Companion forms and ordinary prismatic F-gauges

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Abstract

We introduce a cohomology theory of tame modular forms on the modular curves in positive characteristics and use it to give a uniform proof of the existence of companion forms, including the remaining open case $p = 2$ and Frobenius has repeated eigenvalues. We also construct certain companion forms on the Siegel threefold. This relies on the description of the obstruction to lift Frobenius modulo p^2 on Siegel Shimura varieties and recent advances in integral p-adic Hodge theory.

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1 Introduction

The first aim of this paper is to revisit the theory of companion forms for the modular curves, due to Gross [Gro90], Coleman-Voloch [CV92] and Faltings-Jordan [FJ95]. We let p be a prime number, N be an integer $(p, N) = 1$, and $\epsilon_p : G_{\mathbb{Q}} \to \mathbb{F}_p^{\times}$ be the mod p cyclotomic character. The main theorem is the following.

Theorem 1.1. *Let $X \to \mathrm{Spec}\ \mathbb{F}_p$ be the compactified modular curve of level $\Gamma_1(N)$. Let $2 \leq k \leq p$. Let $f \in \mathrm{H}^0(X, \omega^k)$ be an eigenform, and let $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be the attached Galois representation. Then $\rho_f|_{I_{\mathbb{Q}_p}} = 1 \oplus \epsilon_p^{1-k}$ if and only if there exists an eigenform $g \in \mathrm{H}^0(X, \omega^{p+1-k})$ such that $\rho_g \otimes \epsilon_p^{1-k}$.*

Gross proved the theorem under the hypothesis that the eigenvalues of a Frobenius at p are distinct when $k = p$. Coleman and Voloch proved the theorem under the assumption that $k > 2$. Faltings and Jordan proved the theorem when $k < p$. In this paper we give a uniform proof and settle the remaining case when $p = 2$ and the Frobenius has repeated eigenvalues (see Corollary 4.7).

The methods of Gross and Coleman-Voloch are specific to curves and rely on the p-adic Hodge theory of ordinary p-divisible groups and on Dwork's conjecture proved by Katz. Faltings-Jordan apply more general étale and crystalline integral comparison theorem but they exclude the limit case $k = p$ which is very interesting : it gives a criterium for the existence of a mod p weight one form.

There has been few generalizations of the geometric construction of companion forms to other Shimura varieties. We are aware of the case of Shimura curves [Gee08], as well as the partial results on the Siegel threefold [Til12].

On the other hand, modular lifting technics have been extremely powerful in proving the existence of companion forms for other Shimura varieties [Gee07], [GG12]. However these results only apply in regular weight at the moment, as they rely on the possibility to lift the Galois representations to characteristic zero in that prescribed weight.

In this paper we use a geometric method to study companion forms. As in Faltings-Jordan, we use étale and crystalline integral comparison theorem. Using the theory of prismatization, we are able to treat the case $k = p$ as well. In fact, by studying F-gauges arising from the cohomology of modular curves, we are led to introduce (see Definition 4.1) new sheaves $\omega^{k,tm}$ on modular curves, of *tame* weight k-modular forms for $2 \leq k \leq p$. We prove the following theorem regarding the cohomology of these sheaves (which easily implies Theorem 1.1):

Theorem 1.2 (Coro 4.4, Thm 4.4)**.** *Let $2 \leq k \leq p$. There is a vector bundle $\omega^{k,tm}$ on X, whose cohomology is equipped with an action of the Hecke algebra. Moreover, the following holds:*

1. *(modular description) There is a natural injective map*

$$\mathrm{H}^0(X, \omega^{k,tm}) \to \mathrm{H}^0(X, \omega^k) \times \mathrm{H}^0(X, \omega^{k+1-p}).$$

   Eigenforms in $\mathrm{H}^0(X, \omega^{k,tm})$ are pairs of companion forms (f, g) of weight k and $p+1-k$, $(0, \mathrm{Ha})$ when $k = 2$, $(\mathrm{Ha}, 0)$ when $k = p-1$, $(\mathrm{Ha}h, 0)$ when $k = p$, where Ha is the Hasse invariant and h is a weight one eigenform.

2. *(Galois description) Let $f \in \mathrm{H}^0(X, \omega^k)$ be an eigenform, and assume that ρ_f is irreducible. Let $\mathfrak{m}$ be the maximal ideal of the Hecke algebra corresponding to f. Let $\mathbb{T}_{\mathfrak{m}}$ be the Hecke algebra acting faithfully on $\mathrm{H}^0(X, \omega^k)_{\mathfrak{m}}$ and $\mathbb{T}^{tm}_{\mathfrak{m}}$ be its quotient acting faithfully on $\mathrm{H}^0(X, \omega^{k,tm})_{\mathfrak{m}}$. Let $\rho_{\mathbb{T}_{\mathfrak{m}}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}})$ be the universal modular Galois representation. The induced Galois representation $\rho_{\mathbb{T}^{tm}_{\mathfrak{m}}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}^{tm}_{\mathfrak{m}})$ is tamely ramified on inertia at p. If $k \neq p$ or $k = p$ but $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is not a scalar representation, $\mathbb{T}^{tm}_{\mathfrak{m}}$ is the initial quotient of $\mathbb{T}_{\mathfrak{m}}$ with this property. In the remaining case $k = p$ and $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is a scalar representation, $\mathbb{T}^{tm}_{\mathfrak{m}} \neq 0$.*

Remark 1.1. When $k = p$, tamely ramified means unramified. We prove that $\mathrm{R}\Gamma(X, \omega^{p,tm}) = \mathrm{R}\Gamma(X, \omega^1)^{\oplus 2}$ (Prop 4.3). Thus, Theorem 1.2 2. implies that the Galois representation supported by $\mathrm{R}\Gamma(X, \omega^1)$ is unramified (this is known by [Wie14], [CG18], [CS19], but our proof is completely geometric).

Remark 1.2. We construct in Corollary 4.2 a Serre duality pairing $\mathrm{H}^0(X, \omega^{k,tm}) \times \mathrm{H}^1(X, \omega^{k,tm}) \to \mathbb{F}_p$ which induces a perfect pairing on the non-Eisenstein localization of cohomology. The property that $\omega^{k,tm}$ has cohomology in two degrees is consistent with the fact that tamely ramified local deformation rings have codimension one inside ordinary deformation rings.

Remark 1.3. We are informed that Yicheng Yang has independently found a definition of the sheaves $\omega^{k,tm}$ in his PhD thesis and has lifted them to the modular curves over $\mathbb{Z}_p$. One could further ask for a p-adic variation in the weight k. The corresponding cohomology should then recover the bi-ordinary complex of [CWE25].

To further illustrate our method, we construct some companion forms for the group GSp_4.

Theorem 1.3 (Thm 4.7)**.** *Let $X \to \mathrm{Spec}\ \mathbb{F}_p$ be a toroidal compactification of the Siegel threefold of prime-to-p level. Let $f \in \mathrm{H}^0(X, \omega^3)$ be an eigenform. Assume that:*

1. *$p \geq 5$,*
2. *The Galois representation $r_f : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\bar{\mathbb{F}}_p)$ is irreducible.*
3. *There is an unramified prime ℓ such that $r_f(\mathrm{Frob}_\ell)$ is decomposed generic.*
4. *The form f is ordinary at p and Klingen generic.*
5. *We have*

$$r_f|_{I_{\mathbb{Q}_p}} \simeq \begin{pmatrix} 1 & 0 & \star & \star \\ 0 & \epsilon_p^{-1} & \star & \star \\ 0 & 0 & \epsilon_p^{-2} & 0 \\ 0 & 0 & 0 & \epsilon_p^{-3} \end{pmatrix}$$

Then there exists a companion form $g \in \mathrm{H}^0(X, \Omega^2_X(\log\ D) \otimes \omega^{(p-1)})$ which satisfies $r_g \otimes \epsilon_p^{-1} = r_f$.

Let us briefly sketch the argument for modular curves and $k = 2$. Let $\mathfrak{X}$ be the modular curve over $\mathbb{Z}_p$. There is a mod p prismatic F-gauge $\mathrm{H}^1_{syn}(\mathfrak{X}) \otimes \mathbb{F}_p$ (this is equivalent to the p-torsion of the Jacobian of $\mathfrak{X}$, the theory of F-gauges is necessary to treat general weights). The étale realization functor recovers $\mathrm{H}^1_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, \mathbb{F}_p)$ which contains the representation $\rho_f|_{G_{\mathbb{Q}_p}}$. The idea is

to find a simple invariant of the F-gauge which controls the wild ramification of $\mathrm{H}^1_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, \mathbb{F}_p)$. By [TVX25], we can attach to this F-gauge a Fontaine-Laffaille module $\mathrm{H}^1_{dR}(X)$ with Frobenius ϕ, filtration $\mathrm{H}^0(X, \Omega^1_X)$, and divided Frobenius ϕ_1. There is a subtlety when $p = 2$ (the most interesting case) as there are two Fontaine-Laffaille modules naturally attached to the F-gauge, but only one of them will carry enough information. The invariant we look for is simply the map $\bar{\phi}_1 : \mathrm{H}^0(X, \Omega^1_X) \xrightarrow{\phi_1} \mathrm{H}^1_{dR}(X) \to \mathrm{H}^1(X, \mathscr{O}_X)$ obtained by applying ϕ_1 and reducing modulo the Hodge filtration. The map $\bar{\phi}_1$ originates from the map of sheaves $\Omega^1_X \to F_\star \mathscr{O}_X[1]$ which is the obstruction to lift Frobenius modulo p^2. We thus define $\omega^{2,tm} = \mathrm{Fib}(\Omega^1_X \to F_\star \mathscr{O}_X[1])$ (up to some cuspidality conditions). By construction, the cohomology of $\omega^{2,tm}$ will lead to tame Galois representations. It remains to describe the cohomology groups and make the connection with companion forms. In order to achieve this, we give a description of the obstruction to lift Frobenius modulo p^2. In fact we prove the following general result on Siegel modular varieties. Note the appearance of the companion weight $p-1$ of 2 in this theorem.

Theorem 1.4 (Thm 3.3, Thm 3.4)**.** *Let $X \to \mathrm{Spec}\ \mathbb{F}_p$ be a toroidal compactification of the Siegel modular variety of level prime-to-p. Let $Z \hookrightarrow X$ be the non-ordinary locus. There is a factorization of the obstruction to lift Frobenius $F^\star\Omega^1_X \to \mathscr{O}_X[1]$ on the Siegel modular variety modulo p^2 as*

$$F^\star\Omega^1_X \to F^\star\Omega^1_X|_Z \to \omega^{(p-1)}|_Z \to \mathscr{O}_X[1]$$

where $F^\star\Omega^1_X \to F^\star\Omega^1_X|_Z$ is the restriction map, $\omega^{(p-1)}|_Z \to \mathscr{O}_X[1]$ is the co-restriction map. Moreover, the composition $F^\star\omega^{-(p-1)}|_Z \xrightarrow{F^\star \mathrm{d}} F^\star\Omega^1_X|_Z \to \omega^{(p-1)}|_Z$ (where the first map is the differential) is the weight p^2-1 Hasse invariant of Z.

This is our main tool for understanding the cohomology of the sheaves $\omega^{k,tm}$ in the case of the modular curve, and also for the generalization to GSp_4.

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2 Comparison theorems

Let k be a perfect field of characteristic p. We have a category $\mathrm{Vect}_{[0,a]}(W(k)^{syn} \otimes \mathbb{F}_p)$ of mod p prismatic F-gauges over $W(k)$, with Hodge-Tate weights in the interval $[0, a]$ and there is an étale realization functor $\mathcal{T}_{et} : \mathrm{Vect}_{[0,a]}(W(k)^{syn} \otimes \mathbb{F}_p) \to \mathrm{Rep}_{G_K}(\mathbb{F}_p)$ where G_K is the Galois group of $K = W(k)[1/p]$ ([BL22], [Dri24], [Bha22]). By [TVX25], there is an equivalence of categories $\Phi^{Maz} : \mathrm{Vect}_{[0,p-2]}(W(k)^{\mathrm{syn}} \otimes \mathbb{F}_p) \to \mathcal{MF}_{[0,p-2]}(k)$ to the category of Fontaine-Laffaille modules with weights in the range $[0, p-2]$. The functor $\mathcal{T}_{et} \circ (\Phi^{Maz})^{-1}$ is already considered in [FL82] and is fully faithfull. Let $\mathfrak{X} \to \mathrm{Spec}\ W(k)$ be a proper and smooth scheme of dimension $d \leq p-2$, with special fiber X. The syntomic cohomology $\mathrm{H}^i_{syn}(\mathfrak{X}) \otimes \mathbb{F}_p$ belong to $\mathrm{Vect}_{[0,p-2]}(W(k)^{syn} \otimes \mathbb{F}_p)$. We have $\Phi^{Maz}(\mathrm{H}^i_{syn}(\mathfrak{X}) \otimes \mathbb{F}_p) = \mathrm{H}^i_{dR}(X)$ equipped with its Hodge filtration and divided crystalline Frobenius (or equivalently Deligne-Illusie's divided Frobenius, see section 2.5.4). The isomorphism $\mathcal{T}_{et} \circ (\Phi^{Maz})^{-1}(\mathrm{H}^i_{dR}(X)) = \mathrm{H}^i_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{F}_p)$ is proven already in [Fal89].

The main purpose of this section is to investigate the case of Hodge-Tate weights in the range $[0, p-1]$. We recall that [TVX25] have constructed a functor $\Phi^{DI} : \mathrm{Vect}_{[0,p-1]}(W(k)^{\mathrm{syn}} \otimes \mathbb{F}_p) \to \mathcal{MF}_{[0,p-1]}(k)$ which is not too far from being an equivalence, and is equal to Φ^{Maz} on

$\mathrm{Vect}_{[0,p-2]}(W(k)^{syn} \otimes \mathbb{F}_p)$. In geometric context, this functor uses the Deligne-Illusie divided Frobenius on $\mathrm{H}^i_{dR}(X_k)$, which may differ from the divided crystalline Frobenius if $\dim(X) = p-1$.

Our main result is the following. Let $\epsilon_p : G_K \to \mathbb{F}_p^\times$ be the mod p cyclotomic character. We let V_1, V_2 be unramified G_K-representations on finite dimensional $\mathbb{F}_p$-vector spaces. Let $1 \leq a \leq p-1$. We consider the group $\mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-a}, V_2)$ of ordinary Galois representation (which are also finite flat when $a = 1$). This group is isomorphic to the group of ordinary F-gauges $\mathrm{Ext}^1_{W(k)^{syn}\otimes\mathbb{F}_p}(\mathcal{V}_1 \otimes \mathscr{O}\{-a\}, \mathcal{V}_2)$. Using the functor Φ^{DI} we construct a map (abusively denoted again Φ^{DI}):

$$\Phi^{DI} : \mathrm{Ext}^1_{W(k)^{syn}\otimes\mathbb{F}_p}(\mathcal{V}_1 \otimes \mathscr{O}\{-a\}, \mathcal{V}_2) \to \mathrm{Hom}_k(M(V_1)^{(p)}, M(V_2))$$

with $M(V_i) = (V_i \otimes_{\mathbb{F}_p} \bar{k})^{G_K}$ the unramified Fontaine-Laffaille module of V_i. We prove in Theorem 2.4 that the above map is:

1. an isomorphism if $1 \leq a \leq p-2$,
2. surjective with kernel the unramified extensions when $a = p-1$ and $\epsilon_p^{p-1} = 1$.

In other words, we attach a simple invariant to an ordinary F-gauge of weight 0 and a, which detects wether the corresponding Galois representation is split on inertia. Besides proving Theorem 2.4, we also explain how to compute, in geometric context, the functor Φ^{DI} of [TVX25] by considering crystalline cohomology over $W_p(k)$ where p is equipped with the trivial divided power (given by $\gamma_p(p) = 0$) (Section 2.3) and how this relates to the Deligne-Illusie theory (Proposition 2.3).

2.1 Fontaine-Laffaille modules

We let F denote the Frobenius of the ring of Witt vectors $W(k)$ of the perfect field k of characteristic p. We let $\mathbb{A}^1_- = \mathrm{Spf}\ W(k)[v_-]$. We let $\mathbb{G}_m$ act in such a way that v_- has weight -1. We have a map $\mathbb{A}^1_-/\mathbb{G}_m \to B\mathbb{G}_m$, classifying a line bundle $\mathscr{O}(1)$. We have a universal map $v_- : \mathscr{O}(1) \to \mathscr{O}$ over $\mathbb{A}^1_-/\mathbb{G}_m$. The category $\mathcal{D}_{qc}(\mathbb{A}^1_-/\mathbb{G}_m)$ identifies with the derived category of p-complete garded modules over $W(k)[v_-]$ via

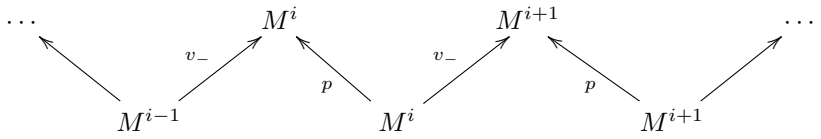
$$\mathcal{M} \mapsto (\oplus_i M^i, v_- : M^i \to M^{i-1})$$

with $M^i = \mathrm{R}\Gamma(\mathbb{A}^1_-/\mathbb{G}_m, \mathcal{M} \otimes \mathscr{O}(i))$.

For any $a \leq b \in \mathbb{Z}$, we let $\mathcal{D}_{qc,[a,b]}(\mathbb{A}^1_-/\mathbb{G}_m)$ for the sub-category of those $\mathcal{M}$ for which M^i is acyclic for $i \geq b$ and $M^i \to M^{i-1}$ is a quasi-isomorphism for $i \leq a$.

Definition 2.1 ([TVX25], Def 2.4)**.** *A small Fontaine-Laffaille module is an object $\mathcal{M} \in \mathrm{Coh}(\mathbb{A}^1_-/\mathbb{G}_m)$, together with an isomorphism $\phi : F^\star i_p^\star \mathcal{M} \to i_1^\star \mathcal{M}$, where $i_p, i_1 : \mathrm{Spf}\ W(k) \to \mathbb{A}^1_-$ are the two points given by $v_- = p$ and $v_- = 1$. We denote this category by $\mathcal{MF}(W(k))$.*

One can attach to $(\mathcal{M}, \phi)$ more familiar objects as in [FL82]. We let $M^i = \mathrm{H}^0(\mathbb{A}^1_-/\mathbb{G}_m, \mathcal{M}(i))$. We have maps $v_- : M^i \to M^{i-1}$, and we let $M = \mathrm{colim}_{i\to-\infty} M^i$. Then M is a finitely generated $W(k)$-module, with a finite, exhaustive filtration M^i. We have $i_1^\star \mathcal{M} = M$, while $i_p^\star \mathcal{M}$ identifies with the colimit of the diagram



Thus, we can attach to ϕ the F-linear maps $\phi_i : M^i \to M$ satisfying $\phi_i|_{M^{i+1}} = p\phi_{i+1}$. One sees that the following extra properties hold:

1. $v_- : M^i \to M^{i-1}$ is a split injection,
2. $M = \oplus_i \phi_i(M^i)$.

The association $(\mathcal{M},\phi) \to (M, M^i, \phi_i)$ induces an equivalent description of $\mathcal{MF}(W(k))$ ([TVX25, Rem 2.5]). We let $\mathcal{MF}(k)$ be the subcategory of $\mathcal{MF}(W(k))$ consisting of p-torsion objects. In that case, we denote by $\bar{M}^i = M^i/M^{i+1}$. Note that $\oplus_i \phi_i : \oplus_i \bar{M}^i \to M$ is an isomorphism.

Following [TVX25, Def 2.6], we let $\mathcal{DMF}^{big}(W(k)) = \mathrm{Eq}(\mathcal{D}_{qc}(\mathbb{A}^1_-/\mathbb{G}_m) \rightrightarrows \mathcal{D}_{qc}(\mathrm{Spec}\ W(k)))$ where the first pullback is the fiber at 1 and the second pullback is given by the fiber at p composed with Frobenius. By [TVX25, Prop 2.7], the category $\mathcal{MF}(W(k))$ is abelian, there is a natural functor $\mathcal{MF}(W(k)) \to \mathcal{DMF}^{big}(W(k))$ which extends to a fully faithfull functor $\mathcal{D}^b(\mathcal{MF}(W(k))) \to \mathcal{DMF}^{big}(W(k))$ with essential image those pairs $(\mathcal{M},\phi)$ with $\mathcal{M} \in \mathcal{D}^b(\mathrm{Coh}(\mathbb{A}^1_-/\mathbb{G}_m))$. By $\mathcal{MF}_{[a,b]}(W(k))$, $\mathcal{DMF}^{big}_{[a,b]}(W(k))$ ... we mean the subcategories consisting of pairs $(\mathcal{M},\phi)$ with $\mathcal{M} \in \mathcal{D}_{qc,[a,b]}(\mathbb{A}^1_-/\mathbb{G}_m)$.

2.2 Syntomification

We will very briefly summarize some constructions of [BL22], [Dri24], [Bha22]. We work in the category of $W(k)$-algebras over which p is nilpotent. We let W denote the ring scheme of p-typical Witt vectors. We let $\mathbb{G}_a$ be the additive group scheme. There is a ring scheme map $W \to \mathbb{G}_a$ (called the 0-th ghost component). We let $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ be the PD-envelop of the 0-section in $\mathbb{G}_a$. Concretely, if $\mathbb{G}_a = \mathrm{Spf}\ W(k)[x]$, then $\mathbb{G}_a^\sharp = \mathrm{Spf}\ W(k)[\frac{x^n}{n!}, n \geq 0]$. Note that $\mathbb{G}_a$ acts on $\mathbb{G}_a^\sharp$ and the map $\mathbb{G}_a^\sharp \to \mathbb{G}_a$ is a quasi-ideal. Moreover, we have an isomorphism of quasi-ideals $W[F] \to \mathbb{G}_a \simeq \mathbb{G}_a^\sharp \to \mathbb{G}_a$ ([Bha22, Lem 2.6.1]). For any p-nilpotent ring R, $\mathbb{A}^1_-/\mathbb{G}_m(R)$ parametrizes an invertible R-module $L = \mathscr{O}(1)|_R$ and an R-module map $v_- : L \to R$. We let $V(L) = \mathrm{Spec}(\oplus_{i\geq 0} L^{-i})$ be the geometric realization of L. We let $V(L)^\sharp$ be the PD-envelop of $V(L)$ along the zero section.

2.2.1 De Rham stack

The ring stack $\mathbb{G}_a^{dR}$ is the sheafification of $R \mapsto \mathrm{cone}(\mathbb{G}_a^\sharp(R) \to R)$. Let X be a p-adic formal scheme. We let $X^{dR}(R) = X(\mathbb{G}_a^{dR}(R))$. There is another model for $\mathbb{G}_a^{dR}$ (see [Bha22, Coro 2.6.8]). Namely, we have the following commutative diagram where all vertical complexes are quasi-isomorphic:

$$\begin{array}{ccccc}
\mathbb{G}_a^\sharp & \longleftarrow & \mathbb{G}_a^\sharp \oplus F_\star W & \longrightarrow & F_\star W \\
\downarrow & & \downarrow{\scriptstyle can,V} & & \downarrow{\scriptstyle p} \\
\mathbb{G}_a & \longleftarrow & W & \xrightarrow{F} & F_\star W
\end{array}$$

We refer to $\mathrm{cone}(F_\star W \xrightarrow{p} F_\star W)$ as the Witt vector model for $\mathbb{G}_a^{dR}$. This second model exhibits more automorphisms of the de Rham stack. We consider the subgroup $W^\times\{p\}$ of $W^\times$ of elements y satisfying $py = p$. This group acts on $\mathbb{G}_a^{dR}$. Indeed, for any $y \in W^\times\{p\}$, the diagram:

$$\begin{array}{ccc}
F_\star W & \xrightarrow{y} & F_\star W \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
F_\star W & \xrightarrow{Id} & F_\star W
\end{array}$$

is commutative. We therefore get an action of $W^\times\{p\}$ on X^{dR}.

Remark 2.1. The group $W^\times\{p\}$ restricted to k-algebras is the group $W^\times[F] = \{x \in W, F(x) = 1\} = \mathbb{G}_m^\sharp$. Indeed, if $F(x) = 1$ then $px = VF(x) = V(1) = p$. Conversely, if $px = p$ then $V(F(x)) = V(1)$. Since V is injective, $F(x) = 1$. The action of $W^\times[F]$ on X^{dR} (viewed as a stack on k-algebras) is considered by Drinfeld ([Bha22, Rem 2.7.13]).

There is a filtered version of the de Rham stack over $\mathbb{A}^1_-/\mathbb{G}_m$ ([Bha22, Def 2.5.3]). We let $\mathbb{G}_a^{dR,+} = \mathrm{cone}(V(L)^\sharp \oplus F_\star W \xrightarrow{v_-^\sharp, V} W)$. We let $X^{dR,+} = X(\mathbb{G}_a^{dR,+})$. We have a canonical isomorphism $X^{dR,+} \times_{\mathbb{A}^1_-/\mathbb{G}_m} \mathbb{G}_m/\mathbb{G}_m = X^{dR}$.

2.2.2 Crystalline stack

We consider $\mathbb{G}_a^\sharp \times_{\mathbb{G}_a, x=p} \mathrm{Spf}\ W(k) = \mathrm{Spf}\ W(k)[\frac{x^i}{i!}, i \geq 0]/(x-p)$. An R-point of this space is equivalent to the data of divided powers $\gamma(p) = \{\gamma_i(p), i \geq 0\}$ on p, or to an R-linear map $\gamma : pR \to \mathbb{G}_a^\sharp(R)$, lifting the map $pR \to R$.

Lemma 2.1. *There is a natural bijection $\psi : \mathbb{G}_a^\sharp \times_{\mathbb{G}_a, x=p} \mathrm{Spf}\ W(k) \simeq W^\times\{p\}$ given by $p - \gamma(p) = V(\psi(\gamma))$.*

Proof. Let γ be divided powers on pR for a p-nilpotent ring R. We have $\gamma(p) \in \mathbb{G}_a^\sharp(R) \simeq W[F](R)$. Since $p - \gamma(p) \in W(R)$ has trivial 0-th ghost component, we deduce that $V(y) = p - \gamma(p)$ for a unique $y \in W(R)$. Applying Frobenius, we deduce that $py = p$. We now check that y is a unit. It suffices to see that the 0-th ghost component y_0 of y is a unit. Let $\mathfrak{m}$ be a maximal ideal of R. We can reduce the identity $py = p$ in $W(R/\mathfrak{m})$ from which we deduce that $y = 1$ in $W(R/\mathfrak{m})$. Therefore, y_0 is a unit modulo $\mathfrak{m}$ for all maximal ideal $\mathfrak{m}$, hence y is a unit. We therefore get a map $\gamma \mapsto y$. Conversely, for any $y \in W^\times\{p\}$, we let $\gamma(p) = p - V(y)$. It is clear that $\gamma(p) \in W[F](R)$ and this defines divided powers on p. □

The standard divided powers are given by $\gamma_i^{std}(p) = \frac{p^i}{i!} \in W(k)$ for all $i \geq 0$. These are the only divided powers on $pW(k) \to W(k)$. We can also define trivial divided powers on $pW_p(k) \to W_p(k)$, by $\gamma_i^{tri}(p) = \frac{p^i}{i!}$ if $i \leq p-1$ and $\gamma_i^{tri}(p) = 0$ if $i \geq p$. Moreover, $\gamma^{tri} = \gamma^{std}$ on $pW_{p-1}(k)$.

Lemma 2.2. *We have $\gamma^{std}(p) = p - V(1) \in W[F](W(k))$ and $\psi(\gamma^{std}) = 1$. We have $\gamma^{tri}(p) = [p] \in W[F](W_p(k))$ and $\psi(\gamma^{tri}) = 1 - \frac{[p^p]}{p}$ where $\frac{[p^p]}{p}$ is the image in $W(W_p(k))$ of the unique element in $W(W(k))$ defined by the equation $p.\frac{[p^p]}{p} = [p^p]$.*

Proof. We first observe that $p - V(1) \in W[F](W(k))$ (since $FV(1) = p$), and has image p via the 0-th ghost component. Since the map $\mathbb{G}_a^\sharp(W(k)) \to \mathbb{G}_a(W(k))$ is injective, we deduce that $\gamma^{std}(p) = p - V(1)$. We turn to the second case. We recall from [Bha22, Lem 2.6.1] how the isomorphism $\mathbb{G}_a^\sharp \simeq W[F]$ is defined. First, $W = \mathrm{Spec}\ W(k)[x_i,\ i \in \mathbb{Z}_{\geq 0}]$, where x_0 is the 0-th ghost component, $F(x_i) = x_i^p + px_{i+1}$, and the n-th ghost component is given $F^n(x_0)$. In these coordinates, the Teichmuller lift of $x \in \mathbb{G}_a$ is given $x_0 = x$, $x_i = 0$ for all $i \geq 1$. In $W[F]$, we have $x_i^p + px_{i+1} = 0$. As a result x_0 has divided powers, $\gamma_{p^n}(x_0) = \epsilon_n x_n$ for a unit ϵ_n. We see that $[p] \in W[F](W_p(k))$, and since $[p] = (p, 0, 0, \cdots)$ in the x_i-coordinates, it follows that this corresponds to the trivial divided powers. It is clear that $\psi(\gamma^{std}) = 1$. We have $[p] - \gamma^{std}(p) = V(y)$ for a unique $y \in W(W(k))$. Applying Frobenius, we get $[p^p] = py$, hence $y = \frac{[p^p]}{p}$. We deduce that $p - V(1) = [p] - V(\frac{[p^p]}{p})$. As a result, $p - \gamma^{tri}(p) = V(1 - \frac{[p^p]}{p})$. □

Let $X \to \mathrm{Spec}\ k$ be scheme. Consider the thickening $W_n(k) \to k$ with kernel (p). Let $\gamma = \{\gamma_i : (p) \to W_n(k), i \geq 0\}$ be divided powers. Using divided powers, we can equip $\mathbb{G}_a^{dR}$ with a structure of k-algebra stack as follows. For any $W_n(k)$-algebra R, we consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{G}_a^\sharp(R) & \longrightarrow & R \\ \uparrow & & \uparrow \\ W(k) & \xrightarrow{p} & W(k) \end{array}$$

where the map $W(k) \xrightarrow{\gamma} \mathbb{G}_a^\sharp(W_n(k)) \to \mathbb{G}_a^\sharp(R)$ sends 1 to the divided powers γ on p. We let $(X/(W_n(k), \gamma))^{crys} = X(\mathbb{G}_a^{dR}(R))$. In case $\gamma = \gamma^{std}$, we can pass to the limit over n and define $(X/W(k))^{crys} = \lim_n (X/(W_n(k), \gamma^{std}))^{crys}$.

Proposition 2.1. *Let $\tilde{X}$ be a flat lift of X to $W_n(k)$, we have a canonical isomorphism of stacks $(X/(W_n(k), \gamma))^{crys} = \tilde{X}^{dR}$.*

Proof. We reduce to the affine case, $X = \operatorname{Spec} A$, $\tilde{X} = \operatorname{Spec} \tilde{A}$. We find that $\operatorname{Hom}_{W_n(k)}(\tilde{A}, \mathbb{G}_a^{dR}(R)) = \operatorname{Hom}_k(A, \mathbb{G}_a^{dR}(R))$ by adjunction. □

Proposition 2.2. *For any two choices of divided powers γ and γ', there is a canonical isomorphism $(X/(W_n(k), \gamma))^{crys} \to (X/(W_n(k), \gamma'))^{crys}$.*

Proof. We let $\gamma(p) \in \mathbb{G}_a^\sharp(W_n(k)) = W[F](W_n(k))$ be the image of p. By Lemma 2.1, $p - \gamma(p) = V(\psi(\gamma))$. The k-algebra structure on $\mathbb{G}_a^{dR}$ is given in the Witt vector model by the map

$$\begin{array}{ccc} F_\star W & \xrightarrow{p} & F_\star W \\ {\scriptstyle \psi(\gamma)}\uparrow & & \uparrow \\ W(k) & \xrightarrow{p} & W(k) \end{array}$$

This follows by considering the map $W(k) \to \mathbb{G}_a^\sharp \oplus F_\star W$, $1 \mapsto (\gamma(p), \psi(\gamma))$, which gives the k-algebra structure in the model $\operatorname{cone}(\mathbb{G}_a^\sharp \oplus F_\star W \to W)$. Similarly the k-algebra structure for the choice of γ' is given by an element $\psi(\gamma')$. Multiplication by $\psi(\gamma')\psi(\gamma)^{-1}$ on $\mathbb{G}_a^{dR}$ induces an isomorphism $(X/(W_n(k), \gamma))^{crys} \to (X/(W_n(k), \gamma'))^{crys}$. □

Remark 2.2. Let $\tilde{X}$ be a flat lift of X to $W_n(k)$ and let γ, γ' be divided powers on $pW_n(k)$. Then we get a commutative square (where the vertical maps are those of Proposition 2.1 and the top horizontal one is from Proposition 2.2):

$$\begin{array}{ccc} (X/W_n(k), \gamma)^{crys} & \longrightarrow & (X/W_n(k), \gamma')^{crys} \\ \downarrow & & \downarrow \\ \tilde{X}^{dR} & \xrightarrow{\psi(\gamma')\psi(\gamma)^{-1}} & \tilde{X}^{dR} \end{array}$$

2.2.3 Syntomification

We refer to [Bha22, Chap 5] for details and more references. There is a stack $W(k)^\triangle$ which parametrizes Cartier-Witt divisors $I \to W(R)$ for a p-nilpotent $W(k)$-algebra R. We have a ring stack $\mathbb{G}_a^\triangle \to W(k)^\triangle$ given by taking $I \to W(R)$ to $\operatorname{cone}(I \to W(R))$. For any p-adic formal scheme, we let $X^\triangle(R) = X(\mathbb{G}_a^\triangle(R))$. There is a map $k^\triangle \to W(k)^\triangle$ induced by functoriality from $\operatorname{Spec} k \to \operatorname{Spf} W(k)$. Moreover $k^\triangle = \operatorname{Spf} W(k)$ corresponds to the Cartier-Witt divisor $W(k) \xrightarrow{p} W(k)$.

There is a stack $W(k)^\mathcal{N}$, equipped with a Rees map $t : W(k)^\mathcal{N} \to \mathbb{A}^1_-/\mathbb{G}_m$. For any p-nilpotent algebra R, we recall that $W(k)^\mathcal{N}(R)$ parametrizes an admissible W_R-module:

$$0 \to V(L)^\sharp \to M \to F_\star M' \to 0$$

and a map $\xi : M \to W_R$ such that the induced map $\bar{\xi} : M' \to W_R$ is a Cartier-Witt divisor. The map $\xi|_{V(L)^\sharp} : V(L)^\sharp \to \mathbb{G}_a^\sharp$ arises from a map $L \to R$ (giving the Rees map). The map $\bar{\xi}$ gives a structural map $\pi : W(k)^\mathcal{N} \to W(k)^\triangle$. There are two open immersions $j_{+/-} : W(k)^\triangle \to W(k)^\mathcal{N}$. The open immersion j_+ is given by sending $I \to W(R)$ to $I \otimes_{W(R)} W_R \to W_R$. The open immersion j_- is given by sending $I \to W(R)$ to $N \to W_R$ defined by the pull-back diagram:

$$\begin{array}{ccc} N & \longrightarrow & F_\star(I \otimes_{W(R)} W_R) \\ \downarrow & & \downarrow \\ W_R & \longrightarrow & F_\star W_R \end{array}$$

We define a ring stack $\mathbb{G}_a^{\mathcal{N}}(R) = \mathrm{R}\Gamma(R, \mathrm{cone}(M \to W_R))$ over $W(k)^{\mathcal{N}}$. For X a p-adic formal scheme, we define $X^{\mathcal{N}} = X(\mathbb{G}_a^{\mathcal{N}})$, over $W(k)^{\mathcal{N}}$. We define X^{syn} by the pushout:

$$\begin{array}{ccc} X^{\triangle} \coprod X^{\triangle} & \xrightarrow{j_+, j_-} & X^{\mathcal{N}} \\ \downarrow & & \downarrow \\ X^{\triangle} & \longrightarrow & X^{syn} \end{array}$$

2.2.4 The crystalline point

There is a map $\mathfrak{p}_{cris} : k^{\mathcal{N}} \to W(k)^{\mathcal{N}}$ induced by functoriality from Spec $k \to$ Spf $W(k)$. In fact $k^{\mathcal{N}} =$ Spf $W(k)[v_+, v_-]/(v_+v_- - p)/\mathbb{G}_m$. The identification is as follows. Let Spf $R \to$ Spf $W(k)[v_+, v_-]/(v_+v_- - p)/\mathbb{G}_m$. We have $M_{cris,R} = \mathrm{coker}(\mathbb{G}_a^\sharp \xrightarrow{v_+^\sharp, -can} V(L)^\sharp \oplus W_R)$, and $\xi : M_{cris,R} \to W_R$ is induced by $V(L)^\sharp \oplus W_R \xrightarrow{can \circ v_-^\sharp, p} W_R$. There is a structural map:

$$\begin{array}{ccc} W(k) & \xrightarrow{p} & W(k) \\ \downarrow & & \downarrow \\ M_{cris,R} & \xrightarrow{\xi} & W_R \end{array}$$

where the vertical map are given by $W(k) \xrightarrow{0, can} V(L)^\sharp \oplus W_R/\mathbb{G}_a^\sharp$ and $W(k) \xrightarrow{can} W_R$.

Lemma 2.3. *Let X be a k-scheme. We have $X^{\mathcal{N}} \times_{\mathbb{A}^1_-/\mathbb{G}_m} \mathbb{G}_m/\mathbb{G}_m = (X/W(k))^{crys}$.*

Proof. The fiber of $\mathbb{G}_a^{\mathcal{N}}$ over $k^{\mathcal{N}} \times_{\mathbb{A}^1_-/\mathbb{G}_m} \mathbb{G}_m/\mathbb{G}_m =$ Spf $W(k)$ (given by $v_- = 1$) is $\mathbb{G}_a^{dR}$, with k-algebra structure given by the standard divided powers. □

For a k-scheme X, the stack $X^{\mathcal{N}}$ (and X^{syn}) geometrizes crystalline cohomology equipped with the Nygaard filtration. For an object $\mathcal{M} \in \mathcal{D}_{qc}(k^{\mathcal{N}})$, we let $M^i = \mathrm{H}^0(k^{\mathcal{N}}, \mathcal{M}(i))$. We have maps $v_+ : M^i \to M^{i+1}$, $v_- : M^i \to M^{i-1}$. We let $\mathcal{D}_{qc,[a,b]}(k^{\mathcal{N}})$ for the category of those $\mathcal{M}$ with $v_+ : M^i \to M^{i+1}$ a quasi-isomorphism for $i \geq b$ and $v_- : M^i \to M^{i-1}$ a quasi-isomorphism for $i \leq a$. We let $\mathcal{D}_{qc,[a,b]}(W(k)^{syn})$ for the category of objects which pull back to $\mathcal{D}_{qc,[a,b]}(k^{\mathcal{N}})$.

2.2.5 The de Rham point

There is a map $\mathfrak{p}_{dR} : \mathbb{A}^1_-/\mathbb{G}_m \to W(k)^{\mathcal{N}}$. This map can be described as follows. Let Spf $R \to \mathbb{A}^1_-/\mathbb{G}_m$. We have $M_{dR,R} = V(L)^\sharp \oplus F_\star W_R$, and $\xi : M_{dR,R} \to W_R$ is given by $V(L)^\sharp \oplus F_\star W_R \xrightarrow{can \circ v_-^\sharp, V} W_R$. The map $\mathfrak{p}_{dR}$ is a section of the Rees map $t : W(k)^{\mathcal{N}} \to \mathbb{A}^1_-/\mathbb{G}_m$.

Lemma 2.4. *Let $X \to$ Spf $W(k)$, inducing $X^{\mathcal{N}} \to W(k)^{\mathcal{N}}$. We have $X^{\mathcal{N}} \times_{W(k)^{\mathcal{N}}, \mathfrak{p}^{dR}} \mathbb{A}^1_-/\mathbb{G}_m = X^{dR,+}$.*

Proof. The fiber of $\mathbb{G}_a^{\mathcal{N}}$ over $\mathbb{A}^1_-/\mathbb{G}_m \to W(k)^{\mathcal{N}}$ is $\mathbb{G}_a^{dR,+}$. □

2.3 From prismatic F-gauges to Fontaine-Laffaille modules

The main theorems of [TVX25] are the following:

Theorem 2.1 ([TVX25], Thm 3)**.** *1. There is a functor*

$$\Phi^{Maz} : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}) \to \mathcal{DMF}^{big}_{[0,p-1]}(W(k))$$

which restricts to an equivalence of categories $\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}) \to \mathcal{DMF}^{big}_{[0,p-2]}(W(k))$.

2. This functor restricts to an equivalence $\mathrm{Coh}_{[0,p-2]}(W(k)^{syn}) \to \mathcal{MF}_{[0,p-2]}(W(k))$.

They also prove a second result which gives some information when the weights are in $[0, p-1]$. There are line bundles $\mathscr{O}\{i\}$ over $W(k)^{syn}$ and there is a section $v_1 \in \mathrm{H}^0(W(k)^{syn} \otimes \mathbb{F}_p, \mathscr{O}\{p-1\})$. We let $W(k)^{syn}_{red}$ be the vanishing locus of this section.

Theorem 2.2 ([TVX25], Prop 5.13)**.** *There is a functor* $\Phi^{DI} : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn} \otimes \mathbb{F}_p) \to \mathcal{DMF}^{big}_{[0,p-1]}(k)$ *which factors as* $\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn} \otimes \mathbb{F}_p) \overset{red}{\to} \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) \overset{\Phi^{DI}_{red}}{\to} \mathcal{DMF}^{big}_{[0,p-1]}(k)$ *where* Φ^{DI}_{red} *is an equivalence. Moreover, the restriction of* Φ^{DI} *and of* $\Phi^{Maz} \otimes \mathbb{F}_p$ *to* $\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn} \otimes \mathbb{F}_p)$ *coincide.*

We will briefly recall how these functors are constructed and how to compute them in practice.

2.3.1 Preliminaries

We consider the fiber product :

$$\mathfrak{D} = \mathbb{A}^1_-/\mathbb{G}_m \times_{W(k)^{\mathcal{N}}} k^{\mathcal{N}}$$

By [TVX25, Prop 3.11], this fiber product can be described as $\tilde{\mathfrak{D}}/\mathbb{G}_m$, where $\tilde{\mathfrak{D}} = \mathrm{Spf}\ B^b$, for $B^b = W(k)[\frac{v_+^n}{n!}, n \geq 0, v_-]/(v_+v_- - p)$. By construction $v_+ \in \mathbb{G}_a(B^b)$ lifts to an element $w \in \mathbb{G}_a^\sharp(B^b)$. We also have an element $\nu = \Psi([v_-]w) \in W^\times\{p\}(B^b)$ solving the equation $p - V(\nu) = [v_-]w$ (since $[v_-]w$ corresponds to divided powers on p, see Lemma 2.1).

Over $\tilde{\mathfrak{D}}(R)$ we have an isomorphism $M_{cris,R} \to M_{dR,R}$. This map is induced by the map $\mathbb{G}_a^\sharp \oplus W_R \to \mathbb{G}_a^\sharp \oplus F_\star W_R$ with matrix $\begin{pmatrix} 1 & w \\ 0 & \nu F \end{pmatrix}$.

Let X be a flat formal scheme over $W(k)$. Over $\mathfrak{D}$ we have a tautological isomorphism

$$\Theta_{crys,dR} : (X_k)^{\mathcal{N}} \times_{k^{\mathcal{N}}} \mathfrak{D} \simeq X^{dR,+} \times_{\mathbb{A}^1_-/\mathbb{G}_m} \mathfrak{D}.$$

We can pass to the generic fiber of the Rees map $\mathbb{G}_m/\mathbb{G}_m \to \mathbb{A}^1_-/\mathbb{G}_m$. The map $\nu : \mathfrak{D} \times_{\mathbb{A}^1_-/\mathbb{G}_m} \mathbb{G}_m/\mathbb{G}_m = \tilde{\mathfrak{D}} \times_{\mathbb{A}^1_-, v_-=1} \mathrm{Spf}\ W(k) \to W^\times\{p\}$ is an isomorphism and the fiber of the isomorphism $\Theta_{crys,dR}$ is the universal isomorphism $(X_k/W(k))^{crys} \times W^\times\{p\} = X^{dR} \times W^\times\{p\}$ given by composing the canonical isomorphism $(X_k/W(k))^{crys} \to X^{dR}$ of Proposition 2.1, with the action of $W^\times\{p\}$ on X^{dR}.

Let B be the Rees algebra of $W(k)$ for the filtration $\mathrm{Fil}^i W(k) = \langle \gamma_s^{std}(p), s \geq i \rangle = p^{\langle i \rangle} W(k)$ where $\langle i \rangle = \inf_{s \geq i}\{v(\frac{p^s}{s!})\}$. In other words, $\mathrm{Fil}^\bullet W(k) = (pW(k))^{[\bullet]}$ is the filtration induced by the PD-ideal $pW(k)$ equipped with the (standard) divided power. There is a map of filtered algebras $p^\bullet W(k) \to \mathrm{Fil}^\bullet W(k)$ inducing a $\mathbb{G}_m$-equivariant map $A \to B$. In fact B is the subalgebra of $A[1/p]$ generated by $v_+^i \frac{p^{\langle i \rangle}}{p^i}$. In particular, v_+ has divided powers $\frac{v_+^i}{i!} = v_+^i \frac{p^{\langle i \rangle}}{p^i} \frac{i! p^i}{i!}$ in B. We denote by $w \in W[F](B)$ the corresponding element lifting v_+. This defines a map $\Psi_{Maz} : \mathrm{Spf}\ B \to \mathfrak{D}$. We also observe that $\nu = 1$ as $w[v_-]$ corresponds to the trivial divided powers.

In B/v_+^p, we can take the trivial divided powers $w' = [v_+] \in W[F](B/v_+^p)$ on v_+. This defines a second map $\Psi_{DI} : \mathrm{Spec}\ B/v_+^p \to \mathfrak{D}$. We have $\nu' = 1 - \frac{[p^p]}{p}$ since $[v_+][v_-] = [p]$.

2.3.2 Description of the two functors

We would like to explain concretely how to compute the functor $\Phi^{Maz} : Vect_{[0,p-1]}(W(k)^{syn}) \to \mathcal{MF}^f_{[0,p-1]}(W(k))$. Let $\mathcal{M} \in Vect_{[0,p-1]}(W(k)^{syn})$. By pull-back to k^{syn} we obtain an F-isocrystal (M_{crys}, ϕ). We attach to this F-isocrystal an F-gauge by taking the Nygaard filtration : for $n \geq 0$, we let $M^n_{crys} = \{m \in M_{crys}, \phi(m) \in p^n M_{crys}\}$. For $n \leq 0$ we let $M^n_{crys} = M_{crys}$. We let $v_- : M^n_{crys} \to M^{n-1}_{crys}$ be the canonical inclusion and $v_+ : M^n_{crys} \to M^{n+1}_{crys}$ is induced by multiplication by p. For $n \geq 0$, v_- is an isomorphism and for $n \leq p-1$, v_+ is an isomorphism. That way $M^\bullet_{crys}$ is a graded module over $W(k)[v_+, v_-]/(v_+v_- - p)$. By pullback of $\mathcal{M}$ to $\mathbb{A}^1_-/\mathbb{G}_m$ we get a filtered vector space M^i_{dR}. Using the point Ψ_{Maz}, we get an isomorphism $\Theta_{Maz} : M_{dR} \simeq M_{crys}$.

Moreover, we have a filtered isomorphism : $M^\bullet_{crys} \otimes_{W(k),p^\bullet} (W(k), \mathrm{Fil}^\bullet) = M^\bullet_{dR} \otimes_{W(k)} (W(k), \mathrm{Fil}^\bullet)$. This informs us that $M^i_{dR} \subseteq M^i_{crys}$ for $0 \leq i \leq p-1$. We can thus define a Fontaine-Laffaille module $\Phi^{Maz}(\mathcal{M}) = M_{dR} = M_{crys}$, with filtration $\Phi^{Maz}(\mathcal{M})^i = M^i_{dR}$ and divided Frobenius $\phi_i = p^{-i}\phi : \Phi^{Maz}(\mathcal{M})^i \to \Phi^{Maz}(\mathcal{M})$. In other words, $\Phi^{Maz}(\mathcal{M})$ is constructed by using the Hodge filtration M^i_{dR}, the Frobenius on M_{crys} and the isomorphism $M_{crys} \simeq M_{dR}$ (using some divisibility of the Frobenius on the Hodge filtration).

We now turn to the functor $\Phi^{DI} : Vect_{[0,p-1]}(W(k)^{syn}) \to \mathcal{MF}_{[0,p-1]}(k)$. The point Ψ_{DI} induces an isomorphism $\Theta_{DI} : M_{crys}/p^p \to M_{dR}/p^p$.

Remark 2.3. This isomorphism is not the reduction modulo p^p of Θ_{Maz} since $\nu' = 1 - \frac{[p^p]}{p}$. However, $\Theta_{Maz} = \Theta_{DI}$ modulo p^{p-1}.

Moreover, Frobenius on M^i_{dR}/p^p restricts to map $\phi : M^i_{dR}/p^p \to p^i M_{crys}/p^p$ which factors through a map $\phi_i : M^i_{dR}/p \to p^i M_{crys}/p^{i-1} = M_{crys}/p$. Thus, $\Phi^{DI}(\mathcal{M}) = M_{crys}/p \simeq M_{dR}/p$, with $\Phi^{DI}(\mathcal{M})^i = M^i_{dR}/p$ and divided Frobenius ϕ_i.

Remark 2.4. Let $\mathfrak{X} \to \mathrm{Spf}\ W(k)$ be a proper smooth formal scheme with special fiber X, assume that $\mathrm{H}^i_{syn}(\mathfrak{X}) \in Vect_{[0,p-1]}(W(k)^{syn})$. The isomorphism $\Theta_{Maz} : (\mathrm{H}^i_{syn}(\mathfrak{X}))_{dR} = \mathrm{H}^i_{dR}(\mathfrak{X}) \simeq (\mathrm{H}^i_{syn}(\mathfrak{X}))_{crys} = \mathrm{H}^i_{crys}(X/W(k))$ is the one provided by Proposition 2.1. The isomorphism $\Theta_{DI} : (\mathrm{H}^i_{syn}(\mathfrak{X}) \otimes W_p(k))_{dR} = \mathrm{H}^i_{dR}(\mathfrak{X} \otimes W_p(k)) \simeq (\mathrm{H}^i_{syn}(\mathfrak{X}) \otimes W_p(k))_{crys} = \mathrm{H}^i_{crys}(X/W_p(k))$ is the one obtained by considering crystalline cohomology $\mathrm{H}^i_{crys}(X/W_p(k))$ with trivial divided powers, and the isomorphism of Proposition 2.1 (see Remark 2.2).

2.4 Etale realization

2.4.1 The étale realization functor

Let $K = W(k)[1/p]$ and G_K be the absolute Galois group. Let $\mathrm{Rep}_{G_K}(\mathbb{Z}_p)$ be the category of continuous representations of G_K on finite type $\mathbb{Z}_p$-modules. We let $\mathrm{Rep}^{crys,f}_{G_K}(\mathbb{Z}_p)$ be the category of lattices in crystalline representations and $\mathrm{Coh}^{refl}(W(k)^{syn})$ be the category of reflexive coherent sheaves. There is a functor $\mathcal{T}_{et} : \mathrm{Perf}(W(k)^{syn}) \to D^b(\mathrm{Rep}_{G_K}(\mathbb{Z}_p))$ ([Bha22, Constr. 6.3.2]) which induces a functor $\mathrm{Coh}(W(k)^{syn}) \to \mathrm{Rep}_{G_K}(\mathbb{Z}_p)$.

Theorem 2.3 ([Bha22], Thm. 6.6.13; [FL82], Chap 6; [TVX25], Rem. 5.25). 1. *The functor $\mathcal{T}_{et}$ induces an equivalence*

$$\mathrm{Coh}^{refl}(W(k)^{syn}) \to \mathrm{Rep}^{crys,f}_{G_K}(\mathbb{Z}_p).$$

2. *The functor $\mathrm{Coh}_{[0,p-2]}(W(k)^{syn}) \to \mathrm{Rep}_{G_K}(\mathbb{Z}_p)$ is fully faithful.*

2.4.2 Ordinarity

The relevant categories of unramified objects are respectively unramified Galois representations $\mathrm{Rep}^{ur}_{G_K}(\mathbb{Z}_p)$, $\mathrm{Coh}_{[0,0]}(W(k)^{syn})$ and $\mathcal{MF}_{[0,0]}(W(k))$. Moreover, we have equivalences

$$\mathrm{Rep}^{ur}_{G_K}(\mathbb{Z}_p) \overset{\mathcal{T}_{et}}{\leftarrow} \mathrm{Coh}_{[0,0]}(W(k)^{syn}) \overset{\Phi^{Maz}}{\to} \mathcal{MF}_{[0,0]}(W(k)).$$

The equivalence $M : \mathrm{Rep}^{ur}_{G_K}(\mathbb{Z}_p) \to \mathcal{MF}_{[0,0]}(W(k))$ is given by $(V \mapsto M(V) = (V \otimes_{\mathbb{Z}_p} W(\bar{k}))^{G_k}$ and a quasi-inverse is given by $M \mapsto (M \otimes_{W(k)} W(\bar{k}))^{\phi \otimes F}$.

Let $\chi_p : G_K \to \mathbb{Z}_p^\times$ be the cyclotomic character. We have a line bundle $\mathcal{O}\{1\}$ on $W(k)^{syn}$. We let $W(k)(1)$ be the rank one Fontaine-Laffaille module with $\mathrm{Fil}^0 W(k)(1) = 0$, $\mathrm{Fil}^{-1} W(k)(1) = W(k)(1)$ and $W(k)(1)$ is the rank one $W(k)$-module (we fix an isomorphism $W(k)(1) \simeq W(k)$) with Frobenius ϕ_{-1} given by the Frobenius F of $W(k)$. We have $\mathcal{T}_{et}(\mathcal{O}\{1\}) = \chi_p$, $\mathcal{O}\{1\} \in \mathrm{Coh}_{[-1,-1]}(W(k)^{syn})$, and $\Phi^{Maz}(\mathcal{O}\{-1\}) = W(k)(-1)$.

For an object V of $\mathrm{Rep}_{G_K}(\mathbb{Z}_p)$, we let $V(i) = V \otimes \chi_p^i$ be the Tate twist. For an object $\mathcal{M}$ of $W(k)^{syn}$, we let $\mathcal{M}\{i\} = \mathcal{M} \otimes \mathcal{O}\{i\}$ be the Breuil-Kisin twist. Let M be an object of $\mathcal{MF}(W(k))$. We let $M(i) = M \otimes W(k)(i)$. This object is obtained by shifting the filtration: $\mathrm{Fil}^k M(i) = \mathrm{Fil}^{k+i} M$.

Definition 2.2. *1. An object V of $\mathrm{Rep}_{G_K}(\mathbb{Z}_p)$ is ordinary if it admits an increasing filtration $\mathrm{Fil}_i^{ord}V$ with the property that $\mathrm{Gr}_i^{ord}V(i)$ is an unramified representation.*

2. An object $\mathcal{M}$ of $\mathrm{Coh}(W(k)^{syn})$ is ordinary if it admits an increasing filtration $\mathrm{Fil}_i^{ord}\mathcal{M}$ with the property that $\mathrm{Gr}_i^{ord}\mathcal{M}\{i\}$ is an unramified F-gauge.

3. An object M of $\mathcal{MF}(W(k))$ is ordinary if it admits an increasing filtration $\mathrm{Fil}_i^{ord}M$ with the property that $\mathrm{Gr}_i^{ord}M(i)$ is an unramified Fontaine-Laffaille module.

In the range $[0, p-2]$, Φ^{Maz} is an equivalence between ordinary F-gauges and ordinary Fontaine-Laffaille modules, the ordinary filtration is unique with $\mathrm{Fil}_{-1}^{ord}M = 0$ and $\mathrm{Fil}_{p-2}^{ord}M = M$. The functor $\mathcal{T}_{et}$ is fully faithfull (but not essentially surjective) towards the category of ordinary Galois representations whose (unique) filtration satisfies $\mathrm{Fil}_{-1}^{ord}V = 0$ and $\mathrm{Fil}_{p-2}^{ord}V = V$.

2.4.3 Ordinarity in weight $0, a$

We now let $1 \leq a \leq p-1$ and let $\mathrm{Coh}_{a,0}^{ord}(W(k)^{syn})$ be the category of ordinary F-gauges, with $\mathrm{Fil}_{-1}^{ord}\mathcal{M} = 0 \subseteq \mathrm{Fil}_0^{ord}\mathcal{M} = \cdots = Fil_{a-1}^{ord}\mathcal{M} \subseteq \mathrm{Fil}_a^{ord}\mathcal{M} = \mathcal{M}$. We define similary $\mathcal{MF}_{a,0}^{ord}(W(k))$ and $\mathrm{Rep}_{G_K,a,0}^{ord}(\mathbb{Z}_p)$. For example $1 \oplus \chi_p^{-a}$ is an object of $\mathrm{Rep}_{G_K,a,0}^{ord}(\mathbb{Z}_p)$.

The category $\mathcal{MF}_{a,0}^{ord}(k)$ can be described very explicitely as follows. Consider the category $\mathcal{C}$ consisting of triples $((V,\phi_V),(W,\phi_W),\psi : V \to W)$ where V, W are finite dimensional k-vector spaces, equipped with F-linear automorphisms ϕ_V and ϕ_W, and $\psi : V \to W$ is an F-linear endomorphism.

Let M be an object of $\mathcal{MF}_{a,0}^{ord}(k)$. The jumps of the Hodge filtration occur at 0 and a. Moreover, we have an isomorphism $M^a \oplus \mathrm{Fil}_0^{ord}M \to M$. As a result $M^a \simeq \mathrm{Gr}_a^{ord}M$ and $\mathrm{Fil}_0^{ord}M \simeq \bar{M}^0$. We define a functor $\mathcal{MF}_{a,0}^{ord}(k) \to \mathcal{C}$, by sending M to $(V = \mathrm{Fil}_0^{ord} = \bar{M}^0, W = M^a = \mathrm{Gr}_a^{ord}M, \psi : M^a \xrightarrow{\phi_a} M \to \bar{M}^0)$.

Lemma 2.5. *The functor $\mathcal{MF}_{a,0}^{ord}(k) \to \mathcal{C}$ is an equivalence.*

Proof. We provide a quasi-inverse $\mathcal{C} \to \mathcal{MF}_{a,0}^{ord}(k)$, by defining $M = V \oplus W$, $M^a = W$, $\phi_0 = \phi_V \oplus 0$, $\phi_a = \psi \oplus \phi_W$. Moreover $V = \mathrm{Fil}_0^{ord}M$ provides the ordinary filtration. □

We let $\epsilon_p : G_K \to \mathbb{F}_p^\times$ be the mod-p cyclotomic character. Let $V_1, V_2 \in \mathrm{Rep}_{G_K}^{un}(\mathbb{F}_p)$. Let $\mathcal{V}_1, \mathcal{V}_2 \in \mathrm{Coh}_{[0,0]}(W(k)^{syn} \otimes \mathbb{F}_p)$ be the corresponding F-gauges. We can consider the group of extensions $\mathrm{Ext}_{G_K}^1(V_1 \otimes \epsilon_p^{-a}, V_2)$.

Lemma 2.6. *If $2 \leq a \leq p-1$, the functor $\mathcal{T}_{et}$ induces an isomorphism: $\mathrm{Ext}^1_{W(k)^{syn}\otimes\mathbb{F}_p}(\mathcal{V}_1 \otimes \mathscr{O}\{-a\}, \mathcal{V}_2) \to \mathrm{Ext}^1_{G_K}(V_1 \otimes \epsilon_p^{-a}, V_2)$.*

Proof. Let $E = \mathcal{V}_1^\vee \otimes \mathcal{V}_2 \otimes \mathscr{O}\{a\}$. By [Bha22, Coro 6.4.6] we have an exact triangle:

$$\mathrm{R}\Gamma(W(k)^{syn}, E) \to \mathrm{R}\Gamma(G_K, \mathcal{T}_{et}(E)) \to \mathrm{RHom}_{\mathbb{F}_p}(\mathrm{R}\Gamma(W(k)^{syn}, E^\vee\{1\}[2]), \mathbb{F}_p) \to$$

It suffices to see that $\mathrm{R}\Gamma(W(k)^{syn}, E^\vee\{1\}[2]) = 0$. But $E^\vee\{1\}$ has weight $a-1 > 0$. So its syntomic cohomology vanishes by [Bha22, Ex 6.5.13]. □

When $a = 1$, we consider the subgroup $\mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-1}, V_2) \hookrightarrow \mathrm{Ext}^1_{G_K}(V_1 \otimes \epsilon_p^{-1}, V_2)$ of finite flat cohomology ([Mil06, Chap 3]). If $k = \bar{k}$, by Kummer theory $\mathrm{Ext}^1_{G_K}(V_1 \otimes \epsilon_p^{-1}, V_2) = V_1^\vee \otimes_{\mathbb{F}_p} V_2 \otimes_{\mathbb{F}_p} K^\times/(K^\times)^p$ and $\mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-1}, V_2)$ consists of the classes in $V_1^\vee \otimes_{\mathbb{F}_p} V_2 \otimes_{\mathbb{F}_p} W(k)^\times/(W(k)^\times)^p$. For general k, $\mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-1}, V_2)$ is the subgroup of $\mathrm{Ext}^1_{G_K}(V_1 \otimes \epsilon_p^{-1}, V_2)$ of classes which restrict to $\mathrm{Ext}^1_{f,G_{K^{nr}}}(V_1 \otimes \epsilon_p^{-1}, V_2)$.

Lemma 2.7. *The functor $\mathcal{T}_{et}$ induces an isomorphism : $\mathrm{Ext}^1_{W(k)^{syn}\otimes\mathbb{F}_p}(\mathcal{V}_1 \otimes \mathscr{O}\{-1\}, \mathcal{V}_2) \to \mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-1}, V_2)$.*

Proof. We set $E = \mathcal{V}_1^\vee \otimes \mathcal{V}_2 \otimes \mathscr{O}\{1\}$. This is an F-gauge with weight -1 corresponding to a truncated Barsotti-Tate group scheme G which is an unramified twist of copies of μ_p. We have $\mathrm{R}\Gamma(W(k)^{syn}, E) = \mathrm{R}\Gamma_{fppf}(W(k), G)$ by [MM25, Thm B] and $\mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-1}, V_2) = \mathrm{H}^1_{fppf}(W(k), G)$ by definition. □

To have uniform notations, let us denote $\mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-a}, V_2) = \mathrm{Ext}^1_{G_K}(V_1 \otimes \epsilon_p^{-a}, V_2)$ when $2 \leq a \leq p-2$. When $a = p-1$, $\epsilon_p^{p-1} = 1$. We have a (possibly non-trivial) Hoschild-Serre exact sequence:

$$0 \to \mathrm{Ext}^1_{G_k}(V_1, V_2) \to \mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{1-p}, V_2) \to \mathrm{H}^0(G_k, \mathrm{Ext}^1_{f,G_K nr}(V_1 \otimes \epsilon_p^{1-p}, V_2)) \to 0.$$

Let $V_1, V_2 \in \mathrm{Rep}^{un}_{G_K}(\mathbb{F}_p)$. We define a map

$$\Phi^{DI} : \mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-a}, V_2) \to \mathrm{Hom}_k(M(V_1)^{(p)}, M(V_2))$$

(by abuse of notation) which is the composition of $\Phi^{DI} \circ \mathcal{T}_{et}^{-1}$ and the functor of Lemma 2.5.

Theorem 2.4. *Let $V_1, V_2 \in \mathrm{Rep}^{un}_{G_K}(\mathbb{F}_p)$.*

1. *If $a \neq p-1$. We have an isomorphism:*

$$\Phi^{DI} : \mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{-a}, V_2) \to \mathrm{Hom}_k(M(V_1)^{(p)}, M(V_2)).$$

2. *If $a = p-1$, we have a map :*

$$\Phi^{DI} : \mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{1-p}, V_2) \to \mathrm{Hom}_k(M(V_1)^{(p)}, M(V_2))$$

which factors through an isomorphism :

$$\mathrm{H}^0(G_k, \mathrm{Ext}^1_{f,G_K nr}(V_1 \otimes \epsilon_p^{1-p}, V_2)) \to \mathrm{Hom}_k(M(V_1)^{(p)}, M(V_2)).$$

Proof. When $1 \leq a \leq p-2$, this is a direct consequence of Theorem 2.1 (but of course, this is already known by [FL82]). We turn to the case $a = p-1$. By Theorem 2.2 we have an isomorphism $\Phi^{DI}_{red} : \mathrm{Ext}^1_{W(k)^{syn}_{red}}(\mathcal{V}_1 \otimes \mathscr{O}\{1-p\}, \mathcal{V}_2) \to \mathrm{Hom}_k(M(V_1)^{(p)}, M(V_2))$. Moreover, we have an exact triangle:

$$\mathrm{Ext}_{W(k)^{syn} \otimes \mathbb{F}_p}(\mathcal{V}_1 \otimes \mathscr{O}\{1-p\}, \mathcal{V}_2 \otimes \mathscr{O}\{1-p\}) \to \mathrm{Ext}_{W(k)^{syn} \otimes \mathbb{F}_p}(\mathcal{V}_1 \otimes \mathscr{O}\{1-p\}, \mathcal{V}_2)$$
$$\to \mathrm{Ext}_{W(k)^{syn}_{red}}(\mathcal{V}_1 \otimes \mathscr{O}\{1-p\}, \mathcal{V}_2) \to$$

which on degree 1 cohomology indentifies with the Hoschild-Serre exact sequence:

$$0 \to \mathrm{Ext}^1_{G_k}(V_1, V_2) \to \mathrm{Ext}^1_{f,G_K}(V_1 \otimes \epsilon_p^{1-p}, V_2) \to \mathrm{H}^0(G_k, \mathrm{Ext}^1_{f,G_K nr}(V_1 \otimes \epsilon_p^{1-p}, V_2)) \to 0$$

□

Remark 2.5. (see also [TVX25, Rem 5.23]) The above theorem is well known in the range $1 \leq a \leq p-2$, but seems to be new when $a = p-1$. We would like to sktech a more elementary proof of this result when $p = 2$ and we are dealing (after twisting by ϵ_p) with ordinary (truncated) Barsotti-Tate groups. For simplicity let us assume that $k = \bar{k}$ and consider the p-divisible group G_q which is an extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by μ_{p^∞} with Serre-Tate parameter $q \in 1 + pW(k)$. The class of the extension $G_q[p](\bar{K}) \in \mathrm{Ext}^1_{f,G_K}(1, \epsilon_p)$ is given by the image $\bar{q} \in W(k)^\times/(W(k)^\times)^p$ of q. Moreover, the map $W(k)^\times/(W(k)^\times)^p \to k$, $\bar{q} \mapsto \frac{1-\bar{q}}{p}$ is a group isomorphism. We can compute the functor Φ^{Maz} applied to $G_q \otimes \mathscr{O}\{-1\}$ (viewed as a mod p prismatic F-gauge with weights 0 and 1). We first have $\mathrm{Lie}\mathbf{E}(G_q^D)(W(k) \to k) \simeq W(k)e_1 \oplus W(k)e_2$ with Frobenius $\mathrm{diag}(1, p)$ (here $\mathbf{E}(-)$ is the universal vector extension of [Mes72], we could also take the crystalline cohomology of an elliptic curve with p-divisible group G_q). Moreover, by Katz's theorem, the Hodge filtration is given by $W(k)(e_2 + \log(q)e_1)$. Since $\log(q) \in p^2W(k)$, we deduce that the functor Φ^{Maz} applied

to $G_q[p]\otimes\mathscr{O}\{-1\}$ yields the trivial Fontaine-Laffaille module $k\oplus k(1)$. We now turn to Φ^{DI}. This time we compute $\mathrm{Lie}\mathbf{E}(G_q^D)(W_2(k)\to k,\gamma^{triv})\simeq W_2(k)e_1\oplus W_2(k)e_2$ with Frobenius $\mathrm{diag}(1,p)$. Moreover, the Hodge filtration is now given by $W_2(k)(e_2+(q-1)e_1)$. Indeed, we are using trivial divided powers so all higher terms in the logarithm disappear. As a result, $\Phi^{DI}(G_q[p]\otimes\mathscr{O}\{-1\})$ is the Fontaine-Laffaille module $ke_1\oplus ke_2$, with $\Phi^{DI}(G_q[p]\otimes\mathscr{O}\{-1\})^1=ke_2$, $\phi_0(e_1)=e_1$, $\phi_0(e_2)=0$, $\phi_1(e_2)=e_2+\frac{\bar{q}-1}{p}e_1$. So we can recover $\frac{\bar{q}-1}{p}$ which is the class of the extension $G_q[p](\bar{K})$ from $\Phi^{DI}(G_q[p]\otimes\mathscr{O}\{-1\})$.

2.5 Deligne-Illusie theory

2.5.1 Cartier isomorphism

Let $X\to\mathrm{Spec}\ k$ be a smooth scheme. We let $F_X:X\to X^{(p)}$ be the relative Frobenius. When the context is clear, we simply denote F_X by F. The complex $F_\star\Omega_X^\bullet$ is a complex of coherent sheaves over $X^{(p)}$. Moreover, we have the following theorem:

Theorem 2.5 ([Ill96], Thm 3.5)**.** *There is a graded isomorphism:*

$$\oplus_i\Omega^i_{X^{(p)}}\to\oplus_i\mathcal{H}^i(\Omega_X^\bullet).$$

The map $\mathcal{O}_{X^{(p)}}\to\mathcal{H}^0(\Omega_X^\bullet)$ is induced by the map $\mathcal{O}_{X^{(p)}}\to F_\star\mathcal{O}_X$. The map $\Omega^1_{X^{(p)}}\to\mathcal{H}^1(\Omega_X^\bullet)$ is induced by the inverse Cartier map $C^{-1}:\Omega^1_{X^{(p)}}\to\Omega^1_X/\mathrm{d}(\mathcal{O}_X)$ which sends $\mathrm{d}x$ to the class of $x^{p-1}\mathrm{d}x$.

2.5.2 The obstruction to lift Frobenius and the Deligne-Illusie map

We assume that X has a flat lift $\tilde{X}$ to Spec $W_2(k)$. The following theorem is a consequence of general deformation theory. See for example [Ill96, Prop. 2.11].

Theorem 2.6. *There is a canonical map $F^\star\Omega^1_{X^{(p)}}\to\mathscr{O}_X[1]$ which vanishes if and only if $F:X\to X^{(p)}$ admits a lift $\tilde{F}:\tilde{X}\to\tilde{X}^{(p)}$.*

Since we will need an explicit description of this obstruction, we include the short proof.

Proof. We pick a finite cover of X by affines $X=\cup_iU_i$ with $U_i=\mathrm{Spec}\ R_i$. Let $U_{i,j}=U_i\cap U_j$. Let Spec $\tilde{R}_i$ be the open in $\tilde{X}$ corresponding to U_i. We can pick local lifts $\tilde{F}_i:\tilde{R}_i^{(p)}\to\tilde{R}_i$. Then $\tilde{F}_i-\tilde{F}_j$ corresponds to a map $\mathrm{Hom}_k(\Omega^1_{U_{i,j}^{(p)}},\mathscr{O}_{U_{i,j}})$. The data $(\tilde{F}_i-\tilde{F}_j)_{i,j}$ defines a 1-cocycle and its class is the desired obstruction. □

In the course of the proof we have used the following classical result in commutative algebra.

Lemma 2.8. *Let A be a ring. Let R be a smooth A-algebra. Let S be an A-algebra and let I be an ideal in S with $I^2=0$. Let $\phi:R\to S/I$ be an A-algebra map. There exists a lift $\phi':R\to S$ of ϕ. Moreover, the set of all lifts is a principal homogeneous set under* $\mathrm{Hom}_A(\Omega^1_{R/A},I)$.

Remark 2.6. Having a lift of Frobenius is very rare when X is proper. If there is a lift of Frobenius, we have a morphism $F^\star\Omega^d_{X^{(p)}}\to\Omega^d_X$ lifting the Cartier map. Applying Serre duality, this shows that X is Frobenius split. This also proves that $(\Omega^d_X)^{(1-p)}$ as a non-zero section, hence that the Kodaira dimension of X is ≤ 0.

We let $\tau_{\leq 1}F_\star\Omega_X^\bullet$ be the canonical troncation of the de Rham complex. Observe that there are two canonical maps (induced by the conjugate filtration and the Hodge filtration) $\tau_{\leq 1}F_\star\Omega_X^\bullet\to\mathcal{H}^1(F_\star\Omega_X^\bullet)[-1]$ and $\tau_{\leq 1}F_\star\Omega_X^\bullet\to\mathscr{O}_X$.

Theorem 2.7 ([DI87], Thm 2.1 and Rem 2.2, (iii))**.** *There is a map $\phi_1:\Omega^1_{X^{(p)}}[-1]\to\tau_{\leq 1}F_\star\Omega_X^\bullet$ with the following property :*

1. *The induced map $\Omega^1_{X^{(p)}}\to\mathcal{H}^1(F_\star\Omega_X^\bullet)$ is the inverse Cartier isomorphism of Theorem 2.5.*

2. *The map $O : \Omega^1_{X^{(p)}}[-1] \to F_\star \mathscr{O}_X$ is the inverse to the obstruction to lift Frobenius on $\tilde{X}$ of Theorem 2.6.*

If $n < p$, Deligne and Illusie construct a map $\phi_n : \Omega^n_{X^{(p)}}[-n] \to \tau_{\leq n} F_\star \Omega^\bullet_X$ using ϕ_1 and the algebra structure on $\Omega^\bullet_X$ as follows (see [DI87], p. 251):

$$\begin{array}{ccc} \Omega^n_{X^{(p)}}[-n] & \longrightarrow & \Omega^1_{X^{(p)}}[-1] \otimes \cdots \otimes \Omega^1_{X^{(p)}}[-1] \\ & & \downarrow \\ & & F_\star \tau_{\leq 1} \Omega^\bullet_X \otimes \cdots \otimes F_\star \tau_{\leq 1} \Omega^\bullet_X \longrightarrow F_\star \tau_{\leq n} \Omega^\bullet_X \end{array}$$

Corollary 2.1 ([DI87], Thm 2.1)**.** *Assume that $n < p$. We have a quasi-isomorphism $\oplus_i \phi_i : \oplus_{i \leq n} \Omega^i_{X^{(p)}}[-i] \to F_\star \tau_{\leq n} \Omega^\bullet_X$.*

2.5.3 A variant with logarithmic poles

Let $D \hookrightarrow X$ be a normal crossing divisor. This means that D is a Cartier divisor in X, and X admits a Zariski cover $X = \cup U_i$, and for each i, there is an étale map $p_i : U_i \to \mathbb{A}^d = \text{Spec } k[T_1, \cdots, T_d]$ and $0 \leq r \leq d$ such that $D|_{U_i} = p_i^{-1} V(T_1 \cdots T_r)$. One can define a de Rham complex with logarithmic poles along D (see [Ill96, Section 7.A]): $\Omega^\bullet_X(\log\, D)$. Over U_i as above, this is the free $\mathscr{O}_{U_i}$-module with basis $\frac{\mathrm{d}T_1}{T_1}, \cdots, \frac{\mathrm{d}T_r}{T_r}, \mathrm{d}T_{r+1}, \cdots, \mathrm{d}T_d$. There is a logarithmic variant of the Cartier isomorphism ([Kat70, 15 7.2]): $\oplus_i \Omega^i_{X^{(p)}}(\log\, D^{(p)}) \to \oplus_i \mathcal{H}^i(\Omega^\bullet_X(\log\, D))$. Suppose we have a lift $(\tilde{X}, \tilde{D})$ to $W_2(k)$ of (X, D) where $\tilde{D}$ is still a normal crossing divisor. We now construct, similar to Theorem 2.6, the following obstruction:

Theorem 2.8. *There is an obstruction $F^\star \Omega^1_{X^{(p)}}(\log\, D^{(p)}) \to \mathscr{O}_X[1]$, which vanishes if and only if we can find a lift of Frobenius $\tilde{F}$ on $\tilde{X}$ with $\tilde{F}^{-1}(\tilde{D}) = p\tilde{D}^{(p)}$.*

Proof. We first assume that $X = \text{Spec } R$ is affine, that $\tilde{X} = \text{Spec } \tilde{R}$ and that we have an étale map $W_2(k)[T_1, \cdots, T_d] \to \tilde{R}$, with $\tilde{D} = V(T_1 \cdots T_r)$. We now show that there is a Frobenius lift with the expected properties. Consider the Frobenius lift $\tilde{F}'$ on the affine space given by $T_i \mapsto T_i^p$. It extends by deformation theory to a Frobenius lift $\tilde{F} : \tilde{R} \to \tilde{R}$. We now check that lifts with the property that $\tilde{F}^{-1}(\tilde{D}) = p\tilde{D}^{(p)}$ are a homogeneous set under $\text{Hom}(\Omega^1_{R^{(p)}}(\log\, D^{(p)}), F_\star R)$. We consider two lifts $\tilde{F}_1$ and $\tilde{F}_2$ of Frobenius with this property. Observe that $\tilde{F}_i(T_j) = T_j^p(1 + pu_{i,j})$ for all $1 \leq j \leq r$. The difference $\tilde{F}_1 - \tilde{F}_2$ is thus a derivation $\Theta : R \to R$ which verifies $\Theta(T_j) = T_j^p(u_{1,j} - u_{2,j})$. Therefore, it induces a map $\Omega^1_R(\log\, D) \to R$, sending $\frac{\mathrm{d}T_j}{T_j}$ to $u_{1,j} - u_{2,j}$. Conversely, one sees that one can modify a lift by adding an element of $\text{Hom}(\Omega^1_{R^{(p)}}(\log\, D^{(p)}), F_\star R)$. It is now clear that one can construct the obstruction $F^\star \Omega^1_{X^{(p)}}(\log\, D^{(p)}) \to \mathscr{O}_X[1]$ as in the proof of Theorem 2.6. □

Remark 2.7. The restriction of the map $F^\star \Omega^1_{X^{(p)}}(\log\, D^{(p)}) \to \mathscr{O}_X[1]$ of Theorem 2.8 along the natural map $F^\star \Omega^1_{X^{(p)}} \to F^\star \Omega^1_{X^{(p)}}(\log\, D^{(p)})$ is the map $F^\star \Omega^1_{X^{(p)}} \to \mathscr{O}_X[1]$ of Theorem 2.6.

By [DI87, 4.2.3] there is a map $\phi_1 : \Omega^1_{X^{(p)}}(\log\, D^{(p)})[-1] \to \tau_{\leq 1} F_\star \Omega^\bullet_X(\log\, D)$ with the following properties :

1. The induced map $\Omega^1_{X^{(p)}}(\log\, D^{(p)}) \to \mathcal{H}^1(F_\star \Omega^\bullet_X(\log\, D))$ is the inverse logarithmic Cartier isomorphism.

2. The map $O : \Omega^1_{X^{(p)}}(\log\, D^{(p)})[-1] \to F_\star \mathscr{O}_X$ is the inverse to the obstruction of Theorem 2.8.

One defines maps ϕ_n for $n < p$ and we have that for $n < p$, $\oplus_i \phi_i : \oplus_{i \leq n} \Omega^i_{X^{(p)}}(\log\, D^{(p)})[-i] \to F_\star \tau_{\leq n} \Omega^\bullet_X(\log\, D)$ is a quasi-isomorphism.

2.5.4 Relation with crystalline Frobenius

Let X be a quasi-projective scheme over Spec k. We equip $pW_p(k) \to W_p(k)$ with the trivial divided powers. We consider the crystalline topos $(X/W_p(k))_{crys}$ and its map $u : (X/W_p(k))_{crys} \to X_{zar}$ to the Zariski topos ([BO78, Prop 5.18]). We let $\mathscr{O}_{X/W_p(k)}$ be the structural sheaf, mapping to $\mathscr{O}_X$, with kernel $J_{X/W_p(k)}$. We have a filtration $\{J^{[m]}_{X/W_p(k)}\}_{m\geq 0}$ on $\mathscr{O}_{X/W_p(k)}$.

Let $X \hookrightarrow Y$ be an embedding of X into a smooth $W_p(k)$-scheme Y. Let $D = D_{X,\gamma}(Y)$ be the relative spectrum of $D_{\mathscr{O}_Y,\gamma}(\mathscr{I}_X)$ (the PD-envelop of the ideal $\mathscr{I}_X$ of X in Y).

We denote by $\Omega^\bullet_D := \mathscr{O}_D \otimes_{\mathscr{O}_Y} \Omega^\bullet_Y$ (where $\Omega^\bullet_Y$ is the de Rham complex of Y relative to $W_p(k)$). For any $m \geq 0$, we let $\mathrm{F}^m\Omega^\bullet_D$ be the sub-complex of $\Omega^\bullet_D$ with $\mathrm{F}^m\Omega^n_D = J_X^{[m-n]}\Omega^n_D$ for J_X the ideal of X in D. In particular, if X has a smooth lift $\tilde{X}$ to $W_p(k)$, we have that $F^m\Omega^\bullet_{\tilde{X}} = [p^{[m]}\mathscr{O}_{\tilde{X}} \to p^{[m-1]}\Omega^1_{\tilde{X}} \to \cdots]$ (moreover $(p)^{[k]} = (p^{\max\{k,0\}})$, since we use trivial divided powers). By [BO78, Thm. 7.2], we have $\mathrm{R}u_\star J^{[m]}_{X/W_p(k)} = F^m\Omega^\bullet_{D/W_p(k)}$.

We let $\phi_{crys} : \mathrm{R}u_\star\mathscr{O}_{X^{(p)}/W_p(k)} \to F_\star\mathrm{R}u_\star\mathscr{O}_{X/W_p(k)}$ be the Frobenius morphism. By [BO78, Thm. 8.20], we get a filtered map $\phi_{crys} : \mathrm{R}u_\star J^{[\bullet]}_{X/W_p(k)} \to p^\bullet F_\star\mathrm{R}u_\star\mathscr{O}_{X/W_p(k)}$. Let us briefly sketch the argument as we are not in the exact setting of [BO78, Thm. 8.20] which works over $W(k)$ with the standard divided powers. Since X is quasi-projective, we can find an embedding $X \hookrightarrow Y$ with Y smooth and admitting a lift $\tilde{F}$ of Frobenius (we can take Y to be a projective space). Let $D = D_{X,\gamma}(Y)$. One checks that $\tilde{F}(\Omega^n_D) \subseteq p^n\Omega^n_D$ and $\tilde{F}(J^{[m]}) \subseteq (p)^{[m]}\mathscr{O}_D$. Since $(p)^{[m]} = p^m$ if $m \geq 0$ we conclude.

We now assume that X has dimension d at most $p-1$ and has a flat lift $\tilde{X}$. For every $0 \leq m \leq d$, we have that $\mathrm{R}u_\star\mathrm{Gr}^m J^{[\bullet]}_{X/W_p(k)} = [\mathscr{O}_X \overset{0}{\to} \cdots \overset{0}{\to} \Omega^m_X]$. And we get an induced map:

$$\phi_{crys} : \mathrm{R}u_\star\mathrm{Gr}^m J^{[\bullet]}_{X/W_p(k)} \to \mathrm{Gr}^m p^\bullet F_\star\mathrm{R}u_\star\mathscr{O}_{X/W_p(k)} = F_\star\Omega^\bullet_X$$

This map is a sum of maps $(\phi_{crys})_i = \Omega^i_X[-i] \to F_\star\Omega^\bullet_X$ for $0 \leq i \leq m$ (so eventually, taking $m = d$ we can define maps $(\phi_{crys})_i$ for $0 \leq i \leq d$).

Proposition 2.3. *We have $(\phi_{crys})_i = \phi_i$.*

Proof. Using the multiplicative structure, it suffices to prove that $(\phi_{crys})_0 = \phi_0$ and $(\phi_{crys})_1 = \phi_1$. But $(\phi_{crys})_0 = \phi_0$ is simply the Frobenius of $\mathscr{O}_X$. We now turn to $(\phi_{crys})_1$. It suffices to work on $W_2(k)$. We first consider the case that X is affine. We take a lift $\tilde{X}$. Let $\tilde{F}_1$ and $\tilde{F}_2$ be lifts of Frobenius. We get two maps $\phi_{crys,i} : \Omega^\bullet_{\tilde{X}} \to \Omega^\bullet_{\tilde{X}}$ representing the crystalline Frobenius and they are homotopic by the general formalism. We actually claim that there is a canonical homotopy. First, these maps actually factor through maps : $[\mathscr{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}}] \to \Omega^\bullet_{\tilde{X}}$. Second, we have that $\tilde{F}_1 = \tilde{F}_2 + p\Theta$ for a derivation $\Theta : \Omega^1_X \to \mathscr{O}_X$. We claim that a canonical homotopy between $\phi_{crys,1}$ and $\phi_{crys,2}$ is given by $p\Theta : \Omega^1_{\tilde{X}} \to \mathscr{O}_{\tilde{X}}$. Once this is proved we deduce that $(\phi_{crys})_1 = \phi_1$ by the construction of ϕ_1 and $(\phi_{crys})_1$. See [DI87], page 251 and 252.

We now explain how to get the homotopy. First we need a version of the Poincaré Lemma. Let $D = \mathrm{Spec}\ W_2(k)[T^{[i]}, i \geq 0]$ be the PD-envelop of 0 in Spec $W_2(k)[T]$, where $T^{[i]} = \gamma_i(T)(= \frac{T^i}{i!})$. We consider the map $\int_0^T : \Omega^1_D \to \mathscr{O}_D$ given by integration between 0 and T. Explicitely, this is the map $\omega \mapsto \sum_{n\geq 0} \partial_T^n\langle\partial_T, \omega\rangle|_{T=0}T^{[n+1]}$. Then $\int_0^T$ induces an homotopy between $Id^\star_D$ and $ev_0^\star$ where Id_D is the identity in D, and ev_0 is the evaluation at 0. Specializing at $T^{[i]} = p^{[i]} = p^i$, we get a homotopy between evaluation at 0 and p, simply given by $\omega \mapsto \langle\partial_T, \omega\rangle|_{T=0}p$. At this place the choice of trivial divided powers when $p = 2$ is important, since for the standard divided powers, there would be higher order terms.

Let $\tilde{X} = \mathrm{Spec}\ \tilde{A}$. Consider the map $\tilde{F}(T) : \tilde{A} \to \tilde{A}[T]/T^2$, $f \mapsto F_1(f) + T\Theta(f)$. We have $\tilde{F}(0) = \tilde{F}_1$ and $\tilde{F}(p) = \tilde{F}_2$. By smoothness of $\tilde{A}$ we can lift it to a map $\tilde{F}(T) : \tilde{A} \to \tilde{A}[[T]] \to \tilde{A}\otimes\mathscr{O}_D$.

We therefore have a map $\tilde{F}(T) : \tilde{X} \times D \to \tilde{X}$ with $\tilde{F}(0) = \tilde{F}_1$ and $\tilde{F}(p) = \tilde{F}_2$. By the Poincaré Lemma, the two maps $ev_p, ev_0 : \Omega^\bullet_{\tilde{X}\times D} \to \Omega^\bullet_X$ are homotopic.

The homotopy is given by the map $\Omega^k_{\tilde{X}\times D} = \Omega^k_{\tilde{X}} \otimes \mathscr{O}_D \oplus \Omega^{k-1}_{\tilde{X}} \otimes \Omega^1_D \to \Omega^{k-1}_{\tilde{X}}$ which is $(a, b) \mapsto \langle\partial_T, b\rangle|_{T=0}p$. We deduce that the homotopy between $\phi_{crys,1}$ and $\phi_{crys,2}$ is given by $p\Theta$. Indeed we

compute the composition $\Omega^1_{\tilde{X}} \overset{\tilde{F}(T)^\star}{\to} \Omega^1_{\tilde{X}} \oplus \mathscr{O}_{\tilde{X}} \otimes \Omega^1_D \overset{(0,\int_0^p)}{\to} \mathscr{O}_{\tilde{X}}$. Since $\tilde{F}(T) \mod T^2 = F_1 + T\Theta$, we deduce that the second component of the first map is $\Theta dT \mod TdT$. But $\int_0^p$ kills all higher order terms and $\int_0^p \Theta dT = p\Theta$. All other homotopies $\Omega^k_{\tilde{X}} \to \Omega^{k-1}_{\tilde{X}}$ for $k \geq 2$ are 0.

□

Remark 2.8. This proposition was stated in [DI87, Rem 2.2, (iv)] without proof for the standard divided powers, in which case there is a restriction on the dimension $d \leq p-2$.

3 The obstruction to lift Frobenius on Siegel varieties

In this section, we develop a method to characterize the obstruction to lift Frobenius on a scheme, assuming a suitable lift exists on a dense open. We then apply this material to Siegel Shimura varieties.

3.1 Overconvergent localization

We introduce a certain form of intermediate localization. Let $\tilde{R}$ be a flat $W_2(k)$ algebra and let $f \in \tilde{R}$ be a non-zero divisor. We define the subring $\tilde{R}[\frac{p}{f}]$ of $\tilde{R}[\frac{1}{f}]$ generated by $\tilde{R}$ and $\frac{p}{f}$. We let $R = \tilde{R}/p$. It is clear that the ring $\tilde{R}[\frac{p}{f}]$ only depends on the image of f in R.

Lemma 3.1. *We have:*

1. $\tilde{R}[\frac{p}{f}] = \tilde{R}[X]/(X^2, fX - p, pX)$.
2. *We have a pushout diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{p} & \tilde{R} & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & R & \xrightarrow{\frac{p}{f}} & \tilde{R}[\frac{p}{f}] & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

3. $\tilde{R}[\frac{p}{f}]/f = R/(f)[X]/(X^2)$.

Proof. Consider the map $\tilde{R}[X] \to \tilde{R}[1/f]$, $X \to \frac{p}{f}$, whose image is $\tilde{R}[\frac{p}{f}]$. It factors through a map $\tilde{R}[X]/X^2 \to \tilde{R}[1/f]$. If an element $a_0 + Xa_1 \in \tilde{R}[X]/X^2$ is in the kernel, then $a_0 + \frac{p}{f}a_1 = 0$, equivalently (since f is not a zero divisor) $fa_0 + pa_1 = 0$ and it follows that there exists $a_0' \in \tilde{R}$ such that $a_0 = pa_0'$ and that there exists $a_1' \in \tilde{R}$ such that $a_1 = fa_0' + pa_1'$. As a result, $a_0 + Xa_1 = a_0'(pX - f) + a_1' pX$. This proves the first point. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{p} & \tilde{R} & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle fX} & & \downarrow & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & XR & \longrightarrow & \tilde{R}[X]/(X^2, fX - p, pX) & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R[1/f] & \xrightarrow{p} & \tilde{R}[1/f] & \longrightarrow & R[1/f] & \longrightarrow & 0 \end{array}$$

which proves the second point. The last point is obvious from the presentation in (1). □

This construction globalizes. Let $\tilde{X}$ be a flat scheme over Spec $W_2(k)$ and let $X = \tilde{X} \times_{\text{Spec } W_2(k)}$ Spec k. Let $\mathcal{I} \to \mathscr{O}_X$ be an invertible ideal defining a Cartier divisor $Z = V(\mathcal{I})$. We let $\tilde{X}^{ord} = \tilde{X} \backslash Z$ and we let $X^{ord} = X \setminus Z$.

Remark 3.1. In our applications, X will be a Siegel Shimura variety and X^{ord} will be the ordinary locus.

We have an affine morphism $j : \tilde{X}^{ord} \to \tilde{X}$. We define a scheme $\tilde{X}^\dagger$ with affine maps: $\tilde{X}^{ord} \overset{j_1}{\to} \tilde{X}^\dagger \overset{j_2}{\to} \tilde{X}$. The algebra $(j_2)_\star \mathscr{O}_{\tilde{X}^\dagger}$ is defined via the pushout diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_X & \longrightarrow & \mathscr{O}_{\tilde{X}} & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Id} & & \\ 0 & \longrightarrow & \mathcal{I}^{-1} & \longrightarrow & (j_2)_\star \mathscr{O}_{\tilde{X}^\dagger} & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \end{array}$$

By construction, we have an ideal $p\mathcal{I}^{-1}$ in $\mathscr{O}_{\tilde{X}^\dagger}$, whose associated closed subscheme is X. It follows from Lemma 3.1 (3), that $\tilde{X}^\dagger \times_{\tilde{X}} Z$ is an affine scheme relatively to Z equal to $\underline{\mathrm{Spec}}_Z(\mathscr{O}_Z \oplus \mathcal{I}^{-1}/\mathscr{O}_X)$.

Lemma 3.2. *There is a natural surjective map $\Omega^1_{\tilde{X}^\dagger} \to \mathcal{I}^{-1}/\mathscr{O}_X$.*

Proof. There is an augmentation map $e : Z \to \tilde{X}^\dagger \times_{\tilde{X}} Z$ and $e^\star \Omega^1_{\tilde{X}^\dagger \times_{\tilde{X}} Z/Z} = \mathcal{I}^{-1}/\mathscr{O}_X$. Therefore we have a chain of surjective maps : $\Omega^1_{\tilde{X}^\dagger} \to \Omega^1_{\tilde{X}^\dagger/\tilde{X}} \to \Omega^1_{\tilde{X}^\dagger \times_{\tilde{X}} Z/Z} \to \mathcal{I}^{-1}/\mathscr{O}_X$. □

We now assume that X is smooth. There is a natural map $j_1 : \tilde{X}^{ord} \to \tilde{X}^\dagger$. We have an induced map $\mathrm{d}j_1 : \Omega^1_{\tilde{X}^\dagger} \to (j_1)_\star \Omega^1_{\tilde{X}^{ord}}$. Observe that $\Omega^1_{\tilde{X}^{ord}} = \Omega^1_{\tilde{X}} \otimes_{\mathscr{O}_{\tilde{X}}} \mathscr{O}_{\tilde{X}^{ord}}$. Moreover, the image of $\mathrm{d}j_1$ is contained inside $\Omega^1_{\tilde{X}} + \Omega^1_{\tilde{X}} p\mathcal{I}^{-1} + \Omega^1_{\tilde{X}} p\mathcal{I}^{-2}$. Since

$$(\Omega^1_{\tilde{X}} + \Omega^1_{\tilde{X}} p\mathcal{I}^{-1} + \Omega^1_{\tilde{X}} p\mathcal{I}^{-2})/(\Omega^1_{\tilde{X}} + \Omega^1_{\tilde{X}} p\mathcal{I}^{-1}) \simeq \Omega^1_X \mathcal{I}^{-2}/\mathcal{I}^{-1}$$

we can therefore construct from the map $\mathrm{d}j_1$ a map $\mathrm{d}j_1' : \Omega^1_{\tilde{X}^\dagger} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathcal{I}^{-2}/\mathcal{I}^{-1}$. This map can be related to the map of Lemma 3.2 as follows.

Lemma 3.3. *1. The differential induces a map $\mathrm{d} : \mathcal{I}^{-1}/\mathscr{O}_X \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathcal{I}^{-2}/\mathcal{I}^{-1}$.*

2. The map d is injective if Z is smooth.

3. The map $\mathrm{d}j_1'$ can be factored through the map of Lemma 3.2 as follows:

$$\Omega^1_{\tilde{X}^\dagger} \to \mathcal{I}^{-1}/\mathscr{O}_X \overset{\mathrm{d}}{\to} \Omega^1_X \otimes_{\mathscr{O}_X} \mathcal{I}^{-2}/\mathcal{I}^{-1}$$

Proof. For the first point, we have the usual sequence $\mathcal{I}/\mathcal{I}^2 \to \Omega^1_X \otimes \mathscr{O}_X/\mathcal{I} \to \Omega^1_Z \to 0$ (see [Sta13, Tag 06BB]). Tensoring the first map with $\mathcal{I}^{-2}/\mathcal{I}^{-1}$ gives the map of (1). Moreover, the first map is injective if Z is also smooth. We are left to prove the last point. We can reduce to the affine case, and we use the same notations as in Lemma 3.1. We have a map $\Omega^1_{\tilde{R}[p/f]} \to \Omega^1_{\tilde{R}[p/f]/\tilde{R}} \to \Omega^1_{(R/f[X]/X^2)/R/f} \to R/fdX$ which is the map of Lemma 3.2. We have a map $\Omega^1_{\tilde{R}[p/f]} \to \Omega^1_{\tilde{R}[1/f]} \to \Omega^1_{\tilde{R}[1/f]}/(\Omega^1_{\tilde{R}} + pf^{-1}\Omega^1_{\tilde{R}})$. Note that $\Omega^1_{\tilde{R}[p/f]}$ is mapped to $\tilde{R}[p/f]\Omega^1_{\tilde{R}} + pf^{-2}\mathrm{d}f\tilde{R}$ inside $\Omega^1_{\tilde{R}[1/f]}$. The image of the map $\Omega^1_{\tilde{R}[p/f]} \to \Omega^1_{\tilde{R}[1/f]}/(\Omega^1_{\tilde{R}} + pf^{-1}\Omega^1_{\tilde{R}})$ lands inside $pf^{-2}\Omega^1_R/pf^{-1}\Omega^1_R$, and this is how we get the map $\mathrm{d}j_1' : \Omega^1_{\tilde{R}[p/f]} \to pf^{-2}\Omega^1_R/pf^{-1}\Omega^1_R$. We finally have a map $R/fdX \to pf^{-2}\Omega^1_R/pf^{-1}\Omega^1_R$, mapping X to $-pf^{-2}df$. We claim that there is a factorization $\mathrm{d}j_1' : \Omega^1_{\tilde{R}[p/f]} \to R/fdX \to pf^{-2}\Omega^1_R/pf^{-1}\Omega^1_R$. It suffices to see that the kernel of $A : \Omega^1_{\tilde{R}[p/f]} \to R/fdX$ is also in the kernel of the map $\Omega^1_{\tilde{R}[p/f]} \to \Omega^1_{\tilde{R}[1/f]}/(\Omega^1_{\tilde{R}} + pf^{-1}\Omega^1_{\tilde{R}})$. The kernel of A is generated by $\Omega^1_{\tilde{R}}$, $f\Omega^1_{\tilde{R}[p/f]}$ and $pf^{-1}\Omega^1_{\tilde{R}[p/f]}$. It is clear that these go to zero. □

3.2 Overconvergent localization and lift of Frobenius

We use the notations $X, \tilde{X}, \tilde{X}^\dagger, X^{ord}$... as in the previous section, and X is assumed to be smooth. The best situation is to have a lift of Frobenius $\tilde{F} : \tilde{X} \to \tilde{X}^{(p)}$. We work under the weaker assumption that there exists a lift of Frobenius $\tilde{F}^\dagger : \tilde{X}^\dagger \to \tilde{X}^{(p)}$. This means that the map $\tilde{F}^\dagger$ reduces modulo p to the map $\tilde{X}^\dagger \times_{\mathrm{Spec}\ W_2(k)} \mathrm{Spec}\ k \to X \overset{F}{\to} X^{(p)}$.

Remark 3.2. Of course, if there is a lift $\tilde{F} : \tilde{X} \to \tilde{X}^{(p)}$, then there is a lift $\tilde{F}^\dagger : \tilde{X}^\dagger \to \tilde{X} \stackrel{\tilde{F}}{\to} \tilde{X}^{(p)}$. Moreover, the lift $\tilde{F}^\dagger : \tilde{X}^\dagger \to \tilde{X}^{(p)}$ induces a lift $\tilde{F}^{ord} : \tilde{X}^{ord} \to \tilde{X}^\dagger \stackrel{\tilde{F}^\dagger}{\to} \tilde{X}^{(p)}$.

We also let $\tilde{D}$ be a divisor with normal crossing in $\tilde{X}$. We let D be the reduction of $\tilde{D}$ modulo p. We assume that D has a transversal intersection with Z. The divisor $\tilde{D}$ pulls back to divisors $\tilde{D}^{ord}$ and $\tilde{D}^\dagger$ on $\tilde{X}^{ord}$ and $\tilde{X}^\dagger$ respectively. We assume that the lift of Frobenius $\tilde{F}^\dagger : \tilde{X}^\dagger \to \tilde{X}^{(p)}$ has the property that $(\tilde{F}^\dagger)^{-1}\tilde{D}^{(p)} = p\tilde{D}^\dagger$.

3.2.1 Definition of a map O'

There is a map

$$O'_0 : \Omega^1_{X^{(p)}} \to F_\star \mathcal{I}^{-1}/\mathscr{O}_X$$

which is obtained as follows. The differential of $\tilde{F}^\dagger$ is a map $\mathrm{d}\tilde{F}^\dagger : \Omega^1_{\tilde{X}^{(p)}} \to \tilde{F}^\dagger_\star \Omega^1_{\tilde{X}^\dagger}$. Composing with the map of Lemma 3.2 gives a map $\Omega^1_{\tilde{X}^{(p)}} \to \tilde{F}^\dagger_\star \mathcal{I}^{-1}/\mathscr{O}_X$ factorizing through the map $O'_0 : \Omega^1_{X^{(p)}} \to F_\star \mathcal{I}^{-1}/\mathscr{O}_X$.

Lemma 3.4. *The map O'_0 extends canonically to a map $O' : \Omega^1_{X^{(p)}}(\log D^{(p)}) \to F_\star \mathcal{I}^{-1}/\mathscr{O}_X$.*

Proof. We have an exact sequence $\mathscr{O}_{\tilde{X}^\dagger} \otimes_{\mathscr{O}_X} \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{X}^\dagger} \to \Omega^1_{\tilde{X}^\dagger/X} \to 0$. We take the pushout along the map $\mathscr{O}_{\tilde{X}^\dagger} \otimes_{\mathscr{O}_X} \Omega^1_{\tilde{X}} \to \mathscr{O}_{\tilde{X}^\dagger} \otimes_{\mathscr{O}_X} \Omega^1_{\tilde{X}}(\log \tilde{D})$ to get a new exact sequence: $\mathscr{O}_{\tilde{X}^\dagger} \otimes \Omega^1_{\tilde{X}}(\log \tilde{D}) \to \Omega^1_{\tilde{X}^\dagger}(\log \tilde{D}) \to \Omega^1_{\tilde{X}^\dagger/X} \to 0$. We claim that the differential $\mathrm{d}\tilde{F}^\dagger$ naturally extends to a map $\Omega^1_{\tilde{X}}(\log \tilde{D}) \to \tilde{F}^\dagger_\star \Omega^1_{\tilde{X}^\dagger}(\log \tilde{D})$. The source is a locally free module. Let $T_1 \cdots T_r = 0$ be a local equation for $\tilde{D}$. By assumption, $\tilde{F}^\dagger(T_i) = T_i^p u_i$ for a unit u_i. It follows that we can define $\mathrm{d}\tilde{F}^\dagger(\frac{\mathrm{d}T_i}{T_i}) = p\frac{\mathrm{d}T_i}{T_i}u_i + \mathrm{d}u_i$. Composing with $\Omega^1_{\tilde{X}^\dagger}(\log \tilde{D}) \to \Omega^1_{\tilde{X}^\dagger/X} \to \mathcal{I}^{-1}/\mathscr{O}_X$ gives the map of the lemma. □

Proposition 3.1. *The map $\Omega^1_{X^{(p)}}(\log D^{(p)}) \to F_\star \mathcal{I}^{-1}/\mathscr{O}_X$ vanishes if and only if the lift of Frobenius $\tilde{F}^\dagger : \tilde{X}^\dagger \to \tilde{X}^{(p)}$ comes from a lift $\tilde{F} : \tilde{X} \to \tilde{X}^{(p)}$.*

Proof. We may reduce to the affine situation and we use the notations of Lemma 3.1. We have a map $\tilde{R}^{(p)} \to \tilde{R}[p/f]$. We reduce this map modulo f, and this yields a map : $\tilde{R}^{(p)} \to R/f[X]/X^2$, factorizing through a map $R^{(p)} \to R/f[X]/X^2$. The projection to the X component is a derivation $R^{(p)} \to XR/f$, thus a linear map $\Omega^1_{R^{(p)}} \to XR/f$. Moreover, this map is zero if and only if the map $\tilde{R}^{(p)} \to \tilde{R}[p/f]$ factors through a map $\tilde{R}^{(p)} \to \tilde{R}$ as we have an exact sequence of $\tilde{R}$-modules $0 \to \tilde{R} \to \tilde{R}[p/f] \to XR/f \to 0$ by Lemma 3.1 (2). □

3.2.2 Definition of the maps O'' and O'''

Using natural adjunctions, the map $O' : \Omega^1_{X^{(p)}}(\log D^{(p)}) \to F_\star \mathcal{I}^{-1}/\mathscr{O}_X$ can be further factored as follows. Let us denote by $i : Z \to X$ the canonical inclusion. Observe that $\mathcal{I}^{-1}/\mathscr{O}_X = i_\star i^\star \mathcal{I}^{-1}$. Let us also denote by $F_Z : Z \to Z^{(p)}$ the relative Frobenius of Z. We have a commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow{\scriptstyle F_Z} & & \downarrow{\scriptstyle F} \\ Z^{(p)} & \xrightarrow{i^{(p)}} & X^{(p)} \end{array}$$

There is a unique map $O'' : (i^{(p)})^\star \Omega^1_{X^{(p)}}(\log D^{(p)}) \to (F_Z)_\star i^\star \mathcal{I}^{-1}$ such that we have

$$O' : \Omega^1_{X^{(p)}}(\log D^{(p)}) \to i^{(p)}_\star (i^{(p)})^\star \Omega^1_{X^{(p)}}(\log D^{(p)}) \stackrel{i^{(p)}_\star O''}{\to} i^{(p)}_\star (F_Z)_\star i^\star \mathcal{I}^{-1} = F_\star \mathcal{I}^{-1}/\mathscr{O}_X.$$

By adjunction, O'' is equivalently a map $F_Z^\star (i^{(p)})^\star \Omega^1_{X^{(p)}}(\log D^{(p)}) \to i^\star \mathcal{I}^{-1}$ of locally free sheaves on Z. The source has rank the dimension of X, the target has dimension 1. We can in fact extract from O'' a simpler map.

We have a natural map $(i^{(p)})^\star\mathcal{I}^{(p)} \xrightarrow{\mathrm{d}} (i^{(p)})^\star\Omega^1_{X^{(p)}}$. We deduce a map $O'' \circ \mathrm{d} : (i^{(p)})^\star\mathcal{I}^{(p)} \to (F_Z)_\star i^\star\mathcal{I}^{-1}$. By adjunction, this map is equivalently a map $F_Z^\star(i^{(p)})^\star\mathcal{I}^{(p)} \to i^\star\mathcal{I}^{-1}$ or a section $O''' \in \mathrm{H}^0(Z, \mathcal{I}^{-p-1})$.

3.2.3 The map O' and the differential of Frobenius

We now explain that the map $O' : \Omega^1_{X^{(p)}}(\log D^{(p)}) \to F_\star\mathcal{I}^{-1}/\mathscr{O}_X$ can often be read directly from the map $\tilde{F}^{ord} : \tilde{X}^{ord} \to \tilde{X}^{(p)}$. The differential of the map $\tilde{F}^{ord}$ fits into the following diagram:

$$\begin{array}{ccc}
(F^{ord})^\star\Omega^1_{X^{(p)}} & \xrightarrow{0} & \Omega^1_{X^{ord}} \\
\uparrow & \searrow & \uparrow \\
(\tilde{F}^{ord})^\star\Omega^1_{\tilde{X}^{(p)}} & \longrightarrow & \Omega^1_{\tilde{X}^{ord}} \\
{\scriptstyle p}\uparrow & \searrow & \uparrow{\scriptstyle p} \\
(F^{ord})^\star\Omega^1_{X^{(p)}} & \xrightarrow{0} & \Omega^1_{X^{ord}}
\end{array}$$

which therefore factors as a map $(F^{ord})^\star\Omega^1_{X^{(p)}} \to \Omega^1_{X^{ord}}$, and induces, by adjunction, a map $\Omega^1_{X^{(p)}} \to F^{ord}_\star\Omega^1_{X^{ord}}$. Adding logarithmic poles, we also have a map $\Omega^1_{X^{(p)}}(\log D^{(p)}) \to F^{ord}_\star\Omega^1_{X^{ord}}(\log D)$.

Lemma 3.5. *We have a factorization of the differential of Frobenius :*

$$\Omega^1_{X^{(p)}}(\log D^{(p)}) \xrightarrow{O'} F_\star\mathcal{I}^{-1}/\mathscr{O}_X \xrightarrow{\mathrm{d}} F^{ord}_\star\Omega^1_{X^{ord}}(\log D)/F_\star\Omega^1_X(\log D) \otimes \mathcal{I}^{-1}.$$

Proof. This follows from Lemma 3.3 (3). □

3.2.4 The map O' and the obstruction to lift Frobenius

We can make the connection between the map O' and the obstruction to find a global lift of Frobenius. We have an exact triangle, $\mathscr{O}_X \to \mathcal{I}^{-1} \to \mathcal{I}^{-1}/\mathscr{O}_X \xrightarrow{+1}$ which yields a map $\mathcal{I}^{-1}/\mathscr{O}_X \to \mathscr{O}_X[1]$.

Proposition 3.2. *The composite map $\Omega^1_{X^{(p)}}(\log D^{(p)}) \to F_\star(\mathcal{I}^{-1}/\mathscr{O}_X) \to F_\star\mathscr{O}_X[1]$ is the map O opposite to the obstruction to lift Frobenius of Theorem 2.8.*

Proof. We consider an open covering $\cup_i \mathrm{Spec}\ \tilde{R}_i = U_i$ of $\tilde{X}$. Let $U_{i,j} = U_i \cap U_j$ be the intersection. We can assume that there are local lifts $\tilde{F}_i : \tilde{R}_i^{(p)} \to \tilde{R}_i$ with the property that $\tilde{F}_i^{-1}(\tilde{D}) = p\tilde{D}$. We also have the global lift $\tilde{F}^\dagger : \tilde{R}_i^{(p)} \to \tilde{R}_i[p/f_i]$ (for f_i a generator of $\mathcal{I}$). The difference $\tilde{F} - \tilde{F}_i$ yields a map $\Omega^1_{R_i^{(p)}}(\log D^{(p)}) \to XR_i$ (where $X = p/f_i$ following the notations of Lemma 3.1) and is a lift of the map $\Omega^1_{R_i^{(p)}}(\log D^{(p)}) \to X(R_i/f_i)$ (which is the restriction to Spec R_i of the map O' of Lemma 3.4). The differences $(\tilde{F}^\dagger - \tilde{F}_i - \tilde{F}^\dagger + \tilde{F}_j = \tilde{F}_j - \tilde{F}_i)_{i,j}$ is a cocycle which represents the map $\Omega^1_{X^{(p)}}(\log D^{(p)}) \to \mathcal{I}^{-1}/\mathscr{O}_X \to \mathscr{O}_X[1]$ and is opposite to the obstruction to lift Frobenius. □

3.2.5 Duality

The duality functor on $X^{(p)}$ is given by $\mathrm{D}(-) = \underline{\mathrm{RHom}}(-, \Omega^d_{X^{(p)}})$.

Lemma 3.6. *The dual of the map $O : \Omega^1_{X^{(p)}}(\log D^{(p)}) \to F_\star\mathscr{O}_X[1]$ gives a map*

$$O^\vee : F_\star\Omega^d_X(\log D)[-1] \to \Omega^{d-1}_{X^{(p)}}(\log D^{(p)}).$$

Proof. We have a perfect pairing $\Omega^1_{X^{(p)}}(\log D^{(p)}) \otimes \Omega^{d-1}_{X^{(p)}}(\log D^{(p)}) \to \Omega^d_{X^{(p)}}(\log D^{(p)}) = \Omega^d_{X^{(p)}}(D^{(p)})$. We deduce that $\mathrm{D}(\Omega^1_{X^{(p)}}(\log D^{(p)})) = \Omega^{d-1}_{X^{(p)}}(\log D^{(p)})(-D^{(p)})$. Moreover, we have that

$$\underline{\mathrm{RHom}}_{X^{(p)}}(F_\star\mathscr{O}_X, \Omega^d_{X^{(p)}}) = F_\star\underline{\mathrm{RHom}}_X(\mathscr{O}_X, F^!\Omega^d_{X^{(p)}}) = F_\star\underline{\mathrm{RHom}}_X(\mathscr{O}_X, \Omega^d_X) = F_\star\Omega^d_X.$$

So the dual of O is the map $F_\star\Omega^d_X[-1] \to \Omega^{d-1}_{X^{(p)}}(\log D^{(p)})(-D^{(p)})$ which induces the map $O^\vee$. □

By Proposition 3.2, the map $O : \Omega^1_{X^{(p)}}(\log\ D^{(p)}) \to F_\star \mathscr{O}_X[1]$ admits a factorization:

$$\Omega^1_{X^{(p)}}(\log\ D^{(p)}) \to i^{(p)}_\star (i^{(p)})^\star \Omega^1_{X^{(p)}}(\log\ D^{(p)}) \overset{i^{(p)}_\star O''}{\to} i^{(p)}_\star (F_Z)_\star i^\star \mathcal{I}^{-1} = F_\star \mathcal{I}^{-1}/\mathscr{O}_X \to F_\star \mathscr{O}_X[1].$$

Lemma 3.7. *The map $O^\vee[1]$ admits a factorization*

$$O^\vee[1] : F_\star \Omega^d_X(\log\ D) \to i^p_\star (F_Z)_\star i^\star \Omega^d_X(\log\ D) \overset{i^{(p)}_\star (O'')^\vee[1]}{\longrightarrow}$$

$$i^{(p)}_\star (i^{(p)})^\star (\Omega^{d-1}_{X^{(p)}}(\log\ D^{(p)}) \otimes_{\mathscr{O}_X} \mathcal{I}^{-1}) \to \Omega^{d-1}_{X^{(p)}}(\log\ D^{(p)})[1]$$

where $(O'')^\vee[1] : (F_Z)_\star i^\star \Omega^d_X(\log\ D) \to (i^{(p)})^\star (\Omega^{d-1}_{X^{(p)}}(\log\ D^{(p)}) \otimes \mathcal{I}^{-1})$.

Proof. Easy and left to the reader □

Let K_Z be the canonical sheaf of Z ([Har66, p 144], where K_Z is denoted by $\omega_{Z/\mathrm{Spec}\ k}$). Recall that $K_Z = i^\star(\Omega^d_X \otimes \mathcal{I}^{-1})$. In case Z is smooth, we simply have that $K_Z = \Omega^{d-1}_Z$. The dual of the map $i^\star\mathcal{I} \overset{\mathrm{d}}{\to} i^\star\Omega^1_X$ is a map $i^\star\Omega^{d-1}_X \otimes \mathcal{I}^{-1} \to K_Z \otimes \mathcal{I}^{-1}$.

We therefore have a commutative diagram :

$$\begin{array}{ccc} (F_Z)_\star i^\star \Omega^d_X(\log\ D) = (F_Z)_\star K_Z(\log\ D) \otimes_{\mathscr{O}_Z} i^\star\mathcal{I} & \xrightarrow{(O'')^\vee[1]} & (i^{(p)})^\star \Omega^{d-1}_{X^{(p)}}(\log\ D^{(p)}) \otimes \mathcal{I}^{-1} \\ & \overset{\gamma}{\searrow} & \downarrow \\ & & K_{Z^{(p)}}(\log(D^{(p)}) \otimes i^\star\mathcal{I}^{-1} \end{array}$$

The map γ is a twist of the Cartier map $(F_Z)_\star K_Z \to K_{Z^{(p)}}$. Namely, this is the map $(F_Z)_\star K_Z(\log D) \otimes_{\mathscr{O}_Z} i^\star\mathcal{I} \overset{O'''}{\to} (F_Z)_\star K_Z(\log\ D) \otimes_{\mathscr{O}_Z} i^\star\mathcal{I}^{-p} \to K_{Z^{(p)}}(\log\ D^{(p)}) \otimes_{\mathscr{O}_Z} i^\star\mathcal{I}^{-1}$ where $O''' \in \mathrm{H}^0(Z, \mathcal{I}^{-p-1})$ is defined in Section 3.2.2.

3.3 The obstruction to lift Frobenius on Siegel Shimura variety

Let $g \in \mathbb{Z}_{\geq 1}$. We will investigate the obstruction to lift Frobenius on the moduli spaces of polarized abelian varieties of dimension g, equipped with a level structure (known as Siegel Shimura varieties). We let $G = \mathrm{GSp}_{2g}$, $\prod_\ell K_\ell = K^p K_p \subseteq G(\mathbb{A}_f)$ be a neat compact open subgroup with $K_p = \mathrm{GSp}_{2g}(\mathbb{Z}_p)$. We let $Sh^{tor}_{K^pK_p} \to \mathrm{Spec}\ \mathbb{Z}_{(p)}$ be a smooth toroidal compactification of the integral model of the Siegel Shimura variety of level K^pK_p, with normal crossing boundary divisor $D_{K^pK_p}$ (see [FC90]). We let $\mathcal{G} \to Sh^{tor}_{K^pK_p}$ be the semi-abelian scheme and we let $\Omega_{\mathcal{G}}$ be its conormal sheaf. We let $\omega = \det(\Omega_{\mathcal{G}})$ be the Hodge bundle. We denote by X the reduction modulo p of $Sh^{tor}_{K^pK_p}$. We let $\mathrm{Ha} \in \mathrm{H}^0(X, \omega^{(p-1)})$ be the Hasse invariant. The vanishing locus of Ha is the non-ordinary locus Z. Its complement is the ordinary locus X^{ord}. The map $\omega^{(1-p)} \overset{\mathrm{Ha}}{\to} \mathscr{O}_X$ identifies $\omega^{(1-p)}$ with the ideal sheaf $\mathcal{I}$ of Z. We let $\tilde{X}$ be the natural lift to $\mathbb{Z}/p^2\mathbb{Z}$ of X, obtained by reduction of $S^{tor}_{K^pK_p}$ modulo p^2. We let D (resp. $\tilde{D}$) be the boundary divisor in X (resp. $\tilde{X}$). We let $\tilde{X}^\dagger$ be the overconvergent localization of $\tilde{X}$ with respect to $\mathcal{I}$ as in Section 3.1.

We have a Frobenius map $F : X \to X$. We abuse notation and still denote by $\mathcal{G}$ the restriction of the semi-abelian scheme over X. Let $\mathcal{G}^{(p)} = X \times_{X,F} \mathcal{G}$. The Frobenius isogeny is $F_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}^{(p)}$.

Remark 3.3. The map $F : X \setminus D \to X \setminus D$ is induced by the Frobenius $F_{\mathcal{G}}$, given the modular interpretation of $X \setminus D$.

By [Str10, Sect 4] (for the treatment of the compactification), the Frobenius isogeny and the Frobenius of X lift modulo p^2 (and even modulo all powers of p) over the ordinary locus. There is a map $\tilde{F}^{ord} : \tilde{X}^{ord} \to \tilde{X}$ and an isogeny $\tilde{F}^{ord}_{\mathcal{G}} : \mathcal{G}|_{\tilde{X}^{ord}} \to (\tilde{F}^{ord})^\star \mathcal{G}$, whose kernel is a finite flat group scheme H_{can} called the canonical subgroup, and equal to the connected part of $\mathcal{G}[p]|_{\tilde{X}^{ord}}$. Moreover, $(\tilde{F}^{ord})^{-1}\tilde{D}^{ord} = p\tilde{D}^{ord}$. We observe that (up to subtleties regarding the boundary) a lift for the Frobenius isogeny of $\mathcal{G}$ induces a lift of the Frobenius map.

Theorem 3.1. *There exists a lift of Frobenius $\tilde{F}^\dagger : \tilde{X}^\dagger \to \tilde{X}$ such that $\tilde{F}^{ord} : \tilde{X}^{ord} \to \tilde{X}^\dagger \xrightarrow{\tilde{F}^\dagger} \tilde{X}$. Moreover, there is an isogeny $\tilde{F}_{\mathcal{G}|_{\tilde{X}^\dagger}} : \mathcal{G}|_{\tilde{X}^\dagger} \to (\tilde{F}^\dagger)^\star \mathcal{G}$ which lifts the Frobenius isogeny modulo the ideal $p\mathcal{I}^{-1}$ (defining $X \hookrightarrow \tilde{X}^\dagger$). Its kernel is a finite flat group scheme.*

Remark 3.4. We should regard this theorem as an overconvergence of the canonical subgroup modulo p^2. We observe that the canonical subgroup overconverges a lot more modulo p^2. Namely, let $\mathcal{O}$ be a complete valued ring for a rank one valuation v extending the p-adic valuation (and normalized by $v(p) = 1$). Let $s : \mathrm{Spec}\ \mathcal{O} \to S^{tor}_{K^p K_p}$ be a map. Classical results on the canonical subgroup ([Sch15, Coro III.2.2], [Far11]) imply that if $v(\mathrm{Ha}(s^\star\mathcal{G})) < \frac{1}{2}$, then there is a canonical subgroup inside $s^\star\mathcal{G}[p]$. Our result shows that if $v(\mathrm{Ha}(s^\star\mathcal{G})) < 1$, then there is a canonical subgroup inside $s^\star\mathcal{G}[p]|_{\mathrm{Spec}\ \mathcal{O}/p^2}$, which lifts the kernel of Frobenius modulo $p^{1-v(\mathrm{Ha}(s^\star\mathcal{G}))}$.

Our proof of Theorem 3.1 is directly inspired by [Sch15, Coro III.2.2]. We use the following theorem of Illusie regarding deformations of finite flat group schemes:

Theorem 3.2 ([Ill72], Thm VII.4.2.5)**.** *Let R be a commutative ring, and G, H be flat and finitely presented commutative group schemes over A, with a group morphism $u : H \to G$. Let $B_1, B_2 \to R$ be two square-zero thickenings with a morphism $B_1 \to B_2$ over R. Let $J_i \subset B_i$ be the augmentation ideal. Let $\tilde{G}_1$ be a lift of G to B_1, and $\tilde{G}_2$ the induced lift to B_2. Let K be the cone of the map $\ell^\vee_H \to \ell^\vee_G$ of Lie complexes.*

1. *For $i = 1, 2$, there is an obstruction class $o_i \in \mathrm{Ext}^1(H, K \otimes^L J_i)$ which vanishes precisely when there exists a lifting $(\tilde{H}_i, \tilde{u}_i)$ of (H, u) to a flat commutative group scheme $\tilde{H}_i$ over B_i, with a morphism $u_i : H_i \to G_i$ lifting $u : H \to G$.*
2. *The obstruction $o_2 \in \mathrm{Ext}^1(H, K \otimes^L J_2)$ is the image of $o_2 \in \mathrm{Ext}^1(H, K \otimes^L J_1)$ under the map $J_1 \to J_2$.*

Proof of Theorem 3.1. We first deal with the good reduction locus. Let Spec $\tilde{R}$ be an open affine in $\tilde{X}$ which sits in the good reduction locus. We also assume that the sheaf $\omega^{(p-1)}$ is trivial over $\tilde{R}$ and identify Ha with an element of $\tilde{R}$. We let $B_1 = \tilde{R}$ and $B_2 = \tilde{R}[p/\mathrm{Ha}]$. Let $\tilde{G} = \mathcal{G}[p]$ be the p-torsion of the universal abelian variety over Spec $\tilde{R}$. Let $u : H = G^{(p)} \to G$ be the Vershiebung. We claim that the obstruction class o_2 is zero since it is the image over o_1 under the map $J_1 \to J_2$ which identifies with $R \to R$ given by multiplication by Ha. To justify the claim, note that Ha is the determinant of the map $\ell^\vee_{G^{(p)}} \to \ell^\vee_G$, so that multiplication by Ha is homotopic to zero on the complex : $\ell^\vee_{G^{(p)}} \to \ell^\vee_G$. By Theorem 3.2, we get a lift $\tilde{H} \to \tilde{G}$ over Spec $\tilde{R}[p/Ha]$. The dual map gives the lift $\tilde{G} \to \tilde{H}^D$ of Frobenius whose kernel we denote by $H^\dagger_{can}$. We observe that $H^\dagger_{can}$ is uniquely caracterized by the property that $H^\dagger_{can}|_{\mathrm{Spec}\ R[1/Ha]} = H_{can}$. Indeed, if another finite flat subgroup H has the same property, we can consider the two surjective maps $A : \mathscr{O}_{G[p]} \to \mathscr{O}_H$ and $B : \mathscr{O}_{G[p]} \to \mathscr{O}_{H^\dagger_{can}}$ of locally free modules over $R[p/H_a]$. Consider the map $A|_{\mathrm{Ker}(B)} : \mathrm{Ker}(B) \to \mathscr{O}_H$. This map is zero on $R[1/\mathrm{Ha}]$, and thus already zero. Reversing the roles of A and B, we deduce that $\mathrm{Ker}(A) = \mathrm{Ker}(B)$ so that $H = H^\dagger_{can}$. This local construction glues to give a subgroup $H^\dagger_{can}$ on $\tilde{X}^\dagger \setminus \tilde{D}^\dagger$. We now deal with the boundary with an extension trick (certainly a full analysis would also be possible). We use the notations as before but we don't assume anymore that $\tilde{R}$ sits in the good reduction locus. We assume that the boundary divisor is generated by an element z. The sequence (Ha, z) is regular. In particular $R[1/z] \times_{R[1/\mathrm{Ha}z]} R[1/\mathrm{Ha}] = R$ by [Sta13, Tag 031T] (R is regular hence normal). This implies that $\tilde{R}[p/H_a][1/z] \times_{\tilde{R}[1/\mathrm{Ha}z]} \tilde{R}[1/\mathrm{Ha}] = \tilde{R}[p/\mathrm{Ha}]$. Indeed, by Lemma 3.1 we have exact sequence $0 \to R \xrightarrow{p/\mathrm{Ha}} \tilde{R}[p/\mathrm{Ha}] \to R \to 0$, and by dévissage we conclude easily.

We have a lift of Frobenius $\tilde{F}^{ord} : \tilde{R} \to \tilde{R}[1/\mathrm{Ha}]$ and another lift just constructed $\tilde{F}' : \tilde{R} \to \tilde{R}[p/\mathrm{Ha}][1/z]$ which both agree inside $\tilde{R}[1/\mathrm{Ha}z]$. We deduce that they arise from a map $\tilde{F}^\dagger : \tilde{R} \to \tilde{R}[p/\mathrm{Ha}]$. We now claim that the finite flat subgroup $H^\dagger_{can}$ is defined over $\tilde{R}[p/\mathrm{Ha}]$. We already have its existence over $\tilde{R}[p/\mathrm{Ha}][1/z]$ and $\tilde{R}[1/\mathrm{Ha}]$. Consider the semi-abelian schemes $\mathcal{G}|_{\mathrm{Spec}\ \tilde{R}[p/\mathrm{Ha}]}$ and $(\tilde{F}^\dagger)^\star\mathcal{G}$. We want to construct an isogeny $\tilde{F}_\mathcal{G} : \mathcal{G}|_{\mathrm{Spec}\ \tilde{R}[p/\mathrm{Ha}]} \to (\tilde{F}^\dagger)^\star\mathcal{G}$ with kernel $H^\dagger_{can}$. As a map of topological spaces, this is just the identity. In order to define a scheme map, we are left to define

the map on each open affine $U = \mathrm{Spec}\ A$ of $\mathcal{G}$ (over $\mathrm{Spec}\ \tilde{R}$). In other words, we want to define the map $A \otimes_{R,\tilde{F}^\dagger} R[p/\mathrm{Ha}] \to A \otimes_R R[p/\mathrm{Ha}]$. The map exists over $R[p/\mathrm{Ha}][1/z]$ and $R[1/\mathrm{Ha}]$ and we are done by descent (since A is flat) as follows. We claim that the functor sending flat modules M over $\tilde{R}[p/Ha]$ to the category of glueing data (M_1, M_2, α) where M_1 is a module over $\tilde{R}[p/Ha][1/z]$, M_2 is a module over $\tilde{R}[1/\mathrm{Ha}]$ and $\alpha : M_1 \otimes \tilde{R}[1/\mathrm{Ha}z] \xrightarrow{\sim} M_2 \otimes \tilde{R}[1/\mathrm{Ha}z]$ is fully faithful. This follows easily from the property that if M is flat, then the sequence $0 \to M \to M_1 \oplus M_2 \to M_3$ is exact. This map is now clearly a group scheme map (again by a descent argument). The kernel of $\tilde{F}_\mathcal{G}$ is a subgroup $H^\dagger_{can}$. Since it lifts the kernel of Frobenius modulo p/Ha, it is finite flat (as an application of the flatness by fiber criterium [Sta13, Tag 00MP]). □

Theorem 3.3. *The map O opposite to the obstruction to lift Frobenius of Theorem 2.8 factors through a map:*

$$\Omega^1_X(\log\ D) \to F_\star \omega^{(p-1)}/\mathscr{O}_X \to F_\star \mathscr{O}_X[-1].$$

Proof. This is a consequence of Section 3.2, notably of Proposition 3.2. □

3.4 Computing the map O'''

At the end of Section 3.2.2, we have attached to the obstruction O a section $O''' \in \mathrm{H}^0(Z, \mathcal{I}^{-1-p} = \omega^{(p^2-1)})$. We will see that this section is the Hasse invariant of the non-ordinary locus.

3.4.1 The second Hasse invariant

We denote by Z^{ord} the p-rank $g-1$ locus of Z (this is the most ordinary locus of Z). Over Z^{ord}, the group scheme $\mathcal{G}[p^\infty]$ sits in an exact sequence: $0 \to \mathcal{G}[p^\infty]^m \to \mathcal{G}[p^\infty]^o \to \mathcal{G}[p^\infty] \to 0$. Passing to conormal sheaves, we deduce that $\Omega_\mathcal{G}$ sits in an exact sequence $0 \to \Omega^{oo}_\mathcal{G} \to \Omega_\mathcal{G} \to \Omega^m_\mathcal{G} \to 0$.

The Frobenius map on the relative log de Rham cohomology over Z^{ord} yields maps $F_m : (\Omega^m_\mathcal{G})^\vee)^{(p)} \to \Omega^\vee_{\mathcal{G}^m}$ and $F_{oo} : (\Omega^m_\mathcal{G})^\vee)^{-(p)} \to \Omega_{\mathcal{G}^m}$ (see below for more details). Let us denote by $\mathrm{Ha}_m = \det F_m \in \mathrm{H}^0(Z^{ord}, (\det \Omega^m_\mathcal{G})^{p-1})$ and by $\mathrm{Ha}_{oo} = F_{oo} \in \mathrm{H}^0(Z^{ord}, (\Omega^{oo}_\mathcal{G})^{p-1})$.

It follows that $(\mathrm{Ha}_m)^{p+1} \otimes (\mathrm{Ha}_{oo})^{p-1} \in \mathrm{H}^0(Z^{ord}, \omega^{(p^2-1)})$. One proves that this section extends to a section $\mathrm{Ha}' \in \mathrm{H}^0(Z, \omega^{(p^2-1)})$ whose non-vanishing locus is exactly Z^{ord}. See [Pil20, Prop 6.3.3.1] (the argument is given for $g = 2$, but it works for all g). We note that this is a particular case of generalized Hasse invariants constructed in [GK19] and [Box15].

Theorem 3.4. *We have $O''' = \mathrm{Ha}'$.*

3.4.2 Proof of Theorem 3.4

It suffices to prove the equality of both sections over the good reduction locus of Z^{ord} (using logarithmic structure we could also discuss the boundary). We fix an open affine $U = \mathrm{Spec}\ B$ of $X^{ord} \cup Z^{ord}$ in the good reduction locus, and we let $\hat{B}$ be its formal completion along the ideal $I = \mathcal{I}(U)$ of $Z \cap U$. The relative de Rham cohomology of $\mathcal{G}$ over $\mathrm{Spf}\hat{B}$ is denoted by $\mathcal{H}$. Shrinking B if necessary, we can assume that this is a free module of rank $2g$. The p-divisible group $\mathcal{G}[p^\infty]$ over $\mathrm{Spf}\ \hat{B}$ sits in the multiplicative-connected filtration (since it has constant multiplicative and étale rank) : $0 \subseteq \mathcal{G}[p^\infty]^m \subseteq \mathcal{G}[p^\infty]^o \subseteq \mathcal{G}[p^\infty]$ by [HT01, Coro II.1.2]. By taking the Lie algebra of the universal extension, we have a filtration

$$0 \subseteq \mathcal{H}^{et} \subseteq \mathcal{H}^o \subseteq \mathcal{H}$$

and the graded pieces are called $\mathcal{H}^{et}$, $\mathcal{H}^{oo}$, $\mathcal{H}^m$. We can also assume that they are free. We have a Frobenius $F : \mathcal{H}^{(p)} \to \mathcal{H}$ respecting the filtration, a connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\hat{B}}$ and the Hodge filtration $\Omega_\mathcal{G} \to \mathcal{H}$. We also assume that $\Omega_\mathcal{G}$ is free. There is a polarization $\langle , \rangle : \mathcal{H} \times \mathcal{H} \to \hat{B}$ and the filtration $\mathcal{H}^{et} \subseteq \mathcal{H}^o \subseteq \mathcal{H}$ is preserved under this pairing. We take a symplectic basis $e_1, \cdots, e_{2g}$ of $\mathcal{H}$ (such that $\langle e_i, e_j \rangle = 0$ if $i + j \neq 2g+1$, $\langle e_i, e_{2g-i+1} \rangle = 1$ if $1 \leq i \leq g$), such that $\mathcal{H}^{et} = \langle e_1, \cdots, e_{g-1} \rangle \subset \mathcal{H}^o = \langle e_1, \cdots, e_{g+1} \rangle$. The Hodge filtration is $\langle e_{g+1}, \cdots, e_{2g} \rangle$ where e_{g+1} generates $\Omega^0_\mathcal{G}$.

The connection induces a Kodaira-Spencer map $KS : \Omega_{\mathcal{G}} \to \Omega_{\mathcal{G}}^{\vee} \otimes \Omega^1_{\hat{B}}$ and an isomorphism (that we still sometimes denote by KS) $\mathrm{Sym}^2\Omega_{\mathcal{G}} \to \Omega^1_{\hat{B}}$. We let $d_{i,j}$ be the image of $e_{g+i}.e_{g+j}$ for $1 \leq i, j \leq g$. It follows that we have the formula $KS(e_{g+i}) = \sum_{j=1}^{g} e_j d_{i,g+1-j}$.

The Frobenius matrix has the shape :

$$\begin{pmatrix} A_1 & c & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where A_1 is an invertible $g-1 \times g-1$ matrix, and represents the map F_m over B/I, a_2 is a scalar which generates I, and b is a scalar and represents the map F_{oo} over B/I.

We see that $F(e_g) = a_2 e_g + b e_{g+1} + \sum_{i=1}^{g-1} c_i e_i$. By differentiating, we get that

$$\nabla(F(e_g)) = 0 = \mathrm{d}(a_2)e_g + a_2\nabla(e_g) + d(b)e_{g+1} + b\nabla(e_{g+1}) + \nabla(\sum_{i=1}^{g-1} c_i e_i).$$

We observe that $\nabla(\sum_{i=1}^{g-1} c_i e_i) \in \mathcal{H}^{et} \otimes \Omega^1_{\hat{B}}$. We project modulo the Hodge filtration and evaluate at Z^{ord} to get that

$$\mathrm{d}(a_2)|_{Z^{ord}} e_g + b|_{Z^{ord}} \sum_{j=1}^{g} e_j d_{1,g+1-j} + \nabla(\sum_{i=1}^{g-1} c_i e_i)|_{Z^{ord}} = 0$$

This implies that $\mathrm{d}(a_2)|_{Z^{ord}} + b|_{Z^{ord}} d_{1,1} = 0$.

Lemma 3.8. *The following diagram is commutative:*

$$\begin{array}{ccc} (\Omega_{\mathcal{G}}^{oo})^2 & \xrightarrow{KS} & \Omega^1_X(\log\ D)|_{Z^{ord}} \\ {\scriptstyle -\mathrm{Ha}_{oo}}\uparrow & & \uparrow{\scriptstyle \mathrm{d}} \\ (\Omega_{\mathcal{G}}^{oo})^{1-p} & \xrightarrow{\mathrm{Ha}_m} & \omega^{(1-p)}|_{Z^{ord}} \end{array}$$

Proof. This follows from the formula $\mathrm{d}(a_2)|_{Z^{ord}} + b|_{Z^{ord}} d_{1,1} = 0$. Namely, in our trivializations, $KS \circ \mathrm{Ha}_{oo} = b|_{Z^{ord}} d_{1,1}$. The composite $\mathrm{d} \circ \mathrm{Ha}_m$ is $\mathrm{d}(a_2)|_{Z^{ord}}$. Note that Ha_m is an isomorphism and the ideal $\omega^{(1-p)}$ of $\mathscr{O}_X$ is generated by a_2. □

We now use the construction of Section 3.2.3. We have a map $F^\star\Omega^1_X \to \Omega^1_{X^{ord}}$ that we need to analyze.

Lemma 3.9. *The map $F^\star\Omega^1_X \to \Omega^1_{X^{ord}}$ factors through a map $F^\star\Omega^1_X \to \Omega^1_X \otimes \omega^{2(p-1)}$. Moreover, if we restrict this map to Z^{ord} we have a commutative diagram:*

$$\begin{array}{ccc} F^\star\Omega^1_X(\log\ D)|_{Z^{ord}} & \longrightarrow & \Omega^1_X(\log\ D) \otimes \omega^{2(p-1)}|_{Z^{ord}} \\ {\scriptstyle KS}\uparrow & & \uparrow{\scriptstyle KS} \\ F^\star(\Omega_{\mathcal{G}}^{oo})^2 & \longrightarrow & (\Omega_{\mathcal{G}}^{oo})^2 \otimes \omega^{2(p-1)}|_{Z^{ord}} \\ & \searrow & \uparrow{\scriptstyle (\mathrm{Ha}_m)^2} \\ & & (\Omega_{\mathcal{G}}^{oo})^2 \otimes (\Omega_{\mathcal{G}}^{oo})^{2(p-1)} \end{array}$$

where all the maps in the bottom triangle are isomorphisms, in particular, the map $F^\star(\Omega_{\mathcal{G}}^{oo})^2 = (\Omega_{\mathcal{G}}^{oo})^{2p} \to (\Omega_{\mathcal{G}}^{oo})^2 \otimes (\Omega_{\mathcal{G}}^{oo})^{2(p-1)}$ is the identity.

Proof. Let $\mathcal{H}$ be the relative log de Rham cohomology of $\mathcal{G}$ over $\tilde{X}^{ord}$. It carries a Frobenius and we have a commutative diagram:

$$\begin{array}{ccc}
(\tilde{F}^{ord})^\star\Omega_{\mathcal{G}}^\vee & \xrightarrow{(H\tilde{W})^\vee} & \Omega_{\mathcal{G}}^\vee \\
\uparrow & & \uparrow \\
(\tilde{F}^{ord})^\star\mathcal{H} & \longrightarrow & \mathcal{H} \\
\uparrow & & \uparrow \\
(\tilde{F}^{ord})^\star\Omega_{\mathcal{G}}^\vee & \xrightarrow{p(H\tilde{W})^{-1}} & \Omega_{\mathcal{G}}
\end{array}$$

where $H\tilde{W}$ is a lift of the classical Hasse-Witt matrix HW. We deduce that there is a commutative diagram:

$$\begin{array}{ccc}
(\tilde{F}^{ord})^\star\Omega^1_{\tilde{X}^{ord}}(\log\ \tilde{D}) & \longrightarrow & \Omega^1_{\tilde{X}^{ord}}(\log\ \tilde{D}) \\
{\scriptstyle KS}\uparrow & & \uparrow{\scriptstyle KS} \\
(\tilde{F}^{ord})^\star\mathrm{Sym}^2\Omega_{\mathcal{G}} & \xrightarrow{p\mathrm{Sym}^2 H\tilde{W}^{-1}} & \mathrm{Sym}^2\Omega_{\mathcal{G}}
\end{array}$$

It follows that $F^\star\Omega^1_{X^{ord}} \to \Omega^1_{X^{ord}}$ is given by $\mathrm{Sym}^2 HW^{-1}$. Since the determinant of HW is Ha, we deduce that $\mathrm{Sym}^2 HW^{-1}$ has poles of order at most 2 along Z and we thus have a map $F^\star\Omega_{\mathcal{G}} \to \Omega_{\mathcal{G}} \otimes \omega^{2(p-1)}$. We similarly have an induced map $(F)^\star\Omega_{\mathcal{G}} \to \Omega_{\mathcal{G}} \otimes \omega^{(p-1)}$. Restricting this map to Z^{ord}, we deduce that it fits in the following diagram :

$$\begin{array}{ccccc}
(F)^\star\Omega^m_{\mathcal{G}} & \xrightarrow{\mathrm{F}_m^{-1}} & \Omega^m_{\mathcal{G}} & \xrightarrow{\mathrm{Ha}|_{Z^{ord}}=0} & \Omega^m_{\mathcal{G}} \otimes \omega^{p-1} \\
\uparrow & & & & \uparrow \\
(F)^\star(\Omega_{\mathcal{G}}|_{Z^{ord}}) & & \longrightarrow & & \Omega_{\mathcal{G}}|_{Z^{ord}} \otimes \omega^{p-1} \\
\uparrow & & & & {\scriptstyle \mathrm{Ha}_m}\uparrow \\
(F)^\star\Omega^{oo}_{\mathcal{G}} & & \longrightarrow & & \Omega^{oo}_{\mathcal{G}} \otimes (\Omega^{oo}_{\mathcal{G}})^{p-1}
\end{array}$$

where the bottom map is the identity. This finishes the proof. □

We can now consider the following big diagram that assembles the diagrams coming from Lemmas 3.8 and 3.9.

$$\begin{array}{ccccc}
F^\star\Omega^1_X(\log\ D)|_{Z^{ord}} & & \longrightarrow & & \Omega^1_X(\log\ D)\otimes\omega^{2(p-1)}|_{Z^{ord}} \\
 & \overset{KS}{\nwarrow} & & & {\scriptstyle KS}\uparrow \\
\mathrm{d}\uparrow & & F^\star(\Omega^{oo}_{\mathcal{G}})^2 & \longrightarrow & (\Omega^{oo}_{\mathcal{G}})^2\otimes\omega^{2(p-1)}|_{Z^{ord}} \\
 & & {\scriptstyle -(\mathrm{Ha}_{oo})^p}\uparrow & \searrow & {\scriptstyle (\mathrm{Ha}_m)^2}\uparrow \\
F^\star\omega^{(1-p)}|_{Z^{ord}} & \xrightarrow{(\mathrm{Ha}_m)^p} & F^\star(\Omega^{oo}_{\mathcal{G}})^{(1-p)} & & (\Omega^{oo}_{\mathcal{G}})^2\otimes(\Omega^{oo}_{\mathcal{G}})^{2(p-1)} \\
 & & & & {\scriptstyle -\mathrm{Ha}_{oo}\otimes(\mathrm{Ha}_m)^{-1}}\uparrow \\
 & & & & \omega^{(p-1)}|_{Z^{ord}}
\end{array}$$

By Lemma 3.5, we find that $O''' = \mathrm{Ha}_m^p \mathrm{Ha}_{oo}^p \mathrm{Ha}_{oo}^{-1} \mathrm{Ha}_m = \mathrm{Ha}'$.

4 Companion forms

4.1 Companion forms for modular curves

Let $K = \prod_\ell K_\ell \subseteq \mathbb{G}(\mathbb{A}_f)$ be a compact open subgroup, with $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$. Let $Sh_K^{tor} \to \mathrm{Spec}\ \mathbb{Z}_{(p)}$ be the compactified modular curve and let $\mathfrak{X} = Sh_K^{tor} \times \mathrm{Spec}\ \mathbb{Z}_p$ and $X = \mathfrak{X} \times \mathrm{Spec}\ \mathbb{F}_p$. We now set up some notations regarding Hecke correspondences. Let ℓ be a prime with $K_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$. Let $\mathfrak{X}_0(\ell)$ be the compactified modular curve of level $\Gamma_0(\ell)$ at ℓ and level K_s at $s \neq \ell$. Away from the boundary, it parametrizes a finite flat group scheme H of order ℓ in $\mathcal{G}[\ell]$. We have two maps $p_1, p_2 : \mathfrak{X}_0(\ell) \to \mathfrak{X}$, informally given by $p_1(\mathcal{G}, H) = \mathcal{G}$ and $p_2(\mathcal{G}, H) = \mathcal{G}/H$. By convention, we attach to the double class $G(\mathbb{Z}_\ell)\mathrm{diag}(\ell, 1)G(\mathbb{Z}_\ell)$ the Hecke correspondence $\mathfrak{X}_0(\ell)$. Its reduction modulo p is denoted $X_0(\ell)$. We also attach to the double class $G(\mathbb{Z}_\ell)\mathrm{diag}(\ell, \ell)G(\mathbb{Z}_\ell)$ the map $S_\ell = \langle \ell \rangle : \mathfrak{X} \to \mathfrak{X}$ which is induced by mutliplication by ℓ on the universal $\mathcal{G}$. These are the traditional conventions (see [Gro90, Sect 3]), which however differ with modern conventions in the theory of Shimura varieties (where everything is transposed, see [FP21, Rem 5.6]).

4.1.1 Preliminaries on the supersingular locus

We let Z be the supersingular locus. We have defined in Section 3.4.1 $\mathrm{Ha}_{oo} \in \mathrm{H}^0(Z, \omega^{(p+1)})$ and $\mathrm{Ha}' = \mathrm{Ha}_{oo}^{p-1}$. We denote by F_Z the Frobenius map of Z.

Lemma 4.1. *We have $F_Z^2 = \langle p \rangle$ as automorphisms of Z.*

Proof. For any supersingular elliptic curve E, there is an isomorphism $E^{(p^2)} \simeq E$, identifying $F^2 : E \to E^{(p^2)}$ with multiplication by p. This follows from the observation that $\mathrm{Ker} F^2 = E[p]$. See [KM85, Thm 2.9.4] □

Lemma 4.2. *For any $k \in \mathbb{Z}$, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{H}^0(Z, \omega^k) & \xrightarrow{F_Z^2} & \mathrm{H}^0(Z, \omega^{p^2 k}) \\ & \searrow^{\langle p \rangle} & \downarrow{\scriptstyle (-\mathrm{Ha}')^{-k}} \\ & & \mathrm{H}^0(Z, \omega^k) \end{array}$$

Proof. Let E be a supersingular elliptic curve. We can find a basis e_1, e_2 for the Dieudonné module of E where e_1 lifts the Hodge filtration, such that the matrix of F has the shape $\begin{pmatrix} 0 & b \\ -pb^{-1} & 0 \end{pmatrix}$ for b a lift of Ha_{oo}. As a result $F^2 = \begin{pmatrix} -pb^{1-p} & 0 \\ 0 & -pb^{p-1} \end{pmatrix}$. We deduce that $\omega_{E^{(p^2)}} = \omega_E^{p^2} \simeq \omega_E$ induced by $p^{-1}F^2$ is given by $-\mathrm{Ha}_{oo}^{1-p}$. □

4.1.2 The sheaf $\omega^{k,tm}$ and its cohomology

We can explicit the maps O' and O in the case of modular curves.

Theorem 4.1. *The map $O' : F^\star \Omega^1_X(\log\ D) \to \omega^{(p-1)}|_Z$ is given by the composition:*

$$F^\star \Omega^1_X(\log\ D) \simeq \omega^{2p} \longrightarrow \omega^{2p}|_Z \xrightarrow{-\mathrm{Ha}_{oo}^{-1}} \omega^{(p-1)}|_Z$$

The map O is given by the composition:

$$F^\star \Omega^1_X(\log\ D) \simeq \omega^{2p} \longrightarrow \omega^{2p}|_Z \xrightarrow{-\mathrm{Ha}_{oo}^{-1}} \omega^{(p-1)}|_Z \longrightarrow \mathscr{C}_X[1]$$

Proof. By Theorem 3.4 and Lemma 3.8, we have a commutative diagram:

$$\begin{array}{ccc} F^\star\Omega^1_X(\log D)|_Z \simeq \omega^{2p}|_Z & \xrightarrow{O'} & \omega^{(p-1)}|_Z \\ {\scriptstyle -\mathrm{Ha}^p_{oo}}\uparrow & \nearrow_{\mathrm{Ha}^{p-1}_{oo}} & \\ F^\star\omega^{(1-p)} & & \end{array}$$

□

Let $2 \leq k \leq p$. We can define a map $O_k : F^\star\omega^k \to \omega^{2-k}[1]$ as follows:

$$F^\star\omega^k \to F^\star\omega^k|_Z \xrightarrow{(-\mathrm{Ha}_{oo})^{1-k}} \omega^{2-k} \otimes \omega^{p-1}|_Z \to \omega^{2-k}[1]$$

Definition 4.1. *We let* $\omega^{k,tm} = \mathrm{Fib}(O_k : \omega^k \to F_\star\omega^{2-k}[1])$.

By definition, $\omega^{k,tm}$ sits in an extension:

$$0 \to F_\star\omega^{2-k} \to \omega^{k,tm} \to \omega^k \to 0 \tag{4.1.A}$$

which is the pull back of the extension $0 \to F_\star\omega^{2-k} \xrightarrow{\mathrm{Ha}} F_\star\omega^{p+1-k} \to F_\star\omega^{p+1-k}|_Z \to 0$ along the map $\omega^k \to F_\star\omega^{2-k} \otimes \omega^{p-1}|_Z$. As a result,

$$\omega^{k,tm} = \mathrm{Ker}(\omega^k \oplus F_\star\omega^{p+1-k} \to F_\star\omega^{p+1-k}|_Z)$$

where the map is $(f, g) \mapsto f|^p_Z(-\mathrm{Ha}_{oo})^{k-1} - g|_Z$. We deduce that $\omega^{k,tm}$ can also be written as an extension in a different way:

$$0 \to \omega^{k+1-p} \to \omega^{k,tm} \to F_\star\omega^{p+1-k} \to 0. \tag{4.1.B}$$

Proposition 4.1. *We have* $\mathrm{H}^0(X, \omega^{k,tm}) = \{(f, g) \in \mathrm{H}^0(X, \omega^k) \times \mathrm{H}^0(X, \omega^{p+1-k}), f|^p_Z(-\mathrm{Ha}_{oo})^{1-k} = g|_Z\}$.

Proof. This follows from the description of $\omega^{k,tm}$. □

Remark 4.1. The formula $f|^p_Z(-\mathrm{Ha}_{oo})^{1-k} = g|_Z$ is equivalent to $\langle p\rangle f = g|^p_Z(-\mathrm{Ha}_{oo})^{k-p}$ by Lemma 4.2.

Proposition 4.2. *1. We have an exact sequence* $0 \to \mathbb{F}_p\mathrm{Ha} \to \mathrm{H}^0(X, \omega^{2,tm}) \to \mathrm{H}^0(X, \omega^2)$.

2. If $3 \leq k \leq p$, *we have an inclusion* $\mathrm{H}^0(X, \omega^{k,tm}) \to \mathrm{H}^0(X, \omega^k)$.

3. If $2 \leq k \leq p-2$, *we have an inclusion* $\mathrm{H}^0(X, \omega^{k,tm}) \to \mathrm{H}^0(X, \omega^{p+1-k})$.

4. We have an exact sequence $0 \to \mathbb{F}_p\mathrm{Ha} \to \mathrm{H}^0(X, \omega^{p-1,tm}) \to \mathrm{H}^0(X, \omega^2)$.

Proof. This follows from Proposition 4.1, and from the vanishing of $\mathrm{H}^0(X, \omega^{-k})$ for $k < 0$. □

Proposition 4.3. *When* $k = p$, *the extension 4.1.B splits:* $\omega^{p,tm} = F_\star\omega \oplus \omega$, *and therefore* $\mathrm{H}^0(X, \omega^{p,tm}) = \mathrm{H}^0(X, \omega) \oplus \mathrm{H}^0(X, \omega)$.

Proof. We have a twisted Cartier map $C : F_\star\omega^p \to \omega^p$. We claim that the map $F_\star\omega \to \omega^p \oplus F_\star\omega$, $f \mapsto (C(\mathrm{Ha}f), f)$ factors through $\omega^{p,tm}$. This gives the desired splitting. We therefore need to see that $(C(\mathrm{Ha}f)|_Z)^p\mathrm{Ha}^{1-p}_{oo} = f|_Z$. We claim that there is a commutative diagram:

$$\begin{array}{ccc} F_\star\omega & \xrightarrow{C(\mathrm{Ha}-)} & \omega^p \\ \downarrow & & \downarrow \\ F_\star\omega|_Z & \longrightarrow & \omega^p|_Z \end{array}$$

where the bottom map is simply the Cartier map on Z, that is the inverse of the Frobenius map (as Z is zero dimensional) $\omega^p \to F_\star\omega^{p^2} \simeq F_\star\omega$, from which the identity $(C(\mathrm{Ha}f)|_Z)^p \mathrm{Ha}_{oo}^{1-p} = f|_Z$ follows. We recall the definition of the map C. We have a usual Cartier map $C : F_\star K_X \to K_X$ for the canonical sheaf K_X on X. This map can be twisted to a map $F_\star(K_X(D) \otimes \omega^{p(p-2)}) \to K_X(D) \otimes \omega^{p-2}$ and we get a Cartier map in weight p, $C : F_\star\omega^p \overset{\mathrm{Ha}^{(p-2)}}{\to} K_X \otimes \omega^{p(p-2)} \to \omega^p$ (using Kodaira-Spencer). From weight 1, we define $C(\mathrm{Ha}-) : F_\star\omega \overset{\mathrm{Ha}^{(p-1)}}{\to} F_\star K_X(D) \otimes \omega^{p(p-2)}(-(p-1)Z) \to K_X \otimes \omega^{p(p-2)} \to \omega^p$. By [BCGP21, Prop 3.8.17], this map restricts to a map $F_\star\omega|_Z \simeq F_\star K_Z \otimes \omega^{p(p-2)}(-pZ)|_Z \to K_Z \otimes \omega^{(p-2)}(-Z)$ which is simply the Cartier map for Z, twisted by the sheaf $\omega^{(p-2)}(-Z)$. □

Remark 4.2. In the exact sequence 4.1.A $0 \to F_\star\omega^{2-p} \to \omega \oplus F_\star\omega \to \omega^p \to 0$, the first map is $f \mapsto (\frac{C(\mathrm{Ha}^2 f)}{\mathrm{Ha}}, -\mathrm{Ha}f)$ and the second map is $(f, g) \mapsto (\mathrm{Ha}f + C(\mathrm{Ha}g))$.

4.1.3 Conditions at the cusps

It is possible to impose vanishing conditions at the boundary on the sheaves $\omega^{k,tm}$. Namely, there is a commutative square:

$$\begin{array}{ccc} \omega^k & \longrightarrow & F_\star\omega^{2-k}[1] \\ \uparrow & & \uparrow \\ \omega^k(-D) & \longrightarrow & F_\star\omega^{2-k}(-D)[1] \end{array}$$

We deduce from this square the existence of four extensions which only differ by boundary conditions. We therefore define the subsheaves $\omega^{k,tm}(-D_k)$, $\omega^{k,tm}(-D_{2-k})$ and $\omega^{k,tm}(-D_k - D_{2-k})$ where the subscript on D is specifying on which sheaf of 4.1.A there is a cuspidal condition. For example $\omega^{k,tm}(-D_k)$ is an extension $0 \to F_\star\omega^{2-k} \to \omega^{k,tm}(-D_k) \to \omega^k(-D) \to 0$. Observe that the second exact sequence 4.1.B becomes $0 \to \omega^{k-p+1}(-D) \to \omega^{k,tm}(-D_k) \to F_\star\omega^{p+1-k} \to 0$.

4.1.4 The sheaf $\mathcal{L}^{k,tm}$ on $X_0(p)$

Recall that $X_0(p)$ is the compactified modular curve of level $\Gamma_0(p)$ at p and level K_ℓ at $\ell \neq p$. Away from the boundary, it parametrizes a finite flat group scheme H of order p in $\mathcal{G}[p]$. We have two maps $p_1, p_2 : X_0(p) \to X$, informally given by $p_1(\mathcal{G}, H) = \mathcal{G}$ and $p_2(\mathcal{G}, H) = \mathcal{G}/H$.

Proposition 4.4. *There is a line bundle $\mathcal{L}^{k,tm}$ over $X_0(p)$ such that $(p_1)_\star\mathcal{L}^{k,tm} = \omega^{k,tm}$.*

Proof. By [DR73, Thm 1.16 on p 243] we have $X_0(p) = X_{et} \cup X_m$ where X_{et} is the union of irreducible components where the group H is generically étale and X_m the union of irreducible components where the group H is generically multiplicative. We fix isomorphisms $X \simeq X_m$, by $\mathcal{G} \mapsto (\mathcal{G}, \mathrm{Ker}F)$ and $X \simeq X_{et}$ by $\mathcal{G} \mapsto (\mathcal{G}/\mathrm{Ker}F, \mathcal{G}[p]/\mathrm{Ker}F)$. We deduce that $p_1|_{X_m} : X_m \to X$ is the identity, while $p_1|_{X_{et}} : X_{et} \to X$ identifies with the Frobenius of X. As a result $X_0(p)$ is the pushout (where $i : Z \to X$ is the obvious closed immersion):

$$\begin{array}{ccc} Z \coprod Z & \xrightarrow{i, i\circ F_Z} & X \coprod X \\ \downarrow{\scriptstyle Id, Id} & & \downarrow{\scriptstyle F, Id} \\ Z & \longrightarrow & X_0(p) \end{array}$$

We deduce that $(p_1)_\star\mathscr{O}_{X_0(p)} = \mathscr{O}_X \times_{\mathscr{O}_Z} F_\star\mathscr{O}_X = \{(a, b), a^p|_Z = b|_Z\}$. By pushforward, coherent sheaves on $X_0(p)$ identify with coherent sheaves of $(p_1)_\star\mathscr{O}_{X_0(p)}$-modules over X. We see from the discussion after Definition 4.1 that $\omega^{k,tm}$ can be endowed with an action of $(p_1)_\star\mathscr{O}_{X_0(p)}$, induced by the action of $\mathscr{O}_X \oplus F_\star\mathscr{O}_X$ on $\omega^k \oplus F_\star\omega^{p+1-k}$. Therefore $\omega^{k,tm} = (p_1)_\star\mathcal{L}^{k,tm}$ for a coherent sheaf $\mathcal{L}^{k,tm}$ on $X_0(p)$ which is clearly locally isomorphic to $\mathscr{O}_{X_0(p)}$. □

Remark 4.3. The sheaf $\mathcal{L}^{k,tm}$ restricts to ω^{p+1-k} on $X_{et} = X$ and to ω^k on $X = X_m$. The exact sequences 4.1.A and 4.1.B are obtained by restricting the sheaf $\mathcal{L}^{k,tm}$ to X_m or X_{et}. The various cuspidality conditions on the sheaf $\omega^{k,tm}$ (Section 4.1.3) correspond to being non-cuspidal, cuspidal only on X_m, cuspidal only on X_{et}, or cuspidal on X_m and X_{et} for $\mathcal{L}^{k,tm}$.

4.1.5 Duality

We define a dual map $F_\star\omega^k \to \omega^{2-k}[1]$ as the composition

$$F_\star\omega^k \to (F_Z)_\star\omega^k|_Z \overset{(-\mathrm{Ha}_{oo})^{p-k}}{\longrightarrow} \omega^{2-k} \otimes \omega^{p-1}|_Z \to \omega^{2-k}[1]$$

where the middle map is the twisted Cartier map $(F_Z)_\star\omega^k|_Z \to (F_Z)_\star\omega^{p(p+1-k)}|_Z \to \omega^{p+1-k}|_Z$. We let $\omega^{k,tm,\vee} = \mathrm{Fib}(F_\star\omega^k \to \omega^{2-k}[1])$.

Lemma 4.3. *The twisted Cartier map $(F_Z)_\star\omega^k|_Z \to \omega^{p+1-k}|_Z$ is given by the map $\mathrm{H}^0(Z,\omega^k) \to \mathrm{H}^0(Z,\omega^{p+1-k})$, $f \mapsto \langle p\rangle^{-1} f^p(-\mathrm{Ha}_{oo})^{1-k}$.*

Proof. The map $(F_Z)_\star\omega^k|_Z \to \omega^{p+1-k}$ is a twisted Cartier map on Z, inverse to the Frobenius map. We can thus apply Lemma 4.2. □

We deduce that $\omega^{k,tm,\vee}$ is an extension $0 \to \omega^{2-k} \to \omega^{k,tm,\vee} \to F_\star\omega^k \to 0$. It can be described by:

$$\omega^{k,tm,\vee} = \mathrm{Ker}(F_\star\omega^k \oplus \omega^{p+1-k} \to i_\star\omega^{p+1-k}|_Z)$$

where the map is $(f,g) \mapsto \langle p\rangle^{-1} f|_Z^p(-\mathrm{Ha}_{oo})^{1-k} - g|_Z$. We have $\mathrm{H}^0(X,\omega^{k,tm,\vee}) = \{(f,g) \in \mathrm{H}^0(X,\omega^k) \times \mathrm{H}^0(X,\omega^{p+1-k}), \langle p\rangle^{-1} f|_Z^p(-\mathrm{Ha}_{oo})^{1-k} = g|_Z\}$.

Remark 4.4. The identity $\langle p\rangle^{-1} f|_Z^p(-\mathrm{Ha}_{oo})^{1-k} = g|_Z$ is equivalent to $f|_Z(-\mathrm{Ha}_{oo})^{p-k} = g|_Z^p$.

Proposition 4.5. *If $2 \le k \le p-1$, we have a canonical isomorphism $\omega^{k,tm} = \omega^{p+1-k,tm,\vee}$. We also have an isomorphism $\omega^{p,tm,\vee} = \omega^{p,tm} = F_\star\omega \oplus \omega$.*

Proof. The first point is clear from the definition and Remark 4.4. For $p = k$, we have $\omega^{p,tm,\vee} = F_\star\omega \oplus \omega$, provided by the map $F_\star\omega \oplus \omega \to F_\star\omega^p \oplus \omega$, $(f,g) \mapsto (\mathrm{Ha}f + g^p, g)$ (factorizing through $\omega^{p,tm,\vee}$). The map $\omega^{2-p} \to F_\star\omega \oplus \omega$ is given by $f \mapsto (f^p\mathrm{Ha}^{p-1}, -f\mathrm{Ha})$. □

Let us denote by $\mathrm{D}(-) = \mathrm{RHom}(-,\Omega_X^1)$ the Serre duality functor on X. We can define cuspidal conditions on $\omega^{k,tm,\vee}$ as in Section 4.1.3.

Lemma 4.4. *We have an isomorphism $D(\omega^{k,tm}) \to \omega^{k,tm,\vee}(-D_k - D_{2-k})$.*

Proof. Dualizing the map O_k and applying the shift [1] gives :

$$F_\star\omega^k(-D) \to F_\star\omega^k|_Z \overset{-\mathrm{Ha}_{oo}}{\to} F_\star\omega^{p+1+k}|_Z \overset{(-\mathrm{Ha}_{oo})^{1-k}}{\to}$$

$$F_\star\omega^{-pk}|_Z \to \omega^{-k}|_Z \overset{-\mathrm{Ha}_{oo}}{\to} \omega^{p+1-k}|_Z \to \omega^{2-k}(-D)[1]$$

and the fiber of this map is $\omega^{k,tm,\vee}(-D_k - D_{2-k})$. □

Proposition 4.6. *Let $\mathcal{L}^{k,tm}$ be the sheaf on $X_0(p)$ from Proposition 4.4. Let w be the Weil involution on $X_0(p)$.*

1. *If $2 \le k \le p-1$, we have $w^\star\mathcal{L}^{k,tm} = \mathcal{L}^{p+1-k,tm}$.*
2. *If $2 \le k \le p$, $(p_2)_\star\mathcal{L}^{k,tm} = \omega^{k,tm,\vee}$.*

Proof. We remark that $p_2 : X_m \to X$ identifies with Frobenius and $p_2 : X_{et} \to X$ identifies with $\langle p\rangle$. The Weil involution is the automorphism (which is not $\mathscr{O}_X$-linear) of $(p_1)_\star\mathscr{O}_{X_0(p)}$, $(a,b) \mapsto (b, \langle p\rangle a)$. Moreover, $p_2 = p_1 \circ w$. The sheaf $\mathcal{L}^{k,tm}$ is the subsheaf of $\omega^k|_{X_m=X} \oplus \omega^{p+1-k}|_{X_{et}=X}$ of sections (f,g) which satisfy the condition $f^p|_Z(-\mathrm{Ha}_{oo})^{1-k} = g|_Z$. The w-linear map $(f,g) \mapsto (g, \langle p\rangle f)$ induces an isomorphism between $w^\star\mathcal{L}^{k,tm}$ and the subsheaf of $\omega^{p+1-k} \oplus \omega^k$ of sections (f',g') satisfying $(f')^p|_Z(-\mathrm{Ha}_{oo})^{p-k} = g'|_Z$ by Remark 4.1. It follows that when $2 \le k \le p-1$, this sheaf is $\mathcal{L}^{p+1-k,tm}$ and that for $2 \le k \le p$, $(p_1)_\star w^\star\mathcal{L}^{k,tm} = (p_2)_\star\mathcal{L}^{k,tm} = \omega^{k,tm,\vee}$. □

Remark 4.5. The isomorphism of Proposition 4.5 for $2 \leq k \leq p-1$ is induced by combining 1. and 2. in Proposition 4.6. Lemma 4.4 and Proposition 4.6 imply that the Serre dual of the line bundle $\mathcal{L}^{k,tm}$ over $X_0(p)$ is isomorphic to its pullback via the Weil involution on $X_0(p)$ (up to the conditions at cusps).

Corollary 4.1. *There is a canonical isomorphism of exact triangles:*

$$\begin{array}{ccccccc} \mathrm{R}\Gamma(X,\omega^{2-k}) & \longrightarrow & \mathrm{R}\Gamma(X,\omega^{k,tm}) & \longrightarrow & \mathrm{R}\Gamma(X,\omega^{k}) & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ \mathrm{R}\Gamma(X,\omega^{2-k}) & \longrightarrow & \mathrm{R}\Gamma(X,\omega^{k,tm,\vee}) & \longrightarrow & \mathrm{R}\Gamma(X,\omega^{k}) & \longrightarrow \end{array}$$

Proof. The sheaf $\mathcal{L}^{k,tm}$ sits in an exact sequence $0 \to \mathcal{L}^{k,tm} \otimes \mathscr{I}_{X_m} \to \mathcal{L}^{k,tm} \to \mathcal{L}^{k,tm}|_{X_m} \to 0$. Both triangle identify with $\mathrm{R}\Gamma(X_0(p), \mathcal{L}^{k,tm} \otimes \mathscr{I}_{X_m}) \to \mathrm{R}\Gamma(X_0(p), \mathcal{L}^{k,tm}) \to \mathrm{R}\Gamma(X_0(p), \mathcal{L}^{k,tm}|_{X_m}) \to$ according to Proposition 4.6, 2. □

Corollary 4.2. *We have a canonical perfect pairing* $\mathrm{H}^0(X,\omega^{k,tm}) \times \mathrm{H}^1(X,\omega^{k,tm}(-D_k - D_{2-k})) \to \mathbb{F}_p$.

Proof. We apply Serre duality on X, and use $\mathrm{D}(\omega^{k,tm}(-D_k - D_{2-k})) = \omega^{k,tm,\vee}$ by Proposition 4.5, together with the isomorphism $\mathrm{H}^0(X,\omega^{k,tm}) = \mathrm{H}^0(X,\omega^{k,tm,\vee})$ from Corollary 4.1. □

4.1.6 Hecke action away from p

We let $\mathcal{H}^p$ be the abstract prime-to-p Hecke algebra of level K^p. The Hecke algebra $\mathcal{H}^p$ acts naturally on the cohomology $\mathrm{R}\Gamma(X,\omega^k)$. For example, we attach to T_ℓ the correspondence $p_1, p_2 : X_0(\ell) \to X$, and we have natural cohomological correspondences $p_2^\star\omega^k \to p_1^!\omega^k$. We review the construction (see [BP22, Sect 3.1]). We have a fundamental class $B : p_1^\star\mathscr{O}_X \to p_1^!\mathscr{O}_X$ given by the trace of p_1. We have a map $A : p_2^\star\omega \to p_1^\star\omega$ induced by the differential of the universal isogeny. We construct a map $T_\ell^{naive} : p_2^\star\omega^k \overset{A^{\otimes k}}{\to} p_1^\star\omega^k \overset{B}{\to} p_1^!\omega^{-k}$ and set $T_{\ell,k} = p^{-\inf\{1,k\}}T_\ell^{naive}$. We sometimes only write T_ℓ for $T_{\ell,k}$ if the context is clear.

Lemma 4.5. *Let E, E' be two supersingular elliptic curves and let $f : E \to E'$ be an étale isogeny. We have $f^\star\mathrm{Ha}_{oo}(E') = \deg(f)\mathrm{Ha}_{oo}(E)$.*

Proof. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^1_{dR}(E')^{(p)} & \xrightarrow{(f^{(p)})^\star} & \mathrm{H}^1_{dR}(E)^{(p)} \\ \downarrow{\scriptstyle F^\star} & & \downarrow{\scriptstyle F^\star} \\ \mathrm{H}^1_{dR}(E') & \xrightarrow{f\star} & \mathrm{H}^1_{dR}(E) \end{array}$$

Considering the Hodge filtration, we extract the following diagram:

$$\begin{array}{ccc} \omega_{E'}^{-p} & \xrightarrow{(f^{\vee(p)})^\star} & \omega_E^{-p} \\ \downarrow{\scriptstyle \mathrm{Ha}_{00}(E')} & & \downarrow{\scriptstyle \mathrm{Ha}_{00}(E)} \\ \omega_{E'} & \xrightarrow{f^\star} & \omega_E \end{array}$$

Tensoring with $(f^\star)^{\otimes p} : \omega_{E'}^p \to \omega_E^p$ and using $ff^\vee = \deg(f)$, we get $f^\star\mathrm{Ha}_{oo}(E') = \deg(f)\mathrm{Ha}_{oo}(E)$. □

Proposition 4.7. *There is an action of T_ℓ on $\mathrm{R}\Gamma(X,\omega^{k,tm})$. This action respects the triangle $\mathrm{R}\Gamma(X,\omega^{2-k}) \to \mathrm{R}\Gamma(X,\omega^{k,tm}) \to \mathrm{R}\Gamma(X,\omega^k) \to$ and induces the actions $T_{\ell,2-k}$ and $T_{\ell,k}$ on the left and right terms respectively. This action respects the triangle $\mathrm{R}\Gamma(X,\omega^{k+1-p}) \to \mathrm{R}\Gamma(X,\omega^{k,tm}) \to \mathrm{R}\Gamma(X,\omega^{p+1-k}) \to$ and induces the actions $\ell^{k-1}T_{\ell,k+1-p}$ and $\ell^{k-1}T_{\ell,p+1-k}$ on the left and right terms respectively.*

Proof. We claim that the following diagram commutes:

$$\begin{array}{ccc} p_2^\star\omega^k & \longrightarrow & p_2^\star F_\star\omega^{p+1-k}[1] \\ \downarrow{\scriptstyle T_\ell^{naive}} & & \downarrow{\scriptstyle \ell^{k-1}T_\ell^{naive}} \\ p_1^!\omega^k & \longrightarrow & p_1^! F_\star\omega^{p+1-k}[1] \end{array}$$

inducing a cohomological correspondence T_ℓ^{naive} on $\omega^{k,tm}$. This boils down to the property that the following diagram commutes :

$$\begin{array}{ccc} p_2^\star(F_Z)_\star\omega^{pk}|_Z & \xrightarrow{{p_2^\star\mathrm{Ha_{oo}}}^{1-k}} & p_2^\star(F_Z)_\star\omega^{p+1-k}|_Z \\ \downarrow & & \downarrow{\scriptstyle \ell^{k-1}} \\ p_1^\star(F_Z)_\star\omega^{pk}|_Z & \xrightarrow{{p_2^\star\mathrm{Ha_{oo}}}^{1-k}} & p_1^\star(F_Z)_\star\omega^{p+1-k}|_Z \end{array}$$

which follows from the property that ${p_2^\star\mathrm{Ha_{oo}}}^{1-k} = \ell^{1-k}{p_1^\star\mathrm{Ha_{oo}}}^{1-k}$ by Lemma 4.5. If we normalize this action by ℓ^{-1} we get the expected T_ℓ-action. Here is another way to think about this action. Consider the action of $T_{\ell,k}\oplus\ell^{k-1}T_{\ell,p+1-k}$ on $\omega^k\oplus F_\star\omega^{p+1-k}$. This induce the action of T_ℓ on $\omega^{k,tm}$. If (f,g) are sections of $\omega^{k,tm}$ satisfying $f^p|_Z(-\mathrm{Ha})_{oo}^{1-k} = g$, we find $T_\ell^{naive}(f^p|_Z(-\mathrm{Ha}_{oo})^{1-k}) = \ell^{1-k}T_\ell^{naive}(f^p|_Z)(-\mathrm{Ha}_{oo})^{1-k} = T_\ell^{naive}(g)$. □

Corollary 4.3. *The Hecke algebra $\mathcal{H}^p$ acts on $\mathrm{R}\Gamma(X,\omega^{k,tm})$, with T_ℓ acting as in Proposition 4.7.*

Proof. We briefly reintroduce the level $K = K^pK_p$ in the notations. We consider the modular curves X_K modulo p and the sheaf of modular forms ω_K^k. We can consider the complex $\mathrm{colim}_{K^p}\omega^k_{K^pK_p} \to F_\star\omega^{2-k}_{K^pK_p}$ over $\lim_{K^p} X_{K^pK_p}$ and equip it with a $G(\mathbb{A}_f^p)$-equivariant action, suitably normalized as explained before. We thus get a $G(\mathbb{A}_f^p)$-action on $\mathrm{R}\Gamma(\mathrm{Fib}(\mathrm{colim}_{K^p}\omega^k_{K^pK_p} \to F_\star\omega^{2-k}_{K^pK_p})$ and after taking K^p invariants we obtain an action of the Hecke algebra. □

Corollary 4.4. *Eigenforms in $\mathrm{H}^0(X,\omega^{k,tm})$ are pairs of companion forms (f,g) of weight k and $(p+1-k)$, $(\mathrm{Ha},0)$ if $k=p-1$, $(0,\mathrm{Ha})$ if $k=2$, and $(\mathrm{Hah},0)$ with h a weight 1 eigenform if $k=p$.*

Proof. Clear from Proposition 4.2 and Proposition 4.3. □

4.1.7 Relation to crystalline cohomology

Let $\bar\eta : \bar{\mathcal{G}} \to \mathfrak{X}$ be the universal generalized elliptic curve ([DR73]). We have a filtered log-F-crystal $\mathcal{H} = \mathrm{R}^1\bar\eta_{\star,\log-crys}\mathscr{O}_{\bar{\mathcal{G}}}$. For $2 \le k \le p-1$, we consider the filtered log-F-crystal $\mathrm{Sym}^{k-2}\mathcal{H}$. We now construct an object of $\mathcal{D}^b(\mathcal{MF}_{[0,k-1]}(\mathbb{F}_p))$. The underlying filtered object is $\mathrm{R}\Gamma_{\log-dR}(X,\mathrm{Sym}^{k-2}\mathcal{H})$ with filtration given by the map coming from the Hodge filtration: $\mathrm{R}\Gamma(X,\omega^k)[-1] \to \mathrm{R}\Gamma_{\log-dR}(X,\mathrm{Sym}^{k-2}\mathcal{H})$ with cofiber $\mathrm{R}\Gamma(X,\omega^{2-k})$. We next construct the Frobenius. We consider $\mathrm{R}\Gamma_{\log-crys}(X/(W_k(\mathbb{F}_p),\gamma^{triv}),\mathrm{Sym}^{k-2}\mathcal{H})$ which comes equipped with a Frobenius ϕ_{crys}. Let $\tilde X$ be the modular curve over $W_k(\mathbb{F}_p)$. We have an isomorphism:

$$\mathrm{R}\Gamma_{\log-crys}(X/(W_k(\mathbb{F}_p),\gamma^{triv}),\mathrm{Sym}^{k-2}\mathcal{H}) \simeq \mathrm{R}\Gamma_{\log-dR}(\tilde X,\mathrm{Sym}^{k-2}\mathcal{H}).$$

Reducing modulo p this isomorphism, ϕ_{crys} induces an endomorphism ϕ_0 of $\mathrm{R}\Gamma_{\log-dR}(X,\mathrm{Sym}^{k-2}\mathcal{H})$, which factors through a map $\mathrm{R}\Gamma(X,\omega^{2-k}) \to \mathrm{R}\Gamma_{\log-dR}(X,\mathrm{Sym}^{k-2}\mathcal{H})$ (see the proof of Proposition 4.8 below). On the other hand, the restriction of ϕ_{crys} to $\mathrm{R}\Gamma(\tilde X,\omega^k)[-1]$ factors through a map (see also the proof of Proposition 4.8 below):

$$\mathrm{R}\Gamma(\tilde X,\omega^k)[-1] \to p^{k-1}\mathrm{R}\Gamma_{\log-crys}(X/(W_k(\mathbb{F}_p),\gamma^{triv}),\mathrm{Sym}^{k-2}\mathcal{H}) \simeq \mathrm{R}\Gamma_{\log-dR}(X,\mathrm{Sym}^{k-2}\mathcal{H})$$

which induces a map $\phi_{k-1} : \mathrm{R}\Gamma(X, \omega^k)[-1] \to \mathrm{R}\Gamma_{\log-dR}(X, \mathrm{Sym}^{k-2}\mathcal{H})$. We can now consider the composition

$$\bar{\phi}_{k-1} : \mathrm{R}\Gamma(X, \omega^k) \stackrel{\phi_{k-1}}{\to} \mathrm{R}\Gamma_{\log-dR}(X, \mathrm{Sym}^{k-2}\mathcal{H})[1] \to \mathrm{R}\Gamma(X, \omega^{2-k})[1]$$

where the second map is projection along the Hodge filtration. The following proposition is inspired by [FJ95, p 62-64]. When $k = 2$, this is also a particular case of Proposition 2.3.

Proposition 4.8. *We have a quasi-isomorphism* $\mathrm{Fib}(\bar{\phi}_{k-1} : \mathrm{R}\Gamma(X, \omega^k) \to \mathrm{R}\Gamma(X, \omega^{2-k})[1]) \simeq \mathrm{R}\Gamma(X, \omega^{k,tm})$.

Proof. We need to compute the crystalline Frobenius

$$\phi_{crys} : [\mathrm{Sym}^{k-2}\mathcal{H} \to \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D)] \to F_\star[\mathrm{Sym}^{k-2}\mathcal{H} \to \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D)]$$

We consider the fppf covering $\tilde{X} = \tilde{X}^{ord} \cup \hat{Z}$ where $\tilde{X}^{ord}$ is the ordinary locus, and $\hat{Z}$ is the completion of $\tilde{X}$ at supersingular points. Note that $\hat{Z}$ is a disjoint union of formal discs Spec $W_k(k(x))[[z]]$ parametrized by supersingular points $x \in Z$. Let us denote by $\hat{Z}^{ord}$ the intersection of $\hat{X}^{ord}$ and $\hat{Z}$, equal to a disjoint union of punctured discs Spec $W_k(k(x))((z))$. We have a canonical lift of Frobenius F^{ord} on $\tilde{X}^{ord}$. It induces a lift $F^{ord} : \mathrm{Sym}^{k-2}\mathcal{H}|_{\tilde{X}^{ord}} \to F^{ord}_\star \mathrm{Sym}^{k-2}\mathcal{H}|_{\tilde{X}^{ord}}$. This lift respects the Hodge filtration (because F^{ord} arises from an isogeny of the universal elliptic curve), and induces on each graded pieces the usual Frobenius map on spaces of p-adic modular forms. Observe that $F^{ord} : \omega^{2-k+2r}|_{\tilde{X}^{ord}} \to \omega^{2-k+2r}|_{\tilde{X}^{ord}}$ is divisible by p^r. We also fix the lift F_Z, $z \to z^p$ over $\hat{Z}$. We get another lift: $F_Z : \mathrm{Sym}^{k-2}\mathcal{H}|_{\hat{Z}} \to (F_Z)_\star \mathrm{Sym}^{k-2}\mathcal{H}|_{\hat{Z}}$ The complex $[\mathrm{Sym}^{k-2}\mathcal{H} \to \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D)]$ is quasi-isomorphic to the complex $C^\bullet$:

$$[\mathrm{Sym}^{k-2}\mathcal{H}|_{\tilde{X}^{ord}} \oplus \mathrm{Sym}^{k-2}\mathcal{H}|_{\hat{Z}} \to$$

$$\mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D)|_{\tilde{X}^{ord}} \oplus \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D)|_{\hat{Z}} \oplus \mathrm{Sym}^{k-2}\mathcal{H}|_{\hat{Z}^{ord}} \to \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D)|_{\hat{Z}^{ord}}]$$

where the first map is $(f, g) \mapsto (\mathrm{d}f, \mathrm{d}g, g - f)$ and the second map is $(a, b, c) \mapsto (a - b + \mathrm{d}c)$. We can now represent the Frobenius map

$$\phi_{crys} : [\mathrm{Sym}^{k-2}\mathcal{H} \to \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D)] \to C^\bullet$$

by sending $f \in \mathrm{Sym}^{k-2}\mathcal{H}$ to $(F^{ord}(f), F_Z(f)) \in C^0$ and sending $g \in \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D)$ to $(F^{ord}(g), F_Z(g), \eta(g)) \in C^1$. Here $\eta : \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\tilde{X}}(\log\, D) \to \mathrm{Sym}^{k-2}\mathcal{H}|_{\hat{Z}^{ord}}$ is the homotopy between the two Frobenius F_Z and F^{ord}, it sends a section g to $(\eta_x(g))_{x \in Z}$ where

$$\eta_x(g) = \sum_{n=1}^{\infty} F^{ord}(\nabla^{n-1}_{\partial_z}(\langle g, \partial_z \rangle))(F_Z(z) - F^{ord}(z))^{[n]}.$$

We now take $g \in \omega^k \otimes \Omega^1_{\tilde{X}}(\log\, D)$. We study the p-divisiblity of $\eta_x(g)$. We see that $F^{ord}(\nabla^{n-1}_{\partial_z}(\langle g, \partial_z \rangle))$ is divisible by $p^{\max\{k-1-n,0\}}$ by Griffith transversality and the divisiblity of F^{ord} on each graded piece of the Hodge filtration. We also see that $(F_Z(z) - F^{ord}(z))^{[n]} \in p^{[n]} W_k(k(z))((z)) = p^n W_k(k(z))((z))$. We deduce that $\eta_x(g)$ is divisible by p^{k-1}, confirming that ϕ_{crys} induces a map $\omega^k \otimes \Omega^1_{\tilde{X}}(\log\, D)[-1] \to p^{k-1}C^\bullet$. We now identify $p^{k-1}C^\bullet \simeq C^\bullet/p$ and project along the Hodge filtration. Only the term $n = k - 1$ survives (by Griffith transversality), and we get an induced map

$$\begin{aligned} B_x : \omega^k \otimes \Omega^1_X(\mathrm{Spec}\ k(x)[[z]]) &\to \omega^{2-k}(\mathrm{Spec}\ k(x)((z))) \\ g(z) &\mapsto F^{ord}(\langle g(z), \partial_z \rangle (KS^{-1}(\mathrm{d}z))^{-(k-2)}(k-2)!) \frac{(F_Z(z) - F^{ord}(z))^{k-1}}{k-1! p^{k-1}} \end{aligned}$$

The composition

$$\omega^k \otimes \Omega^1_X \to \oplus_x \omega^k \otimes \Omega^1_X(\mathrm{Spec}\ k(x)[[z]]) \stackrel{\sum B_x}{\to} \oplus_x \omega^{2-k}(\mathrm{Spec}\ k(x)((z)))$$

represents the map $\bar{\phi}_{k-1} : \omega^k \otimes \Omega^1_X \to \omega^{2-k}[1]$. We now explain how this map relates with O_k. We have the Frobenius map $F^{ord} : F^\star\omega^{2-k} \to \omega^{2-k} \otimes (\omega^{(1-p)})^{k-2}$, $f \mapsto f^p \mathrm{Ha}^{k-2}$. Identifying Ha with the coordinate z, we deduce that

$$F^{ord}(\langle g(z), \partial_z\rangle (KS^{-1}(\mathrm{d}z))^{-(k-2)}) = (\langle g(z), \partial_z\rangle (KS^{-1}(\mathrm{d}z))^{-(k-2)})^p z^{k-2}.$$

On the other hand, by Theorem 3.4,

$$F^{ord}(z) = z^p - \frac{p}{z}\mathrm{Ha}_{oo}(z) \bmod p^2 W_k(k(z))((z)) + pW_k(k(z))[[z]].$$

We deduce that $\frac{(F_Z(z) - F^{ord}(z))^{k-1}}{p^{k-1}} = z^{1-k}\mathrm{Ha}_{oo}(z)^{k-1}$ up to terms of degree $\geq 2-k$. As a result

$$B_x(g(z)) = \frac{1}{(k-1)z}(\langle g(z), \partial_z\rangle (KS^{-1}(\mathrm{d}z))^{-(k-2)})^p \mathrm{Ha}_{oo}(z)^{k-1}$$

up to regular terms. This proves that $\bar{\phi}_{k-1} = \frac{O_k}{k-1}$. □

We now turn to Hecke operators at p. We recall that there is an operator $T_{p,k}$ on acting on the cohomology of ω^k and that we have congruence relations that $T_{p,k}$ acts like U_p if $k \geq 2$ and like Frobenius if $k \leq 2$ ([BP21, Lem 4.1]). We will denote by $U_{p,k}$ the operator $T_{p,k}$ acting on $\mathrm{R}\Gamma(X, \omega^k)$ if $k \geq 2$ and by $U_{p,k}$ the operator $T_{p,2-k}$ acting on $\mathrm{R}\Gamma(X, \omega^{2-k})$. We think this is a suggestive notation in our context.

Corollary 4.5. *We have an action of $U_{p,k}$ on $\mathrm{R}\Gamma(X, \omega^{k,tm})$, commuting with the action of $\mathcal{H}^p$ and the triangle $\mathrm{R}\Gamma(X, \omega^{2-k}) \to \mathrm{R}\Gamma(X, \omega^{k,tm}) \to \mathrm{R}\Gamma(X, \omega^k) \to$ is $U_{p,k}$ equivariant.*

Proof. We have commuting actions of T_p and $\mathcal{H}^p$ on the log-crystalline cohomology

$$\mathrm{R}\Gamma_{\log-crys}(X/(W_p(k), \gamma^{tri}), \mathrm{Sym}^{k-2}\mathcal{H})$$

respecting the Hodge filtration and commuting with Frobenius. We can use Proposition 4.8 and we denote by $U_{p,k}$ the corresponding action of T_p and $\mathrm{R}\Gamma(X, \omega^{k,tm})$. □

Corollary 4.6. *If $2 \leq k \leq p-1$ we have an action of $U_{p,p+1-k}$ on $\mathrm{R}\Gamma(X, \omega^{k,tm})$ and the triangle $\mathrm{R}\Gamma(X, \omega^{k-p+1}) \to \mathrm{R}\Gamma(X, \omega^{k,tm}) \to \mathrm{R}\Gamma(X, \omega^{p+1-k}) \to$ is $U_{p,p+1-k}$-equivariant.*

Proof. By Proposition 4.6, we have $\mathrm{R}\Gamma(X, \omega^{k,tm}) \simeq \mathrm{R}\Gamma(X, \omega^{p+1-k,tm})$. □

Remark 4.6. We will see below (using Galois representations) that $U_{p,k}U_{p,p+1-k} = \langle p\rangle$ (at least after a non-Eisenstein localization).

Since $\mathrm{R}\Gamma(X, \omega^{p,tm}) = \mathrm{R}\Gamma(X, \omega) \oplus \mathrm{R}\Gamma(X, \omega)$, we also have an action of $T_p = T_{p,1}$. The following is another approach to the construction of a U_p-operator.

Proposition 4.9. *There is an operator U_p acting on $\mathrm{R}\Gamma(X, \omega^{p,tm})$ which satisfies the relation : $U_p^2 - U_pT_p + \langle p\rangle = 0$. Moreover, the triangle $\mathrm{R}\Gamma(X, \omega^{2-p}) \to \mathrm{R}\Gamma(X, \omega^{p,tm}) \to \mathrm{R}\Gamma(X, \omega^p) \to$ is U_p-equivariant and we have $U_{p,p} = U_p$ on the left and top term.*

Proof. It is actually easier to work with $\omega^{p,tm,\vee}$ (using Corollary 4.1). We consider the exact sequence : $0 \to \omega^{2-p} \to \omega \oplus F_\star\omega \to F_\star\omega^p \to 0$. The maps are given by $f \mapsto (\mathrm{Ha}f, -\mathrm{Ha}^{p-1}f^p)$ and $(f, g) \mapsto f^p + \mathrm{Ha}g$. We can now define an endomorphism U_p which is $f \mapsto \mathrm{Ha}^{p-2}f^p = U_{p,p}(f)$ on ω^{2-p}, given by $(f, g) \mapsto (-g, T_p(g) + \langle p\rangle f)$ on $\mathrm{R}\Gamma(X, \omega \oplus F_\star\omega)$ and $f \mapsto U_{p,p}(f)$ on $F_\star\omega^p$. The identity $U_p^2 - U_pT_p + \langle p\rangle = 0$ is clear. □

Remark 4.7. If $p \neq 2$, we deduce easily from the property that $\mathrm{H}^0(X, \omega^{2-p}) = \mathrm{H}^1(X, \omega^p) = 0$ that $U_p = U_{p,p}$ on $\mathrm{R}\Gamma(X, \omega^{p,tm})$. When $p = 2$ we can obtain this result after a non-Eisenstein localization.

4.1.8 Galois representations and companion forms

Let $k \geq 2$. Let $f \in \mathrm{H}^0(\mathfrak{X}, \omega^k(-D))$ be a weight k cuspidal eigenform. Let $\Theta_f : \mathcal{H} \to \bar{\mathbb{Z}}_p$ be the associated system of eigenvalues. By [Del71], there is an associated Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Z}}_p)$. It is irreducible, unramified away from $S \cup \{p\}$ and we have that $\det(\rho_f(\mathrm{Frob}_\ell) - X\mathrm{Id}) = X^2 - \Theta_f(T_\ell)X + \Theta_f(S_\ell)\ell^{k-1}$ for a geometric Frobenius Frob_ℓ at $\ell \notin S \cup \{p\}$. Moreover by [Sai97], ρ_f is crystalline at p, with Hodge-Tate weights $0, k-1$. It is ordinary if and only if $\Theta_f(T_p)$ is a p-adic unit. In this case, $\rho_f|_{G_{\mathbb{Q}_p}} = \begin{pmatrix} ur(\alpha) & \star \\ 0 & ur(\frac{\beta}{p^{k-1}})\chi_p^{1-k} \end{pmatrix}$ where α and β are the roots of the polynomial $X^2 - \Theta_f(T_p)X + \Theta_f(S_p)p^{k-1}$ with valuation 1 and p^{k-1} respectively.

We let $\eta : \mathcal{G} \to Sh_{K,\mathbb{Q}}$ be the universal elliptic curve over the open modular curve over Spec $\mathbb{Q}$. We define a local system $\underline{\mathrm{Sym}^{k-2}\mathbb{Z}_p^2} = \mathrm{Sym}^{k-2}\mathrm{R}^1\eta_\star\mathbb{Z}_p$ with mod p reduction $\underline{\mathrm{Sym}^{k-2}\mathbb{F}_p^2}$. We also let $j : Sh_{K,\mathbb{Q}} \to Sh^{tor}_{K,\mathbb{Q}}$ be the inclusion of the open modular curve inside its compactification. We also remind the reader that $\mathfrak{X} = Sh^{tor}_{K,\mathbb{Z}_p}$.

Theorem 4.2 ([Lan73])**.** *Let us fix an isomorphim $\bar{\mathbb{Q}}_p \simeq \mathbb{C}$. We have a $G_{\mathbb{Q}} \times \mathcal{H}$-equivariant isomorphism :*

$$\mathrm{H}^1_{et}(Sh^{tor}_{K,\bar{\mathbb{Q}}}, j_\star\underline{\mathrm{Sym}^{k-2}\mathbb{Z}_p}) \otimes \bar{\mathbb{Q}}_p = \bigoplus_\pi \rho_\pi \otimes (\pi_f)^K$$

where the sum runs over all cuspidal automorphic forms π of weight k.

We assume from now on that X has level $\Gamma_1(N)$ for some $N \geq 3$, $(N, p) = 1$. This gives cleaner results as we can use multiplicity one. Let $\mathbb{T}$ be the image of the Hecke algebra generated by $\{T_\ell\}_{\ell\nmid Np}$, $\{U_\ell\}_{\ell|N}$, $\{S_\ell\}$ and $U_{p,k}$ in $\mathrm{End}(\mathrm{H}^0(X, \omega^k(-D)))$. Let $\mathbb{T}^{tm}$ be its image in $\mathrm{End}(\mathrm{H}^0(X, \omega^{k,tm}(-D_k)))$.

Proposition 4.10. *The operator $U_{p,k}$ is invertible in $\mathbb{T}^{tm}$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{tm}$. We consider the Fontaine-Laffaille module

$$M = \mathrm{Im}\big(\mathrm{H}^1_{\log-dR}(X, \mathrm{Sym}^{k-2}\mathcal{H}(-D)) \to \mathrm{H}^1_{\log-dR}(X, \mathrm{Sym}^{k-2}\mathcal{H})\big)$$

(the category of Fontaine-Laffaille modules is abelian), with its Hodge filtration M^{k-1}, Frobenius ϕ_0 and divided Frobenius ϕ_{k-1}. This is interior cohomology. We see that $M^{k-1}[\mathfrak{m}] = \mathrm{H}^0(X, \omega^k(-D))[\mathfrak{m}]$ is one dimensional (since $\mathrm{H}^0(X, \omega^k(-D)) = \mathbb{T}^\vee$), and that $M[\mathfrak{m}]/M^{k-1}[\mathfrak{m}] = \mathrm{H}^1(X, \omega^{2-k})[\mathfrak{m}]$ (this uses that $\mathrm{Ext}^1_{\mathbb{T}}(\mathbb{T}/\mathfrak{m}, \mathrm{H}^0(X, \omega^k(-D))) = 0$) is non-zero by Serre duality. The map $\bar{\phi}_{k-1} : M^{k-1}[\mathfrak{m}] \to \bar{M}^0[\mathfrak{m}]$ is zero by Proposition 4.8. Therefore, $\phi_{k-1}(M^{k-1}[\mathfrak{m}]) = M^{k-1}[\mathfrak{m}]$. By the condition of being a Fontaine-Laffaille module $\phi_{k-1}(M^{k-1}[\mathfrak{m}]) \oplus \phi_0(\bar{M}^0[\mathfrak{m}]) = M[\mathfrak{m}]$, so ϕ_0 induces an injective (and hence bijective map) of $\phi_0(\bar{M}^0[\mathfrak{m}])$. We deduce that ϕ_0 is not nilpotent. By the Eichler-Shimura relation $\phi_0^2 - U_{p,k}\phi_0 = 0$, we deduce that $U_{p,k}$ is invertible. □

Theorem 4.3. *Let $\mathfrak{m}$ be an ordinary maximal ideal of $\mathbb{T}$.*

1. *There is an exact sequence of $\mathbb{T}_{\mathfrak{m}} \times G_{\mathbb{Q}_p}$-modules*

$$0 \to \mathrm{H}^1(X, \omega^{2-k})_{\mathfrak{m}} \to \mathrm{H}^1_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, j_\star\underline{\mathrm{Sym}^{k-2}(\mathbb{F}_p^2)})_{\mathfrak{m}} \to \mathrm{H}^0(X, \omega^k(-D))_{\mathfrak{m}} \otimes \epsilon_p^{1-k} \to 0$$

   where $\mathrm{H}^1(X, \omega^{2-k})_{\mathfrak{m}}$ is an unramified representation with Frob_p acting like $U_{p,k}$, and $\mathrm{H}^0(X, \omega^k(-D))_{\mathfrak{m}}$ is an unramified representation with Frob_p acting like $U_{p,k}^{-1}S_p$. Moreover, $\mathrm{H}^1(X, \omega^k)_{\mathfrak{m}} \simeq \mathbb{T}_{\mathfrak{m}}$, $\mathrm{H}^0(X, \omega^k(-D))_{\mathfrak{m}} \simeq (\mathbb{T}_{\mathfrak{m}})^\vee$.

2. *The extension in 1. is classified by a map $\mathrm{H}^0(X, \omega^k(-D))_{\mathfrak{m}} \to \mathrm{H}^1(X, \omega^{2-k})_{\mathfrak{m}}$ and this map viewed as a complex in degree 0 and 1 represents $\mathrm{R}\Gamma(X, \omega^{k,tm}(-D_k))_{\mathfrak{m}}$.*

3. We have a short exact sequence of $G_{\mathbb{Q}_p} \times \mathcal{H}$-modules,

$$0 \to \mathrm{H}^1(X, \omega^{k,tm}(-D_k))_{\mathfrak{m}} \to \mathrm{H}^1_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, j_\star \underline{\mathrm{Sym}^{k-2}(\mathbb{F}_p^2)})^{tm}_{\mathfrak{m}} \to \mathrm{H}^0(X, \omega^{k,tm}(-D_k))_{\mathfrak{m}} \otimes \epsilon_p^{1-k} \to 0$$

deduced from the exact sequence in 1. by pull-back along $\mathrm{H}^0(X, \omega^{k,tm}(-D_k))_{\mathfrak{m}} \to \mathrm{H}^0(X, \omega^k(-D))_{\mathfrak{m}}$ and push-out along $\mathrm{H}^1(X, \omega^{2-k})_{\mathfrak{m}} \to \mathrm{H}^1(X, \omega^{2-k,tm}(-D_k))_{\mathfrak{m}}$. This short exact sequence is split as $I_{\mathbb{Q}_p}$-module. Moreover, $\mathrm{H}^1(X, \omega^{k,tm}(-D_k))_{\mathfrak{m}} \simeq \mathbb{T}^{tm}_{\mathfrak{m}}$, $\mathrm{H}^0(X, \omega^{k,tm}(-D_k))_{\mathfrak{m}} \simeq (\mathbb{T}^{tm}_{\mathfrak{m}})^\vee$.

Proof. The representation $\mathrm{H}^1_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, j_\star \underline{\mathrm{Sym}^{k-2}(\mathbb{Z}_p^2)})_{\mathfrak{m}}$ is a lattice in an ordinary crystalline representation and corresponds to an ordinary F-gauge $\mathcal{M} \in Vect_{[0,k-1]}(\mathbb{Z}_p^{syn})$ by Theorem 2.3. Here is a geometric construction of this F-gauge If $k = 2$, we let $\mathcal{M} = \mathrm{H}^1_{syn}(\mathfrak{X})_{\mathfrak{m}} \in \mathrm{Vect}_{[0,1]}(\mathbb{Z}_p^{syn})$. There are canonical isomorphisms $T_{et}(\mathcal{M}) = \mathrm{H}^1_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, \mathbb{Z}_p)_{\mathfrak{m}}$, while $\Phi^{DI}(\mathcal{M}) = \mathrm{H}^1_{dR}(X)_{\mathfrak{m}}$ with its Fontaine-Laffaille module structure coming from the Deligne-Illusie map. If $3 \leq k \leq p$, we follow ideas of [Sch90], [Sch96] (we could certainly have used log prismatic cohomology with coefficients instead). We let $\widetilde{\bar{\mathcal{G}}^{k-2}} \to \mathfrak{X}$ be the Deligne desingularization of the $k-2$-fold product of the generalized elliptic curve $\bar{\mathcal{G}} \to \mathfrak{X}$ (see [Del71, Lem 5.5]). There is an action of the group $(\mathbb{Z}/N\mathbb{Z} \rtimes \mu_2)^{k-2} \rtimes \mathcal{S}_{k-2}$ on $\widetilde{\bar{\mathcal{G}}^{k-2}}$. We let $\epsilon : (\mathbb{Z}/N\mathbb{Z} \rtimes \mu_2)^{k-2} \rtimes \mathcal{S}_{k-2} \to \mu_2$ be the character which is trivial on $(\mathbb{Z}/N\mathbb{Z})^{k-2}$, the tautological character on each μ_2 and the sign on $\mathcal{S}_{k-2}$. We consider the syntomic cohomology $\mathrm{R}\Gamma_{syn}(\widetilde{\bar{\mathcal{G}}^{k-2}})[\epsilon]$. This is an object of $\mathrm{Perf}_{[0,p-1]}(\mathbb{Z}_p^{syn})$. Indeed, the group $(\mathbb{Z}/N\mathbb{Z} \rtimes \mu_2)^{k-2} \rtimes \mathcal{S}_{k-2}$ is of order prime to p, so this is a direct factor of the total cohomology. We claim that this cohomology is concentrated in degree $k-1$. It suffices to prove this for the de Rham realization (for example by Theorem 2.2). As in [Sai97, Lem 3], we find that there is a long exact sequence $0 \to \mathrm{H}^{k-1}_{dR}(\widetilde{\bar{\mathcal{G}}^{k-2}})[\epsilon] \to \mathrm{H}^1_{\log-dR}(\mathfrak{X}, \mathrm{Sym}^{k-2}\mathcal{H}) \to \mathrm{H}^0(D, \mathscr{O}_D) \to \mathrm{H}^k_{dR}(\widetilde{\bar{\mathcal{G}}^{k-2}})[\epsilon] \to \mathrm{H}^2_{\log-dR}(\mathfrak{X}, \mathrm{Sym}^{k-2}\mathcal{H}) \to 0$, as well as isomorphisms $\mathrm{H}^{k-2+i}_{dR}(\widetilde{\bar{\mathcal{G}}^{k-2}})[\epsilon] \to \mathrm{H}^i_{\log-dR}(\mathfrak{X}, \mathrm{Sym}^{k-2}\mathcal{H})$ for $i \neq 1, 2$. By the integral BGG decomposition (which holds in our range of weights),

$$[\mathrm{Sym}^{k-2}\mathcal{H} \to \mathrm{Sym}^{k-2}\mathcal{H} \otimes \Omega^1_{\mathfrak{X}}(\log\, D)] \simeq [\omega^{2-k} \overset{\Theta^{k-1}}{\to} \omega^k].$$

Therefore, $\mathrm{H}^i_{\log-dR}(\mathfrak{X}, \mathrm{Sym}^{k-2}\mathcal{H})$ is concentrated in degree 1. The BGG complex has a sub-complex $[\omega^{2-k} \overset{\Theta^{k-1}}{\to} \omega^k(-D)]$ with quotient $[0 \to \omega^k|_D]$. This gives the map $\mathrm{H}^1_{\log-dR}(\mathfrak{X}, \mathrm{Sym}^{k-2}\mathcal{H}) \to \mathrm{H}^0(D, \mathscr{O}_D)$ since ω is trivial on the cusps.

We therefore define $\mathcal{M} = \mathrm{H}^{k-1}_{syn}(\widetilde{\bar{\mathcal{G}}^{k-2}})[\epsilon]_{\mathfrak{m}}$. Moreover we have a canonical isomorphisms $T_{et}(\mathcal{M}) = \mathrm{H}^1_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, j_\star \underline{\mathrm{Sym}^{k-2}(\mathbb{Z}_p^2)})_{\mathfrak{m}}$, while

$$\Phi^{DI}(\mathcal{M}) = \mathrm{Im}(\mathrm{H}^1_{\log-dR}(X, \mathrm{Sym}^{k-2}\mathcal{H}(-D))_{\mathfrak{m}} \to \mathrm{H}^1_{\log-dR}(X, \mathrm{Sym}^{k-2}\mathcal{H})_{\mathfrak{m}})$$

with Frobenius coming from the crystalline Frobenius of $\mathrm{Im}(\mathrm{H}^1_{\log-crys}(X/W_p(k), \mathrm{Sym}^{k-2}\mathcal{H}(-D))_{\mathfrak{m}} \to \mathrm{H}^1_{\log-crys}(X/W_p(k), \mathrm{Sym}^{k-2}\mathcal{H})_{\mathfrak{m}})$ for the trivial divided power on $pW_p(k)$ (see Section 2.3.2). Since $T_{et}(\mathcal{M})$ is ordinary (because only ordinary modular forms contribute), we deduce that $\mathcal{M}$ is an ordinary F-gauge. Then 1. follows from the general comparison theorem for unramified F-gauge (up to cyclotomic twist), which is a simple instance of Theorem 2.3. Point 2. follows from Theorem 2.4 and Proposition 4.8, and 3. is obvious from 2. □

Remark 4.8. The $G_{\mathbb{Q}_p}$-representations $\mathrm{H}^1_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, j_\star \underline{\mathrm{Sym}^{k-2}(\mathbb{F}_p^2)})_{\mathfrak{m}}$ gives a class in $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\mathbb{T}^\vee_{\mathfrak{m}}\epsilon_p^{1-k}, \mathbb{T}_{\mathfrak{m}})$. Theorem 4.3 2. shows that $\mathbb{T}^{tm}_{\mathfrak{m}}$ is the initial quotient of $\mathbb{T}_{\mathfrak{m}}$ splitting this class.

Corollary 4.7 (Existence of companion forms)**.** *Let $f \in \mathrm{H}^0(X, \omega^k)$ be an eigenform. Assume that $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is irreducible and split on inertia. Then $f \in \mathrm{H}^0(X, \omega^{k,tm})$ admits a companion form $g \in \mathrm{H}^0(X, \omega^{p+1-k})$.*

Proof. Let $\tilde{f}$ be a lift of f to an eigenform. We have an exact sequence of $G_{\mathbb{Q}_p}$-modules : $0 \to \tilde{\mathbb{T}}_{\mathfrak{m}} \to \mathrm{H}^1_{et}(Sh^{tor}_{K,\bar{\mathbb{Q}}}, j_\star \underline{\mathrm{Sym}^{k-2}\mathbb{Z}_p^2})_{\mathfrak{m}} \to \tilde{\mathbb{T}}^\vee_{\mathfrak{m}} \otimes \chi_p^{1-k} \to 0$ lifting the exact sequence of Theorem 4.3, 2. We

can find a surjective map of Galois representations $\mathrm{H}^1_{et}(Sh^{tor}_{K,\bar{\mathbb{Q}}}, j_\star\underline{\mathrm{Sym}^{k-2}\mathbb{Z}_p^2})_{\mathfrak{m}} \to \rho_{\tilde{f}}$ by Theorem 4.2. Moreover, the image of $\tilde{\mathbb{T}}_{\mathfrak{m}}$ will be a line in $\rho_{\tilde{f}}$ (by considering the action of $I_{\mathbb{Q}_p}$) providing an ordinary filtration on $\rho_{\tilde{f}}$ an ρ_f. Reducing modulo p, we obtain a map $\mathrm{H}^1_{et}(Sh^{tor}_{K,\bar{\mathbb{Q}}}, j_\star\underline{\mathrm{Sym}^{k-2}\mathbb{F}_p^2})_{\mathfrak{m}} \to \rho_f$ of ordinary representations. This proves that the map $\mathbb{T}^\vee_{\mathfrak{m}} \to \mathbb{T}_{\mathfrak{m}}$ classifying the extension from Theorem 4.3 is not an isomorphism (since ρ_f is split). Hence $\mathbb{T}^{tm}_{\mathfrak{m}} \neq 0$ and $f \in \mathrm{H}^0(X, \omega^{k,tm})$. □

We assume that $\bar{\rho}_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{m})$ is irreducible. We have a universal modular Galois representation $\rho_{\mathbb{T}_{\mathfrak{m}}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}})$. Passing to the quotient along $\mathbb{T}_{\mathfrak{m}} \to \mathbb{T}^{tm}_m$, we get a Galois representation $\rho_{\mathbb{T}^{tm}_{\mathfrak{m}}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}^{tm}_{\mathfrak{m}})$.

Theorem 4.4. *The Galois representation $\rho_{\mathbb{T}^{tm}_{\mathfrak{m}}}|_{G_{\mathbb{Q}_p}}$ is tamely ramified. More precisely :*

1. *If $2 \leq k \leq p-1$, $\rho_{\mathbb{T}^{tm}_{\mathfrak{m}}}|_{G_{\mathbb{Q}_p}} = ur(U_{p,k}) \oplus ur(U_{p,p+1-k})\epsilon_p^{1-k}$ and $U_{p,k}U_{p,p+1-k} = S_p$.*
2. *If $k = p$, $\rho_{\mathbb{T}^{tm}_{\mathfrak{m}}}|_{G_{\mathbb{Q}_p}}$ is unramified.*

Moreover, unless possibly if $k = p$ and $\bar{\rho}_{\mathfrak{m}}$ is unramified with scalar Frobenius, $\mathbb{T}^{tm}_{\mathfrak{m}}$ is the initial ring among all quotients $\mathbb{T}_{\mathfrak{m}} \to R$ with the property that ρ_R is tamely ramified.

Proof. By [Sch18, Prop 5.4], we have

$$\mathrm{H}^1_{et}(Sh^{tor}_{K,\bar{\mathbb{Q}}}, j_\star\underline{\mathrm{Sym}^{k-2}\mathbb{F}_p^2})_{\mathfrak{m}} \simeq M \otimes \rho_{\mathbb{T}_{\mathfrak{m}}}$$

for a $\mathbb{T}_{\mathfrak{m}}$-module M. Moreover, by Theorem 4.3 1, we have an exact sequence $(ES1) : 0 \to \mathbb{T}_{\mathfrak{m}} \to M^2 \to \mathbb{T}^\vee_{\mathfrak{m}} \to 0$. Unless $k = p$ and $\bar{\rho}_{\mathfrak{m}}$ is unramified at p with scalar Frobenius, we claim that the representation $\rho_{\mathbb{T}_{\mathfrak{m}}}$ sits in an exact sequence $(ES2) : 0 \to \mathbb{T}_{\mathfrak{m}} \to \rho_{\mathbb{T}_{\mathfrak{m}}} \to \mathbb{T}_{\mathfrak{m}} \to 0$ where $\mathbb{T}_{\mathfrak{m}}$ is the unique unramified $G_{\mathbb{Q}_p}$-stable line where Frobenius acts as $U_{p,k}$. Indeed, let $\tilde{\mathbb{T}}_{\mathfrak{m}}$ be the Hecke algebra acting on $\mathrm{H}^0(\mathfrak{X}, \omega^k)_{\mathfrak{m}}$, so that $\mathbb{T}_{\mathfrak{m}} = \tilde{\mathbb{T}}_{\mathfrak{m}}/p$. We have a Galois representation $\rho_{\tilde{\mathbb{T}}_{\mathfrak{m}}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\tilde{\mathbb{T}}_{\mathfrak{m}})$ and $\tilde{\mathbb{T}}_{\mathfrak{m}}[1/p] = \prod_g K(g)$ is the product of the Hecke fields over all eigenforms g in $\mathrm{H}^0(\mathfrak{X}, \omega^k)_{\mathfrak{m}}$. These are ordinary eigenforms, and we deduce that there is a p-torsion free quotient $\rho_{\tilde{\mathbb{T}}_{\mathfrak{m}}} \to L$ where inertia acts on L via χ_p^{1-k} and Frobenius via $S_pU_{p,k}^{-1}$. Reducing modulo $\mathfrak{m}$, we find that $L/\mathfrak{m}$ is one dimensional. This implies that L is generated by one element, and finally that $L \simeq \mathbb{T}_{\mathfrak{m}}$.

We now claim that $M \otimes (ES2) = (ES1)$. It is clear that the map $\mathbb{T}_{\mathfrak{m}} \hookrightarrow M^2$ in $(ES1)$ factors through $M \hookrightarrow M^2$ in $M \otimes (ES2)$. By comparing length, we find that $\mathbb{T}_{\mathfrak{m}} \simeq M$. It follows that $\mathrm{H}^1_{et}(Sh^{tor}_{K,\bar{\mathbb{Q}}}, j_\star\underline{\mathrm{Sym}^{k-2}(\mathbb{F}_p^2)})^{tm}_{\mathfrak{m}} \simeq \rho_{\mathbb{T}^{tm}_{\mathfrak{m}}}$. The theorem is clear in that case. It remains to consider the case $k = p$ and $\bar{\rho}_{\mathfrak{m}}$ is unramified at p with scalar Frobenius. The exact sequence $0 \to \mathbb{T}_{\mathfrak{m}} \to M^2 \to \mathbb{T}^\vee_{\mathfrak{m}} \to 0$ arises by reduction modulo p of an exact sequence :

$$0 \to \tilde{\mathbb{T}}_{\mathfrak{m}} \xrightarrow{\begin{pmatrix}\tilde{m}_1\\ \tilde{m}_2\end{pmatrix}} \tilde{M}^2 \xrightarrow{\begin{pmatrix}\tilde{l}_1 & \tilde{l}_2\end{pmatrix}} \tilde{\mathbb{T}}^\vee_{\mathfrak{m}} \to 0$$

obtained by studying $\mathrm{H}^1_{et}(Sh^{tor}_{K,\bar{\mathbb{Q}}}, j_\star\underline{\mathrm{Sym}^{p-2}\mathbb{Z}_p^2})_{\mathfrak{m}}$. We fix an element σ of $I_{\mathbb{Q}_p}$. Write the matrix of $\rho_{\tilde{\mathbb{T}}_{\mathfrak{m}}}(\sigma) - \mathrm{diag}(1, \chi_p^{1-p}(\sigma)) = \begin{pmatrix}\tilde{a}_\sigma & \tilde{b}_\sigma\\ \tilde{c}_\sigma & \tilde{d}_\sigma\end{pmatrix}$. Letting $\rho_{\tilde{\mathbb{T}}_{\mathfrak{m}}}(\sigma) - \mathrm{diag}(1, \chi_p^{1-p}(\sigma))$ act on $\tilde{M}^2$ we find that there is a morphism $\tilde{\alpha}_\sigma : \tilde{\mathbb{T}}^\vee_{\mathfrak{m}} \to \tilde{\mathbb{T}}_{\mathfrak{m}}$ such that (as endomorphisms of $\tilde{M}^2$)

$$\begin{pmatrix}\tilde{a}_\sigma & \tilde{b}_\sigma\\ \tilde{c}_\sigma & \tilde{d}_\sigma\end{pmatrix} = \begin{pmatrix}\tilde{m}_1\tilde{\alpha}_\sigma\tilde{l}_1 & \tilde{m}_1\tilde{\alpha}_\sigma\tilde{l}_2\\ \tilde{m}_2\tilde{\alpha}_\sigma\tilde{l}_1 & \tilde{m}_1\tilde{\alpha}_\sigma\tilde{l}_2\end{pmatrix}.$$

Let $\alpha_\sigma, a_\sigma ...$ denote the reduction modulo p of $\tilde{\alpha}_\sigma$, $\tilde{a}_\sigma ...$ We deduce from Remark 4.8 that

$$\mathbb{T}^{tm}_{\mathfrak{m}} = \mathbb{T}_{\mathfrak{m}}/(\alpha_\sigma(\mathbb{T}^\vee_{\mathfrak{m}}), \sigma \in I_{\mathbb{Q}_p}).$$

We want to compare it to the ring $\mathbb{T}_{\mathfrak{m}}/(a_\sigma, b_\sigma, c_\sigma, d_\sigma, \sigma \in I_{\mathbb{Q}_p})$ which is the initial quotient where $\rho_{\mathbb{T}_{\mathfrak{m}}}$ becomes unramified. We claim that $a_\sigma = \alpha_\sigma(l_1(m_1))$, $c_\sigma = \alpha_\sigma(l_1(m_2))$, $b_\sigma = \alpha_\sigma(l_2(m_1))$ and

$d_\sigma = \alpha_\sigma(l_2(m_2))$. This implies that $(a_\sigma, b_\sigma, c_\sigma, d_\sigma, \sigma \in I_{\mathbb{Q}_p}) \subseteq (\alpha_\sigma(\mathbb{T}^\vee_{\mathfrak{m}}), \sigma \in I_{\mathbb{Q}_p})$ and therefore that $\rho_{\mathbb{T}^{tm}_{\mathfrak{m}}}$ is unramified. Let us prove for example that $\tilde{a}_\sigma = \tilde{\alpha}_\sigma(\tilde{l}_1(\tilde{m}_1))$. We apply the identity $\tilde{a}_\sigma = \tilde{m}_1\tilde{\alpha}_\sigma\tilde{l}_1$ to $\tilde{\mathbb{T}}_{\mathfrak{m}} \hookrightarrow \tilde{M}^2$. We find that $(\tilde{m}_1\tilde{a}_\sigma, \tilde{m}_2\tilde{a}_\sigma) = (\tilde{m}_1\tilde{\alpha}_\sigma\tilde{l}_1(\tilde{m}_1), \tilde{m}_1\tilde{\alpha}_\sigma\tilde{l}_1(\tilde{m}_2))$. We can identify $\tilde{M}[1/p] \simeq \tilde{\mathbb{T}}_{\mathfrak{m}}[1/p] = \prod_g K(g)$ is a finite product of fields. On each component $K(g)$ where $\tilde{m}_1$ is invertible, we find that $\tilde{a}_\sigma = \tilde{\alpha}_\sigma\tilde{l}_1(\tilde{m}_1)$. On each component $K(g)$ where $\tilde{m}_1$ is zero and $\tilde{m}_2$ must be non-zero, we find that $\tilde{a}_\sigma = 0 = \tilde{\alpha}_\sigma\tilde{l}_1(\tilde{m}_1)$. Thus, the desired identity is proven.

□

4.2 Companion forms for GSp_4

We now let $g = 2$. We let $Sh^{tor}_{K_pK^p} \to \mathrm{Spec}\ \mathbb{Z}_{(p)}$ be a toroidal compactification of the Siegel threefold with spherical level at p ([FC90]), we let $\mathfrak{X}$ be its base change to Spec $\mathbb{Z}_p$, and X be its special fiber. We have automorphic vector bundles $\omega^{(k_1,k_2)} = \mathrm{Sym}^{k_1-k_2}\Omega_{\mathcal{G}} \otimes \det^{k_2}\Omega_{\mathcal{G}}$ for $(k_1, k_2) \in \mathbb{Z}^2$, $k_1 \geq k_2$. If $k = k_1 = k_2$ we simply let $\omega^{(k,k)} = \omega^k$. We also recall the Kodaira-Spencer isomorphism $\omega^{(2,0)} = \Omega^1_X(\log\ D)$, implying $\omega^{(3,1)} = \Omega^2_X(\log\ D)$ and $\omega^3 = \Omega^3_X(\log\ D)$. In Section 3.2.5 we introduced the map dual to the obstruction to lift Frobenius $O^\vee : F_\star\Omega^3_X(\log\ D) \to \Omega^2_X(\log\ D)[1]$

Definition 4.2. *We let* $\omega^{3,tm} = \mathrm{Fib}(O^\vee : F_\star\Omega^3_X(\log\ D) \to \Omega^2_X(\log\ D)[1])$.

Remark 4.9. Following the modular curve case, it would make sense to call this coherent sheaf $\omega^{3,tm,\vee}$. But to simplify the notations, we keep the notation $\omega^{3,tm}$ as there will not be any $\omega^{3,tm,\vee}$ in the GSp_4 context.

This sheaf sits in an exact sequence:

$$0 \to \Omega^2_X(\log\ D) \to \omega^{3,tm} \to F_\star\Omega^3_X(\log\ D) \to 0$$

and admits the description:

$$\omega^{3,tm} = \mathrm{Ker}(\Omega^2_X(\log\ D) \otimes \omega^{(p-1)} \oplus F_\star\Omega^3_X(\log\ D) \to i_\star(\Omega^2_X(\log\ D) \otimes \omega^{(p-1)}|_Z)$$

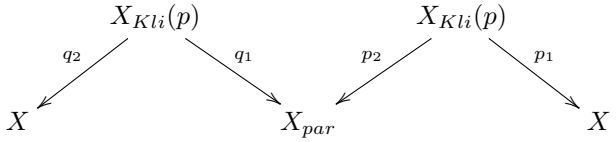
for the map given by $(f, g) \mapsto (\mathcal{C}(f|_Z) - g|_Z)$ where in Section 3.2.5, we introduced a map $(O'')^\vee[-1] := \mathcal{C} : (F_Z)_\star\Omega^3_X(\log\ D)|_Z \to \Omega^2_X(\log\ D) \otimes \omega^{(p-1)}|_Z$. We deduce that

$$\mathrm{H}^0(X, \omega^{3,tm}) = \{(f, g) \in \mathrm{H}^0(X, \Omega^3_X(\log\ D)) \times \mathrm{H}^0(X, \Omega^2_X(\log\ D) \otimes \omega^{(p-1)}), \mathcal{C}(f|_Z) = g|_Z\}.$$

It seems difficult to detect whether $\mathcal{C}(f|_Z)$ is zero or not (unlike in the modular curve case), but this is of crucial importance in order to understand the cohomology of $\omega^{3,tm}$. Our strategy in Section 4.2.1 is to relate $\mathcal{C}(f|_Z)$ with an Hecke operator that detects Siegel ordinarity (which means that the restriction to $G_{\mathbb{Q}_p}$ of the GSp_4-valued Galois representation attached to eigenforms can be conjugated to the Siegel parabolic). We will anyway need this ordinarity condition in our study of companion forms, so this is not a surprise. This strategy is inspired by [BCGP21, Sect 5].

4.2.1 Hecke operators and the map $\mathcal{C}$

We briefly recall constructions from [Pil20, Sect 7], of an operator T which acts on $\mathrm{R}\Gamma(X, \omega^3)$. We let $X_{Kli}(p)$ be the Klingen moduli space and X_{par} be the paramodular level moduli space. We have two correspondences:



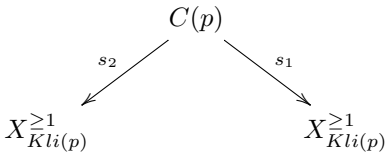
Let us informally explain what are the maps (ignoring subtilities from the boundary). Let us denote by $\mathcal{G}$ the principally polarized semi-abelian scheme over X, and by $\mathcal{G}'$ the semi-abelian scheme with a degree p^2 polarization λ over X_{par}. Let $H \subset \mathcal{G}[p]$ be a rank p subgroup and

$H' \subset \ker(\lambda)$ be a rank p subgroup. We can think of $X_{Kli}(p)$ as the moduli space of pairs $(\mathcal{G}, H)$. We can also think of it as the moduli of pairs $(\mathcal{G}', H' \subseteq \ker(\lambda))$, via the involution $(\mathcal{G}, H) \mapsto (\mathcal{G}/H^\perp, \mathcal{G}[p]/H^\perp)$. The maps in the correspondence are given by $p_1(\mathcal{G}, H) = \mathcal{G}$, $p_2(\mathcal{G}, H) = \mathcal{G}/H^\perp$, $q_1(\mathcal{G}', H') = \mathcal{G}'$, $q_2(\mathcal{G}', H') = \mathcal{G}'/H'$.

We have cohomological correspondences ([Pil20, Lem 7.1.1, Lem 7.1.2]) $T_1 : p_2^\star\omega^3 \to p_1^!\omega^3$ and $T_2 : q_2^\star\omega^3 \to q_1^!\omega^3$. They induce maps $T_1 : \mathrm{R}\Gamma(X_{par}, \omega^3) \to \mathrm{R}\Gamma(X, \omega^3)$ and $T_2 : \mathrm{R}\Gamma(X, \omega^3) \to \mathrm{R}\Gamma(X_{par}, \omega^3)$. We let $T = T_1 \circ T_2$.

We let $X^{\geq 1}$ be the p-rank at least one locus, X^{ord} the ordinary locus and $X^{=1} = X^{\geq 1} \setminus X^{ord}$ the complement (it was denoted by Z^{ord} in Section 3.4.1). We let $X_{Kli}(p)^{\geq 1} \to X^{\geq 1}$ be the space parametrizing a subgroup H of rank p and multiplicative type. This map is étale, and restricts to an isomorphism $X_{Kli}(p)^{=1} \to X^{=1}$.

We now consider a Hecke correspondence $C(p)$ (see [Pil20], Sect 10.1 and 10.2). On the complement of the boundary, this is the moduli space of $L \subseteq \mathcal{G}[p^2]$ which are totally isotropic and satisfy $L \cap H = \{1\}$.



Proposition 4.11. *1. On the complement of the boundary, the group L sits in an exact sequence $0 \to L \cap H^\perp \to L \to L/(L \cap H^\perp) \to 0$, where $L \cap H^\perp$ is a BT_1 of height 2 and dimension 1, and $L/(L \cap H^\perp)$ is an étale group, locally isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$.*

2. We have an natural isomorphism $s_2^\star\omega \to s_1^\star\omega$.

3. Under the isomorphism $s_2^\star\omega^{p-1} \to s_1^\star\omega^{p-1}$ deduced from 2., we have $s_1^\star \mathrm{Ha} = s_2^\star \mathrm{Ha}$.

4. There is a normalized fundamental class $\mathcal{O}_{C(p)} \to s_1^! \mathcal{O}_{X^{\geq 1}_{Kli(p)}}$.

5. Combining maps in 2. and 4., for any sheaf ω^k, we have a cohomological correspondence $U : s_2^\star\omega^k \to s_1^\star\omega^k \to s_1^!\omega^k$. These commute with multiplication by the Hasse invariant.

Proof. For the first point, see [Pil20, Lem 10.1.1]. For the remaining points, see [Pil20, Sect 10.4] and [Pil20, Lem 10.5.2.1]. □

We now would like to study the reduction of the cohomological correspondence U along the rank 1 locus $X^{=1}$.

Proposition 4.12. *1. Via the map s_2, the space $C(p) \times_{s_1, X^{\geq 1}_{Kli(p)}} X^{=1}$ indentifies with $X^{=1}$ and the map $s_1 : C(p) \times_{s_1, X^{\geq 1}_{Kli(p)}} X^{=1} \to X^{=1}$ indentifies with the square of the Frobenius of $X^{=1}$.*

2. The cohomological correspondence $U : s_2^\star\omega^3 \to s_1^!\omega^3$ restricts to a cohomological correspondence on $X^{=1}$ and admits the following description : $K_{X^{=1}}(D) \otimes \omega^{-(p-1)} \to K_{X^{=1}}(p^2D) \otimes \omega^{-p^2(p-1)}$, given by multiplication by $(\mathrm{Ha}')^{-(p-1)}$ and the canonical map $K_{X^{=1}}(D) \to K_{X^{=1}}(p^2D)$.

Proof. For the first point, see [BCGP21, Prop 5.7.3]. We have an exact sequence $0 \to \omega^{3-(p-1)} \to \omega^3 \to \omega^3|_{X^{=1}} \to 0$. Using Proposition 4.11, 5. as well as the first point, we get a cohomological correspondence $\omega^3|_{X^{=1}} \to (F^2)^!\omega^3|_{X^{=1}}$. If we let $K_{X^{=1}}$ be the canonical sheaf, we observe that $\omega^3|_{X^{=1}} = K_{X^{=1}}(D) \otimes \omega^{-(p-1)}$ (by the Kodaira-Spencer isomorphism). We therefore see that U restricts on $X^{=1}$ to a morphism: $K_{X^{=1}}(D) \otimes \omega^{-(p-1)} \to K_{X^{=1}}(p^2D) \otimes \omega^{-p^2(p-1)}$. We identify this map as in [BCGP21, Prop 5.7.6]. □

We now return to the map $\mathcal{C}$ (or its restriction to $X^{=1}$). It writes : $(F_Z)_\star K_{X^{=1}}(\log\ D) \otimes \omega^{-(p-1)} \to \Omega^2_X(\log D)\otimes\omega^{(p-1)}|_{X^{=1}}$. We can compose with the projection $\Omega^2_X(\log D)\otimes\omega^{(p-1)}|_{X^{=1}} \to K_{X^{=1}}(D)\otimes\omega^{(p-1)}$ and we get a diagram :

$$\begin{array}{ccc} (F_Z)_\star K_{X^{=1}}(D)\otimes\omega^{-(p-1)} & \xrightarrow{\mathcal{C}} & \Omega^2_X(\log\ D)\otimes\omega^{(p-1)}|_{X^{=1}} \\ & \searrow^{\gamma} & \downarrow \\ & & K_{X^{=1}}(D)\otimes\omega^{(p-1)} \end{array}$$

where γ is the composition of $(F_{X^{=1}})_\star K_{X^{=1}}(D)\otimes\omega^{-(p-1)} \overset{\mathrm{Ha}'}{\to} (F_{X^{=1}})_\star K_{X^{=1}}(pD)\otimes\omega^{p(p-1)} \to K_{X^{=1}}(D)\otimes\omega^{(p-1)}$. By adjunction, this is just the map $\gamma : K_{X^{=1}}(D)\otimes\omega^{-(p-1)} \overset{\mathrm{Ha}'}{\to} K_{X^{=1}}(pD)\otimes\omega^{p(p-1)} = F^!_{X^{=1}}(K_{X^{=1}}(D)\otimes\omega^{(p-1)})$. Since Ha' is invertible we can further compose this map with $\gamma' : K_{X^{=1}}(pD)\otimes\omega^{p(p-1)} \overset{(\mathrm{Ha}')^{-p}}{\to} K_{X^{=1}}(p^2D)\otimes\omega^{-p^2(p-1)} = (F^2_{X^{=1}})^!(K_{X^{=1}}(D)\otimes\omega^{-(p-1)})$. We finally find:

Lemma 4.6. *We have that $\gamma'\circ\gamma = U$. Consequently we have a commutative diagram where the vertical maps are restriction to $X^{=1}$:*

$$\begin{array}{ccc} \mathrm{H}^0(X^{\geq 1}_{Kli(p)},\omega^3) & \xrightarrow{U} & \mathrm{H}^0(X^{\geq 1}_{Kli(p)},\omega^3) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(X^{=1},\omega^3) & \xrightarrow{U} & \mathrm{H}^0(X^{=1},\omega^3) \\ & \searrow^{\mathcal{C}} \quad \nearrow & \\ & \mathrm{H}^0(X,\Omega^2_X\otimes\omega^{(p-1)}|_{X^{=1}}) & \end{array}$$

Corollary 4.8. *If $f\in\mathrm{H}^0(X,\omega^3)$ satisfies that $\mathcal{C}f = 0$, then Uf vanishes over $X^{=1}$.*

4.2.2 Recollections on spherical representations of $\mathrm{GSp}_4(\mathbb{Q}_p)$

We recall that $K_p = \mathrm{GSp}_4(\mathbb{Z}_p)$. We let $T_{p,2} = K_p\mathrm{diag}(p,p,1,1)K_p$, $T_{p,1} = K_p\mathrm{diag}(p^2,p,p,1)K_p$ and $T_{p,0} = K_p\mathrm{diag}(p,p,p,p)K_p$. We let $T'_{p,1} = T_{p,1} + (p^3+p^2+p+1)T_{p,0}$. The spherical Hecke algebra at p is $\mathcal{H}_p = \mathbb{Z}[T_{p,0},T^{-1}_{p,0},T_{p,1},T_{p,2}]$.

Let π_p be an irreducible smooth admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ defined over $\bar{\mathbb{Q}}_p$, such that $\pi_p^{K_p}\neq 0$. Via the Satake isomorphism, we can attach to $\pi_p\otimes|\nu|^{-\frac{3}{2}}$ a semi-simple conjugacy class $\mathrm{diag}(\alpha,\beta,\gamma,\delta)\in\mathrm{GSp}_4(\bar{\mathbb{Q}}_p)$.

Proposition 4.13. *The eigenvalues of $T_{p,0},T_{p,1},T_{p,2}$ on $\pi_p^{K_p}$ are respectively $p^{-3}\alpha\delta, p^{-1}(\alpha\beta+\alpha\gamma+\beta\delta+\gamma\delta)-p^{-3}\alpha\delta, \alpha+\beta+\gamma+\delta$.*

Proof. See [Pil20, Sect 5.1.5]. □

Definition 4.3. *We say that the parameter $\mathrm{diag}(\alpha,\beta,\gamma,\delta)$ is ordinary of weight 0, if after a permutation, $v_p(\alpha)=0$, $v_p(\beta)=1$, $v_p(\gamma)=2$, $v_p(\delta)=3$.*

Remark 4.10. In the ordinary case, the eigenvalues of $T_{p,0},T'_{p,1}$ and $T_{p,2}$ are in $\bar{\mathbb{Z}}_p$ and congruent respectively to $p^{-3}\alpha\delta, p^{-1}(\alpha\beta),\alpha$.

Proposition 4.14. *There is an action of $\mathcal{H}_p$ on $\mathrm{H}^0(\mathfrak{X},\omega^3)$ and $\mathrm{H}^0(X,\omega^3)$. The operator T corresponds to $T^{-2}_{p,0}T'_{p,1}$.*

Proof. The construction of the action is done in two steps. We first construct an action of the generators of the Hecke algebra. We already have recalled the definition of the action of T. In [FP21, Thm 5.6], we also defined an action of $T_{p,2}$. The action of $T_{p,0}$ is straightforward to define. The commutativity of the operators is checked on q-expansions. The operator T corresponds to $T_{p,0}^{-2}T'_{p,1}$ as we are now using the standard normalization when attaching correspondences to a double class. □

Let $\mathrm{diag}(\alpha,\beta,\gamma,\delta)$ be an ordinary parameter in $\mathrm{GSp}_4(\mathcal{O})$ for $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$. We consider the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}\otimes_{\mathbb{Z}}\mathcal{H}_p$, given by $(\mathfrak{m}_{\mathcal{O}}, T'_{p,1}-p^{-1}\alpha\beta, T_{p,2}-\bar{\alpha}, T_{p,0}-p^{-3}\alpha\delta)$. We now base change all spaces and cohomologies to $\mathcal{O}$.

4.2.3 An injectivity theorem

In this section we prove that the map $\mathcal{C}$ is injective under certain conditions. We begin by collecting a few results needed for the proof of the injectivity theorem.

Proposition 4.15. *We have* $\mathrm{H}^0(X^{\geq 1}_{Kli(p)},\omega^k)=0$ *for all* $k<0$.

Proof. This is [LPSZ21, Prop 3.10]. □

Definition 4.4. *We say that an ordinary parameter* $\mathrm{diag}(\alpha,\beta,\gamma,\delta)$ *is mod p Klingen generic if* $p^{-1}\beta\neq p^{-2}\gamma \bmod \mathfrak{m}_{\mathcal{O}}$. *In that case, we also say that* $\mathfrak{m}_p$ *is Klingen generic.*

Remark 4.11. We are not assuming that $\alpha\neq p^{-1}\beta \bmod \mathfrak{m}_{\mathcal{O}}$. This is some form of genericity, but not the strongest form.

Lemma 4.7. *Assume that* $\mathrm{diag}(\alpha,\beta,\gamma,\delta)$ *is mod p Klingen generic. The map* $f\mapsto Uf$, $\mathrm{H}^0(X,\omega^3)_{\mathfrak{m}_p}\to \mathrm{H}^0(X^{\geq 1}_{Kli(p)},\omega^3)$ *is injective.*

Proof. We will introduce some Hecke operators and relations between these Hecke operators in $\mathrm{H}^0(X^{ord},\omega^3)$ and $\mathrm{H}^0(X^{ord}_{Kli(p)},\omega^3)$. It is sometimes convenient to give the formula for these operators in $\mathrm{H}^0(\mathfrak{X}^{ord},\omega^3)$ and $\mathrm{H}^0(\mathfrak{X}^{ord}_{Kli(p)},\omega^3)$, where $\mathfrak{X}^{ord}$ and $\mathfrak{X}^{ord}_{Kli(p)}$ are the p-adic formal scheme, as one wants to divide by powers of p. We observe that T in $\mathrm{H}^0(\mathfrak{X}^{ord},\omega^3)[1/p]$ given by the formula $f(\mathcal{G},w)\mapsto \frac{1}{p^6}\sum_L f(\mathcal{G}/L,w')$ where L runs through all subgroups of $L\subseteq\mathcal{G}[p^2]$ which are totally isotropic and have multiplicative rank exactly 1 and w' is a basis of $\omega_{\mathcal{G}/L}$ which pulls back to w under the isogeny $\mathcal{G}\to\mathcal{G}/L$. This formula preserves the lattice $\mathrm{H}^0(\mathfrak{X}^{ord},\omega^3)$. We also have that U is given by the formula $f(\mathcal{G},H,w)\mapsto \frac{1}{p^6}\sum_L f(\mathcal{G}/L,\mathcal{G}[p]/L,w')$ where L runs through all subgroups of $L\subseteq\mathcal{G}[p^2]$ which are totally isotropic and have multiplicative rank exactly 1 and verify $L\cap H=\{0\}$. We can define F by the formula $f(\mathcal{G},H,w)\mapsto \frac{1}{p^6}\sum_L f(\mathcal{G}/L,\mathcal{G}[p]/L,w')$ where L runs through all subgroups of $L\subseteq\mathcal{G}[p^2]$ which are totally isotropic and have multiplicative rank exactly 1 and verify $L\cap H\neq\{0\}$. It follows that for $f\in\mathrm{H}^0(X^{ord},\omega^3)$, $Tf=Uf+Ff$. We now check that for $f\in\mathrm{H}^0(X^{ord},\omega^3)$, $UFf=0$. Actually, we can define an Hecke operator $T_{p,0}^{-1}U_{p,2}$ on $\mathrm{H}^0(\mathfrak{X}^{ord},\omega^3)$ (and $\mathrm{H}^0(X^{ord},\omega^3)$ by reduction) via the formula $f(\mathcal{G},\omega)\mapsto\frac{1}{p^3}\sum_C f(\mathcal{G}/C,\omega')$ where C runs through all subgroups of $C\subseteq\mathcal{G}[p]$ which are totally isotropic and have multiplicative rank 0. We remark that $U_{p,2}$ is congruent modulo p to $T_{p,2}$. We claim that $UFf=pT_{p,0}^{-3}U_{p,2}f$ for $f\in\mathrm{H}^0(\mathfrak{X}^{ord},\omega^3)$ or $\mathrm{H}^0(X^{ord},\omega^3)$. This follows from an identification of the correspondences: Any isogeny $\mathcal{G}\to\mathcal{G}/L\to\mathcal{G}/L+L'$ where L is totally isotropic and has multiplicative rank 1, and verifies $L\cap H=\{0\}$ and where L' is totally isotropic and has multiplicative rank 1, and verifies $L\cap\mathcal{G}[p]/L\neq\{0\}$ is uniquely of the shape $\mathcal{G}\xrightarrow{p}\mathcal{G}\to\mathcal{G}/C'$ where $C'\subseteq\mathcal{G}[p^2]$ is totally isotropic and has multiplicative rank 0. Conversely, any such isogeny $\mathcal{G}\xrightarrow{p}\mathcal{G}\to\mathcal{G}/C'$ can be factored in the shape $\mathcal{G}\to\mathcal{G}/L\to\mathcal{G}/L+L'$. However, there are p choices L, which explains the multiple p in the formula $UFf=pT_{p,0}^{-3}U_{p,2}f$ (one computes easily that the degree of U is p^4, the degree of F is p^3 and the degree of $U_{p,2}$ is p^3). We deduce that the equation $U^2+UT(f)=0$ holds for any form in $\mathrm{H}^0(X,\omega^3)$. We also claim that $F^2(f)=T_{p,0}^{-1}U_{p,2}f$ (this can be checked as before).

We deduce that the kernel of the map $f \mapsto Uf$ is stable under T. Indeed, $UTf = U^2 f$. It is also stable under the Hecke operators $T_{p,0}$, $T_{p,2}$. We can therefore suppose that f is a non-zero eigenform in the kernel. Let us define $\mathrm{diag}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) = \mathrm{diag}(\alpha, p^{-1}\beta, p^{-2}\gamma, p^{-3}\delta) \bmod \mathfrak{m}_{\mathcal{O}}$. Thus, if $Uf = 0$, we deduce that $Tf = Ff = (\bar{\alpha}\bar{\delta})^{-2}\bar{\alpha}\bar{\beta}f$. On the other hand, $F^2 f = \bar{\alpha}^2(\bar{\alpha}\bar{\delta})^{-3}f$. We thus get that $(\bar{\alpha}\bar{\delta})^{-4}\bar{\alpha}\bar{\beta}^2 = \bar{\alpha}^2(\bar{\alpha}\bar{\delta})^{-3}$ which implies that $\bar{\beta} = \bar{\gamma}$. This is a contradiction. □

Remark 4.12. We can give some intuition for our genericity condition. Let π_p be a spherical irreducible representation, and assume $\pi_p \otimes |\nu|^{-\frac{3}{2}}$ has ordinary parameter $\mathrm{diag}(\alpha, \beta, \gamma, \delta)$. If π_p is generic, then π_p^{Kli} is 4 dimensional and has an eigenvector for the operator $U_{p,1}$ with eigenvalue $p^{-1}\alpha\beta$. However, if $p\beta = \gamma$, then π_p^{Kli} has no such eigenvector. See [Sch05, Table 2 and 3].

Theorem 4.5. *Assume that $p \geq 5$ and $\mathfrak{m}_p$ is Klingen generic. The map $\mathcal{C} : \mathrm{H}^0(X, \omega^3)_{\mathfrak{m}_p} \to \mathrm{H}^0(Z, \Omega_X^2 \otimes \omega^{(p-1)})$ is injective.*

Proof. Let f be in the kernel. We have $Uf = 0$ on $X^{=1}$ by Corollary 4.8. Thus $Uf\mathrm{Ha}^{-1}$ defines a section of $\mathrm{H}^0(X^{\geq 1}_{Kli(p)}, \omega^{4-p})$, which vanishes by Proposition 4.15. Thus, $Uf = 0$. By Lemma 4.7, U is injective, so $f = 0$. □

Remark 4.13. We feel that Theorem 4.5 should hold for $p = 3$ as well (but we are unable to control the space $\mathrm{H}^0(X^{\geq 1}_{Kli(p)}, \omega^1)$).

Let us define $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_p} \subseteq \mathrm{H}^0(X, \omega^{3,tm})$ by the formula

$$\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_p} = \mathrm{H}^0(X, \omega^{3,tm}) \times_{\mathrm{H}^0(X,\omega^3)} \mathrm{H}^0(X, \omega^3)_{\mathfrak{m}_p}.$$

Corollary 4.9. *Assume that $p \geq 5$ and $\mathfrak{m}_p$ is Klingen generic. The map $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_p} \to \mathrm{H}^0(X, \Omega_X^2 \otimes \omega^{(p-1)})$ is injective.*

Remark 4.14. When $p = 2$, we see that $\mathrm{H}^0(X, \omega^2) \hookrightarrow \mathrm{Ker}(\mathcal{C})$ (via the map $\mathrm{H}^0(X, \omega^2) \to \mathrm{H}^0(X, \omega^3)$ which is multiplication by Ha). Moreover, if $\mathfrak{m}_p$ is Klingen generic, there is an injective map $U : \mathrm{Ker}(\mathcal{C})_{\mathfrak{m}_p} \to \mathrm{H}^0(X^{\geq 1}_{Kli(p)}, \omega^2)$. This target cohomology group (or more precisely its ordinary part) is the degree 0, weight 2 cohomology of the Higher Hida theory considered in [Pil20]. This shows that $\mathrm{Ker}(\mathcal{C})$ is intimately related with the space of (singular) weight 2 modular forms. There is an exact sequence $0 \to \mathrm{Ker}(\mathcal{C}) \to \mathrm{H}^0(X, \omega^{3,tm}) \to \mathrm{H}^0(X, \Omega_X^2 \otimes \omega^1)$. Here $\mathrm{H}^0(X, \Omega_X^2 \otimes \omega^1)$ is the space of Siegel modular forms of weight $(3, 2)$ also of singular weight.

4.2.4 Companion forms

Let $f \in \mathrm{H}^0(X, \omega^3)$ be an eigenform. There is (see for example [BCGP21, Thm 2.7.1, 2.7.2]) an associated Galois representation $r_f : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\bar{\mathbb{F}}_p)$. We say that f is ordinary at p if the eigenvalues of $T_{0,p}$, $T'_{1,p}$ and $T_{2,p}$ are units. In that case, there is a semi-simple element $\mathrm{diag}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) \in \mathrm{GSp}_4(\bar{\mathbb{F}}_p)$ such that $T_{0,p}f = \bar{\alpha}\bar{\delta}f$, $T'_{1,p}f = \bar{\alpha}\bar{\beta}f$ and $T_{2,p}f = \bar{\alpha}f$. Moreover,

$$r_f|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} ur(\bar{\alpha}) & \star & \star & \star \\ 0 & ur(\bar{\beta})\epsilon_p^{-1} & \star & \star \\ 0 & 0 & ur(\bar{\delta})\epsilon_p^{-2} & \star \\ 0 & 0 & 0 & ur(\bar{\gamma})\epsilon_p^{-3} \end{pmatrix}.$$

Definition 4.5. *We say that a prime $\ell \neq 2$ is decomposed generic for r_f if r_f is unramified at ℓ, $r_f(\mathrm{Frob}_\ell) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $\lambda_i\lambda_j^{-1} \notin \{1, \ell\}$ for all $i \neq j$.*

Let $\mathfrak{m}_f^p$ be the maximal ideal corresponding to f in the prime-to-p Hecke algebra. We have the following vanishing theorem.

Theorem 4.6. *If there exists an ℓ which is decomposed generic for r_f, and r_f is irreducible, then the étale cohomology $\mathrm{R}\Gamma(S_{K^pK_p,\bar{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathfrak{m}_f^p}$ is concentrated in degree 3.*

Proof. This is [HL25], together with the property that the boundary cohomology is Eisenstein [BCGP25, Thm 4.9.4]. □

Corollary 4.10. *Assume that $p \geq 5$. The coherent cohomology $\mathrm{R}\Gamma(X, \omega^3)_{\mathfrak{m}_f^p}$ is concentrated in degree 0 and the coherent cohomology $\mathrm{R}\Gamma(X, \Omega^2_X(\log\, D))_{\mathfrak{m}_f^p}$ is concentrated in degree 1.*

Proof. This follows from Theorem 4.6 and Theorem 2.3 that the de Rham cohomology is concentrated in the middle degree. By Corollary 2.1, this implies the concentration of Hodge cohomology. □

The prime-to-p Hecke algebra $\mathcal{H}^p$ acts on $\mathrm{H}^0(X, \Omega^2_X(\log\, D) \otimes \omega^{(p-1)})$. There is the usual normalized action. We will also use the un-normalized action, so that the normalized action of an operator gets multiplied by the similitude factor ν. For example $T_{\ell,0}$ acts like ℓ^2 multiplied by the normalized action. For any eigenform g in $\mathrm{H}^0(X, \Omega^2_X(\log\, D) \otimes \omega^{(p-1)})$, we have a Galois representation (for the normalized action) $r_g : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\bar{\mathbb{F}}_p)$ which satisfies that $\mathrm{Tr}(\mathrm{Frob}_\ell) = T_{\ell,2}$ for all ℓ not dividing p and the level. For the un-normalized action, the attached Galois representation is $\epsilon_p^{-1} \otimes r_g$. If g is ordinary, then

$$r_g|_{I_{\mathbb{Q}_p}} \simeq \begin{pmatrix} 1 & \star & \star & \star \\ 0 & \epsilon_p & \star & \star \\ 0 & 0 & \epsilon_p^{-2} & \star \\ 0 & 0 & 0 & \epsilon_p^{-1} \end{pmatrix}.$$

The Hecke algebra $\mathcal{H}^p$ acts naturally on $\mathrm{H}^0(X, \omega^{3,tm})$. This action is compatible with the maps $\mathrm{H}^0(X, \omega^{3,tm}) \to \mathrm{H}^0(X, \Omega^3_X(\log\, D))$ and $\mathrm{H}^0(X, \omega^{3,tm}) \to \mathrm{H}^0(X, \Omega^2_X(\log\, D) \otimes \omega^{(p-1)})$ where the action is the un-normalized one on $\mathrm{H}^0(X, \Omega^2_X(\log\, D) \otimes \omega^{(p-1)})$.

Corollary 4.11. *Assume $p \geq 5$, r_f is irreducible, there exists an ℓ which is decomposed generic for r_f, and f is Klingen generic at p. Then:*

1. $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_f^p} \hookrightarrow \mathrm{H}^0(X, \Omega^3_X(\log\, D))_{\mathfrak{m}_f^p}$
2. $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_f^p} \hookrightarrow \mathrm{H}^0(X, \Omega^2_X(\log\, D) \otimes \omega^{(p-1)})_{\mathfrak{m}_f^p}$.

Moreover, if $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_f^p} \neq 0$, then $r_f|_{I_{\mathbb{Q}_p}} \simeq \begin{pmatrix} 1 & 0 & \star & \star \\ 0 & \epsilon_p^{-1} & \star & \star \\ 0 & 0 & \epsilon_p^{-2} & 0 \\ 0 & 0 & 0 & \epsilon_p^{-3} \end{pmatrix}$.

Proof. The kernel of $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_f^p} \to \mathrm{H}^0(X, \Omega^3_X(\log\, D))_{\mathfrak{m}_f^p}$ sits inside $\mathrm{H}^0(X, \Omega^2_X(\log\, D))_{\mathfrak{m}_f^p} = 0$ by Corollary 4.10. We now consider the kernel of $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_f^p} \hookrightarrow \mathrm{H}^0(X, \Omega^2_X(\log\, D) \otimes \omega^{(p-1)})_{\mathfrak{m}_f^p}$. This is the kernel of $\mathcal{C}$ on $\mathrm{H}^0(X, \Omega^3_X(\log\, D))_{\mathfrak{m}_f^p}$. It follows from the existence and properties of the Galois representation r_f that $\mathrm{H}^0(X, \Omega^3_X(\log\, D))_{\mathfrak{m}_f^p} = \mathrm{H}^0(X, \Omega^3_X(\log\, D))_{\mathfrak{m}_f^p \mathfrak{m}_p}$. Thus we see that $\mathcal{C}$ is injective by Theorem 4.5. Finally, if $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_f^p} \neq 0$, we deduce that there exist (f, g) such that $r_f = \epsilon_p^{-1} r_g$. The partial splitting follows as r_f has an unramified stable line under $I_{\mathbb{Q}_p}$, while $\epsilon_p^{-1} r_g$ has a stable line acted on by ϵ_p^{-1}. □

We can now state our theorem about companion forms. Some version of it has been established (under slightly different assumptions) in [GG12, Thm 7.6.9] by a completely different (and much more general!) method.

Theorem 4.7. *Let $f \in \mathrm{H}^0(X, \omega^3)$ be an eigenform. Assume that*

1. *$p \geq 5$,*
2. *The Galois representation $r_f : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\bar{\mathbb{F}}_p)$ is irreducible.*

3. *There is an unramified prime ℓ such that $r_f(\mathrm{Frob}_\ell)$ is decomposed generic.*

4. *The form f is ordinary at p and Klingen generic.*

5. *We have*

$$r_f|_{I_{\mathbb{Q}_p}} \simeq \begin{pmatrix} 1 & 0 & \star & \star \\ 0 & \epsilon_p^{-1} & \star & \star \\ 0 & 0 & \epsilon_p^{-2} & 0 \\ 0 & 0 & 0 & \epsilon_p^{-3} \end{pmatrix}$$

Then there exists a companion form $g \in \mathrm{H}^0(X, \Omega_X^2(\log\ D) \otimes \omega^{(p-1)})$ which satisfies $r_g \otimes \epsilon_p^{-1} = r_f$.

Proof. Let $\mathcal{M} = \mathrm{H}^3_{syn}(\mathfrak{X})_{\mathfrak{m}_f^p} \in \mathrm{Vect}_{[0,3]}(\mathbb{Z}_p^{syn})$. We use that the syntomic cohomology is concentrated in degree 3 as a consequence of Theorem 4.6 and Theorem 2.3. This is an ordinary F-gauge. The étale realization of $\mathcal{M} \otimes \mathbb{F}_p$ is $\mathrm{H}^3_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, \mathbb{F}_p)_{\mathfrak{m}_f^p}$. It has an ordinary filtration $\mathrm{Fil}_\bullet \mathrm{H}^3_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, \mathbb{F}_p)_{\mathfrak{m}_f^p}$ with $\mathrm{Gr}_3 = \mathrm{H}^0(X, \Omega_X^3(\log\ D))_{\mathfrak{m}_f^p} \otimes \epsilon_p^{-3}$, $\mathrm{Gr}_2 = \mathrm{H}^1(X, \Omega_X^2(\log\ D))_{\mathfrak{m}_f^p} \otimes \epsilon_p^{-2}$. The extension $\mathrm{H}^3_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, \mathbb{F}_p)_{\mathfrak{m}_f^p}/\mathrm{Fil}_1$ is classified by a map $\mathrm{H}^0(X, \Omega_X^3(\log\ D))_{\mathfrak{m}_f^p} \to \mathrm{H}^1(X, \Omega_X^2(\log\ D))_{\mathfrak{m}_f^p}$ by Theorem 2.4. We can view this map as a complex in degree 0 and 1, and it computes $\mathrm{R}\Gamma(X, \omega^{3,tm})_{\mathfrak{m}_f^p}$ by Proposition 2.3. We let $\rho_f = r_f^\vee \otimes \epsilon_p^{-3}$. By [BCGP25, Thm 1.8.3.1] we have a map of Galois representations $\rho_f \to \mathrm{H}^3_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, \mathbb{F}_p)_{\mathfrak{m}_f^p}$ with the property that the ordinary filtration on $\mathrm{H}^3_{et}(\mathfrak{X}_{\bar{\mathbb{Q}}_p}, \mathbb{F}_p)_{\mathfrak{m}_f^p}$ induces a full flag on ρ_f (this is clear by considering the $I_{\mathbb{Q}_p}$-action). The partial splitting of ρ_f implies that $\mathrm{H}^0(X, \Omega_X^3(\log\ D))_{\mathfrak{m}_f} \to \mathrm{H}^1(X, \Omega_X^2(\log\ D))_{\mathfrak{m}_f}$ is not injective, and therefore $\mathrm{H}^0(X, \omega^{3,tm})_{\mathfrak{m}_f} \neq 0$. Corollary 4.11 implies the existence of a companion. □

Remark 4.15. It would be desirable to prove a similar result when $p = 2$ as we would construct forms of singular weight (see Rem 4.14). The major obstruction is to show $\mathrm{R}\Gamma_{syn}(\mathfrak{X})_{\mathfrak{m}_f^p}$ is concentrated in the middle degree. This is does not follow from the vanishing of étale cohomology (as the Hodge-Tate weights are too big compared to $p = 2$).

References

[BCGP21] George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni, *Abelian surfaces over totally real fields are potentially modular*, Publ. Math. Inst. Hautes Études Sci. **134** (2021), 153–501.

[BCGP25] George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni, *Modularity theorems for abelian surfaces*, 2025.

[Bha22] Bhargav Bhatt, *Prismatic f-gauges*, 2022.

[BL22] Bhargav Bhatt and Jacob Lurie, *Absolute prismatic cohomology*, 2022.

[BO78] Pierre Berthelot and Arthur Ogus, *Notes on crystalline cohomology. (mn-21)*, Princeton University Press, 1978.

[Box15] George Boxer, *Torsion in the Coherent Cohomology of Shimura Varieties and Galois Representations*, preprint, July 2015.

[BP21] George Boxer and Vincent Pilloni, *Higher Coleman theory*, arXiv 2110.10251, 2021.

[BP22] ———, *Higher hida and coleman theories on the modular curve*, Epijournal de Géométrie Algébrique **Volume 6** (2022).

[CG18] Frank Calegari and David Geraghty, *Modularity lifting beyond the Taylor–Wiles method*, Inventiones mathematicae **211** (2018), no. 1, 297–433.

[CS19] Frank Calegari and Joel Specter, *Pseudorepresentations of weight one are unramified*, Algebra & Number Theory **13** (2019), no. 7, 1583–1596.

[CV92] Robert F. Coleman and José Felipe Voloch, *Companion forms and Kodaira–Spencer theory*, Invent. Math. **110** (1992), no. 2, 263–281.

[CWE25] Francesc Castella and Carl Wang-Erickson, *Critical Λ-adic modular forms and bi-ordinary complexes*, Tunisian Journal of Mathematics **7** (2025), no. 3-4, 677–754.

[Del71] Pierre Deligne, *Formes modulaires et représentations ℓ-adiques*, Séminaire Bourbaki : vol. 1968/69, exposés 347-363, Séminaire Bourbaki, no. 11, Springer-Verlag, Berlin/Heidelberg, 1971, talk:355, pp. 139–172 (fr).

[DI87] Pierre Deligne and Luc Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham. (Lifting modulo p^2 and decomposition of the de Rham complex)*, Invent. Math. **89** (1987), 247–270 (French).

[DR73] Pierre Deligne and Michael Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 143–316. Lecture Notes in Math., Vol. 349.

[Dri24] Vladimir Drinfeld, *Prismatization*, 2024.

[Fal89] Gerd Faltings, *Crystalline cohomology and p-adic Galois-representations*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80.

[Far11] Laurent Fargues, *La filtration canonique des points de torsion des groupes p-divisibles*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 6, 905–961, With collaboration of Yichao Tian.

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford.

[FJ95] Gerd Faltings and Bruce W. Jordan, *Crystalline cohomology and* $\mathrm{GL}(2, \mathbf{Q})$, Israel J. Math. **90** (1995), no. 1-3, 1–66.

[FL82] Jean-Marc Fontaine and Guy Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 4, 547–608 (1983).

[FP21] Najmuddin Fakhruddin and Vincent Pilloni, *Hecke operators and the coherent cohomology of Shimura varieties*, Journal of the Institute of Mathematics of Jussieu (2021), 1–69.

[Gee07] Toby Gee, *Companion forms over totally real fields. II*, Duke Math. J. **136** (2007), no. 2, 275–284.

[Gee08] ———, *Companion forms over totally real fields*, manuscripta mathematica **125** (2008), no. 1, 1–41.

[GG12] Toby Gee and David Geraghty, *Companion forms for unitary and symplectic groups*, Duke Math. J. **161** (2012), no. 2, 247–303.

[GK19] Wushi Goldring and Jean-Stefan Koskivirta, *Strata Hasse invariants, Hecke algebras and Galois representations*, Invent. Math. **217** (2019), no. 3, 887–984.

[Gro90] Benedict H. Gross, *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Math. J. **61** (1990), no. 2, 445–517.

[Har66] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by Pierre Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York, 1966.

[HL25] Linus Hamann and Si Ying Lee, *Torsion vanishing for some shimura varieties*, 2025.

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[Ill72] L. Illusie, *Complexe cotangent et deformations ii.*, Lecture notes in mathematics, Springer, 1972.

[Ill96] Luc Illusie, *Frobenius et dégénérescence de Hodge*, Introduction à la théorie de Hodge, Panor. Synthèses, vol. 3, Soc. Math. France, Paris, 1996, pp. 113–168.

[Kat70] Nicholas M. Katz, *Nilpotent connections and the monodromy theorem : applications of a result of Turrittin*, Publications Mathématiques de l'IHÉS **39** (1970), 175–232 (en).

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, no. 108, Princeton University Press, Princeton, NJ, 1985.

[Lan73] R. P. Langlands, *Modular forms and ℓ-adic representations*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 361–500. Lecture Notes in Math., Vol. 349.

[LPSZ21] David Loeffler, Vincent Pilloni, Christopher Skinner, and Sarah Livia Zerbes, *Higher Hida theory and p-adic L-functions for* GSp_4, Duke Math. J. **170** (2021), no. 18, 4033–4121.

[Mes72] William Messing, *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, Vol. 264, Springer-Verlag, Berlin-New York, 1972.

[Mil06] J.S. Milne, *Arithmetic duality theorems*, Kea books, BookSurge, 2006.

[MM25] Keerthi Madapusi and Shubhodip Mondal, *Perfect F-gauges and finite flat group schemes*, arXiv [math.NT] (2025).

[Pil20] Vincent Pilloni, *Higher coherent cohomology and p-adic modular forms of singular weights*, Duke Math. J. **169** (2020), no. 9, 1647–1807.

[Sai97] Takeshi Saito, *Modular forms and p-adic Hodge theory*, Invent. Math. **129** (1997), no. 3, 607–620.

[Sch90] A.J. Scholl, *Motives for modular forms.*, Inventiones mathematicae **100** (1990), no. 2, 419–430.

[Sch96] Anthony J. Scholl, *Vanishing cycles and non-classical parabolic cohomology*, Inventiones mathematicae **124** (1996), no. 1, 503–524.

[Sch05] Ralf Schmidt, *Iwahori-spherical representations of* $\mathrm{GSp}(4)$ *and Siegel modular forms of degree 2 with square-free level*, J. Math. Soc. Japan **57** (2005), no. 1, 259–293.

[Sch15] Peter Scholze, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) **182** (2015), no. 3, 945–1066.

[Sch18] ———, *On the p-adic cohomology of the Lubin-Tate tower*, Annales scientifiques de l'École Normale Supérieure **Ser. 4, 51** (2018), no. 4, 811–863 (en).

[Sta13] The Stacks Project Authors, *Stacks Project*, http://stacks.math.columbia.edu, 2013.

[Str10] Benoît Stroh, *Compactification de variétés de Siegel aux places de mauvaise réduction*, Bulletin de la Société Mathématique de France **138** (2010), no. 2, 259–315 (fr).

[Til12] J. Tilouine, *Formes compagnons et complexe BGG dual pour GSp_4*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 4, 1383–1436.

[TVX25] Gleb Terentiuk, Vadim Vologodsky, and Yujie Xu, *Prismatic f-gauges and fontaine–laffaille modules*, 2025.

[Wie14] Gabor Wiese, *On galois representations of weight one*, Documenta Mathematica **19** (2014), 689–707.