2. **The $fpqc$ and $fppf$ topologies.** Recall that and R-module M is flat if the functor on $Mod(R):\ M\otimes_R -$ is exact.

Definition 3.3. *A morphism $f: X\to S$ is flat if for all $x\in X$, $\mathscr{O}_{X,x}$ is flat over $\mathscr{O}_{S,f(x)}$.*

Proposition 3.1. *A morphism of affine schemes $X = \operatorname{Spec} A \to S = \operatorname{Spec} R$ is flat if and only if A is R-flat.*

Proof. If $R\to A$ is flat then for all $x\in \operatorname{Spec} A$ mapping to $y\in \operatorname{Spec} R$ and any R_y-module M we have that $A_x\otimes_{R_y} M = A_x\otimes_A A\otimes_R M$. Thus $A_x\otimes_{R_y} -$ is exact. Conversely, assume that A_x is R_y-flat for all x. Let $0\to I\to R$ be an inclusion. Let $0\to K\to I\otimes_R A\to A$. We see that for all $x\in \operatorname{Spec} A$, $K_x = 0$ thus $K=0$. □

Definition 3.4. *A family of morphisms $\{\phi_i : U_i \to X\}_{i\in I}$ is an $fppf$ covering if each ϕ_i is flat and locally of finite presentation and $X = \cup_i \phi_i(U_i)$.*

Definition 3.5. *A family of morphisms $\{\phi_i : U_i \to X\}_{i\in I}$ is an $fpqc$ covering if each ϕ_i is flat, $X = \cup_i\phi_i(U_i)$ and for each open affine $T\subseteq X$, there is a finite set K, a map $\alpha : K\to I$, and for all $k\in K$ and open affines $V_{\alpha(k)}\subset U_{\alpha(k)}$ such that $\cup_{k\in K}\phi_{\alpha(k)}(V_{\alpha(k)}) = T$.*

Definition 3.6. *We can define the big fppf (resp. fpqc) sites. It has Sch as its underlying category and its coverings are the fppf (resp. fpqc) coverings.*

Proposition 3.2. *Sch_{fppf} and Sch_{fpqc} are sites.*

Proof. This follows from the fact that a composition of flat morphisms is flat and that the base change of a flat morphism is flat. □

Proposition 3.3. *An $fppf$ covering is an $fpqc$ covering.*

Proof. Let $\{\phi_i : U_i\to X\}_{i\in I}$ be an $fppf$ covering. Let $U\hookrightarrow X$ be an open affine. Let $U\times_X U_i = \cup_j U_{i,j}$ be an affine covering. We have that $U = \cup_{i,j} f(U_{i,j})$. By the proposition below, f is open, therefore since U is quasi-compact, the last covering admits a finite refinement. □

Proposition 3.4. *Let $f : X\to Y$ be a flat morphism, locally of finite presentation. Then f is open.*

Proof. We reduce to the affine case. By Chevalley's theorem [Sta13], Tag 020K, the image of a constructible subset of X is constructible in Y. Moreover, the morphism $X\to Y$ is generalizing. Therefore, the image of a quasi-compact open subset of X is constructible and stable under generization. This is an open subset of X. □

3.3. **The étale topology.**

3.3.1. *Smoothness.* Let R be a ring and let A be an R-algebra. For any A-module M and R-derivation from A to M is an R-linear map $D: A\to M$ such that $D(ab) = aD(b)+bD(a)$ for all $(a,b)\in A^2$. There is a universal A-module Ω^1_A/R equipped with a derivation $d : A\to \Omega^1_{A/R}$ for which $\operatorname{Der}_R(A,M) = \operatorname{Hom}_A(\Omega^1_{A/R}, M)$ for any A-module M. There is a construction by generators and relations

$$\Omega^1_{A/R} = \oplus_{a\in A} Ada/\langle d(ra) = rda\ \forall (r,a)\in R\times A,\ d(ab) = adb + bda,\ \forall (a,b)\in A\times A\rangle.$$

Here is a second construction. We can also consider the exact sequence $0\to I\to A\otimes_R A\to A\to 0$ and we let $\Omega^1_{A/R} = I/I^2$, and let $d: A\to I/I^2$ be $d(f) = 1\otimes f - f\otimes 1$. To see that $d : A\to I/I^2$ is universal, let M be an A-module and let $D : A\to M$ be a derivation. Consider $1\otimes D : A\otimes_R A\to M$ be the linearization. One checks that $1\otimes D(I^2) = 0$ and we can consider the A-linear map $1\otimes D : I/I^2\to M$. We recover D as the composition $A\to I/I^2\to M$.

3.3.2. *Two exact sequences.*

Lemma 3.1. *If $A \to B$ is a map of R-algebras, we have an exact sequence :*

$$\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

Proof. It suffices to check that for any B-module M, the sequence :

$$0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)$$

is exact. □

Lemma 3.2. *If $A \xrightarrow{\alpha} B$ is a surjective map with kernel I, we have :*

$$I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B \to \Omega^1_{B/R} \to 0$$

Proof. It suffices to check that for any B-module M, the sequence :

$$0 \to \mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M) \to \mathrm{Hom}_A(I/I^2, M)$$

is exact. □

Example 7. We have that $\Omega^1_{R[T_1,\cdots,T_n]/R} = \oplus_{i=1}^n R[T_1, \cdots, T_n]dT_i$. Indeed, one checks that the map $\oplus_{i=1}^n R[T_1, \cdots, T_n]dT_i \to \Omega^1_{R[T_1,\cdots,T_n]/R}$ is surjective using the presentation. We have the derivation $\partial_{T_i} : R[T_1, \cdots, T_n] \to R[T_1, \cdots, T_n]$ and they give linear maps : $\partial_{T_i} : \Omega^1_{R[T_1,\cdots,T_n]/R} \to R[T_1, \cdots, T_n]$ with the property that $\partial_{T_i}(dT_j) = \delta_{i,j}$. We deduce that $\{dT_1, \cdots, dT_n\}$ are indeed a basis of the differentials.

Example 8. Let $A = R[T_1, \cdots, T_n]/(P_1, \cdots, P_r)$. Then $\Omega^1_{A/R} = \oplus_{i=1}^n AdT_i/(dP_1, \cdots, dP_r)$.

3.3.3. *The naive cotangent complex.* Let B be an R-algebra of finite presentation. This means that we have an exact sequence $0 \to I \to A \xrightarrow{\alpha} B \to 0$ where A is a polynomial algebra over R and I is a finitely generated ideal. To any such presentation, we can associate the complex : $C(\alpha) : I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B$.

Lemma 3.3. *For any two presentations α, α', the complexes $C(\alpha)$ and $C(\alpha')$ are homotopic.*

Proof. We first prove that if we have a map of presentations :

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & A \arrow[r, "\alpha"] \arrow[d, "\lambda"] & B \arrow[d, "Id"] \\
0 \arrow[r] & I' \arrow[r] & A' \arrow[r, "\alpha'"] & B
\end{tikzcd}$$

we get a map $\lambda : C(\alpha) \to C(\alpha')$.

Second we show that if λ and λ' are two maps of presentation, λ and λ' are homotopic from $C(\alpha)$ to $C(\alpha')$. The homotopy is provided by the map $\lambda - \lambda' : A \to I'/(I')^2$ which is a derivation.

Third, we show that given any two presentations, there is a map between them. It follows that we have maps $C(\alpha) \to C(\alpha')$ and $C(\alpha') \to C(\alpha)$ and both compositions are homotopic to the identity. □

Definition 3.7. *A ring morphism $R \to B$ is smooth if it is of finite presentation and for any presentation α, the complex $C(\alpha) : I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B$ is injective with projective cokernel. A ring morphism $R \to B$ is étale if it is smooth and the Naive cotangent complex is quasi-isomorphic to* 0.

Proposition 3.5. (1) *Let $R \to B$ and $B \to B'$ be smooth (resp. étale) morphisms. Then $R \to B'$ is smooth (resp. étale).*

(2) *Let $R \to B$ and $R \to B'$ be smooth (resp. étale) morphisms. Then $R \to B \otimes_R B'$ is smooth (resp. étale).*

Proof. Take a presentation $\alpha : R[T_1, \cdots T_n] \to B$ with kernel I, and a presentation $\beta : R[T_1, \cdots, T_n, X_1, \cdots, X_r] \to B'$ with kernel J inducing a presentation $\gamma : B[X_1, \cdots, X_r] \to B'$ with kernel K.

We get a commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \oplus_i B' dT_i \arrow[r] & \oplus_i B' dT_i \oplus \oplus_j B' dX_j \arrow[r] & \oplus_j B' dX_j \arrow[r] & 0 \\
& I/I^2 \arrow[r] \arrow[u] & J/J^2 \arrow[r] \arrow[u] & K/K^2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

From which we deduce that the middle map is injective with projective cokernel. The second point is left to the reader.

□

Proposition 3.6. *A smooth morphism $R \to B$ is flat.*

Proof. See [Sta13] TAG 00TA. Note that syntomic morphisms are flat by definition. □

Proposition 3.7. (1) *Let R be a field. A morphism $R \to B$ is étale if and ony if B is a product of finitely many finite separable field extensions of R.*

(2) *Let R be a ring. A morphism $R \to B$ is étale if and only if it is of finite presentation, flat, and for all prime ideal $\mathfrak{p}$ in R, $k(\mathfrak{p}) \to B \otimes_R k(\mathfrak{p})$ is étale.*

Proof. First, assume that R is a field and $B = R[x]/P(x)$ with $(P(x), P'(x)) = 1$. Then $R \to B$ is étale (the naive cotangent complex is given by $B \overset{P'(x)}{\to} B$). In the other direction, we may assume that R is algebraically closed. Then one needs to see that if $R \to B$ is étale, then B is finite over R and reduced. See [Sta13] TAG 00U3. For the second point, see [Sta13] TAG 00U6. □

3.3.4. *Smooth morphism.* If $X \to S$ is a map of schemes, we let $\Omega^1_{X/S}$ be the quasi-coherent sheaf over X of relative differentials. One possible definition is to consider the locally closed immersion $\Delta : X \to X \times_S X$, factor it as the composite of a closed immersion, with ideal $\mathcal{I}$ and open immersion $X \hookrightarrow W \hookrightarrow X \times_S X$ and to let $\Omega^1_{X/S} = \Delta^\star \mathscr{I}/\mathscr{I}^2$. We can also check that for $R \to A$ and $f \in A$, $\Omega^1_{A/R} \otimes_A A_f = \Omega^1_{A_f/R}$, so that the construction of $\Omega^1_{A/R}$ is compatible with Zariski localization.

Definition 3.8. *A morphism $f : X \to S$ is smooth at $x \in X$ is x has an affine neighboorhood* Spec$B$ *over an open* Spec R *of S containing $f(x)$ and $R \to B$ is a smooth map of rings.*

Definition 3.9. *A morphism is smooth if it is smooth at all points.*

The rank of $\Omega^1_{X/S}$ is called the relative dimension of f.

Definition 3.10. *A morphism is étale if it is smooth of relative dimension zero.*

Proposition 3.8. *A morphism $f : X \to S$ is étale if*

(1) *it is locally of finite presentation,*
(2) *it is flat,*
(3) *for all $s \in S$, the fiber X_s is a disjoint union of spectra of finite separable extension of $k(s)$.*

Definition 3.11. *Let X be a scheme. We define its étale site X_{et}. The objects are $U \in Ob(Sch/X)$ which are étale over X. The coverings are the étale coverings.*

Definition 3.12. *We can also define the big étale site Sch_{et}. It has underlying category Sch and the coverings are the étale coverings.*

We can finally compare the various topologies:

Proposition 3.9. *A Zariski covering is an étale covering, an étale covering is an $fppf$ covering, and $fppf$ covering is an $fpqc$ covering.*

In the next lecture we will prove:

Theorem 3.1. *Let X be a scheme. The functor of points $X(-)$ is a sheaf for the fpqc topology.*

4. Lecture III : Descent

4.1. **Descent for sets.** In this introductory section, we consider descent problems in the category SET. Let S be a set. An S-set is a set T equipped with a map $T \to S$. A map of S-sets is a map of sets which respects the structural morphisms. Let S be a set, and let $\{T_i \to S\}$ be a covering. Here are two classical descent problems :

(1) (Descent of maps) Let X, Y be S-sets. Let $\{f_i : X \times_S T_i \to Y \times_S T_i\}_i$ be maps of T_i-sets. Is there a map $f : X \to Y$ of S-sets such that $f_i = f \times Id_{T_i}$?
(2) (Descent of objects) Let $\{X_i \to T_i\}_i$ be T_i-sets. Is there a set $X \to S$ such that $X_i = X \times_S T_i$?

In order to address these questions, it is harmless to replace T_i by $\coprod T_i = T$ and we reduce to the situation where the covering is given by one single surjective morphism $T \to S$. We let $p_1, p_2 : T \times_S T \to T$ be the two projections. Let $f_T : X \times_S T \to Y \times_S T$ be a morphism of T-sets. We can define maps $p_i^\star f_T : X \times_S T \times_S T \to X \times_S T \times_S T$ as follows. The map f_T writes $f_T(x,t) = (f_t(x), t)$. We let $p_1^\star f_T(x,t,t') = (f_t(x), t, t')$ and $p_2^\star f_T(x,t,t') = (f_{t'}(x), t, t')$.

Proposition 4.1. *Let X, Y be two S-sets, there is a canonical bijection between :* $\mathrm{Hom}_S(X,Y)$ *and maps $f_T \in \mathrm{Hom}_T(X_T, Y_T)$ such that $p_1^\star f_T = p_2^\star f_T$.*

Proof. A map from $\mathrm{Hom}_S(X,Y)$ to $\mathrm{Hom}_T(X_T, Y_T)$ is defined by $f \mapsto f_T = f \times Id_T$. It lands in the subset of maps for which $p_1^\star f_T = p_2^\star f_T$. Conversely, take f_T such that $p_1^\star f_T = p_2^\star f_T$. The equality means that $f_t(x) = f_{t'}(x)$ for any $(t,t') \in T \times T$ such that t and t' have the same image as x in S. We can safely define $f(x) = f_t(x)$ for any $t \in T$ with the same image as x in S. $\square$

Remark 4.1. Given the map $f_T \in \mathrm{Hom}_T(X_T, Y_T)$, the data of the isomorphism $p_1^\star f_T = p_2^\star f_T$ is called a descent datum. Since f exists, we say that the descent datum is effective.

We now consider the question of descent of objects. Let $p : X_T \to T$ be a T-set. We define the $T \times_S T$-sets : $p_1^\star X_T = X_T \times_S T$ and $p_2^\star X_T = T \times_S X_T$. Given a map $\phi : p_1^\star X_T \to p_2^\star X_T$ of $T \times_S T$-sets (which we write $(x,t) \mapsto (p(x), \phi_t(x))$), we can define three maps of $T \times_S T \times_S T$-sets:

(1) $p_{1,2}^\star \phi := \phi_{1,2} : X_T \times_S T \times_S T \to T \times_S X_T \times_S T$, $(x,t,t') \mapsto (p(x), \phi_t(x), t')$,
(2) $p_{2,3}^\star \phi := \phi_{2,3} : T \times_S X_T \times_S T \to T \times_S T \times_S X_T$, $(t,x,t') \mapsto (t, p(x), \phi_{t'}(x))$,
(3) $p_{1,3}^\star \phi := \phi_{1,3} : X_T \times_S T \times_S T \to T \times_S T \times_S X_T$, $(x,t,t') \mapsto (p(x), t, \phi_{t'}(x))$.

Proposition 4.2. *There is a bijection between S-sets and T-sets $X_T \to T$ equipped with bijective maps of $T \times_S T$-sets $\phi : p_1^\star X_T \to p_2^\star X_T$ such that $p_{1,3}^\star \phi = p_{2,3}^\star \phi \circ p_{1,2}^\star \phi$.*

Proof. Given an S-set X, we get a T-set X_T and the map ϕ satisfying the conditions. Conversely, let $X_T \to T$. To descend it, we need for any $(t,t') \in T \times_S T$ an isomorphism $\phi_{t,t'} : X_t \to X_{t'}$ which are compatible in the sense that if $(t,t',t'') \in T \times_S T \times_S T$, we have: $\phi_{t'',t'} \circ \phi_{t,t'} = \phi_{t'',t}$. We then pick a section $c : S \to T$. We can then safely define $X \to S$ by setting $X_s = X_c(t)$. We have an isomorphim $X \times_S T \to X_T$ by $(x,t) \mapsto \phi_{c(t),t}(x)$. The data of the $\phi_{t,t'}$ is equivalent to the data of a map $\phi : X_T \times_S T \to T \times_S X_T$ via $(x,t) \mapsto (x, \phi_{p(x),t}(x))$. The compatibility $\phi_{t'',t'} \circ \phi_{t,t'} = \phi_{t'',t}$ is the condition $p_{1,3}^\star \phi = p_{2,3}^\star \phi \circ p_{1,2}^\star \phi$. □

Remark 4.2. The data of ϕ satisfying $p_{1,3}^\star \phi = p_{2,3}^\star \phi \circ p_{1,2}^\star \phi$ is a descent datum for X_T. The existence of X tell us that the descent datum is effective.

We can reformulate both propositions as follows. Let us define the category of T-sets with descent datum as the category whose objects are $(X_T \to T, \psi)$ where X_T is a T-set and $\psi : X_T \times_S T \to T \times_S X_T$ is an isomorphism of $T \times_S T$-sets, satisfying the cocycle condition $\psi_{2,3} \circ \psi_{1,2} = \psi_{1,3}$ and maps $\phi : (X_T, \psi) \to (X'_T, \psi')$ are maps $\phi : X_T \to X'_T$ of T-sets, such that the following diagram commutes :

$$\begin{tikzcd} X_T \times_S T \arrow[r, "\psi"] \arrow[d, "\phi \times Id"] & T \times_S X_T \arrow[d, "Id \times \phi"] \\ X'_T \times_S T \arrow[r, "\psi'"] & T \times_S X'_T \end{tikzcd}$$

Remark that if X is an S-set, then $X_T = X \times_S T$ comes with a canonical isomorphism $\psi : X_T \times_S T \to T \times_S X_T$, given by $(x,t,t') \mapsto (t,x,t')$.

Theorem 4.1. *The category of S-sets is equivalent to the category of T-sets with descent datum.*

4.2. **An example of descent in topological space.** We give here another example. Let $f : T \to S$ be a continuous map of topological space. We say that f is a quotient map if f is surjective and a subset $U \subseteq S$ is open if and only if $f^{-1}(U)$ is open. We remark that closed or open surjective morphisms are quotient maps.

Proposition 4.3. *Consider the site $CompTop$. Let X be an object of the site. The functor* $\mathrm{Hom}(-, X)$ *is a sheaf.*

Proof. Let $\cup_i T_i \to S$ be a covering. Again, it is harmless to consider $\coprod T_i = T$. We claim that the diagram :

$$\mathrm{Hom}(S, X) \to \mathrm{Hom}(T, X) \rightrightarrows \mathrm{Hom}(T \times_S T, X)$$

is exact. Clearly, a map $T \to X$ which is constant along the fibers of $T \to S$ comes from a map $S \to X$. This map is continuous because $T \to S$ is a quotient morphism. □

4.3. **Descent for modules.** Let $A \to B$ be a map of rings.

Definition 4.1. *We say that $A \to B$ is faithfully flat if B is a flat A-module and* Spec $B \to$ Spec A *is surjective.*

Lemma 4.1. *Let $A \to B$ be a faithfully flat map. A complex*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of A-modules is exact if and only if

$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0$$

is exact.

Proof. Since B is a flat A-module, we know that if the first complex is exact, then so is the second one. Conversely, let

$$C := 0 \xrightarrow{u} M_1 \xrightarrow{v} M_2 \xrightarrow{w} M_3 \longrightarrow 0$$

be a complex of A-modules such that the complex $C \otimes_A B$ is exact. Since B is flat, we have $0 = \ker(u \otimes B) \simeq \ker(u) \otimes B$ and similar equalities for the kernels, images and cokernels of u, v and w.
Assume that $\ker(u) \neq 0$, i.e. there is $a \in \mathrm{Ker}(u)$, $a \neq 0$. So $I := \{x \in A \mid x.a = 0\}$ is a proper ideal of A and let $\mathfrak{m}$ be a maximal ideal containing I. From the injection $A/I \hookrightarrow \ker(u)$ we get $(A/I) \otimes B \hookrightarrow \ker(u) \otimes B = 0$ by flatness of B. It follows that $(A/\mathfrak{m}A) \otimes B = 0$, which is a contradiction because the spectrum of that last ring is in bijection with the fiber of $\mathfrak{m}$ under the surjective map Spec $B \to$ Spec A. Therefore we must have $\ker(u) = 0$.
We show in the same manner that $\ker(v)/\mathrm{Im}(u) = \mathrm{coker}(w) = 0$. So C is exact.

□

Let $\phi : A \to B$ be a map of rings. We can form the complex :

$$C^\bullet := 0 \to A \overset{\phi}{\to} B \overset{d_1}{\to} B \otimes_A B \overset{d_2}{\to} B \otimes_A B \otimes_A B \cdots$$

where $d_i(b_1 \otimes \cdots \otimes b_n) = 1 \otimes b_1 \otimes \cdots \otimes b_n - b_1 \otimes 1 \otimes b_2 \otimes \cdots \otimes b_n + \cdots$.

Proposition 4.4. *If ϕ is faithfully flat, the above complex is exact. If M is any A-module, the complex $M \otimes_A C^\bullet$ is exact.*

Proof. We first assume that the map ϕ has a section $s : B \to A$. Define $s_n : B^{\otimes n} \to A$ by $s_n(b_1 \otimes \cdots \otimes b_n) = s(b_1)b_2 \otimes \cdots \otimes b_n$. One checks that $\mathrm{Id}_{C^\bullet} = ds_\bullet + s_\bullet d$. The same applies to $M \otimes_A C^\bullet$. In general we reduce to the case where there is a section. Indeed we tensor with B, and the map $B \to B \otimes_A B$ $b \mapsto b \otimes 1$ has a section $b_1 \otimes b_2 \mapsto b_1 b_2$. □

Remark 4.3. If $A \to B$ isn't flat, then the complex above need not to be exact. Take $A = \mathbb{C}[t^3, t^5]$ and $B = \mathbb{C}[t]$. In $B \otimes_A B$, we have :

$$t^7 \otimes_A 1 = t^2 t^5 \otimes_A 1 = t^2 \otimes_A t^5 = t^2 \otimes_A t^3 t^2 = t^5 \otimes_A t^2 = 1 \otimes_A t^7$$

yet t^7 isn't an element of A.

We now consider the category $DDMod(B)$ of B-modules equipped with a descent datum. Its objects are pairs (N, ψ) where N is a B-module and $\psi : N \otimes_A B \to B \otimes_A N$ is a $B \otimes_A B$-module isomorphism such that the diagram:

$$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[rr, "\psi_{0,2}"] \arrow[dr, "\psi_{0,1}"'] & & B \otimes_A B \otimes_A N \\ & B \otimes_A N \otimes_A B \arrow[ur, "\psi_{1,2}"'] & \end{tikzcd}$$

commutes.
Here we set $\psi_{0,1}(n \otimes b_2 \otimes b_3) = \psi(n \otimes b_2) \otimes b_3$, $\psi_{1,2}(b_1 \otimes n \otimes b_3) = b_1 \otimes \psi(n \otimes b_3)$ and $\psi_{0,2}(n \otimes b_2 \otimes b_3) = \sum b_i' \otimes b_2 \otimes n_i'$ where $\psi(n \otimes b_3) = \sum b_i' \otimes n_i'$.
Maps $(N, \psi) \to (N', \psi')$ are given by B-module maps $f : N \to N'$ such that the following diagram commutes :

$$\begin{array}{ccc} N \otimes B & \xrightarrow{\psi} & B \otimes N \\ \downarrow{\scriptstyle f\otimes 1} & & \downarrow{\scriptstyle 1\otimes f} \\ N' \otimes B & \xrightarrow{\psi'} & B \otimes N' \end{array}$$

We have a natural functor $Mod(A) \to DDMod(B)$, obtained by sending M to $N = M \otimes_A B$ together with $\psi : N \otimes_A B \to B \otimes_A N$ given by $\psi((m \otimes b_1) \otimes b_2) = (b_1 \otimes (m \otimes b_2))$.

We have a natural functor $DDMod(B) \to Mod(A)$ obtained by sending N to $M = \{n \in N, \psi(n \otimes 1) = 1 \otimes n\}$.

Theorem 4.2. *Assume that $A \to B$ is faithfully flat. The two functors induce equivalences of categories $Mod(A) \leftrightarrow DDMod(B)$.*

Proof. Let us denote by $F : Mod(A) \to DDMod(B)$ and by $G : DDMod(B) \to Mod(A)$ the two functors. We first check that $G \circ F = Id_{Mod(A)}$ because if $M \in Mod(A)$, the sequence : $0 \to M \to M \otimes_A B \to B \otimes_A M \otimes_A B$ where the second map is $m \otimes b \mapsto 1 \otimes m \otimes b - b \otimes m \otimes 1$ is exact by the previous proposition.

We now prove that there is a map $F \circ G \to Id_{DDMod(B)}$. Let $N \in DDMod(B)$. Let $M = \{n \in N, \psi(n \otimes 1) = 1 \otimes n\}$.

We define a map $\theta : M \otimes_A B \to N$ by $b \otimes m \mapsto bm$. We also have the map $\psi_M : M \otimes_A B \otimes_A B \to B \otimes_A M \otimes_A B$, given by $m \otimes b \otimes b' \mapsto b \otimes m \otimes b'$. We check easily that the following diagram commutes:

$$\begin{array}{ccc} B \otimes_A M \otimes_A B & \xrightarrow{Id_B \otimes \theta} & B \otimes_A N \\ \uparrow{\scriptstyle \psi_M} & & \uparrow{\scriptstyle \psi} \\ M \otimes_A B \otimes_A B & \xrightarrow{\theta \otimes Id_B} & N \otimes_A B \end{array}$$

This proves that there is a map $F \circ G \to Id_{DDMod(B)}$. We now prove that this is an isomorphism. Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_A B & \xrightarrow{d_1} & N \otimes_A B & \xrightarrow{d_2} & B \otimes_A N \otimes_A B \\ & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_{1,2}} \\ 0 & \longrightarrow & N & \xrightarrow{d_3} & B \otimes N & \xrightarrow{d_4} & B \otimes_A B \otimes_A N \end{array}$$

The first horizontal line is $(0 \to M \to N \to B \otimes N) \otimes_A B$ where the first map is the inclusion, the second map is $m \otimes 1 \mapsto 1 \otimes m \otimes 1 - \psi(m \otimes 1) \otimes 1$. The second horizontal line is the beginning of the exact sequence $0 \to N \to B \otimes N \to B \otimes B \otimes N$ from proposition 4.4. We claim that this diagram is commutative, and this allows us to conclude that θ is an isomorphism.

Here is the proof that the first square is commutative :

$$\begin{aligned} d_3\theta(n \otimes 1) &= 1 \otimes n \\ &= \psi d_1(n \otimes 1) \end{aligned}$$

Here is the proof that the second square is commutative :

$$\begin{aligned} d_4\psi(m\otimes 1) &= \psi_{1,2}(1\otimes m\otimes 1)-\psi_{0,2}(m\otimes 1\otimes 1)\\ d_2(m\otimes 1) &= 1\otimes m\otimes 1-\psi(m\otimes 1)\otimes 1\\ \psi_{1,2}d_2(m\otimes 1) &= \psi_{1,2}(1\otimes m\otimes 1)-\psi_{1,2}\psi_{0,1}(1\otimes m\otimes 1)\\ &= \psi_{1,2}(1\otimes m\otimes 1)-\psi_{0,2}(m\otimes 1\otimes 1)\end{aligned}$$

□

We can consider a slight variant. Let $Alg(A)$ be the category of A-algebras. Let $DDAlg(B)$ be the category of pairs (C,ψ) where C is a B-algebra and $\psi : C\otimes_A B\to B\otimes_A C$ is a $B\otimes_A B$-algebra isomorphism satisfying the same cocycle condition as before.

Theorem 4.3. *We have an equivalence of categories $Alg(A)\to DDAlg(B)$.*

Proof. This follows easily from theorem 4.2. □

4.4. **Application 1 : the functor of points of a scheme is an $fpqc$ sheaf.**

Lemma 4.2. *Let $\mathscr{F}$ be a contravariant functor from the category of schemes to the category of Sets. Assume that*

(1) *$\mathscr{F}$ is a sheaf for the Zariski topology,*
(2) *For any faithfully flat morphism of affine schemes $U\to V$, the following diagram is exact :*

$$\mathscr{F}(U)\to\mathscr{F}(V)\rightrightarrows\mathscr{F}(V\times_U V)$$

Then $\mathscr{F}$ is a sheaf for the fpqc topology.

Proof. Let $\{U_i\to X\}_{i\in I}$ be an $fpqc$ covering. Let $s_i\in\mathscr{F}(U_i)$ be sections, such that $s_i|_{U_i\times_X U_j}=s_j|_{U_j\times_X U_j}$ in $\mathscr{F}(U_i\times_X U_j)$. We need to see that there is a unique $s\in\mathscr{F}(X)$ such that $s|_{U_i}=s_i$. First, assume that X is affine. We can find J finite with a map $\phi : J\to I$ and for each $j\in J$, we can find $V_j\hookrightarrow U_{\phi(j)}$ affine such that $\{V_j\to X\}_{i\in J}$ is a covering. We get by restriction from $s_{\phi(j)}$ the sections s_{V_j}. Finally, since $\mathscr{F}$ is a Zariski sheaf, we have that $\prod_j\mathscr{F}(V_j)=\mathscr{F}(\coprod_j V_j)$. The map $\coprod_j V_j\to X$ satisfies the second axiom, so we deduce that there is a unique $s\in\mathscr{F}(X)$ which restricts to $\prod s_{V_j}$.

Consider the diagram :

$$\begin{tikzcd} \coprod_i U_i \arrow[d] & \coprod_{i,j} U_i\times_X V_j \arrow[l] \arrow[d] \\ X & \coprod_j V_j \arrow[l] \end{tikzcd}$$

The section s pulls back to sections $s_{U_i}\in\mathscr{F}(U_i)$ and we need to see that $s_{U_i}=s_i$. This holds true over $\coprod_j U_i\times_X V_j$ by construction. We can check that $s_{U_i}=s_i$ Zariski locally on U_i (by the first axiom). So we can reduce to the case that U_i is affine, but then the covering $\coprod_j U_i\times_X V_j\to U_i$ is of the type of the second axiom and so we conclude.

This settles the case that X is affine. In the general case, consider an affine Zariski covering $X=\cup_j X_j$, and from the covering $\{U_i\to X\}$ we get covering $\{U_i\times X_j\to X_j\}$. The sections $s_{i,j}$ over $U_i\times X_j$ come from a unique section s_j over X_j. The sections s_k and s_j coincide on $X_k\times_X X_j$ (cover $X_k\times_X X_j$ by open affines U, the sections s_k and s_j coincide over U since they coincide on $U_i\times_X U$). Finally the sections s_i glue to a unique section s over X. □

Corollary 4.1. *Let X be a scheme. The functor of points $X(-)$ is an fpqc sheaf.*

Proof. We first check that $X(-)$ is a Zariski sheaf. Let T be a scheme. Let $T = \cup_i T_i$ be a Zariski cover. Let $f_i : T_i \to X$ be a collection of maps coinciding over the intersections $T_i \times_T T_j$. Then the maps f_i glue to a map of topological space $T \to X$. This is a map of locally ringed spaces (this last condition being of local nature). Then, let us assume that $X = \text{Spec } R$ is affine. Let $T' = \text{Spec } B \to T = \text{Spec } A$ be a faithfully flat map of affine schemes. We want to see that the diagram

$$\text{Hom}(R, A) \to \text{Hom}(R, B) \rightrightarrows \text{Hom}(R, B \otimes_A B)$$

is exact. For any $f_B : R \to B$ with equal pull backs to maps $R \to B \otimes_A B$, we deduce from proposition 4.4 , that $f_B(r) \in A$ for all $r \in R$. We have therefore completed the proof in the case that X is an affine scheme.

We now deal with the case where X is not affine. Let $T' = \text{Spec } B \to T = \text{Spec } A$ be a faithfully flat map of affine schemes. The map of topological spaces $T' \to T$ is a quotient map by the lemma below. If X is a scheme, let $|X|$ be the underlying topological space. The map of underlying sets $|T' \times_T T'| \to |T'| \times_{|T|} |T'|$ is surjective. Let $f' : T' \to X$ be a map whose two pullbacks to $T' \times_T T'$ coincide. Then we get a unique continuous map : $f : T \to X$. Let $X = \cup X_j$ be an affine Zariski cover. Then $\{T' \times_X X_j \to T \times_X X_j\}$ is an fpqc cover, and we get a map of schemes $f|_{T \times_X X_j} : T \times_X X_j \to X_j \to X$. These glue to f which is therefore a scheme map. □

Lemma 4.3. *Let $f : \text{Spec } A \to \text{Spec } B$ be a faithfully flat morphism. Then f is a quotient map.*

Proof. We first see that f is generalizing because f is flat. This means that if $x \in \text{Spec } A$ and $f(x) \in \text{Spec } B$ and if y is a generization of $f(x)$, there exists $z \in \text{Spec } A$ with $f(z) = y$. Let $T \subseteq \text{Spec } B$. We assume that $f^{-1}(T) = \text{Spec } A/J$ is closed in Spec A. We have that $ff^{-1}(T) = T$ and we need to prove that this set is closed. First, we observe that T is closed under specialization. Indeed, since $f^{-1}(T)$ is closed, it is stable under specialization and thus its complementary is stable under generalization. Furthermore, f is surjective and generalizing so $\text{Spec } B \setminus T = f(\text{Spec } A \setminus f^{-1}(T))$ is stable under generalization, hence T is stable under specialization.

Now, let $I = B \cap J$. We claim that $T = \text{Spec } B/I$. Clearly $T \subseteq \text{Spec } B/I$. It suffice to see that T contains all generic points of Spec B/I. For any generic point $x \in \text{Spec } B/I$ corresponding to an ideal $\mathfrak{p}_x$, the localization $(A/J)_{\mathfrak{p}_x} \neq 0$, so there is a point $z \in \text{Spec } A/I = f^{-1}(T)$ with $f(z) = x$. □

4.5. **Application : descent for quasi-coherent sheaves.** Let X be a scheme. Let $\{U_i \to X\}_{i \in I}$ be an fpqc cover. Let $\mathscr{F}_i$ be a quasi-coherent sheaf over U_i.

A descent datum for $\{U_i \to X\}_{i \in I}$ and $\mathscr{F}_i$ is the data of $U_i \times_X U_j$-isomorphisms

$$\psi_{i,j} : p_1^\star \mathscr{F}_i \simeq p_2^\star \mathscr{F}_j$$

(where $p_1 : U_i \times_X U_j \to U_i$ and $p_2 : U_i \times_X U_j \to U_j$ are the projections) satisfying the following cocycle relation. Let (i, j, k) be a triple of indices. Let

$$\begin{aligned}
q_{1,2} : U_i \times_X U_j \times_X U_k &\to U_i \times_X U_j \\
q_{2,3} : U_i \times_X U_j \times_X U_k &\to U_j \times_X U_k \\
q_{1,3} : U_i \times_X U_j \times_X U_k &\to U_i \times_X U_k \\
q_1 : U_i \times_X U_j \times_X U_k &\to U_i \\
q_2 : U_i \times_X U_j \times_X U_k &\to U_j \\
q_3 : U_i \times_X U_j \times_X U_k &\to U_k
\end{aligned}$$

The cocycle condition writes :

$$q_{2,3}^{\star}\psi_{j,k} \circ q_{1,2}^{\star}\psi_{i,j} = q_{1,3}^{\star}\psi_{i,k}.$$

In other words we have :

$$q_{1,3}^{\star}\psi_{i,k} : q_1^{\star}\mathscr{F} \xrightarrow{q_{1,2}^{\star}\psi_{i,j}} q_2^{\star}\mathscr{F} \xrightarrow{q_{2,3}^{\star}\psi_{j,k}} q_3^{\star}\mathscr{F}.$$

We denote by $DDQCoh(\{U_i \to X\}_i)$ the category whose objects are $((\mathscr{F}_i)_i, (\psi_{i,j})_{i,j})$ consisting of quasi-coherent sheaves $\mathscr{F}_i$ over U_i and descent datum $\psi_{i,j}$. A morphism $\phi : ((\mathscr{F}_i)_i, (\psi_{i,j})_{i,j}) \to ((\mathscr{F}_i')_i, (\psi_{i,j}')_{i,j})$ is a collection of morphisms of quasi-coherent sheaves $\phi_i : \mathscr{F}_i \to \mathscr{F}_i'$ such that the following diagrams of morphisms of sheaves over $U_i \times_X U_j$ commute :

$$\begin{tikzcd} p_1^{\star}\mathscr{F}_i \arrow[r, "p_1^{\star}\phi_i"] \arrow[d, "\psi_{i,j}"] & p_1^{\star}\mathscr{F}_i' \arrow[d, "\psi_{i,j}'"] \\ p_2^{\star}\mathscr{F}_j \arrow[r, "p_2^{\star}\phi_j"] & p_2^{\star}\mathscr{F}_j' \end{tikzcd}$$

Theorem 4.4. *The natural functor from $QCoh(X)$ to the category $DDQCoh(\{U_i \to X\}_i)$ is an equivalence of category.*

Proof. The case of a Zariski cover is essentially a tautology. The case where X is affine and I is finite and the U_i are affine is a restatement of theorem 4.2. The general case follows by combining both. See [Sta13], Tag 023T. □

4.6. **Application : descent for (quasi-affine) schemes.** Let X be a scheme. Let Aff/X be the category of schemes over X, $p : Y \to X$ with p an affine morphism. This means that for any open affine $U \subseteq X$, $p^{-1}(U)$ is open affine in Y.

Let $\{U_i \to X\}_{i \in I}$ be an fpqc cover. We now define the category of affine schemes with descent datum for $\{U_i \to X\}$ that we call $DDAff(\{U_i \to X\})$. Objects of this category are $((V_i)_i, (\psi_{i,j})_{i,j})$ that we describe. We have $f_i : V_i \to U_i$ be affine morphisms of scheme. We have $U_i \times_X U_j$-isomorphisms of schemes $\psi_{i,j} : V_i \times_X U_j \to U_i \times_X V_j$ satisfying the cocycle relation. Namely the following diagram is commutative :

$$\begin{tikzcd} V_i \times U_j \times U_k \arrow[rr, "q_{1,3}^{\star}\psi_{i,k}"] \arrow[dr, "q_{1,2}\psi_{i,j}"'] & & U_i \times U_j \times V_k \\ & U_i \times V_j \times U_k \arrow[ur, "q_{2,3}^{\star}\psi_{j,k}"'] & \end{tikzcd}$$

A morphism between $\{V_i, \psi_{i,j}\}$ and $\{V_i', \psi_{i,j}'\}$ is the data of morphisms in U_i-schemes $f_i : V_i \to V_i'$ compatible with the $\psi_{i,j}$ and $\psi_{i,j}'$ in a natural sense.

The descent datum is called effective if there is $f : V \to X$ and isomorphisms $V \times_X U_i = V_i$, such that $\psi_{i,j}$ becomes the canonical isomorphism. $V \times_X U_i \times U_j \to U_i \times_X V \times_X U_j$. The basic result is:

Theorem 4.5. *The functor $Aff/X \to DDAff(\{U_i \to X\})$ is an equivalence of categories.*

A map $V \to X$ of schemes is called quasi-affine if locally on X, there is an affine scheme $X' \to X$ such that $V \hookrightarrow X'$ is a quasi-compact open immersion. We can define the category $quasi - Aff/X$ and similarly a category $DDquasi - Aff(\{U_i \to X\})$

Theorem 4.6. *The functor* $quasi - Aff/X \to DDquasi - Aff(\{U_i \to X\})$ *is an equivalence of categories.*

Proof. See [Sta13], Tag 0246. □

5. Lecture IV : curves over a field

5.1. **Discrete valuations.** A (discrete) valuation on K is a non-constant function $v : K^\times \to \mathbb{Z}$ such that:

(1) $v(x+y) \geq \inf\{v(x), v(y)\}$,
(2) $v(xy) = v(x) + v(y)$.

We often extend a valuation v to K by putting $v(0) = +\infty$. Two valuations v and v' are equivalent if there is a rational number s such that $sv = v'$. We can normalize the valuation by asking that v is surjective. We let $\mathscr{O}_v = \{f \in K, v(k) \geq 0\}$. This is the valuation ring of the valuation v.

A discrete valuation ring in K is a subring A such that :

(1) for any $f \in K^\times$, we have f or $f^{-1} \in A$,
(2) $K^\times/A^\times \simeq \mathbb{Z}$.

Lemma 5.1. *The map* $v \mapsto \mathscr{O}_v$ *induces a bijection between equivalence classes of dicrete valuations on* K *and discrete valuation subrings of* K*.*

5.2. **Dedekind rings.**

Definition 5.1. *A domain* R *is called Dedekind if it satisfies one of the equivalent conditions :*

(1) *R is noetherian, integrally closed and all non-zero prime ideals are maximal,*
(2) *R is noetherian, and for all non zero prime ideal $\mathfrak{m}$, $R_\mathfrak{m}$ is a discrete valuation ring,*
(3) *All fractional ideals of R are invertible.*

Proof. [Fr7], 2, p.7. □

Remark 5.1. Let K be the fraction field of R. A fractional ideal of R is a sub-R-module M of K for which there exists $d \in R \setminus \{0\}$ such that $dM \subseteq R$. Given a fractional ideal M, we let $M^{-1} = \{n \in K, nM \subseteq R\}$. This is again a fractional ideal. We say that M is invertible if $M.M^{-1} = R$. A fractional ideal is invertible if and only if it is projective.

Proposition 5.1. *Let* R *be a Dedekind ring. Let* M *be an* R*-module. Then* M *is flat if and only if* M *is torsion free.*

Proof. For a domain, we always have M flat implies M torsion free. For the converse implication, we use that M is R-flat if and only if all its localizations $M_\mathfrak{m}$ are $R_\mathfrak{m}$-flat. But $R_\mathfrak{m}$ is a discrete valuation ring. It suffices to check that for any ideal $I \subseteq R_\mathfrak{m}$ the map $I \otimes_R M \to M \otimes_R R_\mathfrak{m}$ is injective. But $I = aR_\mathfrak{m}$ for some $a \in R_\mathfrak{m}$. □

Proposition 5.2. *Let* R *be a Dedekind ring with fraction field* K*. Let* L *be a finite extension of* K*. Let* R' *be the normalization of* R *in* L*. Then* R' *is a Dedekind ring. Moreover, the map* $R \to R'$ *is finite flat in the following two cases :*

(1) *L is separable over K,*
(2) *R is a finite type algebra over a field.*

Proof. First assume that the extension is separable. We can find a basis $x_1, \cdots, x_n$ of L over R with $x_i \in R'$. Let $\{x_i^\star\}$ be the dual basis for the Trace map. We have that $\oplus_i Rx_i \subseteq R' \subseteq (R')^\star \subseteq \oplus_i Rx_i^\star$, where $(R')^\star$ is the set of elements x such that $Tr(xR') \subseteq R$.

It follows that R' is of finite type over R, and is therefore Noetherian. Let $\mathfrak{m}'$ be a non-zero ideal of R'. Then $\mathfrak{m} = R \cap \mathfrak{m}'$ is non-zero and therefore is maximal. Since $R \to R'$ is finite, we deduce that $\mathfrak{m}'$ is maximal. Next assume the extension is totally inseparable. We see that $K \subseteq L \subseteq K^{(q^{-1})}$ where $K^{(q^{-1})}$ is the extension of K obtained by adjoining all q^{th}-roots of K for $q = p^f$ (and p the characteristic). (We can also think as $K^{(q^{-1})}$ being the field K viewed as a K-algebra through the map $x \mapsto x^q$.). We deduce that $R \subseteq R' \subseteq R^{(q^{-1})}$. Let I be an ideal in R. Then $IR^{(q^{-1})}$ is invertible, since $R^{(q^{-1})} \simeq R$ is Dedekind. It follows that $1 = \sum a_i b_i$ with $a_i \in I$ and $b_i \in (IR^{(q^{-1})})^{-1}$. But $1 = \sum a_i^q b_i^q = \sum a_i a_i^{q-1} b_i^q$. Now $b_i^q \in I^{-1}$ and therefore $a_i^{q-1} b_i^q \in I^{-1}$. Therefore I is invertible. In general, L is a totally inseparable extension of a separable extension. It remains to see that R' is of finite type over R if R is of finite type over a field k. This is a classical commutative algebra result (it is easy to prove if k is perfect). □

5.3. **Projective schemes.**

Definition 5.2. *Let S be a scheme. An S-scheme X is projective if it can be embedded as a closed subscheme of a projective scheme $\mathbb{P}^N_S$.*

We want to recall a criterium for X to be a projective scheme. Recall that over $\mathbb{P}^N_S$ we have the invertible sheaf $\mathscr{O}_{\mathbb{P}^N_S}(1)$ and it comes with a surjective map $\oplus_{i=0}^N \mathscr{O}_{\mathbb{P}^N_S} \xrightarrow{X_0, \cdots, X_n} \mathscr{O}_{\mathbb{P}^N_S}(1)$.

Theorem 5.1 ([Har77], II, thm. 7.1)**.**
(1) *If $\phi : X \to \mathbb{P}^N_S$ is a morphism, then $\phi^\star \mathscr{O}(1)$ is an invertible sheaf, generated by the section $\phi^\star X_0, \cdots, \phi^\star X_N$.*
(2) *Conversely, let $\mathcal{L}$ be an invertible sheaf over X together with a surjective map $\mathscr{O}_X^N \xrightarrow{s_0, \cdots, s_n} \mathcal{L}$. Then there is a canonical morphism $\phi : X \to \mathbb{P}^N_S$ such that $\phi^\star \mathscr{O}(1) \simeq \mathcal{L}$ and $\phi^\star X_i = s_i$.*

Theorem 5.2 ([Har77], II, prop. 7.2)**.** *An morphism $\phi : X \to \mathbb{P}^N_S$, corresponding to an invertible sheaf $\mathcal{L}$ and sections $s_0, \cdots, s_N$ is a closed immersion if :*
(1) *For all i, the open $D(s_i)$ defined by $s_i \neq 0$ in X is affine,*
(2) *The map $\mathscr{O}_S[X_1/X_i, \cdots, X_N/X_i] \to \mathrm{H}^0(D(s_i), \mathscr{O}_X)$ sending X_j/X_i to s_j/s_i is surjective.*

Remark 5.2. A map $X \to S$ is proper if :
(1) It is of finite type,
(2) It is universally closed,
(3) it is separated.

If the scheme $X \to S$ is projective over S, then the map $X \to S$ is proper.

5.4. **Curves.**

Definition 5.3. *A curve C over* Spec k *is a scheme $C \to$* Spec k *of pure dimension* 1 *over* Spec k*.*

Here the dimension is defined as the maximal length of a chain of irreducible subsets. Pure dimension 1 means that all irreducible components have dimension 1.

It is reasonable to add a few more assumptions.

Definition 5.4. *A Dedekind scheme is a quasi-compact, separated scheme which is covered by affines* Spec A *where A is a Dedekind ring.*

Definition 5.5. *A non-singular curve C over* Spec k *is an irreducible, quasi-compact, separated, Dedekind scheme over* Spec k*.*

Let $K = \mathscr{O}_{C,\eta}$ be the fonction field of an irreducible curve.

Definition 5.6. *We say that C is geometrically connected if k is algebraically closed in K.*

5.5. **Function fields of dimension 1.** Let k be a field.

Definition 5.7. *A function field of dimension one over k is a field K of finite type, transcendance degree 1 and such that k is algebraically closed in K.*

This means that there exists an element $x \in K$ such that K is a finite algebraic extension of $k(x)$. Actually, this holds true for any $x \in K \setminus k$ (indeed such an element is not algebraic over k by assumption).

We now attach a set X (or ${}_K X$) to K: the set of isomorphism classes of valuations $v : K^\times \to \mathbb{Z}$ which are trivial on $k^\times$ ($v(k^\times) = 0$). We put a topology on X as follows : the opens are $\emptyset$ and the complements of a finite set of points. We will also add to X a generic point η, which belongs to all non-empty open subsets.

We now equip X with a sheaf of rings $\mathscr{O}_X$. If U is some open, we let $\mathscr{O}_X(U) = \{f \in K, v(f) \geq 0\ \forall v \in U\} = \cap_{v\in U}\mathscr{O}_v$, so that $(X, \mathscr{O}_X)$ is a locally ringed space.

Theorem 5.3. *Let K be a function field over k. The locally ringed space $(X, \mathscr{O}_X)$ is a geometrically connected, non-singular, projective curve over* Spec k.

Proposition 5.3. *The locally ringed space $(X, \mathscr{O}_X)$ is a non singular curve. Let $x \in K\setminus k$. We consider $U = \{v \in X, v(x) \geq 0\}$. Then U is open in X and $\mathscr{O}_X(U)$ is the normalisation of $k[x]$ in K. Moreover, $(U, \mathscr{O}_X|_U) = (\mathrm{Spec} B, \mathscr{O}_{\mathrm{Spec}\ B})$ for $B = \mathscr{O}_X(U)$.*

Proof. Let B be the normalization of $k[x]$ in K. This is a Dedekind ring. We first show that Spec $B = U$. Let $v \in U$. The map $k[x] \to \mathscr{O}_v$ factors over B since $\mathscr{O}_v$ is integrally closed. Consider the map $B \to \mathscr{O}_v$. We claim that $B \cap \mathfrak{m}_{\mathscr{O}_v} = \mathfrak{m}$ is a maximal ideal in B. Otherwise, $B \cap \mathfrak{m}_{\mathscr{O}_v} = \{0\}$ and therefore $B \setminus \{0\} \hookrightarrow \mathscr{O}_v^\times$ and $Frac(B) = K \hookrightarrow \mathscr{O}_v$, a contradiction. We also claim that $B_\mathfrak{m} = \mathscr{O}_v$. Indeed, let $y \in \mathscr{O}_v \setminus B_\mathfrak{m}$. Then $y^{-1} \in B_\mathfrak{m}$ (since $B_\mathfrak{m}$ is a valuation ring) so that $y^{-1} \in \mathscr{O}_v^\times \cap B_\mathfrak{m} = B_\mathfrak{m}^\times$. So $y \in B_\mathfrak{m}$. We have a map $U \to \mathrm{Spec}\ B$. We can define a map in the other direction : let $\mathfrak{m}$ be a maximal ideal of B. Then $B_\mathfrak{m}$ is a valuation ring, so there is an associated valuation $v_\mathfrak{m}$ and $v_\mathfrak{m}(x) \geq 0$. The bijection Spec $B \to U$ is continuous. It remains to prove that the sheaves coincide. Let $V = \mathrm{Spec}\ R \hookrightarrow \mathrm{Spec}\ B$ an open affine subset. It suffices to see that $R = \cap_{\mathfrak{m}\in \mathrm{Spec}\ R} R_\mathfrak{m}$. This is clear by Zariski descent.

Now let $V = \{v \in X, v(x^{-1}) \geq 0\}$. Then $X = U \cup V$ and moreover $U \setminus (V \cap U) = \{v, v(x) > 0\} = V((x)) \subseteq \mathrm{Spec}\ B$ is closed. We deduce that U, V are open in X and that X is a scheme. Let us see that $U \cap V$ is affine. This is clearly $\{v, v(x) = 0\} = D(x) = \mathrm{Spec}\ B[1/x]$. □

It actually follows from the proof that the curve associated to $k(x)$ is $\mathbb{P}^1_k$ and the choice of $x \in K \setminus k$ provides us with a finite morphism $X \to \mathbb{P}^1_k$. We will now prove that the curve X is projective.

Proposition 5.4. *The curve X is projective.*

Proof. Let us pick a finite morphism $\pi : X \to \mathbb{P}^1$. Let $\mathscr{O}(1)$ be the tautological sheaf on $\mathbb{P}^1$. We claim that $\pi^\star\mathscr{O}_X(d')$ for d' large enough is generated by global sections and provides a closed immersion $X \to \mathbb{P}^N_k$.

First, let us look at $\pi_\star\mathscr{O}_X$. This is a coherent sheaf on $\mathbb{P}^1$. Over Spec $k[x_0/x_1] = U$, $\pi_\star\mathscr{O}_X(U)$ is a finite type k-algebra generated by $s_0, \cdots, s_N$. Moreover, there exists d such that the sections $s_0x_1^d, \cdots, s_Nx_1^d$ extend to global section of $\mathscr{O}_X(d)$. Similarly, over

Spec $k[x_1/x_0] = U'$, $\pi_\star \mathscr{O}_X(U')$ is a finite type k-alegbra generated by $t_0, \cdots, t_{N'}$ (we can arrange that $N = N'$). Moreover, there exists d' (we can arrange that $d = d'$) such that the sections $t_0 x_0^d, \cdots, t_N x_0^d$ extend to global section of $\pi_\star \mathscr{O}_X(d)$.

We have a surjective map $\mathscr{O}_{\mathbb{P}^1}^{2N+4} \to \pi_\star \mathscr{O}_X(d)$ given by $x_1^d, x_0^d, s_0 x_1^d, \cdots, s_N x_1^d, t_0 x_0^d, \cdots, t_N x_0^d$.

By adjunction, this is a map $\mathscr{O}_X^{2N+4} \to \pi^\star \mathscr{O}(d)$ and we claim that the corresponding map $X \to \mathbb{P}^{2N+3}$ is a closed embedding.

Clearly the image is included in the two affine opens $D(X_0)$ and $D(X_1)$ of $\mathbb{P}^{2N+3}$.

The map $k[X_0/X_1, \cdots, X_{2N+4}/X_1] \to \mathscr{O}_X(U)$ is surjective since $\mathscr{O}_X(U)$ is generated by $s_0 = X_2/X_1, \cdots, s_N = X_{N+1}/X_1$. Therefore $X \to \mathbb{P}^{2N+3}$ is a closed embedding.

□

Proposition 5.5. $\mathrm{H}^0(X, \mathscr{O}_X) = k$.

Proof. Let $x \in K \setminus k$. We need to find $v \in X$ such that $v(x) < 0$. Let $V = \{v \in X, v(x^{-1}) \geq 0\}$. Then $\mathscr{O}_X(V) = B$ and $k[x^{-1}] \to B$ is finite flat. We can find a prime ideal above (x^{-1}) in B and it corresponds to a valuation v for which $v(x^{-1}) > 0$. □

5.6. **An equivalence of category.** We now prove that the last construction exhausts all projective non-singular curves.

Lemma 5.2. *Let C be a projective non-singular curve over* Spec k. *Then there is an isomorphism $C \to {}_K X$ where K is the function field of C.*

Proof. We first define a morphism. To any closed point x of C, we have a local ring $\mathcal{O}_{C,x} \hookrightarrow K$ which is a discrete valuation ring because the curve is non-singular. Therefore we have a map $C \to {}_K X$. This map is injective (the curve C is separated). The map extends to a locally ringed space map $(C, \mathscr{O}_C) \to ({}_K X, \mathscr{O}_{{}_K X})$, since for any open U of C, $\mathscr{O}_C(U) = \cap_{x \in U} \mathscr{O}_{C,x}$. The map $C \to {}_K X$ is therefore a map of algebraic curve. Its image is closed since C is projective, it is all of ${}_K X$. □

Let X and Y be two schemes. A morphism $f : X \to Y$ is finite flat if for any affine Spec $A \subset Y$, $f^{-1}(\text{Spec } A) = \text{Spec } B$ is affine and $A \to B$ is a finite flat map.

Lemma 5.3. *Let $f : X \to Y$ be a non-constant morphism between projective non-singular algebraic curves. Then f maps the generic point η_X of X to the generic point η_Y of Y. The morphism f is finite flat and is determined by the morphism $\mathscr{O}_{Y,\eta_Y} \to \mathscr{O}_{X,\eta_X}$ on generic points.*

Proof. The image of f is a connected closed subset of Y. It is either Y or a closed point of Y. It is therefore Y and the generic point of X maps to the generic point of Y. Therefore we have a map $K \to L$ where K is the function field of Y and L is the function field of X. Let $x \in K$ be an element which is not algebraic over k. The we have finite flat maps $k[x] \to A \to B$ where A is the normalization of $k[x]$ in K and B the normalization of $k[x]$ in B. And $\text{Spec}(B) = D(f^\star(x)) \to D(x)$ is finite flat. □

Combining everything, we arrive at the following theorem (compare with [Har77], corollary 6.12.):

Theorem 5.4. *The functor "generic point" induces an equivalence of categories between:*

$\{$*Non-singular, geometrically connected projective curves on* Spec k, *non constant morphisms*$\}$

and

$\{$*Function fields of one variable over k*$\}$.

5.7. **Invertible sheaves and divisors.**

Definition 5.8. *We let $Div(X)$ be the free abelian group generated by the closed points $x \in X$.*

We have a partial order on $Div(X)$. If $D = \sum n_x x$ and $D' = \sum m_x x$, we say that $D \geq D'$ is $n_x \geq m_x$ for all x. We say that a divisor D is effective if $D \geq 0$.

If $f \in K^\times$, we let $div(f) = \sum_{x \in X} v_x(f)x$. These divisors are called principal. We let $deg : Div(X) \to \mathbb{Z}$ which maps $\sum n_x x$ to $\sum n_x [k(x) : k]$. We let $Div^0(X)$ be the kernel of deg.

Lemma 5.4. *For all $f \in K^\times$, $deg(div(f)) = 0$.*

Proof. [Ser88], prop. 1, p. 8. □

Let $D \in Div(X)$. We let $\mathscr{O}_X(D)$ be the invertible sheaf defined by $\mathscr{O}_X(D)(U) = \{x \in K, v(x) + v(D) \geq 0, \ \forall v \in U\}$.

Lemma 5.5. *There is a bijection between :*

$$\{\textit{Locally free sheaves of rank one } \mathcal{L} \ + \ \textit{non-zero rational section } f \in \mathcal{L}_\eta \setminus \{0\}\}/\textit{isom}$$

and $Div(X)$.

Proof. To $D \in Div(X)$ we associate $\mathscr{O}_X(D)$ equipped with the rational section 1. Conversely let $(\mathcal{L}, f)$. Then for all $x \in X$, we consider $\mathcal{L}_x \subset \mathcal{L}_\eta$. This is an $\mathscr{O}_{X,x}$- rank one module inside a K-vector space of dimension 1. The module $f.\mathscr{O}_{X,x}$ is another rank 1 submodule inside $\mathcal{L}_\eta$. Let t_x be a uniformizing parameter at x. Then we have $f.\mathscr{O}_{X,x} = t_x^{n_x} \mathcal{L}_x$ for a unique integer n_x. We let $D = \sum n_x x$. The map $\mathscr{O}_X(D) \stackrel{\times f}{\to} \mathcal{L}$ is an isomorphism which sends 1 to f. □

Corollary 5.1. *There is an isomorphism : $Pic(X) = Div(X)/div(K^\times)$*

By lemma 5.4, the map deg passes to the quotient and defines a map $\deg : Pic(X) \to \mathbb{Z}$. We let $Pic^r(X) = \deg^{-1}(r)$.

5.8. **Cohomology.**

5.8.1. *Cohomology of line bundles.*

Theorem 5.5.
1. *The k-vector spaces $\mathrm{H}^i(X, \mathcal{L})$ are finite dimensional. Let $g = \dim_k \mathrm{H}^1(X, \mathscr{O}_X)$ be the genus of the curve.*
2. *We have $\dim_k \mathrm{H}^0(X, \mathcal{L}) - \dim_k \mathrm{H}^1(X, \mathcal{L}) = \deg(\mathcal{L}) - g + 1$.*
3. *Assume that X/k is smooth. There is an invertible line bundle $\Omega^1_{X/k}$ of degree $2g-2$, and a canonical isomorphism $\mathrm{H}^1(X, \Omega^1_{X/k}) \to k$.*
4. *We have a Serre duality perfect pairing :*
$$\mathrm{H}^0(X, \mathcal{L}) \times \mathrm{H}^1(X, \Omega^1_{X/k} \otimes \mathcal{L}^{-1}) \to k.$$

Proof. See [Ser88], prop. 2 and thm. 1, p. 10 and corollary p. 17, we sketch below the argument. □

Remark 5.3. We notice that if $\deg \mathcal{L} < 0$, then $\mathrm{H}^0(X, \mathcal{L}) = 0$. Using the duality theorem, we deduce that if $\deg \mathcal{L} > 2g - 2$, $\mathrm{H}^1(X, \mathcal{L}) = 0$ and $\dim_k \mathrm{H}^0(X, \mathcal{L}) = deg\mathcal{L} - g + 1$.

Remark 5.4. A non-singular curve needs not necessarily be smooth in caracteristic p. For example let $k = \mathbb{F}_p(t)$, and consider the curve of equation $Y^2 = X^p - t$. This curve is regular at $Y = 0$ but not smooth.

5.8.2. *Explicit definition of the cohomology.* For all $x \in X$, we let $\mathcal{O}_x$ be the completed local ring at x and K_x be its fraction field and t_x be a uniformizing element in $\mathcal{O}_x$.

We let $\mathbb{A}_K$ be the ring of adeles of K. For a divisor $D = \sum n_x x$, we let $\hat{\mathcal{O}}(D) = \{(f_x) \in \mathbb{A}_K,\ v_x(f_x) + n_x \geq 0\} = \prod_x t_x^{-n_x}\mathcal{O}_x$.

Then we have an exact sequence :

$$0 \to \mathrm{H}^0(X, \mathscr{O}_X(D)) \to K \to \mathbb{A}_K/\hat{\mathcal{O}}(D) \to \mathrm{H}^1(X, \mathscr{O}_X(D)) \to 0$$

Indeed, we can consider the following resolution of the sheaf $\mathscr{O}_X(D)$ by skyscraper sheaves (which are acyclic):

$$0 \to \mathscr{O}_X(D) \to (\iota_\eta)_\star K \to \oplus_{x\in X}(\iota_x)_\star K_x/t_x^{-n_x}\mathcal{O}_x \to 0$$

where $\mathcal{O}_x = \hat{\mathscr{O}_{X,x}}$ and t_x is a uniformizing element, $\iota_\eta : \eta \to X$ is the inclusion of the generic point and $\iota_x : x \to X$ is the inclusion of the closed point x.

Remark 5.5. One can therefore interpret $\mathrm{H}^1(X, \mathscr{O}_X)$ as measuring the obstruction to construct a global rational function whose polar part has been given at a finite set of points.

Lemma 5.6. *We have* $\mathrm{H}^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = 0$.

Proof. We need to see that the map $K \to \mathbb{A}_K/\prod_x \mathscr{O}_x = \oplus K_x/\mathscr{O}_x$ is surjective. We may assume that $k = \bar{k}$. Let $x \in \mathbb{P}^1$. If $x \in \mathbb{A}^1$, then $K_x/\mathscr{O}_x \simeq \{\frac{a_1}{(T-x)} + \frac{a_2}{(T-x)^2} + \cdots, a_i \in k\}$. If $x = \infty$, $K_x/\mathscr{O}_x = \{a_1 T + a_2 T^2 + \cdots, a_i \in k\}$. Moreover, we see that $\frac{a_1}{(T-x)} + \frac{a_2}{(T-x)^2} + \cdots \in K_x/\mathscr{O}_x \subseteq \mathbb{A}_K/\prod_x \mathscr{O}_x$ is the image of $\frac{a_1}{(T-x)} + \frac{a_2}{(T-x)^2} + \cdots \in K$ and similarly $\frac{a_1}{(T-x)} + \frac{a_2}{(T-x)^2} + \cdots \in K_x/\mathscr{O}_x \subseteq \mathbb{A}_K/\prod_x \mathscr{O}_x$ is the image of $a_1 T + a_2 T^2 + \cdots \in K$. $\square$

Lemma 5.7. *The k-vector spaces* $\mathrm{H}^i(X, \mathcal{L})$ *are finite dimensional. Let* $g = \dim_k \mathrm{H}^1(X, \mathscr{O}_X)$ *be the genus of the curve. We have* $\dim_k \mathrm{H}^0(X, \mathcal{L}) - \dim_k \mathrm{H}^1(X, \mathcal{L}) = \deg(\mathcal{L}) - g + 1$.

Proof. Id $D \geq D'$ and we set $D - D' = \sum n_x x$, we have an exact sequence :

$$0 \to \mathscr{O}_X(D') \to \mathscr{O}_X(D') \to \oplus_x(\iota_x)_\star \mathscr{O}_x/t_x^{n_x} \to 0$$

It follows that $\mathrm{H}^i(X, \mathscr{O}_X(D))$ is finite dimensional if and only if $\mathrm{H}^i(X, \mathscr{O}_X(D'))$ is and in this case,

$$\dim_k \mathrm{H}^0(X, \mathscr{O}_X(D)) - \dim_k \mathrm{H}^1(X, \mathscr{O}_X(D)) =$$
$$\dim_k \mathrm{H}^0(X, \mathscr{O}_X(D')) - \dim_k \mathrm{H}^1(X, \mathscr{O}_X(D')) - \deg(D) + \deg(D').$$

This settles the case of $\mathbb{P}^1$. In the case of a general X, it suffices to prove that $\mathrm{H}^1(X, \mathcal{L})$ is finite dimensional. Take a finite map $f : X \to \mathbb{P}^1$. We have that $\mathrm{H}^1(X, \mathcal{L}) = \mathrm{H}^1(\mathbb{P}^1, f_\star \mathcal{L})$ and we will see later that $f_\star \mathcal{L}$ is a direct sum of line bundles, so it is finite dimensional. $\square$

5.8.3. *Duality.* We follow here [Tat68]. We first construct the dualizing sheaf $J_{X/k}$ as follows. At the generic point, this is the sheaf of continuous linear forms :

$$\ell : K\backslash\mathbb{A}_K \to k$$

On some open U, we let $J_{X/k}(U) = \{\ell : K\backslash\mathbb{A}_K/\prod_{x\in U}\mathcal{O}_x \to k\}$. We see that by definition, $\mathrm{H}^0(X, J_{X/k}(-D)) = \mathrm{H}^1(X, \mathscr{O}_X(D))^\vee$. We now assume that the curve is smooth over k. In such a case, the following holds

Theorem 5.6 ([Tat68], [Ser88]).
(1) *For all $x \in X$, there is a local residue map : $res_x : \Omega^1 K_x/k \to k$,*
(2) *For all $x \in X(k)$, we have $K_x = k((t_x))$ and $res_x(\sum a_n t_x^n dt_x) = a_{-1}$,*
(3) *For all $\omega \in \Omega^1_{K/k}$, $\sum_x res_x(w) = 0$.*

This theorem implies that there is an isomorphism given by the residue :

$$\begin{array}{rcl} \Omega^1_{X/k} & \to & J_{X/k} \\ \omega & \mapsto & \sum res_x(f_x\omega) \end{array}$$

6. Lecture V : Algebraic spaces

References : [Sta13], Tag 025R, [LMB00], chapitre I. Let Space be the category of functors $Sch^{opp} \to SET$, which are sheaves for the fppf topology. The objects of this category will be called spaces. We also note that these functors are completely determined by their value on affine schemes.

Remark 6.1. Some authors use instead the étale topology.

As we have seen, there is a fully faithfull functor $Sch \to \text{Space}$, $X \mapsto X(-)$.

Remark 6.2. There is also a variant where one fixes a base scheme S. We let S-Space be the category of functors $Sch/S^{opp} \to SET$ which are sheavs for the fppf topology. For simplicity in this lecture we assume that $S = \text{Spec } \mathbb{Z}$.

6.1. **Second definition of a Scheme.** We can give a more categorical definition of schemes. The basic objects are affine schemes.

6.1.1. *Restricting to affine schemes as test objects.* Let $Ring^{op}$ be the category of affine schemes. We turn it into a site by using the $fppf$ topology (for affine schemes). We can consider the category Space$'$ of sheaves on $Ring^{op}$ for the $fppf$ topology. We have a functor Space $\to$ Space$'$.

Lemma 6.1. *The functor* Space $\to$ Space$'$ *is an equivalence.*

Proof. We produce a quasi-inverse. Let $F \in \text{Space}'$. We need to find the value $F(X)$ on an arbitrary scheme X. Take an $fppf$ cover $\{U_i \to X\}$ by affine schemes. Take an $fppf$-cover $\{U_{i,j,k} \to U_i \times_X U_j\}$ by affine scheme. Define $F(X)$ by the exact sequence :

$$0 \to F(X) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j,k} F(U_{i,j,k})$$

We need to prove this is independant of choices. We can reduce to the situation where we have a covering $\{V_l \to X\}_l$ and a covering $\{V_{l,m,n} \to V_l \times_X V_m\}$ and maps $V_l \to U_{\phi(l)}$ and $V_{l,m,n} \to U_{\phi(l),\phi(m),\phi(n)}$ (for a map ϕ on indices).

We may define $F'(X)$ as follows : $0 \to F'(X) \to \prod_l F(V_l) \rightrightarrows \prod_{l,m,n} F(V_{l,m,n})$. There is an injective map $F(X) \to F'(X)$. This map is an isomorphism since we have an exact diagram

$$0 \to F(U_i) \to \prod_{l,\phi(l)=i} F(V_l) \rightrightarrows \prod_{l,m} F(V_l \times_{U_i} V_m)$$

as well as maps $V_l \times U_i V_m \to V_l \times_X V_m$, we can prove that any section $(s_l) \in F'(X) \subset \prod_l F(V_l)$ will descend to a section $(s_i) \in \prod_i F(U_i)$ and then to $F(X)$. ☐

6.1.2. *Open subscheme of an affine scheme.* We now define open subschemes of an affine schemes. Let $X = \text{Spec } A$ be an affine scheme. An open subscheme U of X is a subsheaf $U \hookrightarrow X$, with the property that there is a set I, elements $f_i \in A$, and maps $\text{Spec } A[1/f_i] \to U \to X$ (where the composite $\text{Spec } A[1/f_i] \to X$ is the natural map) and the map $\coprod_i \text{Spec } A[1/f_i] \to U$ is an epimorphism.

6.1.3. *Open subspace of a space.* We can now define open subspaces of a space. Let X be a space. Let $U \hookrightarrow X$ be a subsheaf. We say that U is an open subspace of X if for any ring A and map $\mathrm{Spec}\ A \to X$, the fiber product $U \times_X \mathrm{Spec}\ A$ is an open subscheme of $\mathrm{Spec}\ A$.

6.1.4. *Schemes.* We next define schemes. A scheme X is a space such that there exists a set I, rings A_i for $i \in I$, and maps $\mathrm{Spec}\ A_i \to X$ such that :

(1) $\mathrm{Spec}\ A_i$ is an open subspace of X,
(2) The map $\coprod_i \mathrm{Spec}\ A_i \to X$ is an epimorphism.

6.2. **Algebraic spaces.**

6.2.1. *Representable maps.* Let $X \to X'$ be a map in Space. We say that the map is representable if for any scheme Y and map $Y \to X'$, the fiber product $X \times_{X'} Y$ is representable by a scheme.

Let F be a space. We often ask that the diagonal map $F \to F \times F$ is representable for the following reason :

Lemma 6.2. *Let F be a space with representable diagonal map. Then for any scheme X, any map $X \to F$ is representable.*

Proof. Let $X \to F$ be a map. Let $X' \to F$ be another map from a scheme X'. Then we find that there is a Cartesian diagram :

$$\begin{tikzcd} X \times_F X' \arrow[r] \arrow[d] & F \arrow[d] \\ X \times X' \arrow[r] & F \times F \end{tikzcd}$$

□

Given a property (P) of a map of schemes that is stable under base change and $fppf$-local on the base, we say that a map of spaces $f : X \to X'$ has property (P) if it is representable and for any ring A and any map $\mathrm{Spec}\ A \to X'$, $\mathrm{Spec}\ A \times_{X'} X \to \mathrm{Spec}\ A$ has this property.

6.2.2. *Algebraic spaces.* We say that a space X is an algebraic space if :

(1) The diagonal of X is representable,
(2) there exists a scheme X', and a map :
$$X' \to X$$
of spaces which is surjective and étale.

The map $X' \to X$ is called a presentation of X.

Remark 6.3. Some authors add the condition that the diagonal morphism is quasi-compact.

6.2.3. *Equivalence relations.* Here is another way to think about an algebraic space. Let X be a scheme. Let $R \subseteq X \times X$ be a monomorphism of schemes (this means that the map of corresponding spaces is injective). We let s and t be the two projections $R \to X$. We say that R is an étale equivalence relation on X if :

(1) For all $T \in Sch$, $R(T)$ is an equivalence relation on $X(T)$,
(2) The maps s and t are étale.

Consider the following data :

(1) A scheme X,

(2) An étale equivalence relation $R \hookrightarrow X \times X$.

Let us consider the space X/R associated to the presheaf $S \mapsto X(S)/R(S)$ where $X(S)/R(S)$ means the quotient of the set $X(S)$ by the equivalence relation $R(S)$.

Theorem 6.1 ([Sta13], Theorem Tag 02WW)**.** *The space X/R is an algebraic space and $X \to X/R$ is a presentation of X/R.*

Proof. We will give a complete proof in the case that X and R are affine. For the general case see [Sta13]. The map $X \to X/R$ is surjective. Let us prove that it is representable by an étale map. Let $Y = \operatorname{Spec} A \to X/R$. We need to prove that $Y \times_{X/R} X$ is a scheme and that the map $Y \times_{X/R} X$ is an étale map. First assume that $Y \to X/R$ lifts to a map $Y \to X$. Then $Y \times_{X/R} X = Y \times_X X \times_{X/R} X = Y \times_X R$ is a scheme and the map $Y \times_X R \to Y$ is a base change of an étale map, hence is étale. In general, we can find an fppf covering $\{\phi_i : Y_i \to Y\}$ with the property that $Y_i \to X/R$ lifts to a map $Y_i \to X$. We see that $Y_i \times_{X/R} X := T_i$ is a scheme and that the map $T_i \to Y_i$ is an étale map. Moreover, the maps $T_i \to Y_i$ have a descent datum relatively to $\{\phi_i : Y_i \to Y\}$. Namely, we have an obvious isomorphism $\psi_{i,j} : T_i \times_Y Y_j = Y_i \times_{X/R} X \times_Y Y_j \to Y_i \times_Y T_j = Y_i \times_Y Y_i \times_{X/R} X$. By descent (the affine case), the space $Y \times_{X/R} X$ is a scheme. Moreover, this is an étale scheme as the property of being étale is fpqc local on the target ([Sta13], Tag 02YJ). Hence the space $Y \times_{X/R} X$ is a scheme, étale over Y. We finally need to prove that the map $X/R \to X/R \times X/R$ is representable. We have a diagram :

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & X \times X \arrow[d] \\
X/R \arrow[r] & X/R \times X/R
\end{tikzcd}$$

Let $Y \to X/R \times X/R$ be a map. We need to see that $Y \times_{X/R \times X/R} X$ is representable by a quasi-compact scheme. We first suppose that we have a lift $Y \to X \times X$. We then reduce to this case by fppf descent.

$\square$

Let F be an algebraic space and $X \to F$ be a presentation. Let $R = X \times_F X$.

Proposition 6.1. *The scheme R is an étale equivalence relation on X. Moreover, $F = X/R$.*

Proof. We have a cartesian square :

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & X \arrow[d] \\
X \arrow[r] & F
\end{tikzcd}$$

which proves that s, t are étale. Clearly, R is an equivalence relation, since $R(T) = \{(x, y) \in X(T) \times X(T),\ x = y \in F(T)\}$. Finally the sheaf F is the quotient of X by the equivalence relation R. In other words, we claim that F is the sheafification of $T \mapsto X(T)/R(T)$. The map $X \to F$ is an epimorphism. It factors through an epimorphism $X/R \to F$. We prove that this is a monomorphism. Let $s_1, s_2 \in X/R(T)$. Assume that they have the same image in $F(T)$. There is $\{T_i \to T\}_i$ an $fppf$ covering such that we have lifts $\tilde{s_1}|_{T_i}$ and $\tilde{s_2}|_{T_i} \in X(T_i)$. They have the same image in $F(T_i)$, so $(\tilde{s_1}|_{T_i}, \tilde{s_2}|_{T_i}) \in R(T_i)$. It follows that $s_1|_{T_i} = s_2|_{T_i} \in X/R(T_i)$. Since X/R is a sheaf $s_1 = s_2$.

$\square$

6.2.4. *Example 1.* Consider the étale morphism $\mathbb{Z}_p \to \mathbb{Z}_{p^2}$, with group $\mathbb{Z}/2\mathbb{Z}$. Let $X = \mathrm{Spec}\ \mathbb{Z}_p$, $X' = \mathrm{Spec}\ \mathbb{Z}_{p^2}$. Let $R = X' \times_X X' \hookrightarrow X' \times X'$. Then $R = X' \times \mathbb{Z}/2\mathbb{Z}$. We now modify the equivalence relation as follows : we consider $R' \subseteq R$ to be the union of $X' \times \{1\}$ and $\mathrm{Spec}\ \mathbb{Q}_{p^2} \times \{\sigma\}$. This is an equivalence relation. The quotient $X'/R' = X''$ is an algebraic space and not a scheme. If X'' where a scheme, it would be an affine scheme (it has only two points, a special point and a generic point !). But then R' should be closed in $X' \times X'$, which is not the case. Let A be a $\mathbb{Z}_p$-algebra. If A is a $\mathbb{Q}_p$-algebra, we have that $X''(A) = \mathrm{Hom}(\mathbb{Q}_p, A) = A$. If $\mathrm{Hom}(\mathbb{Q}_p, A) = 0$ (for example if A is a p^n-torsion ring), we have that $X''(A) = \mathrm{Hom}(\mathbb{Z}_{p^2}, A)$.

We have a chain of étale maps $X' \to X'' \to X$.

6.2.5. *Example 2.* Let $X = \mathrm{Spec}\ \bar{\mathbb{F}}_p$. We have the Frobenius $\phi : X \to X$. We consider $X/\phi^{\mathbb{Z}}$. Remark that $X \times X = \mathrm{Spec}\ \mathcal{C}^0(\hat{\mathbb{Z}}, \bar{\mathbb{F}}_p) = \lim_N \mathbb{Z}/N\mathbb{Z}_{\mathrm{Spec}\ \bar{\mathbb{F}}_p}$ is a profinite set. Thus we consider the equivalence relation which is given by the embedding of $\mathbb{Z}_{\mathrm{Spec}\ \bar{\mathbb{F}}_p} \hookrightarrow \hat{\mathbb{Z}}_{\mathrm{Spec}\ \bar{\mathbb{F}}_p}$. We have a chain of maps $X \to X/\phi^{\mathbb{Z}} \to \mathrm{Spec}\ \mathbb{F}_p$. Thus, the Galois group of $X/\phi^{\mathbb{Z}}$ is $\mathbb{Z}$!

Similarly, we can consider $\mathrm{Spec}\ W(\bar{\mathbb{F}}_p)[1/p]/\phi^{\mathbb{Z}}$. The Galois group of this algebraic space is the Weil group $W_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}_p}$.

7. Lecture VI : Generalities on Vector bundles

7.1. **Vector bundles and locally free sheaves.**

Definition 7.1. *Let X be a scheme. Let $r > 0$ be an integer. A geometric vector bundle of rank r over X is a scheme $p : V \to X$ such that there is a Zariski covering $X = \cup_i U_i$ and isomorphisms of U_i-schemes $c_i : V|_{U_i} := V \times_X U_i \xrightarrow{\sim} \mathbb{A}^r_{U_i}$ such that for every affine open subset $U = \mathrm{Spec}A \subseteq U_i \cap U_j$, the automorphism $c_i \circ c_j^{-1}$ of $\mathbb{A}^r_U$ is linear, i.e., there is a matrix $(a_{l,k}) \in \mathrm{GL}_r(A)$ such that the map between the global sections of $c_i \circ c_j^{-1}$ is given by*

$$\begin{array}{rcl} A[T_1, \cdots, T_r] & \to & A[T_1, \cdots, T_r] \\ T_k & \mapsto & \sum a_{l,k} T_l. \end{array}$$

Remark 7.1. Clearly, $\mathbb{A}^r_X$ is a geometric vector bundle (by choosing the Zariski covering X itself). We call $\mathbb{A}^r_X$ the trivial vector bundle of rank r. In this case, for every scheme $T \to X$, $\mathbb{A}^r_X(T) = H^0(T, \mathscr{O}_T)^r$.

From now, we denote X be a scheme and $\mathcal{E}$ be a locally free sheaf of finite rank r over X.

Definition 7.2.
1. *We define $\mathcal{E}^\vee = \underline{\mathrm{Hom}}(\mathcal{E}, \mathscr{O}_X)$ to be the dual of $\mathcal{E}$.*
2. *We define $\mathrm{Sym}(\mathcal{E})$ (resp. symmetric powers $\mathrm{Sym}^k\mathcal{E}$) be the sheafification of the presheaf $U \mapsto \mathrm{Sym}\ \mathcal{E}(U)$ (resp. $U \mapsto \mathrm{Sym}^k\mathcal{E}(U)$).*

Remark 7.2.
1. These definitions can be defined for every sheaf over X.
2. If $U = \mathrm{Spec}A \hookrightarrow X$ an open immersion, then $\mathcal{E}^\vee(\mathrm{Spec}A) \simeq \mathrm{Hom}(\mathcal{E}(\mathrm{Spec}A), A)$.
3. We have $\mathrm{Sym}\ \mathcal{E} \simeq \bigoplus_{k\geq 0} \mathrm{Sym}^k\mathcal{E}$ and this is a sheaf of $\mathscr{O}_X$-algebra.
4. Clearly, if $\mathcal{E}$ is a finite locally free $\mathscr{O}_X$-module, then so is $\mathcal{E}^\vee$ and $(\mathcal{E}^\vee)^\vee \simeq \mathcal{E}$. Moreover, if $h : T \to X$ be a morphism of schemes, then $h^*(\mathcal{E}^\vee) = (h^*\mathcal{E})^\vee$.
5. For every sheaf of $\mathscr{O}_X$-algebra $\mathcal{B}$, $\mathrm{Hom}_{\mathscr{O}_X\text{-alg}}(\mathrm{Sym}\ \mathcal{E}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{E}, \mathcal{B})$.

Lemma 7.1 (see [Gro61], Proposition 1.3.1)**.** *Let X be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of $\mathscr{O}_X$-algebra. Then there exists a unique X-scheme up to a unique isomorphism over X, denoted by $p : \mathrm{Spec}\mathcal{A} \to X$ such that $p_*(\mathscr{O}_{\mathrm{Spec}\mathcal{A}}) = \mathcal{A}$.*

Proof. (Sketch) Let $X = \cup_i U_i$ be a affine covering. Since $\mathcal{A}(U_i)$ is an $\mathscr{O}_X(U_i)$-algebra, there is a map $\mathrm{Spec}\mathcal{A}(U_i) \to U_i$. By using quasi-coherent property, we can glue these maps to get a X-scheme $p : \mathrm{Spec}\mathcal{A} \to X$. □

Lemma 7.2 (see [GW10], Proposition 11.1)**.** *Let X be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of $\mathscr{O}_X$-algebra. Then for every X-scheme $f : T \to X$,*

$$\mathrm{Hom}_X(T, \mathrm{Spec}\mathcal{A}) \simeq \mathrm{Hom}_{\mathscr{O}_X\text{-}alg}(\mathcal{A}, f_*\mathscr{O}_T).$$

Definition 7.3. *We denote $\mathbb{V}(\mathcal{E}) := \mathrm{Spec}(\mathrm{Sym}(\mathcal{E}^\vee))$.*

Remark 7.3. If $U = \mathrm{Spec}A \hookrightarrow X$ is an open immersion, then $\mathbb{V}(\mathcal{E})_{|U} = \mathrm{Spec}(\mathrm{Sym}\ \mathcal{E}^\vee(U))$.

Lemma 7.3. $\mathbb{V}(\mathcal{E})$ *is a geometric vector bundle.*

Proof. Since $\mathcal{E}$ is locally free of finite rank, we can choose an affine covering $X = \cup U_i$, $U_i = \mathrm{Spec}A_i$ such that $\mathcal{E}|_{U_i} \simeq \mathscr{O}_{U_i}^r \simeq \widetilde{A_i^r}$. Then $\mathcal{E}^\vee(U_i) \simeq \mathcal{E}(U_i)^\vee \simeq (A_i^r)^\vee \simeq A_i^r$. Then there is a isomorphisms

$$c_i : \mathbb{V}(\mathcal{E})|_{U_i} = \mathrm{Spec}(\mathrm{Sym}\ \mathcal{E}^\vee(U_i)) \simeq \mathrm{Spec}(\mathrm{Sym}\ A_i^r) \simeq \mathrm{Spec}(A_i[T_1, \dots, T_r]) = \mathbb{A}^r_{U_i}.$$

For each $\mathrm{Spec}A \hookrightarrow \mathrm{Spec}A_i \times_X \mathrm{Spec}A_j$, $c_j \circ c_i^{-1}$ is given by a linear change of coordinate. Thus, $\mathbb{V}(\mathcal{E})$ is a geometric vector bundle. □

Lemma 7.4. *Let $h : T \to X$ be a morphism. We have:*

$$\mathrm{Hom}_X(T, \mathbb{V}(\mathcal{E})) = \mathrm{H}^0(T, h^\star(\mathcal{E})).$$

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_X(T, \mathbb{V}(\mathcal{E})) &= \mathrm{Hom}_{\mathscr{O}_X-alg}(\mathrm{Sym}(\mathcal{E}^\vee), h_\star\mathscr{O}_T) \quad \text{(by Lemma 7.2)} \\
&= \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{E}^\vee, h_\star\mathscr{O}_T) \quad \text{(by Remark 7.2 (5))} \\
&= \mathrm{Hom}_{\mathscr{O}_T}(h^\star(\mathcal{E}^\vee), \mathscr{O}_T) \\
&= (h^*(\mathcal{E}^\vee))^\vee(T) \\
&= \mathrm{H}^0(T, h^\star(\mathcal{E})) \quad \text{(by Remark 7.2 (4))}.
\end{aligned}$$

□

Proposition 7.1. *The assignment $\mathcal{E} \mapsto \mathbb{V}(\mathcal{E})$ defines a covariant functor from the category of finite locally free sheaves of $\mathscr{O}_X$-modules of constant rank to the category of geometric vector bundles with linear morphisms. This functor is an equivalence of category.*

Remark 7.4. The definition of linear morphisms between geometric vector bundles will be defined in the proof of the proposition.

Proof. We construct a functor in the other direction. Let $p : V \to X$ be a geometric vector bundle of rank r. Let $\mathcal{S}(V)$ be the sheaf on X defined by $\mathcal{S}(V)(U) = \{s : U \to V|_U, p \circ s = \mathrm{Id}_U\}$. This is a locally free sheaf of $\mathscr{O}_X$-module. Indeed, there is a Zariski cover $X = \cup U_i$ such that $V_{|U_i} = \mathbb{A}^r_{U_i}$. Then we have $\mathcal{S}(V)_{|U_i} \simeq \mathscr{O}^r_{X|U_i}$ since for every open subset U in U_i,

$$\begin{aligned}
\mathcal{S}(V)_{|U_i}(U) &\simeq \{s : U \to \mathbb{A}^r_U, p \circ s = \mathrm{Id}_U\} \\
&\simeq \{t : U \to \mathbb{A}^r \simeq \mathrm{Spec}\ \mathbb{Z}[T_1, \dots, T_r]\} \\
&\simeq \{t^* : \mathrm{Spec}\ \mathbb{Z}[T_1, \dots, T_r] \to \mathscr{O}_X(U)\} \simeq \mathscr{O}_X(U)^r.
\end{aligned}$$

We now check that $\mathcal{S}(\mathbb{V}(\mathcal{E})) = \mathcal{E}$. By Lemma 7.4, for every open subset $U \subset X$,

$$\mathcal{S}(\mathbb{V}(\mathcal{E}))(U) = \{s : U \to \mathbb{V}(\mathcal{E})_{|U}, p \circ s = \mathrm{Id}_U\} \simeq \{s : U \to \mathbb{V}(\mathcal{E}_{|U})\} \simeq \mathcal{E}(U).$$

Conversely, let $p : V \to X$ be a geometric vector bundle. Since the construction of $\mathcal{S}(V)$ is competible with base change, i.e., for a morphism $h : V' \to X$,

$$h^*\mathcal{S}(V) \simeq \mathcal{S}(V \times_X V'),$$

we have

$$\mathrm{Hom}_X(V, \mathbb{V}(\mathcal{S}(V)) \simeq H^0(V, p^*\mathcal{S}(V)) \simeq H^0(V, \mathcal{S}(V \times_X V)) \simeq \{s : V \to V \times_X V, p \circ s = \mathrm{Id}_V\}.$$

Then the diagonal map $V \to V \times_X V$ which amounts to a map $\mathrm{H}^0(V, p^\star \mathcal{S}(V))$ or equivalently to a map $V \to \mathbb{V}(\mathcal{S}(V))$. And this map is an isomorphism. This also shows that every geometric vector bundle over X can be of the form $\mathbb{V}(\mathcal{E})$ for some finite locally free sheaf of $\mathscr{O}_X$-module $\mathcal{E}$. Then we call a morphism between two vector bundles $\mathbb{V}(\mathcal{E})$ and $\mathbb{V}(\mathcal{E}')$ is linear if it is induced by a morphism $\mathcal{E} \to \mathcal{E}'$. This defined a category of vector bundles with linear morphisms.

It is left to check $\mathrm{Hom}_{\mathscr{O}_X}(\mathcal{E}, \mathcal{E}') \simeq \mathrm{Hom}(\mathbb{V}(\mathcal{E}), \mathbb{V}(\mathcal{E}'))$. Surjectivity is clear by the definition of linear morphisms between vector bundles. Now let $\varphi, \psi : \mathcal{E} \to \mathcal{E}'$ such that $\mathbb{V}(\varphi) = \mathbb{V}(\psi) : \mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{E}')$. This induces that $\mathrm{Sym}(\varphi^\vee) = \mathrm{Sym}(\psi^\vee) : \mathrm{Sym}(\mathcal{E}'^\vee) \to \mathrm{Sym}(\mathcal{E}^\vee)$. We have the following diagram:

$$\begin{tikzcd}
\mathcal{E}' \arrow[r, "\varphi^\vee{,}\psi^\vee"] \arrow[d] & \mathcal{E}^\vee \arrow[d, "\mathrm{Sym}"] \\
\mathrm{Sym}(\mathcal{E}'^\vee) \arrow[r] & \mathrm{Sym}(\mathcal{E}^\vee)
\end{tikzcd}$$

Then $\mathrm{Sym} \circ \varphi^\vee = \mathrm{Sym} \circ \psi^\vee$. On the other hand, by checking at stalks and property that stalk of a direct sum is the sum of stalks, $\mathrm{Sym} : \mathcal{E}^\vee \to \mathrm{Sym}(\mathcal{E}^\vee)$ is injective. So $\varphi^\vee = \psi^\vee : \mathcal{E}'^\vee \to \mathcal{E}^\vee$. Hence $\varphi = \psi$. □

We let $Bun(X)$ be the set of isomorphism classes of vector bundles on X. We let $Bun_n(X)$ be the set of isomorphism classes of rank n vector bundles on X.

7.2. **Operation on vector bundles.** We can perform basic algebra operations on vector bundles. For instance if $\mathcal{E}_1$ and $\mathcal{E}_2$ are vector bundles, so is $\mathcal{E}_1 \otimes \mathcal{E}_2$ and $\mathcal{E}_1 \oplus \mathcal{E}_2$. If $\mathcal{E}$ is a vector bundle, we can consider its dual $\mathcal{E}^\vee$, its exterior powers $\Lambda^i\mathcal{E}$ and symmetric power $\mathrm{Sym}^i\mathcal{E}$. Very important is $\det \mathcal{E} = \Lambda^n \mathcal{E}$ where n is the rank of $\mathcal{E}$.

7.3. **Invertible sheaves.** Vector bundles of rank $r = 1$ on X are called invertible sheaves. In this case we have a monoidal structure : if $\mathcal{E}$ and $\mathcal{E}'$, are invertible sheaves, $\mathcal{E} \otimes_{\mathscr{O}_X} \mathcal{E}'$ is an invertible sheaf. Moreover, if $\mathcal{E}$ is an invertible sheaf, $\mathcal{E}^\vee$ also and $\mathcal{E} \otimes \mathcal{E}^\vee \to \mathscr{O}_X$ is an isomorphism. We let $\mathrm{Pic}(X) = Bun_1(X)$ be the group of isomorphisms classes of invertible sheaves on X.

7.4. **Torsors.** Let X be a topological space and G a sheaf of groups over X. Let T be a sheaf over X. An action of G on T is a morphism of sheaves $G \times T \to T$ such that for any open U of T, $G(U) \times T(U) \to T(U)$ is a group action of $G(U)$ on $T(U)$.

Definition 7.4. *A G-torsor on X is a sheaf T with a G-action such that :*

1. *for all U such that $T(U) \neq \emptyset$, $G(U)$ acts simply transitvely on $T(U)$,*
2. *There is a cover $X = \cup_i U_i$, such that $T(U_i) \neq \emptyset$ for all i.*

Remark 7.5. The condition (1) is equivalent to for all U such that $T(U) \neq \emptyset$, there exists $t \in T(U)$ such that $G(U) \to T(U), g \mapsto g.t$ is a bijection. The condition (2) is equivalent to $T_x \neq 0$ for all $x \in X$.

Definition 7.5. *A morphism of G-torsors is a morphism of sheaves $\varphi : T \to T'$ such that for all open subgroup $U \subset X$, the map $\varphi(U) : T(U) \to T'(U)$ is $G(U)$-equivarent.*

Remark 7.6. (1) Clearly, G is a G-torsor itself by the action $G(U)\times G(U)\to G(U)$, $(g,g')\mapsto gg'$. This torsor is called the trivial G-torsor.

(2) A G-torsor T isomorphic to the trivial torsor if and only if $T(X)\neq\emptyset$. Indeed, the sufficient condition is clear since $T(X)\simeq G(X)\neq\emptyset$. Conversely, for any $t\in T(X)$, we have an isomorphism $G(U)\to T(U), g\mapsto gt_{|U}$ for every open subset U of X.

(3) Morphisms of G-torsors are isomorphisms. Indeed, let $\varphi : T\to T'$ be a morphism of G-torsor. For all open subsets U such that $T(U)\neq\emptyset$, we have the following diagram,

$$\begin{tikzcd} G(U) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & T(U) \\ T'(U) \arrow[ur, "\varphi(U)"'] & \end{tikzcd}$$

Thus, $\varphi : T\to T'$ is an isomorphism.

Let X be a scheme and let $\mathcal{E}$ be a vector bundle of rank r on X. Define

$$T(\mathcal{E})=\underline{\mathrm{Isom}}_{\mathscr{O}_X}(\mathscr{O}_X^r,\mathcal{E}),$$

with action of $G=\mathrm{GL}_r=\underline{\mathrm{Isom}}_{\mathscr{O}_X}(\mathscr{O}_X^r,\mathscr{O}_X^r)$ by $(g,\phi)\mapsto\phi\circ g^{-1}$.

Lemma 7.5. *$T(\mathcal{E})$ is a* GL_r*-torsor.*

Proof. Let U be an open subset of X such that $T(U)\neq\emptyset$ and $\phi_1,\phi_2\in T(U)=\mathrm{Isom}_{\mathscr{O}_X}(\mathscr{O}^r_{X|U},\mathcal{E}_{|U})$ then there exists uniquely $g:=\phi_2^{-1}\circ\phi_1$ in $G(U)$ such that $\phi_1\circ g^{-1}=\phi_2$. And since $\mathcal{E}$ is a vector bundle of rank r, there is a covering $X=\cup U_i$ such that $\mathcal{E}_{|U_i}\simeq\mathscr{O}^r_{X|U_i}$. Then $T(U_i)\simeq\mathrm{Isom}_{\mathscr{O}_X}(\mathscr{O}^r_{X|U_i},\mathscr{O}^r_{X|U_i})$ contains the trivial morphism, hence is not empty. $\square$

We can also think of T as the space of isomorphisms $\mathrm{Isom}_X(\mathbb{V}(\mathscr{O}_X^r),\mathbb{V}(\mathcal{E}))$ between the trivial vector bundles and $\mathbb{V}(\mathcal{E})$.

Conversely, let T be a GL_r-torsor. Then T is representable by a scheme. Indeed, let $X=\cup U_i$ be an open cover such that $T(U_i)\neq\emptyset$ for all i. Then using $T_{|U_i}\xrightarrow{\sim}G_{|U_i}$, we can glue them to get a scheme. We can define a geometric vector bundle $\mathcal{V}(T)$ on X via the following rule :

$$\mathcal{V}(T)=(T\times_X\mathbb{V}(\mathscr{O}_X^r))/\mathrm{GL}_r$$

for the diagonal action $(\phi,v)g=(\phi\circ g,g^{-1}v)$.

Lemma 7.6. *The space $\mathcal{V}(T)$ is a geometric vector bundle.*

Proof. By working locally on X, we may assume that $T=\mathrm{GL}_r$. In this case, we have a morphism : $T\times\mathbb{V}(\mathscr{O}_X^r)\to T\times\mathbb{V}(\mathscr{O}_X^r)$, $(t,v)\mapsto(t,tv)$. This isomorphism intertwins the diagonal action of GL_r and the action of GL_r on T. We deduce that $\mathcal{V}(T)\simeq T/\mathrm{GL}_r\times\mathbb{V}(\mathscr{O}_X^r)$. $\square$

Proposition 7.2. *The above rule defines an equivalence of categories between :*

$$\begin{array}{rcl} \{\textit{vector bundles of rank } r\textit{, maps are isomorphisms}\} & \to & \{\mathrm{GL}_r\textit{-torsors}\} \\ \mathcal{E} & \mapsto & T(\mathcal{E}) \\ \mathcal{V}(T) & \leftarrow\!\shortmid & T \end{array}$$

Proof. Let $T=T(\mathcal{E})$. To any point $t\in T$, we have a morphism $\phi(t): V_{\mathrm{triv}}\to\mathbb{V}(\mathcal{E})$. Moreover, $\phi(t\circ g)=\phi(t)\circ g$. Thus, there is a map

$$\begin{array}{rcl} T\times_X V_{triv} & \to & T\times_X\mathbb{V}(\mathcal{E}) \\ (t,x) & \mapsto & (t,\phi(t).x) \end{array}$$

This map is equivariant for the diagonal GL_r-action on the left hand side and the GL_r-action on T on the right hand side. Passing to the quotient, we deduce that $\mathcal{V}(T) = \mathbb{V}(\mathcal{E})$. □

7.5. **Čech cohomology.** One can describe the set of isomorphism classes of vector bundles using Čech cohomology.

Definition 7.6. *Let G be a sheaf of groups over X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. A Čech 1-cocycle on $\mathcal{U}$ is a collection $\theta = (g_{i,j})_{i,j\in I}$, where $g_{i,j} \in G(\mathcal{U}_i \cap U_j)$ such that*

$$g_{i,j}g_{j,k} = g_{i,k} \text{ on } U_i \cap U_j \cap U_k \quad \forall i,j,k.$$

Two cocycles $(g_{i,j})$ and $(g'_{i,j})$ are cohomologous when there exists $(h_i)_{i\in I}$ with $h_i \in G(U_i)$ such that $g_{i,j} = h_i^{-1}g'_{i,j}h_j$ on $U_i \cap U_j$. This is an equivalence relation. We define :

$$\check{\mathrm{H}}^1(\mathcal{U}, G) = \{cocycles\}/\sim$$

Remark 7.7. (1) If $(g_{i,j})_{i,j\in I}$ is a Čech-cocycle, then $g_{i,i} = 1$ and $g_{i,j} = g_{j,i}^{-1}$ for all i,j.
(2) $\check{\mathrm{H}}^1(\mathcal{U}, G)$ is a pointed set. The point being given by the cocycle $g_{i,j} = \mathrm{Id}$.

Lemma 7.7. *There is a bijection:*

$$\{\text{Isomorphism classes of } G\text{-torsors, trivial with respect to } \mathcal{U}\} \to \check{\mathrm{H}}^1(\mathcal{U}, G).$$

Remark 7.8. A G-torsor T is called trivial with respect to $\mathcal{U} = \{U_i\}_{i\in I}$ if $T_{|U_i}$ is isomorphic the trivial $G_{|U_i}$-torsor for all i. By Remark 7.6 (2), this equivalent to $T(U_i) \neq \emptyset$ for all i.

Proof. Let T be the G-torsor is trivial with respect to $\mathcal{U}$. Let $t_i \in T(U_i)$. Since $G(U_i \cap U_j)$ acts simply transitively on $T(U_i \cap U_j)$, there exists unique $g_{i,j} \in G(U_i \cap U_j)$ such that

$$g_{i,j}t_j = t_i \quad \forall i,j.$$

This shows that $g_{i,j}g_{j,k} = g_{i,k}$ on $U_i \cap U_j \cap U_k$ for all i,j,k. Then the collection $(g_{i,j})$ is a cocycle. Different choices of t_i give cohomologous cocycles. Indeed, let $t_i, t'_i \in T(U_i)$ for $i \in I$ then there exists unique $h_i \in G(U_i)$ such that $h_it_i = t'_i$ for all i. Moreover, there exist unique $g_{i,j}$ and $g'_{i,j}$ in $G(U_i, \cap U_j)$ such that $g_{i,j}t_j = t_i$ and $g'_{i,j}t'_j = t'_i$. Thus,

$$g_{i,j} = h_i^{-1}g'_{i,j}h_j \text{ in } U_i \cap U_j \cap U_k \ \forall i,j,k.$$

Conversely, given a cocycle $(g_{i,j})$, we define a torsor T by the following,

$$T(V) = \{(t_i) \in \prod_i G(V \cap U_i), t_it_j^{-1} = g_{i,j}\} \text{ for } V \text{ open subset of } X.$$

This definition does not depend on the representation of $(g_{i,j})$. Indeed, let $(g_{i,j})$ and $(g'_{i,j})$ are cohomologous and T, T' be the corresponding torsors. There exists $h_i \in G(U_i)$ such that $g_{i,j} = h_i^{-1}g'_{i,j}h_j$. Then for evey open subset V in X, $T(V) \xrightarrow{\sim} T'(V), (t_i) \mapsto (h_it_i)$. The G-action is given by $(g,(t_i)) = (t_ig^{-1})$. Thus $T|_{U_i} = G|_{U_i}$ via $t_i \mapsto t_i$. Hence T is a torsor trivial on $\mathcal{U}$.

Now we show that each map is the inverse of each other. Let T be a G-torsor trivial on $\mathcal{U}$. Let $t_i \in T(U_i)$ and $(g_{i,j})$ be the cocycle given by (t_i). Denote T' be the G-torsor corresponding to $(g_{i,j})$. In order to prove T is isomorphic to T', it suffices to construct a morphism between them. And to define maps $T'_{|U_i} \to T_{|U_i}$, it is enough to define the maps on global section $T'(U_i) \to T(U_i)$ because T and T' are trivial on $\mathcal{U}$. For all $i \in I$, we define

$$\varphi_i : T'(U_i) \subset \prod_j G(U_i \cap U_j) \to T(U_i), (v_j)_j \mapsto v_i^{-1}t_i.$$

This map is G-equivariant since for any $g \in G$, $(v_j) \in T'(U_i)$,

$$\varphi_i(g(v_j)) = \varphi_i((v_j g^{-1})) = (v_i g^{-1})^{-1} t_i = g v_i^{-1} t_i = g\varphi_i((v_j)).$$

For $i, j \in I$, we have $\varphi_{i|U_i \cap U_j} = \varphi_{j|U_i \cap U_j}$. Indeed, for every $(v_k)_k \in T'(U_i \cap U_j)$,

$$\varphi_i((v_k)) = v_i^{-1} t_i = v_i^{-1} g_{i,j} t_j = v_i^{-1} v_i v_j^{-1} t_j = v_j^{-1} t_j = \varphi_j((v_k)).$$

So we can glue these map to get a morphism $\varphi : T' \to T$ and this is an isomorphism.

Conversely, let $(g_{i,j})$ is a cocycle and define T be a G-torsor as above. Let $v = (v_j) \in T(U_i)$ with $v_i = 1$ and $v_j = g_{i,j}^{-1}$ for all $j \neq i$. Let $\tilde{v} = (\tilde{v}_j) \in T(U_k)$ with $\tilde{v}_k = 1$ and $\tilde{v}_j = g_{k,j}^{-1}$ for all $j \neq k$. Now let $g'_{i,k} \in G(U_i \cap U_k)$ such that $g'_{i,k}\tilde{v} = v$. Then $\tilde{v}_k {g'}_{i,k}^{-1} = v_k$, then ${g'}_{i,k}^{-1} = g_{i,k}^{-1}$. Hence $g_{i,j}$ and $g'_{i,k}$ are cohomologous. □

We say that $\mathcal{U}'$ is a *refinement* of $\mathcal{U}$ if $\mathcal{U}' = \{U'_j\}_{j \in J}$ and there is a map $\alpha : J \to I$ with $U'_j \subseteq U_{\alpha(j)}$. This induces a map

$$\check{\mathrm{H}}^1(\mathcal{U}, G) \to \check{\mathrm{H}}^1(\mathcal{U}', G), (g_{i,j})_{i,j \in I} \mapsto \left(g_{\alpha(i),\alpha(j)|U'_i \cap U'_j} \right)_{i,j \in J}$$

Two coverings $\mathcal{U}$ and $\mathcal{U}'$ are called *equivalent* if each one is a refinement of the other. Clearly, if $\mathcal{U}$ and $\mathcal{U}'$ are equivalent, then $\check{\mathrm{H}}^1(\mathcal{U}, G) \simeq \check{\mathrm{H}}^1(\mathcal{U}', G)$. We get a commutative diagram :

$$\begin{CD}
\{\text{Isomorphism classes of } G\text{-torsors, trivial with respect to } \mathcal{U}\} @>>> \check{\mathrm{H}}^1(\mathcal{U}, G) \\
@VVV @VVV \\
\{\text{Isomorphism classes of } G\text{-torsors, trivial with respect to } \mathcal{U}'\} @>>> \check{\mathrm{H}}^1(\mathcal{U}', G)
\end{CD}$$

Definition 7.7. *We call* $\check{\mathrm{H}}^1(X, G) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^1(\mathcal{U}, G)$ *be the Čech cohomology of* G *on* X*, where* U *runs through the set of equivalence classes of open covering of* X*.*

Proposition 7.3. *There is a bijection:*

$$\{\textit{Isomorphism classes of } G\textit{-torsors}\} \to \check{\mathrm{H}}^1(X, G).$$

In particular we deduce that $Bun_n(X) = \check{\mathrm{H}}^1(X, \mathrm{GL}_n)$*.*

Proof. By above commutative diagram, we have a map

$$\varinjlim_{\mathcal{U}} \{\text{Isomorphism classes of } G\text{-torsors, trivial on } \mathcal{U}\} \to \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^1(U, G) = \check{\mathrm{H}}^1(X, G).$$

Then by Lemma 7.7, this is actually a bijection. And by the fact that

$$\{\text{Isomorphism classes of } G\text{-torsors}\} = \varinjlim_{\mathcal{U}} \{\text{Isomorphism classes of } G\text{-torsors, trivial on } \mathcal{U}\},$$

we have the first statement. The second statement is immediately induced by the first statement and Proposition 7.2. □

Corollary 7.1. $\mathrm{Pic}(X) = \check{\mathrm{H}}^1(X, \mathbb{G}_m)$.

Remark 7.9. For every abelian sheaf $\mathcal{F}$ on X, we have

$$\mathrm{H}^1(X, \mathcal{F}) \simeq \check{\mathrm{H}}^1(X, \mathcal{F}).$$

See Lemma 20.4.3 (tag 02FQ), Stack Project to see that

$$\mathrm{H}^1(X, \mathcal{F}) \simeq \{\text{Isomorhism classes of } \mathcal{F}\text{-torsors}\},$$

then by Proposition 7.3, we get the result.

8. Lecture VII : Vector bundles on curves

8.1. **The category** $Coh(X)$**.** We now assume that X is a projective, geometrically connected, non-singular curve over Spec k.

Lemma 8.1. *Let X be a Dedekind scheme. The following are equivalent for a coherent $\mathscr{O}_X$-module $\mathcal{E}$:*

1. *$\mathcal{E}$ is a vector bundle,*
2. *$\mathcal{E}$ is a torsion free $\mathscr{O}_X$-module.*

Proof. All $\mathscr{O}_{X,x}$ are DVR or fields. If $\mathcal{E}$ is a vector bundle, then for all $x \in X$, $\mathcal{E}_x$ is free over $\mathscr{O}_{X_x}$ which is a DVR thus $\mathcal{E}_x$ is torsion free, so $\mathcal{E}$ is torsion free. Conversely, if $\mathcal{E}$ is torsion free, as $\mathscr{O}_{X,x}$ is a PID, $\mathcal{E}_x$ is flat, and of finite presentation ($\mathcal{E}$ is coherent), so it is projective of finite rank, and $\mathscr{O}_{X,x}$ is local so it is free: $\mathcal{E}$ is a vector bundle. $\square$

Let X be a non-singular curve. Let $\mathcal{E}$ be a coherent $\mathscr{O}_X$-module. We have an exact sequence :

$$0 \to \mathcal{E}_{tors} \to \mathcal{E} \to \mathcal{E}^{cotors} \to 0$$

where $\mathcal{E}_{tors}$ is the maximal torsion sub-sheaf of $\mathcal{E}$ and $\mathcal{E}^{cotors}$ is a vector bundle.

We have $\mathrm{Sup}(\mathcal{E}_{tors}) = \{x_1, \cdots, x_n\}$ because $\mathrm{Sup}(\mathcal{E}_{tors})$ is a closed subset (see Stacks tag 01BA) of dimension 0 of X as it does not contain η (the only torsion $\mathscr{O}_{X,\eta}$-module is 0) and $\mathcal{E}_{tors} = \oplus_{i=1}^{n} (\iota_{x_i})_\star M_i$ where M_i is a finite length $\mathscr{O}_{X,x_i}$-module ($A = \mathscr{O}_{X,x_i}$ is a DVR so taking a sequence $0 \subset M_0 \subset \cdots \subset M_n = M := \mathcal{E}_{x_i}$ such that M_{j+1}/M_j is isomorphic to $A/\mathfrak{p}_j$ for $\mathfrak{p}_j \in \mathrm{Spec} A_i$, as A is a DVR, each $\mathfrak{p}_j$ is either 0 or $\mathfrak{m} := \mathfrak{m}_{x_i}$, but a 0 would give a non torsion element, so we get a composition sequence of M_i, which is therefore of finite lenght)

8.2. **Subbundles.** We let $VB(X)$ the full subcategory of $Coh(X)$ whose objects are vector bundles.

Definition 8.1. *Let $\mathcal{E}$ be a vector bundle on X. A subbundle $\mathcal{E}' \subseteq \mathcal{E}$ is a subsheaf such that $\mathcal{E}/\mathcal{E}'$ is a vector bundle.*

We now explain an important construction. Let $\mathcal{E}' \hookrightarrow \mathcal{E}$ be a subsheaf. First, $\mathcal{E}'$ is a vector bundle since it is torsion free. We define a subbundle $(\mathcal{E}')^\sharp$ of $\mathcal{E}$ called the saturation of $\mathcal{E}'$. This is the kernel of the map $\mathcal{E} \to (\mathcal{E}/\mathcal{E}')^{cotors}$. Observe that we have $\mathcal{E}' \subseteq (\mathcal{E}')^\sharp \subseteq \mathcal{E}$.

Lemma 8.2. *We have a bijection :*

$$\{\textit{subbundles of } \mathcal{E}\} \leftrightarrow \{\textit{subvector spaces of the } k(X)\textit{-vector space } \mathcal{E}_\eta\}$$

Proof. Let $\mathcal{E}' \subseteq \mathcal{E}$ be a subbundle. Taking the fiber at the generic point gives a subvector space $\mathcal{E}'_\eta \subseteq \mathcal{E}_\eta$. Conversely, let $V \subseteq \mathcal{E}_\eta$. We define $\mathcal{E}'$ by the short exact sequence :

$$0 \to \mathcal{E}' \to \mathcal{E} \to (\iota_\eta)_\star \mathcal{E}_\eta / V$$

We see that $\mathcal{E}'$ is coherent and saturated in $\mathcal{E}$. $\square$

Corollary 8.1. *Any rank r vector bundle $\mathcal{E}$ on X admits a filtration by subbundles $\{\mathrm{Fil}^i\mathcal{E}\}_{0 \le i \le r}$ with $\mathrm{Fil}^i\mathcal{E}$ of rank $r - i$.*

Proof. Take a full flag of the vector space $\mathcal{E}_\eta$ and apply the lemma. $\square$

8.3. **The category** $VB(X)$**.** The category $VB(X)$ is not abelian.

Theorem 8.1. *The category $VB(X)$ is an additive category which admits kernels and cokernels.*

Let $f : \mathcal{E} \to \mathcal{G}$ be a morphism in the category $VB(X)$. We let $\ker(f)$ be the Kernel of f in the category $Coh(X)$.

Lemma 8.3. *The coherent sheaf $\ker(f)$ is a subbundle of $\mathcal{E}$ and is the kernel of f in $VB(X)$.*

Proof. It is clear that $\ker(f)$ is a vector bundle and it is therefore the kernel of f in $VB(X)$. The map f passes to the quotient to an injective map : $\mathcal{E}/\ker(f) \to \mathcal{G}$. Consider $(\mathcal{E}/\ker(f))^{tors} \to \mathcal{G}$. This map is trivial (there are no nonzero maps from a torsion sheaf to a vector bundle because its image would be a torsion submodule). This proves that $(\mathcal{E}/\ker(f))^{tors} = 0$. $\square$

We let $\mathrm{coim}(f) = \mathcal{E}/\ker(f)$.

Lemma 8.4. *Let $\mathrm{im}(f)$ be the image of f in the category $Coh(X)$. This is a vector bundle. Let $\mathrm{im}(f)^\sharp$ be the saturation of the image of f in $\mathcal{E}$. Then $\mathrm{im}(f)^\sharp$ is the image of f in $VB(X)$. Let $\mathrm{coker}(f)$ be the cokernel of f in the category $Coh(X)$. Then $\mathrm{coker}(f)^{cotors}$ is a vector bundle and is the cokernel in the category $VB(X)$.*

Proof. It suffices to prove the statement regarding $\mathrm{coker}(f)$. We see that we have a map $\mathcal{G} \to \mathrm{coker}(f)$. For any vector bundle $\mathcal{F}$ with a map $\mathcal{G} \to \mathcal{F}$, such that $\mathcal{E} \to \mathcal{F}$ is zero, we have a factorization $\mathcal{G} \to \mathrm{coker}(f) \to \mathrm{coker}(f)/\mathrm{coker}(f)^{tors} \to \mathcal{F}$:

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "f"] \arrow[dr, "0"'] & \mathcal{G} \arrow[r, two heads] \arrow[d, "g"] & \mathrm{coker} f \arrow[r] \arrow[dl, dotted, "\exists ! \varphi \text{ in } Coh(X)"] & \mathrm{coker} f^{cotors} \arrow[dll, dashed, bend left, "\text{exists because } (\mathrm{Im}\varphi)_{tors}=0"] \\
& \mathcal{F} & &
\end{tikzcd}
$$

$\square$

Remark 8.1. If $f : \mathcal{E} \to \mathcal{F}$ is a morphisme in $VB(X)$, we have the commutative diagram with exact columns :

$$
\begin{tikzcd}
0 \arrow[d] & 0 \\
\ker f \arrow[d] & (\mathrm{coker} f)^{cotors} \arrow[u] \\
\mathcal{E} \arrow[r, "f"] \arrow[d] & \mathcal{G} \arrow[u] \\
\mathrm{coIm} f \arrow[r, blue] \arrow[d] & (\mathrm{Im} f)^\sharp \arrow[u] \\
0 & 0 \arrow[u]
\end{tikzcd}
$$

The blue arrow is an isomorphism if and only $\mathrm{Im} f = (\mathrm{Im} f)^\sharp$. In this case we say that f is *strict* morphism.

8.4. **Elementary modifications.** Let $\mathcal{E}$ be a vector bundle of rank r on X. Let $x \in X$ be a closed point and let $\mathscr{O}_x = \widehat{\mathscr{O}_{X,x}} \simeq k(x)[[t_x]]$ (by [Ser68], II, 4, thm 2 in the case that k is perfect, and the reference given there for the general case). Let also $K_x = \mathrm{Frac}(\mathscr{O}_x)$.

A modification of $\mathcal{E}$ at x is a vector bundle $\mathcal{E}'$ such that $\mathcal{E}|_{X\setminus x} = \mathcal{E}'|_{X\setminus x}$.
For A an integral domain of fraction field K, a A-lattice of a K-vector space V is a free A-submodule which spans V as a K-vector space.

Lemma 8.5. *There is a canonical bijection:*

$$\{\textit{modifications of } \mathcal{E} \textit{ at } x\} \leftrightarrow \{\mathscr{O}_x\textit{-lattices in } \mathcal{E}_x \otimes_{\mathscr{O}_{X,x}} K_x\}$$

Proof. Given a modification $\mathcal{E}'$, we get a lattice $\mathcal{E}'_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_x$. Conversely, let Λ_x be a lattice. Replacing $\mathcal{E}$ by $\mathcal{E}(nx)$ for n large enough. We can assume that $\Lambda \subseteq \mathcal{E}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_x$. We consider the short exact sequence :

$$0 \to \mathcal{E}' \to \mathcal{E} \to (i_x)_\star \mathcal{E}_x/\Lambda$$

so that $\mathcal{E}' = \mathcal{E}$ outside of $\{x\}$. □

We observe that the set of lattices in K_x^n is in bijection with $\mathrm{GL}_n(K_x)/\mathrm{GL}_n(\mathscr{O}_x)$.

Corollary 8.2. *We have a bijection :*

$$Bun_n(\mathbb{P}^1) = GL_n(k[x])\backslash GL_n(k((x^{-1})))/GL_n(k[[x^{-1}]]).$$

Proof. Let $\mathcal{E}$ be a rank n vector bundle over $\mathbb{P}^1$. We know that $\mathcal{E}|_{\mathbb{A}^1}$ is trivial, since $k[X]$ is principal. We fix an isomorphism $\phi : \mathcal{E}|_{\mathbb{A}^1} \simeq \mathscr{O}^n_{\mathbb{A}^1}$. Having fixed this isomorphism, $\mathcal{E}$ becomes a modification of the trivial bundle at ∞. Therefore it is given by an element in $GL_n(k((x^{-1})))/GL_n(k[[x^{-1}]])$. Forgetting the trivialization amounts to taking the quotient by $GL_n(k[x])$. □

Corollary 8.3. *The map* $Pic(\mathbb{P}^1) \overset{deg}{\simeq} \mathbb{Z}$ *is an isomorphism.*

8.5. **Weil's formula.**

Theorem 8.2. *There is a bijection :*

$$Bun_n(X) = GL_n(K)\backslash GL_n(\mathbb{A}_K)/\prod_x GL_n(\mathcal{O}_x).$$

Proof. Let $\mathscr{F}$ be a locally free sheaf of rank n. Let $s_1, \cdots, s_n$ be a basis of sections at η (this basis is defined modulo $\mathrm{GL}_n(K)$). Then for all closed points $x \in X$, there is a unique element $f_x \in GL_n(K_x)/GL_n(\mathcal{O}_x)$ and an isomorphism $f_x O_x^n = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_x \hookrightarrow \mathscr{F}_\eta \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_x = K_x^n$ because $\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_x$ is a lattice of $\mathscr{F}_\eta \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_x = K_x^n$. Conversely, given a collection $(f_x) \in GL_n(\mathbb{A}_K)$ we can define the subsheaf of $(\iota_\eta)_\star K^n$ by $\mathscr{F}(U) = \{s \in K^n,\ \forall x \in U,\ s \in f_x \mathcal{O}_x^n\}$. □

Here is a similar, but slightly simpler formula for $\mathbb{P}^1$.

Theorem 8.3.

$$Bun_n(\mathbb{P}^1) = GL_n(k[x^{-1}])\backslash GL_n(k[x,x^{-1}])/GL_n(k[x])$$

Proof. Since $k[x]$ and $k[x^{-1}]$ are principal, any locally free sheaf $\mathscr{F}$ is trivial on Spec $k[x]$ or Spec $k[x^{-1}]$. Elements in $GL_n(k[x,x^{-1}])$ give the gluing data. Namely, we can take a basis $e_1, \cdots, e_n$ of $\mathscr{F}(\text{Spec } k[x])$ and a basis $f_1, \cdots, f_n$ of $\mathscr{F}(\text{Spec } k[x^{-1}])$. Restricting to Spec $k[x,x^{-1}]$, we find a matrix in $GL_n(k[x,x^{-1}])$ which passes from the basis (e_i) to the basis (f_i). □

We let $\mathscr{O}(n)$ be a sheaf of degree n. We have the following theorem of Grothendieck :

Theorem 8.4. *Any vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles $\mathscr{O}(n)$.*

Proof. By theorem 8.3, we are reduced to certain matrix computations. See [HM82]. ☐

8.6. **Cohomology of coherent sheaves on a curve.** Attached to a coherent sheaf $\mathcal{E}$, we have the cohomology groups $\mathrm{H}^0(X,\mathcal{E})$ and $\mathrm{H}^1(X,\mathcal{E})$.

8.6.1. *Rank and degree.* Let $\mathcal{E}$ be a coherent sheaf. We let $\mathrm{rk}(\mathcal{E}) = \dim_{k(X)}(\mathcal{E}_\eta)$ be the rank of the vector bundle $\mathcal{E}^{cotors}$.

We now define the degree. If $\mathcal{E}$ is a torsion sheaf, we have that $\mathrm{H}^0(X,\mathcal{E})$ is a finite k-vector space and we let $\deg(\mathcal{E}) = \dim_k \mathrm{H}^0(X,\mathcal{E})$. If $\mathcal{E}$ is a vector bundle, we let $\deg(\mathcal{E}) = \deg(\det \mathcal{E})$.

Lemma 8.6. *Assume that $\mathcal{E}$ is a vector bundle which corresponds to an element of $(f_x)_{x\in X} \in GL_n(K)\backslash GL_n(\mathbb{A}_K)/\prod_x GL_n(\mathcal{O}_x)$. Then we have*

$$\deg(\mathcal{E}) = -\sum_x [k(x):k] v_x(\det(f_x)).$$

Proof. We have a map $\det : GL_n(K)\backslash GL_n(\mathbb{A}_K)/\prod_x GL_n(\mathcal{O}_x) \to GL_1(K)\backslash GL_1(\mathbb{A}_K)/\prod_x GL_1(\mathcal{O}_x)$. This reduces to the case that $n=1$. By construction, we see that $\mathcal{L}$ corresponding to (f_x) is isomorphic to the $\mathscr{O}(-\sum_x v_x(f_x)x)$. We have that $\deg(-\sum_x v_x(f_x)x) = -\sum_x v_x(f_x)[k(x):k]$. ☐

In general we let $\deg \mathcal{E} = \deg \mathcal{E}_{tors} + \deg \mathcal{E}^{cotors}$.

Lemma 8.7. *The functions* rk *and* deg *are additive on short exact sequences.*

Proof. For the function rank, this is obvious : any short exact sequence :

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$$

induces a short exact sequence :

$$0 \to \mathcal{E}_\eta \to \mathcal{E}'_\eta \to \mathcal{E}''_\eta \to 0$$

For the function degree, we first observe that additivity is obvious in the following case: $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$ is a short exact sequence of torsion sheaves and of locally free sheaves. In this second case, we use that $\det \mathcal{E}' = \det \mathcal{E}'' \otimes \det \mathcal{E}$. We deal with the general case. First, by modding out by $\mathcal{E}'_{tors}$ we have a diagram :

$$\begin{tikzcd}
\mathcal{E}/\mathcal{E}_{tors} \arrow[r] & \mathcal{E}'/\mathcal{E}'_{tors} \arrow[r] & \mathcal{E}''/\mathrm{Im}(\mathcal{E}'_{tors}) \\
\mathcal{E} \arrow[r] \arrow[u] & \mathcal{E}' \arrow[r] \arrow[u] & \mathcal{E}'' \arrow[u] \\
\mathcal{E}_{tors} \arrow[r] \arrow[u] & \mathcal{E}'_{tors} \arrow[r] \arrow[u] & \mathrm{Im}(\mathcal{E}'_{tors}) \arrow[u]
\end{tikzcd}$$

where all raws and columns are exact. We see that deg is additive on all columns and on the bottom raw. Therefore additivity of deg on the top and middle raw is equivalent. We therefore can reduce to the case that $\mathcal{E}$ and $\mathcal{E}'$ are vector bundles.

Next by modding out by the inverse image in $\mathcal{E}'$ of $\mathcal{E}''_{tors}$ we get a diagram :

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{E}'/\pi^{-1}(\mathcal{E}''_{tors}) & \longrightarrow & \mathcal{E}''/\mathcal{E}''_{tors} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}' & \xrightarrow{\pi} & \mathcal{E}'' \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{E} & \longrightarrow & \pi^{-1}\mathcal{E}'_{tors} & \longrightarrow & (\mathcal{E}'_{tors}) \end{array}$$

with exact raws and columns. We see that deg is additive on all columns and on the top raw. Therefore the additivity of the middle line is equivalent to that of the bottom line. We thus reduce to the case that $\mathcal{E}$ and $\mathcal{E}'$ are vector bundles and $\mathcal{E}''$ is a torsion sheaf. Therefore $\mathcal{E}'$ is a modification of $\mathcal{E}$ at a finite set of points $x_1, \cdots, x_n$ and the map $: \mathcal{E}_{x_i} \to \mathcal{E}'_{x_i}$ has elementary divisors $t_{x_i}^{n_{i,1}}, \cdots, t_{x_i}^{n_{i,r}}$ so that

$$\mathrm{length}_k(\mathcal{E}'_{x_i}/\mathcal{E}_{x_i}) = [k(x_i) : k](\sum_{j=1}^{r} n_{i,j}) = [k(x_i) : k]v_{x_i}(\det(\mathcal{E}_{x_i} \to \mathcal{E}'_{x_i})).$$

□

8.6.2. *The Riemann-Roch formula.* We let $\chi(\mathcal{E})$ be the Euler characteristic of $\mathcal{E}$, defined by:

$$\chi(\mathcal{E}) = \dim_k \mathrm{H}^0(X, \mathcal{E}) - \dim_k \mathrm{H}^1(X, \mathcal{E})$$

Theorem 8.5. *For any coherent sheaf $\mathcal{E}$, we have: $\chi(\mathcal{E}) = \deg \mathcal{E} + \mathrm{rk}(\mathcal{E})(1-g)$*

Proof. Since χ, deg and rk are additive on short exact sequences, if we have an exact sequence :

$$0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$$

in $Coh(X)$, and the formula holds for two out of three then it holds for all. The formula holds for torsion sheaves. It also holds for $\mathscr{O}_X$. It remains to prove that it holds for vector bundles. Since any vector bundle is a successive extension of line bundles, it is enough to check the formula for line bundles. Let D be an effective divisor. Since $0 \to \mathscr{O}_X \to \mathscr{O}_X(D) \to \mathscr{O}_X(D)/\mathscr{O}_X \to 0$ is exact and $\mathscr{O}_X(D)/\mathscr{O}_X$ is torsion, the formula holds for any effective divisor. In general, let $D = D_1 - D_2$ with D_1, D_2 effective. We have $0 \to \mathscr{O}_X(D) \to \mathscr{O}_X(D_1) \to \mathscr{O}_X(D_1)/\mathscr{O}_X(D) \to 0$ is exact and $\mathscr{O}_X(D_1)/\mathscr{O}_X(D)$ is torsion. We conclude. □

9. Lecture VIII : Stacks

9.1. **Motivation.** We are interested in constructing geometric objects using moduli problems. Here are some examples. Let S be a base scheme.

Example 9. Let $r \geq d \geq 1$. We consider the functor $(Sch/S)^{opp} \to SET$, sending $X \to S$ to pairs $(\mathcal{L}, \phi)$ consisting of (isomorphism classes) of a locally free sheaf $\mathcal{L}$ of rank d over X, and a surjective map $\phi : \mathscr{O}_X^r \to \mathcal{L}$. This functor is a Space and is actually representable by a projective S-scheme $GR(r, d)$, called a Grassmanian.

Example 10. Let Ell be the functor $(Sch/S)^{opp} \to SET$ sending $X \to S$ to the set of isomorphism classes of elliptic curves $E \to X$. An elliptic curve $E \to T$ being a proper, smooth curve of genus 1 such that each fibers are geometrically connected, which admits a section $T \to S$.
Ell is not representable for the following reason. Let $E_1 : y^2 = x^3+1$ and $E_2 : 2y^2 = x^3+1$ be two elliptic curves defined over $\mathbb{Q}$. We see that they define elements of $Ell(\mathbb{Q})$. These

two curves are not isomorphic over $\mathbb{Q}$, but they become isomorphic over $\mathbb{Q}(\sqrt{2})$. Indeed, we can define over $\mathbb{Q}(\sqrt{2})$ the map $\lambda : E_1 \to E_2$ which sends (x, y) to $(x, \sqrt{2}^{-1} y)$. This map is not defined over $\mathbb{Q}$ since for σ the non-trivial element of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ we have $\lambda^\sigma = -\lambda$. This implies that the functor Ell is not separated (hence not representable) since otherwise the morphism $Ell(Spec\ \mathbb{Q}) \to Ell(Spec(\mathbb{Q}(\sqrt{2})))$ would be injective, since $Spec(\mathbb{Q}(\sqrt{2})) \to Spec\ \mathbb{Q}$ is an fpqc cover, which is not the case as E_1, E_2 are distinct points of $Ell(\mathrm{Spec}\mathbb{Q})$ with the same image in $Ell(\mathrm{Spec}\mathbb{Q}(\sqrt{2}))$.

Example 11. Let X be a projective, non-singular, geometrically connected curve over $\mathrm{Spec}\ k = S$. We define $Bun_{X,n}$ the functor sending $T \to S$ to $Bun_n(X \times_S T)$. This is not representable. Indeed, take $T = \mathbb{P}^1$ and let $p : X \times_S T \to T$ be the projection. Consider $p^\star \mathscr{O}(1)$ and $p^\star \mathscr{O}$. These are non-isomorphic line-bundles, thus define two elements of $Bun_{X,1}(T)$. They are not isomorphic : indeed, after extending scalars, we may assume that X has a k-rational point. And taking pull back under the section we recover $\mathscr{O}$ and $\mathscr{O}(1)$. Let $U \coprod U' \to T$ be the standard affine covering of $\mathbb{P}^1$ by two affine lines. We see that the two points become equal on the pull back to $U \coprod U'$ since $\mathscr{O} \simeq \mathscr{O}(1)$ over $U \coprod U'$. This implies once again that our functor is not separated, hence not representable.

The common feature of the last two examples, is that the objects we are trying to classify have automorphisms. Any elliptic curve has the automorphism -1. Similarly, any vector bundle has scalar automorphisms by $\mathbb{G}_m$. More precisely, if $\mathcal{U} = \{T_i \to T\}_{i \in I}$ is a fpqc covering of T, then it is easy to see that local isomorphisms (such as the one over $U \coprod U'$, or the one over $\mathrm{Spec}\mathbb{Q}(\sqrt{2})$ will glue to a global isomorphism, ensuring sepratness on the cover, if and only if the Cech cohomology $H^1_{\mathcal{U}}(T, G) = 1$ where G is the automorphism group of the object over T.
One first idea is to rigidify the functors to kill automorphisms. For example, we could consider elliptic curves with an invariant differential (at least in characteristic $\neq 2, 3$, or with a basis of the N-torsion for $N \geq 3$). Nevertheless this idea does not work all the time. The other idea is to consider also the possibility of objects having automorphisms.

Namely, one can upgrade Ell has a "functor" whose value on X is the category of elliptic curves with morphisms being isomorphisms of elliptic curves (rather than the isomorphism classes of objects). Similarly, we now redefine $Bun_n(T)$ for any scheme T as the category whose objects are rank n vector bundles and whose morphisms are isomorphisms of vector bundles (rather than considering $Bun_n(T)$ as the set of isomorphism classes of vector bundles). We let $Bun_{X,n}$ be the functor sending T to the category $Bun_n(X \times_S T)$.

There is actually a foundational question of what is a functor valued in a category. We address this below using the formalism of fibered category. We then define Stacks which are the analogue of Spaces, and then algebraic Stacks which are the analogue of algebraic spaces.

9.2. **Fibered categories.** Let $\mathcal{S}$, $\mathcal{C}$ be two categories and let $F : \mathcal{S} \to \mathcal{C}$ be a functor between them. Informally, we say that $\mathcal{S}$ is fibered over $\mathcal{C}$ if there is a good notion of base change. Namely, let $x \in Ob(\mathcal{S})$, and let $U = F(x) \in Ob(\mathcal{C})$. Let $f : V \to U$ be a map in $\mathcal{C}$, then we would like to have an element $f^\star x \in Ob(\mathcal{S})$, called the base change of x to V, with $F(f^\star x) = V$.

Definition 9.1. *A strongly cartesian morphism $\phi : y \to x$ in $\mathcal{S}$ is an arrow with the property that :*

$$\mathrm{Hom}_{\mathcal{S}}(z, y) = \mathrm{Hom}_{\mathcal{S}}(z, x) \times_{\mathrm{Hom}_{\mathcal{C}}(F(z), F(x))} \mathrm{Hom}_{\mathcal{C}}(F(z), F(y))$$

Where the fibered product is given by :

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{S}}(z,y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{S}}(z,x) \arrow[d, "F(-)"] \\
\mathrm{Hom}_{\mathcal{S}}(F(z),F(y)) \arrow[r, "F(\phi\circ -)"'] & \mathrm{Hom}_{\mathcal{C}}(F(z),F(x))
\end{tikzcd}$$

And the bijection is given by $\psi \mapsto \eta = (\phi\circ\psi, F(\psi))$. This means that for all map $\eta : z \to x$ such that $F(\eta)$ factors through $F(y)$, η factors through an unique $\psi : z \to y$:

$$\begin{tikzcd}
z \arrow[r, dashed, "\exists!\psi"'] \arrow[rr, bend left, "\eta"] \arrow[d, squiggly, "F"] & y \arrow[r, "\phi"] \arrow[d, squiggly, "F"] & x \arrow[d, squiggly, "F"] & \mathcal{C} \arrow[d, squiggly, "F"] \\
F(z) \arrow[r] & V = F(y) \arrow[r, "F(\phi)"] & U = F(x) & \mathcal{S}
\end{tikzcd}$$

By the Yoneda lemma, given $f : V \to U$ and $x \in \mathcal{S}$ with $F(x) = U$, if there is a strongly cartesian $\phi : y \to x$ with $F(\phi) = f$, then (y, ϕ) is unique up to a unique isomorphism.

Lemma 9.1. *The composition of two strongly Cartesian morphisms is strongly cartesian.*

Proof. Let $\phi : x' \to x$ and $\phi' : x'' \to x$ be two strongly cartesian arrows. Let U, V, W be the images of x, x', x''. We have that

$$\mathrm{Hom}_{\mathcal{S}}(z,x'') = \mathrm{Hom}_{\mathcal{S}}(z,x') \times_{\mathrm{Hom}_{\mathcal{C}}(F(z),F(x'))} \mathrm{Hom}_{\mathcal{C}}(F(z),F(x''))$$

$$\mathrm{Hom}_{\mathcal{S}}(z,x') = \mathrm{Hom}_{\mathcal{S}}(z,x) \times_{\mathrm{Hom}_{\mathcal{C}}(F(z),F(x))} \mathrm{Hom}_{\mathcal{C}}(F(z),F(x'))$$

So that

$$\mathrm{Hom}_{\mathcal{S}}(z,x'') = \mathrm{Hom}_{\mathcal{S}}(z,x) \times_{\mathrm{Hom}_{\mathcal{C}}(F(z),F(x))} \mathrm{Hom}_{\mathcal{C}}(F(z),F(x'')).$$

□

Definition 9.2. *Let $F : \mathcal{S} \to \mathcal{C}$. We say that $\mathcal{S}$ is fibered over $\mathcal{C}$ if for any $x \in Ob(\mathcal{S})$, and $U = F(x) \in Ob(\mathcal{C})$, and any map $f : V \to U$, there is a strongly cartesian morphism $\phi : f^\star x \to x$ in $\mathcal{S}$ with $F(\phi) = f$.*

Example 12. Let $\mathcal{C}$ be a category, then $Arr(\mathcal{C})$ the category of arrows of $\mathcal{C}$ whose morphisms are the commutative squares is a category over $\mathcal{C}$ with the functor "target". We can see that cartesian morphism are just cartesian squares, and that $Arr(\mathcal{C})$ is fibered over $\mathcal{C}$ if and only if $\mathcal{C}$ has fiber product.

Let $F : \mathcal{S} \to \mathcal{C}$ be a fibered category. For all $U \in \mathcal{C}$ we let $\mathcal{S}_U$ be the fiber category, its objects are $x \in \mathcal{C}$ with $F(x) = U$ and arrows $f : x \to y$ in $\mathcal{S}_U$ are arrows in $\mathcal{C}$ with the property that $F(f) = \mathrm{id}_U$.

Lemma 9.2. *For any arrow $f : V \to U$, we can construct a pull back functor $f^\star : \mathcal{S}_U \to \mathcal{S}_V$.*

Proof. By the axiom of choice, we can choose for any $x \in Ob(\mathcal{S}_U)$ a pull back $f^\star x \in Ob(\mathcal{S}_V)$. Let $g : x \to x'$ be an arrow in $\mathcal{S}_U$. Then we get an arrow $f^\star g : f^\star x \to f^\star x'$ by using the universal property of strongly cartesian morphism. □

Lemma 9.3. *Let $F : \mathcal{S} \to \mathcal{C}$ be a fibered category. We choose pull back functors $f^\star : \mathcal{S}_U \to \mathcal{S}_V$ for any arrow $f : U \to V$. Then the following properties hold :*

1. *For any maps $f : U \to V$ and $g : V \to W$, there is a unique isomorphism of functors $\mathcal{S}_W \to \mathcal{S}_U$, $\alpha_{f,g} : (f \circ g)^\star \simeq f^\star \circ g^\star$.*
2. *If $f = \mathrm{Id}_U$, there is a canonical isomorphism $Id_U^\star = Id_{\mathcal{S}_U}$.*

Proof. For any $x \in \mathcal{S}_W$, $(f \circ g)^\star x$ and $f^\star g^\star x$ are two pull backs of x. Therefore there is a unique isomorphism $(f \circ g)^\star x \to f^\star g^\star x$. This gives the functor. Similarly, there is a unique isomorphism $Id_U^\star x = x$ giving the functor of point two. □

Remark 9.1. Thus the fibered category gives rise to $U \mapsto \mathcal{S}_U$ which is a contravariant pseudo functor from the category $\mathcal{C}$ to the category of categories. The category of categories is a 2-category, with objects being categories, 1-morphisms being functors, and 2-morphisms being natural transformations. The 2-morphisms $\alpha_{f,g}$ appear when trying to compare various pull backs. All the identities satisfied by the 2-morphisms $\alpha_{f,g}$ (when varying f and g) are a little bit long to record. For this reason, one prefers the language of fibered categories.

9.3. **Fibred categories in groupoid.**

9.3.1. *Groupoid, setoid, discrete categories.*

Definition 9.3. *A groupoid is a category C such that all morphisms in C are invertible.*

Definition 9.4. *A setoid is a groupoid C such that any object x has a unique endomorphism : the identity.*

Definition 9.5. *A discrete category C is a category whose only morphisms are the identity morphisms* Id_x *for* $x \in Ob(C)$.

We can attach to a groupoid C a discrete category C^{disc} obtained as follows : its objects are equivalence classes of object of C where $x, y \in Ob(C)$ are equivalent if $\mathrm{Hom}_C(x,y) \neq \emptyset$. There is a functor $C \to C^{disc}, X \mapsto \overline{X}$.

Lemma 9.4. *Let C be a setoid. Then C is equivalent to C^{disc}.*

Proof. As C is a setoid, there are at most one morphism between two of its objects. We define a quasi-inverse : if $\overline{X} \in C^{disc}$, we send it to X a chosen representant of the class, and we send the only morphism $\overline{X} \to \overline{X}$ to Id_X. □

9.3.2. *S-groupoids.* Let S be a scheme.

Definition 9.6. *An S-groupoid is a fibered category $\mathfrak{X}$ over Sch/S, $p : \mathfrak{X} \to Sch/S$ such that, for $U \in Ob(Sch/S)$, $\mathfrak{X}_U$ is a groupoid.*

Example 13. $Bun_{n,X} \to (Sch/S)$ is an S-groupoid, where we have defined the category $Bun_{n,X}$ with $Ob(Bun_{n,X}) = \{T \to S \in (Sch/S), \mathcal{E}$ a vector bundle of rank n over $X_T\}$ and a morphism $(T', \mathcal{E}') \to (T, \mathcal{E})$ is the data of an S-morphism $f : T' \to T$, and an isomophism of vector bundle over $X_{T'}$: $\psi : \mathcal{E}' \to f^*\mathcal{E}$.

We have a trivial "forgetfull" functor $F : Bun_{n,X} \to (Sch/S)$. It is straightforward that for an S-scheme U, the fiber of F at U is a groupoid.
Furthermore, if we take $(T'', \mathcal{E}'')$ and $(T, \mathcal{E})$ two objects in $Bun_{n,X}$, T' an S-scheme, $(f, \varphi) : (T'', \mathcal{E}'') \to (T, \mathcal{E})$ a morphism in $Bun_{n,X}$ and $g : T' \to T$ such that f factors through g. If we let $\mathcal{E}' = g^*\mathcal{E}$, we can see that φ factors as well through $\mathcal{E}'$ and the morphism inducing the factorization is uniquely determined by (f, φ). This show that $Bun_{n,X}$ is indeed an S-groupoid.

Remark 9.2. Let $x \to y$ be a map in $\mathfrak{X}$. Let $f : p(x) \to p(y)$. Let $f^\star y \to y$ be a pull back of y. Then the morphism $x \to y$ factors as $x \to f^\star y \to y$ where $x \to f^\star y$ is a map in $\mathfrak{X}_U$.

Remark 9.3. One proves that equivalently, and S-groupoid $\mathfrak{X}$ is a category over Sch/S, $p : \mathfrak{X} \to Sch/S$ such that

(1) For any arrow $\phi : V \to U$ and any object $x \in \mathcal{X}$ such that $p(x) = U$, there exists an object $y \in \mathfrak{X}$ and an arrow $\Psi : y \to x$ such that $p(\Psi) = \phi$.
(2) For any diagram in $\mathfrak{X}$:

$$z \xrightarrow{h} x \xleftarrow{f} y$$

in $\mathfrak{X}$ with image in (Sch/S) :

$$W \xrightarrow{\chi} U \xleftarrow{\phi} V$$

and any morphism $\psi : W \to V$, such that $\chi = \phi \circ \psi$, there is a unique arrow $z \to y$ in $\mathfrak{X}$ such that $h = f \circ g$ and $p(g) = \psi$.

Proof. First, if we take an S-groupoid $\mathfrak{X}$, the first point is true, if we take Ψ the strongly cartesian morphism coming from ϕ. The second point is similarly true, since it is just the definition rewritten if we take $f = \Psi$.
Conversly, we see as before that $\mathfrak{X}$ must be a fiber category over Sch/S. Now, taking U an S-scheme, $x, y \in \mathfrak{X}_U$ and $f : x \to y$ taking $U = V = W$ and the identity morphism for χ and ϕ and for h yields the existence of an unique arrow $y \to x$ which is a right inverse of f. Using the unicity, f is invertible. $\square$

The category of S-groupoids denoted Gr/S is a 2-category. The 1-morphisms are functors between the objects (the categories). The 2-morphisms are natural transformations between 1-morphisms (the functors).

Definition 9.7. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be two objects of Gr/S. Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a 1-morphism. We say that F is a monomorphism if F is fully faithfull. We say that F is an isomorphism if F is an equivalence of categories.*

Remark 9.4. We see that F is a monomorphism if for all $U \in Ob(Sch/S)$, if $F_U = \mathfrak{X}_U \to \mathfrak{Y}_U$ is fully faithfull. We see that F is an isomorphism if $F_U : \mathfrak{X}_U \to \mathfrak{Y}_U$ is an equivalence.

9.3.3. *Fiber products.* We want to explain the fiber product construction. Let $\mathfrak{X} \xrightarrow{p} \mathfrak{Y} \xleftarrow{q} \mathfrak{Z}$ be a diagram of 1-morphisms. We construct an S-groupoid $\mathfrak{X} \times_{p,\mathfrak{Y},q} \mathfrak{Z}$ as follows:

For any $U \in Sch/S$, the fiber category $(\mathfrak{X} \times_{p,\mathfrak{Y},q} \mathfrak{Z})_U$ has objects the triples (x, z, g) where $x \in \mathfrak{X}_U$, $z \in \mathfrak{Z}_U$ and $g : p(x) \to p(z)$ is an arrow in $\mathfrak{Y}_U$.

A map $(x, z, f) \to (x', z', f')$ is the data of arrow $g : x \to x'$ and $h : z \to z'$ such that the following diagram commutes :

$$\begin{tikzcd} p(x) \arrow[r, "f"] \arrow[d, "p(g)"] & q(z) \arrow[d, "q(h)"] \\ p(x') \arrow[r, "f'"] & q(z') \end{tikzcd}$$

9.4. **Presheaves and groupoids.** Let $X : (Sch/S)^{opp} \to SET$ be a presheaf. We can turn it into a groupoid over S by letting $X_U = X(U)$ (this is the discrete category having objects $X(U)$ and the only arrows being the identity of the objects) and for any maps $\phi : U \to V$, we let $\phi^\star : X(V) \to X(U)$ be the natural restriction map.

Lemma 9.5. *Let $\mathfrak{X}$ be an S-groupoid such that for all U, $\mathfrak{X}_U$ is a discrete category. Then $\mathfrak{X}$ is the groupoid associated to a presheaf.*

Proof. The groupoid is entirely described by the pullback maps $f^\star : \mathfrak{X}_U \to \mathfrak{X}_V$ (which are unambiguously defined) for any $f : V \to U$. We can therefore consider the presheaf F defined by $F(U) = \mathfrak{X}_U$. $\square$

Lemma 9.6. *Let $\mathfrak{X}$ be an S-groupoid. The following are equivalent :*

(1) *$\mathfrak{X}$ is isomorphic to a S-groupoid associated to a presheaf.*
(2) *The diagonal morphism $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is a monomorphism.*

Proof. Let $x \in \mathfrak{X}_U$. We have $\Delta_U : \mathrm{Hom}_{\mathfrak{X}_U}(x, x) \to \mathrm{Hom}_{\mathfrak{X} \times_S \mathfrak{X}_U}((x, x, Id_U), (x, x, Id_U)) = \mathrm{Hom}_{\mathfrak{X}_U}(x, x) \times \mathrm{Hom}_{\mathfrak{X}_U}(x, x)$ given by the diagonal map. This map is an isomorphism if and only if $\mathrm{Hom}_{\mathfrak{X}_U}(x, x)$ is $\{id_x\}$. We can define an S-groupoid $\mathfrak{X}^{disc}$ by letting $\mathfrak{X}^{disc}_U = (\mathfrak{X}_U)^{disc}$. The map $\mathfrak{X} \to \mathfrak{X}^{disc}$ is an equivalence of categories. □

9.4.1. *Schemes and groupoids.* Any S-scheme T determines a presheaf and therefore an S-groupoid. Concreteley, one considers the S-groupoid $Sch/T \to Sch/S$.

We have the 2-yoneda lemma :

Lemma 9.7. *Let T be an S-scheme. There is an equivalence of categories :*

$$\mathrm{Hom}_{Gr/S}(T, \mathfrak{X}) \to \mathfrak{X}_T$$

It sends a 1-morphism $F : T \to \mathfrak{X}$ to $F(T \to T) \in Ob(\mathfrak{X}_T)$. It sends a 2 morphism $G : F \to F'$ to the map $F(T \to T) \to F'(T \to T)$.

Proof. Let $x \in \mathfrak{X}_T$. We define a functor $Sch/T \to \mathfrak{X}$ as follows. We send any $f : U \to T$, to $f^\star x$. □

We therefore get a fully faithful functor $Sch/S \to Gr/S$.

9.5. Stacks.

Definition 9.8. *A groupoid $\mathfrak{X}$ over S is an S-stack if :*

(1) *For all $(x, y) \in \mathfrak{X}_U$, the presheaf $(Sch/U)^{opp} \to SET$, $V \mapsto \mathrm{Hom}_{\mathfrak{X}_V}(x_V, y_V)$ is a sheaf,*
(2) *For all $fppf$ covering $\{U_i \to U\}_{i \in I}$, any descent datum $(x_i, f_{i,j})$ is effective.*

Let us spell out the two conditions. We can define the category $DD(\{U_i \to U\})$ whose objects are $(x_i, f_{i,j})$ where $x_i \in Ob(\mathfrak{X}_{U_i})$. For all i, j, we have maps $p_1 : U_i \times_S U_j \to U_i$ and $p_2 : U_i \times_S U_j \to U_j$. We have maps $f_{i,j} : p_1^\star x_i \to p_2^\star x_j$ in $\mathfrak{X}_{U_i \times_S U_j}$ satisfying the cocycle condition in $\mathfrak{X}_{U_i \times_S U_j \times_S U_k}$. Morphisms $(x_i, f_{i,j}) \to (x_i', f_{i,j}')$ are collection of maps $x_i \to x_i'$ compatible with the maps $f_{i,j}$ and $f_{i,j}'$ We have a functor $\mathfrak{X}_U \to DD(\{U_i \to U\})$. Let $x \in Ob(\mathfrak{X}_U)$, we can define $p_{U_i}^\star x = x_i$ (for the projection $p_{U_i} : U_i \to U$) and we have canonical maps $f_{i,j} : p_1^\star x_i \to p_2^\star x_j$. Then condition 1) is asking that the functor $\mathfrak{X}_U \to DD(\{U_i \to U\})$ is fully faithful, and condition 2) that it is essentially surjective.

Lemma 9.8. *Let $\mathfrak{X}$ be an S-stack. Let $T \to \mathfrak{X} \times_S \mathfrak{X}$ (corresponding to $(x_1, x_2) \in \mathfrak{X}_T^2$. Then the fiber product :*

$$T \times_{\mathfrak{X} \times_S \mathfrak{X}} \mathfrak{X}$$

is the space over T given by $V \mapsto \mathrm{Hom}(x_1|_V, x_2|_V)$. Assume that $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is a monomorphism. Then $\mathfrak{X}$ is equivalent to an S-space.

Proof. An object of $(T \times_{\mathfrak{X} \times_S \mathfrak{X}} \mathfrak{X})_V$ is $(x_1|_V, x_2|_V, x \in \mathfrak{X}_V, f_1 : x_1|_V \to x, f_2 : x_2|_V \to x)$. A morphism $(x_1|_V, x_2|_V, x, f_1, f_2) \to (x_1|_V, x_2|_V, x', f_1', f_2')$ arises from a map $h : x \to x'$ such that $h \circ f_i = f_i'$. This category is a Setoid, equivalent to the discrete category of maps $f : x_1|_V \to x_2|_V$. □

Example 14. Let X be an S-scheme. Let G be an affine S-group scheme which is acting on X. We consider the quotient stack $[X/G]$. Its objects are (U, T, ϕ) :

(1) An S-scheme $U \to S$.
(2) A U-scheme $p : T \to U$ which is a $G \times_S U$-torsor for the fppf topology.

(3) A G-équivariant morphisme $\phi : T \to X$.

The morphisms are $(U, T, \phi) \to (U', T', \phi')$ are given by maps $f : U \to U'$, maps of torsors $F : T \to T'$, such that $p' \circ F = f \circ p$ and $\phi' \circ F = \phi$.

We prove that this is indeed a Stack. Indeed, $fppf$-descent of G-torsors is effective (because G is affine) and the functor of points of X is an $fppf$ sheaf.

We can justify this construction. Let Spec $k \to S$ be a geometric point. We see that $[X/G]_{\mathrm{Spec}\ k}$ is the category with objects $X(k)$ and morphisms $\mathrm{Hom}(x, x') = \{g \in G(k), gx = x'\}$.

Example 15. We see $Bun_{n,X}$ is a stack. This follows from descent theory for coherent sheaf, granting the fact that the condition of being a vector bundle of rank n for a coherent sheaf is local in the $fppf$-topology.

In order to prove this we first recall the following lemma :

Lemma 9.9 ([Sta13], TAG 00NX)**.** *Let M be an A-module. The following conditions are equivalent :*

(1) *M is finitely presented and flat,*
(2) *M is finite and locally free,*
(3) *M is finite and projective.*

Lemma 9.10. *Let M be an A-module. Let $A \to B$ be a faithfully flat morphism. Then :*

(1) *M is finitely presented if and only if $M \otimes_A B$ is finitely presented.*
(2) *M is flat if and only if $M \otimes_A B$ is flat.*

9.6. **Morphisms of stacks.** The category of S-stacks is a 2-category that we denote by $Stack/S$.

Definition 9.9. *We say that a morphism $F : \mathfrak{X} \to \mathfrak{X}'$ of S-stacks is an epimorphism if for any $U \in Ob(Sch/S)$ and any $x \in \mathfrak{X}'_U$ there is a covering $\{U_i \to U\}$ and objects $y_i \in Ob(\mathfrak{X}_{U_i})$ such that $F(y_i)$ is isomorphic to x_{U_i}. We say that F is a monomorphism, or an isomorphism, if the morphism of S-groupoids is a monomorphism, or an isomorphism.*

We now discuss representability.

Definition 9.10. *An S-stack $\mathfrak{X}$ is representable by algebraic spaces if there is an S-algebraic space X and an isomorphism $X \to \mathfrak{X}$.*

Definition 9.11. *An S-stack $\mathfrak{X}$ is representable by a scheme if there exists an S-scheme X and an isomorphism $X \to \mathfrak{X}$.*

Definition 9.12. *Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of S-stack. We say that F is representable by algebraic spaces if for any S-scheme T and any map $T \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} T$ is an algebraic space.*

Let $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ be the diagonal.

Lemma 9.11. *The morphism Δ is representable by algebraic spaces (resp. schemes)if and only if, for any S-scheme T and any $x, y \in Ob(\mathfrak{X}_T)$, the sheaf $V \mapsto \mathrm{Hom}(x_V, y_V)$ is an algebraic space (resp. a scheme).*

Let $f : X \to Y$ be a map of algebraic spaces. Recall that f is called representable if for any scheme T, the fiber product $T \times_Y X$ is a scheme. Let P be a property of a morphism of schemes which is stable under base change. We declared that f has property P is for any scheme T, $T \times_Y X \to T$ has property P.

Definition 9.13. *A property P of morphisms of schemes is said to be étale local on source and target if :*

(1) *If $f : X \to Y$ is étale and $g : Y \to Z$ has P, then $g \circ f$ has P,*
(2) *If $f : X \to Y$ has P and $Y' \to Y$ is étale, then $X \times_Y Y' \to Y'$ has P,*
(3) *Given a morphism $f : X \to Y$, then f has P if and only if, for any $x \in X$, there is a commutative diagram*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "p"] & V \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with p, q étale, $x \in p(U)$ and g has P.

Definition 9.14. *Let $f : X \to Y$ be a map of algebraic spaces. Let P be a property of morphism of schemes which is étale local on source and target. Then we declare that f has property P if, for any commutative diagram :*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "p"] & V \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with U, V schemes, p, q étale morphisme, g has property P.

Lemma 9.12. *In the situation as above, f has property P if and only if g has property P for p, q presentations of X and Y respectively.*

Definition 9.15. *Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of S-stacks, representable by algebraic spaces.*

(1) *Let P be a property of morphisms of schemes which is étale local on source and target, stable under base change and $fppf$-local on the base. Then, we say that F has property P if for any scheme T, $F_T : \mathfrak{X} \times_{\mathfrak{Y}} T \to T$ has property P.*
(2) *Assume that F is representable by schemes. Let P be a property of morphisms of schemes stable under base change and $fppf$-local on the base. Then we sat that F has property P if for any scheme T, $F_T : \mathfrak{X} \times_{\mathfrak{Y}} T \to T$ has property P.*

9.7. **Algebraic stacks.**

Definition 9.16. *A 1-morphism $\mathfrak{X} \to \mathfrak{Y}$ in $Stack/S$ is said to be surjective if for all $U \in Ob(Sch/S)$ and all $Y \in Ob(\mathfrak{Y}_U)$, there exists a fppf covering $\{U_i \to U \mid i \in I\}$ and objects $X_i \in Ob(\mathfrak{X}_{U_i})$ such that $F(X_i) \simeq \mathfrak{Y}_{U_i}$*

Remark 9.5. Compare this to the notion of surjectifity for sheaves.

Definition 9.17. *An S-stack $\mathfrak{X}$ is an algebraic stack if*

(1) *The morphism $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable by algebraic spaces,*
(2) *There is an S-algebraic space X and a smooth surjective morphism $P : X \to \mathfrak{X}$.*

Example 16. Let X be an S-scheme and let G be a smooth affine group scheme acting on X $(a : G \times X \to X)$. Then $[X/G]$ is an algebraic stack. A presentation is given by the morphism $X \to [X/G]$, mapping $f : U \to X$ to $(U, T = G \times U, \phi = a \times f : G \times U \to X)$. We see that $X \times_{[X/G]} X = G \times_S X$.

10. Lecture IX : Harder-Narashiman filtration

10.1. **Generalities.** All quasi-coherent modules will be on a non singular geometrically connected projective curve X. Let $\mathcal{E}$ be a vector bundle. The slope of $\mathcal{E}$ is :

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$$

Proposition 10.1. *Let $\mathcal{E} \to \mathcal{E}'$ be a map of vector bundles which is a generic isomorphism. Then $\deg(\mathcal{E}') \geq \deg(\mathcal{E})$ and equality holds if and only if the map is an isomorphism.*

Proof. Consider a map $f : \mathcal{E} \to \mathcal{E}'$. This map induces an isomorphism at the generic point η. Therefore $\ker(f) = 0$ and $\mathrm{coker}(f)$ is a torsion sheaf. We consider the sequence $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'/\mathcal{E} \to 0$. Since deg is additive, we have that $\deg(\mathcal{E}') = \deg(\mathcal{E}) + \deg(\mathcal{E}'/\mathcal{E})$. Since $\mathcal{E}'/\mathcal{E}$ is a torsion sheaf, we see that $\deg(\mathcal{E}'/\mathcal{E}) \geq 0$ and is zero if and only if f is an isomorphism. $\square$

Proposition 10.2. *Let $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$ be a short exact sequence of vector bundles. We have*

$$\mu(\mathcal{E}') \in [\inf\{\mu(\mathcal{E}), \mu(\mathcal{E}'')\}, \max\{\mu(\mathcal{E}), \mu(\mathcal{E}'')\}].$$

Moreover, if $\mu(\mathcal{E}) \neq \mu(\mathcal{E}'')$ then $\mu(\mathcal{E}')$ lies in the interior of this interval.

Proof. Let $\mu(\mathcal{E}) = \frac{a}{b}$, $\mu(\mathcal{E}) = \frac{a'}{b'}$, $\mu(\mathcal{E}'') = \frac{a''}{b''}$ with $a' = a + a''$ and $b' = b + b''$. Without loss of generality (switching the names of $\mathcal{E}$ and $\mathcal{E}''$), we can assume that $\mu(\mathcal{E}) \leqslant \mu(\mathcal{E}'')$ so that $ba'' - ab'' \geqslant 0$. We have $\frac{a+a''}{b+b''} - \frac{a}{b} = \frac{ba''-ab''}{(b+b')b} \geqslant 0$ and $\frac{a+a''}{b+b''} - \frac{a''}{b''} = \frac{b''a-ab''}{(b+b')b''} \leqslant 0$, with equality if and only $\frac{a}{b} = \frac{a''}{b''}$. $\square$

Definition 10.1. *A vector bundle $\mathcal{E}$ is called stable if for all subbundle $0 \neq \mathcal{E}' \subsetneq \mathcal{E}$, we have $\mu(\mathcal{E}') < \mu(\mathcal{E})$. A vector bundle $\mathcal{E}$ is called semi-stable if for all subbundle $0 \neq \mathcal{E}' \subseteq \mathcal{E}$, we have $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$.*

Remark 10.1. One can also phrase the definition using quotients of $\mathcal{E}$: A vector bundle $\mathcal{E}$ is called stable if for all quotient $\mathcal{E} \to \mathcal{E}' \neq \mathcal{E}$, we have $\mu(\mathcal{E}') > \mu(\mathcal{E})$. A vector bundle $\mathcal{E}$ is called semi-stable if for all quotient $\mathcal{E} \to \mathcal{E}'$, we have $\mu(\mathcal{E}') \geq \mu(\mathcal{E})$.

Remark 10.2. An invertible sheaf is stable as it is simple.

Proposition 10.3. *If $\mathcal{E}$, $\mathcal{F}$ are vector bundles over X, then $\mu(\mathcal{E} \otimes \mathcal{F}) = \mu(\mathcal{E}) + \mu(\mathcal{F})$. Also, $\mu(\mathcal{E}^\vee) = -\mu(\mathcal{E})$.*

Proof. It results from the formula $\deg(\mathcal{E} \otimes \mathcal{F}) = \deg \mathcal{E} \mathrm{rk} \mathcal{F} + \deg \mathcal{F} \mathrm{rk} \mathcal{E}$, which results, denoting $n = \mathrm{rk}\mathcal{E}$ and $m = \mathrm{rk}\mathcal{F}$ from $\det \mathcal{E} \otimes \mathcal{F} \simeq (\det \mathcal{E})^{\otimes m} \otimes (\det \mathcal{F})^{\otimes n}$, if $(e_1, \dots, e_n)$ and $(f_1, \dots, f_m)$ are local basis of $\mathcal{E}$ and $\mathcal{F}$, then the isomorphism is $(e_1 \otimes f_1) \wedge (e_2 \otimes f_1) \wedge \cdots (e_n \otimes f_m) \mapsto [1 \wedge \cdots \wedge e_n]^{\otimes m} \otimes [f_1 \wedge \cdots \wedge f_m]^{\otimes n}$, which is well defined because for endomorphism of free modules we have $\det(f \otimes g) = \det f \times \det g$ so that the map does not depends on the local basis chosen.
For the statement on the slope of the dual, it suffices to show it for the degree. If $\mathcal{E}$ correspond to $(f_x)_{x \in X} \in GL_n(K) \backslash GL_n(\mathbb{A}_K) / \prod_x GL_n(\mathcal{O}_x)$. Then $\deg \mathcal{E} = \deg \mathcal{L}$ where $\mathcal{L}$ corresponds to $(\det f_x)_x \in GL_1(K) \backslash GL_1(\mathbb{A}_K) / \prod_x GL_1(\mathcal{O}_x)$, then $\deg \mathcal{E}^\vee = \deg \mathcal{L}^{-1}$, si by using $\deg(\mathcal{E}) = -\sum_x [k(x) : k] v_x(\det(f_x))$, we get $\deg \mathcal{E} = -\deg \mathcal{E}^\vee$. $\square$

Remark 10.3. A vector bundle $\mathcal{E}$ is stable or semi-stable if and only if $\mathcal{E}^\vee$ is stable or semi-stable, if and only if $\mathcal{E} \otimes \mathcal{L}$ is stable or semi-stable. It follows from the previous proposition

Lemma 10.1. *Let $\mathcal{E}, \mathcal{E}'$ be semi-stable vector bundles with $\mu(\mathcal{E}) > \mu(\mathcal{E}')$. Then* $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') = 0$.

Proof. Let f be a non-zero morphism. Let $\mathcal{E}''$ be the strict image of $\mathcal{E}$ in $\mathcal{E}'$. Then $\mu(\mathcal{E}'') \geq \mu(\mathcal{E})$ and we have a contradiction because $\mu(\mathcal{E}) = \mu(\mathcal{E}'')$. □

Lemma 10.2. *Let $\mathcal{E}, \mathcal{E}'$ be semi-stable vector bundles with $\mu(\mathcal{E}) + 2g - 2 < \mu(\mathcal{E}')$. Then* $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}') = 0$.

Proof. We use the corrolary for Serre duality for coherent sheaves : $\mathrm{H}^0(X, \Omega^1_{X/k} \otimes \mathcal{E}^\vee) \simeq \mathrm{H}^1(X, \mathcal{E})$, then $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}') = \mathrm{H}^1(X, \mathcal{E}^\vee \otimes \mathcal{E}') \simeq \mathrm{H}^0(X, \Omega^1_{X/k} \otimes (\mathcal{E}^\vee \otimes \mathcal{E}')^\vee) = \mathrm{H}^0(X, \Omega^1_{X/k} \otimes \mathcal{E} \otimes \mathcal{E}'^\vee) = H^0(X, \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{E}', \Omega^1_{X/k} \otimes \mathcal{E})) = 0$ because $\mu(\Omega^1_{X/k} \otimes \mathcal{E}) = 2g - 2 + \mu(\mathcal{E}) < \mu(\mathcal{E}')$ and we use the previous lemma. □

10.2. **The category of semi-stable vector bundles.**

Theorem 10.1. *Let $\mu \in \mathbb{Q}$. The full subcategory of $Coh(X)$ of semi-stable vector bundles of slope μ is an abelian category stable by extension.*

Proof. Let $\mathcal{E}$ and $\mathcal{E}'$ be two semi-stable vector bundles of slope μ. Let $f : \mathcal{E} \to \mathcal{E}'$ be a map. We see that
$$\mu(\mathcal{E}) \leq \mu(\mathrm{coim}(f)) \leq \mu(\mathrm{im}(f)) \leq \mu(\mathcal{E}')$$
and therefore, $\mathrm{coim}(f) = \mathrm{im}(f)$ are of slope μ. The morphism f is therefore strict. We deduce that $\mathrm{im}(f)$ is semi-stable of slope μ. It follows that $\ker(f)$ and $\mathrm{coker}(f)$ are also semi-stable of slope μ.

The part about extension follow immediately from the convexity property □

10.3. **The Harder-Narashiman filtration.**

Theorem 10.2. *Let $\mathcal{E}$ be a vector bundle. Then $\mathcal{E}$ has a unique increasing filtration (for some integer n) :*
$$0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_n = \mathcal{E}$$
where $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable of slope μ_i and $\mu_1 > \mu_2 > \cdots > \mu_n$.

Proof. For any subbundle $\mathcal{E}' \subseteq \mathcal{E}$, we have $\deg \mathcal{E}' + \mathrm{rk}(\mathcal{E}')(1-g) \leq \dim_k \mathrm{H}^0(\mathcal{E}') \leq \dim_k \mathrm{H}^0(\mathcal{E})$. We deduce that $\deg \mathcal{E}'$ is bounded by $\dim_k \mathrm{H}^0(\mathcal{E}) - \mathrm{rk}(\mathcal{E})(1-g)$ (if $g \geq 1$), and by $\dim_k \mathrm{H}^0(\mathcal{E})$ otherwise. Thus, the set $\{\mu(\mathcal{E}'),\ \mathcal{E}' \subseteq \mathcal{E}\}$ is bounded above and the maximum is reached. Let μ_1 be the maximum. We claim that there is a maximal subbundle with slope μ_1. Indeed, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two subbundles with slope μ_1. They are semi-stable. Then the subbundle $\mathcal{E}_1 + \mathcal{E}_2$ (defined as the image of $\mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E}$) has slope less or equal than μ_1. Since $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semi-stable of slope μ_1, it has slope greater or equal than μ_1. We deduce that $\mathcal{E}_1 \oplus \mathcal{E}_2$ has slope μ_1. Let $\mathcal{E}_1$ be the maximal subbundle with slope μ_1. Consider the short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_1 \to 0$. By the maximality of μ_1, any subbundle of $\mathcal{E}/\mathcal{E}_1$ has slope $< \mu_1$. By induction, we deduce that $\mathcal{E}$ has a filtration as in the theorem. It remains to prove uniqueness.

We shall prove that if $\mathcal{F}$ is a subbundle of $\mathcal{E}$, then $\mu(\mathcal{F}) \leq \mu_1$ and that equality implies that $\mathcal{F} \subseteq \mathcal{E}_1$. Indeed, the filtration induces a filtration on
$$\mathcal{F} : 0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_n = \mathcal{F}$$
with $\mathcal{F}_n = \mathcal{E}_n \cap \mathcal{F}$. Moreover, $\mathcal{F}_i/\mathcal{F}_{i-1} \hookrightarrow \mathcal{E}_i/\mathcal{E}_{i-1}$. We deduce that $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \leq \mu_i$. It follows from the convexity property that $\mu(\mathcal{F}) \leq \mu_1$ with strict inequality if $\mathcal{F}_i/\mathcal{F}_{i-1} \neq 0$ for some $i \geq 2$.

Let $0 \subsetneq \mathcal{E}'_1 \subsetneq \cdots \subsetneq \mathcal{E}'_{n'} = \mathcal{E}$ be another filtration. We deduce that $\mu'_1 \leq \mu_1$ and that $\mu_1 \leq \mu'_1$ by symmetry. It follows that $\mathcal{E}_1 = \mathcal{E}'_1$. By induction on the rank of $\mathcal{E}$ we will conclude. □

Corollary 10.1. *Let $\mathcal{E}$ be a vector bundle with HN-filtration*

$$0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_n = \mathcal{E}.$$

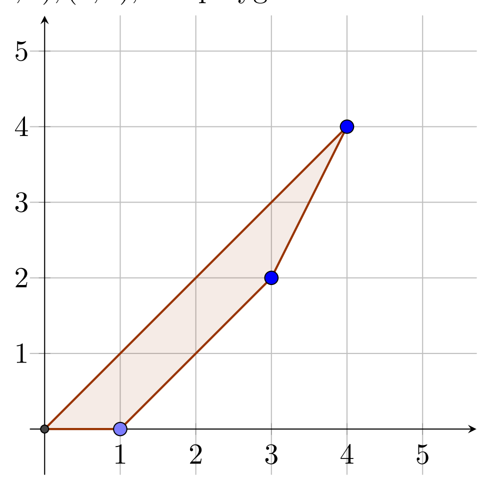
Let $\mathcal{E}' \subseteq \mathcal{E}$ be a subbundle which is a step of the HN filtration of $\mathcal{E}$. The HN-filtration of $\mathcal{E}'$ is $\{\mathcal{E}' \cap \mathcal{E}_i\}$. The HN-filtration of $\mathcal{E}/\mathcal{E}'$ is $\{\mathcal{E}_i/(\mathcal{E}' \cap \mathcal{E}_i)\}$.

We define the HN-polygon of the filtration to be the convex polygon in the plan with vertices $(0,0), (\deg(\mathcal{E}_1), \mathrm{rk}(\mathcal{E}_1)), \cdots, (\deg(\mathcal{E}), \mathrm{rk}(\mathcal{E}))$.

Proposition 10.4. *The HN-polygon is the convex hull of the set of point $(\mathrm{rk}(\mathcal{E}'), \deg(\mathcal{E}'))$ for $\mathcal{E}' \subseteq \mathcal{E}$.*

Proof. Let $\mathcal{E}' \subseteq \mathcal{E}$. Consider the HN-filtration $0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_n = \mathcal{E}$ of $\mathcal{E}$. Let $r_i = \mathrm{rk}(\mathcal{E}_i/\mathcal{E}_{i-1})$. We get a filtration $\{\mathcal{E}'_i = \mathcal{E}_i \cap \mathcal{E}'\}$ for which $\mathcal{E}'_i/\mathcal{E}'_{i-1}$ has slope $\mu'_i \leq \mu_i$ and rank $r'_i \leq r_i$. We have $\deg(\mathcal{E}') = \sum r'_i \mu'_i$. This proves that the point $(\mathrm{rk}(\mathcal{E}'), \deg(\mathcal{E}'))$ is below the HN-polygon. □

Example 17. If one take a vector bundle $\mathcal{E}$ of rank 4 of degree 4, with a filtratrion with ranks and degrees $(1,2), (3,3), (4,4)$, the polygon is :



We can actually read some informations on the polygon. For example, if it has a "slope jump" greater to $2g-2$ at an edge (r_i, d_i) (which means that the difference of the slope after i and before i is greater than $2g-g$, then $\mathcal{F}$ is not indecomposable. Indeed, Lemma 1.2 would give $Ext^1(\mathcal{F}_j/\mathcal{F}_{j-1}, \mathcal{F}_k/\mathcal{F}_{k-1})$ for all $k \leqslant i < j$ so that the HN filtration splits at i : $\mathcal{F} = \mathcal{F}_{j>i} \oplus \mathcal{F}_{j \leqslant i}$.
Also, as there are only finitely many convex polygones with integer coordinates with slope jump less than $2g-2$, going from $(0,0)$ to (r,d) we see that there are only finitely many possible sequences of (r_i, d_i) that are the ranks and degrees of the HN filtration containing an indecomposable vector bundle. It can be seen that the only degree of freedom left is the data of the $Ext^1(\mathcal{F}_j/\mathcal{F}_{j-1}, \mathcal{F}_k/\mathcal{F}_{k-1})$, which are in finite number, and if k is finite, are finite, so we get only a finite amount of semi-simple vector bundle of given rank and degree. Therefore we have only indecomposable ones too.

11. Lecture X : $Bun_{n,X}$ is an algebraic stack

In this lecture, all schemes are noetherian unless explicitly specified.

11.1. **Coherent sheaves on projective space.**

Lemma 11.1 ([Har77], II, lem. 5.14)**.** *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $f \in \mathrm{H}^0(X, \mathcal{L})$. Let $X_f = D(f)$. Let $\mathscr{F}$ be a coherent sheaf on X.*

(1) *If X is quasi-compact, and $s \in \mathrm{H}^0(X, \mathscr{F})$ is such that $s|_{X_f} = 0$. Then there is $n \in \mathbb{N}$, $f^n s = 0$ in $\mathrm{H}^0(X, \mathscr{F} \otimes \mathcal{L})$.*
(2) *If X is quasi-compact and quasi-separated, and if $s \in \mathrm{H}^0(X_f, \mathscr{F}$, there exists $n \in \mathbb{N}$ such that $f^n s \in \mathrm{H}^0(X, \mathscr{F} \otimes \mathcal{L})$.*

Let A be a noetherian ring, $S = \operatorname{Spec} A$. Let $R = A[X_0, \cdots X_n]$. Let $M = \oplus M_n$ be a graded R-module. Then we define $\tilde{M}$ a quasi-coherent sheaf on $\mathbb{P}^n_S$. We have $\tilde{M}(D(X_i)) = (M[1/X_i])_0$ (the subscript 0 means the elements of degree 0). Conversely, let $\mathscr{F}$ be a quasi-coherent sheaf on $\mathbb{P}^n_S$, we define $M(\mathscr{F}) = \oplus_n \mathrm{H}^0(\mathbb{P}^n_S, \mathscr{F}(n))$.

Proposition 11.1 ([Har77], II, prop. 5.15)**.** *The canonical map $\tilde{M(\mathscr{F})} \to \mathscr{F}$ is an isomorphism.*

Corollary 11.1 ([Har77], II, thm. 5.17, coro. 5.18)**.** *Let $\mathscr{F}$ be a coherent sheaf over the projective space. There exists n such that we have a surjective map $\mathscr{O}^N_{\mathbb{P}^n_S} \to \mathscr{F}(n)$.*

11.2. **Cohomology of coherent sheaves.**

11.2.1. *Generalities.* Let X be a scheme. The functor $\Gamma(X, -) : Mod(X) \to Ab$ is left exact. The category $Mod(X)$ has enough injectives. It follows that the functor $\Gamma(X)$ can be derived into : $\mathrm{R}\Gamma(X, -) : \mathcal{D}^+(Mod(X)) \to \mathcal{D}^+(Ab)$.

Let $\mathscr{F} \in Mod(X)$. We can compute the cohomology as follows : pick an injective resolution $\mathscr{F} \to I^\bullet$. Then $\mathrm{R}\Gamma(X, -)$ is represented by $\mathrm{H}^0(X, I^\bullet)$. Actually, we can take any $\Gamma(X, -)$ acyclic resolution. Another way to express the cohomology is using Chech cohomology.

Let X be a scheme and let $\mathcal{U} = \{U_i \to X\}$ be a Zariski cover. Then we let

$$\check{C}^\bullet(\mathcal{U}, \mathscr{F}) = \prod_i \mathrm{H}^0(U_i, \mathscr{F}) \to \prod_{i,j} \mathrm{H}^0(U_i \cap U_j, \mathscr{F}) \to \cdots$$

We let $\check{\mathrm{H}}^i(\mathcal{U}, \mathscr{F}) = \mathrm{H}^i(\check{C}^\bullet(\mathcal{U}, \mathscr{F}))$. We let $\check{\mathrm{H}}^i(X, \mathscr{F}) = \mathrm{colim}_{\mathcal{U}} \check{\mathrm{H}}^i(\mathcal{U}, \mathscr{F})$. This is the Chech cohomology.

Lemma 11.2 ([Sta13], TAG 01EW, TAG 01ET)**.** *Let X be a scheme. Assume that there is a basis $\mathcal{B}$ of open of X stable under intersection such that for each $U \in \mathcal{B}$, $\check{\mathrm{H}}^i(U, \mathscr{F}) = 0$ for all $i > 0$. Then $\mathrm{H}^i(U, \mathscr{F}) = 0$ for all $i > 0$ and $\mathrm{H}^i(X, \mathscr{F}) = \check{\mathrm{H}}^i(X, \mathscr{F}) = \check{\mathrm{H}}^i(\mathcal{U}, \mathscr{F})$ where $\mathcal{U}$ is a coverning of elements of $\mathcal{B}$.*

Theorem 11.1 ([Har77], III, thm. 3.7)**.** *If X is an affine scheme and $\mathscr{F} \in QCoh(X)$, $\mathrm{H}^i(X, \mathscr{F}) = 0$ for all $i > 0$.*

Proof. We first check that Chech cohomology vanishes on any affine. The theorem follows by the above lemma. □

Corollary 11.2 ([Har77], III, thm. 4.5)**.** *Let X be a separated scheme. Let $\mathscr{F}$ be a quasi-coherent sheaf. Let $\cup_i U_i$ be an affine cover. The Cech complex :*

$$\prod_i \mathrm{H}^0(U_i, \mathscr{F}) \to \prod_{i,j} \mathrm{H}^0(U_i \cap U_j, \mathscr{F}) \to \cdots$$

represents the cohomology $\mathrm{R}\Gamma(X, \mathscr{F})$.

Proof. The resolution $\mathscr{F} \to \prod_i \mathscr{F}|_{U_i} \to \prod_{i,j} \mathscr{F}|_{U_i \cap U_j} \to \cdots$ is $\Gamma(X, -)$ acyclic. □

11.2.2. *Relative cohomology.* [[Har77], III, section 8] Let $f : X \to Y$ be a morphism of schemes. We have the functor $f_\star : Mod(X) \to Mod(Y)$ and it can be derived into $Rf_\star : \mathcal{D}^+(Mod(X)) \to \mathcal{D}^+(Ab)$. Concretely, we can take an injective resolution $\mathscr{F} \to I^\bullet$ and $Rf_\star\mathscr{F}$ is represented by $f_\star I^\bullet$. We let $R^if_\star\mathscr{F} = \mathcal{H}^i(f_\star I^\bullet)$.

Lemma 11.3. *$\mathrm{R}^if_\star\mathscr{F}$ is the sheaf associated to the presheaf $U \mapsto \mathrm{H}^i(X \times_Y U, \mathscr{F})$.*

Proof. We have $\mathcal{H}^i(f_\star I^\bullet)$ is the sheaf associated to $U \mapsto \mathrm{H}^i(\Gamma(X_U, I^\bullet))$. We see that $I^\bullet|_{X_U}$ is a complex of flasque sheaves (hence $\Gamma(X_U, -)$-acyclic). So $\mathrm{H}^i(X_U, I^\bullet) = \mathrm{H}^i(X_U, \mathscr{F})$. $\square$

Theorem 11.2. *Assume that $\mathscr{F}$ is quasi-coherent. Then $\mathrm{R}^if_\star\mathscr{F}$ is a quasi-coherent sheaf and for any open affine U of Y, $\mathrm{R}^if_\star\mathscr{F}(U) = \mathrm{H}^i(X \times_Y U, \mathscr{F})$.*

Proof. We only give the proof for a separated, quasi-compact morphism. We can assume that $Y = \mathrm{Spec}\ A$ is affine. We take a finite affine cover $\mathcal{U}$ of X. Then $\check{C}(\mathcal{U}, \mathscr{F})$ represents $\mathrm{R}\Gamma(X, \mathscr{F})$. For any $A \to B$ localization, $\check{C}(\mathcal{U}, \mathscr{F}) \otimes_A B$ represents $\mathrm{R}\Gamma(X \times_Y \mathrm{Spec}\ B, \mathscr{F})$. And since $A \to B$ is flat, $\mathrm{H}^i(\check{C}(\mathcal{U}, \mathscr{F}) \otimes_A B) = \mathrm{H}^i(\check{C}(\mathcal{U}, \mathscr{F})) \otimes_A B$. $\square$

11.2.3. *Finiteness theorem.*

Theorem 11.3 ([Har77], III, thm. 5.2)**.** *Let $S = \mathrm{Spec}\ A$ be a noetherian affine scheme. Let $\mathscr{F}$ be a coherent sheaf on $X = \mathbb{P}^n_S$. Then*

(1) *For all $i \geq 0$, $\mathrm{H}^i(X, \mathscr{F})$ is a finitely generated A-module.*
(2) *There is an integer n_0 such that for all $n \geq n_0$, $\mathrm{H}^i(X, \mathscr{F}(n)) = 0$ for all $i > 0$ and all $n \geq n_0$.*

Remark 11.1. Let $X \to S$ be a projective scheme. Let $\iota : X \hookrightarrow \mathbb{P}^n_S$ be the closed immersion. The functor $i_\star : Mod(X) \to Mod(\mathbb{P}^n_S)$ is exact and sends injectives to injectives. It follows that for any sheaf $\mathscr{F}$, we have $\mathrm{R}\Gamma(X, \mathscr{F}) = \mathrm{R}\Gamma(\mathbb{P}^n_S, \iota_\star\mathscr{F})$.

Theorem 11.4. *Let $S = \mathrm{Spec}\ A$ be a noetherian affine scheme. Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n_S = X$. Then $M(\mathscr{F})$ is finitely generated : there exists n_0 such that for all $n \geq n_0$ the map :*

$$\mathrm{H}^0(X, \mathscr{O}_X(n - n_0)) \otimes \mathrm{H}^0(X, \mathscr{F}(n_0)) \to \mathrm{H}^0(X, \mathscr{F}(n))$$

is surjective.

Proof. Pick an exact sequence $0 \to \mathscr{G} \to \mathscr{O}_X^s \to \mathscr{F}(n_1) \to 0$. By twisiting we may assume that for all $n \geq n_2$, the map $\mathscr{O}_X^s(n) \to \mathscr{F}(n_1 + n)$ induces a surjective map $\mathrm{H}^0(X, \mathscr{O}_X^s(n)) \to \mathrm{H}^0(X, \mathscr{F}(n_1 + n))$. $\square$

Remark 11.2. This implies in particular that $\mathrm{H}^0(X, \mathscr{F}(r)) \otimes \mathscr{O}_X \to \mathscr{F}(r)$ is surjective for any $r \geq r_0$.

Proposition 11.2. *Let X be a smooth projective curve over a field k. Let $\mathcal{L}$ be an invertible sheaf. Assume that $\deg(\mathcal{L}) > 2g - 2$, then $\mathrm{H}^i(X, \mathcal{L}(n))) = 0$ for all $i > 0$ and all $n \geq 0$. Moreover, $\mathcal{L}(n)$ is generated by global sections for all for all $n \geq 1$.*

Proof. By the duality theorem, we know that $\mathrm{H}^1(X, \mathcal{L}) = \mathrm{H}^0(X, \Omega^1_X \otimes \mathcal{L}^\vee) = 0$ if $\deg \mathcal{L} > 2g - 2$. This proves the first point. For the second point, let us assume that k is infinite (we may extend scalars). We can find an hyperplane $H \subseteq \mathbb{P}^N$ such that $X \cap H = D$ is an effective divisor (just take $x \in X(k)$ and find an hyperplane H such that $x \notin H$).

We thus have an exact sequence $0 \to \mathcal{L}(n) \to \mathcal{L}(n+1) \to \mathcal{L}(n+1)|_D \to 0$. It induces an exact sequence $0 \to \mathrm{H}^0(X, \mathcal{L}(n)) \to \mathrm{H}^0(X, \mathcal{L}(n+1)) \to \mathrm{H}^0(X, \mathcal{L}(n+1)|_D) \to 0$.

We consider the map $\mathrm{H}^0(\mathbb{P}^N, \mathscr{O}(1)) \otimes_k \mathrm{H}^0(X, \mathcal{L}(n)) \to \mathrm{H}^0(X, \mathcal{L}(n+1))$.

For all $n \geq 1$, the map $\mathrm{H}^0(X, \mathcal{L}(n)) \to \mathrm{H}^0(D, \mathcal{L}(n))$ is surjective, therefore $\mathrm{H}^0(\mathbb{P}^N, \mathscr{O}(1)) \otimes_k \mathrm{H}^0(X, \mathcal{L}(n)) \to \mathrm{H}^0(D, \mathcal{L}(n+1))$ is surjective. We deduce that $\mathrm{H}^0(\mathbb{P}^N, \mathscr{O}(1)) \otimes_k \mathrm{H}^0(X, \mathcal{L}(n)) \to \mathrm{H}^0(X, \mathcal{L}(n+1))$ is surjective for all $n \geq 1$. $\square$

11.2.4. *The semi-continuity theorem.*

Theorem 11.5 ([Har77], III, prop. 12.2)**.** *Let $X \to S$ be a projective scheme of relative dimension n. Let $\mathscr{F}$ be a coherent sheaf on X, flat over A. Then there is a bounded complex $K^\bullet$ of finite flat A-modules, of amplitude $[0, n]$ such that for any A-module M, $\mathrm{R}\Gamma(X, \mathscr{F} \otimes M)$ is represented by :*

$$K^\bullet \otimes_A M$$

Proof. We only give a proof in the curve case. Assume $X = U_1 \cup U_2$ is covered by two affines. We let $L^\bullet = [\mathrm{H}^0(U_1, \mathscr{F}) \oplus \mathrm{H}^0(U_2, \mathscr{F}) \to \mathrm{H}^0(U_1 \cap U_2, \mathscr{F})]$. This represents $\mathrm{R}\Gamma(X, \mathscr{F})$ and $L^\bullet \otimes_A M$ represents $\mathrm{R}\Gamma(X, \mathscr{F} \otimes_A M)$. We let $x_1, \cdots, x_r$ be generators of $\mathrm{H}^1(L^\bullet)$ in L^1. We let $K^1 = A^r$ and we consider the map $g^1 : K^1 \to L^1$ given by these generators. Let $y_1, \cdots, y_s$ be generators of $\mathrm{Ker}(K^1 \to \mathrm{H}^1(L^\bullet)$. We choose lifts $\hat{y}_i$ of $g(y_i)$ in L^0. We let $K_1^0 = A^s$ we have maps $K_1^0 \to K^1$ (given by $y_1, \cdots, y_s$) and $K_1^0 \to L^0$ given by $\hat{y}_i$. We let $z_0, \cdots, z_t$ be generators of $\mathrm{H}^0(L^0)$. We $K_2^0 = A^t$ we have a map $K_2^0 \to L^0$ given by $z_0, \cdots, z_t$.

We see that there is a commutative diagram.

$$\begin{tikzcd} L^0 \arrow[r] & L^1 \\ K_1^0 \oplus K_2^0 \arrow[r] \arrow[u] & K^1 \arrow[u] \end{tikzcd}$$

If we let $(K')^\bullet$ be the bottom complex, the map $\mathrm{H}^0((K')^\bullet) \to \mathrm{H}^0(L^\bullet)$ is surjective, let K_3^0 be its Kernel.

Finally, we let $K^0 = K_1^0 \oplus K_2^0 / K_3^0$ and we get a diagram :

$$\begin{tikzcd} L^0 \arrow[r] & L^1 \\ K^0 \arrow[r] \arrow[u] & K^1 \arrow[u] \end{tikzcd}$$

Since the map $K^\bullet \to L^\bullet$ is a quasi-isomorphism, the cone $0 \to K^0 \to K^1 \oplus L^0 \to L^1 \to 0$ is exact. It follows that K^0 is flat.

We see that $K^\bullet \otimes_A M \to L^\bullet \otimes_A M$ is a quasi-isomorphism for all A-module M since the cone : $0 \to K^0 \otimes_A M \to K^1 \otimes_A M \oplus L^0 \otimes_A M \to L^1 \otimes_A M \to 0$ is exact. ☐

Corollary 11.3. (1) *The function $s \to \dim_k \mathrm{H}^i(X_s, \mathscr{F}_s)$ is upper semi-continuous.*
(2) *The function $s \to \sum_{i \geq 0} (-1)^i \dim_{k(s)} \mathrm{H}^i(X_s, \mathscr{F}_s)$ is locally constant.*

Proof. For a map $A^n \to A^s$, the locus where the rank is $\geq i$ is open. One deduces the first claim. For the second we have $\sum_{i \geq 0} (-1)^i \dim_{k(s)} \mathrm{H}^i(X_s, \mathscr{F}_s) = \sum (-1)^i \dim_{k(s)} K^i \otimes_A k(s)$. ☐

Corollary 11.4. *The following conditions are equivalent :*
(1) $\mathrm{H}^i(X, \mathscr{F}) = 0$ *for all* $i > 0$,
(2) $\mathrm{H}^i(X_s, \mathscr{F}_s) = 0$ *for all* $i > 0$,

Moreover, if this is the case, $\mathrm{H}^0(X, \mathscr{F})$ *is a flat A-module and the map* $\mathrm{H}^0(X, \mathscr{F}) \otimes_A k(x) \to \mathrm{H}^0(X_x, \mathscr{F}_x)$ *is an isomorphism.*

Proof. In case of 1, we see that $0 \to \mathrm{H}^0(X, \mathscr{F}) \to K^0 \to K^1 \to \cdots$ is exact. It follows that $\mathrm{H}^0(X, \mathscr{F})$ is flat and tensoring with $k(x)$, the sequence remains exact. In case of 2. Let i be the smallest degree such that $\mathrm{H}^i(X, \mathscr{F}) \neq 0$. We claim that $\mathrm{H}^i(X, \mathscr{F}) \otimes_A k(x) \to \mathrm{H}^i(X_x, \mathscr{F}_x)$ is an isomorphism. Consider the sequence $0 \to \ker(d_i) \to K^i \to \mathrm{Im}(d_i) \to 0$. Since $\mathrm{Im}(d_i)$ is flat (because $0 \to \mathrm{Im}(d_i) \to K^{i+1} \to \cdots$ is exact) we see that $\ker(d_i) \otimes$

$k(x) = \ker(K^i \otimes k(x) \to K^{i+1} \otimes k(x))$. Since $\mathrm{Im}(d_{i-1}) \to \ker(d_i) \to \mathrm{H}^i(X, \mathscr{F}) \to 0$ is exact, tensoring with $k(x)$ shows the claim. We deduce that if $i > 0$, $\mathrm{H}^i(X, \mathscr{F}) = 0$ by Nakayama. Therefore $i = 0$. $\square$

11.2.5. *Hilbert Polynomials.* Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n_k$ where k is a field. We let $P_{\mathscr{F}}(m) = \chi(\mathscr{F}(m)) = \sum_{i \geq 0} (-1)^i \dim_k \mathrm{H}^i(\mathbb{P}^n_k, \mathscr{F}(m))$.

Theorem 11.6. *$P_{\mathscr{F}}(m)$ is a polynomial of degree the dimension of the support of $\mathscr{F}$.*

Example 18. If $\mathscr{F}$ is supported on a non-singular curve of genus g, $P_{\mathscr{F}}(m) = m\deg\mathscr{O}(1)\mathrm{rk}(\mathscr{F}) + \deg(\mathscr{F}) + \mathrm{rk}(\mathscr{F})(1-g)$.

Corollary 11.5. *Let $X \to S$ be a projective scheme. Let $\mathscr{F}$ be a coherent sheaf on X, flat over A. Then the function* Spec $A \to \mathbb{Q}[X]$ *which sends x to $P_{\mathscr{F}_x}(m)$ is locally constant.*

11.2.6. *Castelnuovo-Mumford regularity theorem.*

Theorem 11.7. *Let k be a field and let $\mathscr{F} \hookrightarrow \mathscr{O}^{\ell}_{\mathbb{P}^n_k}$ be a coherent sheaf on $\mathbb{P}^n_k$. With Hilbert polynomial P. Then, there is $r_0 \geq 0$, depending only on P, n, ℓ such that :*

(1) *For all $r \geq r_0$, for all $i > 0$, $\mathrm{H}^i(\mathbb{P}^n_k, \mathscr{F}(r)) = 0$,*
(2) *For all $r \geq r_0$, $\mathrm{H}^0(X, \mathscr{O}_{\mathbb{P}^n_k}(r - r_0)) \otimes \mathrm{H}^0(\mathbb{P}^n_k, \mathscr{F}(r_0)) \to \mathrm{H}^0(\mathbb{P}^n_k, \mathscr{F}(r))$ is surjective.*

Proof. See [Nit05], thm. 2.3. $\square$

11.3. **Schemes of homomorphisms of coherent sheaves.**

Theorem 11.8. *Let $\pi : X \to S$ be a projective scheme. Let $\mathscr{F}$ be a coherent sheaf which is flat over S. There exists a coherent sheaf $\mathscr{Q}$ on S and a functorial isomorphism for any quasi-coherent sheaf $\mathscr{G}$ on S :*

$$\pi_\star(\mathscr{F} \otimes_{\mathscr{O}_X} \pi^\star \mathscr{G}) = \underline{\mathrm{Hom}}_{\mathscr{O}_S}(\mathscr{Q}, \mathscr{G})$$

Proof. Let $\mathscr{G}$ be attached to the A-module G. We have $\mathrm{H}^0(\mathscr{F} \otimes_{\mathscr{O}_X} \pi^\star \mathscr{G}) = \mathrm{Ker}(K^0 \otimes_A G \to K^1 \otimes_A G)$. Take $Q = \mathrm{coker}(K_1^\vee \to K_0^\vee)$. $\square$

Let $\mathscr{Q}$ be a coherent sheaf over S. Let $\mathbf{V}(\mathscr{Q}) = \mathrm{Spec}_S \, \mathrm{Sym}_{\mathscr{O}_S} Q$. We have $\mathrm{Hom}(Q, \pi_\star \mathscr{O}_T) = \mathbf{V}(\mathscr{Q})(T)$.

Theorem 11.9. *Let $\mathscr{E}$, $\mathscr{F}$ be coherent sheaves over $X \to S$ a projective scheme. Assume that $\mathscr{F}$ is flat over S. Consider the functor $\mathbf{Hom}(\mathscr{E}, \mathscr{F}) : (Sch/S)^{opp} \to SET$ which sends T to $\mathrm{Hom}_{X_T}(\mathscr{E}|_{X_T}, \mathscr{F}|_{X_T})$. Then this is representable by a linear scheme $\mathbf{V}$.*

Proof. First assume that $\mathscr{E}$ is locally free. Then we want to represent the functor $T \mapsto \mathrm{H}^0(X_T, \mathscr{E}^\vee \otimes \mathscr{F})$ and by the above theorem this is representable by $\mathbf{V}(\mathscr{Q})$. In general, pick a resolution $\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{E} \to 0$ by locally free sheaves. Consider the map $\mathbf{V}(\mathscr{Q}_0) \to \mathbf{V}(\mathscr{Q}_1)$. Then $\mathbf{V}$ is the kernel of this map. $\square$

11.4. **The diagonal of $Bun_{n,X}$ is representable by schemes.**

Proposition 11.3. *The diagonal of $Bun_{n,X}$ is representable by schemes.*

Proof. Let T be a k-scheme. Let $T \to Bun_{n,X} \times_S Bun_{n,X}$ be a morphism. Let $\mathcal{E}, \mathcal{E}'$ be the vector bundles over X_T corresponding to this morphism.

Consider the functor $(Sch/T)^{opp} \to SET$ sending U to $\mathrm{Isom}_{X_U}(\mathcal{E}_{X_U}, \mathcal{E}'_{X_U})$. We want to prove that it is representable. We know that $\mathbf{Hom}(\mathcal{E}, \mathcal{E}')$, $\mathbf{Hom}(\mathcal{E}', \mathcal{E}')$, $\mathbf{Hom}(\mathcal{E}, \mathcal{E})$, $\mathbf{Hom}(\mathcal{E}', \mathcal{E})$ are representable and we have a map : $\mathbf{Hom}(\mathcal{E}, \mathcal{E}') \times \mathbf{Hom}(\mathcal{E}', \mathcal{E}) \to \mathbf{Hom}(\mathcal{E}, \mathcal{E}) \times \mathbf{Hom}(\mathcal{E}', \mathcal{E}')$ given by composing.

The preimage of $Id_{\mathcal{E}} \times Id_{\mathcal{E}'}$ is the scheme representing $U \mapsto \mathrm{Isom}_{X_U}(\mathcal{E}_{X_U}, \mathcal{E}'_{X_U})$. $\square$

11.5. **Flatness.**

11.5.1. *Preliminary results.*

Lemma 11.4. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^N_S$. Then $\mathscr{F}$ is flat over S if and only if there exists r_0 such that for all $r \geq r_0$, $\pi_\star \mathscr{F}(r)$ is locally free over S. Moreover $r_0 = \inf\{r, \mathrm{R}^i\pi_\star\mathscr{F}(r') = 0 \ \forall r' \geq r\}$ works.*

Proof. We see that $\mathscr{F}$ is flat if and only if $\mathscr{F}(r)$ is flat for some r. If $\pi_\star\mathscr{F}(r)$ if flat for all $r \geq r_0$ we deduce that $M(\mathscr{F}(r_0))$ is flat and therefore $M(\tilde{\mathscr{F}}(r_0))$ is flat. Conversely, we can use theorem 11.3. □

Lemma 11.5. *Let $X \to S$ be a finite type morphism where S is an integral scheme. Let $\mathscr{F}$ be a coherent sheaf over X. There is a non-empty open subset $U \subseteq S$ such that $\mathscr{F}|_U$ is flat over U.*

This is a consequence of the following proposition :

Proposition 11.4. *Let A be an integral ring, let B be an A-algebra of finite type. Let M be a finite B-module. There exists $f \in A \setminus \{0\}$ such that M_f is free over A_f.*

Proof. Let $K = \mathrm{Frac}(A)$. The proof is by induction on the dimension of the support of $M \otimes K$. We also note that if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ and the lemma holds for M_1 and M_3 it holds for M_2. Suppose that $M \otimes K$ is zero. Let $m_1, \cdots, m_n$ be generators of M as a B-module. There exists $f \in A \neq 0$ such that $fm_1 = \cdots = fm_n = 0$. Therefore $M \otimes_A A_f = 0$. In general, we recall that M is a successive extension of modules of the form $B/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal ([Sta13], lem. TAG 00L0). We reduce to the case that $M = B$ is a domain. By Noether normalization, there exists $f \in A$ and $b_1, \cdots, b_n \in B$ such that $A_f \to A_f[b_1, \cdots, b_n] \to B_f$ where $A_f[b_1, \cdots, b_n]$ is a polynomial algebra and $A_f[b_1, \cdots, b_n] \to B$ is finite. We let r be the generic rank of M over $A_f[b_1, \cdots, b_n]$. We have a map $0 \to A_f[b_1, \cdots, b_n]^r \to M \to T \to 0$ and the dimension of the support of $T \otimes K$ is less than n. □

Corollary 11.6. *Let $X \to S$ be a finite type morphism where S is a Noetherian scheme. Let $\mathscr{F}$ be a coherent sheaf over X. There are finitely many locally closed subschemes S_i of S such that $|S| = \cup_i |S_i|$ and $\mathscr{F}|_{S_i}$ is flat.*

Proof. We first assume that S is reduced. Then we can find an open subscheme $U_1 \subseteq S$ such that $\mathscr{F}|_{U_1}$ is smooth. We then consider the reduced complement of U_1 and so on... The process ends by Noetherianity. □

Corollary 11.7. *Let $X \to S$ be a finite type morphism where $S = \mathrm{Spec}\ A$ is a Noetherian scheme. Let $\mathscr{F}$ be a coherent sheaf over X. Then there exists r_0 such that for all $r \geq r_0$, for all $s \in S$, $\mathrm{H}^i(X_s, \mathscr{F}(r)) = 0$ for all $i > 0$ and the map*

$$\mathrm{H}^0(X, \mathscr{F}(r)) \otimes_A k(s) \to \mathrm{H}^0(X_s, \mathscr{F}(r))$$

is an isomorphism.

Proof. We consider the finitely many S_j from the previous corollary. We can assume that they are closed in affine open of S. The first statement follows by taking r_0 such that for all $r \geq r_0$, $\mathrm{H}^i(X_{S_i}, \mathscr{F}(r)) = 0$ for all $i > 0$ and all j. For the second statement write $S_j = \mathrm{Spec}\ A_{f_j}/I_j$. The map $\mathrm{H}^0(X, \mathscr{F}(r)) \otimes_A A_{f_i} \to \mathrm{H}^0(X \times_S \mathrm{Spec}\ A_{f_j}, \mathscr{F}(r))$ are isomorphisms. Consider the maps :

$$\mathrm{H}^0(X \times_S \mathrm{Spec}\ A_{f_j}, I_j\mathscr{F}(r)) \to \mathrm{H}^0(X \times_S \mathrm{Spec}\ A_{f_j}, \mathscr{F}(r))$$
$$\to \mathrm{H}^0(X \times_S \mathrm{Spec}\ A_{f_j}, \mathscr{F}(r)/I_j) \to \mathrm{H}^1(X \times_S \mathrm{Spec}\ A_{f_j}, I_j\mathscr{F}(r))$$

We can take r_0 such that for all $r \ \geq r_0$, all j, $\mathrm{H}^1(X \times_S \mathrm{Spec}\ A_{f_j}, I_j\mathscr{F}(r)) = 0$ and the map $I_j \otimes_A \mathrm{H}^0(X \times_S \mathrm{Spec}\ A_{f_j}, \mathscr{F}(r)) \to \mathrm{H}^0(X \times_S \mathrm{Spec}\ A_{f_j}, I_j\mathscr{F}(r))$ is surjective. □

11.5.2. *Flattening stratification.* We begin by Fitting ideals. Let R be a noetherian ring. Let M be a finite R-module. We can choose a presentation : $R^{n'} \to R^n \to M \to 0$. We let A be the $n' \times n$ matrix giving the map.

Definition 11.1. *We let* $\mathrm{Fitt}_i(M)$ *be the ideal generated by the* $n-i \times n-i$*-minors of* A.

Proposition 11.5. *The ideals* $\mathrm{Fitt}_i(M)$ *are independent of the choice of the presentation. Moreover, we have* $0 \subseteq \mathrm{Fitt}_0(M) \subseteq \cdots \subseteq \mathrm{Fitt}_n(M) = R$.

Let R^n be the free module. We see that $\mathrm{Fitt}_{n-1}(R^n) = 0$ and $\mathrm{Fitt}_n(R^n) = R$.

Lemma 11.6. *Let* $R \to R'$ *be a ring map. Then* $\mathrm{Fitt}_i(M \otimes_R R') = \mathrm{Fitt}_i(M)R'$.

Proposition 11.6. *A module* M *is locally free of rank* k *if and only if* $\mathrm{Fitt}_{k-1}(M) = 0$ *and* $\mathrm{Fitt}_k(M) = R$.

Proof. If M is locally free of rank k, we reduce to the free case by Zariski localization. Conversely, assume that $\mathrm{Fitt}_k(M) = R$. Let m be a maximal ideal. We see that $M/\mathfrak{m}M$ is generated by k-elements. Therefore, (after localization) we may assume that we have a surjection $R^k \to M$. Take a presentation $R^{k'} \to R^k \to M$. We see that the ideal generated by the coefficients of the matrix is zero. Therefore the map $R^{k'} \to R^k$ is zero and $M = R^k$. □

Theorem 11.10. *Let* S *be a Noetherian scheme. Let* $\mathscr{F}$ *be a coherent sheaf on* $\mathbb{P}^N_S$. *Then the function* $S \to \mathbb{Q}[X]$ *which maps* $s \in S$ *to* $P_{\mathscr{F}_s}$ *takes finitely many values. Let* I *be the set of values. For any* $f \in I$, *there exists a locally closed subscheme* S_f *of* S *with the following property*

1. *The points of* S_f *are exactly the points of* S *with Hilbert polynomial equal to* f. *In particular,* S *is the disjoint union of the* S_f.
2. *Let* $S' = \coprod S_f$. *Then the sheaf* $\mathscr{F}_{S'}$ *is flat over* S'. *Moreover, a morphism* $T \to S$ *factors through* S' *if and only if* $\mathscr{F}_T$ *is flat over* T.
3. *Define a partial order on* I *by declaring that* $f \geq g$ *if* $f(n) \geq g(n)$ *for all* $n >>$. *Then the closure in* S *of* S_f *is contained in the union of the* S_g *with* $g \geq f$.

Proof. We can construct finitely many locally closed subscheme S' of S such $|S| = \cup|S'|$ and that $\mathscr{F}_{S'}$ is flat. Since the Hilbert Polynomial is locally constant over S', we conclude that the set of possible Hilbert Polynomials is finite.

Next, we consider the case that $n = 0$. Therefore, we have a coherent sheaf over S. Without loss of generality, we can assume that $S =$ Spec A if affine and we let $\mathrm{H}^0(S, \mathscr{F}) = M$. Let us pick a presentation of M : $A^r \to A^s \to M \to 0$, the map $A^r \to A^s$ is represented by an $r \times s$ matrix N. We now consider the ideals $\mathrm{Fitt}_i(M) = \{\text{determinant of size } s-i \text{ of } N\}$. We let $S_{\geq i} = V(\mathrm{Fitt}_{\mathrm{i}-1})$ and we have $S_i = S_{\geq i} \backslash \subseteq S_{\geq i+1}$. In this case, $P_{\mathscr{F}_s} = \dim_{k(s)}(\mathscr{F}|_s)$ and the S_k, $k \in \mathbb{Z}$ are the flattening stratification. We now deal with the general case. We claim that there exists r_0 such that for all $r \geq r_0$, we have :

1. For all $s \in S$, the maps $\mathrm{H}^0(\mathbb{P}^N_S, \mathscr{F}(r)) \to \mathrm{H}^0(\mathbb{P}^N_s, \mathscr{F}_s(r))$ are surjective,
2. For all $s \in S$, $\mathrm{H}^i(\mathbb{P}^N_s, \mathscr{F}_s(r)) = 0$ for all $i > 0$.

We now consider the flattening stratification of $\pi_\star\mathscr{F}(r_0)$: $S = \coprod_{f \in I} S_{f(r_0)}$. Then we consider the flattening stratification of $\pi_\star\mathscr{F}(r_0) \oplus \pi_\star\mathscr{F}(r_0+1) ... \oplus \pi_\star\mathscr{F}(r_0+l)$: $S = \cup_{f \in I} S_{f(r_0), f(r_0+1), \cdots}$ where on $S_{f(r_0), f(r_0+1), \cdots}$, $\pi_\star\mathscr{F}(r_0+i)$ is flat of rank $f(r_0+i)$. These stratifications refine each other. We observe that by property 1) and 2) a point $s \in S$ such that $\mathscr{F}_s$ has hilbert polynomial f lands into the stratum $S_{f(r_0),f(r_0+1),\cdots}$. Now, for $l >>$, the map $I \to \mathbb{Z}^l$, $f \mapsto f(r_0), \cdots, f(r_0+l)$ is injective.

We deduce that the stratification is set theoretically constant for $l >>$. By Noetherianity it is constant for $l >>$.

We therefore have found $S = \coprod S_f$. We see that the map $\pi_\star \mathscr{F}(r)|_{S_f} \to \pi_\star(\mathscr{F}(r)|_{S_f})$ is an isomorphism by property 1). We deduce that $\mathscr{F}|_{S_f}$ is flat.

Let $T \to S$ be a map such that $\mathscr{F}|_T$ is flat. We may assume that the Hilbert polynomial of $\mathscr{F}$ is constant over T equal to f. We see from 2) that for all $r \geq r_0$, $(\pi_T)_\star \mathscr{F}(r)$ is flat for $r \geq r_0$ of rank $f(r_0)$. We deduce that T factors through S_f.

□

11.6. **Quot schemes.** Let $X \to S$ be a projective scheme. Let $\mathscr{F}$ be a coherent sheaf over X, which is flat over S. We let $Quot_{\mathscr{F}/X/S} : (Sch/S)^{op} \to SET$ be the functor which sends T to the set of isomorphism classes of quotients $\mathscr{F}_T \to \mathscr{G}$ which are flat over T.

Let $P \in \mathbb{Q}[X]$. We can consider the locus $Quot^P_{\mathscr{F}/X/S}$ where the Hilbert polynomial of $\mathscr{G}$ is constant equal to P.

Theorem 11.11. *The functor $Quot^P_{\mathscr{F}/X/S}$ is representable by a projective scheme.*

Remark 11.3. Here we understand projective in the following "weak" sense : there is a closed embedding of $Quot^P_{\mathscr{F}/X/S}$ inside $\mathbb{P}(\mathscr{W})$ where $\mathscr{W}$ is a locally free sheaf over S of constant rank. This means that $\mathbb{P}(\mathscr{W})$ is Zariski locally over S a projective space.

11.6.1. *Grassmanians.* We consider the case where $X = S$, therefore $\mathscr{F}$ is a vector bundle of rank r, and P has to be given by an integer $0 \leq d \leq r$. In this case $Quot^P_{\mathscr{F}/X/S} = GR(\mathscr{F}, d)$ is a "twisted" grassmanian. It is the grassmanian when $\mathscr{F}$ is free.

Proposition 11.7. *The sheaf $GR(\mathscr{F}, d)$ is representable by a projective scheme.*

Proof. We reduce to the case that $\mathscr{F}$ is free (work locally on S). In this case, we let $GR(\mathscr{F}, d) = GR(r, d)$. Let $J \subseteq \{1, \cdots, r\}$ be a set of cardinal d. Let $GR(r, d)_J$ be the subfunctor which parametrizes quotients $\mathscr{O}^r_T \xrightarrow{\phi} \mathscr{G}$ such that $\{\phi(e_i)\}_{i \in J}$ generate $\mathscr{G}$. We see that $GR(r, d)_J$ is representable by Spec $A[x_{i,j},\ i \in J^c, j \in J]$ where $\phi(e_i) = \sum_{j \in J} x_{i,j}\phi(e_j)$ for $i \in J^c$. We also see that $GR(r, d)_J$ is an open subsheaf of $GR(r, d)$. Indeed, let $T \to GR(r, d)$ be a map. We see that $T \times_{GR(r,d)} GR(r, d)_J = T \setminus \sup(\mathscr{G}/\langle\phi(e_j),\ j \in J^c\rangle)$. We also see that $\coprod_J GR(r, d)_J \to GR(r, d)$ is a covering. This proves the proposition. □

11.6.2. *Reduction.*

Lemma 11.7. *It suffices to prove the theorem for $Quot_{\mathscr{O}_{\mathbb{P}^N_S}(r)^t/\mathbb{P}^N_S/S}$.*

Proof. Consider a closed immersion $X \hookrightarrow \mathbb{P}^N_S$ and a surjective map $\mathscr{O}_{\mathbb{P}^N_S}(r)^t \to \mathscr{F}$. Then we claim that the map $Quot_{\mathscr{F}/X/S} \hookrightarrow Quot_{\mathscr{O}_{\mathbb{P}^N_S}(r)^t/\mathbb{P}^N_S/S}$ is a closed immersion. Indeed let $\mathscr{F}' = \ker(\mathscr{O}_{\mathbb{P}^N_S}(r)^t \to \mathscr{F})$. Then $Quot_{\mathscr{F}/X/S}$ is the fiber of the map $Quot_{\mathscr{O}_{\mathbb{P}^N_S}(r)^t/\mathbb{P}^N_S/S} \to$ $\mathbf{Hom}(\mathscr{F}', \mathscr{G})$ at the zero section. □

11.6.3. *End of the proof.* We fix the Hilbert Polynomial P. We now assume also that $X = \mathbb{P}^N_S$ and $\mathscr{F} = \mathscr{O}_{\mathbb{P}^N_S}(r)^t$.

We claim that there exists ν such that for any map $T \to Quot^P_{\mathscr{F}/X/S}$ and any associated short exact sequence $0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0$ of X_T, we have (for the projection $\pi : X_T \to T$:

(1) We have $\mathrm{R}^i\pi_\star\mathscr{H}(\nu) = \mathrm{R}^i\pi_\star\mathscr{F}(\nu) = \mathrm{R}^i\pi_\star\mathscr{G}(\nu) = 0$ for all $i > 0$. It follows that $\pi_\star\mathscr{H}(\nu)$, $\pi_\star\mathscr{F}(\nu)$, $\pi_\star\mathscr{G}(\nu)$ are locally free, and the sequence : $0 \to \pi_\star\mathscr{H}(\nu) \to \pi_\star\mathscr{F}(\nu) \to \pi_\star\mathscr{G} \to 0$ is exact.
(2) The map $\pi^\star\pi_\star\mathscr{H}(\nu) \to \mathscr{H}(\nu)$ is surjective.

We deduce that there is a commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \pi^\star\pi_\star\mathscr{H}(\nu) \arrow[r] \arrow[d] & \pi^\star\pi_\star\mathscr{F}(\nu) \arrow[r] \arrow[d] & \pi^\star\pi_\star\mathscr{G}(\nu) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{H}(\nu) \arrow[r] & \mathscr{F}(\nu) \arrow[r] & \mathscr{G}(\nu) \arrow[r] & 0
\end{tikzcd}$$

And the quotient map $\mathscr{F}(\nu) \to \mathscr{G}(\nu)$ is determined by the map $\pi_\star\mathscr{H}(\nu) \to \pi_\star\mathscr{F}(\nu)$.

This gives an embedding $Quot^P_{\mathscr{F}/X/S} \to GR(\pi_\star\mathscr{F}(\nu), P(\nu))$. This identifies $Quot^P_{\mathscr{F}/X/S}$ with a locally closed subscheme of $GR(\pi_\star\mathscr{F}(\nu), P(\nu))$. Indeed, let $0 \to \mathscr{H}_1 \to \pi_\star\mathscr{F}(\nu) \to \mathscr{G}_1 \to 0$ be the universal short exact sequence.

Let us consider the quotient $\pi^\star\mathscr{H}_1 \to \mathscr{F} \to \mathscr{F}/\pi^\star\mathscr{H}_1 \to 0$. Then $Quot^P_{\mathscr{F}/X/S}$ is the locally closed subscheme of $GR(\pi_\star\mathscr{F}(\nu), P(\nu))$ where $\mathscr{F}/\pi^\star\mathscr{H}_1 \to 0$ is flat and has Hilbert polynomial $P(\nu + -)$.

We have proved that $Quot^P_{\mathscr{F}/X/S}$ is a quasi-projective S-scheme. It suffices to prove it is proper. We use the valuative criterion. Let R be a discrete valuation ring with quotient field K. Consider a map $\operatorname{Spec} K \to Quot^P_{\mathscr{F}/X/S}$. And assume that $\operatorname{Spec} R \to S$.

Let $\mathscr{F}_{\operatorname{Spec}\ K} \to \mathscr{G}$ be the quotient. Let $\mathscr{G}'$ be the image of $\mathscr{F}_{\operatorname{Spec} R}$ inside $\mathscr{G}$. This is a flat sheaf over R (because torsion free), and it provides the required unique extension to a point $\operatorname{Spec} R \to Quot^P_{\mathscr{F}/X/S}$.

11.7. $Bun_{n,X}$ **is an algebraic stack.** For any d, N, we let $\mathfrak{U}_{d,N}$ be the substack of $Bun_{n,X}$ defined by $Ob(\mathfrak{U}_{d,N})_S$ is the set of $\mathcal{E} \in Ob(Bun_{n,X})_S$ satisfying :

(1) $\mathrm{R}^1(p_S)_\star\mathcal{E}(d) = 0$,
(2) $p_S^\star(p_S)_\star\mathcal{E}(d) \to \mathcal{E}(d)$ is surjective,
(3) $(p_S)_\star\mathcal{E}(d)$ is of rank N.

The map $\mathfrak{U}_{d,N} \to Bun_{n,X}$ is a schematic open immersion. Indeed let $S \to Bun_{n,X}$. Let us compute the fiber product $S \times_{Bun_{n,X}} U_{d,N}$. This is the locus in S where conditions (1), (2) and (3) hold.

Condition (1) holds over the open $S' = S \setminus \operatorname{Support}(\mathrm{R}^1(p_S)_\star\mathcal{E}(d))$. Condition (1) and (2) holds over $S'' = S' \setminus \operatorname{Support}(\operatorname{coker} p_S^\star(p_S)_\star\mathcal{E}(d) \to \mathcal{E}(d))$. By (1), $(p_S)_\star\mathcal{E}(d)$ is locally free of rank N, thus condition (3) holds over an open and closed subscheme of S''.

Note also that N is related to the degree of $\mathcal{E}$ by the formula $N = d\deg\mathscr{O}(1)n + \deg(\mathcal{E}) + n(1-g)$.

By the vanishing theorems we have : $Bun_{n,X} = \cup_{d,N}\mathfrak{U}_{d,N}$.

Let $Y_{d,N} \to \mathfrak{U}_{d,N}$ be defined by $Ob(Y_{d,N})_S = \{\mathcal{E}, \lambda : \mathscr{O}_S^N \simeq (p_S)_\star\mathcal{E}(d)\}$. The map $Y_{d,N} \to \mathfrak{U}_{d,N}$ is representable by schemes and is a GL_N-torsor.

We have a map $Y_{d,N} \to Quot_{\mathscr{O}_X^N/X/k}$. We claim that this is an open immersion. Let $T \to Quot_{\mathscr{O}_X^N/X/k}$. Let us consider a quotient $\mathscr{O}^N_{X_T} \to \mathscr{G}$ where $\mathscr{G}$ is flat over T. We first claim that the set of $t \in T$ such that $\mathscr{G}_t$ is flat over X_t of rank n is open in T. This follows from the following lemma :

Lemma 11.8 ([Sta13] TAG 00MP)**.** *Let $R \to S$ be a map of local rings. Let M be an S-module of finite type which is flat over R. Assume that $M/\mathfrak{m}_R M$ is flat over $S/\mathfrak{m}_R S$. Then M is flat over S.*

Let U be this open of T. Then $\mathscr{G}_U$ is flat over X_U. Now the condition that U factors through $Y_{d,N}$ are open ($\mathscr{G}_U$ is of rank n + (1), (2), (3)).

We also observe that $Y_{d,N}$ is quasi-projective, since the Hilbert polynomial of $\mathscr{G}_U$ is determined by N. We deduce that $\coprod_{d,N} Y_{d,N} \to Bun_{n,X}$ is a presentation.

12. Lecture XI : The Harder Narashiman filtration and families

12.0.1. *Invariance of the HN filtration under base extension.*

Theorem 12.1. *Let $X \to \operatorname{Spec} k$ be a complete, non-singular geometrically connected curve. Let $\mathcal{E}$ be a vector bundle over X. Let $k \to l$ be a field extension. Let $X_l = X \times_{\operatorname{Spec} k} \operatorname{Spec} l$. Let $p : X_l \to X$. Let $\mathcal{E}_\bullet$ be the HN filtration of $\mathcal{E}$. Then $p^\star\mathcal{E}_\bullet$ is the HN filtration of $p^\star\mathcal{E}$.*

12.0.2. *Partial order.* Let $\mathbb{Q}^n_+$ be the cone in $\mathbb{Q}^n$ of vectors $(x_1, \cdots, x_n)$ with $x_1 \geq x_2 \cdots \geq x_n$. We define a partial order in $\mathbb{Q}^n_+$: we say that $x = (x_1, \cdots, x_n) \geq x' = (x'_1, \cdots, x'_n)$ if $x_1 \geq x'_1, x_1 + x_2 \geq x'_1 + x'_2, \cdots, \sum_{i=1}^{n-1} x_i \geq \sum_{i=1}^{n-1} x'_i, \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x'_i$.

Let $\mathcal{E}$ be a rank n vector bundle with HN filtration $\mathcal{E}_\bullet$. We let $HN(\mathcal{E}) = (\mu(\mathcal{E}_1)^{rk(\mathcal{E}_1)}, \mu(\mathcal{E}_2)^{rk(\mathcal{E}_2)}, \cdots) \in \mathbb{Q}^n_+$ be the HN vector of $\mathcal{E}$.

12.0.3. *Behaviour of the HN filtration in families.* Let $S = \operatorname{Spec} A \to \operatorname{Spec} k$. Let $\mathcal{E}$ be a vector bundle over X_S.

Since the functions rk and deg are locally constant on S we can assume (after replacing S by a connected component) that they are constant. We let $\mu = \mu(\mathcal{E}_s)$. For any $P \in \mathbb{Q}_+$, $P \geq (\mu)^n$, we let $S^{\leq P}$ be the set of points $s \in S$ such that $HN(\mathcal{E}_s) \leq P$. We also let $S^{=P}$ be the set of points $s \in S$ such that $HN(\mathcal{E}_s) \leq P$.

Theorem 12.2. *$S^{\leq P}$ is open in S.*

Lemma 12.1. *The set $S^{=(\mu)^n}$ is open.*

Proof. First assume $S = \operatorname{Spec} l$ is a point. Recall that if $\mathcal{E}' \hookrightarrow \mathcal{E}$ is a subbundle, then $\deg \mathcal{E}' + \operatorname{rk}(\mathcal{E}')(1-g) < \dim \mathrm{H}^0(X, \mathcal{E})$. We deduce that $\mu(\mathcal{E}') \leq (g-1) + \dim \mathrm{H}^0(X, \mathcal{E})$. Let $r = \sup_{s \in S} \dim H^0(X_s, \mathcal{E}_s)$. We deduce that $\mathcal{E}_s$ is semi-stable of slope μ if and only if $\mathcal{E}_s$ has no subbundle $\mathcal{E}'_s$ with slope $\mu < \mu(\mathcal{E}'_s) < (g-1) + r$. One similarly deduces that $\mathcal{E}_s$ is semi-stable of slope μ if and only if $\mathcal{E}_s$ has no quotient $\mathcal{G}_s$ with slope $q < \mu(\mathcal{G}_s) < \mu$ for some q. It follows that to prove that $\mathcal{E}_s$ is semistable, it suffices to show that $\mathcal{E}_s$ has no quotient with Hilbert polynomial in a finite list $P_1, \cdots, P_r$.

Consider the schemes $Quot^{P_i}_{\mathcal{E}/X_S/S} \to S$. Their image are closed subschemes $S_i \subseteq S$. And over $S \setminus \cup_i S_i$, we see that $\mathcal{E}$ is indeed semi-stable of slope μ. ☐

Let η be a generic point of S.

Lemma 12.2. *The set $S^{=HN(\mathcal{E}_\eta)}$ contains an non empty open subset.*

Proof. We have the HN filtration $\mathcal{E}_{\eta,\bullet}$ of $\mathcal{E}_\eta$. By general results, after replacing S by its reduction and then by an open subset containing η, we can extend this to a filtration $\mathcal{E}_\bullet$ on $\mathcal{E}$. We see by the lemma that the locus where all $\mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable is open in S. ☐

Corollary 12.1. *We have a stratification into locally closed subsets $S = \coprod S_i$ where over each S_i the HN filtration is constant.*

Lemma 12.3. *Assume that $S = \{\eta, s\}$ is the spectrum of a DVR. Then $HN(\mathcal{E}_\eta) \leq HN(\mathcal{E}_s)$.*

Proof. Let $\mathcal{E}_{\eta,\bullet}$ be the HN filtration of $\mathcal{E}_\eta$. Then we let $\mathcal{E}_i = \mathcal{E}_{\eta,i} \cap \mathcal{E}$. This is an $\mathscr{O}_S$-flat sheaf and we see that $\mathcal{E}_{i,s} \to \mathcal{E}_s$. Since $\mu(\mathcal{E}_{i,s}) = \mu(\mathcal{E}_{\eta,i})$ we deduce that $HN(\mathcal{E}_\eta) \leq HN(\mathcal{E}_s)$. ☐

Corollary 12.2. *$S^{\leq P}$ is open in S.*

Proof. This is a constructible set which is stable under generization, therefore it is open. $\square$

12.1. **Consequences for $\mathrm{Bun}_{n,X}$.** For any $P \in \mathbb{Q}_+^n$, we define the substack $Bun_{n,X}^{\leq P}$ on $Bun_{n,X}$ whose objets are $(S \to \mathrm{Spec}\ k, \mathcal{E})$ such that for all points $s \in S$, $HN(\mathcal{E}_s) \leq P$.

Theorem 12.3. *The substack $Bun_{n,X}^{\leq P}$ is open in $Bun_{n,X}$ and of finite type.*

Proof. The fact that this is an open substack follows from theorem 12.2. To see that $Bun_{n,X}^{\leq P}$ is quasi-compact it suffices to prove that for any $(S, \mathcal{E})$, there exists d and N such that $Bun_{n,X}^{\leq P} \subseteq \mathfrak{U}_{d,N}$. This follows from the proposition below. $\square$

Proposition 12.1. *Let $\mu \in \mathbb{Q}$. Let $\mathcal{E}$ be a vector bundle on a projective curve $X \to \mathrm{Spec}\ k$ which is semi-stable of slope μ. For all $d \geq -\mu + 2g$:*

1. $\mathrm{H}^1(X, \mathcal{E}(d)) = 0$,
2. $\mathrm{H}^0(X, \mathcal{E}(d)) \otimes \mathscr{O}_X \to \mathcal{E}(d)$ *is surjective.*

Proof. By the duality theorem, we have that $\mathrm{H}^1(X, \mathcal{E}(d))^\vee = \mathrm{H}^0(X, \mathcal{E}^\vee(-d) \otimes \Omega^1_{X/k})$. Since $\mathcal{E}^\vee(-d) \otimes \Omega^1_{X/k}$ is semi-stable of slope $-\mu - d\mathrm{deg}\mathscr{O}_X(1) + 2g - 2$, we see that for $d > -\mu + 2g - 2$, $\mathrm{H}^0(X, \mathcal{E}^\vee(-d) \otimes \Omega^1_{X/k}) = 0$.

For the second point, let us assume that k is infinite (we may extend scalars). We can find an hyperplane $H \subseteq \mathbb{P}^N$ such that $X \cap H = D$ is an effective divisor (just take $x \in X(k)$ and find an hyperplane H such that $x \notin H$).

We thus have an exact sequence $0 \to \mathcal{E}(n) \to \mathcal{E}(n+1) \to \mathcal{E}(n+1)|_D \to 0$. It induces an exact sequence $0 \to \mathrm{H}^0(X, \mathcal{E}(n)) \to \mathrm{H}^0(X, \mathcal{E}(n+1)) \to \mathrm{H}^0(X, \mathcal{E}(n+1)|_D) \to 0$ for all $n > -\mu + 2g - 2$.

We consider the map $\mathrm{H}^0(\mathbb{P}^N, \mathscr{O}(1)) \otimes_k \mathrm{H}^0(X, \mathcal{E}(n)) \to \mathrm{H}^0(X, \mathcal{E}(n+1))$.

For all $n > -\mu + 2g - 1$, the map $\mathrm{H}^0(X, \mathcal{E}(n)) \to \mathrm{H}^0(D, \mathcal{E}(n))$ is surjective, therefore $\mathrm{H}^0(\mathbb{P}^N, \mathscr{O}(1)) \otimes_k \mathrm{H}^0(X, \mathcal{E}(n)) \to \mathrm{H}^0(D, \mathcal{E}(n+1))$ is surjective. We deduce that $\mathrm{H}^0(\mathbb{P}^N, \mathscr{O}(1)) \otimes_k \mathrm{H}^0(X, \mathcal{E}(n)) \to \mathrm{H}^0(X, \mathcal{E}(n+1))$ is surjective for all $n > -\mu + 2g - 1$. $\square$

Corollary 12.3. *Let $P = (\mu_1, \cdots, \mu_n) \in \mathbb{Q}_+^n$. Let $(S, \mathcal{E}) \in Bun_{n,X}^{\leq P}$. Then for all $d \geq -\mu_n + 2g$, we have that :*

1. $\mathrm{R}^1(\pi_S)_\star \mathcal{E}(d) = 0$,
2. $(\pi_S)^\star (\pi_S)_\star \mathcal{E}(d) \to \mathcal{E}(d)$ *is surjective,*
3. $(\pi_S)_\star \mathcal{E}(d)$ *is of rank* $(\sum \mu_i) + dn \deg \mathscr{O}_X(1) + n(1-g)$.

13. Leture XII : Topological properties of algebraic stacks

13.1. The topological space.

13.1.1. *Points of a stack.* Let $\mathfrak{X}$ be an algebraic stack. We define its points $|\mathfrak{X}|$ as the set of equivalence classe of $\coprod_{k,f} Ob(\mathfrak{X}_{\mathrm{Spec}\ k})/\sim$ where (k, f) runs through the maps f : $\mathrm{Spec}\ k \to S$ from the spectrum of field to S. Two points $x_{k,f}$, $x'_{k',f'}$ are equivalent if there is a common field extension k'' of k and k' and a map f' : $\mathrm{Spec}\ k'' \to S$ over the maps f and f' such that $x_{k,f}$ and $x'_{k',f'}$ become isomorphic in $\mathfrak{X}_{\mathrm{Spec}\ k''}$.

If $\mathfrak{X}$ is a scheme then $|\mathfrak{X}|$ is indeed the underlying set of $\mathfrak{X}$.

13.1.2. *Open substacks.* Let $\mathfrak{X}$ be a stack. A substack $\mathfrak{X}'$ of $\mathfrak{X}$ is a strictly full subcategory $\mathfrak{X}'$ of $\mathfrak{X}$ which is a stack. Strictly full means full and that any object of $\mathfrak{X}$ isomorphic to an object of $\mathfrak{X}'$ is an object of $\mathfrak{X}'$.

Definition 13.1. *An open substack $\mathfrak{X}' \hookrightarrow \mathfrak{X}$ is a substack with the property that the map $\mathfrak{X}' \hookrightarrow \mathfrak{X}$ is representable by open immersion.*

Definition 13.2. *An closed substack $\mathfrak{X}' \hookrightarrow \mathfrak{X}$ is a substack with the property that the map $\mathfrak{X}' \hookrightarrow \mathfrak{X}$ is representable by closed immersions.*

The opens of the form $|\mathfrak{U}|$ for $\mathfrak{U} \hookrightarrow \mathfrak{X}$ an open substack are the open of a topology called the Zariski topology.

13.1.3. *The Zariski topology on $|\mathfrak{X}|$.* Let $\mathfrak{X}$ be an algebraic stack. Let $p : X \to \mathfrak{X}$ be a presentation.

Proposition 13.1.
(1) *The map $p : |X| \to |\mathfrak{X}|$ is surjective.*
(2) *To any open $U \hookrightarrow X$ we can associate an open substack of $\mathfrak{U}$ of $\mathfrak{X}$ uniquely determined by the property that $|p(U)| = |\mathfrak{U}|$.*
(3) *The subsets $|\mathfrak{U}|$ of $|\mathfrak{X}|$ where $\mathfrak{U}$ runs through the open substacks of $\mathfrak{X}$ are the open of a topology on $\mathfrak{X}$ called the Zariski topology.*
(4) *The map $p : |X| \to |\mathfrak{X}|$ is surjective, continuous and open.*
(5) *For any morphism of algebraic stacks $F : \mathfrak{X} \to \mathfrak{Y}$ the corresponding map $F : |\mathfrak{X}| \to |\mathfrak{Y}|$ is continuous.*

Proof. For the first point, consider a map $\mathrm{Spec} k \to \mathfrak{X}$. We can find a field extension k'/k such that the map $\mathrm{Spec}\ k' \to \mathrm{Spec}\ k \to \mathfrak{X}$ lifts to $\mathrm{Spec}\ k' \to X$. For the second point, we define $\mathfrak{U}$ to be substack of $\mathfrak{X}$ whose objects are locally isomorphic to objects in the image of $p(U)$. We claim that this is an algebraic stack and an open substack of $\mathfrak{X}$. This is clearly a substack (for x,y in $\mathfrak{U}$, $V \mapsto Hom_{\mathfrak{U}_V}(x_V, y_V)$ is a sheaf because we took a full subcategory of $\mathfrak{X}$ and the fact that the descent is effective in $\mathfrak{X}$ added to the local isomorphism condition we chose makes the descent effective) . It is algebraic because $U \to \mathfrak{U}$ is a presentation. Let us check that $\mathfrak{U} \to \mathfrak{X}$ is open. We take a map $T \to \mathfrak{X}$ and we need to see that $\mathfrak{U} \times_{\mathfrak{X}} T$ is an open of T.

First, this is a space. Second, assume that the map $T \to \mathfrak{X}$ lifts to X. Then $\mathfrak{U} \times_{\mathfrak{X}} T = U \times_X T$ is open. In general, we can find $T' \to T$ an fppf covering such that the map $T' \to \mathfrak{X}$ lifts to X. Then we get an open V' of T' which satisfies a descent datum for $T' \to T$. Therefore it descends to an open V of T which is $\mathfrak{U} \times_{\mathfrak{X}} T$. We deduce that the map $X \to \mathfrak{X}$ induces a quotient map.

For points (3) and (4), since $p : |X| \to |\mathfrak{X}|$ is surjective and $|X|$ being a scheme has a Zariski topology which already coincides with the definition of point (3) by our remark 1.1, we only need to prove that the family of subsets defined by (3) is the final topology associated to p. Explicitly we need to see a substack $\mathfrak{U}$ of $\mathfrak{X}$ is open if and only if it satisfies $|\mathfrak{U}| = |p(U)|$ with U an open subscheme of X. Since one way was proved in point (2) we only need the converse, let $\mathfrak{U} \to \mathfrak{X}$ be an open substack then the projection $X \times_{\mathfrak{X}} \mathfrak{U} \to X$ is an open immersion, and by the same argument we used in Remark 1.1, it is then an open subscheme of X. But $|p(X \times_{\mathfrak{X}} \mathfrak{U})| = |\mathfrak{U}|$ by point (1) because $X \times_{\mathfrak{X}} \mathfrak{U} \to \mathfrak{U}$ is a presentation of $\mathfrak{U}$.
It remains to prove point (5) : let $F : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of algebraic stacks and let $\mathfrak{U} \hookrightarrow \mathfrak{Y}$ be an open substack of $\mathfrak{Y}$. We can find $x : X \to \mathfrak{X}$,$y : Y \to \mathfrak{Y}$ surjective smooth morphisms of stacks and $f : X \to Y$ a morphism of schemes such that the following square commutes in the category of stacks :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ \mathfrak{X} & \xrightarrow{F} & \mathfrak{Y} \end{array}$$

Indeed if we have $x : X \to \mathfrak{X}$ and $y : Y \to \mathfrak{Y}$ smooth presentations we can take the fiber product $X \times_{F\circ x, \mathfrak{Y}, y} Y$. Because $\mathfrak{Y}$ is algebraic, this is an algebraic space, and because it's a base change of such a morphism the projection of this fiber product to X is smooth surjective. Taking X' an étale surjective presentation of this algebraic space we end up with the desired diagram.
Suppose we have such a diagram, by our proof of (3) and (4) $|x|$ and $|y|$ are continuous surjective open. f being a morphism of schemes, using our remark 1.1 we see that $|f|$ is the usual continuous map corresponding to a morphism of schemes. Therefore using continuity : $(|y| \circ |f|)^{-1}(|\mathfrak{U}|)$ is an open of $|X|$. But $(|y| \circ |f|)^{-1}(|\mathfrak{U}|) = |x|^{-1}(|F|^{-1}(|\mathfrak{U}|))$ and $|x|$ is open therefore $|F|^{-1}(|\mathfrak{U}|)$ is an open subset of $|\mathfrak{X}|$. Therefore we proved $|F|$ is continuous. ☐

Remark 13.1. To prove that sets of points of open substacks form a topology we used the final topology associated to a presentation. Therefore one presentation suffices to exhaust all the open subsets of the space of points of the stack, but those open subsets are independent of the choice of the presentation

Lemma 13.1. *Let $\mathfrak{X}$ be an algebraic stack and let $S \subset \mathfrak{X}$ be a closed subset. Then there is a unique reduced closed substack $\mathfrak{Z}$ of $\mathfrak{X}$ with underlying set S.*

Proof. Take a presentation $p : X \to \mathfrak{X}$. Let $Z \subseteq X$ be the reduced closed subscheme corresponding to $p^{-1}(S)$. We define $\mathfrak{Z}$ has the substack whose points are locally in the image of $p : Z \to \mathfrak{X}$. This is an algebraic stack and one proves that $\mathfrak{Z}$ is closed as in the last proof. We can see that this is independant of the presentation. Let $X' \to \mathfrak{X}$ be another presentation. We can assume that $X' \to X \to \mathfrak{X}$. The map $X' \to X$ induces a surjective smooth map $Z' \to Z$. ☐

13.1.4. *Spectral spaces.*

Definition 13.3. *A topological space is called spectral if it is quasi-compact, the quasi-compact opens are a basis of the topology, stable under intersection, and if any irreducible subspace has a unique generic point.*

Remark 13.2. Some authors add the condition for the space to satisfy the T_0 separation axiom, but this is contained in the uniqueness of the generic point of an irreducible subspace.

There is a functor $Ring^{op} \to \{\text{Spectral spaces}\}$ defined by sending R to Spec R.

Theorem 13.1. *(Hochster) The functor $Ring^{op} \to \{Spectral\ spaces\}$ is essentially surjective.*

Definition 13.4. *An algebraic stack is called quasi-compact if it admits a presentation $X \to \mathfrak{X}$ with X quasi-compact.*

Remark 13.3. Equivalently, if $|\mathfrak{X}|$ is quasi-compact. If X is a quasi compact presentation then $|p| : |X| \to |\mathfrak{X}|$ being continuous surjective $|p|(|X|) = |\mathfrak{X}|$ is quasi-compact. If $|\mathfrak{X}|$ is quasi-compact, pick a smooth surjective presentation $p : X \to \mathfrak{X}$, then p being surjective $|p|(U)$ for $U \subset |X|$ quasi-compact open cover $|\mathfrak{X}|$. By quasi-compacity we can pick $U_1, \dots, U_n$ quasi-compact opens of X such that$\cup_{i=1}^n |p|(U_i) = |\mathfrak{X}|$. Then $\cup_{i=1}^n U_i$ is an open subset of X, finite union of quasi-compact topological spaces therefore quasi compact and an open immersion is smooth, therefore the morphism $\cup_{i=1}^n U_i \to \mathfrak{X}$ obtained by composition of p and the open immersion$\cup_{i=1}^n U_i \to X$ is a smooth surjective morphism : a presentation of $\mathfrak{X}$ with a quasi-compact scheme.

Definition 13.5. *An algebraic stack is quasi-separated if the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is quasi-compact.*

We recall that in a topological space, a point x specializes to y if $y \in \overline{\{x\}}$. Equivalently, this means that $x \in \cap_{y \in U, \text{open}} U$.

Theorem 13.2. *Let $\mathfrak{X}$ be a quasi-compact algebraic stack with quasi-compact diagonal. Then $|\mathfrak{X}|$ is a spectral space.*

Proof. Since $\mathfrak{X}$ is quasi-compact, we deduce that the quasi-compact open of $|\mathfrak{X}|$ are a basis of the topology. Since $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is quasi-compact, we deduce that the intersection of two quasi-compact opens is quasi-compact. It remains to prove that every irreducible component has a unique generic point. Consider an irreducible component. By taking the corresponding stack, we may assume that this irreducible component is $\mathfrak{X}$ itself. We now take a generic point $x \in X$ (which we can suppose quasi-compact). Let $p(x) \in \mathfrak{X}$. We first see that $p(x)$ has no non-trivial generization : $\{x\} = \cap_i U_i$ where U_i are open neighborhood of x. Therefore $p(x) = \cap p(U_i)$ which are open neighborhoods of $p(x)$. We now want to see that $p(x)$ is a generic point (i.e. that $p(x)$ belongs to any open subset of $|\mathfrak{X}|$). Let $\mathfrak{U} \subseteq \mathfrak{X}$ be an open. We can assume that it is quasi-compact. Since $|\mathfrak{X}|$ is irreducible, $|\mathfrak{U}|$ is dense. Since $p : X \to \mathfrak{X}$ is open, we deduce that

$$p^{-1}(\overline{|\mathfrak{U}|}) = \overline{p^{-1}(|\mathfrak{U}|)}$$

and therefore, $p^{-1}(|\mathfrak{U}|)$ is dense. Moreover, $p^{-1}(|\mathfrak{U}|)$ is quasi-compact. It follows that its closure is the set of all its specializations, so that $x \in p^{-1}(|\mathfrak{U}|)$. ☐

13.1.5. *The connected components of $Bun_{n,X}$.* We have a well defined map $\deg : |Bun_{n,X}| \to \mathbb{Z}$. It maps $\mathcal{E}$ to $\deg(\mathcal{E})$. Moreover this map is locally constant.

Theorem 13.3. *The map* deg *induces an isomorphism* $\deg : \pi_0(|Bun_{n,X}|) \to \mathbb{Z}$.

14. Lecture XIII : Smoothness and dimension

14.1. **Smoothness.**

14.1.1. *Generalities.* Let R be a ring and let A be an R-algebra. For any A-module M and R-derivation from A to M is an R-linear map $D : A \to M$ such that $D(ab) = aD(b) + bD(a)$ for all $(a, b) \in A^2$. There is a universal A-module Ω^1_A/R equipped with a derivation $d : A \to \Omega^1_{A/R}$ for which $\mathrm{Der}_R(A, M) = \mathrm{Hom}_A(\Omega^1_{A/R}, M)$ for any A-module M. There is a construction by generators and relations

$$\Omega^1_{A/R} = \oplus_{a \in A} A da / \langle d(ra) = r da \ \forall (r, a) \in R \times A, \ d(ab) = a db + b da, \ \forall (a, b) \in A \times A \rangle.$$

Here is a second construction. We can also consider the exact sequence $0 \to I \to A \otimes_R A \to A \to 0$ and we let $\Omega^1_{A/R} = I/I^2$, and let $d : A \to I/I^2$ be $d(f) = 1 \otimes f - f \otimes 1$. To see that $d : A \to I/I^2$ is universal, let M be an A-module and let $D : A \to M$ be a derivation. Consider $1 \otimes D : A \otimes_R A \to M$ be the linearization. One checks that $1 \otimes D(I^2) = 0$ and we can consider the A-linear map $1 \otimes D : I/I^2 \to M$. We recover D as the composition $A \to I/I^2 \to M$. The following lemma is easy.

Lemma 14.1. *Let B be an R algebra and $J \subseteq B$ be a square zero ideal. Let $f_1, f_2 : A \to B$ be R-algebra maps such that $f_1 = f_2 \mod J$. Then $f_1 - f_2 : A \to J$ is an R-linear derivation. Conversely, let $f : A \to B$ and let D be an R-linear derivation $A \to J$. Then $f + D : A \to B$ is an R-algebra map such that $f + D = f \mod J$.*

14.1.2. *Two exact sequences.*

Lemma 14.2. *If $A \to B$ is a map of R-algebras, we have an exact sequence :*

$$\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

Proof. It suffices to check that for any B-module M, the sequence :

$$0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)$$

is exact. □

Lemma 14.3. *If $A \xrightarrow{\alpha} B$ is a surjective map with kernel I, we have :*

$$I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B \to \Omega^1_{B/R} \to 0$$

Proof. It suffices to check that for any B-module M, the sequence :

$$0 \to \mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M) \to \mathrm{Hom}_A(I/I^2, M)$$

is exact. □

Example 19. We have that $\Omega^1_{R[T_1,\cdots,T_n]/R} = \oplus_{i=1}^n R[T_1, \cdots, T_n]dT_i$. Indeed, one checks that the map $\oplus_{i=1}^n R[T_1, \cdots, T_n]dT_i \to \Omega^1_{R[T_1,\cdots,T_n]/R}$ is surjective using the presentation. We have the derivation $\partial_{T_i} : R[T_1, \cdots, T_n] \to R[T_1, \cdots, T_n]$ and they give linear maps : $\partial_{T_i} : \Omega^1_{R[T_1,\cdots,T_n]/R} \to R[T_1, \cdots, T_n]$ with the property that $\partial_{T_i}(dT_j) = \delta_{i,j}$. We deduce that $\{dT_1, \cdots, dT_n\}$ are indeed a basis of the differentials.

Example 20. Let $A = R[T_1, \cdots, T_n]/(P_1, \cdots, P_r)$. Then $\Omega^1_{A/R} = \oplus_{i=1}^n AdT_i/(dP_1, \cdots, dP_r)$.

14.1.3. *The naive cotangent complex.* Let B be an R-algebra of finite presentation. This means that we have an exact sequence $0 \to I \to A \xrightarrow{\alpha} B \to 0$ where A is a polynomial algebra over R and I is a finitely generated ideal.

To any such presentation, we can associate the complex : $C(\alpha) : I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B$.

Lemma 14.4. *For any two presentations α, α', the complexes $C(\alpha)$ and $C(\alpha')$ are homotopic.*

Proof. We first prove that if we have a map of presentations :

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\alpha} & B \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle Id} \\ 0 & \longrightarrow & I' & \longrightarrow & A' & \xrightarrow{\alpha'} & B \end{array}$$

we get a map $\lambda : C(\alpha) \to C(\alpha')$.

Second we show that if λ and λ' are two maps of presentation, λ and λ' are homotopic from $C(\alpha)$ to $C(\alpha')$. The homotopy is provided by the map $\lambda - \lambda' : A \to I'/(I')^2$ which is a derivation.

Third, we show that given any two presentations, there is a map between them. It follows that we have maps $C(\alpha) \to C(\alpha')$ and $C(\alpha') \to C(\alpha)$ and both compositions are homotopic to the identity. □

Definition 14.1. *A ring morphism $R \to B$ is smooth if it is of finite presentation and for any presentation α, the complex $C(\alpha) : I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B$ is injective with projective cokernel.*

14.1.4. *Smooth morphism.* If $X \to S$ is a map of schemes, we let $\Omega^1_{X/S}$ be the quasi-coherent sheaf over X of relative differentials. One possible definition is to consider the locally closed immersion $\Delta : X \to X \times_S X$, factor it as the composite of a closed immersion, with ideal $\mathcal{I}$ and open immersion $X \hookrightarrow W \hookrightarrow X \times_S X$ and to let $\Omega^1_{X/S} = \Delta^\star \mathscr{I}/\mathscr{I}^2$. We can also check that for $R \to A$ and $f \in A$, $\Omega^1_{A/R} \otimes_A A_f = \Omega^1_{A_f/R}$, so that the construction of $\Omega^1_{A/R}$ is compatible with Zariski localization.

Definition 14.2. *A morphism $f : X \to S$ is smooth at $x \in X$ is x has an affine neighboorhood* $\mathrm{Spec} B$ *over an open* $\mathrm{Spec}\ R$ *of S containing $f(x)$ and $R \to B$ is a smooth map of rings.*

Definition 14.3. *A morphism is smooth if it is smooth at all points.*

The rank of $\Omega^1_{X/S}$ is called the relative dimension of f.

Definition 14.4. *A morphism is étale if it is smooth of relative dimension zero.*

14.1.5. *Infinitesimal criterium.*

Proposition 14.1. *Let $f : X \to S$ be a morphism which is locally of finite presentation. Then f is smooth if and only if, for any affine S-scheme $Y =$* $\mathrm{Spec}\ A$ *and any ideal $I \subseteq A$ such that $I^2 = 0$, if we let $Y_0 = \mathrm{Spec} A/I$, then the morphism :*

$$\mathrm{Hom}_S(Y, X) \to \mathrm{Hom}_S(Y_0, X)$$

is surjective.

Proof. We show that smoothness implies the lifting property. We first prove this locally on Y and assume all schemes are affine. We have a diagram :

$$\begin{tikzcd} A/I & R[T_1, \cdots, T_r]/J \arrow[l, "\psi"'] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

We can lift ψ to a map $\tilde{\psi} : R[T_1, \cdots, T_r] \to A$. It induces $\tilde{\psi} : J/J^2 \to I$.

We now have a locally split exact sequence

$$0 \to J/J^2 \xrightarrow{d} \Omega^1_{R[T_1,\cdots T_r]/R} \otimes R[T_1, \cdots, T_r]/J \to \Omega^1_{(R[T_1,\cdots,T_r]/J)/R} \to 0$$

and we can lift $J/J^2 \to I$ to a map $\Omega^1_{R[T_1,\cdots T_r]/R} \to I$ and therefore there is a derivation $D : R[T_1, \cdots, T_r] \to I$ whose restriction to J is $\tilde{\psi}$. We now consider $\tilde{\psi} - D$. This is a ring morphism $R[T_1, \cdots, T_r]/J \to A$ lifting $\tilde{\psi}$. We now consider the global case. On an open cover $Y_0 = \cup U_{0,i}$ such that we can find local lifts $\tilde{\psi}_i : U_i \to X$. The differences between lifts are given by derivation $\tilde{\psi}_i - \tilde{\psi}_j \in \mathrm{Hom}(\psi^\star \Omega^1_{X/S}, \mathscr{I})(U_{0,i} \cap U_{0,j})$. This defines a 1-cocycle. Since the sheaf $\mathrm{Hom}(\psi^\star \Omega^1_{X/S}, \mathscr{I})$ is quasi-coherent this cocycle is a coboundary, and we deduce that there is a global lift. We now prove that the lifting property implies smoothness. Consider a presentation $0 \to I \to R[T_1, \cdots, T_n] \to B \to 0$ of B. By the lifting property, the identity of B lifts to a map $s : B \to R[T_1, \cdots, T_n]/I^2$. It follows that the sequence $0 \to I/I^2 \to R[T_1, \cdots, T_n]/I^2 \to B \to 0$ has a splitting. The two maps $Id : R[T_1, \cdots, T_n]/I^2 \to R[T_1, \cdots, T_n]/I^2$ and $s \circ \alpha$ are equal modulo I (to the identity of B) and therefore their difference is a derivation $\Omega^1_{R[T_1,\cdots,T_n]/R} \to I/I^2$, giving the splitting. $\square$

Remark 14.1. Given a diagram

$$\begin{tikzcd} Y_0 \arrow[r, "\psi"] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

with $Y_0 \to Y$ a closed immersion with ideal $\mathscr{I}$. There is an obstruction $Ob(\psi) \in \mathrm{H}^1(Y_0, \mathrm{Hom}(\psi^\star\Omega^1_{X/S}, \mathscr{I}))$ which vanishes if and only if ψ lifts to $\tilde{\psi} : Y \to X$.

Remark 14.2. It suffices to check the infinitesimal criterium on schemes Y which are spectra of local rings. The point is that to check that the sequence

$$0 \to I/I^2 \to \Omega^1_{R[T_1,\cdots,T_n]/R} \to \Omega^1_{B/R} \to 0$$

is split we can compute it on stalks. This reduces to the case that B is local. It is even enough to consider the case that Y is a spectra of a strictly henselian local ring. Indeed, any local ring A admits a map $A \to A^{sh}$ where A^{sh} is strictly henselian and is a filtered colimit of étale R-algebras. The map $A \to A^{sh}$ is faithfully flat. We use that the formation of $\Omega^1_{B/R}$ commutes with étale localization : if we have $B \to B'$ étale then $\Omega^1_{B'/R} = \Omega^1_{B/R} \otimes_B B'$.

14.1.6. *Jacobian criterium.*

Proposition 14.2. *Let $\iota : X \hookrightarrow Z$ be a closed immersion of S-schemes with ideal $\mathscr{I}$. Assume that Z is smooth. Then X is smooth at a point x if and only if there are generators $s_1, \cdots, s_r$ of the ideal $\mathscr{I}$ in a neighborhood of x and* $\mathrm{d}s_1(x), \cdots, ds_r(x)$ *are independant in $\Omega^1_{Z/S} \otimes_{\mathscr{O}_Z} k(x)$.*

Corollary 14.1. *Let $f : X \to S$ be a smooth map at x. Then there is a neighborhood $U =$* Spec *A of x such that $A = R[T_1, \cdots, T_n]/I$ where $I = (P_1, \cdots P_r)$ and $I/I^2 = \oplus_{i=1}^r P_i A$.*

Proof. We start by taking a neighborhood $U = \mathrm{Spec}\ A$ of x such that $A = R[T_1, \cdots, T_n]/I$. In a neighborhood of x, I is generated by $s_1, \cdots, s_r$ and $ds_1(x), \cdots, ds(x)$ are a basis of $I/I^2 \otimes_A k(x)$. Thus, there exists $f \in R[T_1, \cdots, T_n]$ such that $x \in \mathrm{Spec}\ A_f$, I_f is generated by $s_1, \cdots, s_r$ and I_f/I_f^2 is the free module generated by the s_i's. We now consider $R[T_1, \cdots, T_n, T_{n+1}]$ and $I' = (s_1, \cdots, s_r, T_{r+1}f - 1)$. ☐

14.1.7. *Deformation properties.*

Proposition 14.3. *Given a diagram*

$$\begin{tikzcd} X_0 \arrow[d, "f_0"] & \\ S_0 \arrow[r] & S \end{tikzcd}$$

with $f_0 : X_0 \to S_0$ smooth and $S_0 \hookrightarrow S$ a closed immersion with ideal $\mathscr{I}$ such that $\mathscr{I}^2 = 0$, then there is an obstruction $Ob(f_0) \in \mathrm{H}^2(X_0, \mathrm{Hom}(\Omega^1_{X_0/S_0}, f_0^\star\mathscr{I})$ which vanishes if and only if there is a cartesian diagram :

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"] & X \arrow[d, "f"] \\ S_0 \arrow[r] & S \end{tikzcd}$$

with f smooth.

Corollary 14.2. *If $f_0 : X_0 \to S_0$ is étale, there is a unique étale lift $f : X \to S$.*

14.1.8.

Proposition 14.4 ([Sta13] TAG 02K5). *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes. Let $h = g \circ f$.*

(1) *If f and g are smooth then h is smooth.*
(2) *If f is smooth and surjective, h is smooth, then g is smooth.*
(3) *If the exact sequence $0 \to f^\star\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$ is locally split exact and h is smooth, then f is smooth.*

14.1.9. *Smooth algebraic stack.*

Definition 14.5. *An alegbraic S-stack $\mathfrak{X}$ is smooth if it admits a presentation $X \to \mathfrak{X}$ with X smooth over S.*

Lemma 14.5. *An algebraic S-stack is smooth if for any presentation $X \to \mathfrak{X}$, the scheme X is smooth over S.*

Proof. Let $X \to \mathfrak{X}$ be a presentation with X smooth over S. Let $X' \to \mathfrak{X}$ be another presentation. We let $X' \times_{\mathfrak{X}} X$ be the fiber product. This is an algebraic space. Let $Z \to X' \times_{\mathfrak{X}} X$ be an étale surjective morphism. The map $Z \to X$ is smooth and therefore Z is smooth over S. The map $Z \to X'$ is smooth and surjective. We deduce that X' is smooth over S. □

Proposition 14.5. *An algebraic S-stack $\mathfrak{X}$ is smooth if it is locally of finite presentation and if for any S-scheme $Y =$ Spec A with A a strictly henselian ring, and any ideal $I \subseteq A$ such that $I^2 = 0$, if we let $Y_0 = \mathrm{Spec}A/I$, Then any diagram :*

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

can be completed into a 2-commutative diagram :

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] \arrow[ur] & S \end{tikzcd}$$

Proof. Assume that $\mathfrak{X}$ is smooth. We have a map $Y_0 \to \mathfrak{X}$. The map admits a lift $Y_0 \to X$ because Y_0 is strictly henselian : the map $X \times_{\mathfrak{X}} Y_0 \to Y_0$ is smooth surjective, hence has a section because Y_0 is strictly henselian. The smoothness of X implies the claim. Conversely, assume that $\mathfrak{X}$ satisfies the infinitesimal criterion. Let $Y_0 \to X$ be a map. The map $Y_0 \to \mathfrak{X}$ lifts to a map $Y \to \mathfrak{X}$. We now consider $X \times_{\mathfrak{X}} Y \to Y$ which is a smooth map. We have a map $Y_0 \to X \times_{\mathfrak{X}} Y$ over Y and it lifts to a map $Y \to X \times_{\mathfrak{X}} Y$. Projecting to X gives a lifting of $Y_0 \to X$. □

14.1.10. *$Bun_{n,X}$ is smooth.*

Theorem 14.1. *The stack $Bun_{n,X}$ is smooth.*

Proof. The stack is locally of finite presentation. We check the infinitesimal criterion. Let $\mathcal{E}_0$ be a vector bundle over $X \times_{\mathrm{Spec}\ k} Y_0$. We pick a cover $X_{Y_0} = \cup U_{0,i}$ trivializing $\mathcal{E}_0$. We have $X_Y = \cup_i U_i$. We have $U_i =$ Spec A_i and $U_{0,i} =$ Spec A_i/I_i. Over each A_i we can lift $M_{0,i} = \mathcal{E}(U_{0,i})$ (which is free) to the free A_i module M_i. We have maps $\alpha_{i,j} : M_{0,i} \otimes A_{0,i,j} \to M_{0,j} \otimes A_{0,i,j}$. We lift them to maps $\tilde{\alpha_{i,j}} : M_i \otimes A_{i,j} \to M_j \otimes A_{i,j}$. Now we measure the obstruction to the lifts satisfying the cocycle condition, let $\delta_{i,j,k} =$

$\tilde{\alpha_{j,k}}\tilde{\alpha_{i,j}} \circ \tilde{\alpha_{i,k}^{-1}} - Id$. This is an element of $\mathrm{Hom}(M_{0,i,j,k}, M_{0,i,j,k}) \otimes I$ and it defines a cohomology class in $\mathrm{H}^2(X_{Y_0}, \mathrm{End}(\mathcal{E}_0) \otimes I)$ which vanishes since X is a curve. □

14.2. **Dimension.** Let X be a scheme. We define $\dim(X) \in \mathbb{N} \cup \{\infty\}$ as the Krull dimension of X : maximal length of a chain of closed irreducible subsets of X. We let $\dim_x(X) = \inf_{x \in U} \dim(U)$ where U is open in X. If $x \in X$, we let $\delta_x(X) = \dim(\mathscr{O}_{X,x})$. One proves that $\dim X = \sup_x \delta_x(X)$. If X is a locally of finite type scheme over a field k, we have that $\dim_x(X) = \delta_x(X) + \mathrm{degtr} k(x)/k$.

If $f : X \to Y$ is a morphism, and let $x \in X$. We let $\dim_x(f) = \dim_x(X_{f(x)})$.

Proposition 14.6. *Let $X \to Y$ be two locally noetherian schemes. The we have that* $\dim_x(X) = \dim_x(f) + \dim_{f(x)}(Y)$.

Let $\mathfrak{X}$ be an algebraic stack. Let $P : X \to \mathfrak{X}$ be a presentation. Let $\xi \in \mathfrak{X}$ be a point and $x \in X$ be a point mapping to ξ. Let $\dim_x P$ be the relative dimension of the morphism $X \times_{\mathfrak{X}} X \to X$ at the point (x, x).

We let $\dim_\xi \mathfrak{X} = \dim_x X - \dim_x P$.

14.3. **The dimension of** $\mathrm{Bun}_{n,X}$**.**

Theorem 14.2. *$Bun_{n,X}$ has dimension $n^2(g-1)$.*

Proof. Let $\xi : \mathrm{Spec}\ l \to Bun_{n,X}$ be a point corresponding to a bundle $\mathcal{E}$ over X_l.

We let $P : X \to Bun_{n,X}$ be a presentation. We let $x \in X$ be a point above ξ. We consider the l-vector spaces $T_{X,x}$ and $T_{X \times_{\mathfrak{X}} X,(x,x)}$. We have a functor $(Bun_{n,X})_{\mathrm{Spec}\ l[\epsilon]} \to (Bun_{n,X})_{\mathrm{Spec}\ l}$ and we let $T_{Bun_{n,X},\xi}$ be the subcategory of $(Bun_{n,X})_{\mathrm{Spec}\ l[\epsilon]}$ whose objects map to ξ and morphisms to the identity of ξ.

This category is equivalent to the category whose objects are $T_{X,x}$ and morphisms are given by $s, t : T_{X \times_{\mathfrak{X}} X,(x,x)} \rightrightarrows T_{X,x}$.

Therefore the isomorphisms classes of deformations are given by

$$\mathrm{coker}(s - t : T_{X \times_{\mathfrak{X}} X,(x,x)} \to T_{X,x})$$

We let $T_{x \times_{\mathfrak{X}} X} = \mathrm{Ker}(s)$. We consider the complex $L^\bullet = t : T_{x \times_{\mathfrak{X}} X,(x,x)} \to T_{X,x}$ in degree -1 and 0. Then $\mathrm{H}^0(L^\bullet)$ is the set of isomorphisms classes of deformations of $\mathcal{E}$ to $l[\epsilon]$ and $\mathrm{H}^{-1}(L^\bullet)$ is the set of automorphisms of the trivial deformation of $\mathcal{E}$ to $l[\epsilon]$. Thus we see that $\dim T_{X,x} - \dim T_{x \times_{\mathfrak{X}} X,(x,x)} = \dim \mathrm{H}^1(X, \mathrm{End}(\mathcal{E})) - \dim \mathrm{H}^0(X, \mathrm{End}(\mathcal{E})) = n^2(g-1)$. □

Lemma 14.6. *Let $\mathcal{E}$ be a vector bundle over X_k. Consider the set $Def(\mathcal{E})(k[\epsilon])$ of isomorphism classes of vector bundles $\tilde{\mathcal{E}}$ on $X_{k[\epsilon]}$ together with an isomorphisme $\tilde{\mathcal{E}}|_{X_k} \simeq \mathcal{E}$. Then $Def(\mathcal{E})(k[\epsilon])$ is a k-vector space isomorphic to* $\mathrm{H}^1(X, \mathrm{End}(\mathcal{E}))$.

Proof. Let $\tilde{\mathcal{E}}$ be a deformation. Take a covering $X = \cup_i U_i = \mathrm{Spec} A_i$ such that $\psi_i : \mathcal{E}|_{U_i} = A_i^n$. Then fix $\tilde{\psi}_i : \tilde{\mathcal{E}}|_{U_i} \simeq A_i^n[\epsilon]$ lifting ψ_i. Let $\tilde{\psi}_{i,j} : \psi_j \circ \psi_i^{-1}$. This defines an element of $\mathrm{H}^1(X, \mathrm{End}(\mathcal{E}))$. Alternatively, any deformation $\tilde{\mathcal{E}}$ can be viewed as an extension (using multiplication by ϵ)

$$0 \to \mathcal{E} \to \tilde{\mathcal{E}} \xrightarrow{\epsilon} \mathcal{E} \to 0$$

thus giving a class in $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$. Conversely, any extension

$$0 \to \mathcal{E} \to \tilde{\mathcal{E}} \to \mathcal{E} \to 0$$

can be viewed as a vector bundle over $X_{l[\epsilon]}$ by letting ϵ act as the composition of $\tilde{\mathcal{E}} \to \mathcal{E}$ and $\mathcal{E} \to \tilde{\mathcal{E}}$. □

15. Lecture XV : The Jacobian

15.1. **The relative Picard functor.** In this lecture we study the stack $Bun_{1,X}$. For a scheme Y we let $Pic(Y)$ be the group of isomorphism classes of line bundles over Y

Let us define $P_X : (Sch/S)^{op} \to SET$ as the sheafification of $T \to Pic(X_T)$. We see that $Pic(X_T)$ is $Ob(Bun_{1,X})_T/\sim$.

Let us describe this. We see that an element of $P_X(T)$ is represented by an invertible sheaf $\mathcal{L} \in Pic(X_{T'})$ for an $fppf$ covering $T' \to T$ with the property that there is an $fppf$ covering $T'' \to T' \times_X T'$ and an isomorphism $\alpha : p_1^\star \mathcal{L} \simeq p_2^\star \mathcal{L}$ on T''. Two invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ in $Pic(X_{T_1})$ and $Pic(X_{T_2})$ represent for $T_i \to T$ and $fppf$ covering represent the same object in $P_X(T)$ if they become isomorphic over an X_{T_3} where $T_3 \to T_1 \times_T T_2$ is an $fppf$ covering.

Lemma 15.1. *The following sequence is exact :*

$$0 \to Pic(T) \to Pic(X_T) \to P_X(T)$$

Proof. Let $\mathcal{L} \in Pic(X_T)$. Assume that we have an $fppf$ covering $T' \to T$ such that $\mathcal{L}|_{X_{T'}} \simeq \mathscr{O}_{X_{T'}}$. We let $\pi_T : X_T \to T$ be the projection. We claim that the map $\pi_T^\star(\pi_T)_\star \mathcal{L} \to \mathcal{L}$ is an isomorphism. Using flat base change, it suffices to check it after pull back to T'. But then $\pi_{T'} \mathscr{O}_{X_{T'}} = \mathscr{O}_{T'}$. $\square$

Assume that $X_T \to T$ has a section $s : T \to X_T$. Let us define $Pic(X_T, s)$ as the group of isomorphism classes over $\mathcal{L} \in Pic(X_T)$ and $u : s^\star \mathcal{L} \simeq \mathscr{O}_T$.

Lemma 15.2. *We have an isomorphism :*

$$\begin{array}{rcl} Pic(X_T) & \to & Pic(X_T, s) \times Pic(T) \\ \mathcal{L} & \mapsto & (\mathcal{L} \otimes \pi_T^\star s^\star \mathcal{L}^{-1}, s^\star \mathcal{L}) \end{array}$$

We see that $T' \mapsto Pic(X_{T'}, s_{T'})$ defines an $fppf$ sheaf.

Corollary 15.1. *If $X_T \to T$ has a section $s : T \to X_T$, the sequence :*

$$0 \to Pic(T) \to Pic(X_T) \to P_X(T) \to 0$$

is split exact.

Remark 15.1. Assume that $X(k) \neq \emptyset$. Then we see that $Bun_{1,X} = P_X \times_S \mathrm{B}\mathbb{G}_m$ where $\mathrm{B}\mathbb{G}_m$ is the classifying stack of $\mathbb{G}_m$. That is $(\mathrm{B}\mathbb{G}_m)_T$ is the category of invertible sheaves over T.

15.2. **Representability of the relative Picard functor.** We have the classical theorem:

Theorem 15.1. *The relative Picard functor is representable and P_X^0 is an abelian scheme, called the Jacobian of the curve.*

We sketch the proof. A good reference is [Mil86].

15.2.1. *Relative Cartier divisors.*

Definition 15.1. *An effective relative Cartier divisor D over X_S is a closed subscheme $D \hookrightarrow X_S$ such that $p_S : D \to S$ is finite flat.*

Attached to D, we have the invertible sheaf $\mathscr{O}_{X_S}(-D) = \mathcal{I}_D$, to which we can attach the pair $(\mathscr{O}_{X_S}(D), 1 : \mathscr{O}_{X_S} \to \mathscr{O}_{X_S}(D))$. The set of effective Cartier divisors over S, $Div_{\geq 0}(S)$, is the set of isomorphism classes of pairs $(\mathcal{L} \in Pic(X_S), f : \mathscr{O}_{X_S} \to \mathcal{L})$ such that $\mathcal{L}/\mathscr{O}_{X_S}$ is finite flat over S. This last condition is also equivalent to asking that f is nowhere identically zero over S.

Let $r \geq 0$. We let $Div^r_{\geq 0}(-)$ be the functor which maps S to the set of isomorphism classes of effective relative Cartier divisors of degree r over X_S.

15.2.2. *Quotienting schemes by finite groups.* Let A be a ring and let G be a finite group acting on A. Let $B = A^G$. Let $p : \mathrm{Spec}\, A = X \to \mathrm{Spec}\, B = Y$ be the corresponding morphism.

Proposition 15.1 ([Gro03], exp V, prop. 1.1)**.** (1) *The ring A is integral over B.*
(2) *The morphism p is surjective and closed and the map $X \to Y$ induces an homeomorphism $Y \simeq X/G$ (where X/G carries the quotient topology).*
(3) *For any scheme Z, we have that* $\mathrm{Hom}(Y, Z) = \mathrm{Hom}(X, Z)^G$.

We now let X be a scheme and we assume that G acts on X and that X admits an affine covering stable by G.

Proposition 15.2 ([Gro03], exp V, prop. 1.8)**.** *There is a scheme Y and a surjective morphism $p : X \to Y$ such that :*
(1) *The morphism p is surjective and closed and the map $X \to Y$ induces an homeomorphism $Y \simeq X/G$ (where X/G carries the quotient topology).*
(2) *For any scheme Z, we have that* $\mathrm{Hom}(Y, Z) = \mathrm{Hom}(X, Z)^G$.
(3) *We have $\mathscr{O}_Y = p_\star \mathscr{O}_X^G$.*

The scheme Y of the proposition (which is unique up to a unique isomorphism) is called a categorial quotient of X by G.

15.2.3. *Representing $Div^r_{\geq 0}(-)$.* We will prove that $Div^r_{\geq 0}(-)$ is representable. Let $X^r = X \times \cdots \times X$ be the r-th fold product of the curve. The symmetric group $\mathcal{S}_r$ acts on X^r by permutation of the factors.

Lemma 15.3. *The categorical quotient $X^r/\mathcal{S}_r = X^{(r)}$ exists and is smooth.*

Proof. See [Mil86], prop. 3.2. ☐

We have a map $X^r \to Div^r_{\geq 0}$ which sends $(P_1, \cdots, P_r)$ to $(\mathscr{O}_{X_S}(\sum P_i), \mathscr{O}_{X_S} \to \mathscr{O}_{X_S}(\sum P_i))$. This map pass to the quotient to a map $X^{(r)} \to Div^r_{\geq 0}$.

Proposition 15.3. *$X^{(r)} \to Div^r_{\geq 0}$ is an isomorphism.*

Proof. We need to show injectivity and surjectivity. For surjectivity, it suffices to prove the surjectivity of $X^r \to Div^r_{\geq 0}$. We will show that if $(\mathcal{L} \in Pic(X_S), f : \mathscr{O}_{X_S} \to \mathcal{L})$ is a degree r cartier divisor, there is a finite flat map $T \to S$ and sections $P_1, \cdots, P_r \in X(T)$ such that $(\mathcal{L}, f) \simeq (\mathscr{O}_{X_T}(\sum P_i), 1)$. We prove this by induction on r. The case $r = 1$ is trivial. Let us assume $r \geq 2$. Let $T = V(\mathcal{L}^{-1}) \subset X_S$. The map $T \to S$ is finite flat. Over X_T we have the degree 1 divisor $P : T \overset{\Delta}{\hookrightarrow} T \times_S T \hookrightarrow X_T$. We see that $f_T : \mathscr{O}_{X_T} \to \mathcal{L}_T(-P) \to \mathcal{L}_T$ and $\mathcal{L}_T(-P)$ is now of degree $r-1$. We conclude by induction.

We need to show injectivity. We do this when $r = 2$, the general case is left to the reader. Let P_1, P_2 and Q_1, Q_2 by Spec R-points of X. We assume that $\mathscr{O}_{X_S}(-P_1 - P_2) = \mathscr{O}_{X_S}(-Q_1 - Q_2)$.

After localizing in Spec R, we can find an affine open Spec A of X_S with the property that P_1, P_2, Q_1, Q_2 factor through Spec A. Therefore, we have morphisms $Q_i : A \to R$ with kernel I_i and $P_i : A \to R$ with kernel J_i and by assumption $I_1 I_2 = J_1 J_2$. We want to deduce that the maps $P_1 \otimes P_2 : A \otimes A \to R$ and $Q_1 \otimes Q_2 : A \otimes A \to R$ have the same restriction to $(A \otimes A)^{\Sigma_2}$. We claim that for any $a \in A$, $Q_1(a)Q_2(a) = P_1(a)P_2(a)$ and $Q_1(a) + Q_2(a) = P_1(a) + P_2(a)$ because they can be interpreted as the coefficients of the characteristic polyomial of a acting on $A/I_1I_2 = A/J_1J_2$. We deduce that $P_1 \otimes P_2(a \otimes 1 + 1 \otimes a) = Q_1 \otimes Q_2(a \otimes 1 + 1 \otimes a)$ and $P_1 \otimes P_2(a \otimes a) = Q_1 \otimes Q_2(a \otimes a)$. The elements $a \otimes 1 + 1 \otimes a$ and $a \otimes a$ generate $(A \otimes A)^{\Sigma_2}$ as an algebra. ☐

15.2.4. *The Abel-Jacobi map.* We call the map $AJ_r : X^{(r)} \to P_X^r$ the Abel-Jacobi map. We will use this map to prove the representability of P_X^r.

Let us assume that $r \geq 2g-1$. Let $S \to Spec\ k$ and let $\mathcal{L} \in Pic^r(X_S)$ (corresponding to a point $x : S \to P_X^r$). Then the fiber product $X^{(r)} \times_{AJ_r, P_X^r, x} S$ is the set of nowhere vanishing sections $f \in R^0(p_S)_\star \mathcal{L}$, up to isomorphism.

But since $r \geq 2g-1$, $R^0(p_S)_\star\mathcal{L}$ is a locally free sheaf of rank $r-g+1$, and

$$X^{(r)} \times_{AJ_r, P_X^r, x} S = (R^0(p_S)_\star \mathcal{L} \setminus \{0\})/\mathscr{O}_S^\times$$

is therefore a fibration in projective spaces of dimension $r-g$.

If we had a section $s : P_X^r \to X^{(r)}$, then we would deduce that P_X^r is representable. Indeed, if we let $q : X^{(r)} \to P_X^r \xrightarrow{q} X^{(r)}$ then the morphism p induces an isomorphism between $X^{(r)} \times_{q, X^{(r)}, id} X^{(r)}$ and P_X^r.

We will prove that there are local sections. At this stage, we assume that the field k is separably closed. By Galois descent, we can reduce to this case.

For any $r-g$-uple of points $t = (t_1, \cdots, t_{r-g}) \in X(k)^{r-g}$, we let

$$X_t^{(r)} = \{(P_1, \cdots, P_r), \dim \mathrm{H}^0(\mathscr{O}_X(\sum_{i=1}^r P_i - \sum_{i=1}^{r-g} t_j)) = 1\}.$$

This is an open of $X_t^{(r)}$ and moreover, $X^{(r)} = \cup_t X_t^{(r)}$.

We similarly defined $(P_X^r)_t$ has the subfunctor parametrizing $\mathcal{L}$ with the property that $\dim \mathrm{H}^0(\mathcal{L}(-\sum_{i=1}^{r-g} t_j)) = 1$. The map $X_t^{(r)} \to (P_X^r)_t$ is an isomorphism and therefore $(P_X^r)_t$ is representable. And we have a covering $P_X^r = \cup (P_X^r)_t$.

We finally deduce that P_X^r is smooth and geometrically connected because $X^{(r)}$ is.

Finally for any line bunde $\mathcal{L}$ of degree s over X, the map $-\otimes \mathcal{L} : P_X^r \to P_X^{r+s}$ is an isomorphism. We deduce the representability of P_X^r for all r.

16. Lecture XVI : Geometric class field theory

16.1. **The classical fundamental group.** Let S be a connected, locally arcwise connected, locally simply connected topological space. Let $s \in S$ be a point. We can define $\pi_1(S, s)$, the group of homotopy classes of loops $\gamma : \mathrm{S}^1 \to S$ with $\gamma(0) = s$. Let Cov be the category of coverings of S. Recall that $S' \to S$ is a covering if any point $x \in S$ has a neighborhood U_x such that $p^{-1}(U_x) \simeq U_x \times I$ for a discrete set I. We define a functor $F : Cov \to \mathrm{SET}$ by sending $p : S' \to S$ to $p^{-1}(s)$. Let $\pi_1(S,s) - \mathrm{SET}$ be the category of sets equipped with an action of $\pi_1(S,s)$. We have the following classical theorem :

Theorem 16.1. *The functor F can be enriched to an equivalence of categories $Cov \to \pi_1(S,s) - \mathrm{SET}$.*

Moreover, F is representable functor : let $\tilde{p} : \tilde{S} \to S$ be the universal cover of S, and $\xi \in F(\tilde{S}) = \tilde{p}^{-1}(s)$. Then $F(-) = \tilde{S}(-)$.

Remark 16.1. One can recover $\pi_1(S,s)$ abstractly from the functor F, as the group of automorphisms of F.

16.2. **The fundamental group of a field k.** Let k be a field. A connected covering of k is by definition of finite separable field extension of k. A covering of k will be by definition a finite product of finite separable extension of k (we say also a finite étale extension of k). Let Cov be the category of coverings of k. Let $\overline{k}$ be an algebraic closure of k and $k^{sep} \subset k^{alg}$ be the separable closure. We define a functor $F : Cov \to \mathrm{FSET}$ by mapping ℓ/k to $\mathrm{Hom}(\ell, \overline{k})$ where FSET is the category of finite sets. Let $Gal(k^{sep}/k) = G_k$ and $G_k - \mathrm{FSET}$ the category of finite sets equipped with a continuous left action of G_k.

Theorem 16.2. *The functor F can be upgraded to an equivalence of category $Cov \to G_k - \mathrm{FSET}$.*

Proof. This is a reformulation of Galois theory. We exhibit and inverse functor. If I is a G_k-set. We consider the algebra of functions $f : I \to k^{sep}$ which are G_k-equivariant. □

The functor F is pro-representable. We can write $k^{sep} = \cup_i k_i$ has a filtered union of finite extensions, and $F(-) = \mathrm{colim}_i \mathrm{Hom}(-, k_i)$.

16.3. **The étale fundamental group of a scheme.** The original reference is [Gro03]. Another good reference is [Mur67].

16.3.1. *Etale covers.* We let X be a locally noetherian scheme.

Definition 16.1. *A morphism $p : Y \to X$ of schemes is finite étale if*

1. *For all affine open* $\mathrm{Spec} A \hookrightarrow X$*, the fiber* $\mathrm{Spec}\ A \times_X Y = \mathrm{Spec}\ B$ *is affine and B is a finite projective A-module,*
2. *For all point $x \in X$, Y_x is the spectrum of a finite étale extension of $k(x)$.*

A finite étale cover $p : Y \to X$ is a finite étale map which is surjective. In general, the image of p is an open and closed subscheme of X. In particular, if X is connected, an finite étale morphism is a cover. We assume that X is connected.

We let Cov be the category whose objects are finite étale schemes $Y \to X$ and morphisms are X-morphisms of schemes.

16.3.2. *The main theorem.* We let $x \in X$ be a point and we pick $\bar{x} \to x$ a geometric point above x. We can define a functor $F : Cov \to \mathrm{FSET}$ by mapping Y to the set $Y \times_X \bar{x}$.

Theorem 16.3 ([Mur67], thm. 4.4.1)**.**

1. *There exists a unique profinite group $\pi_1(X, \bar{x})$ such the functor F can be enriched to an equivalence of categories :*
$$Cov \to \pi_1(X, \bar{x}) - \mathrm{FSET}.$$
2. *Let $\overline{x'} \to X$ be another point geometric point of X. There exists a topological isomorphism: $\pi_1(X, \bar{x}) \to \pi_1(X, \bar{x'})$, which is unique up to an inner automorphism.*

If $Y \to X$ is a morphism of schemes, the pull-back of étale covers from X to Y induces a morphism $\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{x})$.

Remark 16.2. We can revisit Frobenius substitution. If $X \to \mathrm{Spec}\ \mathbb{Z}$ is a finite type scheme. Then any closed point $s \in X$ has residue field a finite field. Let $\overline{x}$ be a geometric point of X. For any $s \in X$ and any geometric point $\overline{s} \to s$, we get a morphism (well defined up to conjugacy) $\pi_1(s, \overline{s}) \to \pi_1(X, \overline{x})$. If s is a closed point, $\pi_1(s, \overline{s})$ is topologically generated by the Frobenius.

16.4. $\mathbb{P}^1$ **is geometrically simply connected.** In this section we prove :

Theorem 16.4. *Let k be an algebraically closed field. Then $\pi_1(\mathbb{P}^1_k, \overline{x}) = 1$.*

Proof. Let $f : X \to \mathbb{P}^1_k$ be a finite étale cover. Let $f_\star \mathscr{O}_X$. This is a vector bundle over $\mathbb{P}^1_k$. Therefore, $f_\star \mathscr{O}_X = \oplus_{i=1}^r \mathscr{O}(n_i)$ for integers n_i. We will prove that this is the trivial bundle (all n_i are 0). This will prove that $f_\star \mathscr{O}_X = \mathrm{H}^0(X, \mathscr{O}_X) \otimes_k \mathscr{O}_{\mathbb{P}^1_k}$. Since k is algebraically closed, $\mathrm{H}^0(X, \mathscr{O}_X) = k^r$ (as algebra) and X is the disjoint union of r copies of $\mathbb{P}^1_k$. There is a bilinear trace map : $f_\star \mathscr{O}_X \times f_\star \mathscr{O}_X \to \mathscr{O}_{\mathbb{P}^1_k}$ and this is a perfect pairing. Therefore we deduce that it is enough to prove that for all i, $n_i \geq 0$. Let i be the index for which n_i is minimal and assume that $n_i < 0$. The product map $m : f_\star \mathscr{O}_X \otimes f_\star \mathscr{O}_X \to f_\star \mathscr{O}_X$ restricts to a map $\mathscr{O}(n_i) \otimes \mathscr{O}(n_i) \to f_\star \mathscr{O}_X$. But there are no non-zero maps $\mathscr{O}(2n_i) \to f_\star \mathscr{O}_X$. Therefore $m(\mathscr{O}(n_i) \otimes \mathscr{O}(n_i)) = 0$. But X is a smooth curve and therefore it is reduced. □

16.5. **Descent of étale covers.** We consider the following situation : X is a scheme and Γ is a finite group acting on X. We assume that X has an affine covering stable under Γ. We can define the categorical quotient X/Γ (see [Gro03], exposé V, sect. 1). For a point $x \in X$, we let Γ_x be the inertia group at x. This is the subgroup of Γ of elements which stabilize x and act trivially on the residual field at x, $k(x)$.

Let $Y \to X$ be an étale cover. We assume that Y carries an action of Γ compatible with the action on X.

We can therefore consider the quotient Y/Γ and we have a diagram :

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y/\Gamma \arrow[d] \\ X \arrow[r] & X/\Gamma \end{tikzcd}$$

The following two propositions are [Gro03], exposé IX, rem. 5.8.

Proposition 16.1. *The map $Y/\Gamma \to X/\Gamma$ is finite étale if for all $x \in X$, if we let Γ_x the inertia subgroup at x, then Γ_x acts trivially on Y_x.*

Proposition 16.2. *We have an equivalence between the category of finite étale cover of X/Γ and the finite étale cover of X which carry an action of Γ compatible with the action on X and such that for all $x \in X$, Γ_x act trivially on the fiber.*

16.6. $\mathbb{P}^r$ **is geometrically simply connected.**

Theorem 16.5. *Let k be an algebraically closed field. Then $\pi_1(\mathbb{P}^r_k, \overline{x}) = 1$.*

Proof. We first need to prove that $\pi_1((\mathbb{P}^1_k)^r, \overline{x}) = 1$. We prove this by induction on r. The case $r = 1$ is theorem 16.4. We assume $r \geq 2$ and consider the map $p : (\mathbb{P}^1_k)^r \to (\mathbb{P}^1_k)^{r-1}$ given by the projection on the first $r-1$ coordinates. We now let $f : X \to (\mathbb{P}^1_k)^r$ be a finite étale cover of degree d. We claim that $p_\star f_\star \mathscr{O}_X$ is a locally free sheaf of algebras over $(\mathbb{P}^1_k)^{r-1}$. This follows from corollary **??**. Indeed, for each point $t \in (\mathbb{P}^1_k)^{r-1}$, $p^{-1}(t) = \mathbb{P}^1_{k(t)}$ and $X_t \to \mathbb{P}^1_{k(t)}$ is isomorphic to $\mathbb{P}^1_{k(t)} \times_{\mathrm{Spec}\ k(t)} \mathrm{Spec}\ k(t)'$ for a finite étale extension of $k(t)'$ of degree d. We find that $dim_{k(t)}(\mathbb{P}^1_{k(t)}, (f_t)_\star \mathscr{O}_{X_t}) = d$ is constant. Let X' be the spectrum of this sheaf of algebras. We see that $X' \to (\mathbb{P}^1_k)^{r-1}$ is finite flat and moreover, $X' \times_{(\mathbb{P}^1_k)^{r-1}} (\mathbb{P}^1_k)^r \simeq X$. In other words, X' descends X. We see that $X' \to (\mathbb{P}^1_k)^{r-1}$ is smooth, because $X \to (\mathbb{P}^1_k)^r$ is. Therefore we deduce that X' is a finite étale cover. We also deduce that the map $p : \pi_1((\mathbb{P}^1_k)^r, \overline{x}) \to \pi_1((\mathbb{P}^1_k)^{r-1}, p(\overline{x}))$ is an isomorphism. By induction we deduce that $\pi_1((\mathbb{P}^1_k)^r, \overline{x}) = 1$. Then we use proposition 16.2. Indeed, $\mathbb{P}^r_k = (\mathbb{P}^1_k)^r/\mathcal{S}_r$. Let $X \to \mathbb{P}^r_k$ be an étale cover. Its pullback to $(\mathbb{P}^1_k)^r$ is $\tilde{X}$ and it is isomorphic to $(\mathbb{P}^1_k)^r \times I$ where I is a finite set over which Γ acts. Take a point in the diagonal x. Then the inertia group is $(\mathcal{S}_r)_x = \mathcal{S}_r$ and we deduce that $\mathcal{S}_r$ acts trivially on I. Therefore $X = \mathbb{P}^r_k \times I$. □

16.7. **Descending étale covers under projective fibration.**

Theorem 16.6. *Let $f : X \to Y$ be a projective fibration. Then the map $\pi_1(X, \overline{x}) \to \pi_1(Y, f(\overline{x}))$ is an isomorphism.*

Proof. We have seen a proof in theorem 16.5 in the case of a fibration in projective lines. A similar argument applies. □

16.8. **Geometric class field theory.** Let L be a finite abelian group. Let X be a complete non-singular curve over $\mathbb{F}_q$. We will prove the following theorem :

Theorem 16.7. *There is a canonical bijection :*

$$\begin{array}{rcl}\{\chi : \pi_1(X) \to L\} & \to & \{\rho : Pic(X) \to L\} \\ \chi & \mapsto & \rho\end{array}$$

where ρ is defined by the rule that for all $x \in X$, $\rho(\mathscr{O}(x)) = \chi(Frob_x)$.

As a corollary, we deduce :

Theorem 16.8. *We have a commutative diagram:*

$$\begin{tikzcd} Pic(X) \arrow[r] \arrow[d] & \pi_1(X)^{ab} \arrow[d] \\ \mathbb{Z} \arrow[r] & \hat{\mathbb{Z}} \end{tikzcd}$$

which induces an isomorphism between $Pic(X)$ and $W(X)^{ab}$.

Proof. The theorem 16.7 implies that the profinite completion of $Pic(X)$ is isomorphic to $\pi_1(X)^{ab}$. Now we have an exact sequence $1 \to Pic^0(X) \to Pic(X) \to \mathbb{Z} \to 1$ (which splits non-canonically) and $Pic^0(X)$ is a finite group. Therefore the profinite completion of $Pic(X)$ is $Pic^0(X) \times \hat{\mathbb{Z}}$. □

16.8.1. *Systems of abelian covers over $\{X^{(r)}\}_{r\geq 0}$, compatible with the monoidal structure.* We recall that we have multiplications $m : X^{(r)} \times X^{(r')} \to X^{(r+r')}$, and projections $p_1 : X^{(r)} \times X^{(r')} \to X^{(r)}$ and $p_2 : X^{(r)} \times X^{(r')} \to X^{(r')}$. It will be convenient to consider also $Div_{\geq 0} = \coprod_{r\geq 0} X^{(r)}$. So that we have three maps, $m, p_1, p_2 : Div_{\geq 0} \times Div_{\geq 0} \to Div_{\geq 0}$. We also let $\pi_1(Div_{\geq 0})^{ab} = \oplus_{r\geq 0}\pi_1(X^{(r)})^{ab}$.

Let L be a finite abelian group. Let $\chi_1 : \pi_1(X) \to L$ be a character.

Proposition 16.3. *There is a unique way to attach to χ_1 a character $\chi = \prod_{r\geq 0}\chi_r$: $\pi_1(Div_{\geq 0})^{ab} \to L$ such that :*

$$m^\star\chi = p_1^\star\chi + p_2^\star\chi$$

as characters of $\pi_1(Div_{\geq 0} \times Div_{\geq 0})^{ab}$.

Remark 16.3. We thus claim that there is a unique system of characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r\geq 1}$ which satisfy that the pull backs of $\chi_r + \chi_{r'}$ and $\chi_{(r+r')}$ to characters of $\pi_1(X^{(r)} \times X^{(r')})$ coincide:

$$\begin{tikzcd} \pi_1(X^{(r)} \times X^{(r')}) \arrow[r] \arrow[d] & \pi_1(X^{(r+r')}) \arrow[r, "\chi_{r+r'}"] & L \\ \pi_1(X^{(r)}) \times \pi_1(X^{(r')}) \arrow[d, "\chi_r+\chi_{r'}"] & & \\ L & & \end{tikzcd}$$

Proof. Let $Y \to X$ be the abelian cover with group L corresponding to χ_1. We construct an abelian cover over X^r corresponding to $\chi_1^{\oplus r} : \pi_1(X^r) \to L$. This is $Y^r/H \to X^r$ where $H = \mathrm{Ker}(L^r \stackrel{\Sigma}{\to} L)$. Then we check that the action of $\mathcal{S}_r$ on X^r lifts to Y^r and passes to the quotient Y^r/H. Moreover, the action of the inertia group is trivial on the fibers. Therefore the cover descends to $X^{(r)}$. □

Lemma 16.1. *Let $r_0 \geq 0$. Assume that we have a system of characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq r_0}$ which satisfy that the pull backs $p_1^\star \chi_r + p_2^\star \chi_{r'} = m^\star \chi_{(r+r')}$ has characters of $\pi_1(X^{(r)} \times X^{(r')})$.*

Then, there exists a unique character $\chi_1 : \pi_1(X, \overline{x}) \to L$ such that this system arises from χ_1.

Proof. Let x_0 be a rational point on $X^{(r)}$ for $r \geq r_0$. We get a map $X \to X^{(r+1)}$ by sending x to (x, x_0). We let $\chi_1 : \pi_1(X) \to \pi_1(X^{(r+1)}) \to L$. □

16.8.2. *Systems of abelian covers of P_X, compatible with the monoidal structure.* Recall that $P_X(\mathbb{F}_q) = Pic(X)$. We have maps $m : P_X \times P_X \to P_X$ as well as projections $p_i : P_X \times P_X \to P_X$.

A character $\rho : \pi_1(P_X) \to L$ is compatible with the monoidal structure if we have $p_1^\star \rho + p_2 \star \rho = m^\star \rho$ as characters of $\pi_1(P_X \times P_X)$.

To such a character we can associated a group morphism : $\tilde{\rho} : Pic(X) \to L$ by evaluating on $Frob_x$ for each $x \in Pic(X)$.

Proposition 16.4. *The association $\rho \mapsto \tilde{\rho}$ defines an bijection between characters compatible with the monoidal structure on P_X and characters of $Pic(X)$.*

Proof. Let $P_X \overset{Frob_q - 1}{\to} P_X$ be the Lang isogeny which maps $\mathcal{L}$ to $Frob_q^\star \mathcal{L} \otimes \mathcal{L}^{-1}$. Its kernel is precisely $P_X(\mathbb{F}_q) = Pic(X)$. This provides a map $\rho_{Lang} : \pi_1(P_X) \to Pic(X)$. Moreover, for any $\mathcal{L} \in Pic(X)$, $\rho_{Lang}(Frob_{\mathcal{L}}) = \mathcal{L}$. It is an easy exercise to check that $m^\star \rho_{Lang} = p_1^\star \rho_{Lang} + p_2^\star \rho_{Lang}$.

Let $\rho : \pi_1(P_X) \to L$ be a character compatible with the monoidal structure. We need to find a factorization $\rho : \pi_1(P_X) \overset{\rho_{Lang}}{\to} Pic(X) \to L$. We therefore need to prove that $\pi_1(P_X) \overset{Frob_q - 1}{\to} \pi_1(P_X) \overset{\rho}{\to} L$ is the trivial character. But this is nothing else than $Frob_q^\star \rho - \rho$ (because ρ is compatible with the monoidal structure). And we know that $Frob_q^\star \rho = \rho$.

□

16.8.3. *Proof of theorem 16.7.* We see that the following sets are in natural bijection :

(1) Characters $\chi : \pi_1(X) \to L$,
(2) Characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq 0}$, compatible with the monoidal structure,
(3) Characters $\{\chi_r : \pi_1(X^{(r)}) \to L\}_{r \geq r_0}$, compatible with the monoidal structure,
(4) Characters $\{\rho_r : \pi_1(P_X^r) \to L\}_{r \geq r_0}$, compatible with the monoidal structure,
(5) Characters $\{\rho_r : \pi_1(P_X^r) \to L\}_{r \geq 0}$, compatible with the monoidal structure,
(6) Characters $\tilde{\rho} : Pic(X) \to L$.

- (1) $\Leftrightarrow$ (2) is proposition 16.3,
- (2) $\Leftrightarrow$ (3) is lemma 16.1,
- (3) $\Leftrightarrow$ (4) is proposition 16.6,
- (4) $\Leftrightarrow$ (5) is similar to lemma 16.1,
- (5) $\Leftrightarrow$ (6) is proposition 16.4

Remark 16.4. We can restate our theorem as follows. Given a character $\chi : \pi_1(X) \to L$, there exists a unique character $\rho : \pi_1(P_X) \to L$ such that for $m : X \times P_X \to P_X$ the map which sends $(x, \mathcal{L})$ to $\mathcal{L}(x)$, we have $m^\star \rho = p_1^\star \chi + p_2^\star \rho$.

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