GEOMETRIC REPRESENTATION THEORY

1. Introduction

In this course we will use techniques from algebraic geometry in order to understand representations of algebraic groups.

Let us consider in this introduction the group $\mathrm{SL}_2 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1\}$. We let B be the upper triangular Borel. This group naturally acts on the right on the projective line $\mathbb{P}^1$ by the formula

$$[X, Y].\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [aX + cY, bX + dY].$$

In fact, $\mathbb{P}^1 = B\backslash \mathrm{SL}_2$. The Picard group of $\mathbb{P}^1$ is $\mathbb{Z}$ and for every $n \in \mathbb{Z}$, we can construct a line bundle $\mathscr{O}_{\mathbb{P}^1}(n)$. For example, using the Proj construction, we have that $\mathbb{P}^1 = \mathrm{Proj}\ k[X, Y]$ and $\mathscr{O}_{\mathbb{P}^1}(n)$ is associated to the graded module $\oplus_{k\geq 0} M_k$ where $M_k = k[X, Y]_{n+k}$ is the set of homogeneous polynomials of degree $n + k$. The group SL_2 also acts on these modules and therefore acts equivariantly on the sheaf. When $n \geq 0$, $\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = \oplus_{p+q=n} kX^pY^q$. This is the irreducible $n + 1$-dimensional representation of SL_2. In fact, we have :

Theorem 1.0.1. *For any $n \geq -1$, $\mathrm{H}^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = 0$. For any $n \leq -1$, and $\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = 0$. Moreover, we have a (non-canonical) isomorphism of SL_2-representations : $\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = \mathrm{H}^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2 - n))$.*

There is a connection between the sheaves $\mathscr{O}_{\mathbb{P}^1}(n)$, $\mathscr{O}_{\mathbb{P}^1}(2 - n)$ and the representation $\mathrm{Sym}^n k^2$. As we have seen, we can obtain the representation by taking cohomology. In fact it is also possible to recover the sheaves from the representation by applying a certain localization functor.

Theorem 1.0.2. *There are two sheaves of "twisted" differential operators $\mathcal{D}_{-n}$ and $\mathcal{D}_{n-2}$ on $\mathbb{P}^1$ (locally, they look like the Weyl algebra $k[X, \partial_X]$ but the gluing data is non trivial) such that*

$$\mathcal{D}_{-n} \otimes_{U(\mathfrak{sl}_2)} \mathrm{Sym}^n k^2 = \mathscr{O}_{\mathbb{P}^1}(n)$$

and

$$\mathcal{D}_{n-2} \otimes^L_{U(\mathfrak{sl}_2)} \mathrm{Sym}^n k^2 = \mathscr{O}_{\mathbb{P}^1}(2 - n)[1]$$

We next want to illustrate that having a sheaf on a space, rather than just a representation allows for many interesting constructions. We consider the stratification by B-orbits where $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$:

$$\mathbb{P}^1 = B\backslash Bw_0B \coprod B\backslash B = \mathbb{A}^1 \coprod \{\infty\}.$$

Let $\pi : \mathrm{SL}_2 \to \mathbb{P}^1$ be the projection. Here $\mathbb{A}^1 = \pi(w_0U)$ (where U is the unipotent radical in B), and $\{\infty\} = \pi(1)$. In terms of coordinates, $\mathbb{A}^1 = \mathrm{Spec}\ k[Y/X]$, and $\infty = [0, 1]$. We have an exact triangle :

$$(i_\infty)_\star i^!_\infty \mathscr{O}_{\mathbb{P}^1}(n) \to \mathscr{O}_{\mathbb{P}^1}(n) \to j_\star j^\star \mathscr{O}_{\mathbb{P}^1}(n) \xrightarrow{+1}$$

for $i_\infty : \{\infty\} \to \mathbb{P}^1 \leftarrow \mathbb{A}^1 : j$. We deduce that the following complex computes the cohomology:

$$\mathcal{C}ous(n) : 0 \to \mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to 0$$

If $n \geq 0$, we have a short exact sequence $0 \to \mathrm{Sym}^n k^2 \to \mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to 0$.
If $n \leq -2$, we have a short exact sequence $0 \to \mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{Sym}^n k^2 \to 0$.

We claim that there is an action of $\mathfrak{sl}_2$ on $\mathcal{C}ous(n)$ and the two above exact sequence are the "famous" dual BGG and BGG resolution of $\mathrm{Sym}^n k^2$.

Let us in fact compute everything. We have an isomorphism $\mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = X^n k[Y/X]$. One easily computes the action of T. For $t = \mathrm{diag}(t, t^{-1}) \in T$, we have $t.X^n(X/Y)^s = t^{n-2s}X^n(X/Y)^s$. Therefore the weights of T on $\mathrm{H}^0(\mathbb{A}^1, \mathscr{O}_{\mathbb{P}^1}(n))$ are $n, n-2, n-4, \cdots$.

We can also compute $\mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)))$. Let $\bar{U}$ be the opposite unipotent radical. We find that $\bar{U}$ maps isomorphically via π to a neighborhood $(\mathbb{A}^1)'$ of $\{\infty\} \in \mathbb{P}^1$. We have a short exact sequence:

$$0 \to \mathrm{H}^0((\mathbb{A}^1)', \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^0((\mathbb{A}^1)' \setminus \{\infty\}, \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n))) \to 0$$

Moreover, $\mathrm{H}^0((\mathbb{A}^1)', \mathscr{O}_{\mathbb{P}^1}(n)) = Y^n k[X/Y]$ and $\mathrm{H}^0((\mathbb{A}^1)' \setminus \{\infty\}, \mathscr{O}_{\mathbb{P}^1}(n)) = Y^n k[X/Y, Y/X]$ so that $\mathrm{H}^1_\infty(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(n)) = Y^n k[X/Y, Y/X]/k[X/Y]$. The weights of T are $-n-2, -n-4, \cdots$.

We deduce that $\mathcal{C}ous(n)$ is given by the following complex:

$$0 \to X^n k[Y/X] \to Y^n k[X/Y, Y/X]/k[X/Y] \to 0$$

Let us finally examine all the actions we have on this complex. We have an action of B, where $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.Y/X = Y/X + t$.

There is no action of $\bar{U}$ since $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.Y/X = \frac{Y/X}{1+tY/X}$. We can however differentiate this action to get an action of $\mathfrak{u}$ where $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = -(Y/X)^2 \partial_{Y/X}$.

The goal of this course will be to generalize these constructions from SL_2 to an arbitrary reductive group G and prove versions of theorems 1.0.1 and 1.0.2 in this setting.

2. Recollections on schemes

2.1. **Affine Schemes.**

Let A be a commutative ring. We define $\mathrm{Spec}\ A = \{\text{prime ideals of } A\}$. We equip $\mathrm{Spec}\ A$ with the Zariski topology. A basis of open are the $\{D(f)\}_{f \in A}$ where $D(f) = \mathrm{Spec}\ A[1/f] \hookrightarrow \mathrm{Spec}\ A$.

We construct a sheaf of rings $\mathscr{O}_{\mathrm{Spec}\ A}$ on the topological space $\mathrm{Spec}\ A$ by putting $\mathscr{O}_{\mathrm{Spec}\ A}(D[f]) = A[1/f]$. That this defines a sheaf follows from the following proposition.

Proposition 2.1.1. *Let $f_1, \cdots, f_n \in A$ be such that $(f_1, \cdots, f_n) = A$. Then the following sequence is exact :*

$$0 \to A \to \prod_i A[1/f_i] \to \prod_{i,j} A[1/f_i f_j]$$

where the first map is the diagonal map $a \mapsto (a)_i$ and the second map if $(f_i) \mapsto (f_{i,j})$ where $f_{i,j} = f_i - f_j$.

The pair $(\mathrm{Spec}\ A, \mathscr{O}_{\mathrm{Spec}\ A})$ is an affine scheme. Any ring morphism $f : A \to B$ induces a map of topological spaces $f : \mathrm{Spec}\ B \to \mathrm{Spec}\ A$ and a map of sheaves $\mathscr{O}_{\mathrm{Spec}\ A} \to f_\star \mathscr{O}_{\mathrm{Spec}\ B}$.

2.2. **Schemes.**

Definition 2.2.1. A locally ringed space $(X, \mathscr{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ over X with the property that for all $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. A map $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of locally ringed spaces is a map $f : X \to Y$ of topological spaces together with a map of sheaves of rings :

$$f^\star \mathscr{O}_Y \to \mathscr{O}_X$$

such that for all $x \in X$, the map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local ring map.

Definition 2.2.2. A scheme is a locally ringed space $(X, \mathscr{O}_X)$ which is locally isomorphic to an affine scheme.

Schemes are therefore a full subcategory of the category of locally ringed spaces. Inside the category of schemes, we have the full subcategory of affine schemes.

Proposition 2.2.3. *The category of affine schemes is equivalent to the opposite category of rings via the quasi-inverse functors* $(X, \mathscr{O}_X) \to \mathrm{H}^0(X, \mathscr{O}_X)$ *and* $A \to (\mathrm{Spec}\ A, \mathscr{O}_{\mathrm{Spec}\ A})$*, which are respectively left and right adjoints of the other.*

Remark 2.2.4. *This proposition explains why we insist on working with locally ringed spaces and not just ringed spaces. Let* k *be a field and let* $\mathrm{Spec}\ k[[T]]$ *be the affine scheme. This has a special point* s *and a generic point* η*. Consider the map* $\mathrm{Spec}\ k((T)) \to \mathrm{Spec}\ k[[T]]$ *obtained by sending* $(0) = \mathrm{Spec}\ k((T))$ *to* s*. This induces a map of ringed spaces, but not of locally ringed spaces. The point is that the map* $k[[T]] = \mathscr{O}_{\mathrm{Spec}\ k[[T]],s} \to k((T)) = \mathscr{O}_{\mathrm{Spec}\ k((T)),0}$ *is not a local map. The good map* $\mathrm{Spec}\ k((T)) \to \mathrm{Spec}\ k[[T]]$ *is the one induced by applying* Spec *to the map* $k[[T]] \to k((T))$ *and it sends* (0) *to* η*.*

One often fixes a base scheme S and consider the category of S-schemes Sch/S. This is the category whose objects are given by a scheme X together with a "structural" morphism $X \to S$. Maps $X \to Y$ between two objects of Sch/S is a map of schemes which respects the structural morphisms.

Remark 2.2.5. $Sch = Sch/\mathbb{Z}$*.*

One is often led to impose finiteness conditions. Here is a brutal list of the most common finiteness conditions:

Finiteness conditions on a scheme :

1. A scheme is quasi-compact if its underlying topological space is quasi compact.
2. Quasi separated if the intersection of two quasi-compact subsets is quasi-compact.
3. Locally noetherian : each point as an open affine neighborhood $\mathrm{Spec}\ R$ with R noetherian.
4. Noetherian : quasi compact and locally noetherian.

Fineteness conditions on a morphism $f : X \to S$.

1. quasi-compact : for any quasi compact open $U \hookrightarrow S$, $f^{-1}(U)$ is quasi-compact.
2. quasi-separated : the diagonal $X \to X \times_S X$ is quasi-compact.
3. separated : the diagonal is a closed immersion.
4. locally of finite type : for every point $x \in X$ there are open affine $x \in \mathrm{Spec}\ R \hookrightarrow X$ and $\mathrm{Spec}\ A \hookrightarrow S$ with $f(\mathrm{Spec}\ R) \subseteq \mathrm{Spec}\ A$ and R is a finite type A-algebra.
5. locally of finite presentation : same as before with R a finite presentation A-algebra.
6. finite type : locally of finite type + quasi-compact.
7. finite presentation : locally of finite presentation + quasi-compact + quasi-separated.

2.3. **Sheaves.** In the case of a ring A, we have the abelian category $\mathrm{Mod}(A)$ of A-modules and its full subcategory $Mod_f(A)$ of finite type A-modules. The category $Mod_f(A)$ is abelian if A is Noetherian. To $M \in \mathrm{Mod}(A)$, we can associate a sheaf of $\mathscr{O}_{\mathrm{Spec}\ A}$-modules over $\mathrm{Spec}\ A$, denoted by $\widetilde{M}$ and defined by the rule that $\widetilde{M}(D(f)) = M \otimes_A A[1/f]$. That this defines a sheaf follows from:

Proposition 2.3.1. *Let* $f_1, \cdots, f_n \in A$ *be such that* $(f_1, \cdots, f_n) = A$*. Then the following sequence is exact :*

$$0 \to M \to \prod_i M[1/f_i] \to \prod_{i,j} M[1/f_i f_j]$$

where the first map is the diagonal map $m \mapsto (m)_i$ *and the second map if* $(m_i) \mapsto (m_{i,j})$ *where* $m_{i,j} = m_i - m_j$*.*

Definition 2.3.2. Let X be a scheme and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. The sheaf $\mathscr{F}$ is quasi-coherent if there is a covering $X = \cup \text{Spec } A_i$ and A_i-modules M_i such that $\mathscr{F}|_{\text{Spec } A_i} = \widetilde{M_i}$. The sheaf is called coherent if theres is a covering as before such that the modules M_i are finite A_i-modules.

We denote by $QCoh(X)$ the category of quasi-coherent sheaves on a scheme X and $Coh(X)$ the category of coherent sheaves on X. This category $QCoh(X)$ is abelian. The category $Coh(X)$ is also abelian if X is locally Noetherian.

Remark 2.3.3. *One finds in the literature several definitions of coherent sheaves on general schemes, which all agree in the locally Noetherian case. We have chosen the simplest one.*

Proposition 2.3.4. *Let* $\text{Spec } A$ *be an affine scheme. The category* $QCoh(\text{Spec } A)$ *is equivalent to the category* $Mod(A)$*, and the category* $Coh(\text{Spec } A)$ *is equivalent to the category* $Mod_f(A)$ *of finite A-modules via the quasi-inverse functors :* $\mathscr{F} \to \mathrm{H}^0(\text{Spec } A, \mathscr{F})$ *and* $M \to \widetilde{M}$.

2.4. **Functor of points.** To any scheme X we attach a functor of points :

$$\begin{array}{rcl} X(-) : Sch^{opp} & \to & SETS \\ T & \mapsto & X(T) \end{array}$$

Lemma 2.4.1 (Yoneda). *The functor* $Sch \to Func(Sch^{opp}, SETS)$ *is fully faithful.*

Definition 2.4.2. A functor $F : Sch^{opp} \to SETS$ is representable if it is in the essential image of the Yoneda functor.

2.5. **Fibre products.** [Reference, [Har77], II, thm. 3.3] Let X, Y, S be schemes and $f : X \to S$, $g : Y \to S$ be maps. Then there is a scheme $X \times_S Y$ called the fibre product of X and Y over S. It fits in a commutative diagram :

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

and satisfies the following universal property:

$$\mathrm{Hom}(-, X \times_S Y) = \mathrm{Hom}(-, X) \times_{\mathrm{Hom}(-,S)} \mathrm{Hom}(-, Y).$$

In the affine case $X = \text{Spec } A, Y = \text{Spec } B, S = \text{Spec } R$ then $X \times_S Y = \text{Spec } (A \otimes_R B)$ which is in particular affine. The general case is obtained by gluing.

2.6. **Sites.**

Definition 2.6.1. A site is a category $\mathcal{C}$ and a collection $Cov(\mathcal{C})$ of families of morphisms with fixed target (called coverings) satisfying the following axioms :

(1) An isomorphism $\phi : V \to U$ is a covering,
(2) If $\{\phi_i : U_i \to U\}_I$ is a covering, and $\{\phi_{i,j} : U_{i,j} \to U_i\}_j$ is a covering then $\{\phi_i \circ \phi_{i,j} : U_{i,j} \to U\}_{i,j}$ is a covering.
(3) If $\{Ui \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then $\forall i$ the fiber product $U_i \times_U V$ exists in $\mathcal{C}$, and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

Definition 2.6.2. A presheaf F on a site $\mathcal{C}$ is a functor $\mathcal{C}^{op} \to SET$. A presheaf F is a sheaf if for any covering $\{\phi_i : U_i \to U\}_{i \in I}$, the diagram:

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact. If the morphism $F(U) \to \prod_i F(U_i)$ is simply injective, the presheaf is said to be separated. A morphism of presheaves is simply a natural transformation of functors. Define $\mathrm{Sh}(\mathcal{C})$ to be the full subcategory of $Func(\mathcal{C}^{op}, SET)$ whose objects are sheaves.

Before giving an example of site in the theory of schemes we mention a few examples:

Example 2.6.3.
1. Let X be a topological space. Let $Op(X)$ be the category of open subsets of X, ordered by inclusion. Coverings are jointly surjective maps. A sheaf on $Op(X)$ is a sheaf in the usual sense, *ie* a topological sheaf.
2. Let $SETS$ be the category of sets. We turn it into a site by declaring that the coverings are the jointly surjective maps.
3. Let Top be the category of topological spaces. Coverings are open coverings.
4. Let $CompTop$ be the category of compact Hausdorff topological spaces. Coverings are finite collections of maps, jointly surjective. A sheaf on $CompTop$ for this topology is called a "condensed set".

2.7. **The** $fppf$ **topology.** Recall that and R-module M is flat if the functor on $Mod(R):\ M \otimes_R -$ is exact.

Definition 2.7.1. A morphism $f : X \to S$ is flat if for all $x \in X$, $\mathscr{O}_{X,x}$ is flat over $\mathscr{O}_{S,f(x)}$.

Proposition 2.7.2. *A morphism of affine schemes $X = \mathrm{Spec}\ A \to S = \mathrm{Spec}\, R$ is flat if and only if A is R-flat.*

Proof. If $R \to A$ is flat then for all $x \in \mathrm{Spec}\ A$ mapping to $y \in \mathrm{Spec}\ R$ and any R_y-module M we have that $A_x \otimes_{R_y} M = A_x \otimes_A A \otimes_R M$. Thus $A_x \otimes_{R_y} -$ is exact. Conversely, assume that A_x is R_y-flat for all x. Let $0 \to I \to R$ be an inclusion. Let $0 \to K \to I \otimes_R A \to A$. We see that for all $x \in \mathrm{Spec}\ A$, $K_x = 0$ thus $K = 0$. ☐

Definition 2.7.3. A family of morphisms $\{\phi_i : U_i \to X\}_{i \in I}$ is an $fppf$ covering if each ϕ_i is flat and locally of finite presentation and $X = \cup_i \phi_i(U_i)$.

Proposition 2.7.4. *Sch_{fppf} is a site.*

Proof. This follows from the fact that a composition of flat morphisms is flat and that the base change of a flat morphism is flat. ☐

Theorem 2.7.5. *Let X be a scheme. The functor of points $X(-)$ is an fppf sheaf.*

2.8. **Differentials and smoothness.**

2.8.1. *The module of differentials.* Let R be a ring and let A be an R-algebra. For any A-module M an R-derivation from A to M is an R-linear map $D : A \to M$ such that $D(ab) = aD(b) + bD(a)$ for all $(a, b) \in A^2$. There is a universal A-module Ω^1_A/R equipped with a derivation $d : A \to \Omega^1_{A/R}$ for which $\mathrm{Der}_R(A, M) = \mathrm{Hom}_A(\Omega^1_{A/R}, M)$ for any A-module M. There is a construction by generators an relations

$$\Omega^1_{A/R} = \oplus_{a \in A} Ada/\langle d(ra) = rda\ \forall (r, a) \in R \times A,\ d(ab) = adb + bda,\ \forall (a, b) \in A \times A\rangle.$$

Here is a second construction. We can also consider the exact sequence $0 \to I \to A \otimes_R A \to A \to 0$ and we let $\Omega^1_{A/R} = I/I^2$, and let $d : A \to I/I^2$ be $d(f) = 1 \otimes f - f \otimes 1$. To see that $d : A \to I/I^2$ is universal, let M be an A-module and let $D : A \to M$ be a derivation. Consider $1 \otimes D : A \otimes_R A \to M$ be the linearization. One checks that $1 \otimes D(I^2) = 0$ and we can consider the A-linear map $1 \otimes D : I/I^2 \to M$. We recover D as the composition $A \to I/I^2 \to M$.

2.8.2. *Two exact sequences.*

Lemma 2.8.1. *If $A \to B$ is a map of R-algebras, we have an exact sequence :*

$$\Omega^1_{A/R} \otimes_A B \to \Omega^1_{B/R} \to \Omega^1_{B/A} \to 0$$

Proof. It suffices to check that for any B-module M, the sequence :

$$0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)$$

is exact. □

Lemma 2.8.2. *If $A \xrightarrow{\alpha} B$ is a surjective map with kernel I, we have :*

$$I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B \to \Omega^1_{B/R} \to 0$$

If $A \to B$ has a splitting $B \to A$ as algebras, then

$$0 \to I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B \to \Omega^1_{B/R} \to 0.$$

Proof. It suffices to check that for any B-module M, the sequence :

$$0 \to \mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M) \to \mathrm{Hom}_A(I/I^2, M)$$

is exact. In case we have a splitting, we check that the map is onto. Indeed, we have $A/I^2 = B \oplus I/I^2$. Given $D \in \mathrm{Hom}_A(I/I^2, M)$, we can extend it to a derivation on $B \oplus I/I^2$ by $D(b+i) = D(i)$. □

Example 2.8.3. We have that $\Omega^1_{R[T_1,\cdots,T_n]/R} = \oplus_{i=1}^n R[T_1, \cdots, T_n]dT_i$. Indeed, one checks that the map $\oplus_{i=1}^n R[T_1, \cdots, T_n]dT_i \to \Omega^1_{R[T_1,\cdots,T_n]/R}$ is surjective using the presentation. We have the derivation $\partial_{T_i} : R[T_1, \cdots, T_n] \to R[T_1, \cdots, T_n]$ and they give linear maps : $\partial_{T_i} : \Omega^1_{R[T_1,\cdots,T_n]/R} \to R[T_1, \cdots, T_n]$ with the property that $\partial_{T_i}(dT_j) = \delta_{i,j}$. We deduce that $\{dT_1, \cdots, dT_n\}$ are indeed a basis of the differentials.

Example 2.8.4. Let $A = R[T_1, \cdots, T_n]/(P_1, \cdots, P_r)$. Then $\Omega^1_{A/R} = \oplus_{i=1}^n AdT_i/(dP_1, \cdots, dP_r)$.

2.8.3. *The naive cotangent complex.* Let B be an R-algebra of finite presentation. This means that we have an exact sequence $0 \to I \to A \xrightarrow{\alpha} B \to 0$ where A is a polynomial algebra over R and I is a finitely generated ideal. To any such presentation, we can associate the complex : $C(\alpha) : I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B$.

Lemma 2.8.5. *For any two presentations α, α', the complexes $C(\alpha)$ and $C(\alpha')$ are homotopic.*

Proof. We first prove that if we have a map of presentations :

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & A \arrow[r, "\alpha"] \arrow[d, "\lambda"] & B \arrow[d, "Id"] \\
0 \arrow[r] & I' \arrow[r] & A' \arrow[r, "\alpha'"] & B
\end{tikzcd}$$

we get a map $\lambda : C(\alpha) \to C(\alpha')$.

Second we show that if λ and λ' are two maps of presentation, λ and λ' are homotopic from $C(\alpha)$ to $C(\alpha')$. The homotopy is provided by the map $\lambda - \lambda' : A \to I'/(I')^2$ which is a derivation.

Third, we show that given any two presentations, there is a map between them. It follows that we have maps $C(\alpha) \to C(\alpha')$ and $C(\alpha') \to C(\alpha)$ and both compositions are homotopic to the identity. □

Definition 2.8.6. A ring morphism $R \to B$ is smooth if it is of finite presentation and for any presentation α, the complex $C(\alpha) : I/I^2 \xrightarrow{d} \Omega^1_{A/R} \otimes B$ is injective with projective cokernel. A ring morphism $R \to B$ is étale if it is smooth and the Naive cotangent complex is quasi-isomorphic to 0.

Proposition 2.8.7. *(1) Let $R \to B$ and $B \to B'$ be smooth (resp. étale) morphisms. Then $R \to B'$ is smooth (resp. étale).*
(2) Let $R \to B$ and $R \to B'$ be smooth (resp. étale) morphisms. Then $R \to B \otimes_R B'$ is smooth (resp. étale).

Proof. Take a presentation $\alpha : R[T_1, \cdots T_n] \to B$ with kernel I, and a presentation $\beta : R[T_1, \cdots, T_n, X_1, \cdots, X_r] \to B'$ with kernel J inducing a presentation $\gamma : B[X_1, \cdots, X_r] \to B'$ with kernel K.

We get a commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \oplus_i B' dT_i \arrow[r] & \oplus_i B' dT_i \oplus \oplus_j B' dX_j \arrow[r] & \oplus_j B' dX_j \arrow[r] & 0 \\
 & I/I^2 \otimes_B B' \arrow[u] \arrow[r] & J/J^2 \arrow[u] \arrow[r] & K/K^2 \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

From which we deduce that the middle map is injective with projective cokernel. The second point is left to the reader. □

2.8.4. *Standard smooth morphisms.*

Definition 2.8.8. An R algebra A is called standard smooth if it has a presentation $R[T_1, \cdots, T_n]/(f_1, \cdots, f_c)$ where the Jacobian matrix $(\partial_{T_i} f_j)_{1 \leq i,j \leq c}$ is invertible in A.

Lemma 2.8.9. *A standard smooth R-algebra A is smooth.*

Proof. Indeed, we observe that $df_1, \cdots, df_c, dT_{c+1}, \cdots, dT_n$ is a basis of $\Omega^1_{R[T_1, \cdots, T_n]} \otimes_R A$. □

Lemma 2.8.10. *Let A be a standard smooth algebra. There is an étale map $R[X_1, \cdots, X_t] \to A$.*

Proof. We consider a standard presentation : $R[T_1, \cdots, T_n]/(f_1, \cdots, f_c)$. We just take $X_1 = T_{c+1}, \cdots, X_t = T_n$. □

Lemma 2.8.11. *A smooth R-algebra A admits a Zariski cover* $\operatorname{Spec} A = \cup_i \operatorname{Spec} A[1/f_i]$ *where $A[1/f_i]$ is a standard smooth R-algebra.*

Proof. Let A be a smooth R-algebra. We take a presentation $0 \to I \to R[T_1, \cdots, T_n] \to A \to 0$. For any $f \in A$, with lift $\widetilde{f}$, $0 \to (I, T_{n+1}\widetilde{f} - 1) \to R[T_1, \cdots, T_n, T_{n+1}] \to A[1/f] \to 0$ is a presentation. We also observe that $(I, T_{n+1}\widetilde{f} - 1)/(I, T_{n+1}\widetilde{f} - 1) = I/I^2 \otimes_A A[1/f] \oplus A[1/f](f dT_{n+1} - df T_{n+1})$. Let $x \in \operatorname{Spec} \neg\dagger A$. Since I/I^2 is projective, there exists $f \in A$ such that $I/I^2 \otimes_A A[1/f]$ is free and $f(x) \neq 0$. We can replace A by $A[1/f]$ and assume I/I^2 is free. It has a basis $(f_1, \cdots, f_c)$. Pick a lift $(h_1, \cdots, h_c)$ in I. By Nakayama, $h \in I$ such that $(1 + h)(h_1, \cdots, h_c) \subseteq I$. We see that $0 \to (h_1, \cdots, h_c, T_{n+1}(1 + h) - 1) \to R[T_1, \cdots, T_{n+1}] \to A \to 0$ is a presentation. Thus we can assume that I is generated by $(f_1, \cdots, f_c)$ which map to a basis of I/I^2. We see that one of the minors of size c of $(\partial_{T_i} f_j)_{1 \leq j \leq c, 1 \leq i \leq n}$ is non-zero. Making one more localization, and possibly reordering, we can assume that $(\partial_{T_i} f_j)_{1 \leq i,j \leq c}$ is invertible in A. □

2.9. **Smoothness and flatness.**

Proposition 2.9.1. *A smooth morphism $R \to B$ is flat.*

Proof. See [Sta13] TAG 00TA. Note that syntomic morphisms are flat by definition. □

Proposition 2.9.2. *(1) Let R be a field. A morphism $R \to B$ is étale if and ony if B is a product of finitely many finite separable field extensions of R.*
(2) Let R be a ring. A morphism $R \to B$ is étale if and only if it is of finite presentation, flat, and for all prime ideal $\mathfrak{p}$ in R, $k(\mathfrak{p}) \to B \otimes_R k(\mathfrak{p})$ is étale.

Proof. First, assume that R is a field and $B = R[x]/P(x)$ with $(P(x), P'(x)) = 1$. Then $R \to B$ is étale (the naive cotangent complex is given by $B \overset{P'(x)}{\to} B$). In the other direction, we may assume that R is algebraically closed. Then one needs to see that if $R \to B$ is étale, then B is finite over R and reduced. See [Sta13] TAG 00U3. For the second point, see [Sta13] TAG 00U6. $\square$

2.9.1. *Smooth morphism.* If $X \to S$ is a map of schemes, we let $\Omega^1_{X/S}$ be the quasi-coherent sheaf over X of relative differentials. If X/S is locally of finite type, this sheaf is coherent. One possible definition is to consider the locally closed immersion $\Delta : X \to X \times_S X$, factor it as the composite of a closed immersion, with ideal $\mathscr{I}$ and open immersion $X \hookrightarrow W \hookrightarrow X \times_S X$ and to let $\Omega^1_{X/S} = \Delta^\star \mathscr{I}/\mathscr{I}^2$. We can also check that for $R \to A$ and $f \in A$, $\Omega^1_{A/R} \otimes_A A_f = \Omega^1_{A_f/R}$, so that the construction of $\Omega^1_{A/R}$ is compatible with Zariski localization.

Definition 2.9.3. A morphism $f : X \to S$ is smooth at $x \in X$ is x has an affine neighboorhood $\operatorname{Spec} B$ over an open Spec R of S containing $f(x)$ and $R \to B$ is a smooth map of rings.

Definition 2.9.4. A morphism is smooth if it is smooth at all points.

The rank of $\Omega^1_{X/S}$ is called the relative dimension of f.

Definition 2.9.5. A morphism is étale if it is smooth of relative dimension zero.

Proposition 2.9.6. *A morphism $f : X \to S$ is étale if*
(1) it is locally of finite presentation,
(2) it is flat,
(3) for all $s \in S$, the fiber X_s is a disjoint union of spectra of finite separable extension of $k(s)$.

2.10. **The Tangent sheaf and the Zariski tangent space.** Assume that $X \to S$ is locally of finite type.

Definition 2.10.1. The tangent sheaf is $T_{X/S} = \underline{Hom}(\Omega^1_{X/S}, \mathscr{O}_{X/S})$.

Proposition 2.10.2. *Assume that $S = \text{Spec } R$. Then $\mathrm{H}^0(X, T_{X/S})$ identifies with the group of automorphisms of $X \times_{\text{Spec } R} \text{Spec } R[\varepsilon]$ which induce the identity on X.*

Proof. Let ϕ be such automorphism. Since $X \times_{\text{Spec } R} \text{Spec } R[\varepsilon]$ and X have the same open subsets, ϕ will preserve any affine cover. We can assume that X is affine, say $X = \text{Spec } A$. We therefore have $\phi : A[\varepsilon] \to A[\varepsilon]$. We have $\phi(a) = a + \varepsilon D(a)$. We check that D is in $\mathrm{Der}_R(A, A) = \mathrm{Hom}_A(\Omega^1_{A/R}, A)$. $\square$

Assume that $S = \operatorname{Spec} R$. Let $X \to S$ be a scheme and let $x : S \to X$ be an S-point. Let I_x be the ideal sheaf of the immersion. We let $V(I_x^2)$ be the first neighborhood of x. We remark that I_x/I_x^2 is supported on S and corresponds to an R-module still denoted by I_x/I_x^2. It follows that $V(I_x^2) = \mathrm{Spec}(R \oplus I_x/I_x^2)$.

Proposition 2.10.3. *Consider the map $r : X(R[\varepsilon]) \to X(R)$ induced by the map $R[\varepsilon] \to R$, $a + \varepsilon b \mapsto a$. Then $r^{-1}(x) = \mathrm{Hom}_R(I_x/I_x^2, R)$.*

Proof. An element of $r^{-1}(x)$ corresponds to a morphism $R \oplus I_x/I_x^2 \to R[\varepsilon]$. $\square$

We call $\mathrm{Hom}(I_x/I_x^2, R)$ the Zariski tangent space at x. We can spell out the connection with $T_{X/S}$.

Lemma 2.10.4. *There is a canonical map $x^\star T_{X/S} \to \mathrm{Hom}_R(I_x/I_x^2, R)$. If X/S is smooth, this map is an isomorphism.*

Proof. Let us assume $X = \mathrm{Spec}\, A$ is affine. We have $x^\star T_{X/S} = \mathrm{Hom}_A(\Omega^1_{A/R}, A) \otimes_A R \to \mathrm{Hom}_A(\Omega^1_{A/R}, R) = \mathrm{Hom}_R(R \otimes_A \Omega^1_{A/R}, R) = \mathrm{Hom}_R(I_x/I_x^2, R)$. In the smooth case, the first map is an isomorphism. □

2.11. **Differential operators.** [Reference : [BO78], section 2]

Let $X \to S$ be a map of schemes. Let $\mathcal{P}_{X/S} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} \mathscr{O}_X$ and $\mathcal{P}^n_{X/S} = \mathcal{P}_{X/S}/I^{n+1}$ where I is the kernel of the map $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} \mathscr{O}_X \to \mathscr{O}_X$. This quasi-coherent sheaf has two structures of $\mathscr{O}_X$-modules given by left and right multiplication.

Lemma 2.11.1. *The map $X \times_S X \times_S X \to X \times_S X$, $(x, y, z) \mapsto (x, y)$ induces a map $\delta : \mathcal{P}^{n+m}_{X/S} \to \mathcal{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathcal{P}^m_{X/S}$, given by $a \otimes b \mapsto a \otimes 1 \otimes 1 \otimes b$.*

Proof. We have a map $\delta : \mathcal{P}_{X/S} \to \mathcal{P}_{X/S} \otimes_{\mathscr{O}_X} \mathcal{P}_{X/S}$, given by $a \otimes b \mapsto a \otimes 1 \otimes 1 \otimes b$. This map sends the ideal I to $I \otimes \mathcal{P}_{X/S} + \mathcal{P}_{X/S} \otimes I$. It sends I^{m+n+1} to $\sum_{a+b=m+n+1} (I \otimes \mathcal{P}_{X/S})^a (\mathcal{P}_{X/S} \otimes I)^b = I^a \otimes I^b$. We see that $I^a \otimes I^b \subseteq I^{n+1} \otimes \mathcal{P}_{X/S} + \mathcal{P}_{X/S} \otimes I^{m+1}$. □

Remark 2.11.2. *Geometrically, the lemma says that if (x, y) are closed to order m and (y, z) are closed to order n, the (x, z) are closed to order $m + n$.*

We want to understand the structure of $\mathcal{P}^m_{X/S}$.

Definition 2.11.3.
1. Let A be a ring. Let $f_1, \cdots, f_c$ be elements of A. We say that the elements $f_1, \cdots, f_c$ define a regular sequence if f_i is not a zero divisor in $A/(f_1, \cdots, f_{i-1})$ for all $0 \leq i \leq c$.
2. Let X be a scheme. An ideal $I \subseteq \mathscr{O}_X$ is called regular, if for any $x \in X$, there is an open affine U and elements $f_1, \cdots, f_c$ which generate $I(U)$ and form a regular sequence in $\mathscr{O}_X(U)$.
3. An immersion of schemes $Z \to X$ is called regular if there exists an open $U \subseteq X$ such that Z can be defined by a regular ideal I in U.

Lemma 2.11.4 ([Sta13], Tag 00LN). *Let A be a ring and $f_1, \cdots, f_c$ be a regular sequence defining an ideal I. The map $A/I[X_1, \cdots, X_c] \to \oplus_{n \geq 0} I^n/I^{n+1}$, sending $\prod X_i^{e_i}$ to $\prod f_i^{e_i} \mod I^{n+1}$ is an isomorphism.*

Proposition 2.11.5. *Assume that X is smooth and that X admits an étale map $X \to \mathbb{A}^n_S$. Let x_i be the coordinates on $\mathbb{A}^n_S$ and $\xi_i = 1 \otimes x_i - x_i \otimes 1 \in \mathcal{P}_{X/S}$. Then $\mathcal{P}^m_{X/S}$ is the free $\mathscr{O}_X$-module with basis the $\prod_i \xi_i^{\alpha_i}$ with $\sum \alpha_i \leq m$.*

Proof.
1. The map $X \to X \times_S X$ is a regular immersion (see [Sta13], Tag 067U). This implies that $\oplus_k I^k/I^{k+1} = \mathrm{Sym}(I/I^2)$.
2. The elements $\xi_i \in I$ map to a basis of I/I^2 by our assumption.
3. We prove by induction on m that the map $\oplus_\alpha \mathscr{O}_X \xi^\alpha \to \mathcal{P}^m_{X/S}$ is an isomorphism.

Here is a more direct argument. We can assume $X = \mathrm{Spec}\ A$ and $R[X_1, \cdots, X_n] \to A$ is étale. Let $J = \mathrm{Ker}(R[X_1, \cdots, X_n] \otimes_R R[X_1, \cdots, X_n] \to R[X_1, \cdots, X_n]$ and $I = \mathrm{Ker}(A \otimes_R A \to A)$. We have that $J \otimes_{R[X_1,\cdots,X_n] \otimes_R R[X_1,\cdots,X_n]} A \otimes_R A$ injects into I and $J/J^2 \otimes_{R[X_1,\cdots,X_n]} A = I/I^2$. This implies easily (Nakayama) that $J/J^n \otimes_{R[X_1,\cdots,X_n] \otimes_R R[X_1,\cdots,X_n]} A \otimes_R A = I/I^n$ for all n. □

Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-coherent sheaves over X. Let $D : \mathcal{E} \to \mathcal{F}$ be an $f^{-1}(\mathscr{O}_S)$-linear operator. One can linearize it by considering $1 \otimes D : \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} \mathcal{E} \to \mathcal{F}$. Alternatively, we see that $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_S} \mathcal{E} = \mathcal{P}_{X/S} \otimes_{\mathscr{O}_X} \mathcal{E}$.

Definition 2.11.6. We say that D is a differential operator of order $\leq n$ is we have a factorization:

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \\ \mathcal{P}^n_{X/S} \otimes \mathcal{E} \arrow[ur] & \end{tikzcd}$$

Remark 2.11.7. *D is of order 0 if and only if it is a linear map.*

Lemma 2.11.8. *Let $D : \mathcal{E} \to \mathcal{E}'$ and $D' : \mathcal{E}' \to \mathcal{E}''$ be differential operators of order $\leq n$ and $\leq m$. Then $D' \circ D$ is of order $\leq m + n$.*

Proof. We have maps $\delta : \mathcal{P}_{X/S} \to \mathcal{P}_{X/S} \otimes_{\mathscr{O}_X} \mathcal{P}_{X/S}$, given by $a \otimes b \mapsto a \otimes 1 \otimes b$. It induces maps $\mathcal{P}^{n+m}_{X/S} \to \mathcal{P}^m_{X/S} \otimes_{\mathscr{O}_X} \mathcal{P}^m_{X/S}$. We now consider the diagram :

$$\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{E}' \arrow[rr] \arrow[dr] & & \mathcal{E}'' \\ & & \mathcal{P}^n \otimes \mathcal{E} \arrow[u] \arrow[d] & & \mathcal{P}^m \otimes \mathcal{E}' \arrow[u] \\ \mathcal{P}^{n+m} \otimes \mathcal{E} \arrow[rr] & & \mathcal{P}^m \otimes \mathcal{P}^n \otimes \mathcal{E} \arrow[urr] & & \end{tikzcd}$$

The top maps are $e \mapsto D(e) \mapsto D'D(e)$. The map $\mathcal{P}^n \otimes \mathcal{E} \to \mathcal{E}$ is $a \otimes b \otimes e \mapsto aD(be)$. The map $\mathcal{P}^n \otimes \mathcal{E} \to \mathcal{P}^m \otimes \mathcal{P}^n \otimes \mathcal{E}$ is $a \otimes b \otimes e \mapsto 1 \otimes a \otimes b \otimes e$. The map $\mathcal{P}^{n+m} \otimes \mathcal{E} \to \mathcal{P}^m \otimes \mathcal{P}^n \otimes \mathcal{E}$ is $a' \otimes b \otimes e \mapsto a' \otimes 1 \otimes b \otimes e$. The map $\mathcal{P}^m \otimes \mathcal{P}^n \otimes \mathcal{E} \to \mathcal{P}^m \otimes \mathcal{E}'$ is $a \otimes b \otimes c \otimes e \mapsto a \otimes b \otimes D(ce)$. $\square$

Definition 2.11.9. We let $\mathcal{D}_{X/S}$ be the ring of differential operators on $\mathscr{O}_X$.

This is a graded ring with $(\mathcal{D}_{X/S})_n = \mathrm{Hom}_{\mathscr{O}_X}(\mathcal{P}^n_{X/S}, \mathscr{O}_X)$ are the differential operators of order $\leq n$. This is a subsheaf of $\mathrm{End}_{f^{-1}(\mathscr{O}_S)}(\mathscr{O}_X)$.

Lemma 2.11.10. *(1) $(\mathcal{D}_{X/S})_0 = \mathscr{O}_X$*
(2) $(\mathcal{D}_{X/S})_1 = \mathscr{O}_X \oplus T_X$.

Proof. The first point is clear. For the second point, we observe that $\mathcal{P}^1_{X/S} = \Omega^1_{X/S} \oplus \mathscr{O}_X$. $\square$

Remark 2.11.11. *We see that a section D of $\mathrm{End}_{f^{-1}(\mathscr{O}_S)}(\mathscr{O}_X)$ is a differential operator of order m if and only if $D((f_1 \otimes 1 - 1 \otimes f_1)....(f_m \otimes 1 - 1 \otimes f_m)) = 0$ for any local sections $(f_i)_{1 \leq i \leq m}$ in $\mathscr{O}_X$. Observe that $D((f_1 \otimes 1 - 1 \otimes f_1)....(f_m \otimes 1 - 1 \otimes f_m)) = [D, f_1](f_2 \otimes 1 - 1 \otimes f_2)....(f_m \otimes 1 - 1 \otimes f_m))$. We deduce that D is of order $\leq m$ if and only if $[D, f]$ is of order $\leq m - 1$ for all $f \in \mathscr{O}_X$.*

Lemma 2.11.12. *If D is of order $\leq n$ and D' is of order $\leq m$, then $[D, D']$ is of order $\leq m + n - 1$.*

Proof. We prove the lemma by induction on the order of D' and D, using the above remark. We have $[[D, D'], f] = [[D, f], D'] + [D, [D', f]]$. $\square$

We define $\mathrm{gr}(\mathcal{D}_{X/S}) = \oplus (\mathcal{D}_{X/S})_n / (\mathcal{D}_{X/S})_{n-1}$.

Corollary 2.11.13. *We have that $\mathrm{gr}(\mathcal{D}_{X/S})$ is a commutative algebra.*

We now put ourselves in the setting of proposition 2.11.5. We assume that X is smooth, and that X admits an étale map $X \to \mathbb{A}^n_S$. Let x_i be the coordinates on $\mathbb{A}^n_S$ and $\xi_i = 1 \otimes x_i - x_i \otimes 1 \in \mathcal{P}_{X/S}$. In the above situation, for any $q = (q_1, \cdots, q_n)$ we let $\xi^q = \prod_i \xi_i^{q_i}$. If $\sum q_i \leq m$, let D_q be the differential operator of order $\leq m$ which satisfies $D_q(\xi^q) = 1$ and $D_q(\xi^{q'}) = 0$ for $q' \neq q$. The D_q form a basis of $\mathcal{D}_{X/S}$.

Remark 2.11.14. *In particular, for $m = 1$, we see that D_q corresponding to $q = (0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in i-th position is ∂_{x_i}. Indeed, $\partial_{x_i}(1 \otimes x_i - x_i \otimes 1) = \partial_{x_i}(x_i) - x_i\partial_{x_i}(1) = 1$.*

We now need to understand the composition.

Lemma 2.11.15. *We have $D_q \circ D_{q'} = \frac{(q+q')!}{q!q'!}D_{q+q'}$.*

Proof. We have $\delta(\xi_i) = x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i = \xi_i \otimes 1 + 1 \otimes \xi_i$. We deduce that $\delta(\xi^p) = \sum_{i+j=p} \frac{q!}{i!(p-i)!}(\xi^{p-i} \otimes 1)(1 \otimes \xi^i)$. Applying $Id \otimes D_{q'}$ first, we find 0, unless $i = q'$. Then apply D_q, we see that we must have $p - q' = q$ otherwise we get 0. □

Proposition 2.11.16. *Assume that X is smooth over S and that S is a $\mathbb{Q}$-scheme. Then $\mathcal{D}_{X/S}$ is the sheaf of algebras generated by $\mathscr{O}_X$ and $T_{X/S}$, subject to the following relations :*

1. *$f_1.f_2 = f_1 f_2$, $f_i \in \mathscr{O}_X$,*
2. *$f.D = fD$, $f \in \mathscr{O}_X$, $D \in T_X$,*
3. *$D_1.D_2 - D_2.D_1 = [D_1, D_2]$,*
4. *$f.D - D.f = D(f)$.*

In particular, $\mathrm{gr}(\mathcal{D}_{X/S}) = \mathrm{Sym}(T_X)$.

Proof. This is a local statement so we can assume that X has an étale map to $\mathbb{A}^n_S$. The above description shows that $\mathcal{D}_{X/S} = \oplus_{(i_1,\cdots,i_n)}\mathscr{O}_X \prod_{l=1}^n \partial_{x_l}^{i_l}$ and

$$(\mathcal{D}_{X/S})_k = \oplus_{(i_1,\cdots,i_n),\sum i_l \leq k}\mathscr{O}_X \prod_{l=1}^n \partial_{x_l}^{i_l}.$$

This implies that the map $T_{X/S} \to \mathrm{gr}((\mathcal{D}_{X/S}))$ induces an isomorphism $\mathrm{gr}((\mathcal{D}_{X/S})) = \mathrm{Sym}(T_X)$. Let $\mathcal{A}$ be the algebra generated by $\mathscr{O}_X$ and $T_{X/S}$ as above. We have a map $\mathcal{A} \to \mathcal{D}_{X/S}$. We can turn $\mathcal{A}$ into a graded algebra by declaring that elements of $T_{X/S}$ have degree ≤ 1 and elements of $\mathscr{O}_X$ have degree ≤ 0. Now we have $\mathrm{Sym}(T_X) \to \mathrm{gr}(\mathcal{A}) \to \mathrm{gr}((\mathcal{D}_{X/S}))$ and the composite is an isomorphism, while the first map is onto. We deduce that $\mathrm{gr}(\mathcal{A}) \to \mathrm{gr}((\mathcal{D}_{X/S}))$ is an isomorphism, hence that $\mathcal{A} \to \mathcal{D}_{X/S}$ is an isomorphism. □

We also need to understand the (left) stalks of $\mathcal{D}_{X/S}$.

Lemma 2.11.17. *Assume that $S = \mathrm{Spec}\ k$ where k is a field. Let $x : S \to X$ be a section. We have a map : $x^\star\mathcal{D}_{X/S} \to \mathrm{colim}_n\mathrm{Hom}_k(\mathscr{O}_{X,x}/\mathfrak{m}_x^n, k)$. This map is an isomorphism if $X \to S$ is smooth.*

Proof. We claim that $x^\star\mathcal{P}^m_{X/S} = \mathscr{O}_{X,x}/\mathfrak{m}_x^{m+1}$. We reduce to the affine case, $X = \mathrm{Spec}\ A$. We have $R \otimes_A (A \otimes_R A) = A$. The ideal $R \otimes_A I$ maps to the maximal ideal $\mathfrak{m}_x$. Now we have $x^\star\mathcal{D}_{X/S} = \mathrm{colim}_n\mathrm{Hom}(\mathcal{P}^m_{X/S}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} k \to \mathrm{colim}_n\mathrm{Hom}_k(\mathscr{O}_{X,x}/\mathfrak{m}_x^n, k)$ where the map is an isomorphism in the smooth case. □

Corollary 2.11.18. *Assume that k is of characteristic 0 and X/S is smooth. We have that $x^\star\mathcal{D}_{X/S} = \oplus_{(i_1,\cdots,i_n)}k\prod_{l=1}^n \partial_{x_l}^{i_l}$ where ∂_{x_l} are a basis of the Tangent space at x. The isomorphism takes $\prod_l \partial_{x_l}^{i_l}$ to the map $f \mapsto (\prod_l \partial_{x_l}^{i_l} f)(x)$ where $f \in \mathscr{O}_{X,x}$.*

2.12. **Dimension.** Let X be a scheme. A closed subset Z is called irreducible if it is non-empty and whenever $Z = Z_1 \cup Z_2$ where Z_1 and Z_2 are closed, then $Z_1 = Z$ or $Z_2 = Z$. Equivalently, this means that any non-empty open subset of Z is dense.

Lemma 2.12.1. *Let U be an open affine subset of X. We have a bijection between irreducible closed subsets Z of X such that $Z \cap U \neq \emptyset$ and irreducible closed subsets Z of U. The maps are given by $Z \mapsto Z \cap U$ and $Z \mapsto \overline{Z}$.*

Proof. Let Z be closed in U. We claim that $\overline{Z} \cap U = Z$. Indeed, let $x \in U \setminus Z$. Then we can find a function f in the ideal defining U such that $x \in D(f)$. Thus, $D(f)$ is open in X. And $\overline{Z} \subseteq D(f)^c$, which proves that $x \notin \overline{Z} \cap U$. If Z is irreducible, then we claim that $\overline{Z}$ is irreducible. Otherwise, $\overline{Z} = Z_1 \cup Z_2$. Then $\overline{Z} \cap U = Z = Z_1 \cap U \cup Z_2 \cap U$. So, we can assume that $Z = Z_1 \cap U$ and $\overline{Z} = Z_1$. Let Z be a irreducible subset of X be such that $Z \cap U \neq \emptyset$. We claim that $Z = \overline{Z \cap U}$. Indeed, $Z = \overline{Z \cap U} \cup U^c \cap Z$ and Z is irreducible. We also deduce that $Z \cap U$ is irreducible. Indeed, if $Z_1 \cup Z_2 = Z \cap U$, then $\overline{Z_1} \cup \overline{Z_2} = Z$ so that $\overline{Z}_1 \subseteq \overline{Z}_2$ (or conversely) and therefore $Z_1 \subseteq Z_2$. □

Lemma 2.12.2. *Any irreducible closed subset of X has a unique generic point.*

Proof. Let Z be an irreducible closed subset. Take U affine open such that $Z \cap U \neq \emptyset$. Let ξ be the generic point of $Z \cap U$. Then the closure of ξ in X is Z.Let ξ and ξ' be two points in X with the same closure. Let U be an affine open containing ξ. If $\xi' \notin U$, we get that $\xi' \in X \setminus U$, a contradiction. So $\xi, \xi' \in U$. But then $\xi = \xi'$ correspond to the same prime ideal. □

We say that X is noetherian if every open subset is quasi-compact.

Lemma 2.12.3. *If X is noetherian, then any closed subset is a finite union of irreducible closed subsets.*

Proof. Let Z be a smallest closed subset which is not a finite union of irreducible closed subset (exists by noetherian assumption). Then Z is not irreducible, hence $Z = Z_1 \cup Z_2$ where Z_i are strictly included in Z. Then Z_i are finite union of irreducible components. Thus Z is a finite union of irreducible closed subsets. □

We let $\dim(X)$ be the maximal length of chain of irreducible closed subsets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ (this chain has length n) in X. When A is a ring we let $\dim(A) = \dim(\mathrm{Spec}\ A)$.

Lemma 2.12.4. *Let X be a scheme. Then* $\dim(X) = \sup_{x \in X} \dim(\mathscr{O}_{X,x})$.

Proof. Let $\xi_0 \to \xi_1 \to \xi_2 \cdots \to \xi_n$ be a chain of specializations in $\mathscr{O}_{X,x}$. They define a chain of prime ideals in any affine open containing x and thus a chain of irreducible closed subsets in X. Therefore $\dim(\mathscr{O}_{X,x}) \leq \dim(X)$. Conversely let $\xi_1 \to \xi_2 \cdots \to \xi_n$ be a chain of specializations in X. These define a chain of specializations in $\mathscr{O}_{X,x}$ with $x = \xi_n$. □

We let $\mathrm{codim}_X(x) = \dim(\mathscr{O}_{X,x})$.

We now assume that $(A, \mathfrak{m})$ be a noetherian local ring. A system of parameters of A is a sequence of elements $(a_1, \cdots, a_n)$ such that $\sqrt{(a_1, \cdots, a_n)} = \mathfrak{m}$.

Theorem 2.12.5 ([Sta13], Tag 00KQ)**.** *The minimal number of elements defining a system of parameters is the dimension of A.*

Corollary 2.12.6. *Let A be a noetherian scheme, then* $\dim A[X] = \dim A + 1$.

Proof. Let $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$ be a sequence of prime ideals in A. Then $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n \subseteq (\mathfrak{p}_n, X)$ is a sequence of lenght $n+1$. Thus, $\dim A[X] \geq \dim A + 1$. Conversely, let $x \in \mathrm{Spec}\ A[X]$ mapping to y. Let $(f_1, \cdots, f_n)$ be a sequence of parameters of y. Observe that $A[X]_x/\mathfrak{m}_y$ is a localization of $k(y)[X]$ at a prime ideal. Thus, it is either a field, if which case $(f_1, \cdots, f_n)$ is a sequence of parameters of x as well, or x defines a closed point corresponding to some irreducible polynomial $P(X) \in k(y)[X]$. In that case, $(f_1, \cdots, f_n, \widetilde{P})$ is a sequence of parameters at x. □

Corollary 2.12.7. *Let k be a field, then* $\dim\ k[X_1, \cdots, X_n] = n$. *More precisely, for any closed point x,* $\dim\ k[X_1, \cdots, X_n]_x = n$.

2.12.1. *Dimensions of k-schemes.* Let $A = k[X_1, \cdots, X_n]/(P_1, \cdots, P_r)$. Let x be a closed point of Spec A.

Lemma 2.12.8. *We have* $\dim A_x \geq n - r$.

Proof. If $f_1, \cdots, f_k$ is a system of parameters of x in Spec A, then $(\widetilde{f_1}, \cdots, \widetilde{f_k}, P_1, \cdots, P_r)$ is a system of parameters of x in Spec $k[X_1, \cdots, X_n]$. Thus $k + r \geq n$. □

Theorem 2.12.9 ([Sta13], Tag 02JN)**.** *If $P_1, \cdots, P_r$ are a regular sequence in $k[X_1, \cdots, X_n]_x$, then* $\dim(k[X_1, \cdots, X_n]/(P_1, \cdots, P_r))_x = n - r$.

Theorem 2.12.10. *Assume that $X \to \mathrm{Spec}\ k$ is a smooth scheme, then $\dim(X) = \dim(\mathscr{O}_{X,x})$ for any closed point x is the rank of $\Omega^1_{X/k}$.*

Proof. Locally, X is standard smooth, of the form Spec $k[X_1, \cdots, X_n]/(P_1, \cdots, P_r)$. Then $P_1, \cdots, P_r$ is a regular sequence by [Sta13] Tag 067U. □

2.12.2. *Constructibility theorem.* Let X be a qcqs scheme. A subset of X is called constructible if it is a finite union of sets of the form $U \cap Z$ where U is a quasi-compact open and Z is a closed subset with quasi-compact complement.

Theorem 2.12.11. *Let $f : X \to Y$ be a morphisms where f is locally of finite presentation, and X, Y are quasi-compact, quasi-separated. Then the image of any constructible set is constructible.*

Corollary 2.12.12. *Assume that X is noetherian. The image $f(X)$ contains an open subset of its closure.*

Proof. This means that $f(X) = \cup_{i \in I} U_i \cap Z_i$ where I is finite, U_i and Z_i are quasi-compact opens and complement of quasi-compact opens respectively. We can also suppose that Z_i is irreducible and U_i dense in Z_i. We claim that $f(X)$ contains an open subset of its closure. Let I' be a minimal set such that $\cup_{i \in I'} Z_i = \cup_{i \in I} Z_i$. Let $U'_i = U_i \setminus \{\cup_{j \neq i \in I'} Z_j\}$. Then U'_i is still dense in Z_i and open in $\cup_i Z_i$. Thus, $\cup_{i \in I'} U'_i = V$ is dense open in the closure of $f(X)$ □

2.12.3. *Generic flatness theorem.* Let us first recall Noether's normalization lemma.

Lemma 2.12.13 ([Sta13], Tag 07NA)**.** *Let $R \to R'$ be an injective map of algebras with R a domain. There exists $f \in R \neq 0$ and an integer d such that we have a factorization $R_f \to R_f[T_1, \cdots, T_d] \hookrightarrow R'_f$ with R'_f finite over $R_f[T_1, \cdots, T_d]$.*

Theorem 2.12.14. *Let S be a noetherian scheme. Let $X \to S$ be a morphism of finite type, with S reduced. There is an open dense $U \subseteq S$ such that $X|_U \to U$ is flat.*

This is a consequence of the following.

Proposition 2.12.15. *Let A be a noetherian integral ring, let B be an A-algebra of finite type. Let M be a finite B-module. There exists $f \in A \setminus \{0\}$ such that M_f is free over A_f.*

Proof. Let $K = \mathrm{Frac}(A)$. The proof is by induction on the dimension of the support of $M \otimes K$. We also note that if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ and the lemma holds for M_1 and M_3 it holds for M_2. Suppose that $M \otimes K$ is zero. Let $m_1, \cdots, m_n$ be generators of M as a B-module. There exists $f \in A \neq 0$ such that $fm_1 = \cdots = fm_n = 0$. Therefore $M \otimes_A A_f = 0$. In general, we recall that M is a successive extension of modules of the form $B/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. We reduce to the case that $M = B$ is a domain. By Noether normalization, there exists $f \in A$ and $b_1, \cdots, b_n \in B$ such that $A_f \to A_f[b_1, \cdots, b_n] \to B_f$ where $A_f[b_1, \cdots, b_n]$ is a polynomial algebra and $A_f[b_1, \cdots, b_n] \to B$ is finite. We let r be the generic rank of M over $A_f[b_1, \cdots, b_n]$. We have a map $0 \to A_f[b_1, \cdots, b_n]^r \to M \to T \to 0$ and the dimension of the support of $T \otimes K$ is less than n. □

Lemma 2.12.16. *Let A be a Noetherian ring. Let M be a finite type A-module. Then M is a finite successive extension of modules of the shape $A/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal.*

Proof. First, we see that M is a finite successive extension of modules of the shape A/I by induction on the number of generators of M. So we reduce to $M = A/I$. Now we consider an ideal I which is maximal among ideals with the property that A/I is not a successive extension of modules of the shape $A/\mathfrak{p}$. By contradiction, assume $I \neq A$. Clearly, I is not prime, so there is a, b with $ab \in I$ but $a, b \notin I$. Replace A/I by A. We have $0 \to A/ann(a) \to A \to A/a \to 0$. But both $A/ann(a)$ and A/a have the property. So does A. $\square$

2.12.4. *Generically smooth.*

Proposition 2.12.17. *Let $X \to \text{Spec } k$ be a finite type morphism. Assume that X is geometrically reduced. Then X is generically smooth.*

Proof. We can suppose $X = \text{Spec } A$ with A a domain. By Noether normalization, we have $k \to k[x_1, \cdots, x_n] \to A$ where $k[x_1, \cdots, x_n] \to A$ is injective and finite. Passing to the generic point, we have $A \otimes_{k[x_1,\cdots,x_n]} k(x_1, \cdots, x_n)$ is an étale $k(x_1, \cdots, x_n)$-algebra. Indeed, $A \otimes_{k[x_1,\cdots,x_n]} k(x_1, \cdots, x_n)$ is reduced, so it must be a product of finite field extensions of $k(x_1, \cdots, x_n)$. Being geometricall reduced implies these are étale extension. We deduce that there is a map $k[x_1, \cdots, x_n, y_1, \cdots, y_r]/[P_1, \cdots, P_r] \to A$ which induces an isomorphism over $k(x_1, \cdots, x_n)$. This implies that there is $f \in k[x_1, \cdots, x_n]$ such that $k[x_1, \cdots, x_n, y_1, \cdots, y_r]/[P_1, \cdots, P_r][1/f] \simeq A[1/f]$. We look at the determinant d of the Jacobian $(\partial_{Y_j} P_i)_{1 \leq i,j \leq r}$. We have $d \in k[x_1, \cdots, x_n, y_1, \cdots, y_r]/[P_1, \cdots, P_r]$ and $V(d)$ is nowhere dense (indeed, d is invertible over $k(x_1, \cdots, x_n)$). We deduce that

$$k[x_1, \cdots, x_n][1/f][y_1, \cdots, y_r, y_{r+1}]/(P_1, \cdots, P_r, y_{r+1}d - 1) = A[1/fd]$$

is standard étale over $k[x_1, \cdots, x_n][1/f]$. $\square$

3. Group schemes

3.1. **Group Schemes.** [Reference : [Jan03], part I, section 2] We now work in Sch/S. A group scheme $G \to S$ is a scheme equipped with the following additional structure : $m : G \times_S G \to G$, $e : S \to G$, $\iota : G \to G$, satisfying associativity, neutral element and inverse axioms. Alternatively, a group scheme is a group functor $Fun(\text{Sch}^{op}, Gr)$ that is representable by a scheme G. A morphism of Group schemes is a morphism of schemes, compatible with the group structure. When G is affine we can completely describe this extra structure in ring theoretic terms.

Definition 3.1.1. Let R be a base ring. An Hopf algebra A over R is a commutative ring equipped with a comultiplication $m^\star : A \to A \otimes_R A$, counit $e^\star : A \to R$ and coinverse $\iota^\star : A \to A$ and satisfy the following axioms :

(1) Co-associative

$$\begin{tikzcd} A \arrow[r, "m^\star"] \arrow[d, "m^\star"] & A \otimes A \arrow[d, "1 \otimes m^\star"] \\ A \otimes A \arrow[r, "m^\star \otimes 1"] & A \otimes A \otimes A \end{tikzcd}$$

(2) Inverse $A \overset{m^\star}{\to} A \otimes A \overset{Id \otimes \iota^\star}{\to} A \otimes A \to A$ and $A \overset{m^\star}{\to} A \otimes A \overset{\iota^\star \otimes Id}{\to} A \otimes A \to A$ are the identity map.
(3) Neutral element $A \overset{m^\star}{\to} A \otimes A \overset{Id \otimes e^\star}{\to} A$ and $A \overset{m^\star}{\to} A \otimes A \overset{e^\star \otimes Id}{\to} A$ are the identity map.

Proposition 3.1.2. *Hopf algebras over R are anti-equivalent to afffine group schemes over* Spec R

Example 3.1.3. (1) We have $\mathbb{G}_a = \text{Spec } R[X]$, with $m^\star(X) = 1 \otimes X + X \otimes 1$.
(2) We have $\mathbb{G}_a = \text{Spec } R[X, X^{-1}]$, with $m^\star(X) = X \otimes X$.
(3) We have $\text{GL}_n = \text{Spec } R[X_{i,j}, 1 \leq i, j \leq n][1/\det]$ with $m^\star(X_{i,j}) = \sum_l X_{i,l} \otimes X_{l,j}$.

3.2. **Action.** Let G be a group scheme and let X be a scheme. A left action of G on X is a morphism : $G \times X \to X$ such that the action is associative and the unit acts trivially. Equivalently, for any S-scheme T, we have an action $G(T) \times X(T) \to X(T)$, functorially in T.

3.3. **Representations.** Let us assume $S = \mathrm{Spec}\ R$. Let M be an R-module. We can associate to M the functor $\underline{M}$ on R-algebras such that $\underline{M}(A) = A \otimes_R M$. A representation of G on M is a map of functors $G \times \underline{M} \to \underline{M}$ such that for all $A \in Alg/R$, $G(A) \times \underline{M}(A) \to \underline{M}(A)$ defines an A-linear action of $G(A)$ on $\underline{M}(A)$. Let us denote by $GL(\underline{M})$ the group functor which sends an R-algebra B to the group $GL(M \otimes_R B)$ of B-linear automorphisms of $M \otimes_R B$. A representation on M is therefore a group functor map $G \to GL(\underline{M})$. If $M = A^n$ is a finite free module, a representation on M is the same as a group scheme homomorphism $G \to \mathrm{GL}_n$. We let $Mod_G(R)$ be the category of representations of G on R-modules. Assume that $G = \mathrm{Spec}\ A$ is affine, with A an Hopf algebra.

Definition 3.3.1. A co-module is an R-module M equipped with an A-linear map $\Delta : M \to M \otimes A$ which satisfies the axioms :

(1)
$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"] & M \otimes A \arrow[d, "\Delta\otimes Id"] \\ M \otimes A \arrow[r, "Id\otimes m^\star"] & M \otimes A \otimes A \end{tikzcd}$$

(2) $M \to M \otimes A \stackrel{Id\otimes e^\star}{\to} M$ is the identity map.

Proposition 3.3.2. *The category $Mod_G(R)$ is equivalent to the category of co-modules.*

Proof. Given a co-module $\Delta : M \to M \otimes A$, and a B and A-algebra, we produce a map $G(B) \to \mathrm{GL}(M \otimes_A B)$ as follows. Let $g \in G(B)$, corresponding to $g : A \to B$. We have an A-linear map $M \to M \otimes_R A \stackrel{Id\otimes g}{\to} M \otimes_R B$ which extends to a B-linear map $\Theta_g : M \otimes_A B \to M \otimes_A B$. The associativity axiom implies that $\Theta_g \circ \Theta_h = \Theta_{gh}$. Let us spell out the details. The following diagram is commutative. The bottom horizontal arrow computes $\Theta_g \circ \Theta_h$. The composition of the right vertical map and upper horizontal map is $M \to M \otimes_R B$ whose linearization is Θ_{gh}:

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "Id_M"] & M \otimes A \arrow[r, "Id\otimes m^\star"] & M \otimes A \otimes A \arrow[d, "Id_{M\otimes A\otimes A}"] \\ M \arrow[r, "\Delta"] \arrow[d] & M \otimes A \arrow[r, "\Delta\otimes Id_A"] \arrow[d, "Id\otimes g"] & M \otimes A \otimes A \arrow[d, "Id\otimes(g\otimes h)"] \\ M \otimes_R B \arrow[r, "\Theta_g"] & M \otimes_R B \arrow[r, "\Theta_h"] & M \otimes_R B \end{tikzcd}$$

In particular, $\Theta_{g^{-1}} \circ \Theta_g = \Theta_e = \mathrm{Id}$ so Θ_g is an automorphism. Thus, the co-module gives a group action on $\underline{M}$. Conversely, assume we have a group action. Let $un \in G(A)$ be the universal element (corresponding to the identity morphism $A \to A$). Then we set $\Delta : M \to M \otimes_R A$ be the action of the universal element. This is the co-module structure. The coassociativity follows as in the diagram above by taking $B = A \otimes A$, g and h the two universal point given by $A \mapsto A \otimes A$, $a \mapsto a \otimes 1$, and $a \mapsto 1 \otimes a$. ☐

Example 3.3.3.
(1) If $G = \mathrm{Spec}\ A$ is affine, we can consider the regular representation : we take $M = A$ itself. We claim that the map $G \to GL(\underline{A})$ is injective (as a map of group functors).
(2) The category $Mod_{\mathbb{G}_m}(R)$ is equivalent to the category of $\mathbb{Z}$-graded modules.

(3) The category $Mod_{\mathbb{G}_a}(R)$ is more complicated. When R is a $\mathbb{Q}$-algebra, a representation of $\mathbb{G}_a$ on a module M is equivalent to the data of an endomorphism E of M which is locally nilpotent.

3.4. **The Lie algebra of a group scheme.** [Reference [DG70], II, section 4] Let $G \to S$ be a group scheme. For any R-algebra B, we let

$$1 \to \mathrm{Lie}(G)(B) \to G(B[\varepsilon]) \to G(B) \to 1$$

We see that this defines a group functor $\mathrm{Lie}(G)(-)$ on R algebras. Let us put $\mathrm{Lie}(G) := \mathrm{Lie}(G)(R)$. Let us also put $\omega_G = e^\star\Omega^1_{G/S}$. We get a functor $\mathrm{Gr}/S \to \mathrm{Mod}_R$, $G \mapsto \mathrm{Lie}(G) := \mathrm{Lie}(G)(R)$.

Remark 3.4.1. *This definition of Lie algebra applies more generally to any group functor (not necessarily representable).*

Theorem 3.4.2. *We have an isomorphism of groups* $\mathrm{Lie}(G)(B) = \mathrm{Hom}_R(\omega_G, B)$.

Proof. By proposition 2.10.3, $\mathrm{Lie}(G)(B) = \mathrm{Hom}_R(\omega_G, B)$. The RHS carries a natural group law (call it $\star$). The multiplication $m : G \times G$ on the group induces a map $\mathrm{Lie}(G)(B) \oplus \mathrm{Lie}(G)(B) \to \mathrm{Lie}(G)(B)$ which gives a second group law $\circ$ compatible with $\star$. To show that these two group law agree, we use the lemma below. □

Lemma 3.4.3. *Let X be a set. We assume that X has two group structures, $\star$ and $\circ$ and that $(a \star b) \circ (a' \star b') = (a \circ a') \star (b \circ b')$. Then $\star = \circ$ are commutative group laws.*

Proof. We first check that the units $1_\star$ and $1_\circ$ agree :

$$\begin{aligned} 1_\star &= (1_\circ \circ 1_\star) \star (1_\star \circ 1_\circ) \\ &= (1_\circ \star 1_\star) \circ (1_\circ \star 1_\circ) \\ &= 1_\circ \end{aligned}$$

We deduce that:

$$\begin{aligned} a \star b &= (a \circ 1) \star (1 \circ b) \\ &= a \circ b \end{aligned}$$

Finally, we have:

$$\begin{aligned} a \star b &= (1 \circ a) \star (b \circ 1) \\ &= b \circ a \end{aligned}$$

□

If $x \in \mathrm{Lie}(G)(B)$, we let $e^{\varepsilon x}$ be its image in $G(B[\varepsilon])$.

Remark 3.4.4. *When ω_G is finite projective, then* $\mathrm{Lie}(G)(B) = \mathrm{Lie}(G) \otimes_R B$. *This is true if R is a field, and G is of finite type. We often simply restrict to this case.*

Example 3.4.5. (1) If we take M an R module, then $\mathrm{Lie}(\underline{M}) = M$.

(2) If M is an R-module, then $\mathrm{End}(M) = \mathrm{Lie}(\mathrm{GL}(\underline{M}))$ via the map sending N to $Id + \varepsilon N$. Remark that $Id + \varepsilon N$ has inverse $Id - \varepsilon N$.

3.5. **Lie bracket.** Consider a linear representation ρ of a group G on a module M. This induces a group morphism $d\rho : \mathrm{Lie}(G) \to \mathrm{End}(M)$, with the property that $\rho(e^{\varepsilon x}) = 1 + \varepsilon \mathrm{d}\rho(x)$.

We now assume that $\mathrm{Lie}(G)(B) = \mathrm{Lie}(G) \otimes_R B$ as in remark 3.4.4. We have a linear adjoint representation of G on $\mathrm{Lie}(G)$ denoted by $Ad : G \to \mathrm{GL}(\underline{\mathrm{Lie}(G)})$. Indeed we look at the exact sequence :

$$1 \to \mathrm{Lie}(G)(B) \to G(B[\varepsilon]) \to G(B) \to 1$$

and the group $G(B)$ acts by conjugation on $\mathrm{Lie}(G)(B)$. We justify that the elements of $G(B)$ act B-linearly. For $g \in G(B)$, we get an map $g : G_B \to G_B$, $h \mapsto ghg^{-1}$. By functoriality, this induces a B-linear map on the tangent space (which is the map we are considering). By derivation, we get $ad : \mathrm{Lie}(G) \to \mathrm{End}(\mathrm{Lie}(G))$. We define a Lie bracket by $ad(x)(y) = [x, y]$.

Consider the ring $R[\varepsilon, \varepsilon'] = R[X, X']/(X^2, (X')^2)$. It contains the subrings $R[\varepsilon]$, $R[\varepsilon']$ and $R[\varepsilon\varepsilon']$. We also have an exact sequence $0 \to \varepsilon' R[\varepsilon] \to R[\varepsilon', \varepsilon] \to R[\varepsilon] \to 0$.

Lemma 3.5.1. *Let $x, y \in \mathrm{Lie}(G)$. We have*

$$e^{\varepsilon x} e^{\varepsilon' y} e^{-\varepsilon x} e^{-\varepsilon' y} = e^{\varepsilon\varepsilon'[x,y]}.$$

Proof. Consider the long exact sequence :

$1 \to \mathrm{Lie}(G)(R[\varepsilon']) \stackrel{e^{\varepsilon'(-)}}{\to} G(R[\varepsilon, \varepsilon']) \to G(R[\varepsilon]) \to 1$, so that $e^{\varepsilon x} \in G(R[\varepsilon])$. We have

$$\begin{aligned} e^{\varepsilon x} e^{\varepsilon' y} e^{-\varepsilon x} &= Ad(e^{\varepsilon x})(e^{\varepsilon' y}) \\ &= (Id + \varepsilon ad(x))(e^{\varepsilon' y}) \\ &= e^{\varepsilon' y + \varepsilon\varepsilon'[x,y]} \end{aligned}$$

□

Corollary 3.5.2. *Let M be a finite projective R-module. The Lie bracket on* $\mathrm{End}(M)$ *is given by* $[x, y] = xy - yx$.

Proposition 3.5.3. *For any representation ρ of G on M, the map $d\rho : Lie(G) \to$* $\mathrm{End}(M)$ *is compatible with the Lie bracket.*

Proof. We work in $R[\varepsilon, \varepsilon']$. We have $e^{\varepsilon x} e^{\varepsilon' y} e^{-\varepsilon x} e^{-\varepsilon' y} = e^{\varepsilon\varepsilon'[x,y]}$. Applying ρ we get $Id + \varepsilon\varepsilon' ad([x, y]) = (Id + \varepsilon ad(x))(Id + \varepsilon' ad(y))(Id - \varepsilon ad(x))(Id - \varepsilon' ad(y))$. □

3.6. **Lie algebra and derivations.** If X is a scheme, we can define a group functor $\mathrm{Aut}(X)$ by $\mathrm{Aut}(X)(B) = \mathrm{Aut}(X \times \mathrm{Spec}\ B/\mathrm{Spec}\ B)$. We can in particular consider $\mathrm{Lie}(\mathrm{Aut})(X)$.

Proposition 3.6.1. $\mathrm{Lie}(\mathrm{Aut})(X) = \mathrm{Der}(X)$.

Proof. See proposition 2.10.2. □

Let G be a group scheme acting on the right on X. Then we get a map $G \to \mathrm{Aut}(X)$ and a corresponding map $\mathrm{Lie}(G) \to \mathrm{Der}(X)$. We can make this more explicit. Let $x \in \mathrm{Lie}(G)$. Then we have a map $e^{\varepsilon x} : X \times_R R[\varepsilon] \to X \times_R R[\varepsilon]$.

If f is a local function on X, and $m \in X$, then $f(me^{\varepsilon x}) = f(m) + \varepsilon D_x(f)(m)$.

Proposition 3.6.2. *The map* $\mathrm{Lie}(G) \to \mathrm{Der}(X)$ *is compatible with Lie bracket.*

Proof. We compute :

$$\begin{aligned} f(me^{-\varepsilon' y}) &= f(m) - \varepsilon' D_y(f)(m) \\ f(me^{-\varepsilon x} e^{-\varepsilon' y}) &= f(m) - \varepsilon' D_y(f)(m) - \varepsilon D_x f(m) + \varepsilon\varepsilon' D_x D_y(f)(m) \\ f(me^{\varepsilon' y} e^{-\varepsilon x} e^{-\varepsilon' y}) &= f(m) - \varepsilon' D_y(f)(m) - \varepsilon D_x f(m) + \varepsilon\varepsilon' D_x D_y(f)(m) + \varepsilon' D_y f(m) - \varepsilon'\varepsilon D_y D_x f(m) \\ &= f(m) - \varepsilon D_x f(m) + \varepsilon\varepsilon' D_x D_y(f)(m) - \varepsilon'\varepsilon D_y D_x f(m) \\ f(me^{\varepsilon x} e^{\varepsilon' y} e^{-\varepsilon x} e^{-\varepsilon' y}) &= f(m) - \varepsilon D_x f(m) + \varepsilon\varepsilon' D_x D_y(f)(m) - \varepsilon'\varepsilon D_y D_x f(m) + \varepsilon D_x f(m) \\ &= f(m) + \varepsilon\varepsilon' [D_x, D_y] f(m) \end{aligned}$$

□

We can consider the action of G on itself by right translation, $\star_r$ and by left translation $\star_l$.
We have a map $G^{op} \to \mathrm{Aut}(G)$, $g \mapsto g \star_r (-)$.

Lemma 3.6.3. *The map $G^{op} \to \mathrm{Aut}(G)$ induces an isomorphism of G^{op} on the subspace of $Aut^{\star_l G}(G)$, of automorphisms which commute with left translation.*

Proof. We define a map $Aut^{\star_l G}(G) \to G$, by $\phi \mapsto \phi(e)$. Observe that $\phi(g) = g\phi(e)$ and $\phi' \circ \phi \mapsto \phi(e)\phi'(e)$. □

Corollary 3.6.4. *The map $\mathrm{Lie}(G) \to \mathrm{Der}(G)$ identifies $\mathrm{Lie}(G)$ with the space of left invariant derivations.*

Assume G is Spec A. Then we can make explicit what are the left invariant derivation. Let $D : A \to A$ be a derivation. For any element $g \in G(T)$. We have a map $g : A \otimes T \to A \otimes T$ given by left translation. Then we ask that $g^{-1} \circ D \otimes 1 \circ g = D \otimes 1$.

3.7. **Lie algebras : general.** A Lie algebra $\mathfrak{g}$ over a ring R is an R-module $\mathfrak{g}$ endowed with a braket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Such that :

(1) $[,]$ is bilinear over R,
(2) $[X, X] = 0$ for all $X \in \mathfrak{g}$.
(3) (Jacobi identity) For all $X, Y, Z \in \mathfrak{g}$, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$.

Example 3.7.1. (1) If M is an R-module, then $\mathrm{End}_R(M)$, endowed with the braket $[X, Y] = XY - YX$ is a Lie algebra.
(2) If A is an R-algebra, then $\mathrm{Der}_R(A)$ is a sub-Lie algebra of $\mathrm{End}_R(A)$.
(3) If X is a Spec R-scheme, then $Der(X) = \mathrm{H}^0(X, T_{X/S})$ is a Lie algebra.

Corollary 3.7.2. *Let G be a group scheme. Then $\mathrm{Lie}(G)$ with its bracket $[,]$ is a Lie algebra.*

Proof. Indeed, $\mathrm{Lie}(G)$ is a sub-Lie algebra of $Der(G)$. □

3.8. **Affine algebraic groups over a field.** We fix a field k. All schemes are over Spec k.

Definition 3.8.1. An algebraic group is a group scheme G which is of finite type over Spec k. An affine algebraic group G is an affine group scheme over Spec k which is of finite type.

Concretely, G is an affine algebraic group if $G = \mathrm{Spec}\ A$, where A is a k-algebra of finite type and an Hopf algebra.

3.8.1. *Smoothness.* The following result is known as Cartier's theorem. See [Mum08], III, sect. 11, thm. on page 101.

Theorem 3.8.2. *Assume that k is of characteristic 0. Let $G \to \mathrm{Spec}\ k$ be a group scheme, locally of finite type. Then G is smooth over Spec k.*

3.8.2. *Subgroups.*

Lemma 3.8.3. *Let $U \subseteq G$ be a dense open subscheme. Then $U \times U$ maps surjectively onto G.*

Proof. We can suppose $k = \bar{k}$. It suffices to see that $U.U(\bar{k}) \to G$ which is open in G, contains all k-points. Let $g \in G(k)$. Then $U \cap gU^{-1}$ is again dense open (indeed, U and gU^{-1} contain all generic points). Thus, there are points $v, w \in U(\bar{k})$ such that $v = gw^{-1}$. □

Let G be an algebraic group and let $i : H \hookrightarrow G$ be an algebraic subgroup.

Lemma 3.8.4. *The image of H is a closed subset of G.*

Proof. We claim that the image $i(H)$ of H in G is a closed subspace. In order to prove this, we can assume that $k = \bar{k}$. The image $i(H)$ in G is constructible. This implies that $i(H)$ contains a subset V which is dense and open in $\overline{i(H)}$. Then $H(\bar{k}).i^{-1}(V) = H$. We deduce that $i(H)$ is open in $\overline{i(H)}$. Note that $\overline{i(H)}$ is a closed subgroup of G. The above lemma shows that $i(H) = \overline{i(H)}$. □

We assume that the map $H \to G$ is a monomorphism of sheaves.

Lemma 3.8.5. *The group H is a closed subgroup of G.*

Proof. We first claim that $i : H \to G$ is injective and induces isomorphisms on residue fields. Consider a point $g \in G$ and look at the fiber $H_g \to g$. Then H_g is a subfunctor of g. In particular $H_g \times_g H_g \to H_g$ is an isomorphism. This means that H_g has a unique point. So $H_g = \mathrm{Spec}\ A$ for some artinian alegbra and $A \otimes_{k(g)} A = A$ which implies that A has dimension 1 (as a $k(g)$-vector space). So $A = k(g)$.

We next claim that there exists a dense open V of G such that $i^{-1}(V) \to V$ is a closed immersion. Let h be a generic point of H mapping to $h \in G$. Consider the map $\mathscr{O}_{G,h} \mapsto \mathscr{O}_{H,h}$. Since $\mathscr{O}_{H,h}$ is Artinian, and they have the same residue field, we deduce that $\mathscr{O}_{G,h} \to \mathscr{O}_{H,h}$ is surjective. Let $\mathrm{Spec}\, A$ be an irreducible open subset of H mapping to $\mathrm{Spec}\, B$ open in G. We have a map $B \to B/I \hookrightarrow A$. Moreover B/I and A have the same generic point ξ and $(B/I)_\xi = A_\xi$. Let $x_1, \cdots, x_n$ be generators of A as a B-algebra. There exists $f \in B$ with $f(\xi) \neq 0$ such that $fx_i \in B$. We deduce that $B[1/f]/I \to A[1/f]/I$ is an isomorphism. By translation, this finally implies that H is a closed subgroup of G. □

3.8.3. *Existence of representations.* Let G be an affine algebraic group and H a subgroup of G (necessarily closed in G and affine).

Lemma 3.8.6. *There exists a finite dimensional representation $\rho : G \to GL(V)$ with the property that H is the stabilizer of a line L.*

Proof. We let A be the algebra of G and I the ideal of H. We consider the representation of G on A. We first claim that H is exactly the stabilizor of I. If $h \in H(R)$, and $f \in I \otimes R$, we have $h.f = f(-h)$. It is clear that if f vanishes on H, then so does $h.f$. Conversely, if $g \in G(R)$ is such that $g.f = f(-g) \in I \otimes R$ for all f, then $f(g) = 0$ for all f. Thus $g \in H(R)$.

We now let $W \subseteq A$ be finite dimesional k-vector space, with the property that V generates A as an algebra and $W \cap I$ generates I as an ideal.

We claim that there exists $W \subseteq X$ such that X is a finite dimensional representation of G. Let (a_i) be a k-basis of A. We have $\Delta(x) = \sum_i x_i \otimes a_i$ and $\sum_i \Delta(x_i) \otimes a_i = \sum_i x_i \otimes \Delta(a_i) = \sum_{i,j} x_i \otimes b_{ij} \otimes a_j$. We deduce that $\Delta(x_i) = \sum x_j \otimes b_{ji}$. Thus, we let X be the space generated by x_i's. Let n be the dimension of $X \cap I$. Then H is stabilizer of $X \cap I$. We finally consider $\Lambda^n X$ and $\Lambda^n X \cap I$.

□

Lemma 3.8.7. *If H is normal, there exists a finite dimensional representation $\rho : G \to GL(V)$ with the property that H is the kernel.*

Proof. Let us take the representation given by the last lemma. Let us consider $\bigoplus_{g \in G(\bar{k})} g.L$. Picking representatives, we can write this space $\oplus_{g_i} g_i L$. This is a representation of G. Moreover, H preserves each of these lines. We can replace V by $\oplus_{g_i} g_i L$. We now consider the composition $Ad \circ \rho : G \to GL(\mathrm{End}(V))$. Clearly, H is in the kernel. Any element in the kernel of Ad is a scalar in $GL(V)$. A scalar in $GL(V)$ preserves L and therefore the kernel of $Ad \circ \rho$ consists of elements of G which stabilize L. This is inside H. □

3.9. **Quotient.** Our goal is to prove the following theorems.

Theorem 3.9.1. *Let G be an algebraic group acting on a scheme of finite type X. For any $x \in X(k)$, the orbit map $G \to X$, $g \mapsto gx$ factors through an immersion $G/H \to X$ where H is the stabilizor of x. Moreover, if G is smooth, the orbit is smooth.*

We only give the proof in characteristic 0. Therefore we know that G is smooth (hence reduced).

Proof. We let H be the pre-image of x under the orbit map. This is obviously a closed subgroup H of G. The map $orb : G \to X$ has constructible image. Thus there exists $V \subseteq \overline{orb(G)}$ dense open such that $V \subseteq orb(G)$. We deduce that $orb^{-1}(V) \subseteq G$ is open. Assuming $k = \bar{k}$ we deduce that $G(k)orb^{-1}(V) = G$ so that $orb(G)$ is open in its closure. We now equip $\overline{orb(G)}$ with the reduced scheme structure and this induces a scheme structure on $orb(G)$. Since G is smooth, the map $G \to X$ factors through a map $G \to \overline{orb(G)} \to X$. By generic flatness, there is a dense open W of $orb(G)$ such that $orb^{-1}(W) \to W$ is flat. Using group translation, we deduce that $G \to orb(G)$ is flat. This is thus an fppf cover. Moreover, $G \times_{orb(G)} G = G \times H$. Thus, $orb(G) = G/H$. Note that $orb(G)$ is geometrically reduded, hence generically smooth. By homogenity it is smooth. $\square$

Theorem 3.9.2. *Let G be an affine algebraic group. Let H be a closed subgroup. The fppf quotient G/H is representable.*

Proof. We pick a representation $\rho : G \to GL(V)$ with V finite dimensional k-vector space. We let n be the dimension of V and we let $\mathrm{Gr}(n, 1)$ be the Grassmanian of lines in V. This is the functor which sends a k-scheme T to isomorphism classes of exact sequence $0 \to \mathcal{L} \to \mathscr{O}_T \otimes_k V \to \mathscr{G} \to 0$ where $\mathcal{L}$ is an invertible sheaf and $\mathscr{G}$ is locally free of rank $n-1$. The group $GL(V)$ acts on $\mathrm{Gr}(n, 1)$ and so does G. The line L in V defines a k-point. And G/H is represented by the orbit of L in $\mathrm{Gr}(n, 1)$. $\square$

Theorem 3.9.3. *Let G be an affine algebraic group. Let H be a normal subgroup, then G/H is an affine subgroup.*

Proof. We consider a representation $\rho : G \to GL(V)$ with kernel H. The image of ρ is closed, so that $G/H \to \rho(G)$ identifies with a closed subgroup of $GL(V)$. Hence it is affine. $\square$

4. Equivariant sheaves

4.1. **Equivariant sheaves.** Let X be an S-scheme. We have a category $QCoh(X)$ of quasi-coherent sheaves. If $X = \mathrm{Spec}\ A$ is affine, this is simply the category of A-modules. Let G be a group schemes acting on X on the right.

We consider the following diagram

$$G \times G \times X \substack{\to\\ \to\\ \to} G \times X \substack{\to\\ \leftarrow\\ \to} X$$

where the maps $G \times X \to X$ are $act((g, x)) = gx$ and $p(g, x) = x$. The map $s : X \to X \times G$ is $s(x) = (x, e)$. The maps $G \times G \times X \to G \times X$ are $m_0((g_1, g_2, x)) = (g_1g_2, x)$, $m_1((g_1, g_2, x)) = (g_2, xg_1)$ and $m_2((g_1, g_2, x)) = (g_1, x)$. We have $act \circ m_0 = act \circ m_1$, $p \circ m_0 = p \circ m_2$, $p \circ m_1 = act \circ m_2$. Moreover $act \circ s = p \circ s$.

Remark 4.1.1. *We give some intuition about this diagram. If $f : X \to Y$ is a map of schemes, we can consider the Chech nerve of this map. This is a simplicial scheme. The first few maps are given by :*

$$X \times_Y X \times_Y X \substack{\to\\ \to\\ \to} X \times_Y X \substack{\to\\ \leftarrow\\ \to} X$$

with $q_{1,2}(x_0, x_1, x_2) = (x_1, x_2)$, $q_{0,2}(x_0, x_1, x_2) = (x_0, x_2)$ and $q_{0,1}(x_0, x_1, x_2) = (x_0, x_1)$, $p_0(x_0, x_1) = x_0$ and $p_1(x_0, x_1) = x_1$. Given X and G, we can form the quotient stack $[X/G]$. There is a map $f : X \to [X/G]$. We can still consider the Chech nerve of this map. Moreover, we have $G \times X = X \times_{[X/G]} X$ (via $(g, x) \mapsto (x, xg)$) and $G \times G \times X = X \times_{[X/G]} X \times_{[X/G]} X$ via $(g_1, g_2, x) \mapsto (x, xg_1, xg_1g_2)$.

Definition 4.1.2. We let $QCoh_G(X)$ be the category whose objects consist of a quasi-coherent sheaf $\mathscr{F}$ on X together with an isomorphism $\Theta : act^\star\mathscr{F} \to p^\star\mathscr{F}$ which satisfies :

(1) (identity acts trivially) $s^\star\Theta = \mathrm{Id}_{\mathscr{F}}$.
(2) (associativity) $m_0^\star\Theta = m_2^\star\Theta \circ m_1^\star\Theta$.

A map $(\mathscr{F}, \Theta) \to (\mathscr{F}', \Theta')$ is a map of quasi-coherent sheaves $\psi : \mathscr{F} \to \mathscr{F}'$ such that the following diagram commutes :

$$\begin{tikzcd} act^\star \mathscr{F} \arrow[r, "act^\star \psi"] \arrow[d, "\Theta"] & act^\star \mathscr{F}' \arrow[d, "\Theta'"] \\ p^\star \mathscr{F} \arrow[r, "p^\star \psi"] & p^\star \mathscr{F}' \end{tikzcd}$$

Remark 4.1.3. *One can see that* (2) *implies* (1)*. Indeed, pulling back the identity $m_0^\star \Theta = m_2^\star \Theta \circ m_1^\star \Theta$ on $\{e\} \times \{e\} \times X$, we get $s^\star \Theta = s^\star \Theta \circ s^\star \Theta$. Since $s^\star \Theta$ is an isomorphism, it must be the identity.*

This notion may be hard to understand. Let us provide some intuition.

Lemma 4.1.4. *Let $x : \mathrm{Spec}\ \ k \to X$. Let $g \in G(k)$. The map Θ induces a map $\Theta_g : \mathscr{F}_{xg} \simeq \mathscr{F}_x$ identifying the stalks at xg and x.*

Proposition 4.1.5. *If G is flat, the category $QCoh_G(X)$ is abelian.*

Proof. Let $\mathscr{F} \to \mathscr{F}'$ be a map in $QCoh_G(X)$. We can form the kernel K and cokernel C in $QCoh(X)$. Then we observe that K and Q inherit a canonical G-equivariant action. Indeed, we have long exact sequences $0 \to act^\star K \to act^\star \mathscr{F} \to act^\star \mathscr{F}' \to act^\star C \to 0$ and $0 \to p^\star K \to p^\star \mathscr{F} \to p^\star \mathscr{F}' \to p^\star C \to 0$ as act and p are flat maps. $\square$

Proposition 4.1.6. *If $\mathscr{F}, \mathscr{G} \in QCoh_G(X)$ then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \in QCoh_G(X)$. If $\mathscr{F}, \mathscr{G} \in Coh_G(X)$ and G is flat then $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}) \in Coh_G(X)$.*

Proposition 4.1.7. *The sheaves $\mathscr{O}_X, \Omega^1_{X/S}, T_{X/S}, \mathcal{D}_{X/S}$ are G-equivariant sheaves.*

Proof. If Z is a scheme and $\psi : Z \to Z$ is a scheme map, then we have a map $\psi^\star \mathscr{O}_Z \to \mathscr{O}_Z$. If ψ is an isomorphism, this map is an isomorphism. Similarly, if $Z \mapsto S$ and ψ is a S-morphism, we have a map $\psi^\star \Omega^1_{Z/S} \to \Omega^1_{Z/S}$. If ψ is an isomorphism, this map is an isomorphism. Consider the map $\lambda : G \times X \to G \times X$, $(g, x) \mapsto (g, xg)$. This map is an automorphism. Moreover, $p \circ \lambda = act$. $\square$

4.2. **Special cases.**

Proposition 4.2.1. *If $X = \mathrm{Spec}\ \ R$ has trivial action, then the category $QCoh_G(X)$ is simply the category $\mathrm{Mod}_G(R)$ of representations of G on R-modules.*

Proof. Note that $act = p$. For any R-algebra B, and any $g \in G(B)$, we have a map $\Theta_g : M \otimes_R B \to M \otimes_R B$ which is B-linear. $\square$

Let $f : X \to Y$ be a morphism. The category of quasi-coherent sheaves on X with descent datum has objects quasi-coherent sheaves $\mathscr{F}$ on X, an isomorphism $\psi : p_0^\star \mathscr{F} \to p_1^\star \mathscr{F}$ such that the diagram commutes:

$$\begin{tikzcd} q_{0,1}^\star p_0^\star \mathscr{F} \arrow[r, "q_{0,1}^\star \psi"] \arrow[d] & q_{0,1}^\star p_1^\star \mathscr{F} = q_{1,2}^\star p_0^\star \mathscr{F} \arrow[r, "q_{1,2}^\star \psi"] & q_{1,2}^\star p_1^\star \mathscr{F} \arrow[d] \\ q_{0,2}^\star p_0^\star \mathscr{F} \arrow[rr, "q_{0,2}^\star \psi"] & & q_{0,2}^\star p_1^\star \mathscr{F} \end{tikzcd}$$

Theorem 4.2.2 ([Sta13], Tag 023T)**.** *Let $f : X \to Y$ is an fppf epimorphism. The functor $\mathscr{F} \mapsto f^\star \mathscr{F}$ identifies $QCoh(Y)$ with the category of quasi-coherent sheaves on X with a descent datum.*

Proposition 4.2.3. *Assume that there is a scheme X/G, such that $X \to X/G$ is an fppf epimorphism with $X \times_{X/G} X = X \times G$. Then $QCoh_G(X) = QCoh(X/G)$.*

We now investigate the case where $X = H\backslash G$ for some subgroup H of G and for simplicity $S = \mathrm{Spec}\ k$ with k a field.

Proposition 4.2.4. *The H-equivariant map $e : \mathrm{Spec}\ k \to X$ induces an equivalence of category $\mathrm{Mod}_H(k) \to QCoh_G(X)$.*

Proof. We have $QCoh_G(X) = QCoh_{H\times G}(G) = QCoh_H(S) = Mod_H(k)$. We can also be more explicit about the construction. The functor $QCoh_G(X) \to Mod_H(k)$ is given by taking the stalk at e. We define a functor $Mod_H(k) \to QCoh_G(X)$ as follows. To M we associate first the sheaf $\mathcal{M} = \pi_\star \mathscr{O}_G \otimes_k M$. This sheaf carries an action $\star_{1,3}$ of H, and a $\star_2$ action of G. We take the H-invariants for $\star_{1,3}$. There remains a $\star_2$ action. Very concretely, the section of $\mathcal{M}$ over some $U \to X$ which lifts to G can be described as function $f : HU \to M$ such that $h^{-1}f(hx) = f(x)$. It is easy to see that $Mod_H(k) \to QCoh_G(X) \to Mod_H(k)$ is the identity. We need to see that any object of $QCoh_G(X)$ arises from an object of $Mod_H(k)$. Let $\mathscr{F}$ in $QCoh_G(X)$. Consider the map $\pi : G \to X$. Then $\pi^\star\mathscr{F}$ is a G-equivariant sheaf on G. We have maps $act, p : G \times X \to X$ and $\Theta : act^\star\mathscr{F} \to p^\star\mathscr{F}$. Restricting to $G \times \{e\}$, this gives $\pi^\star\mathscr{F} = \mathscr{O}_G \otimes_k \mathscr{F}_e$. Concretely, this map is given $s(g) \mapsto \Theta_g s(g)$. The G-equivariant structure is given by $[gg' \mapsto s(gg')] \mapsto [g \mapsto \Theta_{g'} s(gg')]$ and it transforms into $[gg' \mapsto \Theta_{gg'} s(gg')] \mapsto [g \mapsto \Theta_g\Theta_{g'} s(gg') = \Theta_{gg'} s(gg')]$. This is the trivial G-equivariant structure. □

4.3. **Adjunction.** Let $\mathcal{C}, \mathcal{D}$ be two categories. We consider functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. We say that G is a left adjoint to F (or that F is a right adjoint to G) if $\mathrm{Hom}(X, F(Y)) = \mathrm{Hom}(G(X), Y)$ for all $X, Y \in ob(\mathcal{C}) \times ob(\mathcal{D})$. We say that (G, F) are adjoint functors.

Example 4.3.1. Let $f : X \to Y$ be a map of schemes. Then $(f^\star, f_\star)$ are adjoint.

Example 4.3.2. Let H, G be abstract groups. Let $Mod_H(k)$ and $Mod_G(k)$ be the categories of representations of H and G on k-vector spaces. Let $f : H \to G$ be a map. We have a natural map $f^\star : \mathrm{Mod}_G(k) \to Mod_H(k)$ sometimes called inflation. We can also define a map $f_\star : \mathrm{Mod}_H(k) \to Mod_G(k)$ sometimes called induction, such that $(f^\star, f_\star)$ are adjoint. We let $f_\star V = \{f : G \to V, hf(gh) = f(g)\}$.

4.4. **Pull back and pushforward of equivariant sheaves.** Consider a group H acting on a scheme Y and a group G acting on a scheme X. We assume that we have maps $i : H \to G$ and $f : Y \to X$ such that the diagram commutes :

$$\begin{tikzcd} H \times Y \arrow[r, "act_Y"] \arrow[d] & Y \arrow[d] \\ G \times X \arrow[r, "act_X"] & X \end{tikzcd}$$

We want to construct adjoint functors $(f^\star, f_\star)$ between $QCoh_G(X)$ and $QCoh_H(Y)$.

Proposition 4.4.1. *We have a functor $f^\star : QCoh_G(X) \to QCoh_H(Y)$.*

Proof. Let $(\mathscr{F}, \Theta) \in QCoh_G(X)$. We consider $(f^\star\mathscr{F}, i \times f^\star\Theta)$. □

In order to consider the functoriality for direct images, we need the following base change result.

Proposition 4.4.2 ([Sta13], Tag 02KH)**.** *Let $g : X \to S$ be a map of schemes which is quasi-compact and quasi-separated. Let $T \to S$ be a flat morphism. Consider the following diagram:*

$$\begin{tikzcd} X \times_S T \arrow[r, "h'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\ T \arrow[r, "h"] & S \end{tikzcd}$$

Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then we have $h^\star \mathrm{R}^i g_\star \mathscr{F} = \mathrm{R}^i g'_\star (h')^\star \mathscr{F}$.

Proof. We only give the argument when X is quasi-compact and separated. We also assume $S = \mathrm{Spec}\ A$ and $T = \mathrm{Spec}\ B$ are affine. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a finite affinoid cover. Let $\mathcal{C}(\mathcal{U}, \mathscr{F})$ be the Chech complex which computes the cohomology of $\mathcal{F}$. Then $\mathcal{C}(\mathcal{U}, \mathscr{F}) \otimes_A B$ is the Chech complex which computes the cohomology of $\mathscr{F}'$. We see that since B is A-flat, $\mathrm{H}^i(\mathcal{C}(\mathcal{U}, \mathscr{F})) \otimes_A B = \mathrm{H}^i(\mathcal{C}(\mathcal{U}, \mathscr{F}) \otimes_A B)$. □

Proposition 4.4.3. *Assume that $H = G$. We have functors $\mathrm{R}^i f_\star : QCoh_G(Y) \to QCoh_G(X)$. Moreover, $(f^\star, f_\star)$ are adjoint functors.*

Proof. We have $(act_X)^\star \mathrm{R}^i f_\star = \mathrm{R}^i (Id \times f)_\star act_Y^\star$. □

Corollary 4.4.4. *Let G be a group acting on the right on X, let $f : X \to \mathrm{Spec}\ k$. Then we have functors $\mathrm{H}^i(X, -) : QCoh_G(X) \to Mod_G(k)$.*

Let $H \subseteq G$ be a subgroup.

Corollary 4.4.5. *We have functors $\mathrm{H}^i(H \backslash G, -) : \mathrm{Mod}_H(k) \to \mathrm{Mod}_G(k)$.*

Unravelling the definitions, the functor $\mathrm{H}^0(H \backslash G, -)$ is the induction functor which sends M to the space of functions $f : G \to M$ with the property that $hf(h^{-1}g) = f(g)$.

For the sake of completeness we construct the functor $f_\star : QCoh_H(Y) \to QCoh_G(X)$ in general.

We can first construct a functor $QCoh_H(Y) \to QCoh_H(X)$ (by using the preceeding arguments), reducing to the case that $X = Y$. Let $K = \mathrm{Ker}(H \to G)$. Necessarily, K acts trivially on X. We construct a functor $QCoh_H(X) \to QCoh_{H/K}(X)$, as $\mathscr{F} \mapsto \mathscr{F}^K$. This reduces to the case that H is a (closed) subgroup of G.

Then we have the equivalences $QCoh_H(X) = QCoh_{G \times H}(G \times X) = QCoh_G(G \times X/H)$ where H acts by $h(g, x) = h^{-1}g, xh$. We use that $G \times X/H \to G/H$ is representable, hence $G \times X/H$ exists. Finally, we have a G-equivariant map $G \times X/H \to X$ and we have a functor $QCoh_G(G \times X/H) \to QCoh_G(X)$.

5. The Borel-Weil-Bott theorem and the Weyl character formula

5.1. The abelian category $Mod_G(k)$.

Definition 5.1.1. Let $\mathcal{C}$ be an abelian category. An object A of $\mathcal{C}$ is simple if it is non-zero and the only sub objects are 0 and A. An object is semi-simple if it is a direct sum of simple objects

Definition 5.1.2. An abelian category is semi-simple if any object is semi-simple.

Let G be an affine algebraic group. The category $Mod_G(k)$ is the category of representations of G on k-vector spaces. An object of this category is called finite dimensional if its underlying vector space is finite dimensional.

Proposition 5.1.3. *The category $Mod_G(k)$ is abelian. Moreover, any object of $Mod_G(k)$ is a union of finite dimensional sub-objects.*

Proof. We have already seen that $Mod_G(k)$ is abelian. Let V be a representation. Let $X \subseteq V$ be a finite subvector space. We want to prove that there exists $X \subseteq X'$ with X' a finite dimensional representation. Let (a_i) be a k-basis of A. We have $\Delta(x) = \sum_i x_i \otimes a_i$ and $\sum_i \Delta(x_i) \otimes a_i = \sum_i x_i \otimes \Delta(a_i) = \sum_{i,j} x_i \otimes b_{ij} \otimes a_j$. We deduce that $\Delta(x_i) = \sum x_j \otimes b_{ji}$. Thus, we let X' be the space generated by x_i's with $x \in X$. □

Lemma 5.1.4. *Any non-zero finite dimensional representation V admits a filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ with V_i/V_{i-1} simple for $i \geq 1$.*

Proof. We do induction on the dimension of V. The theorem holds if V has dimension 1 since any 1-dimensional representation is simple. Assume V is of dimension n. Then either V is simple or it admits a non-trivial sub-representation V'. We apply the induction hypothesis to V/V' and V'. □

Definition 5.1.5. Let V be a finite dimensional representation. The socle of V, $soc(V)$ is the greatest semi-simple sub-representation of V.

Proposition 5.1.6. *The category* $\mathrm{Mod}_G(k)$ *admits a tensor product, and if* V_1, V_2 *are finite dimensional representations, then* $\mathrm{Hom}_k(V_1, V_2)$ *is also a finite dimensional representation. Moreover, we have* $\mathrm{Hom}_G(V_3 \otimes V_1, V_2) = \mathrm{Hom}_G(V_3, \mathrm{Hom}_k(V_1, V_2))$.

Proof. If V_1, V_2 are finite dimensional, then for every k-algebra B, we have $\mathrm{Hom}_k(V_1, V_2) \otimes_k B = \mathrm{Hom}_k(V_1, V_2 \otimes_k B) = \mathrm{Hom}_B(V_1 \otimes_k B, V_2 \otimes_k B)$. If $g \in G(B)$, we let g act on $\mathrm{Hom}_k(V_1, V_2) \otimes_k B$ via $\psi \mapsto \rho_2(g)\psi(\rho_1^{-1}(g)-)$. □

Let $f : G \to G'$ be a morphism of groups. We have a pair of adjoint functors $(f^\star, f_\star)$ between $Mod_G(k)$ and $Mod_{G'}(k)$. We have

$$\mathrm{Hom}_{Mod_G(k)}(f^\star V, V') = \mathrm{Hom}_{Mod_{G'}(k)}(V, f_\star V').$$

It is particularly interesting to study the special case where G is a subgroup of G'. In this case, $f^\star$ is the restriction which we shall denote by $\mathrm{Res}_G^{G'}$ and $f_\star$ is the induction which we shall denote by $\mathrm{Ind}_G^{G'}$.

5.2. **Representations of a Torus.** We let $T = \mathbb{G}_m^n$ be a split torus.

We let $X^\star(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ be the character group of T.

Lemma 5.2.1. *We have* $X^\star(T) \simeq \mathbb{Z}^n$, $\kappa = (k_1, \cdots, k_n)$ *corresponding to the map* $(t_1, \cdots, t_n) \mapsto \prod_i t_i^{k_i}$.

Proof. We reduce to check that $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$. Any scheme map $\mathbb{G}_m \to \mathbb{G}_m$ will send T to aT^n for some $a \in k$ and $n \in \mathbb{Z}$. A group scheme map will send 1 to 1, so we deduce that $a = 1$. □

For any $\kappa = (k_1, \cdots, k_n) \in \mathbb{Z}^n$, we let $k(\kappa)$ be the one dimensional representation of T where T acts by κ.

Proposition 5.2.2. *The category* $Mod_T(k)$ *is semi-simple with simple objects* $k(\kappa)$.

Proof. Let $V \in Mod_T(k)$. We let $\Delta : V \to V \otimes \mathscr{O}_T$. We write $\Delta(v) = \sum_{(k_i) \in \mathbb{Z}^n} v_{(k_i)} \prod_i T_i^{k_i}$. For $\kappa = (k_i)$, we let $p_\kappa : V \to V$ be the map given by $v \mapsto v_\kappa$. We see that $\Delta(v_\kappa) = v_\kappa \prod_i T_i^{k_i}$. It follows that $p_\kappa^2 = p_\kappa$ and $p_\kappa p_{\kappa'} = 0$ if $\kappa \neq \kappa'$. We also have $v = \sum_\kappa v_\kappa$. Hence, $V = \oplus_\kappa p_\kappa(V)(\kappa)$. □

5.3. **Representations of the additive group.** Let $\mathbb{G}_a$ be the additive group. Let (ρ, V) be a representation of $\mathbb{G}_a$. We attach to ρ the endomorphism $N = \mathrm{d}\rho(1)$ of V.

Theorem 5.3.1. *Assume that* k *is a field of characteristic* 0. *The above functor induces an equivalence of categories between* $Mod_{\mathbb{G}_a}(k)$ *and the category of pairs* (N, V) *where* V *is a* k*-vector space and* N *is a locally nilpotent endomorphism of* V *(i.e. for all* $v \in V$, $\exists n$, $N^n v = 0$*)*.

Proof. Let (ρ, V) be a representation of $\mathbb{G}_a$. We have $\Delta : V \to V \otimes_k k[T]$. We let $\Delta(v) = \sum p_n(v) T^n$. Note that $p_0(v) = v$ and $p_1(v) = N(v)$. We now use associativity of co-multiplication to get that

$$\sum_{m,n \geq 0} p_m(p_n(v)) T^m \otimes T^n = \sum_{a+b=n, a,b \geq 0} p_n(v) T^a \otimes T^b \frac{(a+b)!}{a!b!}.$$

We deduce that $p_{a+b} \frac{(a+b)!}{a!b!} = p_a \circ p_b$. This implies that $p_n(v) = \frac{1}{n!}(p_1)^n$. We deduce that p_1 is locally nilpotent. A quasi-inverse is given by taking (N, V) to V equipped with $\Delta : V \to V \otimes_k k[T]$ defined by $\Delta(v) = \exp(TNv) = \sum_{n \geq 0} \frac{N^n(v)}{n!} T^n$. □

5.4. **Representations of Unipotent radical and Borel.** We let $G = \mathrm{GL}_n$, we let B be the upper triangular Borel, T be the maximal diagonal torus and U be the unipotent radical. We also let $\bar{U}$ be the opposite unipotent radical and $\bar{B}$ be the opposite Borel.

Let $V \in Mod_U(k)$. We let V^U be the sub-vector space of fixed point for the action of U. Concretely, $V^U = \{v \in V, \Delta(v) = v \otimes 1\}$. We let $V^{\mathfrak{u}}$ be the subspace over vectors anihilated by the Lie algebra. Clearly, $V^U \subseteq V^{\mathfrak{u}}$.

Proposition 5.4.1. *Let $V \in Mod_U(k)$ be non-zero. Then $V^U \neq 0$. Moreover, if k is of characteristic 0, $V^U = V^{\mathfrak{u}}$.*

Proof. Let V be a non-zero object of $Mod_U(k)$. We wish to prove that $V^U \neq 0$. We can assume that V is finite dimensional. We consider the projective space $\mathbb{P}^n$ which parametrizes lines in V (where $n+1$ is the dimension of V). The group U acts on $\mathbb{P}^n$. We take a filtration of U by normal subgroups with graded pieces being isomorphic to $\mathbb{G}_a$. By induction, it will suffice to show that $V^{\mathbb{G}_a} \neq 0$.

Let Z be an orbit of minimal dimension for the action of $\mathbb{G}_a$. We claim that $Z = \overline{Z}$. Indeed, $\overline{Z}$ is stable under U and so is $\overline{Z} \setminus Z$ which is of dimension less than the dimension of Z. It follows that Z is a closed, connected, reduced, projective scheme. We claim that it must be a point. The orbit map is a surjective map from $\mathbb{G}_a$ to a connected, reduced, projective scheme. It must be constant. We deduce that Z is a point. It follows that V admits a stable line. $\square$

Corollary 5.4.2. *The only simple objects of $Mod_U(k)$ if k. The simple objects of $Mod_B(k)$ are the $k(\kappa)$.*

5.5. **Weights of representations of G.** Let $V \in \mathrm{Mod}_G(k)$ be a finite dimensional representation, it admits a weight decomposition $V = \oplus_\kappa V_\kappa$.

We let $W = N_G(T)(k)/T(k)$.

Lemma 5.5.1. *We have an exact sequence $0 \to T(k) \to N_G T(k) \to \mathcal{S}_n \to 0$, identifying $\mathcal{S}_n$ with W.*

Proof. Let $e_1, \cdots, e_n$ be the canonical basis of k^n. The lines ke_i are the only fixed lines by the action of T. An element of $N_G(T)(k)$ induces a permetutation of these lines. $\square$

This map has a section. Indeed, we define a map $W \to G(k)$, by sending w to the automorphism $e_i \mapsto e_{w(i)}$.

Corollary 5.5.2. *If κ is a weight of V and $w \in W$, then $w\kappa$ is also a weight of V.*

The group G has an adjoint action on $\mathrm{Lie}(G) = \mathfrak{g}$. As a T representation we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b} \oplus \bar{\mathfrak{b}}$. The characters of T appearing on $\mathfrak{b}$ are called the positive roots. We denote by Φ^+ that set. They are the $\alpha_{i,j}$ with $i > j$, given by $\alpha_{i,j}((t_k)) = t_i t_j^{-1}$. The simple roots $\Delta \subseteq \Phi^+$ are the $\alpha_{i,i+1}$. Given any root α, we let $T_\alpha : \ker(\alpha)$. We consider the centralizor of T_α, G^{T_α}. This is a group isomorphic to GL_2. We let $U_\alpha = U^{T_\alpha}$ and $U_{-\alpha} = \bar{U}^{T_\alpha}$. We pick a generator x_α of $Lie(U_\alpha) = \mathfrak{u}_\alpha$ so that $Id + x_\alpha$ generates U_α. We also define a cocharacter $\check{\alpha} : \mathbb{G}_m \to G^{T_\alpha}$. We let $\rho = \frac{1}{2}(\sum_{\alpha \in \Phi^+} \alpha)$. We have $\rho = (\frac{n-1}{2}, \cdots, \frac{1-n}{2})$. We let W be the Weyl group of G. Here, W identifies with the symmetric group via $w(t_1, \cdots, t_n) = (t_{w^{-1}(1)}, \cdots, t_{w^{-1}(n)})$. We let s_α be the transposition corresponding to $\alpha \in \Phi^+$. The simple transposition generated W. We let $\ell : W \to \mathbb{N}$ be the length function. We let $X^\star(T)^+$ be the dominant cone (given by the condition $\langle \kappa, \check{\alpha} \geq 0$ for all $\alpha \in \Phi^+$). A weight is regular if $\langle \kappa, \check{\alpha} \rangle \neq 0$ for all $\alpha \in \Phi^+$. We also let $X^\star(T)^{++}$ be the cone of regular dominant weights.

Lemma 5.5.3. *Let $V \in Mod_G(k)$. Let $\kappa \in X^\star(T)$ and $\alpha \in \Phi$. Then $\mathfrak{u}_\alpha : V_\kappa \to V_{\kappa+\alpha}$.*

Proof. For $v \in V_\kappa$, $t.x_\alpha v = t x_\alpha t^{-1}.tv = \kappa(t)\alpha(t) x_\alpha v$. $\square$

Definition 5.5.4. Let $V \in Mod_G(k)$ be non-zero. A non-zero vector in V^U is called a highest weight vector. A weight of V^U is called a highest weight.

5.6. **The geometry of $B\backslash G$.**

Lemma 5.6.1. *The Flag variety is proper.*

Proof. It suffices to check the valuation criterium for properness with discrete valuation rings by [Sta13],tag 0CM2. Now, let A be a discrete valuation ring and K be its field of fraction. Let $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n$ be a full flag in K^n. We see that $F_0 \cap A \subseteq F_1 \cap A \subseteq \cdots \subseteq F_n \cap A$ is a full flag in A^n. $\square$

Lemma 5.6.2. *The scheme $B\backslash G$ has dimension $\frac{n(n-1)}{2}$. Its tangent sheaf at Be is $\mathfrak{b}\backslash\mathfrak{g}$.*

Proof. Since the scheme $B\backslash G$ is smooth, its dimension is the rank of its sheaf of differentials. We have maps $G \xrightarrow{f} B\backslash G \to \mathrm{Spec}\ k$. We deduce an exact sequence : $0 \to f^\star\Omega^1_{B\backslash G/k} \to \Omega^1_{G/k} \to \Omega^1_{G/B\backslash G} \to 0$. We take stalks at e. We have $e^\star\Omega^1_{G/k} = \mathfrak{g}^\vee$. We have a cartesian diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & G \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & B\backslash G \end{tikzcd}$$

from which we deduce that $e^\star\Omega^1_{G/B\backslash G} = \mathfrak{b}^\vee$. We deduce that $e^\star\Omega^1_{B\backslash G/k} = (\mathfrak{b}\backslash\mathfrak{g})^\vee$. $\square$

Theorem 5.6.3. *The scheme $B\backslash G$ admits a finite stratification into B-orbits : $\coprod_{w\in W} X_w \to B\backslash G$ where $X_w = B\backslash BwB$. Each X_w is a locally closed subscheme. The map $\coprod_{w\in W} X_w \to B\backslash G$ is surjective.*

Proof. Using the proposition below we see that the various X_w are distinct as w is the only T-fixed point of X_w, so if $X_w = X_{w'}$, then $Bw = Bw'$, which implies $w = w'$. We next show at the level of k-points that $G(k) = \cup_{w\in W} B(k)wB(k)$. This is an easy matrix computation. Let $E_{i,j}$ be the matrix with 1 on the i-th line and j-th column. If M is a matrix then $ME_{i,j}$ is the matrix whose j-th column is the i-th column of M (which has zeroes everywhere else). Similarly, $E_{i,j}M$ is the matrix whose j-th line is the i-th line of M and with zeroes everywhere else. We explain an algorithm to turn $M \in G(k)$ into an element of W by left and right multiplication by elements of B.

1. M has a non-zero coefficient on the first colum. Let $m_{k,1}$ be the first non-zero coefficient.
2. Using torus element and left multiplication by $Id+E_{k,j}$ with $j \geq k$ we may kill all coefficients of M on the first column except for the k-th coefficient. We can now assume that M has zeroes on the first column, except for a $m_{k,1} = 1$.
3. Using multiplication by torus element and right multiplciation by $Id + E_{1,j}$ with $j \geq 1$, we can kill all coefficients on the k-th line, except for the first one. We can now assume that M has zeroes on the first column and the k-th line, except for a $m_{k,1} = 1$.
4. We now restart the algorithm by looking for the first non-zero coefficient on the second column.

We deduce a map $\coprod_w X_w \to B\backslash G$. This map is surjective. Indeed, let $\xi \in B\backslash G$ be a point. Let $\bar{\xi}$ be its closure. We look at $X_w \cap \bar{\xi}$. If none of these contain ξ, we deduce that $\cup X_w \cap \bar{\xi}$ is contained in closed subset of $\bar{\xi}$, thus there must be a k point in the complement. $\square$

Lemma 5.6.4. *The map $\prod_{\alpha\in\Phi^+} U_\alpha \to U$ is an isomorphism (for any order of positive roots).*

Proof. We order the roots $\alpha_0, \alpha_1, \cdots$ with the property that U_{α_i} embedds in the center of $U/(U_{\alpha_0} \times \cdots U_{\alpha_{i-1}})$. Let $\Phi^+_{\geq i} = \{\alpha_j, j \geq i\}$. We see that there is a map $\prod_{\alpha\in\Phi^+_{\geq i}} U_\alpha \to U/(U_{\alpha_0} \times \cdots U_{\alpha_{i-1}})$ and this map is equivariant for the action of U_{α_i} on both factors (which act without fixed point). We deduce that the map $\prod_{\alpha\in\Phi^+_{\geq i}} U_\alpha \to U/(U_{\alpha_0} \times \cdots U_{\alpha_{i-1}})$ is a bijection if and only if the map $\prod_{\alpha\in\Phi^+_{\geq i+1}} U_\alpha \to U/(U_{\alpha_0} \times \cdots U_{\alpha_i})$ is a bijection. $\square$

Proposition 5.6.5. *The orbit X_w is isomorphic (as a T-space) to $\prod_{\alpha \in w^{-1}\Phi^- \cap \Phi^+} U_\alpha$.*

We can also consider the stratification into $\bar{B}$-orbits, with $X'_w = X_w w_0 = B \backslash B w w_0 w_0 B w_0$. Thus, $X'_w = \prod_{\alpha \in w^{-1}\Phi^- \cap \Phi^-} U_\alpha$.

Lemma 5.6.6. *There is a unique open orbit X'_{Id}. The map $B \times \bar{U} \to G$ is a dense open immersion.*

Proof. We consider the orbit map $X'_{Id} = \bar{U} \to B\backslash G$. Since the stabilizor is trivial, this map identifies $\bar{U}$ with a locally closed subscheme of $B\backslash G$. We see that $\bar{U}$ and $B\backslash G$ have the same dimension. Hence, this map must be an open immersion. By pull back to G we get the desired map. □

We next look at codimension one orbits.

Lemma 5.6.7. *The codimension 1 orbits are the X'_{s_α} for α a simple root (s_α is a transposition $(i, i+1)$).*

Proof. The codimension 1 orbits correspond to permutations w with the property that $w^{-1}\Phi^+ \cap \Phi^+$ has cardinality $\frac{n(n-1)}{2} - 1$. This means that there are indexes $i < j$ such that $w(i) > w(j)$, while otherwise $k < l$ implies $w(k) < w(l)$. We see easily that w must be the identity on $\{1, \cdots, i-1\}$ and $\{j+1, \cdots, n\}$. Next we consider $\sum_{k=i}^{j-1} w(k+1) - w(k) = w(j) - w(i)$. If $j \neq i+1$ this quantity would be both negative and positive, a contradiction. Therefore w must be a transposition $(i, i+1)$. □

5.7. **Induction of a character.** [Reference : [Jan03], part II, section 2]

Let $\kappa \in X^\star(T)$. We let $\mathrm{H}^0(\kappa) = \mathrm{Ind}_B^G(\kappa)$.

Proposition 5.7.1. *We have that $\mathrm{H}^0(\kappa)$ is finite dimensional.*

Proof. This follows from properness of $B\backslash G$. □

Lemma 5.7.2. *Let $V \in Mod_G(k)$ be non-zero and finite dimensional. Then there exists κ and a non-zero homomorphism $V \to \mathrm{H}^0(\kappa)$.*

Proof. We have seen that $(V^\vee)^U \neq 0$. Thus there is λ, such that $\mathrm{Hom}_B(\lambda, V^\vee) \neq 0$. Thus $\mathrm{Hom}_B(V, -\lambda) \neq 0$. We take $\lambda = -\kappa$. □

Lemma 5.7.3. *We have that $\dim(\mathrm{H}^0(\kappa)^U) \leq 1$ and $\mathrm{H}^0(\kappa)^U \subseteq \mathrm{H}^0(\kappa)_{w_0\kappa}$.*

Proof. We prove that $\dim(\mathrm{H}^0(\kappa)^{\overline{U}}) \leq 1$. There is an injective map $\mathscr{O}_G \to \mathscr{O}_B \otimes \mathscr{O}_{\overline{U}}$ and it induces an injective map $\mathrm{H}^0(\kappa)^{\overline{U}} \to k f_\kappa$ where f_κ is the function on $U \times T \times \bar{U}$ defined by $f(bt\bar{u}) = \kappa(t)$. Conjugating by w_0 we deduce that $\mathrm{H}^0(\kappa)^U \subseteq \mathrm{H}^0(\kappa)_{w_0\kappa}$. □

Corollary 5.7.4. *Assume that $\mathrm{H}^0(\kappa) \neq 0$. Let λ be a weight of $\mathrm{H}^0(\kappa)$. Then $\kappa \leq \lambda \leq w_0\kappa$.*

Proof. Let λ be a weight and v a vector of weight λ. We see that there is an element of $\mathfrak{u}$, $\prod_\alpha u_\alpha^{k_\alpha}$ such that $u_\alpha^{k_\alpha} v \neq 0$ but $\mathfrak{u}.(u_\alpha^{k_\alpha} v) = 0$. Thus, $u_\alpha^{k_\alpha} v$ has weight $w_0\kappa$. If λ is a weight, $w_0\lambda$ is also a weight. □

Lemma 5.7.5. *Assume that $\mathrm{H}^0(\kappa) \neq 0$. Let $L(\kappa) = soc(\mathrm{H}^0(\kappa))$. Then $L(\kappa)$ is a simple G-module and all simple G-modules are of this form. Moreover, $L(\kappa)$ has highest weight $w_0\kappa$ and lowest weight κ.*

Proof. We see that $\dim(\mathrm{H}^0(\kappa)^U) \leq 1$. We deduce that $L(\kappa)$ is simple. □

We say that a weight $\kappa = (k_1, \cdots, k_n) \in X^\star(T)$ is antidominant if $k_1 \leq k_2 \cdots \leq k_n$.

Theorem 5.7.6. *We have that $\mathrm{H}^0(B\backslash G, \mathscr{O}(\kappa))$ is non-zero if and only if κ is anti-dominant.*

Proof. We consider the function $f_\kappa : B\bar{U} \to \mathbb{A}^1$, given by $f(b\bar{u}) = \kappa(b)$. We will prove that it extends. It suffices to show that it extends in codimension 1. Let α be a simple root. Then f_κ defines an element in the fraction field of the space of functions on $B\bar{U}s_\alpha = Bs_\alpha s_\alpha \bar{U} s_\alpha$.

Let us write $\bar{U} = U_{-\alpha} \times \bar{U}'$ with $\bar{U}' = \prod_{\beta\in\Phi^-,\beta\neq\alpha} U_{-\beta}$.

We recall the following identity in GL_2.

$$\begin{pmatrix} 1 & -X^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X^{-1} & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -X^{-1} & 1 \end{pmatrix}$$

There is a corresponding identity in $\mathrm{GL}_n^{T_\alpha}$.

We denote by $u_\alpha : \mathrm{Spec}\ k[X] \to U_\alpha$ the natural map. We similarly have a natural map $u_{-\alpha} : \mathrm{Spec}\ k[X] \to U_\alpha$.

We deduce an identity of the form $u_\alpha(X^{-1})s_\alpha u_\alpha(X) = \check{\alpha}(X^{-1})u_{-\alpha}(X^{-1})$ in $\mathrm{GL}_n(k[X,X^{-1}])$. It follows that

$$f_\kappa(bs_\alpha u_\alpha(X)u') = f_\kappa(\check{\alpha}(X^{-1})u_\alpha(X^{-1})) = \kappa(\check{\alpha}(X^{-1})) = X^{-\langle\kappa,\check{\alpha}\rangle}.$$

We see that the function f_κ is well defined on $B\bar{U}s_\alpha$ if κ is anti-dominant. $\square$

5.8. **Borel-Weil-Bott theorem and Weyl character formula.** We say that a weight $\kappa = (k_1, \cdots, k_n) \in X^\star(T)_\mathbb{Q}$ is antidominant if $k_1 \leq k_2 \cdots \leq k_n$. We let $X^\star(T)^-_\mathbb{Q}$ be the cone of antidominant weights. We also let $X^\star(T)^+_\mathbb{Q}$ be the cone of dominant weights. We have $\rho = (\frac{n-1}{2}, \cdots, \frac{1-n}{2}) = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. A weight $\kappa \in X^\star(T)_\mathbb{Q}$ is regular if $\langle\check{\alpha},\kappa\rangle \neq 0$ for all $\alpha \in \Phi$. This means that $\kappa = (k_1, \cdots, k_n)$ with $k_i \neq k_j$ for all $i \neq j$. We let $X^\star(T)^{--}$ be the subcone of regular and antidominant weights.

Here are some useful facts.

(1) $X^\star(T)_\mathbb{Q} = \cup_w wX^\star(T)^-_\mathbb{Q}$.
(2) $X^\star(T)^{reg}_\mathbb{Q} = \coprod_w wX^\star(T)^{--}_\mathbb{Q}$.
(3) $X^\star(T)^-_\mathbb{Q} - \rho \subseteq X^\star(T)^{--}_\mathbb{Q}$.

We also have a length function $\ell : W \to \mathbb{N}$. It is defined as follows. Any element $w \in W$ has a minimal expression $w = s_{\alpha_1}\cdots s_{\alpha_k}$ where α_i are simple roots and $\ell(w) = k$.

The following theorem completely describes induction and derived induction.

Theorem 5.8.1. *(Borel-Weil-Bott) Let k be a field of characteristic 0. Let $\kappa \in X^\star(T)$. Then :*

(1) If $\kappa - \rho$ is not regular, then $\mathrm{H}^i(B\backslash G, \mathscr{O}(\kappa)) = 0$ for all κ.
(2) If $\kappa-\rho$ is regular, there is a unique w such that $w(\kappa-\rho)$ is antidominant and $\mathrm{R}\Gamma(B\backslash G, \mathscr{O}(\kappa)) = \mathrm{H}^{\ell(w)}(B\backslash G, \mathscr{O}(\kappa))[-\ell(w)]$ and $\mathrm{H}^{\ell(w)}(B\backslash G, \mathscr{O}(\kappa)) \simeq \mathrm{H}^0(B\backslash G, \mathscr{O}(w(\kappa-\rho)+\rho))$ is the representation with lowest weight $w(\kappa-\rho)+\rho$.

The following theorem describes completely the character of each induction :

Theorem 5.8.2. *(Weyl character formula) The character of $\mathrm{H}^0(\kappa)$ is*

$$\sum_{w\in W}(-1)^{\ell(w)}\frac{(w\cdot(w_0\kappa))}{\prod_{\alpha\in\Phi^-}(1-\alpha)}$$

5.9. **Proof of the theorem for** GL_2**.** Here the group is GL_2.

Let us denote by St the standard 2-dimensional representation, and by det the one dimensional representation. If e_1, e_2 is the basis of St, then $Sym^k St$ has basis $e_1^a e_2^b$, $a+b=k$.

Here is another way to think of $Sym^k St$. Consider the space P_k of homologenous polynomials of degree k in X, Y. We consider the action $gP((X,Y)) = P((X,Y)g)$.

Proposition 5.9.1. *The representations $\mathrm{Sym}^k\mathrm{St}\otimes\det^n$ for $k \geq 0$ and $n \in \mathbb{Z}$ are irreducible. They have weights $(n, n+k), (n+1, n+k-1), \cdots, (n+k, n)$.*

Proof. We first check that $\mathrm{Sym}^k\mathrm{St} \otimes \det^n$ is irreducible. We compute the action of x on Sym^k. It send $e_2^a e_1^b$ to $a e_2^{a-1} e_1^{b+1}$. Similarly, $\bar{x}$ sends $e_1^a e_2^b$ to $a e_1^{a-1} e_2^{b+1}$. Now if V is a subrepresentation of $\mathrm{Sym}^k\mathrm{St}$, it contains a certain weight. Thus it contains a vector $e_2^a e_1^b$. Using x and $\bar{x}$ we can span the entire $\mathrm{Sym}^k\mathrm{St}$. □

We now make the connection with induction. We recall the definition of $\mathscr{O}_{\mathbb{P}^1}(n)$.

Definition 5.9.2. The line bundle $\mathscr{O}_{\mathbb{P}^1}(n)$ is the sheaf of sections on $\mathbb{P}^1$ with a pole of order n at ∞.

In other words, $\mathscr{O}_{\mathbb{P}^1}(n) = \mathscr{O}_{\mathbb{P}^1}(n\infty)$. If $n \geq 0$, we have an exact sequence :

$$0 \to \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(n) \to (i_\infty)_\star t^{-n}k[t]/k[t] \to 0$$

If $n \leq 0$, we have an exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^1}(n) \to \mathscr{O}_{\mathbb{P}^1} \to (i_\infty)_\star k[t]/]t^{-n}k[t] \to 0.$$

Corollary 5.9.3. *We have* $\dim \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}^1_{\mathbb{P}}(n)) = n+1$ *if* $n \geq 0$*, and* 0 *if* $n \neq -1$*. We have* $\dim \mathrm{H}^1(\mathbb{P}^1, \mathscr{O}^1_{\mathbb{P}}(n)) = 0$ *if* $n \geq -1$*, and* $-n-1$ *if* $n \leq -2$*.*

We have a covering $\mathbb{P}^1$ by Spec $k[X] = \mathbb{A}^1$ and Spec $k[X^{-1}] = \mathbb{A}^1$ glued along $\mathrm{Spec}\, k[X, X^{-1}] = \mathbb{G}_m$. We define

$$\begin{array}{rcl} \mathscr{O}_{\mathbb{P}^1}(n) & \to & \mathscr{O}_{\mathbb{A}^1} \oplus \mathscr{O}_{\mathbb{A}^1} \to \mathscr{O}_{\mathbb{G}_m} \\ & & (P(X), Q(X^{-1})) \mapsto P(X) - X^n Q(X^{-1}) \end{array}$$

We see that if $n \geq 1$, then $(1, X^{-n})$ defines a global section of $\mathscr{O}_{\mathbb{P}^1}(n)$ which vanishes at order n.

Lemma 5.9.4. *Let* $\kappa = (k_1, k_2)$*. We have an isomorphism of line bundles* $\mathscr{O}_{\mathbb{P}^1}((\kappa)) = \mathscr{O}_{\mathbb{P}^1}(k_2 - k_1)$*.*

Proof. We have a map $\pi : \mathrm{GL}_2 \to \mathbb{P}^1$. For any open U, we have $\mathscr{O}(\kappa)(U) = \{f : \pi^{-1}(U) \to \mathbb{A}^1, f(bu) = \kappa(b)f(u)\}$. On $\pi(w_0 U)$ we get $\mathscr{O}(\kappa)(\pi(w_0 U)) \simeq k[X]$, via $f \mapsto f(w_0 \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix})$. On $\pi(w_0 U w_0)$, we get $\mathscr{O}(\kappa)(U) \simeq k[X']$, via $f \mapsto f(\begin{pmatrix} 1 & 0 \\ X' & 1 \end{pmatrix})$

We have the following identity :

$$\begin{pmatrix} 1 & X^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X^{-1} & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X^{-1} & 1 \end{pmatrix}$$

We can let $X' = X^{-1}$, and the gluing data is given by $P(X) = X^{k_2 - k_1} Q(X^{-1})$.

□

Lemma 5.9.5. *Assume that* $k_2 \geq k_1$*. We have that* $\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}((k_1, k_2)))$ *is the irreducible representation* $\mathrm{Sym}^{k_2 - k_1} \otimes \det_1^k$*.*

Proof. If $k_1 = k_2$, we have $\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}((k_1, k_2))) = k$. And clearly, $det^{k_1} \in \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}((k_1, k_2)))$. By twisting we reduce to the case where $k_1 = 0$. We construct a map $\mathrm{Sym}^{k_2}\mathrm{St} \to \mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}((0, k_2)))$ by sending $P(X, Y)$ to the function $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto P(c, d)$. This map is injective, and the spaces have the same dimension. □

Corollary 5.9.6. *The category* $\mathrm{Mod}_G(k)$ *is semi-simple.*

Proof. We consider an extension $0 \to \mathrm{H}^0(\kappa) \to E \to \mathrm{H}^0(\lambda) \to 0$. We claim that E^U is two dimensional. We see that $\mathrm{H}^0(\lambda)$ has highest weight $w_0\lambda$. We look at $E_{w_0\lambda}$. If this space is one dimensional, this means that $w_0\lambda$ is not a weight of $\mathrm{H}^0(\kappa)$. We deduce that $w_0\lambda + \alpha$ is not a weight of E. Therefore $w_0\lambda$ is also a highest weight vector. If $w_0\lambda$ is a weight of $\mathrm{H}^0(\kappa)$ but $\lambda \neq \kappa$, we see that either $E_{w_0\kappa+\alpha}$ is one dimensional and therefore the map $u : E_{w_0\kappa} \to E_{w_0\kappa+\alpha}$ has a one dimensional kernel. Finally, if $\lambda = \kappa$, then $E_{w_0\kappa}$ is a two dimensional space of highest weight vectors. $\square$

In order to study H^1 we will use Serre duality.

Lemma 5.9.7. *We have* $\Omega^1_{\mathbb{P}^1/k} = \mathscr{O}_{\mathbb{P}^1}((1,-1))$.

Proof. We have that the stalk at identity of the tangent sheaf is $\mathfrak{g}/\mathfrak{b}$. $\square$

The Serre dual sheaf to $\mathscr{O}_{\mathbb{P}^1}(\kappa)$ is $\mathscr{O}_{\mathbb{P}^1}(-\kappa + (1,-1))$.
Moreover, Serre duality says :

Theorem 5.9.8. *We have a perfect pairing*

$$\mathrm{H}^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(\kappa)) \times \mathrm{H}^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-\kappa + (1,-1)) \to k$$

Corollary 5.9.9. *Assume that* $-1 - k_2 \geq 1 - k_1$. *We have that* $\mathrm{H}^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}((k_1,k_2))) = \mathrm{H}^0((1 - k_1, -1 - k_2))^\vee \simeq \mathrm{H}^0((k_2+1, k_1-1))$.

5.10. Digression : relative cohomology and semi-continuity theorem.

Theorem 5.10.1 ([Har77], III, thm. 5.2)**.** *Let* $S = \mathrm{Spec}\ A$ *be a noetherian affine scheme. Let* $\mathscr{F}$ *be a coherent sheaf on* $X = \mathbb{P}^n_S$. *Then*

(1) *For all* $i \geq 0$, $\mathrm{H}^i(X, \mathscr{F})$ *is a finitely generated* A*-module.*
(2) *There is an integer* n_0 *such that for all* $n \geq n_0$, $\mathrm{H}^i(X, \mathscr{F}(n)) = 0$ *for all* $i > 0$ *and all* $n \geq n_0$.

Remark 5.10.2. *Let* $X \to S$ *be a projective scheme. Let* $\iota : X \hookrightarrow \mathbb{P}^n_S$ *be the closed immersion. The functor* $i_\star : Mod(X) \to Mod(\mathbb{P}^n_S)$ *is exact and sends injectives to injectives. It follows that for any sheaf* $\mathscr{F}$, *we have* $\mathrm{R}\Gamma(X, \mathscr{F}) = \mathrm{R}\Gamma(\mathbb{P}^n_S, \iota_\star\mathscr{F})$.

Theorem 5.10.3 ([Har77], III, prop. 12.2)**.** *Let* $X \to S$ *be a projective scheme of relative dimension* n. *Let* $\mathscr{F}$ *be a coherent sheaf on* X, *flat over* A. *Then there is a bounded complex* $K^\bullet$ *of finite flat* A*-modules, of amplitude* $[0,n]$ *such that for any* A*-module* M, $\mathrm{R}\Gamma(X, \mathscr{F} \otimes M)$ *is represented by :*

$$K^\bullet \otimes_A M$$

Proof. We only give a proof in the curve case. Assume $X = U_1 \cup U_2$ is covered by two affines. We let $L^\bullet = [\mathrm{H}^0(U_1, \mathscr{F}) \oplus \mathrm{H}^0(U_2, \mathscr{F}) \to \mathrm{H}^0(U_1 \cap U_2, \mathscr{F})]$. This represents $\mathrm{R}\Gamma(X, \mathscr{F})$ and $L^\bullet \otimes_A M$ represents $\mathrm{R}\Gamma(X, \mathscr{F} \otimes_A M)$. We let $x_1, \cdots, x_r$ be generators of $\mathrm{H}^1(L^\bullet)$ in L^1. We let $K^1 = A^r$ and we consider the map $g^1 : K^1 \to L^1$ given by these generators. Let $y_1, \cdots, y_s$ be generators of $\mathrm{Ker}(K^1 \to \mathrm{H}^1(L^\bullet)$. We choose lifts $\hat{y}_i$ of $g(y_i)$ in L^0. We let $K^0_1 = A^s$ we have maps $K^0_1 \to K^1$ (given by $y_1, \cdots, y_s$) and $K^0_1 \to L^0$ given by $\hat{y}_i$. We let $z_0, \cdots, z_t$ be generators of $\mathrm{H}^0(L^0)$. We $K^0_2 = A^t$ we have a map $K^0_2 \to L^0$ given by $z_0, \cdots, z_t$.

We see that there is a commutative diagram.

$$\begin{tikzcd} L^0 \arrow[r] & L^1 \\ K^0_1 \oplus K^0_2 \arrow[r] \arrow[u] & K^1 \arrow[u] \end{tikzcd}$$

If we let $(K')^\bullet$ be the bottom complex, the map $\mathrm{H}^0((K')^\bullet) \to \mathrm{H}^0(L^\bullet)$ is surjective, let K^0_3 be its Kernel.

Finally, we let $K^0 = K_1^0 \oplus K_2^0/K_3^0$ and we get a diagram :

$$\begin{tikzcd} L^0 \arrow[r] & L^1 \\ K^0 \arrow[u] \arrow[r] & K^1 \arrow[u] \end{tikzcd}$$

Since the map $K^\bullet \to L^\bullet$ is a quasi-isomorphism, the cone $0 \to K^0 \to K^1 \oplus L^0 \to L^1 \to 0$ is exact. It follows that K^0 is flat.

We see that $K^\bullet \otimes_A M \to L^\bullet \otimes_A M$ is a quasi-isomorphism for all A-module M since the cone : $0 \to K^0 \otimes_A M \to K^1 \otimes_A M \oplus L^0 \otimes_A M \to L^1 \otimes_A M \to 0$ is exact. ☐

Corollary 5.10.4. *(1) The function $s \to \dim_k \mathrm{H}^i(X_s, \mathscr{F}_s)$ is upper semi-continuous.*
(2) The function $s \to \sum_{i\geq 0}(-1)^i \dim_{k(s)} \mathrm{H}^i(X_s, \mathscr{F}_s)$ is locally constant.

Proof. For a map $A^n \to A^s$, the locus where the rank is $\geq i$ is open. One deduces the first claim. For the second we have $\sum_{i\geq 0}(-1)^i \dim_{k(s)} \mathrm{H}^i(X_s, \mathscr{F}_s) = \sum(-1)^i \dim_{k(s)} K^i \otimes_A k(s)$. ☐

Corollary 5.10.5. *The following conditions are equivalent :*
(1) $\mathrm{H}^i(X, \mathscr{F}) = 0$ for all $i > 0$,
(2) $\mathrm{H}^i(X_s, \mathscr{F}_s) = 0$ for all $i > 0$,
Moreover, if this is the case, $\mathrm{H}^0(X, \mathscr{F})$ is a flat A-module and the map $\mathrm{H}^0(X, \mathscr{F}) \otimes_A k(x) \to \mathrm{H}^0(X_x, \mathscr{F}_x)$ is an isomorphism.

Proof. In case of 1, we see that $0 \to \mathrm{H}^0(X, \mathscr{F}) \to K^0 \to K^1 \to \cdots$ is exact. It follows that $\mathrm{H}^0(X, \mathscr{F})$ is flat and tensoring with $k(x)$, the sequence remains exact. In case of 2. Let i be the smallest degree such that $\mathrm{H}^i(X, \mathscr{F}) \neq 0$. We claim that $\mathrm{H}^i(X, \mathscr{F}) \otimes_A k(x) \to \mathrm{H}^i(X_x, \mathscr{F}_x)$ is an isomorphism. Consider the sequence $0 \to \ker(d_i) \to K^i \to \mathrm{Im}(d_i) \to 0$. Since $\mathrm{Im}(d_i)$ is flat (because $0 \to \mathrm{Im}(d_i) \to K^{i+1} \to \cdots$ is exact) we see that $\ker(d_i) \otimes k(x) = \ker(K^i \otimes k(x) \to K^{i+1} \otimes k(x))$. Since $\mathrm{Im}(d_{i-1}) \to \ker(d_i) \to \mathrm{H}^i(X, \mathscr{F}) \to 0$ is exact, tensoring with $k(x)$ shows the claim. We deduce that if $i > 0$, $\mathrm{H}^i(X, \mathscr{F}) = 0$ by Nakayama. Therefore $i = 0$. ☐

5.11. **The general case.**

Proposition 5.11.1. *For any simple root α and weight κ. Assume that $\langle \check{\alpha}, \kappa \rangle \leq 1$ we have*

$$\mathrm{H}^i(B\backslash G, \mathscr{O}(\kappa)) = \mathrm{H}^{i+1}(B\backslash G, \mathscr{O}(\alpha(\kappa - \rho) + \rho))$$

Proof. We let P_α be the minimal parabolic corresponding to α. Let M_α be its Levi quotient (note that $M_\alpha \simeq \mathrm{GL}_2 \times \mathbb{G}_m^{n-2}$. We consider the map $\pi : B\backslash G \to P_\alpha\backslash G$. This is a $\mathbb{P}^1$-bundle. We see that $\pi_\star\mathscr{O}(\kappa)$ and $\mathrm{R}^1\pi_\star(\mathscr{O}(\kappa))$ are two G-equivariant vector bundle. We claim that $\pi_\star\mathscr{O}(\kappa) = \mathrm{R}^1\pi_\star\mathscr{O}(s_\alpha(\kappa - \rho) + \rho)$. We consider the following cartesian diagram :

$$\begin{tikzcd} B\backslash P_\alpha \arrow[r, "j"] \arrow[d, "\pi'"] & B\backslash G \arrow[d, "\pi"] \\ P_\alpha\backslash P_\alpha \arrow[r, "j'"] & P_\alpha\backslash G \end{tikzcd}$$

We note that $B\backslash P_\alpha = (B \cap M_\alpha)\backslash M_\alpha$. It suffices to show that $(j')^\star\pi_\star\mathscr{O}(\kappa) = (j')^\star\mathrm{R}^1\pi_\star\mathscr{O}(s_\alpha(\kappa - \rho) + \rho)$ as P_α-representations (the action factors through M_α). Using the base change formula of corollary 5.10.5, we are left to prove that

$$\mathrm{H}^0((B \cap M_\alpha)\backslash M_\alpha, \mathscr{O}(\kappa)) = \mathrm{H}^1((B \cap M_\alpha)\backslash M_\alpha, \mathscr{O}(s_\alpha(\kappa - \rho) + \rho)).$$

We see that when $\langle \check{\alpha}, \kappa \rangle \leq 0$, $\mathrm{H}^0((B \cap M_\alpha)\backslash M_\alpha, \mathscr{O}(\kappa))$ is the representations with highest weight $s_\alpha\kappa$ and $\mathrm{H}^1((B \cap M_\alpha)\backslash M_\alpha, \mathscr{O}(\kappa))$ is the zero representation. We see that when $\langle \check{\alpha}, \kappa \rangle \geq 2$ they are respectively 0 and the representation with highest weight $s_\alpha(\kappa-\rho)+\rho$. When $\langle\check\alpha,\kappa\rangle = 1$, they are all zero. If we assume that $\langle\check\alpha,\kappa\rangle \leq 0$, then we deduce that there is an isomorphism $\pi_\star\mathscr{O}(\kappa) = \mathrm{R}^1\pi_\star\mathscr{O}(s_\alpha(\kappa-\rho)+\rho)$. We deduce that $\mathrm{H}^i(B\backslash G,\mathscr{O}(\kappa)) = \mathrm{H}^{i+1}(B\backslash G,\mathscr{O}(s_\alpha(\kappa-\rho)+\rho))$. □

Corollary 5.11.2. *(1) If $\kappa-\rho$ is not regular, then $\mathrm{H}^i(B\backslash G,\mathscr{O}(\kappa)) = 0$ for all κ.*
(2) If $\kappa-\rho$ is regular, there is a unique w such that $w(\kappa-\rho)$ is antidominant and $\mathrm{R}\Gamma(B\backslash G,\mathscr{O}(\kappa)) = \mathrm{H}^{\ell(w)}(B\backslash G,\mathscr{O}(\kappa))[-\ell(w)]$ and $\mathrm{H}^{\ell(w)}(B\backslash G,\mathscr{O}(\kappa)) \simeq \mathrm{H}^0(B\backslash G,\mathscr{O}(w(\kappa-\rho)+\rho))$.

Proof. By general vanishing theorems, we know that $\mathrm{H}^i(B\backslash G,\mathscr{O}(\kappa))$ is concentrated in the range $[0,d]$. Our last result shows that given κ antidominant, we have $\mathrm{H}^i(B\backslash G,\mathscr{O}(\kappa)) = \mathrm{H}^{i+\ell(w)}(B\backslash G,\mathscr{O}(w(\kappa-\rho)+\rho))$. Applying this to $w=w_0$, we deduce that κ antidominant has only cohomology in degree 0. For a general κ, write let w be such that $w(\kappa-\rho)$ is antidominant. Then $\mathrm{H}^i(B\backslash G,\mathscr{O}(\kappa)) = \mathrm{H}^{i-\ell(w)}(B\backslash G,\mathscr{O}(w(\kappa-\rho)+\rho))$. If $\kappa-\rho$ is regular, then $w(\kappa-\rho)+\rho$ is antidominant and we conclude. If $\kappa-\rho$ is not regular, then there let $w(\kappa-\rho)+\rho=\kappa'$. We deduce that there exists i such that $k_i = k_{i+1}+1$. We see that $s_\alpha(\kappa'-\rho)+\rho=\kappa'$ where $s_\alpha=(i,i+1)$. We deduce that $\mathrm{H}^i(B\backslash G,\mathscr{O}(\kappa')) = \mathrm{H}^{i+1}(B\backslash G,\mathscr{O}(\kappa'))$. Thus the cohomology of κ', and then κ must vanish.

□

5.12. Semi-simplicity.

Theorem 5.12.1. *Assume that $char(k)=0$. Then the category $Mod_G(k)$ is semi-simple. The simple objects are $\mathrm{H}^0(\kappa)$ for κ anti-dominant. They have highest weight $w_0\kappa$.*

6. Lecture IV : Sheaves of differential operators on the flag variety

6.1. The enveloping algebra.

Let k be a field of characteristic zero. Let V be a k-vector space. We let $T(V)=\oplus_{n\geq 0}V^{\otimes n}$. We have natural maps $V^{\otimes n}\otimes V^{\otimes m}\to V^{\otimes n+m}$. Thus, $T(V)$ is naturally a graded algebra.

We let
$$\mathrm{Sym}^n V = V^{\otimes n}/\langle v_1\otimes\cdots\otimes v_n - v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}, \sigma\in\mathcal{S}_n\rangle.$$

We let $S(V)=\oplus_{n\geq 0}\mathrm{Sym}^n(V)$. This is the symmetric algebra. This is also a graded algebra.

We have a natural map $T(V)\to S(V)$ with kernel the two-sided ideal generated by $v\otimes w - w\otimes v$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. We define $U(\mathfrak{g}) = T(\mathfrak{g})/(x\otimes y - y\otimes x - [x,y], x,y\in\mathfrak{g})$.

Remark 6.1.1. *If $\mathfrak{g}$ is abelian, then $U(\mathfrak{g})=S(\mathfrak{g})$.*

We see that $U(\mathfrak{g})$ is a filtered algebra, with $U(\mathfrak{g})_{\leq i} = \mathrm{Im}(\oplus_{j\leq i}\mathfrak{g}^{\otimes j})$.

We let $\mathrm{Gr}(U(\mathfrak{g}))$ be the associated graded algebra.

Theorem 6.1.2. *There is a natural isomorphism $S(\mathfrak{g})\to\mathrm{Gr}(U(\mathfrak{g}))$.*

Proof. (Sketch) We first observe that $\mathrm{Gr}(U(\mathfrak{g}))$ is commutative, as for $v\in U(\mathfrak{g})_{\leq i}$ and $w\in U(\mathfrak{g})_{\leq j}$, $v.w - w.v\in U(\mathfrak{g})_{\leq i+j-1}$. We also have a map $\mathfrak{g}\to\mathrm{Gr}(U(\mathfrak{g}))$ and it extends to an algebra map $S(\mathfrak{g})\to\mathrm{Gr}(U(\mathfrak{g}))$ which is easily seen to be surjective. □

Corollary 6.1.3. *Let $(g_j)_{j\in J}$ be a k-basis of $\mathfrak{g}$. Then as k-vector space, $U(\mathfrak{g}) = \oplus_{(n_j)\in\mathbb{Z}^J_{\geq 0}} g_j^{n_j}k$.*

Proof. By induction, we check that $U(\mathfrak{g})_{\leq i} = \oplus_{(n_j)\in\mathbb{Z}^J_{\geq 0},\sum n_j\leq i} g_j^{n_j}k$. □

6.2. Examples of $\mathfrak{g}$-modules.

We now let $\mathfrak{g}=\mathfrak{gl}_n$ of $\mathfrak{sl}_n$.

Any representation of GL_n or SL_n gives an element of $\mathrm{Mod}(U(\mathfrak{g}))$. One can also consider infinite dimensional examples.

The most important example is Verma modules.

Let $\mathfrak{b}=\mathrm{Lie}(B)$ and $\mathfrak{h}=\mathrm{Lie}(T)$. Let $\lambda\in\mathrm{Hom}(\mathfrak{h},k)$. We let $M(\lambda)=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}k(\lambda)$ be the Verma module of weight λ.

Proposition 6.2.1. *As a $\bar{\mathfrak{b}}$-module, $M(\lambda) = U(\bar{\mathfrak{u}}) \otimes_k k(\lambda)$.*

Proof. We have $U(\mathfrak{g}) = U(\bar{\mathfrak{u}}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{u})$. □

Proposition 6.2.2. *We have* $\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), N) = \mathrm{Hom}_{\mathfrak{b}}(\lambda, N)$.

Proof. Use adjunction. □

Example 6.2.3. Let us do the $\mathfrak{sl}_2$ example. We take $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h = [x, y] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We see that $M(\lambda) = \oplus_n y^n \otimes 1$. We have $y.y^n \otimes 1 = y^{n+1} \otimes 1$. We have $h.(y^n \otimes 1) = (\lambda(h) - 2n) y^n \otimes 1$. Finally, we have $x.(y^n \otimes 1) = n(\lambda(h) - (n-1)) y^{n-1} \otimes 1$.

If $\lambda(h) = n \in \mathbb{Z}_{\geq 0}$ we have an exact sequence :

$$0 \to M(-n-2) \to M(n) \to Sym^n k^2 \to 0$$

6.3. **Harish-Chandra isomorphism.** We let W be the weyl group. It acts on cocharacters on the left and also on characters on the left. In particular we have $\langle \kappa, t \rangle = \langle w\kappa, wt \rangle$ for any $w \in W$.

The dotted Weyl group action on $X^\star(T)_k$ is given by $\kappa \mapsto w.\kappa = w(\kappa + \rho) - \rho$. We let $U(\mathfrak{h})$ be the envelopping algebra of $\mathfrak{h}$. We have $\mathrm{Spec}\, U(\mathfrak{h})(k) = X^\star(T)_k$. There is also a dotted action on $U(\mathfrak{h})$ which is given by $w.h = w^{-1}h + \langle h, w\rho \rangle - \langle h, \rho \rangle$. We see that $\mathrm{Spec}(U(\mathfrak{h})^{W,\cdot})(k) = X^\star(T)_k/(W, .)$. We let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Here is a description.

Proposition 6.3.1. *We have a map $HC' : Z(\mathfrak{g}) \to U(\mathfrak{h})$ given by $z \otimes 1 = 1 \otimes HC'(z)$ in $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{h})$.*

Proof. We have $M_{un} = U(\bar{\mathfrak{u}}) \otimes_k U(\mathfrak{h})$. We claim that $\mathrm{End}_{U(\mathfrak{g})}(M_{un}) = U(\mathfrak{h})$. We see that $1 \otimes U(\mathfrak{h})$ as weight $\chi_{univ} : \mathfrak{h} \to U(\mathfrak{h})$ while vectors in $\bar{\mathfrak{u}}U(\bar{\mathfrak{u}}) \otimes \otimes_k U(\mathfrak{h})$ have weights of the shape $\chi_{un} - \sum$(positive roots). It follows that any endomorphism will map $1 \otimes 1$ to $k \otimes U(\mathfrak{h})$. One map is $f \mapsto f(1 \otimes 1)$. The other is left multiplication. It follows that we get a map $Z(\mathfrak{g}) \to U(\mathfrak{h})$. □

Theorem 6.3.2. *The above map identifies $Z(\mathfrak{g})$ with $U(\mathfrak{h})^{W,\cdot}$.*

Proof. We only sketch a proof for SL_2. We have $U(\mathfrak{h}) = k[h]$. We define a map $k[h] \to \prod_{n \in \mathbb{Z}} k$, by $P(h) \mapsto P(n)$. The composition $Z(\mathfrak{g}) \to k$ given by $HC(z)(n)$ gives the action of z on $M(n)$. Since $M(-n-2) \hookrightarrow M(n)$ for $n \geq 0$, we deduce that $Z(\mathfrak{g})$ lands in $U(\mathfrak{h})^{W,\cdot}$. To prove surjectivity, we can simply exhibit the Casimir element $h^2 + yx + xy$ which maps to $h(h+2)$ □

Thus, we see that the map HC' is designed in such a way that $Z(\mathfrak{g})$ acts on a module with highest weight μ via the character μ. We will actually consider a variant of this map. We consider the map $HC : Z(\mathfrak{g}) \to U(\mathfrak{h})$ given by $HC(z) \otimes 1 = 1 \otimes z$ in $U(\mathfrak{h}) \otimes_{U(\mathfrak{b})} U(\mathfrak{g})$. With this notation, we see that via this map $Z(\mathfrak{g})$ acts via the character $w_0\mu$ on the representation of highest weight μ.

6.4. **Sheaves of differential operators on the flag variety.** We let $X = B \backslash G$. We recall that we have a functor $\mathrm{Mod}_B(k) \to QCoh_G(X)$.

We let $\mathfrak{g}^0 = \mathscr{O}_X \otimes_k \mathfrak{g}$, $\mathfrak{b}^0$, $\mathfrak{u}^0$ be the images of $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{u}$ via this functor. We see that $\mathfrak{g}^0$ is the constant locally free sheaf, with fiber $\mathfrak{g}$. For any $x \in X$, $\mathfrak{b}^0_x = x^{-1}\mathfrak{b}x$. This is the moving Borel Lie algebra.

Remark 6.4.1. *The G-action on each of these sheaves can be differentiating to a $\mathfrak{g}$-action. On $\mathfrak{g} \otimes \mathscr{O}_X$, the action of $g' \in \mathfrak{g}$ on $g \otimes f$ is $g'.(g \otimes f) = [g', g] \otimes f \oplus g' \otimes g(f)$.*

For any character $\lambda \in X^\star(T)$, we let $\mathscr{O}_X(\lambda)$ be the image of the one dimensional representation λ by this functor.

Lemma 6.4.2. *The map $\mathfrak{g} \to T_X$ induces an isomorphism $\mathfrak{g}^0/\mathfrak{b}^0 \to T_X$.*

Proof. Indeed, we have seen that the fiber of T_X at $B\backslash B$ is $\mathfrak{g}/\mathfrak{b}$. □

We have an action of $U(\mathfrak{g})$ on $\mathscr{O}_X$. We can construct a sheaf of rings $\mathscr{O}_X \otimes_k U(\mathfrak{g})$ where the product is defined by the rule :

$$f \otimes g.f' \otimes g' = ff' \otimes gg' + fg(f') \otimes g'.$$

Remark 6.4.3. *For any $g \in \mathfrak{g}$ and $f \otimes y \in \mathscr{O}_X \otimes_k U(\mathfrak{g})$, we have that $g.(f \otimes y) - (f \otimes y).g = g(f) \otimes y + y \otimes [g, y]$.*

We deduce that there is a map of sheaf of rings $\mathscr{O}_X \otimes_k U(\mathfrak{g}) \to \mathcal{D}_X$.

Lemma 6.4.4. *We have that $\mathfrak{b}^0\mathscr{O}_X \otimes_k U(\mathfrak{g}) = \mathscr{O}_X \otimes_k U(\mathfrak{g})\mathfrak{b}^0$.*

Proof. For any $s \in \mathfrak{g}^0$, we have $s(f \otimes 1) = (f \otimes 1)s = s(f)$ for any $f \in \mathscr{O}_X$, as $\mathfrak{b}_0$ acts trivially on $\mathscr{O}_X$ on derivation, we deduce that if $s \in \mathfrak{b}^0$, $s(f \otimes 1) = (f \otimes 1)s$. Next, we have $x\mathfrak{b}_0 - \mathfrak{b}_0 x \subseteq \mathfrak{b}_0$ for any $x \in \mathfrak{g}$, since $\mathfrak{b}_0$ is $\mathfrak{g}$-equivariant.

□

The map $\mathscr{O}_X \otimes_k U(\mathfrak{g}) \to \mathcal{D}_X$ passes to the quotient to a map $\mathscr{O}_X \otimes_k U(\mathfrak{g})/\mathfrak{b}^0 \to \mathcal{D}_X$.

Proposition 6.4.5. *We have an isomorphism $\mathscr{O}_X \otimes_k U(\mathfrak{g})/\mathfrak{b}^0 \to \mathcal{D}_X$.*

Proof. These are G-equivariant sheaves. It suffices to compare stalks at e. We have $e^\star\mathscr{O}_X \otimes_k U(\mathfrak{g})/\mathfrak{b}^0 = k \otimes_{U(\mathfrak{u})} U(\mathfrak{g}) \simeq U(\bar{\mathfrak{u}}) = e^\star\mathcal{D}_X = \mathrm{colim}_n \mathrm{Hom}(\mathscr{O}_{X,x}/\mathfrak{m}_x^n, k)$. More precisely, the big cell is $B\backslash B\bar{U} = \mathrm{Spec}\, k[X_\alpha, \alpha \in \Phi^{-1}]$ and $\mathcal{D}_X|_{B\backslash B\bar{U}} = k\{X_\alpha, \partial_{X_\alpha}\}$. Taking stalks we get that $e^\star\mathcal{D}_X = k[\partial_{X_\alpha}, \alpha \in \Phi^-] \simeq U(\bar{\mathfrak{u}})$. □

We let $\widetilde{\mathcal{D}}_X = \mathscr{O}_X \otimes_k U(\mathfrak{g})/\mathfrak{u}^0$.

There is a map $\mathfrak{h} \to \mathfrak{b}/\mathfrak{u}$ of B-modules, where in fact, $\mathfrak{h}$ has the trivial B-action. It follows that we have a map $\mathfrak{h} \to \mathfrak{b}^0/\mathfrak{u}^0$. We can think of this map as follows. For any $h \in \mathfrak{h}$, we can consider the map $x \mapsto xhx^{-1}$ which defines a global section of $\mathfrak{b}^0/\mathfrak{u}^0$.

Remark 6.4.6. *For any $x \in X$, $k(x) \otimes_{\mathscr{O}_X} \widetilde{\mathcal{D}}_X = k_x \otimes_{U(\mathfrak{u}_x)} U(\mathfrak{g}) = U(\mathfrak{h}_x) \otimes_{U(\mathfrak{u}_x)} U(\mathfrak{g})$.*

This induces a map $\Theta_{hor} : U(\mathfrak{h}) \to \widetilde{\mathcal{D}}_X$.

Proposition 6.4.7. *The map $U(\mathfrak{h}) \to \widetilde{\mathcal{D}}_X$ is injective. For any $x \in X$, the composite $U(\mathfrak{h}) \to \widetilde{\mathcal{D}}_X \to U(\mathfrak{h}_x) \otimes_{U(\mathfrak{b}_x)} U(\mathfrak{g})$ is given by $h \mapsto x^{-1}hx \otimes 1$. Moreover, $U(\mathfrak{h})$ lies in the center of $\widetilde{\mathcal{D}}_X$.*

Proof. Since $\mathfrak{h} \hookrightarrow \mathfrak{b}^0$, any element of $\mathfrak{h}$ acts trivially on $f \in \mathscr{O}_X$. Also, $\mathfrak{h} = \mathrm{H}^0(X, \mathfrak{b}^0/\mathfrak{u}^0)$ carries the trivial G-action. We deduce that $g\Theta(\mathfrak{h}) - \Theta(\mathfrak{h})g = 0$ for all $g \in \mathfrak{g}$. □

Proposition 6.4.8. *We have that $U(\mathfrak{h}) = \mathrm{H}^0(X, \widetilde{\mathcal{D}}_X)^G$.*

For any character $\lambda \in X^\star(T)_k$, we let $\mathcal{D}_{X,\lambda} = \widetilde{\mathcal{D}}_X \otimes_{U(\mathfrak{h}),\lambda} k$. This is an algebra of twisted differential operators.

Proposition 6.4.9. *(1) We have $\mathcal{D}_{X,0} = \mathcal{D}_X$.*
(2) $\widetilde{D}$ is a locally free $U(\mathfrak{h}) \otimes_k \mathscr{O}_X$-module.

Proof. The first point follows from the definition. Next, we claim that $\widetilde{D}$ is locally free as a $U(\mathfrak{h}) \otimes_k \mathscr{O}_X$-module. For example, consider the big cell $V = B\backslash B\bar{U}$. We have $\mathfrak{g}^0|_V = \mathfrak{u}^0|_V \oplus \bar{\mathfrak{b}} \otimes_k \mathscr{O}_V$. We deduce that

$$U(\mathfrak{g}) \otimes_k \mathscr{O}_V = \mathfrak{u}^0(U(\mathfrak{g}) \otimes_k \mathscr{O}_V) \oplus U(\bar{\mathfrak{b}}) \otimes_k \mathscr{O}_V.$$

As a result,

$$\widetilde{D}|_V \simeq U(\bar{\mathfrak{b}}) \otimes_k \mathscr{O}_V = U(\mathfrak{h}) \otimes_k U(\bar{\mathfrak{u}}) \otimes_k \mathscr{O}_V.$$

□

Lemma 6.4.10. *Let $\lambda \in X^\star(T)$. Then $\mathscr{O}(\lambda)$ is naturally a $\mathcal{D}_\lambda$-module.*

Proof. The G-equivariant action on $\mathscr{O}(\lambda)$ differentiates to a $\mathfrak{g}$-action. This induces a linear action of $\mathfrak{b}^0$. This is a map of G-equivariant sheaves : $\mathfrak{b}^0 \otimes_{\mathscr{O}_X} \mathscr{O}(\lambda) \to \mathscr{O}(\lambda)$. We study this map at the stalk at $B\backslash B$. This is a map $\mathfrak{b} \otimes k(\lambda) \to k(\lambda)$ which is induced by $\mathfrak{b} \to \mathfrak{h} \xrightarrow{\lambda} k$. We deduce that the $\mathfrak{b}^0$ action factors through an action of $\mathfrak{h}$ via the character λ. This implies that the $\mathcal{U}^0$-action on $\mathscr{O}(\lambda)$ factors through an action of $\mathcal{D}_\lambda$. $\square$

Remark 6.4.11. *One can prove that for $\lambda \in X^\star(T)$, we have $\mathcal{D}_\lambda = \mathscr{O}_X(\lambda) \otimes \mathcal{D}_X \otimes \mathscr{O}_X(-\lambda)$ is the ring of differential operators on $\mathscr{O}_X(\lambda)$.*

We also have a natural map $Z(\mathfrak{g}) \to U(\mathfrak{g}) \to \widetilde{\mathcal{D}}_X$.

Proposition 6.4.12. *We have the following factorization :*

$$\begin{tikzcd} U(\mathfrak{h}) \arrow[r] & \widetilde{\mathcal{D}} \\ & Z(\mathfrak{g}) \arrow[ul, "HC"] \arrow[u] \end{tikzcd}$$

Proof. As $\widetilde{\mathcal{D}}$ is a locally free $\mathscr{O}_X$-module, it suffices to check the desired factorization at stalks. For any $x \in X$, we have $x^\star\widetilde{\mathcal{D}} = U(\mathfrak{h}_x) \otimes_{U(\mathfrak{b}_x)} U(\mathfrak{g})$. Moreover, we have $1 \otimes z = x^{-1}HC(z)x \otimes 1$ by definition of the map HC. $\square$

As a corollary, we have a map $\widetilde{U} := U(\mathfrak{h}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g}) \to \mathrm{H}^0(X, \widetilde{\mathcal{D}})$. Similarly, for any $\lambda \in X^\star(T)_k$, we have a map $U_\lambda := k \otimes_{\lambda, U(\mathfrak{h})} U(\mathfrak{h}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g}) \to \mathrm{H}^0(X, \mathcal{D}_\lambda)$.

7. Localization

7.1. **Statement of the localization theorem.** Let $\lambda \in X^\star(T)_k$. We consider the category $Mod(U_\lambda)$ of U_λ-modules, as well as the category $Mod(\mathcal{D}_\lambda)$ whose objects are quasi-coherent sheaves of $\mathscr{O}_X$-modules, equipped with an action of $\mathcal{D}_\lambda$. We define a localization functor :

$$\begin{array}{rcl} \mathrm{Loc} : Mod(U_\lambda) & \to & Mod(\mathcal{D}_\lambda) \\ M & \mapsto & \mathcal{D}_\lambda \otimes_{U_\lambda} M \end{array}$$

We also have a global section functor :

$$\begin{array}{rcl} \Gamma : Mod(\mathcal{D}_\lambda) & \to & Mod(U_\lambda) \\ \mathscr{M} & \mapsto & \Gamma(X, \mathscr{M}) \end{array}$$

Lemma 7.1.1. *The functors* (Loc, Γ) *form a pair of adjoint functors.*

Proof. Concretely, this means that for $M \in Mod(U_\lambda)$ and $\mathscr{N} \in Mod(\mathcal{D}_\lambda)$, we have :

$$\mathrm{Hom}_{U_\lambda}(M, \Gamma(\mathscr{N})) = \mathrm{Hom}_{\mathcal{D}_\lambda}(\mathcal{D}_\lambda \otimes_{U_\lambda} M, \mathscr{N})$$

This is clear ! $\square$

We say that a weight $\nu \in X^\star(T)_k$ is antidominant if for all $w \in W$, $w \neq Id$, we don't have $w(\nu) \leq \nu$. If $\nu = (\nu_1, \cdots, \nu_n) \in X^\star(T)$, then this coincides with the condition that $\nu_1 \leq \nu_2 \cdots \leq \nu_n$.

Theorem 7.1.2. *Let $\lambda \in X^\star(T)_k$. Assume that $\lambda - \rho$ is antidominant.*

(1) The functor Γ is exact and the adjonction $M \mapsto \Gamma \circ \mathrm{Loc}(M)$ is an isomorphism.
(2) If we further assume that $\lambda - \rho$ is regular, then Loc *and* Γ *are inverse equivalences of each other.*

Remark 7.1.3. *We see that if $\lambda \in X^\star(T)^-$, then $\lambda - \rho$ is antidominant and regular. Let $V_{w_0\lambda}$ be the highest weight representation $w_0\lambda$. We have $V_{w_0\lambda} = \mathrm{H}^0(X, \mathscr{O}(\lambda))$. Conversely, we have that $\mathrm{Loc}(V_{w_0\lambda}) = \mathscr{O}(\lambda)$.*

7.2. Global sections and cohomology of $\widetilde{\mathcal{D}}$ and $\mathcal{D}_\lambda$.

Theorem 7.2.1. *We have that :*

(1) $\mathrm{H}^0(\widetilde{\mathcal{D}}) = \widetilde{U}$ *and* $\mathrm{H}^i(\widetilde{\mathcal{D}}) = 0$ *for all* $i > 0$.
(2) $\mathrm{H}^0(\mathcal{D}_\lambda) = U_\lambda$ *and* $\mathrm{H}^i(\mathcal{D}_\lambda) = 0$ *for all* $i > 0$.

For a full proof, see [Mil], sect. II, thm 6.1 and 6.5.

Proof. We only give the proof for SL_2. We consider the following resolution of $\widetilde{\mathcal{D}}$:

$$0 \to U(\mathfrak{g}) \otimes_k \mathfrak{u}^0 \to U(\mathfrak{g}) \otimes_k \mathscr{O}_X \to \widetilde{\mathcal{D}} \to 0$$

Notice that $\mathfrak{u}^0 \simeq \mathscr{O}_{\mathbb{P}^1}(-2)$. We deduce that $\mathrm{H}^1(X, \widetilde{\mathcal{D}}) = 0$ and we have an exact sequence :

$$0 \to \mathrm{H}^0(X, U(\mathfrak{g}) \otimes_k \mathscr{O}_X) \to \mathrm{H}^0(X, \widetilde{\mathcal{D}}) \to \mathrm{H}^1(X, U(\mathfrak{g}) \otimes_k \mathfrak{u}^0) \to 0$$

Also, notice that the exact sequence $0 \to \mathfrak{u}^0 \to \mathfrak{b}^0 \to \mathfrak{b}^0/\mathfrak{u}^0 \to 0$ induces an isomorphism $\mathfrak{h} = \mathrm{H}^1(X, \mathfrak{u}^0)$. So we deduce that we have a commutative diagram where all vertical maps are isomorphisms :

$$\begin{tikzcd}
0 \arrow[r] & U(\mathfrak{g}) \arrow[r] & \mathrm{H}^0(X, \widetilde{\mathcal{D}}_X) \arrow[r] & U(\mathfrak{g}) \otimes \mathrm{H}^1(X, \mathfrak{u}^0) \arrow[r] & 0 \\
0 \arrow[r] & U(\mathfrak{g}) \arrow[r] \arrow[u] & U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{h}) \arrow[r] \arrow[u] & U(\mathfrak{g}) \otimes_k \mathfrak{h} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Next, we claim that $\widetilde{D}$ is locally free as a $U(\mathfrak{h})$-module. For example, consider the big cell $V = B\backslash B\bar{U}$. We have $\mathfrak{g}^0|_V = \mathfrak{u}^0|_V \oplus \bar{\mathfrak{b}} \otimes_k \mathscr{O}_V$. We deduce that

$$U(\mathfrak{g}) \otimes_k \mathscr{O}_V = \mathfrak{u}^0(U(\mathfrak{g}) \otimes_k \mathscr{O}_V) \oplus U(\bar{\mathfrak{b}}) \otimes_k \mathscr{O}_V.$$

As a result,

$$\widetilde{D}|_V \simeq U(\bar{\mathfrak{b}}) \otimes_k \mathscr{O}_V = U(\mathfrak{h}) \otimes_k U(\bar{\mathfrak{u}}) \otimes_k \mathscr{O}_V.$$

We deduce that for any $\lambda \in X^\star(T)_k$ and for $h \in \mathfrak{h}$ a generator, the sequence :

$$0 \to \widetilde{D} \overset{h-\lambda(h)}{\to} \widetilde{D} \to \mathcal{D}_\lambda \to 0$$

is exact. After taking cohomology, we conclude the proof of the theorem. $\square$

7.3. **Translation principle.** For any $\lambda \in X^\star(T)$, we have an invertible sheaf $\mathscr{O}_X(\lambda)$, and we have an equivalence of categories $- \otimes \mathscr{O}_X(\lambda) : Mod(\mathcal{D}_{\lambda'}) \to Mod(\mathcal{D}_{\lambda+\lambda'})$.

Now, let V be a finite dimensional representation of G. We let $V^0 = V \otimes_k \mathscr{O}_X$. This is a G-equivariant $\mathcal{U}^0 = U(\mathfrak{g}) \otimes_k \mathscr{O}_X$-module. We see that V, viewed as a B-module, admits a decreasing filtration $\mathrm{Fil}^i V$, with $\mathrm{gr}^i V = k(\nu_i)$ where ν_i is a weight of V.

Let $\mathscr{G}$ be a $\mathcal{D}_\lambda$-module. We see that $\mathscr{G} \otimes V$ is a $\mathcal{U}^0 = U(\mathfrak{g}) \otimes_k \mathscr{O}_X$-module. Moreover, it carries a filtration with graded pieces the $\mathscr{G}(\nu_i)$. In particular, we deduce that for any $z \in Z(\mathfrak{g})$, we have $\prod_i (z - (\lambda + \nu_i)(z)) = 0$. In particular we can project $\mathscr{G} \otimes V$ onto its $(\lambda+\nu_i)$-generalized eigenspace, denoted by $(\mathscr{G} \otimes V)_{\lambda+\nu_i}$.

Lemma 7.3.1. *Let* $\lambda \in X^\star(T)_k$, $\mu \in X^\star(T)$. *Assume that* $\lambda - \rho$ *and* $-\mu$ *are antidominant. The functor* $\mathscr{G} \to (\mathscr{G}(-\mu) \otimes V_\mu)_\lambda$ *is equivalent to the identity functor of* $Mod(\mathcal{D}_\lambda)$.

Proof. Let ν be a weight of V_μ. Necessarily, $\nu \leq \mu$. We see that $\mathfrak{h}$ acts on $\mathscr{G}(-\mu+\nu)$ via $\lambda - \mu + \nu$. We assume that $\mathscr{G}(-\mu+\nu)$ contributes to the λ-generalized eigenspace. That means that there exists $w \in W$ such that $\lambda - \mu + \nu - \rho = w(\lambda - \rho)$. Since $\lambda - \mu + \nu - \rho \leq \lambda - \rho$, we must have $\lambda - \rho = w(\lambda - \rho)$. As a result $\mu = \nu$. $\square$

Lemma 7.3.2. *Let $\lambda \in X^\star(T)_k$. Assume that $\lambda-\rho$ is antidominant and regular. Let $-\mu \in X^\star(T)$ be antidominant. Let $V_{w_0\mu}$ be the highest weight $w_0\mu$ representation. Let $\mathscr{G} \in \mathcal{D}_\lambda$. The map $\mathscr{G} \otimes V_{w_0\mu} \to \mathscr{G}(-\mu)$ induces an isomorphism $(\mathscr{G} \otimes V_{w_0\mu})_{\lambda-\mu} \to \mathscr{G}(-\mu)$.*

Proof. Let ν be a weight of $V_{w_0\mu}$. Then $\nu+\lambda$ is a weight of $\mathscr{G} \otimes V_{w_0\mu}$. If $\mathscr{G}(\nu)$ maps non-trivially to $(\mathscr{G} \otimes V_{w_0\mu})_{\lambda-\mu}$, this means that $\lambda-\mu-\rho = w(\lambda+\nu-\rho)$. We deduce that $w(\lambda-\rho) = \lambda-\mu-\rho-w\nu$. By $-\mu \geq w\nu$ so $\lambda-\mu-\rho-w\nu \leq \lambda-\rho$. We deduce that $w = Id$ and that $\nu = -\mu$. ☐

7.4. **Proof of the localization theorem.** We recall :

Lemma 7.4.1. *A quasi-coherent sheaf on a noetherian scheme is a filtered colimit of coherent sheaves.*

Proof. See [Sta13] tag 01PJ. ☐

Lemma 7.4.2. *The invertible sheaves $\mathscr{O}(\kappa)$ for $\kappa \in X^\star(T)^-$ are ample.*

Lemma 7.4.3. *Let $\lambda \in X^\star(T)_k$. Assume that $\lambda-\rho$ is antidominant. The functor $\Gamma : Mod(\mathcal{D}_\lambda) \to Mod(U_\lambda)$ is exact.*

Proof. We can write $\mathscr{G}$ as a filtered colimit of coherent sheaves $\mathscr{G} = \mathrm{colim}_i \mathscr{G}_i$. We will prove that $\mathrm{Im}(\mathrm{H}^k(X,\mathscr{G}_i), \mathrm{H}^k(X,\mathscr{G})) = 0$ for all $k>0$. Choose $-\mu$ antidominant such that $\mathrm{H}^k(X,\mathscr{G}_i(-\mu)) = 0$. We look at the following diagram (see lemma 7.3.1) :

$$\begin{tikzcd} \mathscr{G} \arrow[r] & \mathscr{G}(-\mu) \otimes V_\mu \arrow[r] & \mathscr{G} \\ \mathscr{G}_i \arrow[r] \arrow[u] & \mathscr{G}_i(-\mu) \otimes V_\mu \arrow[u] & \end{tikzcd}$$

Taking H^k, we deduce that $\mathrm{Im}(\mathrm{H}^k(X,\mathscr{G}_i), \mathrm{H}^k(X,\mathscr{G})) = 0$. ☐

Corollary 7.4.4. *The functor Γ is exact and the adjonction $M \mapsto \Gamma \circ \mathrm{Loc}(M)$ is an isomorphism.*

Proof. Let $U_\lambda \otimes_k V \to U_\lambda \otimes_k W \to M \to 0$ be a resolution of M by free U_λ-modules. This induces a resolution of $\mathrm{Loc}(M)$: $\mathcal{D}_\lambda \otimes_k V \to \mathcal{D}_\lambda \otimes_k W \to \mathrm{Loc}(M) \to 0$. Since Γ is exact, we deduce that the following is exact $U_\lambda \otimes_k V \to U_\lambda \otimes_k W \to \Gamma \circ \mathrm{Loc}(M) \to 0$. ☐

Lemma 7.4.5. *Assume that $\lambda-\rho$ is regular and anti-dominant. Let $\mathscr{G} \in \mathrm{Mod}(\mathcal{D}_\lambda)$ be non-zero. Then $\mathrm{H}^0(X,\mathscr{G}) \neq 0$.*

Proof. Let $-\mu \in X^\star(T)$ be antidominant and such that $\mathrm{H}^0(X,\mathscr{G}(-\mu)) \neq 0$. Let $V_{w_0\mu}$ be the highest weight $w_0\mu$ representation. By lemma 7.3.2, the map $\mathscr{G} \otimes V_{w_0\mu} \to \mathscr{G}(-\mu)$ induces an isomorphism $(\mathscr{G} \otimes V_{w_0\mu})_{\lambda-\mu} \to \mathscr{G}(-\mu)$ and therefore we get a surjection $\mathrm{H}^0(X,\mathscr{G}) \otimes V_{w_0\mu} \to \mathrm{H}^0(X,\mathscr{G}(-\mu))$. ☐

Corollary 7.4.6. *Assume that $\lambda-\rho$ is regular and anti-dominant. Then the adjunction* $\mathrm{Loc} \circ \Gamma(\mathscr{G}) \to \mathscr{G}$ *is an isomorphism.*

Proof. Consider the long exact sequence : $0 \to K \to \mathrm{Loc} \circ \Gamma(\mathscr{G}) \to \mathscr{G} \to C \to 0$. We deduce that $\Gamma(K) = \Gamma(C) = 0$. It follows that $K = C = 0$. ☐

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