1. Sheaves on the Flag variety

Let k be a field of characteristic zero complete for a rank 1 valuation extending the p-adic valuation. Let $\mathcal{O}_k$ be its ring of integers. All our spaces will be adic spaces over $\mathrm{Spa}(k, \mathcal{O}_k)$.

Let $G = GL_2$, B be the upper Borel, with unipotent radical U and maximal torus T. We let $\mathfrak{g} = Lie(G)$, $\mathfrak{u} = Lie(U)$, $\mathfrak{b} = Lie(B)$ and $\mathfrak{h} = Lie(T)$.

The characters of the torus are denoted by $X^\star(T)$. We identify $X^\star(T)$ with $(k; w) \in \mathbb{Z}^2$ with $k = w \mod 2$ via $(k, w).\mathrm{diag}(tz, t^{-1}z) = t^k z^w$.

Let $FL = B\backslash G$ be the corresponding partial flag variety. It carries an action of G by right translation. We let $\{\infty\} = B\backslash B \in FL$ and $\mathbb{A}^1 = FL \setminus \{\infty\}$ be the Bruhat strata. We let T be the maximal diagonal torus.

1.1. G-equivariant sheaves.

Let $m, p : FL \times G \to FL$ be the action map and the projection map respectively. We let $\mathrm{Coh}_G(FL)$ be the category of G-equivariant coherent sheaves over FL. Its objects consist of a coherent sheaves $\mathscr{F}$ together with an isomorphism $m^\star \mathscr{F} \to p^\star \mathscr{F}$ satisfying certain obvious condition.

We let $\mathrm{Rep}^f(B)$ be the category of finite dimensional algebraic representations of B on k-vector spaces. We define a functor :

$$\begin{array}{rcl} F : \mathrm{Rep}^f(B) & \to & \mathrm{Coh}_G(FL) \\ V & \mapsto & \mathcal{V} = F(V) \end{array}$$

as follows. The group G acts on itself by left and right translation. It follows that $\mathscr{O}_G$ carries two G-actions, denoted $\star_l$ and $\star_r$, given by the rule $g \star_l f(-) = f(g^{-1}-)$ and $g \star_r f(-) = f(-g)$. We consider the projection $\pi : G \mapsto B\backslash G$, $g \mapsto \infty.g$. Let (V, ρ) be an object of $\mathrm{Rep}(B)$. We let $F(V) = (\pi_\star \mathscr{O}_G \otimes_k V)^{\star_l \otimes \rho}$. The action $\star_r$ provides a G-equivariant structure on $F(V)$.

Let $i_\infty : \{\infty\} \to FL$ be the inclusion. We conversely define a functor :

$$\begin{array}{rcl} i_\infty^\star : \mathrm{Coh}_G(FL) & \to & \mathrm{Rep}^f(B) \\ \mathcal{V} & \mapsto & i_\infty^\star \mathcal{V} \end{array}$$

Proposition 1.1.1. *The functors F and $i_\infty^\star$ are quasi-inverse equivalences of categories.*

Example 1.1.2.
1. If $\kappa = (k; w) \in X^\star(T)$, we let $\mathscr{O}_{FL}^\kappa = F(w_0\kappa)$. This twist in the notation is justified by the fact that $\mathscr{O}_{FL}^\kappa$ has degree k, and that $\mathrm{H}^0(FL, \mathscr{O}_{FL}^\kappa)$ is the representation of highest weight κ when $\kappa \in X^\star(T)^+$.
2. If $V \in \mathrm{Rep}(G)$, one checks that $F(V) = \mathscr{O}_{FL} \otimes_k V$.
3. Let St be the standard representation of G. It sits in the exact sequence of B-representations $0 \to k((1; 1)) \to St \to k((-1; 1)) \to 0$. By applying F, we get the "tautological" exact sequence : $0 \to \mathscr{O}_{FL}^{(-1;1)} \to \mathscr{O}_{FL} \otimes St \to \mathscr{O}_{FL}^{(1;1)} \to 0$.
4. We have $\mathfrak{u} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$. We let $\mathfrak{u}^0 := F(\mathfrak{u}) \subseteq \mathfrak{b}^0 := F(\mathfrak{b}) \subseteq F(\mathfrak{g}) := \mathfrak{g}^0 = \mathscr{O}_{FL} \otimes \mathfrak{g}$.
5. We have $T_{FL} = F(\mathfrak{g}/\mathfrak{b}) = \mathfrak{g}^0/\mathfrak{b}^0$. This is the tangent sheaf of FL.
6. We have $\mathfrak{b}^0/\mathfrak{u}^0 \simeq \mathfrak{h} \otimes \mathscr{O}_{\mathcal{FL}}$. This is the horizontal Cartan, with global section $\mathfrak{h}_{hor} \simeq \mathfrak{h}$.
7. We have $\mathfrak{u}^0 = \Omega^1_{FL}$.

1.2. $\mathfrak{g}$-equivariant sheaves.

The Lie algebra $\mathfrak{g}$ acts by derivations on $\mathscr{O}_{FL}$. We let $\mathrm{Coh}_{\mathfrak{g}}(FL)$ be the category of $\mathfrak{g}$-equivariant coherent sheaves over FL. Its objects consist of a coherent sheaf $\mathscr{F}$ and a k-linear map $\mathfrak{g} \to \mathrm{End}_k(\mathscr{F})$ such that :

1. $\forall (g, a, f) \in \mathfrak{g} \times \mathscr{O}_{FL} \times \mathscr{F}$, $g(af) = g(a)f + ag(f)$,
2. $\forall (g, g') \in \mathfrak{g}^2$, $gg' - g'g = [g, g']$ in $\mathrm{End}_k(\mathscr{F})$.

Any such map extends to a map $\mathfrak{g}^0 \to \underline{\mathrm{End}}_k(\mathscr{F})$ and we observe that the induced map $\mathfrak{b}^0 \to \underline{End}_k(\mathscr{F})$ factors through $\underline{End}_{\mathscr{O}_{FL}}(\mathscr{F})$. We observe that if $\mathfrak{u}^0$ acts trivially, $\mathscr{F}$ gets an action of the horizontal cartan $\mathfrak{h}_{hor}$.

Remark 1.2.1. *When $\mathfrak{b}^0$ acts trivially, $\mathscr{F}$ is a D-module. When only $\mathfrak{u}^0$ acts trivially and $\mathfrak{h}_{hor}$ acts via a character λ, $\mathscr{F}$ is a twisted D-module.*

We let $\mathrm{Coh}_{(\mathfrak{g},G)}(FL)$ be the category of $(\mathfrak{g}, G)$-equivariant coherent sheaves over FL. Its objects consist of a G-equivariant and $\mathfrak{g}$-equivariant coherent sheaf $\mathscr{F}$ such that the map $\mathfrak{g} \otimes \mathscr{F} \to \mathscr{F}$ is G-equivariant.

We have a natural functor $\mathrm{Coh}_G(FL) \to \mathrm{Coh}_{(\mathfrak{g},G)}(FL)$, obtained by deriving the action of G.

Exercise 1.2.2. *Prove that $\mathfrak{u}^0$ acts trivially on $\mathscr{O}^{\kappa}_{FL}$ and that $\mathfrak{h}_{hor}$ acts via $w_0\kappa$.*

Exercise 1.2.3. *Prove that $\mathrm{Coh}_{(\mathfrak{g},G)}(FL)$ is equivalent to the category $\mathrm{Rep}^f((\mathfrak{g}, B))$ consisting of a finite dimensional k vector space V equipped with actions of $\mathfrak{g}$ and B, such that $(bgb^{-1}).v = b.g.b^{-1}.v$ for all $(g, b, v) \in \mathfrak{g} \times B \times V$.*

Exercise 1.2.4. *Prove that all the rank one objects of $\mathrm{Coh}_{(\mathfrak{g},G)}(FL)$ are of the form $\mathscr{O}^{\kappa}_{FL}(\nu)$ where $\nu : \mathfrak{g} \to k$ is a twist by a character of $\mathfrak{g}$.*

We will also need to consider infinite dimensional sheaves.

Remark 1.2.5. *There is no abelian category of quasi-coherent sheaves because localizations are not exact in this context (they involve a completion which is rarely exact). There is a derived category of quasi-coherent sheaves defined using the solid formalism. We will below introduce certain exact categories of topological sheaves.*

We let $Ban_{\mathfrak{g}}(FL)$ be the category whose objects are topological sheaves $\mathscr{F}$ of $\mathscr{O}_{FL}$ endowed with an action of $\mathfrak{g}$ by derivations as before and such that the following holds : there is a covering $\{U_i\}$ such that $\mathscr{F}|_{U_i} = \mathscr{O}_{U_i}\hat{\otimes}_{\mathbb{Q}_p} V_i$ for an orthonormalisable Banach space V_i, and there are integers n_i such that the action of $\mathfrak{g}$ integrates to an action of $G_{n_i} = \{g \in G, g = 1 \mod p^{n_i}\}$ of $\mathscr{F}|_{U_i}$.

Remark 1.2.6. *Without loss of generality, we can assume that U_i is affinoid, stable under G_{n_i}. The action is given by a co-module map $\mathscr{F}(U_i) \to \mathscr{F}(U_i)\hat{\otimes}_{\mathscr{O}_{U_i}}\mathscr{O}_{G_{n_i}}$. This last condition hides a smallness condition. Let $\mathscr{F}(U_i)^+$ be an $\mathscr{O}_k$-lattice. Then, after possibly increasing n_i, we can assume that $\mathscr{F}(U_i)^+$ is stable by G_{n_i}, and the action on $\mathscr{F}(U_i)^+/p$ is trivial.*

Let $\mathscr{F} \in Ban_{\mathfrak{g}}(FL)$. We can consider the $\mathfrak{u}^0$-cohomology of $\mathscr{F}$, defined as the cohomology of the complex of amplitude $[0, 1]$:

$$\mathrm{R}\Gamma(\mathfrak{u}^0, \mathscr{F}) = [\mathscr{F} \to \mathscr{F} \otimes (\mathfrak{u}^0)^{\vee}].$$

We let $\mathscr{F}^{\mathfrak{u}^0}$ be the H^0 of this complex. We let $Coh_{\mathfrak{g}}(FL)^{\mathfrak{u}_0}$, $Ban_{\mathfrak{g}}(FL)^{\mathfrak{u}_0}$ be the subcategories of objects killed by $\mathfrak{u}_0$.

Example 1.2.7. We consider the sheaf $\mathscr{O}_{FL}\hat{\otimes}\mathscr{O}_{G_n}$. It is the sheaf of functions on $G_n \times FL$. A section of this sheaf is thus $f(g, x)$ with $g \in G_n$, and $x \in FL$. We have three actions :

(1) $g' \star_1 f(g, x) = f((g')^{-1}g, x)$ for $g' \in G_n$,
(2) $g' \star_2 f(g, x) = f(gg', x)$ for $g' \in G_n$,
(3) $g' \star_3 f(g, x) = f(g, xg')$ for $g' \in G$.

Using the composite $\star_1\star_3$, this becomes a $\mathfrak{g}$-equivariant sheaf over FL. We let $\mathcal{C}^{n-an} = (\mathscr{O}_{FL}\hat{\otimes}\mathscr{O}_{G_n})^{\mathfrak{u}^0}$. Concretely, a section of $\mathcal{C}^{n-an}$ is a function $f(g, x)$ which satisfies $f(n_x g, x) = f(g, x)$ for $n_x \in x^{-1}Ux \cap G_n$. There is also an action $\star_2$ of $\mathfrak{g}$ commuting with the $\star_1\star_3$-action. We let $\mathcal{C}^{la} = \mathrm{colim}_n \mathcal{C}^{n-an}$. This sheaf carries an action $\star_1 \star_2 \star_3$ of G and is thus an object of $LBan_{(\mathfrak{g},G)}(FL)$. It also has a $\star_2$ action of $\mathfrak{g}$.

Remark 1.2.8. *The sheaf $\mathcal{C}^{la}$ can be seen as an infinite jet space over FL.*

1.3. $(\mathfrak{g}, B)$**-equivariant sheaves on Bruhat strata.** We now consider $(\mathfrak{g}, B)$-equivariant sheaves on Bruhat strata.

Proposition 1.3.1. *(1) For all $\kappa \in X^\star(T)_k$, we have a rank one $(\mathfrak{g}, B)$-equivariant sheaf $\mathscr{O}^{\kappa}_{\mathbb{A}^1}$ over $\mathbb{A}^1$. The horziontal Cartan acts via $w_0\kappa$.*

(2) For all $\kappa \in X^\star(T)_k$, we have a rank one $(\mathfrak{g}, B)$-equivariant sheaf $\mathscr{O}^{\kappa}_{\infty}$ over $\{\infty\}^{\dagger}$[1]. The horizontal Cartan acts via $w_0\kappa$.

(3) They are the only rank one objects of $Coh_{(\mathfrak{g},B)}(\mathbb{A}^1)$ and $Coh_{(\mathfrak{g},B)}(\{\infty\}^\dagger)$, up to twist by a character of T.

(4) If $\kappa \in X^\star(T)$, then $\mathscr{O}^{\kappa}_{\mathbb{A}^1}(\kappa)$ and $\mathscr{O}^{\kappa}_{\infty}(w_0\kappa)$ glue to $\mathscr{O}^{\kappa}_{FL}$.

Proof. We sketch the construction of $\mathscr{O}^{\kappa}_{\mathbb{A}^1}$. We have a map $\pi : \hat{U}\backslash\hat{B}w_0U \to \hat{B}\backslash\hat{B}w_0U$. This map is a left $\hat{T}$-torsor. We have a right action of $\hat{G}$ and also an adjoint action of B. We let $\mathscr{O}^{\kappa}_{\mathbb{A}^1} = \pi_\star\mathscr{O}_{\hat{U}\backslash\hat{B}w_0U}[-w_0\kappa]$ (the braket means sections $f(x)$ such that $f(\hat{t}x) = -w_0\kappa(\hat{t})f(x)$). This belongs to $Coh_{(\mathfrak{g},B)}(\mathbb{A}^1)$. □

2. Geometric Sen theory

2.1. **Modular curves and the Hodge-Tate period map.** Let $K = K_pK^p \subseteq G(\mathbb{A}_f)$ be a compact open subgroup. Let $X_K \to \mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ be the compactified modular curve of level K. We let E be the universal semi-abelian scheme.

Let $X_{K^p} = \lim X_{K_pK^p}$ be the perfectoid modular curve of level K^p. It carries an action of $G(\mathbb{Q}_p)$. The (complete) structural sheaf is $\mathscr{O}_{X_{K^p}}$.

We have a diagram ([Sch15]) :

$$\begin{array}{ccccc} & & X_{K^p} & & \\ & \swarrow^{\pi_{K_p}} & & \searrow^{\pi_{HT}} & \\ X_{K_pK^p} & & & & FL \end{array}$$

Here π_{HT} is the $G(\mathbb{Q}_p)$-equivariant map is defined by the property that

$$\pi^\star_{HT}(0 \to \mathscr{O}^{(-1;1)}_{FL} \to St \otimes \mathscr{O}_{FL} \to \mathscr{O}^{(1;1)}_{FL} \to 0) = 0 \to \omega^{-1}_E \to T_pE \otimes \mathscr{O}_{X_{K^p}} \to \omega_{E^t} \to 0$$

The structural sheaf $\mathscr{O}_{X_{K^p}}$ contains the sheaf $\mathscr{O}^{sm}_{X_{K^p}}$ of smooth vectors for the action of $G(\mathbb{Q}_p)$. It has the description $\mathscr{O}^{sm}_{X_{K^p}} = \mathrm{colim}\pi^{-1}_{K_p}\mathscr{O}_{X_{K^pK_p}}$.

2.2. **The functor** VB**.** We now consider a functor

$$\begin{array}{rcl} VB : Ban_{\mathfrak{g}}(FL) & \to & \mathrm{Mod}(\mathscr{O}^{sm}_{X_{K^p}}) \\ \mathscr{F} & \mapsto & (\pi^\star_{HT}\mathscr{F})^{sm} \end{array}$$

This functor is thus constructed in two steps. First we consider $\pi^\star_{HT}\mathscr{F} = \pi^{-1}_{HT}\mathscr{F}\hat{\otimes}_{\pi^{-1}_{HT}\mathscr{O}_{FL}}\mathscr{O}_{X_{K^p}}$ which is a Banach sheaf over $\mathscr{O}_{X_{K^p}}$. We have $(\pi^\star_{HT}\mathscr{F})^{sm} = \mathrm{colim}_{K_p}\mathrm{H}^0(K_p, \pi^\star_{HT}\mathscr{F})$ where the action of any small enough K_p is well defined on the sections of $\pi^\star_{HT}\mathscr{F}$ over any given affinoid subspace.

Remark 2.2.1. *This functor can be enriched to a functor : $VB : Ban_{(\mathfrak{g},G)}(FL) \to \mathrm{Mod}_{G(\mathbb{Q}_p)}(\mathscr{O}^{sm}_{X_{K^p}})$ and even $VB : LBan_{(\mathfrak{g},G)}(FL) \to \mathrm{Mod}_{G(\mathbb{Q}_p)}(\mathscr{O}^{sm}_{X_{K^p}})$.*

Remark 2.2.2. *We also have variants over Bruhat strata.*

[1] $\{\infty\}^\dagger$ is the space $\{\infty\}$, equipped with the sheaf (in fact module) $\mathscr{O}_{FL,\{\infty\}}$

2.3. **Geometric Sen theory.**

2.3.1. *The main theorem.*

Theorem 2.3.1 ([Pan22], [Cam22]). *(1) We have that* $VB(\mathscr{F}) = \mathrm{colim}_{K_p}\mathrm{H}^0(K_p, (\mathrm{H}^0(\mathfrak{u}^0, \pi_{HT}^\star\mathscr{F})))$

(2) If $\mathscr{F} \in Ban_{\mathfrak{g}}(FL)^{\mathfrak{u}_0}$*, then*

$$VB(\mathscr{F})\hat{\otimes}_{\mathscr{O}^{sm}_{X_{K^p}}}\mathscr{O}_{X_{K^p}} = \mathscr{F}\hat{\otimes}_{\mathscr{O}_{FL}}\mathscr{O}_{X_{K^p}}$$

Remark 2.3.2. *More precisely, for all small enough* K_p*, there is a Banach sheaf* $VB(\mathscr{F})_{K_p}$ *over* $X_{K_pK^p}$ *such that*

$$VB(\mathscr{F})_{K_p}\hat{\otimes}_{\mathscr{O}_{X_{K^pK_p}}}\mathscr{O}_{X_{K^p}} = \mathscr{F}\hat{\otimes}_{\mathscr{O}_{FL}}\mathscr{O}_{X_{K^p}}.$$

This isomorphism is K_p*-equivariant. We have* $VB(\mathscr{F}) = \mathrm{colim}_{K_p}VB(\mathscr{F})_{K_p}$.

Remark 2.3.3. *We thus think of objects in* $Ban_{\mathfrak{g}}(FL)^{\mathfrak{u}_0}$ *as admissible objects.*

Remark 2.3.4. *If* $\mathscr{F} \in Coh_{\mathfrak{g}}(FL)$*, we prove in* [Pil22] *that* $\mathrm{R}^iVB(\mathscr{F}) = VB(\mathrm{H}^i(\mathfrak{u}_0, \mathscr{F}))$*. There is also a formula in this spirit in for Banach sheaves, see* [Cam22].

Remark 2.3.5. *We can give a heuristic explanation for the fact that one needs to consider objects killed by* $\mathfrak{u}^0$*. The pro-étale descent along* $\pi_{K_p} : X_{K^p} \to X_{K^pK_p}$ *is not effective because this tower is very ramified integrally (indeed,* X_{K^p} *is a perfectoid space). Basically one is trying to factor the map* π_{K^p} *into* $X_{K^p} \to X_{K^p}/\mathfrak{u}^0 \to X_{K^pK_p}$ *where the second map is of effective descent. An object of* $Ban_{\mathfrak{g}}(FL)^{\mathfrak{u}_0}$ *pulls back to* $X_{K^p}/\mathfrak{u}^0$ *and descends then to* $X_{K^pK_p}$*. One can give a meaning to this factorization over* $\pi_{HT}^{-1}(\infty)$*. Note that* $\mathfrak{u}^0|_\infty = \mathfrak{u}$*. Let us assume that* $K_p = G(\mathbb{Z}_p)$ *and let* $X^{ord}_{K^pG(\mathbb{Z}_p)}$ *be the ordinary locus. Let* $IG \to X^{ord}_{K^pG(\mathbb{Z}_p)}$ *be the Igusa tower. This is a* $T(\mathbb{Z}_p)$*-torsor of trivialization of* $T_p(E)^{et}$ *and* $T_p(E)^m$*. There is a lift* Φ *of Frobenius on* IG*. Let* $IG^{perf} = \lim_\Phi IG$*. Then* $\pi_{HT}^{-1}(\infty)$ *is the closure of* IG^{perf} *and* $IG = IG^{perf}/U(\mathbb{Z}_p)$*. The map* $IG^{perf} \to IG$ *is integrally a perfectization, hence is very ramified integrally. The map* $IG \to X^{ord}_{K^pG(\mathbb{Z}_p)}$ *extends to a* $T(\mathbb{Z}_p)$*-torsor integrally. It is thus of effective pro-étale descent.*

2.3.2. *A few words about the proof.* The proof of the theorem proceeds in roughly two steps. First, one produces locally over X_{K^p} a geometric Sen operator $\Theta_{Sen,geom} \in \mathrm{End}(\pi_{HT}^\star\mathscr{F})$ which has the property that $VB(\mathscr{F}) = \mathrm{H}^0(K_p, (\mathrm{H}^0(\mathfrak{u}^0, \pi_{HT}^\star\mathscr{F})))$. This uses Sen theory as formulated in [BC16] and [BC08]. We then need to identify this operator as a generator of the image of $\mathfrak{u}^0$ in $\mathrm{End}(\pi_{HT}^\star\mathscr{F})$. One checks this in the case of Faltings' extension (see 2.3.4) and deduce it in general.

2.3.3. *Special case of the theorem : sheaves of modular forms.* We can apply the theorem to the sheaves $\mathscr{O}^\kappa_{FL} \in Coh_{(\mathfrak{g},G)}(FL)$. We let $VB(\mathscr{O}^\kappa_{FL}) = \omega^{\kappa,sm}$.

Lemma 2.3.6. *We have* $\mathrm{colim}\omega^\kappa_{K_p}$ *where* $\omega^\kappa_{K_p}$ *is the sheaf of weight* κ *modular forms over* $X_{K_pK^p}$*.*

Proof. Let us justify this in the case that $\kappa = (1;1)$ (the general case follows easily). By construction of π_{HT}, we have $\pi_{HT}^\star\mathscr{O}^{(1;1)}_{FL} = \pi^\star_{K_p}\omega_{E^t}$. □

We can consider $\mathrm{R}\Gamma(X_{K^p}, \omega^{\kappa,sm})$. This is a complex of smooth $G(\mathbb{Q}_p)$-representation, whose H^0 computes classical modular forms.

2.3.4. *Special case of the theorem : Faltings's extension.* We have $\mathrm{H}^0(\mathfrak{u}^0, St \otimes \mathscr{O}_{FL}) = \mathscr{O}^{(-1;1)}_{FL}$, and $\mathrm{H}^1(\mathfrak{u}^0, St \otimes \mathscr{O}_{FL}) = \mathscr{O}^{(1;1)}_{FL} \otimes (\mathfrak{u}^0)^\vee$.

The theorem is thus claiming that : $VB(St\otimes\mathscr{O}_{FL}) = \omega^{(-1;1),sm}$ and $\mathrm{R}^1VB(St\otimes\mathscr{O}_{FL}) = \omega^{(1;1),sm}$. This is actually a reformulation of some classical computations in p-adic Hodge theory. We have the following extension (called Faltings'extension, see [Sch13]) of pro-Kummer étale sheaves on $X_{K_pK^p}$:

$$0 \to \hat{\mathscr{O}}(1) \to \mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{dR} \to \Omega^1_{X_{K^pK_p}}(\log(cusp)) \otimes_{\mathscr{O}_{X_{K_pK^p}}} \hat{\mathscr{O}} \to 0$$

This extension witnesses the fact that $\pi_{pket,an,\star}\hat{\mathscr{O}} = \mathscr{O}_{X_{K_pK^p}}$ and $\mathrm{R}^1\pi_{pket,an,\star}\hat{\mathscr{O}}(1) = \Omega^1_{X_{K_pK^p}}\log(cusp)$ for $\pi_{pket,an} : X_{K_pK^p,pket} \to X_{K_pK^p,an}$ the projection of sites.

We can evaluate $\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{dR}$ on X_{K^p} and we get an extension (on the analytic site of X_{K^p}) :

$$0 \to \mathscr{O}_{X_{K^p}}(1) \to FE \to \Omega^1_{X_{K^pK_p}}(\log(cusp)) \otimes_{\mathscr{O}_{X_{K_pK^p}}} \mathscr{O}_{X_{K^p}} \to 0.$$

Theorem 2.3.7 ([Fal87], [Sch13], [Pan22]). *We have the following isomorphism of exact sequences :*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_{X_{K^p}} & \longrightarrow & T_pE \otimes_{\mathbb{Z}_p} \mathscr{O}_{X_{K^p}} \otimes \omega^{(1;-1),sm} & \longrightarrow & \omega^{(1;1),sm} \otimes \omega^{(1;-1),sm} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow {\scriptstyle -KS} & & \\
0 & \longrightarrow & \mathscr{O}_{X_{K^p}} & \longrightarrow & FE & \longrightarrow & \Omega^1_{X_{K^pK_p}}(\log(cusp)) \otimes_{\mathscr{O}_{X_{K_pK^p}}} \mathscr{O}_{X_{K^p}} & \longrightarrow & 0
\end{array}$$

This isomorphism together with the computation of the projection to the étale site of the Falting's extension recovers our theorem.

2.3.5. *Locally analytic vectors.* Let $n \geq 0$. Let $G_n \subseteq G$ be the analytic subgroup of elements which reduce to 1 modulo p^n. So $G_n(\mathbb{Q}_p) = 1 + p^n\mathrm{M}_2(\mathbb{Z}_p)$. Let V be a $\mathbb{Q}_p$-banach space equipped with a continuous $G_n(\mathbb{Q}_p)$-representation. Then we have an orbit map orb : $V \to \mathcal{C}^0(G_n(\mathbb{Q}_p), V) = \mathcal{C}^0(G_n(\mathbb{Q}_p), \mathbb{Q}_p)\hat{\otimes}V$. And $V = \mathcal{C}^0(G_n(\mathbb{Q}_p), V)^{G_n(\mathbb{Q}_p)}$ for the action $g' \star f(g) = g'f((g')^{-1}g)$.

Definition 2.3.8. We let $V^{n-an} = (\mathscr{O}_{G_n} \otimes V)^{G_n(\mathbb{Q}_p)}$ be the subset of V of analytic vectors. We let $V^{la} = \mathrm{colim}_n V^{n-an}$ be the subset of locally analytic vectors.

Remark 2.3.9. *We have an action of $\mathfrak{g}$ on V^{la}. Indeed let $v \in V^{n-an}$. Then we let $g.v = (g.\mathrm{orb}(v))(1)$.*

By definition $\mathscr{O}^{la}_{X_{K_p}} = (\mathscr{O}_{G,1}\hat{\otimes}_{\mathbb{C}_p}\mathscr{O}_{X_{K_p}})^{sm} = VB(\mathscr{O}_{G,1}\hat{\otimes}_{\mathbb{C}_p}\mathscr{O}_{FL})$.

The theorem therefore reads :

Theorem 2.3.10 ([Pan22]). *We have $VB(\mathcal{C}^{la}) = \mathscr{O}^{la}_{X_{K_p}}$ and $\mathscr{O}^{la}_{X_{K^p}}\hat{\otimes}_{\mathscr{O}^{sm}_{X_{K^p}}}\mathscr{O}_{X_{K^p}} = \mathcal{C}^{la}\hat{\otimes}_{\mathscr{O}_{FL}}\mathscr{O}_{X_{K^p}}$.*

Corollary 2.3.11. *The action of $\mathfrak{g}$ on $\mathscr{O}^{la}_{X_{K_p}}$ induces an action of $\mathfrak{g}^0$ on $\mathscr{O}^{la}_{X_{K_p}}$ and the induced action of $\mathfrak{u}^0$ is trivial.*

2.4. **The arithmetic Sen operator.** Let L be a finite extension of $\mathbb{Q}_p$. We let $X_{K^pK_p,L}$ be the modular curve defined over L, and let $\mathscr{F} \in Ban_{\mathfrak{g}}(FL_L)^{\mathfrak{u}^o}$. By construction, we see that $VB(\mathscr{F})$ carries a semi-linear action of G_L. We can ask wether $VB(\mathscr{F})$ descends to a sheaf $VB(\mathscr{F})_{L'}$ of $\mathscr{O}^{sm}_{X_{K^p,L'}}$-modules for some finite extension L' of L.

Theorem 2.4.1. *We have that* $\mathrm{diag}(1,0) \in \mathfrak{h}_{hor}$*, acting on $VB(\mathscr{F})$ is an arithmetic Sen operator $\Theta_{sen,arith}$. If it vanishes, then $VB(\mathscr{F})$ descends to a sheaf $VB(\mathscr{F})_L$.*

Remark 2.4.2. *More generally, the action of $\Theta_{sen,arith}$ of $VB(\mathscr{F})$ exponentiates to an action of a subgroup $\Gamma_{L'}$ of $\mathbb{Z}_p^\times$. Let L' be the extension of L with the property that $\chi_{cycl}(G_{L'}) \subseteq \Gamma_{L'}$. If we twist the action of $G_{L'}$ on $VB(\mathscr{F})$, using the projection to $\Gamma_{L'}$, we force the vanishing of the Sen operator and the sheaf $VB(\mathscr{F})$ descends to $VB(\mathscr{F})_{L'}$.*

3. Overconvergent modular forms

3.0.1. *Sheaves of overconvergent modular forms.* For all $\kappa \in X^\star(T)_{\mathbb{C}_p}$, we have defined in proposition 1.3.1, sheaves $\mathscr{O}^\kappa_{\mathbb{A}^1}$ and $\mathscr{O}^\kappa_\infty$. These sheaves have vanishing geometric Sen operator. By geometric Sen theory, we thus get $B(\mathbb{Q}_p)$-equivariant sheaves : $\omega^{\kappa,sm}_\infty$ on $\pi^{-1}_{HT}(\infty)$ and $\omega^{\kappa,sm}_{\mathbb{A}^1}$ on $\pi^{-1}_{HT}(\mathbb{A}^1)$. Let us call $\mathrm{H}^0_{Id}(\kappa) = \mathrm{H}^0(\pi^{-1}_{HT}(\infty), \omega^{\kappa,sm}_\infty)$ and $(\mathrm{H}^1_{w_0})(\kappa) = \mathrm{H}^1_c(\pi^{-1}_{HT}(\mathbb{A}^1), \omega^{\kappa,sm}_{\mathbb{A}^1})$.

Proposition 3.0.1. *The LB spaces* $\mathrm{H}^0_{Id}(\kappa)$ *and* $\mathrm{H}^1_{w_0}(\kappa)$ *are locally analytic representations of* $B(\mathbb{Q}_p)$*. The action of* $\mathfrak{b}$ *is via* $-w_0\kappa$ *and* $-\kappa$ *respectively.*

3.0.2. *The Cousin complex.* The following exact sequence is a first justification for considering support conditions in our cohomologies:

Proposition 3.0.2. *Let* $\kappa \in X^\star(T)$*. Then we have a long exact sequence of* $B(\mathbb{Q}_p)$*-smooth representations :*

$$0 \to \mathrm{H}^0(X_{K^p}\omega^{\kappa,sm}) \to \mathrm{H}^0_{Id}(\kappa) \otimes \mathbb{C}_p(w_0\kappa) \to \mathrm{H}^1_{w_0}(\kappa) \otimes \mathbb{C}_p(\kappa) \to \mathrm{H}^1(X_{K^p}, \omega^{\kappa,sm}) \to 0$$

Proof. Let $j : \pi_{HT}^{-1}(\mathbb{A}^1) \hookrightarrow X_{K^p}$ be the open immersion and $i : \pi_{HT}^{-1}(\{\infty\}) \hookrightarrow X_{K^p}$ be the closed immersion. Then we have an exact sequence :

$$0 \to j_! j^\star \omega_{FL}^{\kappa,sm} \to \omega_{FL}^{\kappa,sm} \to i_\star i^{-1} \omega_{FL}^{\kappa,sm} \to 0.$$

□

3.0.3. *Higher Coleman theory.* A reference is [BP20]. Let $\nu : T(\mathbb{Z}_p) \to \mathbb{C}_p^\times$ be a continuous character. Let $\kappa = \log(\nu) \in X^\star(T)_{\mathbb{C}_p}$. We let $M_{Id,\nu} = (\mathrm{H}^0_{Id}(\kappa) \otimes \mathbb{C}_p(w_0\nu))^{B(\mathbb{Z}_p)}$ and $M_{w_0,\nu} = (\mathrm{H}^1_{Id}(\kappa) \otimes \mathbb{C}_p(\nu))^{B(\mathbb{Z}_p)}$.

Proposition 3.0.3. *The spaces* $M_{Id,\nu}$ *and* $M_{w_0,\nu}$ *have an action of* $U_p = B(\mathbb{Z}_p)\mathrm{diag}(1,p^{-1})B(\mathbb{Z}_p)$*. The* U_p*-operator is compact and all the eigenvalues for* U_p *on* $M_{Id,\nu}$ *have slope* ≥ 1*, all the eigenvalues for* U_p *on* $M_{w_0,\nu}$ *have slope* ≥ 0*.*

Corollary 3.0.4. *Let* $\kappa = (k; -k) \in X^\star(T)$*. If* $k > 1$*, any class in* $M_{Id,\kappa}$ *of slope* $< k-1$ *is classical. If* $k < 1$*, any class in* $M_{w_0,\kappa}$ *of slope* $< 1-k$ *is classical.*

3.0.4. *p-adic families.* We can also construct p-adic families. Let $\mathscr{W}$ be the character space of $T(\mathbb{Z}_p)$. Let ν^{univ} be the universal character. We have a logarithm map : $\log : \mathscr{W} \to X^\star(T)_{\mathbb{C}_p}$. Let $\kappa^{univ} = \log(\nu^{univ})$. We can define families of overconvergent modular forms by taking $(\log^\star \mathrm{H}^0_{Id}(\kappa^{univ}) \otimes \mathscr{O}_{\mathcal{W}}(-w_0\nu^{univ}))^{B(\mathbb{Z}_p)} = M_{Id,\nu^{univ}}$ and $(\log^\star \mathrm{H}^1_{w_0}(\nu^{univ}) \otimes \mathscr{O}_{\mathcal{W}}(-\nu^{univ}))^{B(\mathbb{Z}_p)} = M_{w_0,\nu^{univ}}$.

Proposition 3.0.5. $M_{Id,\nu^{univ}}$ *and* $M_{w_0,\nu^{univ}}$ *define sheaf of LBan-modules over* $\mathscr{W}$ *(which are locally of the form* $\mathscr{O}_{U_i}\hat{\otimes}_{\mathbb{C}_p}V_i$ *for* V_i *an inductive limit of orthonormalizable Banach module).*

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