HIGHER COHERENT COHOMOLOGY AND p-ADIC MODULAR FORMS OF SINGULAR WEIGHTS

by

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Abstract. — We investigate the p-adic properties of higher coherent cohomology of automorphic vector bundles of singular weights on the Siegel threefolds.

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Contents

1. Introduction

In this paper we investigate the theory of p-adic families of automorphic forms for the group $\mathrm{GSp}_4/\mathbb{Q}$ whose component at infinity has singular Harish-Chandra parameter and is a non-degenerate limit of discrete series. The automorphic forms we consider can be realized in the coherent cohomology of an appropriate automorphic vector bundle over a Siegel threefold ([**28**]). The Siegel threefolds are finite unions of arithmetic quotients of the three dimensional Siegel upper half space. They have a modular interpretation as moduli spaces of abelian surfaces with polarization and level structure and they have canonical models over number fields. Using this coherent realization one can prove that the Hecke parameters of these automorphic forms are defined over number fields and construct, using congruences, compatible systems of 4-dimensional Galois representations ([**76**], [**63**]).

For the group $\mathrm{GL}_2(\mathbb{R})$ there is (up to twist by a character) one non-degenerate limit of discrete series. Automorphic forms with this component at infinity realize in the weight 1 coherent cohomology of the modular curves and correspond to weight 1 modular forms in the classical terminology. We recall certain special features of weight 1 modular forms compared to modular forms of weight $k \geq 2$: they do not occur in the étale cohomology of a local system of the modular curve; there is no dimension formula for the space of weight 1 modular forms; they occur in degree 0 and degree 1 coherent cohomology of the same weight 1 automorphic locally free sheaf; the Galois representations attached to an eigenform has finite image (and has irregular Hodge-Tate weights $(0,0)$)...

For the group $\mathrm{GSp}_4(\mathbb{R})$ there are lots of non degenerate limits of discrete series (even modulo twist by a character). Their Harish-Chandra parameters lie on certain walls of the character space of a maximal torus of the derived group Sp_4, and these walls are 1-dimensional ! If π is an automorphic form on GSp_4 with component at infinity one of these non degenerate limits of discrete series, the associated compatible system of Galois representations has (conjectural) Hodge-Tate weights of the form $(k+1, k+1, 0, 0)$ or $(k+1, 0, 0, -k-1)$ for $k \in \mathbb{Z}_{\geq 0}$, up to twist. In this paper we will only consider Harish-Chandra parameters which yield Hodge-Tate weights of the form $(k+1, k+1, 0, 0)$. The corresponding automorphic forms realize in the degree 0 and the degree 1 coherent cohomology of a vector bundle that we denote by $\Omega^{(k,2)}$ (and is attached to the representation $\mathrm{Sym}^k\mathrm{St} \otimes \det^2\mathrm{St}$ of the group GL_2 which is the Levi of the Siegel parabolic of Sp_4).

We construct p-adic families of (cuspidal) cohomology classes for the sheaves $\{\Omega^{(k,2)}\}_{k\geq 0}$ in degree 0 and 1. To state precisely the theorems, we need some more terminology. We denote by $X_K \to \mathrm{Spec}\ \mathbb{Z}_p$ a toroidal compactification of the Siegel threefold of level given by an compact open subgroup $K = \prod_\ell K_\ell \subset \mathrm{GSp}_4(\mathbb{A}_f)$ such that $K_p = \mathrm{GSp}_4(\mathbb{Z}_p)$. Attached to the Klingen parahoric subgroup $Kli(p) \subset K_p$, we get a covering $X_{Kli}(p)_K \to X_K$ which parametrizes a subgroup of order p of the semi-abelian scheme (at least when the semi-abelian scheme is abelian). We denote by D the relative Cartier divisor of the boundary in X_K or $X_{Kli}(p)_K$ (no confusion should arise). In the paper we define an Hecke operator U at p associated to the double coset $Kli(p)\mathrm{diag}(p^2, p, p, 1)Kli(p)$ which acts on the cohomology of $X_{Kli}(p)_K$. There is also a corresponding Hecke operator T at p associated to the double coset $K_p\mathrm{diag}(p^2, p, p, 1)K_p$ which acts on the cohomology of X_K. Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the one-dimensional Iwasawa algebra. For each integer k, there is a map $k : \Lambda \to \mathbb{Z}_p$ extending the character $z \mapsto z^k$ of $\mathbb{Z}_p^\times$.

Our main theorem is :

Theorem 1.1. — *There is a perfect complex M of Λ-modules of amplitude $[0,1]$ such that:*

1. *For all $k \in \mathbb{Z}_{\geq 0}$ we have a canonical quasi-isomorphism :*
$$M \otimes^L_{\Lambda,k} \mathbb{Q}_p = \mathrm{R}\Gamma(X_{Kli}(p)_K, \Omega^{(k,2)}(-D) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{U-\mathrm{ord}},$$
where the exponent $U - \mathrm{ord}$ means the ordinary part for the action of U.
2. *For all $k \in \mathbb{Z}$, $k > p+1$, we have a canonical isomorphism :*
$$\mathrm{H}^0(M \otimes^L_{\Lambda,k} \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{H}^0(X_K, \Omega^{(k,2)}(-D) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{T-\mathrm{ord}},$$
where the exponent $T - \mathrm{ord}$ means the ordinary part for the action of T.
3. *The perfect complex M carries an action of the Hecke algebra of level prime-to-p, and the isomorphisms above are equivariant for this action.*

Remark 1.1. — There is a natural compatibility between the first and second point of the theorem : for any $k \in \mathbb{Z}$, $k > p+1$, the natural map (a p-stabilization map) $\mathrm{H}^0(X_K, \Omega^{(k,2)}(-D) \otimes \mathbb{Q}_p)^{T-\mathrm{ord}} \to \mathrm{H}^0(X_{Kli}(p)_K, \Omega^{(k,2)}(-D) \otimes \mathbb{Q}_p)^{U-\mathrm{ord}}$ is an isomorphism.

Remark 1.2. — We also develop a theory of finite slope families in the third part of this work.

Remark 1.3. — In [**33**], Hida initiated the study of ordinary Betti cohomology on locally symmetric spaces associated to GL_n over arbitrary number fields F. When $n \geq 3$ (or $n \geq 2$ and F is not totally real), the non-Eisenstein cohomology is concentrated in more than one degree. To some extent, what we present here is the beginning of a coherent analogue of this theory. The analogy is that in both situations the interesting cohomology is naturally supported in several consecutive degrees. See the introduction of [**11**].

Let N be the product of primes ℓ such that $K_\ell \neq \mathrm{GSp}_4(\mathbb{Z}_\ell)$. The perfect complex M carries an action of the prime-to-pN Hecke algebra. For a maximal ideal $\mathfrak{m}$ of this Hecke algebra, we can consider the direct factor $M_\mathfrak{m}$ of M obtained by localization at $\mathfrak{m}$. We say that $\mathfrak{m}$ is a non-Eisenstein maximal ideal if it has an associated 4-dimensional representation of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (unramified away from pN and satisfying the familiar local-global compatibility conditions at the primes not dividing pN), and this representation is absolutely irreducible. Our second theorem is :

Theorem 1.2. — *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of the prime-to-pN Hecke algebra. The complex $M_\mathfrak{m}$ has trivial Euler characteristic.*

Remark 1.4. — We in fact believe that if $\mathfrak{m}$ is associated to a Galois representation which is absolutely irreducible and stays absolutely irreducible after restriction to Galois groups of real quadratic fields, then the cohomology groups $\mathrm{H}^i(M_\mathfrak{m})$ are torsion Λ-modules.

The perfect complex M is obtained as the U-ordinary part of the cohomology of a huge sheaf of Λ-modules $\mathfrak{F}^\kappa \otimes \Omega^{(0,2)}(-D)$. This sheaf is defined on the open formal subscheme $\mathfrak{X}^{\geq 1}_{Kli}(p)_K$ of the p-adic formal scheme $\mathfrak{X}_{Kli}(p)_K$ attached to $X_{Kli}(p)_K$ where the p-rank of the semi-abelian scheme is at least 1 (and the universal rank p group scheme is multiplicative). This formal scheme contains strictly the ordinary locus which is the locus where the p-rank is 2. Its image in the minimal compactification is covered by two affines, and this explains why the complex M is supported in two degrees. The sheaf $\mathfrak{F}^\kappa(-D)$ "interpolates" the sheaves $\{\Omega^{(k,0)}(-D)\}_{k \in \mathbb{Z}_{\geq 0}}$ in the sense that for all $k \in \mathbb{Z}_{\geq 0}$, we have a canonical map :
$$\Omega^{(k,0)}(-D) \to \mathfrak{F}^\kappa(-D) \otimes_{\Lambda,k} \mathbb{Z}_p.$$

The interpolation property rests on the special shape of the universal p-divisible group over $\mathfrak{X}^{\geq 1}_{Kli}(p)_K$ which contains at least a one-dimensional multiplicative p-divisible group. More precisely, we can define a pro-étale tower : $\mathfrak{X}^{\geq 1}_{Kli}(p^\infty)_K \to \mathfrak{X}^{\geq 1}_{Kli}(p)_K$ which parametrizes the one-dimensional multiplicative p-divisible groups H_∞ inside the universal p-divisible group. The fibers of this last map are isomorphic to $1+p\mathbb{Z}_p \subset \mathbb{P}^1(\mathbb{Z}_p)$ over the ordinary locus, while the map is an isomorphism over the rank one locus. Denote by ω_{H_∞} the dual of the Lie algebra of H_∞, this is a line bundle. Over the space $\mathfrak{X}^{\geq 1}_{Kli}(p^\infty)_K$ we have a canonical surjective map $\Omega^{(k,0)} \to \omega_{H_\infty}^{\otimes k}$ which is an analogue of the projection to the highest weight vectors in representation theory. The sheaf $\mathfrak{F}^\kappa$ is obtained by p-adically interpolating the powers of ω_{H_∞}. This can be done because the $\mathbb{G}_m$-torsor ω_{H_∞} possesses a $\mathbb{Z}_p^\times$-reduction, given by the Hodge-Tate period map :

$$\mathrm{HT} : T_p(H_\infty^D) \to \omega_{H_\infty}.$$

Before taking the ordinary part, the cohomology is enormous. The U-ordinary part cuts the perfect complex inside this enormous cohomology. There is a heuristic explanation for this. We explain it at a spherical level, using the T-operator instead (for technical reasons we sometimes prefer to work at spherical level). Over the complement of $\mathfrak{X}^{\geq 1}_K$ (the supersingular locus), one can prove that the T-operator acts topologically nilpotently on the sheaf $\Omega^{(k,2)}$, when k is large enough. This comes form the following observation. Let $\lambda : A \to A'$ be an isogeny of "type" T between two abelian surfaces defined over a discrete valuation ring $\mathcal{O}_K$. If A and A' have supersingular reduction, one shows that the isogeny on the reduction factors through the Frobenius map of A. As a result, the differential of the isogeny $\mathrm{d}\lambda : \omega_{A'} \to \omega_A$ has to vanish modulo the maximal $\mathfrak{m}_K$ of $\mathcal{O}_K$. This property is special to the supersingular locus.

Making this heuristic argument work requires some efforts. One of the difficulties is to make sense of the Hecke operators U and T on the integral cohomology. We first need to define the correspondence underlying the U and T operator integrally. The formulation of the moduli problem is difficult because it involves the p^2 torsion of the universal abelian variety (the cocharacter of the torus of GSp_4 underlying the double coset is not minuscule). Our approach is to use the factorization $\mathrm{diag}(p^2,p,p,1) = \mathrm{diag}(p,p,p,1).\mathrm{diag}(p,1,1,1)$ and factor accordingly the correspondence into two correspondences U_1 and U_2 and T_1 and T_2. The moduli problems underlying U_1 and U_2 or T_1 and T_2 can be defined integrally, and the moduli spaces can even be described locally using the local model theory. There is another difficulty. The correspondences are not finite flat over the Siegel threefold. Defining the necessary trace maps in cohomology requires some results from Grothendieck-Serre duality in coherent cohomology. There is also a subtle normalization issue. But luckily, all this can be resolved.

Having defined the Hecke operator T, we are able to prove an integral control theorem for $k \gg 0$:

$$\mathrm{H}^0(M \otimes^L_{\Lambda,k} \mathbb{Z}_p) = \mathrm{H}^0(X_K, \Omega^{(k,2)}(-D))^{T-\mathrm{ord}}$$

and to show that M is a perfect complex.

It seems very hard to obtain an integral control theorem for all $k \geq 0$. We will nevertheless be able to obtain a control theorem after inverting p by an indirect method. Over $\mathbb{Q}_p$, we can construct an overconvergent version $M^\dagger$ of M, obtained by taking the ordinary part for U of some overconvergent cohomology of the analytic fiber $\mathcal{X}^{\geq 1}_{Kli}(p)_K$ of $\mathfrak{X}^{\geq 1}_{Kli}(p)_K$ with value in a huge Banach sheaf. We observe that U is compact on this cohomology and we actually develop a theory of finite slope families.

By construction, there is a map $M^\dagger \to M \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p$ which is easily seen to be injective on H^0 and surjective on H^1. This is a "degeneration" of the classical statement that all ordinary p-adic modular forms are overconvergent.

With finite slope overconvergent cohomology classes, we can adapt the argument of analytic continuation and gluing of [**38**] and prove that small slope cohomology classes are classical. In the ordinary case, we obtain that for all $k \geq 0$:

$$M^{\dagger} \otimes^{L}_{\Lambda,k} \mathbb{Q}_p = \mathrm{R}\Gamma(X_{Kli}(p)_K, \Omega^{(k,2)}(-D) \otimes^{L}_{\mathbb{Z}_p} \mathbb{Q}_p)^{U-ord}.$$

Combining everything, we deduces that the map $M^{\dagger} \to M \otimes^{L}_{\mathbb{Z}_p} \mathbb{Q}_p$ is a quasi-isomorphism at weights $k \gg 0$ and then at all weight $k \geq 0$ by some elementary dimension argument.

The cohomology $M \otimes^{L}_{\Lambda,k} \mathbb{Z}_p$ is thus an "integral" structure of the cohomology $\mathrm{R}\Gamma(X_{Kli}(p)_K, \Omega^{(k,2)}(-D) \otimes^{L}_{\mathbb{Z}_p} \mathbb{Q}_p)^{U-ord}$. A very important feature is that $M \otimes^{L}_{\Lambda,k} \mathbb{Z}_p$ is concentrated in degree 0 and 1.

In [**34**] and [**3**] a theory of p-adic modular forms in coherent cohomology is developed for all weights. This means that we consider all possible automorphic vector bundles $\Omega^{(k,r)}(-D)$ for $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ coming from the representations $\mathrm{Sym}^k\mathrm{St} \otimes \det^r\mathrm{St}$ of the group GL_2. In this theory, only the degree 0 cohomology is interpolated. Let Λ_2 be the two dimensional Iwasawa algebra. For each pair $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ we can define a specialization morphism $(k,r) : \Lambda_2 \to \mathbb{Z}_p$. The main theorem of [**34**] for the group GSp_4 (using also the results of [**62**]), states that there exists a finite free Λ_2-module M' such that :

1. for all $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ we have $M' \otimes_{\Lambda_2,(k,r)} \mathbb{Z}_p = \mathrm{H}^0(\mathfrak{X}^{\geq 2}_{Kli}(p)_K, \Omega^{(k,r)}(-D))^{ord'}$,
2. for all $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 4}$, $\mathrm{H}^0(\mathfrak{X}^{\geq 2}_{Kli}(p)_K, \Omega^{(k,r)}(-D))^{ord'}$ is a subspace of the space of classical modular forms of Iwahori level at p.

In this theorem, $\mathfrak{X}^{\geq 2}_{Kli}(p)_K$ is the ordinary locus in $\mathfrak{X}^{\geq 1}_{Kli}(p)_K$ and ord' means the ordinary part for the usual ordinary idempotent attached to the diagonal matrix $\mathrm{diag}(p^3, p^2, p, 1) \in \mathrm{GSp}_4(\mathbb{Q}_p)$. The control theorem holds for weights (k,r) with $r \geq 4$. One can sometimes (after making some localization) improve the control theorem to $r \geq 3$ which is exactly the condition under which the corresponding automorphic forms are discrete series at infinity.

When we specialize M' at singular weights we cannot expect to have a good classicity theorem : we can attach p-adic Galois representations to eigenforms in $M' \otimes_{\Lambda_2,(k,2)} \mathbb{Z}_p$ but these Galois representations may not be de Rham at p. It should be true that classical eigenforms in $M' \otimes_{\Lambda_2,(k,2)} \mathbb{Z}_p$ are exactly those with de Rham associated Galois representation but unfortunately we do not know how to establish this directly.

On the other hand, eigenforms in $\mathrm{H}^0(M \otimes^{L}_{\Lambda,k} \mathbb{Z}_p)$ correspond to classical automorphic forms and one often knows that their associated Galois representation is de Rham ([**54**], prop. 4.16). There is a natural injective map $\mathrm{H}^0(M \otimes^{L}_{\Lambda,k} \mathbb{Z}_p) \to M' \otimes_{\Lambda_2,(k,2)} \mathbb{Z}_p$. It should actually be true that the subspace of $M' \otimes_{\Lambda_2,(k,2)} \mathbb{Z}_p$ spanned by eigenforms with de Rham associated Galois representations is "generated" by the image of $\mathrm{H}^0(M \otimes^{L}_{\Lambda,k} \mathbb{Z}_p)$.

It is conjectured that for every simple abelian surface A over $\mathbb{Q}$, there should exist a cuspidal automorphic form π on $\mathrm{GSp}_4/\mathbb{Q}$ such that the spin L-function of π and the L-function of $\mathrm{H}^1(A)$ coincide. When $\mathrm{End}(A) \neq \mathbb{Z}$ this is known ([**85**], [**42**]). See [**8**] for a precise conjecture in the case $\mathrm{End}(A) = \mathbb{Z}$. These automorphic forms are of the type we have considered so far as their component at infinity should be a limit of discrete series and they should realize in the cuspidal coherent cohomology of the sheaf $\Omega^{(0,2)}$. In [**61**] we were able to prove a modular lifting theorem saying, under many technical assumptions, that an abelian surface whose associated p-adic Galois representation is residually modular arises from a p-adic modular form. In that paper, our Taylor-Wiles system was constructed by letting Galois deformation rings act on the module of ordinary

p-adic modular forms $\mathrm{H}^0(\mathfrak{X}^{\geq 2}_{Kli}(p)_K, \Omega^{(0,2)}(-D))^{ord'}$. Congruences are unobstructed for ordinary p-adic modular forms, while they are for classical modular forms in weight $(0,2)$ because of the non vanishing of H^1. The classical Taylor-Wiles method requires unobstructed congruences. The draw back is that we do not know how to characterize classical modular forms among ordinary p-adic modular forms in weight $(0,2)$. In [**11**] and [**12**], Calegari-Geraghty explained how to modify the Taylor-Wiles method in order to apply it in obstructed situations. They could prove a better (but conditional) modular lifting theorem saying, under technical conditions, that an abelian surface whose associated p-adic Galois representation is residually modular arises from a weight $(0,2)$ modular form by letting the Galois deformation ring act on some localization of $\mathrm{H}^0(X_K, \Omega^{(0,2)}(-D) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ provided one could show that the localized cohomology vanishes in degree greater or equal than 2. Unfortunately, nobody has been able to establish this vanishing for the moment. As a replacement of $\mathrm{H}^0(X_K, \Omega^{(0,2)}(-D) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$, we suggest to use $\mathrm{H}^0(M \otimes^L_{\Lambda,2} \mathbb{Q}_p/\mathbb{Z}_p)$ where M is the complex provided by theorem 1.1. The point is that p-divisible classes in $\mathrm{H}^0(M \otimes^L_{\Lambda,2} \mathbb{Q}_p/\mathbb{Z}_p)$ do come from cohomology classes in $\mathrm{H}^0(X_{Kli}(p)_K, \Omega^{(0,2)}(-D))$ and thus from classical automorphic forms. This strategy will be employed in a future joint work with G. Boxer, F. Calegari and T. Gee.

This paper is organized in four parts. The first part is preliminary. Readers are suggested to skip it on first reading, and come back to it when necessary. We study the existence of projectors on complexes of modules. This will be used to define ordinary projector on cohomology. We present certain technical results on the cohomology of the sheaf $\mathscr{O}_{\mathcal{X}^+}$ on an adic space. These are only used in section 14. We also develop a formalism of cohomological correspondences that is adapted to our situation. Finally we recall some results concerning automorphic forms and Siegel threefolds over $\mathbb{C}$.

The second part of the work is dedicated to the construction of the perfect complex M in theorem 1.1. The definition of the complex itself is not so difficult, but establishing that it is a perfect complex involves a delicate study of the correspondences in characteristic p.

The third part is dedicated to complete the proof of theorem 1.1 and establishing the control theorem in weight $k \geq 0$. The argument is indirect as we have to use overconvergent cohomology. Most of this part is dedicated to develop a theory of finite slope overconvergent cohomology. In some sense this is easier than the integral slope zero theory: we can prove that U is compact and the finiteness of the finite slope cohomology follows easily. There is nevertheless the delicate problem of proving that the cuspidal cohomology is concentrated in degree 0 and 1. Finally we show that small slope cohomology classes are classical. We use the method of [**38**], but need to rephrase it at the sheaf level (one cannot glue higher cohomology classes).

In the fourth part we prove that the Euler characteristic of a non-Eisenstein localization of our perfect complex is zero by using results of Arthur on the theory of automorphic forms.

I thank G. Boxer for suggesting that there should exist a theory of p-adic modular forms for singular weights. The author attended a workshop in McGill Bellairs Research Institute in 2014 where F.Calegari and D. Geraghty explained their modified Taylor-Wiles method (now available in [**12**]). This was a motivation for developing a theory of p-adic modular forms on higher cohomology. We are pleased to thank the organizers and speakers of this workshop. I thank N. Fakhruddin for inviting me to the Tata institute and for helping me to define Hecke operators. In a forthcoming joint work, we will study the problem of defining Hecke operators on the integral coherent cohomology of more general PEL Shimura varieties. I thank G. Chenevier for his help with section 15.2.4. I thank G.

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PART I
PRELIMINARIES

2. Ordinary projectors on complexes

In this section, R is a complete local noetherian ring with maximal ideal $\mathfrak{m}_R$. We assume moreover that $R/\mathfrak{m}_R$ is a finite field. We develop a theory of ordinary (or Hida) projectors for certain complexes of R-modules.

2.1. Locally finite endomorphisms. — Let $\mathbf{Mod}(R)$ be the abelian category of R-modules. Let $\mathbf{Mod}^{comp}(R)$ be the category of $\mathfrak{m}_R$-adically separated and complete R-modules. This is a full subcategory of $\mathbf{Mod}(R)$. The category $\mathbf{Mod}^{comp}(R)$ is not abelian in general. Nevertheless, there is a notion of exact sequence in $\mathbf{Mod}^{comp}(R)$ (a complex of objects in $\mathbf{Mod}^{comp}(R)$ is exact if its image in $\mathbf{Mod}(R)$ is). Also, one sees easily that any arrow $M \to N$ in $\mathbf{Mod}^{comp}(R)$ has a kernel in $\mathbf{Mod}^{comp}(R)$ (its kernel in $\mathbf{Mod}(R)$, which is an object of $\mathbf{Mod}^{comp}(R)$) .

Definition 2.1.1. — *Let M be an object of $\mathbf{Mod}^{comp}(R)$. Let $T \in \mathrm{End}_R(M)$. The action of T on M is locally finite if for all $n \in \mathbb{N}$ and all $v \in M/\mathfrak{m}_R^n$, the elements $\{T^k v\}_{k\in\mathbb{N}}$ generate a finite $R/\mathfrak{m}_R^n$ submodule of $M/\mathfrak{m}_R^n$.*

Thus, the action of T on M is locally finite if for all $n \in \mathbb{N}$, $M/\mathfrak{m}_R^n$ can be written as an inductive limit of finite and T-stable R-modules.

Lemma 2.1.1. — *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathbf{Mod}^{comp}(R)$. Let T be a R-linear homomorphism acting equivariantly on M_1, M_2 and M_3.*

1. *If the action of T is locally finite on M_3 and M_1, it is locally finite on M_2.*
2. *If the action of T is locally finite on M_2, it is locally finite on M_3.*
3. *If there exists $n \in \mathbb{N}$ such that $\mathfrak{m}_R^n.M_2 = 0$ and if T is locally finite on M_2, then it is locally finite on M_1.*

Proof. Point 2 and 3 are obvious. We check point 1. For all $n \in \mathbb{N}$, we have an exact sequence:

$$M_1/\mathfrak{m}_R^n \to M_2/\mathfrak{m}_R^n \to M_3/\mathfrak{m}_R^n \to 0.$$

Let M be the image of $M_1/\mathfrak{m}_R^n$ in $M_2/\mathfrak{m}_R^n$. The action of T on M is locally finite by 2. Let $v \in M_2$. Since T is locally finite on M_3, there is $N \in \mathbb{N}$, $w \in M$, $a_0, \cdots, a_{N-1} \in R$ such that $T^N v = w + \sum_{i=0}^{N-1} a_i T^i v$. Since T is locally finite on M, there is $N' \in \mathbb{N}$, $b_0, \cdots, b_{N'-1} \in R$ such that $T^{N'} w = \sum_{j=0}^{N'-1} b_j T^j w$. The submodule of $M_2/\mathfrak{m}_R^n$ generated by $\{T^i v,\ T^j w,\ 0 \leq i \leq N-1,\ 0 \leq j \leq N'-1\}$ is stable under the action of T. □

Remark 2.1.1. — The assumption that $\mathfrak{m}_R^n.M_2 = 0$ in lemma 2.1.1, 3. is necessary: take $R = \mathbb{Z}_p$, $M_2 = \prod_{i\in\mathbb{Z}_{\geq 0}} \mathbb{Z}_p$, $M_3 = \prod_{i\in\mathbb{Z}_{\geq 0}} \mathbb{Z}/p^{\frac{i(i+1)}{2}}\mathbb{Z}$, $M_1 \simeq \prod_{i\in\mathbb{Z}_{\geq 0}} \mathbb{Z}_p$ the kernel of the natural map $M_2 \to M_3$, and T the endomorphism of M_2 which maps $(a_i)_{i\in\mathbb{Z}_{\geq 0}} \in M_2$ to

$(0, pa_0, p^2a_1, \cdots, p^ia_{i-1}, \cdots)$. One checks that T is locally finite on M_2, that $T(M_1) \subset M_1$, and that T is not locally finite on M_1.

Lemma 2.1.2. — *Let M be an object of $\mathbf{Mod}^{comp}(R)$ and let T be an endomorphism of M. The action of T on M is locally finite if and only if it is on $M/\mathfrak{m}_R$.*

Proof. We prove it by induction on n. Consider the exact sequence :

$$\mathfrak{m}_R^{n-1}/\mathfrak{m}_R^n \otimes_R M \to M/\mathfrak{m}_R^n \to M/\mathfrak{m}_R^{n-1} \to 0$$

By assumption, the action is locally finite on $M/\mathfrak{m}_R^{n-1}$ and on $\mathfrak{m}_R^{n-1}/\mathfrak{m}_R^n \otimes_R M$. It is also on $\mathfrak{m}_R^{n-1}M/\mathfrak{m}_R^n$ and finally on $M/\mathfrak{m}_R^n$ by the above lemma. □

Lemma 2.1.3. — *Assume that T acts locally finitely on an object M of $\mathbf{Mod}^{comp}(R)$. Then there is a unique ordinary projector $e \in \mathrm{End}_R(M)$ such that :*

1. *For all $v \in M$, $ev = \lim_{N\to\infty} T^{N!}v$ where the limit is computed for the $\mathfrak{m}_R$-adic topology.*
2. *e and T commute, we have a T-stable decomposition $M = eM \oplus (1-e)M$ where T is bijective on eM and topologically nilpotent on $(1-e)M$.*

Proof. We reduce to the situation where M is a finite $R/\mathfrak{m}_R^n$-module for some n. Then M is a finite set and we claim that the sequence $\{T^{N!}v\}$ is constant for N large enough. Indeed, the decreasing sequence of modules $T^{N!}M$ is stationnary for $N \geq N_0$. On $T^{N_0!}M$, T acts bijectively, hence has finite order. As a result the projector e is well defined and all the properties are easily deduced. □

Lemma 2.1.4. — *Let $f : M_1 \to M_2$ be a morphism in $\mathbf{Mod}^{comp}(R)$. Let T be a R-linear homomorphism acting equivariantly on M_1 and M_2. Assume that the induced action of T is locally finite on M_1 and M_2 and denote by e the ordinary projector associated to T on M_1 and M_2.*

1. *We have $f(eM_1) \subset eM_2$ and $f((1-e)M_1) \subset (1-e)M_2$.*
2. *Assume that $M_3 = \mathrm{coker} f \in \mathbf{Mod}^{comp}(R)$. Then $\mathrm{coker}(eM_1 \to eM_2) = eM_3$.*
3. *Let $M_0 = \mathrm{ker} f$. Assume that T is locally finite on M_0. Then $\mathrm{ker}(eM_1 \to eM_2) = eM_0$.*

Proof. Since f commutes with T, it also commutes with the projector e given by the formula of lemma 2.1.3. We deduce point 1 which means that f decomposes as the direct sum of the maps ef and $(1-e)f$. Using point 1, we deduce easily point 2 and point 3. □

2.2. Perfect complexes. — Let $\mathbf{D}(R)$ be the derived category of $\mathbf{Mod}(R)$. Let $\mathbf{C}^{comp}(R)$ be the category of bounded complexes of $\mathfrak{m}_R$-adically complete and separated R-modules with morphisms the morphisms of complexes of degree 0. Let $\mathbf{C}^{flat}(R)$ be the full subcategory of $\mathbf{C}^{comp}(R)$ whose objects are bounded complexes of $\mathfrak{m}_R$-adically complete and separated, flat R-modules. Let $\mathbf{D}^{comp}(R)$ and $\mathbf{D}^{flat}(R)$ be the full subcategories of $\mathbf{D}(R)$ generated by the objects of $\mathbf{C}^{comp}(R)$ and $\mathbf{C}^{flat}(R)$. We denote by $\mathbf{C}^{perf}(R)$ the full subcategory of $\mathbf{C}^{flat}(R)$ of complexes of finite free R-modules (also called perfect complexes), and by $\mathbf{K}^{perf}(R)$ the homotopy category. Its objects are the same as $\mathbf{C}^{perf}(R)$ but morphisms are homotopy classes of morphisms in $\mathbf{C}^{perf}(R)$. Let $\mathbf{D}^{perf}(R)$ be the full subcategory of $\mathbf{D}(R)$ generated by $\mathbf{C}^{perf}(R)$. The functor $\mathbf{K}^{perf}(R) \to \mathbf{D}^{perf}(R)$ is an equivalence of category ([**83**], coro. 10.4.7).

The following proposition gives a characterization of $\mathbf{D}^{perf}(R)$ inside $\mathbf{D}^{flat}(R)$.

Proposition 2.2.1. — *Let $M^\bullet$ be an object of $\mathbf{C}^{flat}(R)$, concentrated in degree $[a,b]$. Assume that $M^\bullet \otimes_R R/\mathfrak{m}_R$ has finite cohomology groups. Then $M^\bullet$ is quasi-isomorphic to a perfect complex concentrated in degree $[a,b]$.*

Proof. It suffices to show that $\mathrm{H}^n(M^\bullet)$ is a finite R-module. By [**55**], lem. 1, p. 44, we would then deduce that $M^\bullet$ is quasi-isomorphic to a perfect complex concentrated in degree $[a,b]$.

We have short exact sequences of complexes

$$0 \to \mathfrak{m}_R^n/\mathfrak{m}_R^{n-1} \otimes_R M^\bullet \to M^\bullet/\mathfrak{m}_R^n \to M^\bullet/\mathfrak{m}_R^{n-1} \to 0$$

and by induction, we deduce easily that the cohomology groups $\mathrm{H}^i(M^\bullet/\mathfrak{m}_R^n)$ are finite $R/\mathfrak{m}_R^n$-modules. As a result, the system $\{\mathrm{H}^i(M^\bullet/\mathfrak{m}_R^n)\}$ satisfies the Mittag-Leffler condition. By [**EGA**], III, chap. 0, prop. 13.2.3, we deduce that $\mathrm{H}^i(M^\bullet) = \lim_n \mathrm{H}^i(M^\bullet/\mathfrak{m}_R^n)$. It follows that $\mathrm{H}^i(M^\bullet)$ is complete and separated. The map $\mathrm{H}^i(M^\bullet) \to \lim_n \mathrm{H}^i(M^\bullet)/\mathfrak{m}_R^n$ is an isomorphism. Therefore, $\mathrm{H}^i(M^\bullet)$ is a finite R-module if and only if $\mathrm{H}^i(M^\bullet)/\mathfrak{m}_R$ is a finite R-module by topological Nakayama's lemma. Recall ([**83**], thm. 5.6.4) that there is a spectral sequence

$$E_2^{p,q} = \mathrm{Tor}^R_{-p}(H^q(M^\bullet), R/\mathfrak{m}_R) \Rightarrow \mathrm{H}^{p+q}(M^\bullet \otimes_R R/\mathfrak{m}_R)$$

with $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$. We prove by descending induction on i that $\mathrm{H}^i(M^\bullet)$ is a finite R-module. Assume this holds for $i \geq n+1$ and let us prove it for $i = n$. The map $\mathrm{H}^n(M^\bullet)/\mathfrak{m}_R \to \mathrm{H}^n(M^\bullet/\mathfrak{m}_R)$ has a kernel which admits a surjective map from subquotients of the modules $\mathrm{Tor}_{r+1}(\mathrm{H}^{n+r}(M^\bullet), R/\mathfrak{m}_R)$ for $r \geq 1$. There are only finitely many values of r for which these modules are non-zero and all are finite dimensional by the induction hypothesis. It follows that the kernel is finite dimensional and thus $\mathrm{H}^n(M^\bullet)/\mathfrak{m}_R$ is also finite dimensional and $\mathrm{H}^n(M^\bullet)$ is a finite R-module by Nakayama's lemma. □

The following is a version of Nakayama's lemma for complexes.

Proposition 2.2.2. — *Let $f : M^\bullet \to N^\bullet$ be a map in $\mathbf{C}^{flat}(R)$. We assume that $f \otimes 1 : M^\bullet \otimes_R R/\mathfrak{m}_R \to N^\bullet \otimes_R R/\mathfrak{m}_R$ is a quasi-isomorphism. Then f is a quasi-isomorphism.*

Proof. Consider the cone $C(f)$ of the map f. We need to prove that $C(f)$ is acyclic. $C(f)$ is an object of $\mathbf{C}^{flat}(R)$ and $C(f) \otimes_R R/\mathfrak{m}_R$ is the cone of $f \otimes 1$ and is acyclic. It follows from the previous proposition that $C(f)$ is quasi-isomorphic to a perfect complex and thus, the groups $\mathrm{H}^i(C(f))$ are finite R-modules. We now prove by descending induction on i that $\mathrm{H}^i(C(f)) = 0$. Assume this holds for $i \geq n+1$. Using the spectral sequence $E_2^{p,q} = \mathrm{Tor}^R_{-p}(H^q(M^\bullet), R/\mathfrak{m}_R) \Rightarrow \mathrm{H}^{p+q}(M^\bullet \otimes_R R/\mathfrak{m}_R)$ we see that $\mathrm{H}^n(C(f))/\mathfrak{m}_R \hookrightarrow \mathrm{H}^n(C(f)/\mathfrak{m}_R) = 0$. By Nakayama's lemma, we deduce that $\mathrm{H}^n(C(f)) = 0$. □

2.3. Projectors. — We now consider projectors on complexes.

Definition 2.3.1. — *Let $M^\bullet \in \mathbf{C}^{flat}(R)$. Let $T \in \mathrm{End}_{\mathbf{C}^{flat}(R)}(M^\bullet)$. We say that T is locally finite on $M^\bullet$ if T acts locally finitely on each M^i.*

By lemma 2.1.3, we can attach to T a projector $e \in \mathrm{End}_{\mathbf{C}^{flat}(R)}(M^\bullet)$.

Definition 2.3.2. — *Let $M^\bullet \in \mathbf{D}^{flat}(R)$. Let $T \in \mathrm{End}_{\mathbf{D}^{flat}(R)}(M^\bullet)$. We say that T is locally finite if there exist $M_0^\bullet \in \mathbf{C}^{flat}(R)$ a representative of $M^\bullet$ and $T_0 \in \mathrm{End}_{\mathbf{C}^{flat}(R)}(M_0^\bullet)$ a representative of T which is locally finite.*

The following is a characterization of locally finite morphisms.

Proposition 2.3.1. — *Let $M^\bullet \in \mathbf{D}^{flat}(R)$. Let $T \in \mathrm{End}_{\mathbf{D}^{flat}(R)}(M^\bullet)$. The following are equivalent :*

1. *T is locally finite,*
2. *T is locally finite on the cohomology groups $\mathrm{H}^i(M^\bullet \otimes^L_R R/\mathfrak{m}_R)$ and there exist representatives $M_0^\bullet \in \mathbf{C}^{flat}(R)$ of $M^\bullet$ and $T_0 \in \mathrm{End}_{\mathbf{C}^{flat}(R)}(M_0^\bullet)$ of T.*

Proof. The implication 1. ⇒ 2. follows from lemma 2.1.1. We do the other implication. Let $M_0^\bullet$ and T_0 be representatives of $M^\bullet$ and T. We claim that $M_0^\bullet$ has a subcomplex $N^\bullet \in \mathbf{C}^{flat}(R)$ which has the properties:

1. all the differentials $d : N^i \to N^{i+1}$ are 0 modulo $\mathfrak{m}_R$,
2. the inclusion map $i : N^\bullet \to M_0^\bullet$ has a section $s : M_0^\bullet \to N^\bullet$,
3. the maps i and s are quasi-isomorphisms.

It follows that $N^\bullet$ and $s \circ T_0 \circ i$ are representatives of $M^\bullet$ and T, and moreover $s \circ T_0 \circ i$ acts like T on $\mathrm{H}^i(M^\bullet \otimes^L_R R/\mathfrak{m}_R) = N^i/\mathfrak{m}_R$ and is therefore locally finite.

It remains to prove the claim. Fix some index i. By lemma 2.3.1, we can find decompositions $M_0^i = J^i \oplus K^i$ and $M_0^{i+1} = J^{i+1} \oplus K^{i+1}$ such that $d : M_0^i \to M_0^{i+1}$ preserves these decompositions and induces isomorphisms $J^i \to J^{i+1}$ and the zero map $K^i/\mathfrak{m}_R \to K^{i+1}/\mathfrak{m}_R$. It is easy to check that we get a subcomplex $S^\bullet$ of $M_0^\bullet$ by setting $S^j = M_0^j$ if $j \neq i, i+1$ and $S^j = K^j$ if $j \in \{i, i+1\}$. This subcomplex is a direct factor of $M_0^\bullet$ and the differential $d : S^i \to S^{i+1}$ vanishes modulo $\mathfrak{m}_R$. Repeating the process for all indices will produce a complex $N^\bullet$ with the expected property. □

Lemma 2.3.1. — *Let $f : M \to N$ be a map in $\mathbf{Mod}^{comp}(R)$. Assume that M and N are flat. There is a decomposition $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ such that $f(M_i) \subset N_i$ for $i \in \{1,2\}$, $f|_{M_1} : M_1 \to N_1$ is an isomorphism and $f|_{M_2} : M_2 \to N_2$ is zero modulo $\mathfrak{m}_R$.*

Proof. Let M be a flat object of $\mathbf{Mod}^{comp}(R)$. Let $\{\overline{e}_i\}_{i\in I}$ be a basis of $M/\mathfrak{m}_R$ as an $R/\mathfrak{m}_R$-module. Let $\{e_i\}_{i\in I} \in M^I$ be a lift of $\{\overline{e}_i\}_{i\in I}$. Denote by $\widehat{R^I}$ the $\mathfrak{m}_R$-adic completion of R^I. The map $R^I \to M$ corresponding to $\{e_i\}_{i\in I}$ induces an isomorphism $\widehat{R^I} \to M$ by Nakayama's lemma and the flatness assumption on M. We refer below to $\{e_i\}_{i\in I}$ as a topological basis of M. Let $f : M \to N$ be a map as in the lemma. Let $\{e_i\}_{i\in I}$ be a topological basis of M and let $\{\overline{e}_i\}_{i\in I}$ be its reduction modulo $\mathfrak{m}_R$. We can assume that $I = I' \coprod I''$ and that $\{\overline{e}_i\}_{i\in I''}$ is a basis of $\ker(M/\mathfrak{m}_R \to N/\mathfrak{m}_R)$. We may now take a topological basis for N, denoted by $\{h_j\}_{j\in J}$ with the property that $J = J' \coprod J''$, $J' = I'$ and $h_j = f(e_j)$ for $j \in J'$. In these basis, f is represented by an upper triangular matrix :

$$\begin{pmatrix} 1_{I'\times I'} & C \\ 0 & B \end{pmatrix},$$

where $1_{I'\times I'}$ is the identity matrix of size $I' \times I'$, $C = (c_{i,j})_{(i,j)\in I'\times I''} \in M_{I'\times I''}(R)$ and $B \in M_{I'',J''}(\mathfrak{m}_R)$. These matrices have the property that all their columns tend to 0 in R (for the filter of the complements of the finite subsets). For each $l \in I''$, we can replace e_l by $e_l - \sum_{i\in I'} c_{i,l} e_i$ (one checks that the sum converges). In this new basis, f has the correct shape :

$$\begin{pmatrix} 1_{I'\times I'} & 0 \\ 0 & B \end{pmatrix}.$$

□

Let $M^\bullet \in \mathbf{D}^{flat}(R)$ and $T \in \mathrm{End}_{\mathbf{D}^{flat}(R)}(M^\bullet)$ be a locally finite endomorphism. For each locally finite representative $M_0^\bullet \in \mathbf{C}^{flat}(R)$ of $M^\bullet$, and $T_0 \in \mathrm{End}_{\mathbf{C}^{flat}(R)}(M_0^\bullet)$ of T, we get a projector $e_0 \in \mathrm{End}_{\mathbf{C}^{flat}(R)}(M_0^\bullet)$ and a direct factor $e_0 M_0^\bullet$ of $M_0^\bullet$. We can consider $\bar{e_0}$ the image of e_0 in $\mathrm{End}_{\mathbf{D}^{comp}(R)}(M^\bullet)$ and the associated direct factor $\bar{e_0}M^\bullet$ of $M^\bullet$ in $\mathbf{D}^{comp}(R)$, which is represented with $e_0 M_0^\bullet$. We now discuss the independence on the lift $(M_0^\bullet, T_0)$.

Lemma 2.3.2. — *Let $(M_0^\bullet, T_0)$ and $(M_1^\bullet, T_1)$ be two locally finite representatives of $(M^\bullet, T)$. We denote by e_0 and e_1 the projectors associated to T_0 and T_1, and by $\bar{e_0}$ and $\bar{e_1}$ their images in $\mathrm{End}_{\mathbf{D}^{comp}(R)}(M^\bullet)$. Assume that $\bar{e_0}M^\bullet$ is an object of $\mathbf{D}^{perf}(R)$. Then the canonical map $\bar{e_0}M^\bullet \to \bar{e_1}M^\bullet$ is a quasi-isomorphism.*

Proof. On $\mathrm{H}^i(M^\bullet/\mathfrak{m}_R)$ we have $T_0 = T_1$ and T_0 and T_1 are locally finite by lemma 2.1.1. Moreover, the projector e' on $\mathrm{H}^i(M^\bullet/\mathfrak{m}_R)$ associated to $T_0 = T_1$ acting on $\mathrm{H}^i(M^\bullet/\mathfrak{m}_R)$ is also equal to the projector induced by $\bar{e_0}$ or $\bar{e_1}$ by lemma 2.1.4. It follows that the map $\bar{e_0}\mathrm{H}^i(M^\bullet/\mathfrak{m}_R) \to \bar{e_1}\mathrm{H}^i(M^\bullet/\mathfrak{m}_R)$ is an isomorphism. By proposition 2.2.1, $\bar{e_1}M^\bullet$ is a perfect complex. It follows that the natural map $\bar{e_0}M^\bullet \to \bar{e_1}M^\bullet$ can be represented by a map in $\mathbf{C}^{perf}(R)$. By proposition 2.2.2, this map will be a quasi-isomorphism. □

In the sequel of the paper, and under the assumptions of lemma 2.3.2, we will sometimes speak of the projector associated to a locally finite endomorphism, but one should keep in mind that this projector could depend on the choice of a particular representative, although two representatives give canonically isomorphic direct factors.

Remark 2.3.1. — In [**41**], lem. 2.12, there is a definition of the ordinary projector attached to an element $T \in \mathrm{End}_{\mathbf{D}^{comp}(R)}(M^\bullet)$ in the case where $M^\bullet$ is an object of $\mathbf{D}^{perf}(R)$. In this setting, the condition of being locally finite is automatically satisfied. Our definition in a more general setting is compatible with the definition of *op. cit.*. It is proven in *op. cit.* that the projector is unique. This rests on the property that the algebra $\mathrm{End}_{\mathbf{D}^{comp}(R)}(M^\bullet)$ is finite over R when $M^\bullet$ is a perfect complex.

3. Cohomological preliminaries

This section contains a number of technical results concerning the cohomology of adic spaces. These results are only used in part III of this work.

3.1. Cohomology of $\mathscr{O}_{\mathcal{X}}^+$. — Let k be a complete non-archimedean field with ring of integers $\mathcal{O}_k$ and maximal ideal $m_{\mathcal{O}_k}$. In this section, we will only consider adic spaces $\mathcal{X}$ over $\mathrm{Spa}(k, \mathcal{O}_k)$ which are of finite type (in particular quasi-compact), and separated. The structural sheaf of $\mathcal{X}$ is denoted by $\mathscr{O}_{\mathcal{X}}$. There are a subsheaves $\mathscr{O}_{\mathcal{X}}^+$ and $\mathscr{O}_{\mathcal{X}}^{++}$ of $\mathscr{O}_{\mathcal{X}}$ defined by

$$\mathscr{O}_{\mathcal{X}}^+(U) = \{f \in \mathscr{O}_{\mathcal{X}}(U),\ \forall x \in U\ |f|_x \leq 1\} \text{ and } \mathscr{O}_{\mathcal{X}}^{++}(U) = \{f \in \mathscr{O}_{\mathcal{X}}(U),\ \forall x \in U\ |f|_x < 1\}$$

for all open subsets U of $\mathcal{X}$. If $U = \mathrm{Spa}(A, A^+)$ for a complete Tate algebra topologically of finite type, and A^0 denotes the subring of A of power bounded elements, and A^{00} the ideal of A^0 of topologically nilpotent elements, then $\mathscr{O}_{\mathcal{X}}^+(U) = A^+ = A^0$ and $\mathscr{O}_{\mathcal{X}}^{++}(U) = A^{00}$ ([**35**], lem. 4.4).

Proposition 3.1.1. — *Let $\mathcal{X}$ be a separated adic space of finite type. The natural maps*

$$\check{\mathrm{H}}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) \to \mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+)$$

and

$$\check{\mathrm{H}}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{++}) \to \mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{++})$$

from Čech cohomology to cohomology are isomorphisms.

Proof. There is an isomorphism in the category of locally ringed spaces $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) = \lim_{\mathfrak{X}}(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ where $\mathfrak{X}$ runs over all formal models of $\mathcal{X}$ (see [**69**], thm. 2.22). By [**19**], prop. 3.1.10, we deduce that $\mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) = \lim_{\mathfrak{X}} \mathrm{H}^i(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$. Since $\mathcal{X}$ is quasi-compact, one can compute Čech cohomology using only finite coverings (see [**26**], p. 224). It follows that $\check{\mathrm{H}}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) = \lim_{\mathfrak{X}} \check{\mathrm{H}}^i(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$. Since $\mathcal{X}$ is separated, the formal models $\mathfrak{X}$ are separated ([**7**], prop. 4.7) and $\mathrm{H}^i(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = \check{\mathrm{H}}^i(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$.

We prove the second isomorphism. Let $\mathfrak{X}$ be a formal scheme which is topologically of finite type over Spf $\mathcal{O}_k$. Let $\bar{X}$ be its special fiber over Spec $\mathcal{O}_k/m_{\mathcal{O}_k}$ and $\bar{X}^{red}$ the reduced special fiber. There is a surjective map of coherent sheaves over $\mathfrak{X}$: $\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\bar{X}^{red}}$ and we denote by $\mathfrak{I}_{\mathfrak{X}}$ its kernel. Under the isomorphism of locally ringed spaces $(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) = \lim_{\mathfrak{X}}(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ where $\mathfrak{X}$ runs over all formal models of $\mathcal{X}$, we have $\mathscr{O}_{\mathcal{X}}^{++} = \mathrm{colim}_{\mathfrak{X}} \mathfrak{I}_{\mathfrak{X}}$. The second isomorphism can be proved by repeating the proof of the first isomorphism. $\square$

We now recall a result of Bartenwerfer.

Theorem 3.1.1 **([2])**. — *Let $\mathcal{X}$ be a smooth affinoid adic space of finite type. For all $i > 0$, $\mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+)$ is annihilated by a non-zero element $c(\mathcal{X}) \in \mathcal{O}_k$. If $\mathcal{X}$ admits a smooth affine formal model, then $\mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{++}) = 0$ for all $i > 0$.*

Remark 3.1.1. — We do not known whether $\mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) = 0$ for affinoids which admit a smooth affine formal models. For some results in dimension 1, see [**82**] sect. 3.

Corollary 3.1.1. — *Let $\mathfrak{X}$ be an admissible smooth and separated formal scheme. Let $\mathcal{X}$ be its generic fiber. Then the canonical map $\mathrm{H}^i(\mathfrak{X}, m_{\mathcal{O}_k}\mathscr{O}_{\mathfrak{X}}) \to \mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{++})$ is an isomorphism.*

Proof. Take an affine covering $\mathfrak{U}$ of $\mathfrak{X}$. The cohomology of $m_{\mathcal{O}_k}\mathscr{O}_{\mathfrak{X}}$ is computed by Čech cohomology with respect to this covering : $\mathrm{H}^i(\mathfrak{X}, m_{\mathcal{O}_k}\mathscr{O}_{\mathfrak{X}}) = \check{\mathrm{H}}^i_{\mathfrak{U}}(\mathfrak{X}, m_{\mathcal{O}_k}\mathscr{O}_{\mathfrak{X}})$. Let $\mathcal{U}$ be the generic fiber of $\mathfrak{U}$. Let $\mathfrak{V}$ be an open in $\mathfrak{X}$ with generic fiber $\mathcal{V}$. Since $\mathfrak{X}$ is smooth, $m_{\mathcal{O}_k}\mathscr{O}_{\mathfrak{X}}(\mathfrak{V}) = \mathscr{O}_{\mathcal{X}}^{++}(\mathcal{V})$. We deduce that $\check{\mathrm{H}}^i_{\mathfrak{U}}(\mathfrak{X}, m_{\mathcal{O}_k}\mathscr{O}_{\mathfrak{X}}) = \check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{++})$. By [**26**] corollaire on page 213 and theorem 3.1.1, we have $\check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{++}) = \mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^{++})$. $\square$

3.2. Cohomology of projective limits of sheaves.

— We denote by p a topologically nilpotent unit in k.

Lemma 3.2.1. — *Let $\mathcal{X}$ be a smooth affinoid adic space. The map $\mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \to \lim_n \mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}/p^n\mathscr{O}_{\mathcal{X}}^+)$ is an isomorphism.*

Proof. First assume that $i > 0$. We need to prove that $\lim_n \mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}/p^n\mathscr{O}_{\mathcal{X}}^+) = 0$. Using the exact sequence $0 \to p^n\mathscr{O}_{\mathcal{X}}^+ \to \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{X}}/p^n\mathscr{O}_{\mathcal{X}}^+ \to 0$ and theorem 3.1.1, we deduce that $\mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}/p^n\mathscr{O}_{\mathcal{X}}^+)$ is annihilated by some constant $c \in \mathcal{O}_k \setminus \{0\}$ for all $i, n > 0$. It follows that $\lim_n \mathrm{H}^i(\mathcal{X}, \mathscr{O}_{\mathcal{X}}/p^n\mathscr{O}_{\mathcal{X}}^+)$ is annihilated by c. On the other hand, this group is p-divisible. It follows that it vanishes. The cokernel of the map $\mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}})/p^n\mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) \to \mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}}/p^n\mathscr{O}_{\mathcal{X}}^+)$ is killed by c. It follows that the map $\lim_n \mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}})/p^n\mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) \to \lim_n \mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}}/p^n\mathscr{O}_{\mathcal{X}}^+)$ is surjective : its cokernel is killed by c and both sides are p-divisible. On the other hand, $\mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ is a Banach space and, since $\mathcal{X}$ is reduced, $\mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+)$ is bounded inside this Banach space. It follows that $\cap_n p^n\mathrm{H}^0(\mathcal{X}, \mathscr{O}_{\mathcal{X}}^+) = \{0\}$. $\square$

Let $\mathscr{F}$ be a locally free sheaf of $\mathscr{O}_{\mathcal{X}}$-modules. We assume that there exists $\mathscr{F}^+ \subset \mathscr{F}$ a locally free sheaf of $\mathscr{O}_{\mathcal{X}}^+$-modules such that $\mathscr{F} = \mathscr{F}^+ \otimes_{\mathscr{O}_{\mathcal{X}}^+} \mathscr{O}_{\mathcal{X}}$.

Lemma 3.2.2. — *Assume that $\mathcal{X}$ is a smooth and separated adic space. Let $\mathcal{U}$ be a finite affinoid covering of $\mathcal{X}$, such that $\mathscr{F}^+|_{\mathcal{U}}$ is trivial. There is a non-zero element $c \in \mathcal{O}_k$ depending on $\mathcal{U}$ such that :*

- *for all $n \in \mathbb{N}$, the map $\check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+) \to \mathrm{H}^i(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$ from Čech cohomology relative to $\mathcal{U}$ to cohomology has kernel and cokernel annihilated by c,*
- *the map $\check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}^+) \to \mathrm{H}^i(\mathcal{X}, \mathscr{F}^+)$ has kernel and cokernel killed by c,*
- *the map $\lim_n \check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+) \to \lim_n \mathrm{H}^i(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$ has kernel killed by c and is surjective.*

Proof. Considering the spectral sequence associated to the covering $\coprod_{\mathcal{U}_i \in \mathcal{U}} \mathcal{U}_i \to \mathcal{X}$, we deduce that the kernel and cokernel of the maps $\check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+) \to \mathrm{H}^i(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$ are subquotients of $\mathrm{H}^k(U_J, \mathscr{F}/p^n\mathscr{F}^+)$ for $k > 0$ and U_J some intersection of the affinoids in $\mathcal{U}$. By theorem 3.1.1, both the kernel and cokernel are killed by some non-zero constant c (which does not depend on n). The same applies to the map $\check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}^+) \to \mathrm{H}^i(\mathcal{X}, \mathscr{F}^+)$. It follows that the map $\lim_n \check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+) \to \lim_n \mathrm{H}^i(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$ has kernel killed by c. Let us prove that the cokernel is killed by c^2. Since both modules are p-divisible, this will show the surjectivity. Let $(f_n) \in \lim_n \mathrm{H}^i(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$. Then for all n, there exists g_n in $\check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$ such that the image of g_n is cf_n. One sees that $(cg_n) \in \lim_n \check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$ has image $(c^2 f_n)$. □

Proposition 3.2.1. — *Let $\mathcal{X}$ be a smooth and separated adic space. The map*

$$\mathrm{H}^i(\mathcal{X}, \mathscr{F}) \to \lim_n \mathrm{H}^i(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$$

is surjective. If $\mathcal{X}$ is proper, the map is an isomorphism.

Proof. Let $\mathcal{U}$ be a finite affinoid covering of $\mathcal{X}$, such that $\mathscr{F}^+|_{\mathcal{U}}$ is trivial. The map $\lim_n \mathrm{H}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+) \to \lim_n \mathrm{H}^i(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$ is surjective. To prove the surjectivity of the map of the proposition, it suffices to show that the map $\mathrm{H}^i(\mathcal{X}, \mathscr{F}) = \check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}) \to \lim_n \mathrm{H}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^n\mathscr{F}^+)$ is surjective. Since all groups are p-divisible it is enough to prove that the cokernel is killed by some non-zero element $c \in \mathcal{O}_K$. This follows from the lemma below where $K^\bullet$ is the Čech complex which computes $\check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F})$ and $K^\bullet_\alpha$ is the complex that computes $\check{\mathrm{H}}^i_{\mathcal{U}}(\mathcal{X}, \mathscr{F}/p^\alpha\mathscr{F}^+)$. The fact that $K^\bullet$ is the limit of the $K^\bullet_\alpha$ is a consequence of lemma 3.2.1.

We now prove injectivity in case $\mathcal{X}$ is proper. The kernel of the map of the proposition is

$$\cap p^n \mathrm{Im}(H^i(\mathcal{X}, \mathscr{F}^+) \to \mathrm{H}^i(\mathcal{X}, \mathscr{F})).$$

Since $\mathrm{H}^i(\mathcal{X}, \mathscr{F})$ is a finite dimensional k-vector space, we need to show that

$$\mathrm{Im}(\mathrm{H}^i(\mathcal{X}, \mathscr{F}^+) \to \mathrm{H}^i(\mathcal{X}, \mathscr{F}))$$

is a lattice. This will follow if we can show that that $H^i(\mathcal{X}, \mathscr{F}^+)$ is the sum of a finite type $\mathcal{O}_k$-module and a torsion group. This can be proved as follows. Take a normal proper formal model $\mathfrak{X}$ of $\mathcal{X}$ such that the sheaf $\mathscr{F}^+$ comes from a locally free sheaf $\mathcal{F}$ on $\mathfrak{X}$. We can obtain such a model as follows. By Raynaud's theory, we can find a model $\mathfrak{X}'$ of $\mathcal{X}$ which admits an affinoid covering $\mathfrak{U}'$ whose generic fiber refines $\mathcal{U}$. We can replace $\mathfrak{X}'$ by its normalisation $\mathfrak{X}$ in $\mathcal{X}$. This is still a formal model. The sheaf $\mathscr{F}^+$ comes from a locally free sheaf $\mathcal{F}$ on $\mathfrak{X}'$. By [**52**], lemma 2.6, this model is automatically proper. Let $\mathfrak{V}$ be an affine covering of $\mathfrak{X}$ and $\mathcal{V}$ be its generic fiber. We have a map from Čech cohomology to cohomology $\check{\mathrm{H}}^i_{\mathcal{V}}(\mathcal{X}, \mathscr{F}^+) \to \mathrm{H}^i(\mathcal{X}, \mathscr{F}^+)$ whose kernel and cokernel are killed by a non-zero constant c by lemma 3.2.2. The cohomology $\check{\mathrm{H}}^i_{\mathcal{V}}(\mathcal{X}, \mathscr{F}^+)$ is identified with the cohomology $\mathrm{H}^i(\mathfrak{X}, \mathcal{F})$ and it is a finite $\mathcal{O}_k$-module since $\mathfrak{X}$ is proper.

□

Lemma 3.2.3. — *Let $(K_\alpha^\bullet)_{\alpha\in\mathbb{N}}$ be a projective system of complexes of $\mathcal{O}_k$-modules. Let $K^\bullet = \lim_\alpha K_\alpha^\bullet$. Assume that there is an element $c \in \mathcal{O}_k$ such that the cokernel of the map $K^n \to K_\alpha^n$ is killed by c for all n and α. Then the cokernel of the map $\mathrm{H}^i(K^\bullet) \to \lim_\alpha \mathrm{H}^i(K_\alpha^\bullet)$ is killed by c.*

Proof. For all i we have exact sequences :

$$0 \to B^i(K_\alpha^\bullet) \to Z^i(K_\alpha^\bullet) \to \mathrm{H}^i(K_\alpha^\bullet) \to 0$$

Clearly $Z^i(K^\bullet) = \lim_\alpha Z^i(K_\alpha^\bullet) \hookrightarrow K^i$. Let $(x_\alpha) \in \lim_\alpha \mathrm{H}^i(K_\alpha^\bullet)$. Let $z_\alpha \in Z^i(K_\alpha^\bullet)$ be a lift of x_α. Let $\mathrm{Im}_\alpha(z_{\alpha+1})$ be the image of $z_{\alpha+1}$ in $Z^i(K_\alpha^\bullet)$. Then $\mathrm{Im}_\alpha(z_{\alpha+1}) - z_\alpha = d(w_\alpha) \in B^i(K_\alpha^\bullet)$. Let $t_\alpha \in K^{i-1}$ be a lift of cw_α. The sequence $cz_0, cz_1 + d(t_0), cz_2 + d(t_0+t_1), \cdots$ converges in $Z^i(K^\bullet)$ to a lift of $c(x_\alpha)$.

□

3.3. Base change. — Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact map of finite type adic spaces over $\mathrm{Spa}(k, \mathcal{O}_k)$. Let $i : \mathcal{Z} \to \mathcal{Y}$ be a map of adic spaces over $\mathrm{Spa}(k, \mathcal{O}_k)$ inducing an homeomorphism from $\mathcal{Z}$ to $i(\mathcal{Z})$ and for all $z \in \mathcal{Z}$ a bijective map $(k(i(z)), k(i(z))^+) \to (k(z), k(z)^+)$. We can form the following cartesian diagram :

$$\begin{tikzcd} \mathcal{X}_{\mathcal{Z}} \arrow[r, "i'"] \arrow[d, "f'"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{Z} \arrow[r, "i"] & \mathcal{Y} \end{tikzcd}$$

Lemma 3.3.1. — *For all $n \in \mathbb{N}$, the canonical map $(i')^{-1}\mathscr{O}_{\mathcal{X}}^{++}/p^n \to \mathscr{O}_{\mathcal{X}_{\mathcal{Z}}}^{++}/p^n$ is an isomorphism.*

Proof. The stalk of these sheaves at a point $x \in \mathcal{X}_{\mathcal{Z}}$ is $k(x)^{00}/p^n$ (compare with [**69**], prop. 2.25). □

Proposition 3.3.1. — *For all $n \in \mathbb{N}$, we have the base change formula :*

$$i^{-1}\mathrm{R}f_\star \mathscr{O}_{\mathcal{X}}^{++}/p^n = \mathrm{R}f'_\star \mathscr{O}_{\mathcal{X}_{\mathcal{Z}}}^{++}/p^n.$$

Proof. The sheaf $\mathrm{R}^k f_\star \mathscr{O}_{\mathcal{X}}^{++}/p^n$ is sheaf associated to the presheaf $U \mapsto \mathrm{H}^k(f^{-1}(U), \mathscr{O}_{\mathcal{X}}^{++}/p^n)$. Thus, $i^{-1}\mathrm{R}^k f_\star \mathscr{O}_{\mathcal{X}}^{++}/p^n$ is the sheaf associated to the presheaf $V \mapsto \mathrm{colim}_{V\subset U}\mathrm{H}^k(f^{-1}(U), \mathscr{O}_{\mathcal{X}}^{++}/p^n)$ where U runs over the neighborhoods of V in $\mathcal{Y}$. Using the lemma above, we deduce that $\mathrm{R}^k f'_\star \mathscr{O}_{\mathcal{X}_{\mathcal{Z}}}^{++}/p^n = \mathrm{R}^k f'_\star i'^{-1}\mathscr{O}_{\mathcal{X}}^{++}/p^n$ is the sheaf associated to the presheaf $V \mapsto \mathrm{colim}_{\mathcal{X}_V \subset W}\mathrm{H}^k(W, \mathscr{O}_{\mathcal{X}}^{++}/p^n)$ where W runs along the neighborhoods of $\mathcal{X}_V$ in $\mathcal{X}$.

Since the map f is quasi-compact, we deduce that for V a quasi-compact open in $\mathcal{Z}$, the set of neighborhoods of $\mathcal{X}_V$ of the form $f^{-1}(U)$ for U a neighborhood of V in $\mathcal{Y}$ is cofinal in the set of all neighborhoods of $\mathcal{X}_V$ in $\mathcal{X}$. □

3.4. Cohomology of torus embeddings. — Let T be a split torus over Spec $\mathbb{Z}$. We will denote by $\mathfrak{T}$ the formal torus over Spf $\mathbb{Z}_p$ obtained by taking the completion of T along its special fiber $T \times \mathrm{Spec}\ \mathbb{F}_p$. We denote by $T^{an} \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ the analytification of $T \times \mathrm{Spec}\ \mathbb{Q}_p$ (in other words, $T^{an} = \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \times_{\mathrm{Spec}\ \mathbb{Q}} T$, see [**35**], prop. 3.8). We denote by $T^{rig} \subset T^{an}$ the generic fiber of $\mathfrak{T}$ (see [**35**], prop. 4.2). Let $X_\star(T)$ denote the group of cocharacters of T. Let Σ be a rational polyhedral cone in $X_\star(T)$. Let $T \to T_\Sigma$ be the associated toric embedding defined over Spec $\mathbb{Z}$ ([**40**]). We define obviously T_Σ^{an},

T_{Σ}^{rig} and $\mathfrak{T}_{\Sigma}$. Let Σ' be a refinement of Σ. We can similarly define $T_{\Sigma'}^{an}$, $T_{\Sigma'}^{rig}$ and $\mathfrak{T}_{\Sigma'}$. We say that Σ' is smooth if $T_{\Sigma'}$ is smooth.

Proposition 3.4.1. — *With the notations as above, let $f : T_{\Sigma'}^{an} \to T_{\Sigma}^{an}$ be the natural morphism. Assume that Σ' is smooth. Then we have a quasi-isomorphism :*

$$\mathscr{O}^{++}_{T_{\Sigma}^{an}} \to \mathrm{R}f_\star \mathscr{O}^{++}_{T_{\Sigma'}^{an}}.$$

Proof. We first observe that the result holds true after inverting p by classical results on toroidal embeddings (see [**40**], coro. 1 on page 44) and the comparison theorem stated in [**70**], thm. 9.1. It follows easily that $\mathscr{O}^{++}_{T_{\Sigma}^{an}} \simeq f_\star \mathscr{O}^{++}_{T_{\Sigma'}^{an}}$ and we are left to prove that $\mathrm{R}^i f_\star \mathscr{O}^{++}_{T_{\Sigma'}^{an}} = 0$ for all $i > 0$. It suffices to show that $\mathrm{R}^i f_\star \mathscr{O}^{++}_{T_{\Sigma'}^{an}}/p = 0$ for all $i > 0$ since this will imply that multiplication by p is surjective on $\mathrm{R}^i f_\star \mathscr{O}^{++}_{T_{\Sigma'}^{an}}$ for all $i > 0$ and we know that this sheaf is torsion.

Let $x \in T_{\Sigma}^{an}$. Let $\sigma \in \Sigma$ be the minimal cone such that $x \in T_{\sigma}^{an}$. This means that x belong to the closed stratum in T_{σ}^{an}. Let $\sigma_{\mathbb{R}} \subset X_\star(T)_{\mathbb{R}}$ be the $\mathbb{R}$-span of σ. Define $X_\star(T_2) = X_\star(T) \cap \sigma_{\mathbb{R}}$. This is a saturated submodule of $X_\star(T)$. It follows that $X_\star(T_2)$ is a free $\mathbb{Z}$-module and a direct factor. We choose a direct factor $X_\star(T_1)$. We have $X_\star(T) = X_\star(T_1) \oplus X_\star(T_2)$. Let $T = T_1 \times T_2$ be the associated decomposition of T.

Then we have $T_{\sigma}^{an} \simeq T_1^{an} \times T_{2,\sigma}^{an}$. Moreover, since σ spans $X_\star(T_2)$, we deduce that the closed stratum of $T_{2,\sigma}^{an}$ for the action of T_2^{an} is reduced to a point which we call 0. Then $x = (x', 0) \in T_1^{an} \times T_{2,\sigma}^{an}$. Moreover, $f^{-1}(T_{\sigma}^{an}) \simeq T_1^{an} \times T_{2,\Sigma''}^{an}$ where Σ'' is the polyhedral decomposition $(\sigma \cap \Sigma') \cap X_\star(T_2)$. Let $f_2 : T_{2,\Sigma''}^{an} \to T_{2,\sigma}^{an}$ be the natural projection deduced from f. Let $f_2' : x' \times T_{2,\Sigma''}^{an} \to x' \times T_{2,\sigma}^{an}$ be the map obtained from f_2 by base change.

By proposition 3.3.1, we have

$$\mathrm{R}^i f_\star \mathscr{O}^{++}_{T_{\Sigma'}^{an}}/p|_{(x',0)} = \mathrm{R}^i (f_2')_\star \mathscr{O}^{++}_{x' \times T_{2,\Sigma''}^{an}}/p|_{(x',0)}.$$

First assume that x is a maximal point corresponding to a rank 1 valuation on $k(x)$. Set $U_0 = x' \times T_{2,\sigma}^{rig}$. Fix an isomorphism $T_2 \simeq \mathbb{G}_m^s$ for some integer s. Let $\underline{p} = (p, \cdots, p) \in T_2^{an}(\mathbb{Q}_p)$. Then the $\{U_n = \underline{p}^n U_0\}_{n \in \mathbb{N}}$ form a fundamental system of neighborhoods of x in $x' \times T_{2,\sigma}^{an}$. It is enough to prove that $\mathrm{H}^i(f^{-1}(U_n), \mathscr{O}^{++}_{x' \times T_{2,\Sigma''}^{an}}) = 0$ for all $i > 0$ and all $n \geq 0$. Using the action of $\underline{p}$ we are reduced to the case of U_0. There,

$$\mathrm{H}^i(f^{-1}(U_0), \mathscr{O}^{++}_{x' \times T_{2,\Sigma''}^{an}}) = \mathrm{H}^i(x' \times T_{2,\Sigma''}^{rig}, \mathscr{O}^{++}_{x' \times T_{2,\Sigma''}^{rig}}) = \mathrm{H}^i(\mathfrak{T}_{2,\Sigma''}, k(x)^{00} \hat{\otimes}_{\mathbb{Z}_p} \mathscr{O}_{\mathfrak{T}_{2,\Sigma''}})$$

by corollary 3.1.1 applied over the non-archimedean field $(k(x), k(x)^+)$. By classical results on toroidal embeddings (see [**40**], coro. 1 on page 44) we find that $\mathrm{H}^i(\mathfrak{T}_{2,\Sigma''}, k(x)^{00} \hat{\otimes} \mathscr{O}_{\mathfrak{T}_{2,\Sigma''}}) = \mathrm{H}^i(\mathfrak{T}_{2,\sigma}, k(x)^{00} \hat{\otimes} \mathscr{O}_{\mathfrak{T}_{2,\sigma}})$. But $\mathrm{H}^i(\mathfrak{T}_{2,\sigma}, k(x)^{00} \hat{\otimes} \mathscr{O}_{\mathfrak{T}_{2,\sigma}}) = 0$ for $i > 0$ since $\mathfrak{T}_{2,\sigma}$ is affine.

If x is not a maximal point, let $\tilde{x}$ be the maximal generisation of x. Then

$$\mathrm{R}^i f_\star \mathscr{O}^{++}_{T_{\Sigma'}^{an}}/p|_x = \mathrm{R}^i f_\star \mathscr{O}^{++}_{T_{\Sigma'}^{an}}/p|_{\tilde{x}} = 0$$

by [**82**], prop. 1.4.10 and example 1.5.2. □

4. Correspondences and coherent cohomology

In this section we develop a formalism of cohomological correspondences in coherent cohomology and prove a few results that will allow us to consider Hecke operators on the coherent cohomology of Shimura varieties.

4.1. Preliminaries on residue and duality. — We start by recalling some results of the duality theory for coherent cohomology. Standard references are [**32**] and [**15**]. For a scheme X we let $\mathbf{D}_{qcoh}(\mathscr{O}_X)$ be the subcategory of the derived category $\mathbf{D}(\mathscr{O}_X)$ of $\mathscr{O}_X$-modules whose objects have quasi-coherent cohomology sheaves. We let $\mathbf{D}^+_{qcoh}(\mathscr{O}_X)$ (resp. $\mathbf{D}^-_{qcoh}(\mathscr{O}_X)$) be the full subcategory of $\mathbf{D}_{qcoh}(\mathscr{O}_X)$ whose objects have trivial cohomology sheaves in sufficiently negative (resp. positive) degree. We let $\mathbf{D}^b_{qcoh}(\mathscr{O}_X)$ be the full subcategory of $\mathbf{D}_{qcoh}(\mathscr{O}_X)$ whose objects have trivial cohomology sheaves for all but finitely many degrees. We remark that if X is locally noetherian $\mathbf{D}^+_{qcoh}(\mathscr{O}_X)$ is also the derived category of the category of bounded below complexes of quasi-coherent sheaves on X ([**32**], coro. 7.19). We let $\mathbf{D}^b_{qcoh}(\mathscr{O}_X)_{fTd}$ be the full subcategory of $\mathbf{D}^b_{qcoh}(\mathscr{O}_X)$ whose objects are quasi-isomorphic to bounded complexes of flat sheaves of $\mathscr{O}_X$-modules (see [**32**], def. 4.3 on p. 97). Let us fix for the rest of this section a noetherian affine scheme S.

4.1.1. Embeddable morphisms. — Let X, Y be two S-schemes and $f : X \to Y$ be a morphism of S-schemes. The mophisme f is embeddable if there exists a smooth S-scheme P and a finite map $i : X \to P \times_S Y$ such that f is the composition of i and the second projection (see [**32**], p. 189). A morphism f is projectively embeddable if it is embeddable and P can be taken to be a projective space over S (see [**32**], p. 206).

4.1.2. The functor $f^!$. — Let $f : X \to Y$ be a morphism of S-schemes. There is a functor $\mathrm{R}f_\star : \mathbf{D}_{qcoh}(\mathscr{O}_X) \to \mathbf{D}_{qcoh}(\mathscr{O}_Y)$. By [**32**], thm. 8.7, if f is embeddable, we can define a functor $f^! : \mathbf{D}^+_{qcoh}(\mathscr{O}_Y) \to \mathbf{D}^+_{qcoh}(\mathscr{O}_X)$. If f is projectively embeddable, by [**32**] thm. 10.5, there is a natural transformation (trace map) $\mathrm{R}f_\star f^! \Rightarrow Id$ of endofunctors of $\mathbf{D}^+_{qcoh}(\mathscr{O}_Y)$. Moreover, by [**32**], thm. 11. 1, this natural transformation induces a duality isomorphism:

$$\mathrm{Hom}_{\mathbf{D}_{qcoh}(\mathscr{O}_X)}(\mathscr{F}, f^!\mathscr{G}) \overset{\sim}{\to} \mathrm{Hom}_{\mathbf{D}_{qcoh}(\mathscr{O}_Y)}(\mathrm{R}f_\star\mathscr{F}, \mathscr{G})$$

for $\mathscr{F} \in \mathbf{D}^-_{qcoh}(\mathscr{O}_X)$ and $\mathscr{G} \in \mathbf{D}^+_{qcoh}(\mathscr{O}_Y)$.

The functor $f^!$ for embeddable morphisms enjoys many good properties. Let us record one that will be crucially used.

***Proposition 4.1.2.1* ([32], prop. 8.8).** — *If $\mathscr{F} \in \mathbf{D}^+_{qcoh}(\mathscr{O}_Y)$ and $\mathscr{G} \in \mathbf{D}^b_{qcoh}(\mathscr{O}_Y)_{fTd}$, we have a functorial isomorphism $f^!\mathscr{F} \otimes^L Lf^\star\mathscr{G} = f^!(\mathscr{F} \otimes^L \mathscr{G})$.*

4.1.3. Dualizing sheaf and cotangent complex. — A morphism $f : X \to S$ is called a local complete intersection (abbreviated lci) if locally on X we have a factorization $f : X \overset{i}{\to} Z \to S$ where i is a regular immersion (see [**EGA**] IV, def. 16.9.2) and Z is a smooth S-scheme. If f is lci, we can define the cotangent complex of f denoted by $\mathbb{L}_{X/S}$ (see [**37**], prop. 3.2.9). This is a perfect complex concentrated in degree -1 and 0. Its determinant in the sense of [**43**] is denoted by $\omega_{X/S}$.

Proposition 4.1.3.1. — *If $h : X \to S$ is an embeddable morphism and a local complete intersection of pure relative dimension n, then $f^!\mathscr{O}_X = \omega_{X/S}[n]$ where $\omega_{X/S}$ is the determinant of the cotangent complex $\mathbb{L}_{X/S}$.*

Proof. This follows from the very definition of $f^!$ given in thm 8.7 of [**32**]. $\square$

Corollary 4.1.3.1. — *Let $h : X \to S$, $g : Y \to S$ be embeddable morphisms of S-schemes which are lci of pure dimension n. Let $f : X \to Y$ be an embeddable morphism of S-schemes. Then $f^!\mathscr{O}_Y = \omega_{X/S} \otimes f^\star\omega^{-1}_{Y/S}$ is an invertible sheaf.*

Proof. We have $h^!\mathscr{O}_S = \omega_{X/S}[n]$. On the other hand,

$$\begin{aligned} h^!\mathscr{O}_S &= f^!(g^!\mathscr{O}_S) \\ &= f^!(\omega_{Y/S}[n]) \\ &= f^!(\mathscr{O}_Y \otimes \omega_{Y/S}[n]) \\ &= f^!(\mathscr{O}_Y) \otimes f^\star\omega_{Y/S}[n]. \end{aligned}$$

□

4.2. Fundamental class. — Let X, Y be two embeddable S-schemes and let $f : X \to Y$ be an embeddable morphism. Under certain assumptions, we can construct a natural map

$$\Theta : f^\star\mathscr{O}_Y \to f^!\mathscr{O}_Y$$

which we call the "fundamental class". Our construction of the fundamental class is completely *ad hoc* and rather elementary. The interest of this fundamental class is that if f is projectively embeddable, applying $\mathrm{R}f_\star$ and taking the trace we get a map :

$$\mathrm{Tr} : \mathrm{R}f_\star f^\star\mathscr{O}_Y \to \mathscr{O}_Y.$$

4.2.1. Construction 1. — Assume that X and Y are local complete intersections over S of the same pure relative dimension. Assume that X is normal and that there is an open $V \subset X$ which is smooth over S, whose complement is of codimension 2 in X and an open $U \subset Y$ which is smooth and such that $f(V) \subset U$. In this case, it is enough to specify the fundamental class over V because, by normality, it will extend to X. Then over V, we define the fundamental class to be the determinant of the map $\mathrm{d}f : f^\star\Omega^1_{U/S} \to \Omega^1_{V/S}$.

4.2.2. Construction 2. — Here is another important example. Assume simply that $f : X \to Y$ is a finite flat map. In this situation, $f^!\mathscr{O}_Y = \underline{Hom}(f_\star\mathscr{O}_X, \mathscr{O}_Y)$. We have a trace morphism $tr_f : f_\star\mathscr{O}_X \to \mathscr{O}_Y$ and the fundamental class is defined by $\Theta(1) = tr_f$.

4.2.3. Comparison. — We check that the two constructions coincide in the situation where X, Y are smooth over S and the map $X \to Y$ is finite flat. In this situation, $X \to Y$ is lci [1] and it makes sense to compare our two constructions of the fundamental class.

Lemma 4.2.3.1. — *The cotangent complex $\mathbb{L}_{X/Y}$ is represented by the complex $[\Omega^1_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{O}_X \overset{df}{\to} \Omega^1_{X/S}]$. The determinant $\det(df) \in \omega_{X/Y} = f^!\mathscr{O}_Y$ is the trace map tr_f.*

Proof. We can first assume that S, Y and X are affine because the question is local on X. We have a closed embedding (in fact a regular immersion) $i : X \hookrightarrow X \times_S Y$ of X into the smooth Y-scheme $X \times_S Y$. We have an exact sequence :

$$0 \to \mathcal{I}_Y \to \mathscr{O}_{Y \times_S Y} \to \mathscr{O}_Y \to 0$$

which gives after tensoring with $\mathscr{O}_X$ above $\mathscr{O}_Y$

$$0 \to \mathcal{I}_X \to \mathscr{O}_{X \times_S Y} \to \mathscr{O}_X \to 0$$

where $\mathcal{I}_X$ is the ideal sheaf of the immersion i. It follows that $\mathcal{I}_X/\mathcal{I}_X^2 = \mathcal{I}_Y/\mathcal{I}_Y^2 \otimes_{\mathscr{O}_Y} \mathscr{O}_X = \Omega^1_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{O}_X$.

On the other hand, $i^\star\Omega^1_{X\times_S Y/Y} = \Omega^1_{X/S}$. The cotangent complex is represented by $[\mathcal{I}_X/\mathcal{I}_X^2 \to i^\star\Omega^1_{X\times_S Y/Y}]$ which is the same as $[\Omega^1_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{O}_X \to \Omega^1_{X/S}]$.

1. Observe that $X \hookrightarrow X \times_S Y$ is a regular immersion of X in the smooth Y-scheme $X \times_S Y$.

The morphism $f_\star \det \mathbb{L}_{X/Y} = \underline{\mathrm{Hom}}(f_\star \mathscr{O}_X, \mathscr{O}_Y) \to \mathscr{O}_Y$ is the residue map which associates to $\omega \in f_\star \Omega^1_{X/S}$ and to $(t_1, \cdots, t_n)$ local generators of the ideal $\mathcal{I}_X$ over Y the function $\mathrm{Res}[\omega, t_1, ..., t_n]$. It follows from [**32**], property (R6) on page 198 that the determinant of $[\Omega^1_{Y/S} \otimes_{\mathscr{O}_Y} \mathscr{O}_X \to \Omega^1_{X/S}]$ maps to the usual trace map.

□

4.2.4. Fundamental class and divisors. — Let $D_X \hookrightarrow X$ and $D_Y \hookrightarrow Y$ be two effective, reduced Cartier divisors relative to S. We assume that $f : X \to Y$ restricts to a map $f|_{D_X} : D_X \to D_Y$. We moreover assume that the induced map $D_X \to f^{-1}(D_Y)$ is an isomorphism of topological spaces. We assume that the fundamental class Θ of f is defined, so that we are either in the situation of construction 1 or construction 2.

Lemma 4.2.4.1. — *1. In the setting of construction 1, assume moreover that over the smooth locus X^{sm} of X, $D_X \cap X^{sm}$ is a normal crossing divisor and that over the smooth locus Y^{sm} of Y, $D_Y \cap Y^{sm}$ is a normal crossing divisor. Then the fundamental class $\Theta : \mathscr{O}_X \to f^! \mathscr{O}_Y$ restricts to a morphism : $\mathscr{O}_X(-D_X) \to f^! \mathscr{O}_Y(-D_Y)$.*

2. In the setting of construction 2, the fundamental class $\Theta : \mathscr{O}_X \to f^! \mathscr{O}_Y$ restricts to a morphism : $\mathscr{O}_X(-D_X) \to f^! \mathscr{O}_Y(-D_Y)$.

Proof. We first assume that X and Y are smooth, D_X and D_Y are relative normal crossing divisors. In that case, we have a well-defined differential map $\mathrm{df} : f^\star \Omega^1_{Y/S}(\log D_Y) \to \Omega^1_{X/S}(\log D_X)$. Taking the determinant yields $\det \mathrm{df} : f^\star \det \Omega^1_{Y/S}(D_Y) \to \det \Omega^1_{X/S}(D_X)$ or equivalently $\det \mathrm{df} : \mathscr{O}_X(-D_X) \to f^! \mathscr{O}_Y(-D_Y)$. We work in the setting of construction 1. Let V be an open subset of X. Let $s \in \mathscr{O}_X(-D_X)(V)$ be a section. We deduce that $\Theta(s) \in f^! \mathscr{O}_Y(V)$ actually belongs to $f^! \mathscr{O}_Y(-D_Y)(V \cap U)$ where U is a smooth open in X whose complement is of codimension 2. But then $f^! \mathscr{O}_Y(-D_Y)(V) = f^! \mathscr{O}_Y(-D_Y)(V \cap U)$ and the lemma is proven. We now work in the setting of construction 2. The lemma is then equivalent to the obvious assertion that the trace of a section which vanishes along D_X will vanish along D_Y (since D_Y is reduced). □

4.2.5. Base change. — Assume that we are in the situation of construction 1 or 2. Let $\Theta : f^\star \mathscr{O}_Y \to f^! \mathscr{O}_Y$ be the fundamental class. Consider a cartesian diagram :

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "i"] & Y \end{tikzcd}$$

Proposition 4.2.5.1. — *Assume that i is an open immersion or that f is a finite flat morphism. Then there is a natural isomorphism of sheaves $j^\star f^! \mathscr{O}_Y = (f')^! \mathscr{O}_{Y'}$ [2]. Moreover, if we denote by $\Theta : \mathscr{O}_X \to f^! \mathscr{O}_Y$ the fundamental class of f, then $j^\star \Theta : \mathscr{O}_{X'} \to (f')^! \mathscr{O}_{Y'}$ is the the fundamental class of f'.*

Proof. If i is an open immersion, the formula $j^\star f^! \mathscr{O}_Y = (f')^! \mathscr{O}_{Y'}$ follows from [**32**], thm. 8.7, 5. If f is finite flat, the formula $j^\star f^! \mathscr{O}_Y = (f')^! \mathscr{O}_{Y'}$ is obvious from the definitions: one reduces to the case that $Y = \mathrm{Spec}\ A$, $X = \mathrm{Spec}\ B$, $Y' = \mathrm{Spec}\ A'$, $X' = \mathrm{Spec}\ B'$. We may even assume that B is a free A-module after further localization on Y. Then the claim reduces to the following isomorphism : $A' \otimes_A \mathrm{Hom}_A(B, A) \xrightarrow{\sim} \mathrm{Hom}_{A'}(B', A')$. The compatibility of the fundamental class with base change is obvious from its definition (in

2. In this formula, $j^\star$ is not taken in the derived sense

construction 1 this follows from functorial properties of differentials, in construction 2 this follows from functorial properties of the trace morphism). □

4.3. Cohomological correspondences. — Let X, Y be two S-schemes. We adopt the following definition:

Definition 4.3.1. — *1. A correspondence C over X and Y is a diagram of S-morphisms :*

$$\begin{tikzcd} & C \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ X & & Y \end{tikzcd}$$

2. Let $\mathscr{F}$ be a coherent sheaf over X and $\mathscr{G}$ a coherent sheaf over Y. A cohomological correspondence from $\mathscr{F}$ to $\mathscr{G}$ is a map $T : \mathrm{R}(p_1)_\star p_2^\star \mathscr{F} \to \mathscr{G}$.

Associated with T, we have a map on cohomology which is still denoted by T :

$$\mathrm{R}\Gamma(X, \mathscr{F}) \xrightarrow{p_2^\star} \mathrm{R}\Gamma(C, p_2^\star\mathscr{F}) = \mathrm{R}\Gamma(Y, \mathrm{R}(p_1)_\star p_2^\star \mathscr{F}) \xrightarrow{T} \mathrm{R}\Gamma(Y, \mathscr{G}).$$

Remark 4.3.1. — In practice X, Y and C will have the same pure relative dimension over S and the morphisms p_1 and p_2 will be surjective and generically finite.

Remark 4.3.2. — If we assume that p_1 is projectively embeddable the map T can be seen, by adjunction, as a map $p_2^\star\mathscr{F} \to p_1^!\mathscr{G}$.

4.3.1. Construction of cohomological correspondences. — We now explain how we can construct cohomological correspondences in practice. Let C be a correspondence over X and Y as before, we assume that p_1 is projectively embeddable. Let $\mathscr{F}$ and $\mathscr{G}$ be locally free sheaves of finite rank over X and Y respectively. We assume that we are given a morphism $p_2^\star\mathscr{F} \to p_1^\star\mathscr{G}$. We also assume that we have a map $p_1^\star\mathscr{O}_Y \to p_1^!\mathscr{O}_Y$ (typically a fundamental class). Tensoring by $\mathscr{G}$ the map $p_1^\star\mathscr{O}_Y \to p_1^!\mathscr{O}_Y$ and using prop. 4.1.2.1, we obtain a morphism $p_1^\star\mathscr{G} \to p_1^!\mathscr{G}$ and composing we obtain a cohomological correspondence $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{G}$.

Remark 4.3.3. — In certain cases, one wants to renormalize this morphism. Let $\mathcal{O}$ be a discrete valuation ring with uniformizer ϖ. We assume that $S = \mathrm{Spec}\ \mathcal{O}$, that X, Y, C are flat over S. We further assume that the map $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{G}$ factors through $T : p_2^\star\mathscr{F} \to \varpi^k p_1^!\mathscr{G} \to p_1^!\mathscr{G}$ for some non-negative integer k. Then we can normalize the map T into a map $\varpi^{-k}T : p_2^\star\mathscr{F} \to p_1^!\mathscr{G}$. We will see many situations where this occurs in the sequel.

5. Automorphic forms, Galois representations and Shimura varieties

This section collects a number of classical results concerning automorphic forms, Galois representations and Shimura varieties for the group GSp_4.

5.1. The group GSp_4. — Let $V = \mathbb{Z}^4$ with canonical basis $(e_1, \cdots, e_4)$. Let $J = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$ where A is the anti-diagonal matrix with coefficients equal to 1 on the anti-diagonal. This is the matrix of a symplectic form $<,>$ on V. We let $\mathrm{GSp}_4 \to \mathrm{Spec}\ \mathbb{Z}$ be the group scheme $\mathrm{GSp}(V, <,>)$. The similitude character is denoted by $\nu : \mathrm{GSp}_4 \to \mathbb{G}_m$, and its kernel is the derived subgroup Sp_4 of GSp_4.

5.1.1. The dual group of GSp_4. — Let $\mathrm{T}^{\mathrm{der}} = \{\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}), t_1, t_2 \in \mathbb{G}_m\}$ be the diagonal (maximal) torus of Sp_4 and $\mathrm{Z} = \{\mathrm{diag}(s, s, s, s), s \in \mathbb{G}_m\}$ the center of GSp_4. Let T be the diagonal torus of GSp_4. We have a surjective map (of $fppf$ abelian sheaves) $\mathrm{T}^{\mathrm{der}} \times \mathrm{Z} \to \mathrm{T}$ with kernel the group μ_2. The character group $X^\star(\mathrm{T})$ is identified with

$$\{(a_1, a_2; c), c = a_1 + a_2 \mod 2\} \subset \mathbb{Z}^3,$$

where $(a_1, a_2; c).\mathrm{diag}(st_1, st_2, st_2^{-1}, st_1^{-1}) = s^c t_1^{a_1} t_2^{a_2}$. We pick the following basis of $X^\star(\mathrm{T})$:

$$e_1 = (1, 0; 1),\ e_2 = (0, 1; 1) \text{ and } e_3 = (0, 0; 2).$$

Note that e_3 is the similitude character ν.

We make the following choice of positive roots $\{e_1 - e_2, -2e_1 + e_3, -e_1 - e_2 + e_3, -2e_2 + e_3\}$. Set $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -2e_2 + e_3$. The simple positive roots are $\Delta = \{\alpha_1, \alpha_2\}$. The compact root is α_1. We let $\rho = (-1, -2; 0)$ be half the sum of the positive roots. This choice of positive roots is related to the Shimura datum (see remark 5.2.1.1).

The cocharacter group $X_\star(\mathrm{T})$ is the dual of $X^\star(\mathrm{T})$. We identify it with

$$\{(b_1, b_2; d) \in \frac{1}{2}\mathbb{Z}^3,\ b_1 + d \in \mathbb{Z},\ b_2 + d \in \mathbb{Z}\}$$

via $(b_1, b_2; d).t = \mathrm{diag}(t^{b_1+d}, t^{b_2+d}, t^{-b_2+d}, t^{-b_1+d})$. The following basis of $X_\star(\mathrm{T})$ is dual to e_1, e_2 and e_3 :

$$f_1 = (1, 0; 0),\ f_2 = (0, 1; 0),\ \text{and } f_3 = (-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}).$$

The coroot of α_1 is $\alpha_1^\vee = f_1 - f_2$ and the coroot of α_2 is $\alpha_2^\vee = f_2$. We let $\Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee\}$.

We let $(X^\star(\mathrm{T}), \Delta, X_\star(\mathrm{T}), \Delta^\vee)$ be the based root datum of GSp_4 corresponding to our choices of maximal torus T and positive roots.

By [**65**], lemma 2.3.1 there is an isomorphism of roots datum between

$$(X^\star(\mathrm{T}), \Delta, X_\star(\mathrm{T}), \Delta^\vee) \text{ and } (X_\star(\mathrm{T}), \Delta^\vee, X^\star(\mathrm{T}), \Delta).$$

It is given by a map $i : X^\star(\mathrm{T}) \to X_\star(\mathrm{T})$ whose matrix in the basis e_1, e_2, e_3 and f_1, f_2, f_3 is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

This isomorphism induces an identification of the dual group $\widehat{\mathrm{GSp}_4}$ with $\mathrm{GSp}_4(\mathbb{C})$.

5.1.2. Parabolic subgroups. — If $W \subset V$ is a totally isotropic direct factor, we let P_W be the parabolic subgroup of GSp_4 which stabilizes P_W. We denote by U_W its unipotent radical and by M_W its Levi quotient. The group M_W decomposes as the product $M_{W,l} \times M_{W,h}$ where $M_{W,l}$ is the linear group of automorphisms of W and $M_{W,h}$ is the group of symplectic similitudes of $W^\perp/W$ (with the convention that when $W = W^\perp$, this group is $\mathbb{G}_m$.)

When $W = \langle e_1 \rangle$, then P_W is denoted by P_{Kli} and called the Klingen parabolic. Its Levi quotient is $M_{Kli} \simeq M_{Kli,l} \times M_{Kli,h} \simeq \mathbb{G}_m \times \mathrm{GL}_2$. If $W = \langle e_1, e_2 \rangle$, then P_W is denoted by P_{Si} and called the Siegel parabolic. Its Levi quotient is $M_{Si} \simeq M_{Si,l} \times M_{Si,h} \simeq \mathrm{GL}_2 \times \mathbb{G}_m$.

Remark 5.1.2.1. — Let $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{p}_{Si}$ and $\mathfrak{m}_{Si}$ be the Lie algebras of $\mathrm{GSp}_4/_{\mathbb{C}}$, P_{Si} and M_{Si}. Our positive roots lie in $\mathfrak{m}_{Si}$ (the compact roots), and $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{Si}$ (the non-compact roots).

5.1.3. Spherical Hecke algebra. — Let ℓ be a prime number. The group $\mathrm{GSp}_4(\mathbb{Z}_\ell) \subset \mathrm{GSp}_4(\mathbb{Q}_\ell)$ is a maximal compact subgroup [(3)]. We let $\mathcal{H}_\ell$ be the spherical Hecke algebra

$$\mathcal{C}^0_c(\mathrm{GSp}_4(\mathbb{Q}_\ell)//\mathrm{GSp}_4(\mathbb{Z}_\ell), \mathbb{Z}).$$

This is a commutative algebra isomorphic to $\mathbb{Z}[T_{\ell,0}, T_{\ell,0}^{-1}, T_{\ell,1}, T_{\ell,2}]$, generated by the characteristic functions of the double cosets :

$$T_{\ell,2} = \mathrm{GSp}_4(\mathbb{Z}_\ell)\mathrm{diag}(\ell,\ell,1,1)\mathrm{GSp}_4(\mathbb{Z}_\ell), \quad T_{\ell,1} = \mathrm{GSp}_4(\mathbb{Z}_\ell)\mathrm{diag}(\ell^2,\ell,\ell,1)\mathrm{GSp}_4(\mathbb{Z}_\ell),$$
$$T_{\ell,0} = \ell\mathrm{GSp}_4(\mathbb{Z}_\ell).$$

The Hecke polynomial is by definition $Q_\ell(X) = 1 - T_{\ell,2}X + \ell(T_{\ell,1} + (\ell^2+1)T_{\ell,0})X^2 - \ell^3 T_{\ell,2}T_{\ell,0}X^3 + \ell^6 T_{\ell,0}^2 X^4$.

Consider the twisted Satake isomorphism $\mathcal{H}_\ell \otimes \mathbb{C} \to \mathbb{C}[X_\star(\mathrm{T})]^W$ where W is the Weyl group of GSp_4 acting naturally on $X_\star(\mathrm{T})$ (see [**25**], p. 193, see also remark 5.1.5.1). To any homomorphism $\Theta_\ell : \mathcal{H}_\ell \to \mathbb{C}$ we can associate (using the identification $\widehat{\mathrm{GSp}_4} \simeq \mathrm{GSp}_4(\mathbb{C})$ and the twisted Satake isomorphism) a semi-simple conjugacy class $c_{\Theta_\ell} \in \mathrm{GSp}_4(\mathbb{C})$. Moreover, $\Theta_\ell(Q_\ell(X)) = \det(1 - Xc_{\Theta_\ell})$ ([**25**], rem. 3 on page 196).

5.1.4. A parahoric Hecke algebras. — We denote by $Kli(\ell) \subset \mathrm{GSp}_4(\mathbb{Z}_\ell)$ the Klingen parahoric of elements which belong to $P_{Kli}(\mathbb{F}_\ell)$ modulo ℓ.

We denote by $\mathcal{H}^+_{Kli(\ell)}$ the subalgebra of $\mathcal{C}^0_c(\mathrm{GSp}_4(\mathbb{Q}_\ell)//Kli(\ell), \mathbb{Z})$ generated by the double cosets :

$$U_{Kli(\ell),2} = Kli(\ell)\mathrm{diag}(\ell,\ell,1,1)Kli(\ell), \quad U_{Kli(\ell),1} = Kli(\ell)\mathrm{diag}(\ell^2,\ell,\ell,1)Kli(\ell)$$
$$U_{Kli(\ell),0} = \ell Kli(\ell).$$

This is a polynomial algebra in these variables.

5.1.5. Some local representation theory. — We let ℓ be a prime and let π_ℓ be an irreducible complex smooth admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$. Assume that π_ℓ is spherical : $\pi_\ell^{\mathrm{GSp}_4(\mathbb{Z}_\ell)} \neq 0$ (and necessarily one-dimensional). We let $\theta_{\pi_\ell} : \mathcal{H}_\ell \to \mathbb{C}$ be the corresponding character. We denote by $(\alpha,\beta,\gamma,\delta)$ the roots of the reciprocal of $\Theta_{\pi_\ell}(Q_\ell(X))$, ordered in such a way that $\alpha\delta = \beta\gamma$, so that $\mathrm{diag}(\alpha,\beta,\gamma,\delta)$ represents the semi-simple conjugacy class $c_{\Theta_{\pi_\ell}}$. The Weyl group W acts on the quadruple $(\alpha,\beta,\gamma,\delta)$ and the Weyl group orbit exhausts all diagonal representatives of the conjugacy class $c_{\Theta_{\pi_\ell}}$. We call (the W-orbit of) $(\alpha,\beta,\gamma,\delta)$ the Hecke parameters of π_ℓ.

Remark 5.1.5.1. — The conjugacy class $c_{\Theta_{\pi_\ell}}$ is the one attached to $\pi_\ell \otimes |\nu|^{-\frac{3}{2}}$ by the usual Satake isomorphism (as opposed to the twisted Satake map that we use).

Lemma 5.1.5.1. — *The eigenvalues for $T_{\ell,0}$, $T_{\ell,1}$ and $T_{\ell,2}$ acting of $\pi_\ell^{\mathrm{GSp}_4(\mathbb{Z}_\ell)}$ are respectively :*

$$\ell^{-3}\alpha\delta,\ \ell^{-1}(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\delta + \gamma\delta) - \ell^{-3}\alpha\delta,\ \alpha+\beta+\gamma+\delta.$$

Proof. This is a straightforward computation. □

Let π_ℓ be an irreducible complex smooth admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$ which is spherical. We say that π_ℓ is generic if π_ℓ is an irreducible (unramified) principal series representation. Let $(\alpha,\beta,\gamma,\delta)$ be the Hecke parameters of π_ℓ. This is equivalent to ask that for all $\xi,\xi' \in \{\alpha,\beta,\gamma,\delta\}$, we have $\xi^{-1}\xi' \neq \ell$ ([**24**], prop. 3.2.3).

3. There are two conjugacy classes of maximal compact subgroups in $\mathrm{GSp}_4(\mathbb{Q}_\ell)$. The group $\mathrm{GSp}_4(\mathbb{Z}_\ell)$ is a representative of the standard class and is hyperspecial. The other class is represented by the paramodular group which is not hyperspecial.

Proposition 5.1.5.1. — *Let π_ℓ be a generic spherical representation with associated Hecke parameters $(\alpha,\beta,\gamma,\delta)$. Then $\pi_\ell^{Kli(\ell)}$ is 4 dimensional, and the eigenvalues of $U_{Kli(\ell),0}$, $U_{Kli(\ell),1}$ and $U_{Kli(\ell),2}$ acting on $\pi_\ell^{Kli(\ell)}$ are the Weyl orbit of $\ell^{-3}\alpha\delta$, $\ell^{-1}\alpha\beta$, $\alpha+\beta$.*

Proof. This is [**24**], coro. 3.2.2. □

Let π_ℓ be an irreducible complex smooth admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$ and assume that $\pi_\ell^{\mathrm{Kli}(\ell)} \neq 0$. Then there is a quadruple $(\alpha,\beta,\gamma,\delta)\in\mathbb{C}^4$ satisfying $\alpha\beta=\gamma\delta$ such that $\pi_\ell^{\mathrm{Kli}(\ell)} \neq 0$ contains an eigenvector for $U_{Kli(\ell),0}$, $U_{Kli(\ell),1}$ and $U_{Kli(\ell),2}$ with eigenvalues $\ell^{-3}\alpha\delta$, $\ell^{-1}\alpha\beta$, $\alpha+\beta$. The W-orbit of $(\alpha,\beta,\gamma,\delta)\in\mathbb{C}^4$ is still called the Hecke parameters of π_ℓ.

Proposition 5.1.5.2. — *Let π_ℓ be an irreducible complex smooth admissible representation of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$ and assume that $\pi_\ell^{\mathrm{Kli}(\ell)} \neq 0$ and has Hecke parameters $(\alpha,\beta,\gamma,\delta)$. Assume that for all $\xi,\xi'\in\{\alpha,\beta,\gamma,\delta\}$, we have $\xi^{-1}\xi'\neq\ell$. Then π_ℓ is a generic spherical representation with Hecke parameters $(\alpha,\beta,\gamma,\delta)$.*

Proof. This is again [**24**], prop. 3.2.3. □

Let π_ℓ be a generic spherical representation with Hecke parameters $(\alpha,\beta,\gamma,\delta)$. Let us denote by $\left(\pi_\ell^{Kli(\ell)}\right)_{\alpha\beta}$ the $\ell^{-1}\alpha\beta$ eigenspace in $\pi_\ell^{Kli(\ell)}$ for $U_{Kli(\ell),1}$. Let us denote by $\pi_{\alpha\beta}:\pi_\ell^{Kli(\ell)}\to\left(\pi_\ell^{Kli(\ell)}\right)_{\alpha\beta}$ the projection orthogonal to the other eigenspaces.

Lemma 5.1.5.2. — *Assume that the set $\{\alpha\beta,\beta\delta,\alpha\gamma,\gamma\delta\}$ has 4 distinct elements. The map $\pi_\ell^{\mathrm{GSp}_4(\mathbb{Z}_\ell)}\hookrightarrow\pi_\ell^{Kli(\ell)}\overset{\pi_{\alpha\beta}}{\to}\left(\pi_\ell^{Kli(\ell)}\right)_{\alpha\beta}$ is an isomorphism.*

Proof. It is enough to prove that the map is injective. This follows from corollary 3.2.2 of [**24**]. □

5.1.6. Discrete series. — Given $\lambda=(\lambda_1,\lambda_2;c)\in X^\star(\mathrm{T})+(-1,-2;0)\subset X^\star(\mathrm{T})_{\mathbb{C}}$ which satisfies $-\lambda_1\geq\lambda_2\geq\lambda_1$ and a Weyl chamber C positive for λ we have a (limit of) discrete series $\pi(\lambda,C)$ (see [**28**], 3.3).

Let $\mathfrak{Z}$ be the center of the enveloping algebra $U(\mathfrak{g})$. By Harish-Chandra isomorphism (recalled in [**18**], p. 229 for instance), $\mathfrak{Z}\simeq\mathbb{C}[X_\star(\mathrm{T})]^W$ where W is the Weyl group. The infinitesimal character of $\pi(\lambda,C)$ is the Weyl group orbit of λ.

If $\lambda_2\neq0$ and $\lambda_2\neq-\lambda_1$, λ determines uniquely C and $\pi(\lambda,C)$ is a discrete series. It is natural to normalize the central character c by $c=-\lambda_1-\lambda_2+3$.

If $0\geq\lambda_2>\lambda_1$ and C is the dominant chamber corresponding to our choice of positive roots, then $\pi(\lambda,C)$ is called a holomorphic (limit of) discrete series.

5.1.7. Galois representations attached to automorphic forms. — The following theorem is obtained in [**77**], [**47**], [**84**] and [**80**]. A different proof (for the general type, see below) is given in [**73**], completed by [**54**] using a lift to GL_4 and [**13**].

Theorem 5.1.7.1. — *Let $\pi=\pi_\infty\otimes\pi_f$ be a cuspidal automorphic form for the group GSp_4 such that $\pi_\infty=\pi(\lambda,C)$ is in the discrete series and $\lambda=(\lambda_1,\lambda_2;-\lambda_1-\lambda_2+3)$. Let N be the product of primes ℓ such that π_ℓ is not spherical. We let $\mathcal{H}^N=\otimes'_{\ell\nmid N}\mathcal{H}_\ell$ be the restricted tensor product of the spherical Hecke algebras $\mathcal{H}_\ell$ for all prime numbers $\ell\nmid N$. Let $\Theta_\pi:\mathcal{H}^N\to\mathbb{C}$ be the homomorphism giving the action of $\mathcal{H}^N$ on $\otimes_{\ell\nmid N}\pi_\ell^{\mathrm{GSp}_4(\mathbb{Z}_\ell)}$.*

1. *The image of Θ_π generates a number field E.*

2. *For all finite places λ of E, there is a semi-simple, continuous Galois representation:*
$$\rho_{\pi,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{E_\lambda}),$$
which is unramified away from N and the prime p below λ and such that for all $\ell \nmid Np$, we have
$$\det(1 - X\rho_{\pi,\lambda}(\mathrm{Frob}_\ell)) = \Theta_\pi(Q_\ell(X)).$$
3. *The representation $\rho_{\pi,\lambda}$ is de Rham at p with Hodge-Tate weights* $(0, -\lambda_2, -\lambda_1, -\lambda_1 - \lambda_2)$.
4. *If $p \nmid N$, then $\rho_{\pi,\lambda}$ is crystalline at p and* $\det(1 - X\phi|\mathrm{D}_{crys}(\rho_{\pi,\lambda})) = \Theta_\pi(Q_p(X))$.
5. $\rho_{\pi,\lambda} \simeq \rho^\vee_{\pi,\lambda} \otimes \chi_p^{-\lambda_1-\lambda_2}\omega_{\pi,\lambda}$ *for some finite character $\omega_{\pi,\lambda}$ and the cyclotomic character χ_p.*

Remark 5.1.7.1. — We use geometric Frobenii, the Artin reciprocity law is normalized by sending p to the geometric Frobenius at p, and the Hodge-Tate weight of the cyclotomic character is -1. The Galois representation $\rho_{\pi,\lambda}$ is the Galois representation attached to the L-algebraic automorphic form $\pi \otimes |\nu|^{-\frac{3}{2}}$ as predicted in conjecture 3.2.2 of [**10**]. The twist by $|\nu|^{-\frac{3}{2}}$ corresponds to the twisted Satake isomorphism that we use. The Hodge-Tate weights are given by the infinitesimal character of $\pi_\infty \otimes |\nu|^{-\frac{3}{2}}$ which is $(\lambda_1, \lambda_2; -\lambda_1 - \lambda_2)$. The Hodge cocharacter is given by $t \mapsto \mathrm{diag}(1, t^{-\lambda_1}, t^{-\lambda_2}, t^{-\lambda_1-\lambda_2})$. The central character of $\pi \otimes |\nu|^{-\frac{3}{2}}$ is $|.|^{-\lambda_1-\lambda_2} \otimes \omega_\pi$ for some finite character ω_π. The character $\omega_{\pi,\lambda}$ is the λ-adic character of the Galois group $G_{\mathbb{Q}}$ associated to ω_π by class field theory.

According to Arthur's classification [**1**], the representation π in the theorem can fall into six categories. If π is not of general type [(4)] then $\rho_{\pi,\lambda}$ is reducible. Indeed, it follows from an examination of Arthur's classification that the representation $\rho_{\pi,\lambda}$ can be either the sum of Galois representations attached to algebraic automorphic forms on GL_1 (case e) and f)), the sum of Galois representation attached to algebraic automorphic forms on GL_1 and regular algebraic automorphic forms on GL_2 (case c) and d)), the sum of Galois representations attached to regular algebraic automorphic forms on GL_2 (case b)). On the contrary, if π is of general type then it is expected that $\rho_{\pi,\lambda}$ is irreducible.

5.2. Complex Siegel threefolds. —

5.2.1. Siegel datum. — We let $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{GSp}_4/_{\mathbb{R}}$ be the map given by $h(a+ib) = a1_2 + bJ$. We let $K_\infty \subset \mathrm{GSp}_4(\mathbb{R})$ be the centralizer of the image of h. The quotient $\mathcal{H} = \mathrm{GSp}_4(\mathbb{R})/K_\infty$ is the Siegel space.

Remark 5.2.1.1. — We have a Hodge structure on $\mathfrak{g}_{\mathbb{R}}$ induced by $\mathrm{ad}(h)$. We let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{(0,0)} \oplus \mathfrak{g}^{(-1,1)} \oplus \mathfrak{g}^{(1,-1)}$ be the corresponding Hodge decomposition and we let $\mathfrak{p}_h = \mathfrak{g}^{(0,0)} \oplus \mathfrak{g}^{(1,-1)}$. The parabolic $\mathfrak{p}_h$ is conjugated to $\mathfrak{p}_{Si}$ by some element $g \in \mathfrak{g}_{\mathbb{C}}$, and our positive roots are in $\mathrm{ad}(g).(\mathfrak{g}^{(0,0)} \oplus \mathfrak{g}^{(-1,1)})$.

Let $K \subset \mathrm{GSp}_4(\mathbb{A}_f)$ be a neat compact open subgroup. We let $S_K = \mathrm{GSp}_4(\mathbb{Q})\backslash\mathcal{H} \times \mathrm{GSp}_4(\mathbb{A}_f)/K$. This is the complex analytic Siegel threefold of level K. It can be interpreted as a moduli space of abelian surfaces with additional structures. See [**47**], sect. 3 for example.

4. π is of general type if it has a base change to a cuspidal automorphic representation on GL_4.

5.2.2. Minimal compactification. — Let $S_K^\star$ be the minimal compactification of S_K (see [**64**], sect. 3 for example). There is a stratification of $S_K^\star$:

$$S_K \coprod S_K^{(1)} \coprod S_K^{(0)}.$$

Let $\mathcal{H}^{(1)} = \mathbb{C} \setminus \mathbb{R}$ and $\mathcal{H}^{(0)} = \{1, -1\}$.

$$S_K^{(1)} = P_{Kli}(\mathbb{Q}) \backslash \mathcal{H}^{(1)} \times G(\mathbb{A}_f)/K$$

is a union of modular curves and

$$S_K^{(0)} = P_{Si}(\mathbb{Q}) \backslash \mathcal{H}^{(0)} \times G(\mathbb{A}_f)/K$$

is the union of cusps of these modular curves. The parabolics $P_{Kli}(\mathbb{Q})$ and $P_{Si}(\mathbb{Q})$ act diagonally. They act on $\mathcal{H}^1$ and $\mathcal{H}^0$ through their quotients $M_{Kli,h}(\mathbb{Q})$ and $M_{Si,h}(\mathbb{Q})$. We let $S_K^{(1),\star} = S_K^{(1)} \coprod S_K^{(0)}$. This is a union of compactified modular curves.

5.2.3. Toroidal compactification. — Depending on a certain auxiliary choice of polyhedral cone decomposition Σ, one can also construct toroidal compactifications $S_{K,\Sigma}^{tor}$ of S_K. There is a semi-abelian surface $G \to S_{K,\Sigma}^{tor}$. See [**29**], sect. 2.

5.3. Coherent cohomology and Galois representations. — Over $S_{K,\Sigma}^{tor}$, we have a semi-abelian surface G. We let $\omega_G \to S_{K,\Sigma}^{tor}$ be the conormal sheaf of G at the unit section. This is a locally free sheaf of rank 2. For all pairs of integers $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$, we define an automorphic vector bundle $\Omega^{(k,r)} = \mathrm{Sym}^k \omega_G \otimes \det^r \omega_G$ on $S_{K,\Sigma}^{tor}$. We let $D_{K,\Sigma} = S_{K,\Sigma}^{tor} \setminus S_{K,\Sigma}$. This is a Cartier divisor. We can consider the cuspidal subsheaf $\Omega^{(k,r)}(-D_{K,\Sigma})$ (or simply $\Omega^{(k,r)}(-D)$ if no confusion will arise) of $\Omega^{(k,r)}$.

We will be interested in the coherent cohomology groups $\mathrm{H}^i(S_{K,\Sigma}^{tor}, \Omega^{(k,r)}(-D))$. These cohomology groups are independent of the choice of Σ ([**30**], prop. 2.4). Our main focus will be on the case $r = 2$, $i \in \{0, 1\}$.

If $\pi = \pi_\infty \otimes \pi_f$ and $\pi_\infty = \pi(\lambda, C)$ is a holomorphic (limit of) discrete series with $\lambda = (\lambda_1, \lambda_2; -\lambda_1 - \lambda_2 + 3)$ (and hence $0 \geq \lambda_2 > \lambda_1$), then there is a natural embedding $\pi_f^K \hookrightarrow \mathrm{H}^0(S_{K,\Sigma}^{tor}, \Omega^{(\lambda_2 - \lambda_1 - 1, 2 - \lambda_2)}(-D))$.

It follows from the description of representations having a "lowest weight" given in [**67**], p. 12 diagram (44) that for all $r \geq 2$:

$$\mathrm{H}^0(S_{K,\Sigma}^{tor}, \Omega^{(k,r)}(-D)) = \oplus_{\pi_f} \pi_f^K,$$

where π_f runs through the set of admissible representations of $\mathrm{GSp}_4(\mathbb{A}_f)$ such that $\pi(\lambda, C) \otimes \pi_f$ is cuspidal automorphic for $\lambda = (1 - k + r, 2 - r; k + 2r)$ and $\pi(\lambda, C)$ the holomorphic (limit) of discrete series.

We let N be the product of primes ℓ such that $K_\ell \neq \mathrm{GSp}_4(\mathbb{Z}_\ell)$. We let $\mathcal{H}^N = \otimes'_{\ell \nmid N} \mathcal{H}^\ell$ be the restricted tensor product of all the spherical Hecke algebras.

The Hecke algebra $\mathcal{H}^N$ acts on $\mathrm{H}^i(S_{K,\Sigma}^{tor}, \Omega^{(k,r)})$ and $\mathrm{H}^i(S_{K,\Sigma}^{tor}, \Omega^{(k,r)}(-D))$ (see [**30**], prop. 2.6). Let us briefly recall how the action is defined (and which normalization factors are involved). For certain choices of polyhedral cone decompositions Σ, Σ' and Σ'', we can define Hecke correspondences attached to the double coset $T_{\ell,i}$ (see [**18**], p. 253) :

$$\begin{tikzcd} & S^{tor}_{K(\ell,i),\Sigma''} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ S^{tor}_{K,\Sigma'} & & S^{tor}_{K,\Sigma} \end{tikzcd}$$

where $K(\ell,i) = K \cap d_{\ell,i}^{-1} K d_{\ell,i}$ (for $d_{\ell,i}$ is the diagonal matrix whose associated double coset gives $T_{\ell,i}$). Attached to this data is an isogeny $p_1^\star G \to p_2^\star G$, whose differential provides a natural map $p_2^\star \Omega^{(k,r)} \to p_1^\star \Omega^{(k,r)}$. On the other hand, the map p_1 carries a fundamental class $p_1^\star \mathscr{O}_{S^{tor}_{K,\Sigma}} \to p_1^! \mathscr{O}_{S^{tor}_{K,\Sigma}}$ (see section 4.2.1). Taking the tensor product we get a cohomological correspondence $T'_{\ell,i} : p_2^\star \Omega^{(k,r)} \to p_1^! \mathscr{O}_{S^{tor}_{K,\Sigma}}$. We now set $T_{\ell,2} = \ell^{-3} T'_{\ell,2}$ and $T_{\ell,i} = \ell^{-6} T'_{\ell,i}$ for $i \in \{0,1\}$ and denote in the same way the operators on cohomology.

Remark 5.3.1. — The explanation for the powers of ℓ in the formula defining the Hecke operators is the following. The sheaf $\Omega^{(k,r)}$ is attached to the representation of K_∞ of highest weight $(k+r, r; -k-2r)$ by the vector bundle dictionary and therefore the automorphic forms contributing to the cohomology have infinitesimal character $(1-k-r, 2-r; k+2r)$. We introduce a twist to fix the infinitesimal character to be $(1-k-r, 2-r; k+2r-3)$ because when r is greater than 2, this twist optimizes the integral properties of the operator $T_{\ell,2}$ (it makes it integral on q-expansions) and normalizes the greater Hodge-Tate weight to be 0. Consult also [**18**], p. 258 (the paragraph starting by "what is happening ?") for further explanations.

Let $\Theta : \mathcal{H}^N \to \mathbb{C}$ be a system of eigenvalues for the action of $\mathcal{H}^N$ on the coherent cohomology $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \Omega^{(k,r)})$ and $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \Omega^{(k,r)}(-D))$. The following theorem is deduced from theorem 5.1.7.1 in [**76**] and [**63**], using p-adic interpolation :

Theorem 5.3.1. — *The image of Θ generates a number field E. For all finite place λ of E there is a semi-simple, continuous Galois representation :*

$$\rho_{\Theta,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{E_\lambda}),$$

which is unramified away from N and the prime p below λ and such that for all $\ell \nmid Np$, we have

$$\det(1 - X\rho_{\Theta,\lambda}(\mathrm{Frob}_\ell)) = \Theta(\mathrm{Q}_\ell(X)).$$

Proof. If $k \geq 0$ and $r \geq 3$, then

$$\mathrm{H}^0(S^{tor}_{K,\Sigma}, \Omega^{(k,r)}(-D)) = \oplus \pi_f^K$$

where π_f runs through the set of admissible representations of $\mathrm{GSp}_4(\mathbb{A}_f)$ such that $\pi(\lambda, C) \otimes \pi_f$ is cuspidal automorphic with $\lambda = (1-k-r, 2-r; k+2r)$. Thus, when $k \geq 0, r \geq 3$, we can use theorem 5.1.7.1. The general case follows from the main result of [**63**] (but see already [**76**] for degree 0 cuspidal cohomology) by p-adic interpolation techniques. □

Remark 5.3.2. — One believes that the representations constructed in the theorem are de Rham with Hodge-Tate weights $(0, r-2, r+k-1, k+2r-3)$ and crystalline at p if $(N,p) = 1$. Such a statement seems accessible if the weight is cohomological [(5)] although we do not know a reference. In particular, if $(N,p) = 1$ one believes that if $(\alpha, \beta, \gamma, \delta)$ are

5. The cohomological condition is that $r \neq 2$, $k+r \neq 1$ and $k+2r \neq 3$, so that the Hodge-Tate weights are all distinct. It ensures that the coherent cohomology group appears in the Hodge decomposition of the cohomology of an automorphic local system over the Shimura variety.

the Hecke parameters attached to a local representation π_p contributing to the cohomology $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \Omega^{(k,r)})$ or $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \Omega^{(k,r)}(-D))$, the Newton polygon associated to $(\alpha, \beta, \gamma, \delta)$ is above the Hodge polygon with the same initial and ending point. This is last statement is a consequence of the main theorem of [**45**] if the weight is cohomological.

PART II
HIGHER HIDA THEORY

6. Siegel threefolds over $\mathbb{Z}_p$

6.1. Schemes. — We fix a prime p. We introduce several Siegel threefolds defined over Spec $\mathbb{Z}_p$ and study their p-adic geometry.

6.1.1. The smooth Siegel threefold. — Let $K \subset \mathrm{GSp}_4(\mathbb{A}_f)$ be a neat compact open subgroup. We assume that $K = K^p K_p$ and that $K_p = \mathrm{GSp}_4(\mathbb{Z}_p)$. We let $Y_{K,\mathbb{Z}_{(p)}} \to \mathrm{Spec}\ \mathbb{Z}_{(p)}$ be the moduli space representing the functor which associates to each locally noetherian scheme S over Spec $\mathbb{Z}_{(p)}$ the set of isomorphism classes of triples (G, λ, ψ) where :

1. G is an abelian surface,
2. $\lambda : G \to G^t$ is a $\mathbb{Z}_{(p)}^\times$-multiple of a polarization of degree prime-to-p where G^t stands for the dual abelian scheme of G,
3. ψ is a K^p-level structure : if S is connected and s is a geometric point of S, ψ is a K^p-orbit of symplectic similitudes $\mathrm{H}_1(G_s, \mathbb{A}_f^p) \simeq V \otimes_{\mathbb{Z}} \mathbb{A}_f^p$ that is invariant under the action of $\pi_1(S, s)$ [6] (V is defined in section 5.1).

The triples (G, λ, ψ) are taken up to prime-to-p quasi-isogenies. See [**44**]. There is an isomorphism $(Y_{K,\mathbb{Z}_{(p)}} \times \mathrm{Spec}\ \mathbb{C})^{an} \simeq S_K$. We shall denote by $Y_K = Y_{K,\mathbb{Z}_{(p)}} \times_{\mathrm{Spec}\ \mathbb{Z}_{(p)}} \mathrm{Spec}\ \mathbb{Z}_p$.

6.1.2. Klingen level. — We denote by $p_1 : Y_{Kli}(p)_K \to Y_K$ the moduli space which parametrizes subgroups of order p, $H \subset G[p]$. Over $Y_{Kli}(p)_K$ we have a chain of isogenies of abelian surfaces $G \to G/H \to G/H^\perp \to G$. Here $H^\perp$ is the orthogonal of H for the Weil pairing on $G[p]$ (obtained by the polarization). The total map $G \to G$ is multiplication by p.

6.1.3. Paramodular level. — We also introduce $Y_{\mathrm{par},K} \to \mathrm{Spec}\ \mathbb{Z}_p$, the moduli space of isomorphism classes of triples (G', λ', ψ) where $\lambda' : G' \to (G')^t$ is a $\mathbb{Z}_{(p)}^\times$-multiple of a polarization of degree p^2 and ψ is a K^p-level structure. We have a natural map $p_2 : Y_{Kli}(p)_K \to Y_{\mathrm{par},K}$ which sends (G, λ, H, ψ) to $(G/H^\perp, \lambda', \psi')$ where λ' is the polarization on $G/H^\perp$ obtained by descending the polarization $p^2\lambda$ from G to $G/H^\perp$ and ψ' is induced by the isomorphism $G[N] = G/H^\perp[N]$ for every integer N prime-to-p.

6.1.4. Local properties. — We now investigate the local geometry of these schemes.

Proposition 6.1.4.1. — *The scheme Y_K is smooth over* Spec $\mathbb{Z}_p$. *The schemes $Y_{\mathrm{par},K}$ and $Y_{Kli}(p)_K$ are regular schemes. They are flat, local complete intersections over* Spec $\mathbb{Z}_p$. *The non smooth locus of $Y_{\mathrm{par},K}$ consists of a finite set of characteristic p points.*

6. This definition does not depend on s. When S is not connected, one chooses geometric points on each connected component.

Proof. The smoothness of Y_K over $\mathbb{Z}_p$ results from the deformation theory of abelian varieties with a polarization of degree prime-to-p. For $Y_{Kli}(p)_K$, the local model theory computation is worked out in [**79**], sect. 2.2, thm. 3. For $Y_{\mathrm{par},K}$ we can again use local model theory (see [**17**]). Let $V_1 = pe_1\mathbb{Z} \oplus \bigoplus_{i=2}^4 e_i\mathbb{Z} \subset V$ (V is defined in section 5.1). The local model for $Y_{\mathrm{par},K}$ is the moduli space of totally isotropic direct factors $L \subset V_1$ of rank 2. The only singularity occurs at $L_0 = \langle pe_1, e_4\rangle \subset V_1 \otimes \mathbb{F}_p$. The formal deformation ring at this point has equation $\mathbb{Z}_p[[X,Y,W,Z]]/(XY - WZ + p)$ and the universal deformation of L_0 is the module $\langle pe_1 + Xe_2 + We_3, Ze_2 + Ye_3 + e_4\rangle$ (see also [**86**], theorem 4.4). □

6.1.5. Integral arithmetic compactifications. — We recall results of Faltings-Chai [**18**], Lan [**48**], [**49**], [**50**] and Stroh [**74**].

6.1.5.1. Arithmetic groups. — Let $\Gamma = \mathrm{GSp}_4(\mathbb{Z}_{(p)})^+$ be the group of automorphisms of $(V \otimes \mathbb{Z}_{(p)}, <\,.\,>)$ up to a positive similitude factor. Let $V_1 = pe_1\mathbb{Z} \oplus \bigoplus_{i=2}^4 e_i\mathbb{Z} \subset V$. We let $\mathrm{GSp}_4' \to \mathrm{Spec}\,\mathbb{Z}$ be the group scheme $\mathrm{GSp}(V_1, <\,.\,>)$. This is the paramodular group. Let $\Gamma_{\mathrm{par}} = \mathrm{GSp}_4'(\mathbb{Z}_{(p)})^+$ be the subgroup of $\mathrm{GSp}_4'(\mathbb{Z}_{(p)})$ of elements with positive similitude factor. Let $\Gamma_{Kli}(p)$ be the automorphisms group of $(V_1 \otimes \mathbb{Z}_{(p)} \to V \otimes \mathbb{Z}_{(p)}, <\,.\,>)$ up to a positive similitude factor. Thus, $\Gamma_{Kli}(p)$ is a subgroup of both Γ and Γ_{par}. All are subgroups of $\mathrm{GSp}_4(\mathbb{Q})$.

6.1.5.2. Local charts. — Let $\mathfrak{C}$ be the set of totally isotropic direct factors $W \subset V$. For all $W \in \mathfrak{C}$, let $C(V/W^\perp)$ be the cone of positive symmetric bilinear forms $V/W^\perp \times V/W^\perp \to \mathbb{R}$ with radical defined over $\mathbb{Q}$. Let $\mathcal{C}$ be the conical complex which is the quotient of $\coprod_{W\in\mathfrak{C}} C(V/W^\perp)$ by the equivalence relation induced by the inclusions $C(V/W^\perp) \subset C(V/Z^\perp)$ for $W \subset Z$. This set carries an action of $\mathrm{GSp}_4(\mathbb{Q})$.

Let $W \in \mathfrak{C}$. Recall from section 5.1.2 that P_W is the parabolic subgroup which is the stabilizer of W, that $M_W = M_{W,l} \times M_{W,h}$ is its Levi quotient. There is a projection $P_W \to M_W$ and we let $P_{W,h}$ be the inverse image of $M_{W,h} \in P_W$. Let $\gamma \in \mathrm{GSp}_4(\mathbb{A}_f^p)/K^p$. We can attach to W and γ moduli spaces of 1-motives (see [**74**], sect. 1 and [**48**], sect. 6.2) which only depend on the class of γ in $\mathrm{GSp}_4(\mathbb{A}_f^p)/K^p$:

$$\begin{array}{ccccc}
\mathcal{M}_{W,\gamma} & & \mathcal{M}_{W,\gamma,Kli(p)} & & \mathcal{M}_{W,\gamma,\mathrm{par}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{B}_{W,\gamma} & & \mathcal{B}_{W,\gamma,Kli(p)} & & \mathcal{B}_{W,\gamma,\mathrm{par}} \\
\downarrow & & \downarrow & & \downarrow \\
Y_{W,\gamma} & & Y_{W,\gamma,Kli(p)} & & Y_{W,\gamma,\mathrm{par}}
\end{array}$$

The scheme $\mathcal{M}_{W,\gamma}$ is a moduli space of polarized 1-motives (for a polarization of degree prime-to-p), rigidified by $V/W^\perp$ ([**74**], def. 1.4.3) with a K_p-level structure.

The scheme $\mathcal{M}_{W,\gamma}$ admits the following description : it is a torsor under a torus $T_{W,\gamma}$ isogenous to $\mathrm{Sym}^2(V/W^\perp) \otimes \mathbb{G}_m$ over $\mathcal{B}_{W,\gamma}$. The scheme $\mathcal{B}_{W,\gamma}$ is an abelian scheme over $Y_{W,\gamma}$ which is a moduli space of abelian schemes of dimension $\mathrm{rank}_{\mathbb{Z}} W$ with a polarization of degree prime-to-p and a level structure away from p.

The scheme $\mathcal{M}_{W,\gamma,Kli(p)}$ is a moduli space of polarized 1-motives (for a polarization of degree prime-to-p), rigidified by $V/W^\perp$ with a K_p-level structure and a Klingen level structure.

The scheme $\mathcal{M}_{W,\gamma,Kli(p)}$ admits the following description : it is a torsor under a torus $T_{W,\gamma,Kli(p)}$ isogenous to $\mathrm{Sym}^2(V/W^\perp) \otimes \mathbb{G}_m$ over $\mathcal{B}_{W,\gamma,Kli(p)}$. The scheme $\mathcal{B}_{W,\gamma,Kli(p)}$ is an abelian scheme over $Y_{W,\gamma,Kli(p)}$ which is a moduli space of abelian schemes of genus $\mathrm{rank}_{\mathbb{Z}}W$ with a polarization of degree prime-to-p a level structure away from p and possibly a Klingen level structure at p [7].

The scheme $\mathcal{M}_{W,\gamma,\mathrm{par}}$ is a moduli space of 1-motives with a polarization of degree Np^2 (with $(N,p)=1$, the integer N depends on the tame level K^p). The character group of the toric part is isomorphic to $V_1/W^\perp$. It carries a K_p-level structure.

The scheme $\mathcal{M}_{W,\gamma,\mathrm{par}}$ admits the following description : it is a torsor under a torus $T_{W,\gamma,\mathrm{par}}$ isogenous to $\mathrm{Sym}^2(V/W^\perp)\otimes\mathbb{G}_m$ over $\mathcal{B}_{W,\gamma,\mathrm{par}}$. The scheme $\mathcal{B}_{W,\gamma,\mathrm{par}}$ is an abelian scheme over $Y_{W,\gamma}$ which is a moduli space of either abelian schemes of genus $\mathrm{rank}_{\mathbb{Z}}W$ with a polarization of degree prime-to-p, a level structure away from p or a moduli space of abelian schemes of genus $\mathrm{rank}_{\mathbb{Z}}W$ with a polarization of degree a prime-to-p multiple of p^2 and a level structure away from p.

Let $\sigma\subset C(V/W^\perp)$ be a cone. Associated to this cone we have affine toroidal embedding $T_{W,\gamma}\to T_{W,\gamma,\sigma}$, $T_{W,\gamma,Kli(p)}\to T_{W,\gamma,Kli(p),\sigma}$ and $T_{W,\gamma,\mathrm{par}}\to T_{W,\gamma,\mathrm{par},\sigma}$. We can define $\overline{\mathcal{M}}_{W,\gamma,\sigma}=\mathcal{M}_{W,\gamma}\times^{T_{W,\gamma}}T_{W,\gamma,\sigma}$, $\overline{\mathcal{M}}_{W,\gamma,Kli(p),\sigma}=\mathcal{M}_{W,\gamma,Kli(p)}\times^{T_{W,\gamma,Kli(p)}}T_{W,\gamma,Kli(p),\sigma}$, $\overline{\mathcal{M}}_{W,\gamma,\mathrm{par},\sigma}=\mathcal{M}_{W,\gamma,\mathrm{par}}\times^{T_{W,\gamma,\mathrm{par}}}T_{W,\gamma,\mathrm{par},\sigma}$, and we denote by $Z_{W,\gamma,\sigma}$, $Z_{W,\gamma,Kli(p),\sigma}$ and $Z_{W,\gamma,\mathrm{par},\sigma}$ the closed subschemes that correspond to the closed strata of these respective affine toroidal embeddings.

6.1.5.3. Polyhedral decompositions. — We consider the set $\mathcal{C}\times\mathrm{GSp}_4(\mathbb{A}_f^p)/K^p$. This set carries a diagonal action of $\mathrm{GSp}_4(\mathbb{Q})$ and a left action of $\mathrm{GSp}_4(\mathbb{A}_f^p)$ (by translation on the second factor).

A non-degenerate rational polyhedral cone of $\mathcal{C}\times\mathrm{GSp}_4(\mathbb{A}_f^p)/K^p$ is a subset contained in $\mathcal{C}(V/W^\perp)\times\{\gamma\}$ for some (W,γ) which is of the form $\oplus_{i=1}^k\mathbb{R}_{>0}s_i$ for symmetric pairings $s_i : V/W^\perp\times V/W^\perp\to\mathbb{Q}$.

Let us fix a $\mathbb{Z}$-lattice $L_W\subset\mathrm{Sym}^2(V/W^\perp)\otimes_{\mathbb{Z}}\mathbb{Q}$. Then the cone is called smooth with respect to L_W if the s_i's can be taken to be part of a $\mathbb{Z}$-basis of $\mathrm{Hom}(L_W,\mathbb{Z})$.

A rational polyhedral cone decomposition Σ of $\mathcal{C}\times\mathrm{GSp}_4(\mathbb{A}_f^p)/K^p$ is a partition $\mathcal{C}\times\mathrm{GSp}_4(\mathbb{A}_f^p)/K^p=\coprod_{\sigma\in\Sigma}\sigma$ by non-degenerate rational polyhedral cones σ such that :

1. the closure of each cone is a union of cones,
2. for any $\sigma\in\Sigma$, $\sigma\subset\mathcal{C}(V/W^\perp)\times\{\gamma\}$, we have that $p\sigma\in\Sigma$ for all $p\in P_{W,h}(\mathbb{A}_f^p)$.

For any subgroup $H\subset\mathrm{GSp}_4(\mathbb{Q})$ a rational polyhedral cone decomposition Σ is H-equivariant if for all $h\in H$ and $\sigma\in\Sigma$, $h.\sigma\in\Sigma$. It is H-admissible if $H\backslash\Sigma$ is finite. It is projective if there exists a polarization function (see [**50**], def. 2.4).

For all $(W,\gamma)\in\mathfrak{C}\times\mathrm{GSp}_4(\mathbb{A}_f^p)/K^p$ we have integral structures $X_\star(T_{W,\gamma})$, $X_\star(T_{W,\gamma,\mathrm{par}})$ and $X_\star(T_{W,\gamma,Kli(p)})\subset\mathrm{Sym}^2(V/W^\perp)\otimes_{\mathbb{Z}}\mathbb{Q}$. We say that a rational polyhedral cone decomposition Σ is smooth with respect to one of these integral structures if each cone $\sigma\in\Sigma$ is smooth.

Let H be either Γ, Γ_{par} or $\Gamma_{Kli}(p)$. The H-admissible rational polyhedral cone decompositions exist and are naturally ordered by inclusion ([**18**], p. 97). Any two H-admissible rational polyhedral cone decompositions can be refined by a third one.

The H-admissible rational polyhedral cone decompositions which satisfy the following extra properties form a cofinal subset of the set of all H-admissible rational polyhedral cone decompositions (see [**18**], p. 97) :

1. The decomposition is projective.

7. This depends on the relative position of W with respect to $V_1\subset V$.

2. For all cone σ, let $W \in \mathfrak{C}$ be minimal such that $\sigma \subset \mathcal{C}(V/W^\perp)$. If $h \in H \cap P_W$ satisfies $hp\sigma \cap \sigma \neq \emptyset$ for some $p \in P_{W,h}(\mathbb{A}_f^p)$, then h acts trivially on $\mathcal{C}(V/W^\perp)$.
3. If H is Γ (resp. Γ_{par}, resp. $\Gamma_{Kli}(p)$)-admissible, the decomposition is smooth with respect to the integral structure given by $X_\star(T_{W,\gamma})$, (resp. $X_\star(T_{W,\gamma,\mathrm{par}})$, resp. $X_\star(T_{W,\gamma,Kli(p)})$).

In the sequel of the paper we will consider mostly H-admissible rational polyhedral cone decompositions which satisfy these extra properties unless explicitly stated. We will call them H-admissible good polyhedral cone decompositions or simply good polyhedral cone decompositions.

6.1.5.4. Main theorem on compactification. — The following theorem is a special case of [**50**], thm. 6.1.

Theorem 6.1.5.1. — *1. Let Σ be a good polyhedral cone decomposition which is Γ (resp. $\Gamma_{Kli}(p)$, resp. Γ_{par})-admissible. There is a toroidal compactification $X_{K,\Sigma}$ of Y_K (resp. $X_{Kli}(p)_{K,\Sigma}$ of $Y_{Kli}(p)_K$, resp. $X_{\mathrm{par},K,\Sigma}$ of $Y_{\mathrm{par},K}$). It has a stratification indexed by $\Gamma\backslash\Sigma$ (resp. $\Gamma_{Kli}(p)\backslash\Sigma$, resp. $\Gamma_{\mathrm{par}}\backslash\Sigma$). For each $(\sigma,\gamma) \in \Sigma$, the (σ,γ)-stratum is isomorphic to $Z_{W,\gamma,\sigma}$ (resp. $Z_{W,\gamma,\mathrm{par},\sigma}$, resp. $Z_{W,\gamma,Kli(p),\sigma}$). The completion of $X_{K,\Sigma}$ (resp. $X_{Kli}(p)_{K,\Sigma}$, resp. $X_{\mathrm{par},K,\Sigma}$) along $Z_{W,\gamma,\sigma}$ (resp. $Z_{W,\gamma,Kli(p),\sigma}$, resp. $Z_{W,\gamma,\mathrm{par},\sigma}$) is isomorphic to the completion of $\overline{\mathcal{M}}_{W,\gamma,\sigma}$ along $Z_{W,\gamma,\sigma}$ (resp. $\overline{\mathcal{M}}_{W,\gamma,Kli(p),\sigma}$ along $Z_{W,\gamma,Kli(p),\sigma}$, resp. $\overline{\mathcal{M}}_{W,\gamma,\mathrm{par},\sigma}$ along $Z_{W,\gamma,\mathrm{par},\sigma}$.) The boundary is the reduced complement of Y_K in $X_{K,\Sigma}$ (resp. of $Y_{Kli}(p)_K$ in $X_{Kli}(p)_{K,\Sigma}$, resp. of $Y_{\mathrm{par},K}$ in $X_{\mathrm{par},K,\Sigma}$). This is a relative Cartier divisor.*

2. If $\Sigma' \subset \Sigma$ is a refinement, then there are projective maps $\pi_{\Sigma',\Sigma} : X_{K,\Sigma'} \to X_{K,\Sigma}$ and $(\mathrm{R}\pi_{\Sigma',\Sigma})_\star\mathscr{O}_{X_{K,\Sigma'}} = \mathscr{O}_{X_{K,\Sigma}}$. Let $\mathscr{I}_{X_{K,\Sigma}}$ and $\mathscr{I}_{X_{K,\Sigma'}}$ be the invertible sheaves of the boundary in $X_{K,\Sigma}$ and $X_{K,\Sigma'}$. Then $\pi^\star_{\Sigma',\Sigma}\mathscr{I}_{X_{K,\Sigma}} = \mathscr{I}_{X_{K,\Sigma'}}$. Similar results hold for $X_{\mathrm{par},K,\Sigma}$ and $X_{Kli(p),K,\Sigma}$.

3. If Σ is Γ-admissible and Σ' is a refinement which is $\Gamma_{Kli}(p)$-admissible, then the map $p_1 : Y_{Kli}(p)_K \to Y_K$ extends to a map $X_{Kli}(p)_{K,\Sigma'} \to X_{K,\Sigma}$. If Σ is Γ_{par}-admissible and Σ' is a refinement which is $\Gamma_{Kli}(p)$-admissible, then the map $p_2 : Y_{Kli}(p)_K \to Y_{\mathrm{par},K}$ extends to a map $X_{Kli}(p)_{K,\Sigma'} \to X_{\mathrm{par},K,\Sigma}$.

4. If Σ is Γ (resp. $\Gamma_{Kli}(p)$, resp. Γ_{par})-admissible, then the toroidal compactification $X_{K,\Sigma}$ of Y_K (resp. $X_{Kli}(p)_{K,\Sigma}$ of $Y_{Kli}(p)_K$, resp. $X_{\mathrm{par},K,\Sigma}$ of $Y_{\mathrm{par},K}$) is normal and a local complete intersection over Spec $\mathbb{Z}_p$.

Proof. All points follow from [**50**], thm. 6.1 and prop. 7.5, except for the last point which follows from the description of the local charts, proposition 6.1.4.1 and our knowledge of modular curves. Let us recall that in the case of Y_K, the toroidal compactification is constructed in the book [**18**]. In the case of $Y_{\mathrm{par},K}$, the method of [**49**] and [**50**] is to embed $Y_{\mathrm{par},K}$ in a Siegel moduli space of principally polarized abelian varieties of genus 16 (Zarhin's trick). The latter can be compactified by the methods of [**18**]. The compactification of $Y_{\mathrm{par},K}$ is obtained by normalization. The toroidal compactification of $Y_{Kli}(p)_K$ is constructed in [**74**]. It is also constructed in [**49**], [**50**] by first embedding $Y_{Kli}(p)_K$ in the product $Y_{\mathrm{par},K} \times Y_K$, then considering the toroidal compactification of the product and then normalizing. □

Notation : We often drop the subscript K or Σ and simply write X, X_{par} and $X_{Kli}(p)$ for $X_{K,\Sigma}$, $X_{Kli}(p)_{K,\Sigma}$ and $X_{\mathrm{par},K,\Sigma}$.

6.2. Sheaves. — We recall the definition of the classical automorphic sheaves as well as the vanishing theorem for the projection to the minimal compactification.

6.2.1. Definition. — We now define several sheaves of modular forms. Over X we have a rank 2 locally free sheaf $\omega_G := e^\star\Omega^1_{G/X}$. For all pairs $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ we set $\Omega^{(k,r)} = \mathrm{Sym}^k\omega_G \otimes \det^r \omega_G$. For simplicity, we sometimes write ω^r instead of $\Omega^{(0,r)}$ and Ω^k instead of $\Omega^{(k,0)}$. Similarly, over X_{par} we have a rank 2 locally free sheaf $e^\star\Omega^1_{G'/X_{\mathrm{par}}}$. If no confusion arises, we still denote this sheaf by Ω^1. We define similarly $\Omega^{(k,r)}$. The sheaves $\Omega^{(k,r)}$ satisfy the expected functorialities with respect to change of polyhedral cone decomposition and level structure away from p. It follows from theorem 6.1.5.1, point 2 (and an application of the projection formula) that the cohomology of these sheaves does not depend on the choice of a particular polyhedral cone decomposition.

6.2.2. Vanishing theorems. — According to [**18**], [**49**] and [**50**], we can construct minimal compactifications $X^\star$ and $X^\star_{\mathrm{par}}$ for Y_K and $Y_{\mathrm{par},K}$. They are defined as the Proj of the graded algebras $\oplus_{k\geq 0}\mathrm{H}^0(X,\omega^k)$ and $\oplus_{k\geq 0}\mathrm{H}^0(X_{\mathrm{par}},\omega^k)$. The sheaves ω descend to ample sheaves on $X^\star$ and $X^\star_{\mathrm{par}}$. We have canonical morphisms $\pi : X \to X^\star$ and $\pi_{\mathrm{par}} : X_{\mathrm{par}} \to X^\star_{\mathrm{par}}$.

Theorem 6.2.2.1 **([50], thm. 8.6).** — *For all $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ and $i > 0$, we have*

$$\mathrm{R}^i\pi_\star\Omega^{(k,r)}(-D_X) = 0$$

and

$$\mathrm{R}^i(\pi_{\mathrm{par}})_\star\Omega^{(k,r)}(-D_{X_{\mathrm{par}}}) = 0.$$

6.3. Hasse invariants. — In this section, let S be a scheme over $\mathrm{Spec}\,\mathbb{F}_p$. If $H \to S$ is a group scheme, we denote by ω_H the conormal sheaf of H along the unit section.

6.3.1. The classical Hasse invariant. — Let $G \to S$ be a truncated Barsotti-Tate group of level 1 (BT_1 for short). We have a Verschiebung map $V : G^{(p)} \to G$ with differential $V^\star : \omega_G \to \omega_G^{(p)}$ also called the Hasse-Witt operator. The Hasse invariant is $\mathrm{Ha}(G) := \det V^\star \in \mathrm{H}^0(S, (\det\omega_G)^{p-1})$. We let G^D be the Cartier dual of G. We recall the following result of Fargues.

Proposition 6.3.1.1 **([21], 2.2.3, prop. 2).** — *There is a canonical and functorial isomorphism $LF : (\det\omega_G)^{p-1} \simeq (\det\omega_{G^D})^{(p-1)}$ such that $LF(\mathrm{Ha}(G)) = \mathrm{Ha}(G^D)$.*

Suppose that we have a quasi-polarization $\lambda : G \overset{\sim}{\to} G^D$. By definition, this is an isomorphism that satisfies the extra condition $\lambda^D = -\lambda$.

Lemma 6.3.1.1. — *The composite $(\det\omega_{G^D})^{p-1} \xrightarrow{\lambda^*} (\det\omega_G)^{p-1} \xrightarrow{LF} (\det\omega_{G^D})^{p-1}$ is the identity map.*

Proof. We first assume that G is ordinary. Thus $\mathrm{Ha}(G)\mathscr{O}_S \simeq (\det\omega_G)^{p-1}$ and similarly, $\mathrm{Ha}(G^D)\mathscr{O}_S \simeq (\det\omega_{G^D})^{p-1}$. By functoriality, $\lambda^\star\mathrm{Ha}(G^D) = \mathrm{Ha}(G)$. Since $LF(\mathrm{Ha}(G)) = \mathrm{Ha}(G^D)$ we deduce the claim. The algebraic stack of quasi-polarized truncated Barsotti-Tate group schemes of level 1 is smooth with dense ordinary locus by [**36**]. We can thus deduce the lemma in general. $\square$

6.3.2. Another Hasse invariant. — We assume that S is reduced, that G is a BT_1 of height 4 and dimension 2, and that the étale rank and multiplicative rank of G are constant, both equal to 1. In this setting, the classical Hasse invariant vanishes identically on S. We recall the construction of an other Hasse invariant in this situation (this is a very special case of more general constructions of Boxer [**5**] and Goldring-Koskivirta [**27**]). We have a multiplicative-connected filtration over S :

$$G^m \subset G^o \subset G.$$

We set $G^{oo} = G^o/G^m$. This is a BT_1 of height 2 and dimension 1. Let $\mathcal{E} = Ext^1_{cris}(G^{oo}, \mathscr{O}_{S/\mathrm{Spec}\mathbb{F}_p})_S$. It carries the Hodge filtration:

$$0 \to \omega_{G^{oo}} \to \mathcal{E} \to \omega^{-1}_{(G^{oo})^D} \to 0.$$

There is a map $V^\star : \mathcal{E} \to \mathcal{E}^{(p)}$. The map $V^\star|\omega_{G^{oo}} : \omega_{G^{oo}} \to \omega^p_{G^{oo}}$ is zero (because it is zero pointwise and S is reduced). The map $V^\star|\omega^{-1}_{(G^{oo})^D} : \omega^{-1}_{(G^{oo})^D} \to \omega^{-p}_{(G^{oo})^D}$ is also zero (this map is the differential of Frobenius on the Lie algebra of $(G^{oo})^D$). Passing to the quotient, we get an isomorphism $V^\star : \omega^{-1}_{(G^{oo})^D} \to \omega^p_{G^{oo}}$. We set $\mathrm{Ha}'(G^{oo}) = (V^\star)^{p-1} \in \mathrm{H}^0(S, \omega^{p(p-1)}_{G^{oo}} \otimes \omega^{p-1}_{(G^{oo})^D}) \simeq \mathrm{H}^0(S, \omega^{p^2-1}_{G^{oo}})$. We are using here the isomorphism LF to identify $\omega^{p-1}_{(G^{oo})^D}$ and $\omega^{p-1}_{G^{oo}}$.

We define the following invertible section (which we call the second Hasse invariant):

$$\mathrm{Ha}'(G) = \mathrm{Ha}(G^m)^{p+1} \otimes \mathrm{Ha}'(G^{oo}) \in \mathrm{H}^0(S, (\det \omega_G)^{p^2-1}).$$

Let G^D be the Cartier dual of G. It satisfies the same assumptions as G and we can define $\mathrm{Ha}'(G^D)$. We have a map $LF^{\otimes(p+1)} : (\det \omega_G)^{p^2-1} \simeq (\det \omega_{G^D})^{p^2-1}$.

Lemma 6.3.2.1. — *The following identity holds : $LF^{\otimes(p+1)}(\mathrm{Ha}'(G)) = \mathrm{Ha}'(G^D)$.*

Proof. Since S is reduced, we need only to check the equality on points. Thus, we can reduce to the case where S is the spectrum of an algebraically closed field. In this case, there exists a quasi-polarization $\lambda : G \to G^D$. The composite

$$(\det \omega_{G^D})^{p^2-1} \overset{\lambda^\star}{\to} (\det \omega_G)^{p^2-1} \overset{LF^{\otimes(p+1)}}{\longrightarrow} (\det \omega_G)^{p^2-1}.$$

is the identity map by lemma 6.3.1.1. On the other hand, $\lambda^\star(\mathrm{Ha}'(G^D)) = \mathrm{Ha}'(G)$ by functoriality. It follows that $LF^{\otimes(p+1)}(\mathrm{Ha}'(G)) = \mathrm{Ha}'(G^D)$. □

6.3.3. Extension of the second Hasse invariant. — We are going to prove that the second Hasse invariant can be extended under some hypothesis. This is again a very special case of extensions considered by Boxer [**5**] and Goldring-Koskivirta [**27**]. We now assume that S is a normal reduced scheme and that G is a BT_1 of height 4, dimension 2 over S. We suppose that the generic étale rank and multiplicative rank of G over S are equal to one. We let S' be the dense open subscheme of S where G has étale rank and multiplicative rank one. We moreover assume that over S, the Hasse-Witt map $V^\star : \omega_G \to \omega_G^{(p)}$ has rank 1 : this means that Ker $V^\star$ is an invertible sheaf and locally a direct factor of ω_G. The next lemma shows that G^D satisfies the same hypothesis as G.

Lemma 6.3.3.1. — *The map $V^\star : \omega_{G^D} \to \omega^{(p)}_{G^D}$ has rank one.*

Proof. Let $\mathcal{E} = Ext^1_{cris}(G, \mathscr{O}_{S/\mathbb{F}_p})_S$. As in [**21**], p. 915, one proves that there is a short exact sequence of perfect complexes (the complexes are the horizontal ones) :

$$\begin{tikzcd}
 & \omega_G \arrow[r, "V^\star"] \arrow[d] & \omega_G^{(p)} \arrow[d] \\
(\omega^\vee_{G^D})^{(p)} \arrow[r, "F^\star"] \arrow[d] & \mathcal{E} \arrow[r, "V^\star"] \arrow[d] & \omega_G^{(p)} \\
(\omega^\vee_{G^D})^{(p)} \arrow[r, "F^\star"] & \omega^\vee_{G^D} &
\end{tikzcd}$$

The map $F^\star : (\omega^\vee_{G^D})^{(p)} \to \omega^\vee_{G^D}$ is the dual of the map $V^\star : \omega_{G^D} \to \omega^{(p)}_{G^D}$. Taking the long exact sequence in cohomology shows that this last map has rank one. □

Over S', we have a multiplicative subgroup $H = G^m \subset G[F] := \mathrm{Ker}\ F$.

Lemma 6.3.3.2. — *The group H extends to a finite flat group scheme $H \subset G[F]$ over S.*

Proof. Consider the map $V : G[F]^{(p)} \to G[F]$. We prove that the kernel K of this map is a finite flat rank p group scheme (locally isomorphic to α_p). Note that K is also the kernel of $F : G^{(p)}[V] \to G^{(p^2)}[V]$. The Hodge-Tate map provides a long exact sequence (see [**21**], sect. 2.1.2) :

$$0 \to \mathrm{Ker}F \to G \overset{\mathrm{HT}}{\to} \omega_{G^D} \overset{F-V^\star}{\to} \omega^{(p)}_{G^D}.$$

In this last equation, ω_{G^D} and $\omega^{(p)}_{G^D}$ are taken as vectorial group schemes (so they are twisted forms of $\mathbb{G}_a^2$), we use F to denote the Frobenius on G and ω_{G^D}, and $V^\star$ is the Hasse-Witt map of G^D.

Moreover, $G/\mathrm{Ker}F \simeq G^{(p)}[V]$. It follows that $K \simeq \mathrm{Ker}(\omega_{G^D}[F] \overset{V^\star}{\to} \omega^{(p)}_{G^D}[F])$ (where $\omega_{G^D}[F]$ is a twisted form of α_p^2) is a rank p group. We now set $H = G[F]^{(p)}/K \hookrightarrow G[F]$. This is the extension we are looking for.

□

Remark 6.3.3.1. — The referee suggests another proof of the lemma : because S is reduced, it is enough to check that $K = \mathrm{Ker}(V : G[F]^{(p)} \to G[F])$ is of rank p on geometric points, and this boils down to an elementary computation with Dieudonné modules.

Applying the lemma to G^D, we also get a subgroup $L \subset G^D[F]$. We now consider the chain of maps $G \overset{F}{\to} G^{(p)} \overset{V}{\to} G$. Applying the functor $Ext^1_{cris}(-, \mathscr{O}_{S/\mathbb{F}_p})_S$ and setting $\mathcal{E} = Ext^1_{cris}(G, \mathscr{O}_{S/\mathbb{F}_p})_S$ yields the following diagram (whose columns are short exact sequences giving the Hodge filtration):

(6.3.A)
$$\begin{array}{ccccc}
\omega_G^{(p)} & \xrightarrow{0} & \omega_G & \longrightarrow & \omega_G^{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}^{(p)} & \xrightarrow{F^\star} & \mathcal{E} & \xrightarrow{V^\star} & \mathcal{E}^{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
(\omega^\vee_{G^D})^{(p)} & \longrightarrow & \omega^\vee_{G^D} & \xrightarrow{0} & (\omega^\vee_{G^D})^{(p)}
\end{array}$$

The map $V^\star : \omega_G \to \omega_G^{(p)}$ fits in the diagram (whose columns are short exact sequences):

(6.3.B)
$$\begin{tikzcd}
\omega_{G[F]/H} \arrow[r, "0"] \arrow[d] & \omega^{(p)}_{G[F]/H} \arrow[d] \\
\omega_G \arrow[r, "V^\star"] \arrow[d] & \omega^{(p)}_G \arrow[d] \\
\omega_H \arrow[r, "V^\star_H"] & \omega^{(p)}_H
\end{tikzcd}$$

We retain from this diagram the two maps : $V^\star_H : \omega_H \to \omega_H^{(p)}$ and $W : \omega^{(p)}_{G[F]/H} \to \omega_G^{(p)}/V^\star(\omega_G)$.

Lemma 6.3.3.3. — *The maps $V^\star_H$ and W vanish on the complement of S'. Moreover, they have the same order of vanishing.*

Proof. Let x be a generic point of one component of $S \setminus S'$. We work over the discrete valuation ring $\mathscr{O}_{S,x}$. We take a basis e_1, e_2 for $\omega_{G,x}$ and f_1, f_2 for $\omega^{(p)}_{G,x}$ such that e_1 generates $\omega_{G[F]/H}$ and f_1 generates $\omega^{(p)}_{G[F]/H}$. The matrix of $V^\star$ in this basis has the form

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

where $b \in \mathfrak{m}_{S,x}$ and $a \in \mathscr{O}^\times_{S,x}$ since $V^\star_H$ vanishes at x and $V^\star$ has rank one. The claim is now obvious. $\square$

The map $V^\star$ of diagram 6.3.A induces, after passing to the quotient, a map

$$Z : \omega^\vee_{G^D}/F^\star(\omega^\vee_{G^D})^{(p)} \to \omega^{(p)}_G/V^\star\omega_G.$$

Lemma 6.3.3.4. — *There is a canonical isomorphism $\omega^\vee_{G^D}/F^\star(\omega^\vee_{G^D})^{(p)} = (\omega_{G^D[F]/L})^\vee$.*

Proof. The map $F^\star : (\omega^\vee_{G^D})^{(p)} \to \omega^\vee_{G^D}$ is dual to $V^\star : \omega_{G^D} \to \omega^{(p)}_{G^D}$ and the kernel of $V^\star$ is $\omega_{G^D[F]/L}$ by the analogue of diagram 6.3.B for G^D. $\square$

We can define a rational section $(V^\star_H)^{p+1} \otimes (W^{-1} \circ Z)^{p-1}$ of the sheaf $\omega_H^{p^2-1} \otimes \omega^{p(p-1)}_{G[F]/H} \otimes \omega^{p-1}_{G^D[F]/L}$.

Lemma 6.3.3.5. — *This section is regular and vanishes precisely over $S \setminus S'$.*

Proof. This follows from lemma 6.3.3.3 since $p+1 > p-1$. $\square$

We can finally prove :

Proposition 6.3.3.1. — *The Hasse invariant $\mathrm{Ha}'(G) \in \mathrm{H}^0(S', \omega_G^{p^2-1})$ extends to S. Moreover, it vanishes precisely on $S \setminus S'$.*

Proof. It is enough to prove the claim for $(\mathrm{Ha}'(G))^2 = \mathrm{Ha}'(G) \otimes \mathrm{Ha}'(G^D)$ (see lemma 6.3.2.1) because S is normal. Call $A = (V^\star_H)^{p+1} \otimes (W^{-1} \circ Z)^{p-1}$ the section of the sheaf $\omega_H^{p^2-1} \otimes \omega^{p(p-1)}_{G[F]/H} \otimes \omega^{p-1}_{G^D[F]/L}$ we just constructed. Exchanging the roles of G and G^D, we obtain a section B of $\omega_L^{p^2-1} \otimes \omega^{p(p-1)}_{G^D[F]/L} \otimes \omega^{p-1}_{G[F]/H}$. By definition, the product $A \otimes B$ extends $(\mathrm{Ha}'(G))^2$. $\square$

6.3.4. Functoriality. — Let S be a scheme over Spec $\mathbb{F}_p$. Let $G, G' \to \mathrm{Spec}\, S$ be Barsotti-Tate groups. We recall that if $\lambda : G \to G'$ is an étale isogeny, then $\lambda^\star : \omega_{G'} \to \omega_G$ is an isomorphism and moreover $\lambda^\star \mathrm{Ha}(G') = \mathrm{Ha}(G)$. If we are in a situation where the second Hasse invariant is defined, we also have $\lambda^\star \mathrm{Ha}'(G') = \mathrm{Ha}'(G)$. We want to obtain similar results in the case of non-étale isogeny.

Lemma 6.3.4.1. — *Assume that G and G' are Barsotti-Tate groups of multiplicative type. Let $\lambda : G \to G'$ be an isogeny. Then we can define a canonical isomorphism :*

$$\tilde{\lambda}^\star : \det \omega_{G'} \to \det \omega_G.$$

Moreover, $\tilde{\lambda}^\star \mathrm{Ha}(G') = \mathrm{Ha}(G)$.

Proof. Let p^r be the degree of λ. We have $G = T \otimes_{\mathbb{Z}_p} \mu_{p^\infty}$ and $G' = T' \otimes_{\mathbb{Z}_p} \mu_{p^\infty}$ for two smooth pro-étale sheaves T and T'. The map λ provides a map $\lambda_0 : T \to T'$ which induces an isomorphism $p^{-r} \det \lambda_0 : \det T \to \det T'$. Since $\det \omega_G = \det T \otimes \omega_{\mu_{p^\infty}}$ and $\det \omega_{G'} = \det T' \otimes \omega_{\mu_{p^\infty}}$ we get a canonical isomorphism $\tilde{\lambda}^\star$ between these two. There are canonical trivialisations $\mathbb{F}_p \simeq (\det T/pT)^{p-1}$ and $\mathbb{F}_p \simeq (\det T'/pT')^{p-1}$. In these trivalisations we have $\mathrm{Ha}(G) = 1 \otimes \mathrm{Ha}(\mu_{p^\infty})$ and $\mathrm{Ha}(G') = 1 \otimes \mathrm{Ha}(\mu_{p^\infty})$ which are identified via the map $\tilde{\lambda}^\star$. □

Lemma 6.3.4.2. — *Let G and G' be Barsotti-Tate groups. We assume that they have constant multiplicative rank over S. Let $\lambda : G \to G'$ be an isogeny with kernel $L \subset G[p]$. Assume that for all geometric points $x \to S$, there exists a multiplicative group $H_x \subset G_x[p]$ such that $H_x \oplus L_x = G_x[p]$. Then there is a canonical isomorphism*

$$\tilde{\lambda}^\star : \det \omega_{G'} \to \det \omega_G.$$

Moreover, $\tilde{\lambda}^\star \mathrm{Ha}(G') = \mathrm{Ha}(G)$. If the second Hasse invariant is defined, we also have $\tilde{\lambda}^\star \mathrm{Ha}'(G') = \mathrm{Ha}'(G)$.

Proof. We have filtrations by multiplicative Barsotti-Tate subgroups $G^m \subset G$ and $(G')^m \subset G'$ (see for example corollary II.1.2 of [**31**]). Let $L^m = L \cap G^m$. Then we have a commutative diagram :

$$\begin{tikzcd} G^m \arrow[r] \arrow[d, "\lambda^m"] & G \arrow[r] \arrow[d, "\lambda"] & G/G^m \arrow[d, "p\mu"] \\ (G')^m \arrow[r] & G' \arrow[r] & G'/(G')^m \end{tikzcd}$$

where the left vertical map has kernel L^m. The isogeny $G/G^m \to G'/(G')^m$ can be uniquely written in the form $p\mu$ where μ is an isomorphism. Indeed, $L/L_m \hookrightarrow G/G^m[p]$ is a finite flat group scheme whose rank is equal to the rank of $G/G^m[p]$ by our assumptions, so we deduce that $L/L_m = G/G^m[p]$. The map μ induces $\mu^\star : \det \omega_{G'/(G')^m} \xrightarrow{\sim} \det \omega_{G/G^m}$. The above lemma provides an isomorphism $(\tilde{\lambda}^m)^\star : \det \omega_{(G')^m} \to \det \omega_{G^m}$. The tensor product of these two maps is the isomorphism we are looking for. The other properties are obvious. □

6.4. Stratification of the special fiber. — We will now stratify the special fibers of the Siegel threefolds. We denote by G the semi-abelian scheme over X and by G' the semi-abelian scheme over X_{par}. For all $n \in \mathbb{Z}_{\geq 1}$, we let $X_n \to \mathrm{Spec}\, \mathbb{Z}/p^n\mathbb{Z}$ be the mod p^n reduction of X and $X_{\mathrm{par},n}$ the reduction modulo p^n of X_{par}.

For $r \in \{0, 1, 2\}$, we set :

— $X_n^{=r}$ the locally closed subset of X_n where the multiplicative rank of G is exactly r,

— $X_n^{\leq r}$ the closed subset of X_n where the multiplicative rank of G is less than r,
— $X_n^{\geq r}$, the open subscheme of X_n where the multiplicative rank of G is greater than r.

We define similarly $X_{\mathrm{par},n}^{=r}$, $X_{\mathrm{par},n}^{\leq r}$ and $X_{\mathrm{par},n}^{\geq r}$. We recall that $X_n^{=r}$ is dense open in $X_n^{\leq r}$, that $X_{\mathrm{par},n}^{=r}$ is dense open in $X_{\mathrm{par},n}^{\leq r}$ and they are of dimension $3-r$ (see [**57**]).

We now specify the schematic structure. We let ω denote the invertible sheaf $\det \omega_G$ over X_1 or $\det \omega_{G'}$ over $X_{\mathrm{par},1}$ (no confusion should arise). We have two Hasse invariants $\mathrm{Ha}(G) \in \mathrm{H}^0(X_1, \omega^{p-1})$ and $\mathrm{Ha}(G') \in \mathrm{H}^0(X_{\mathrm{par},1}, \omega^{p-1})$. Their definition was recalled in section 6.3.1 in the context of abelian schemes. The same definition works for semi-abelian schemes (take the determinant of the differential of Verschiebung). Alternatively, we can use Koecher's principle. We let $X_1^{\leq 1}$ be the vanishing locus of $\mathrm{Ha}(G)$ and $X_{\mathrm{par},1}^{\leq 1}$ be the vanishing locus of $\mathrm{Ha}(G')$.

Lemma 6.4.1. — *The schemes $X_1^{\leq 1}$ and $X_{\mathrm{par},1}^{\leq 1}$ carry the reduced schematic structure.*

Proof. The scheme X_1 is smooth, hence normal. The scheme $X_{\mathrm{par},1}$ is smooth up to a dimension 0 set and is Cohen-Macaulay by proposition 6.1.4.1. By Serre's criterion, it is also normal. It follows that it suffices to prove that $\mathrm{Ha}(G)$ and $\mathrm{Ha}(G')$ vanish at order one at each generic point of the non-ordinary locus. Let k be an algebraically closed field of characteristic p and let $x : \operatorname{Spec} k \to X_1^{=1}$ or $x : \operatorname{Spec} k \to X_{\mathrm{par},1}^{=1}$. Let $H \to \operatorname{Spec} k$ be the p-divisible group associated to x. The contravariant Dieudonné module D of H is isomorphic to the 4-dimensional free $W(k)$-module with canonical basis (e_1, e_2, e_3, e_4) and with Frobenius matrix given by :

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is the sum of three direct factors $W(k)e_1 \bigoplus (W(k)e_2 \oplus W(k)e_3) \bigoplus W(k)e_4$, corresponding to the multiplicative-biconnected-étale decomposition. We find that the Hodge filtration is given by $\mathrm{Ker}(F) = \langle \bar{e}_1, \bar{e}_2 \rangle \subset D/pD$.

By [**36**], the universal first order deformation of H is represented by

$$R = k[X, Y, W, Z]/(X, Y, Z, W)^2$$

where the universal Hodge filtration Fil inside $D \otimes_{W(k)} R$ is generated by the columns of the matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ X & Y \\ W & Z \end{pmatrix}.$$

The Hasse-invariant of the universal deformation is the determinant of $F : D \otimes R/Fil \to D \otimes R/Fil$. The matrix of F in the basis $\bar{e}_3, \bar{e}_4$ of $D \otimes R/Fil$ is:

$$\begin{pmatrix} -Y & 0 \\ -Z & 1 \end{pmatrix}.$$

In order to find the universal deformation of x we need to incorporate the polarization. We will show in all cases that the tangent space is not contained in $Y = 0$. This will prove that the Hasse invariant defines a non-zero linear form. There is a unique principal polarization on D, induced by the symplectic form of matrix J (see section 5.1). In the principally polarized case, the tangent space at x is given by the subspace where the filtration is isotropic with respect to this polarization. This condition writes $X = Z$. The principal polarization identifies D with D^t and $Fil(D^t) = Fil^\perp$ is generated by the columns of the matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ Z & Y \\ W & X \end{pmatrix}.$$

In the paramodular case, the polarization $\lambda : D \to D^t$ identifies with $\mathrm{diag}(1, p, p, 1)$ or $\mathrm{diag}(p, 1, 1, p)$. The condition defining the tangent space are $\lambda(Fil) \subset Fil^\perp$ and $\lambda^\vee(Fil^\perp) \subset Fil$. In the first case we find that $Z = 0$, in the second case that $X = 0$. □

In section 6.3.2 we have defined a second Hasse invariant. The construction applies to the open subscheme of $X_1^{=1}$ and $X_{\mathrm{par},1}^{=1}$ where the semi-abelian scheme is an abelian scheme. We check that the second Hasse invariant extends to the boundary. Indeed, we can consider the connected component of the identity in $G[p]$ and $G'[p]$, that we denote by $G[p]^o$ and $G'[p]^o$. These are truncated BT of level 1, height 3, dimension 2 and multiplicative rank 1 to which we can apply the construction of section 6.3.2. As a result, we have two Hasse invariants $\mathrm{Ha}'(G) \in \mathrm{H}^0(X_1^{=1}, \omega^{p^2-1})$ and $\mathrm{Ha}'(G') \in \mathrm{H}^0(X_{\mathrm{par},1}^{=1}, \omega^{p^2-1})$.

Lemma 6.4.2. — *The second Hasse invariants* $\mathrm{Ha}'(G) \in \mathrm{H}^0(X_1^{=1}, \omega^{p^2-1})$ *and* $\mathrm{Ha}'(G') \in \mathrm{H}^0(X_{\mathrm{par},1}^{=1}, \omega^{p^2-1})$ *extend to* $X_1^{\leq 1}$ *and* $X_{\mathrm{par},1}^{\leq 1}$*. Moreover, they vanish on* $X_1^{\leq 0}$ *and* $X_{\mathrm{par},1}^{\leq 0}$*.*

Proof. Recall that an abelian surface is called superspecial if it is isomorphic to the product of two supersingular elliptic curves. There are only finitely many superspecial points on $X_{\mathrm{par},1}$ and X_1 by [**56**]. Call this finite set SS. Since $X_{\mathrm{par},1}^{\leq 1}$ and $X_1^{\leq 1}$ are Cohen-Macaulay, it suffices to construct the extension over the complement of SS. Moreover, since we removed the superspecial points, the Hasse-Witt matrix has rank 1. We now prove the smoothness for $X_1^{\leq 1} \setminus SS$. Over $X_1^{\leq 1} \setminus SS$, we have a canonical filtration $H \subset \mathrm{Ker} F$ where the group H is constructed in lemma 6.3.3.2 . As a result, $X_1^{\leq 1} \setminus SS$ embeds in the moduli space of abelian surfaces with a polarization of degree prime-to-p and with Iwahori level. The local model is computed in detail in [**62**], page 186 to 189. We find that $X_1^{\leq 1} \setminus SS$ is exactly the union of the strata denoted $X_0^{m,e}$ and $X_0^{sg,F}$ in that reference. We see that this union of strata is smooth. We compute that the closure of $X_0^{m,e}$ is locally isomorphic to

$$\mathrm{Spec}\ \mathbb{F}_p[x, y, a, b, c]/(xy, ax + by + abc, a, y, x + bc) \simeq \mathbb{F}_p[b, c]$$

where $X_0^{m,e}$ is corresponds to the stratum $bc \neq 0$ and $X_0^{sg,F}$ corresponds to the stratum $c = 0, b \neq 0$. The extension of $\mathrm{Ha}'(G)$ over $X_1^{\leq 1} \setminus SS$ follows from proposition 6.3.3.1.

We now prove that $X_{\mathrm{par},1}^{\leq 1} \setminus SS$ is locally isomorphic to $\mathrm{Spec}\ \mathbb{F}_p[a, b, c]/(ab)$ with $a \neq 0$ or $b \neq 0$ corresponding to $X_{\mathrm{par},1}^{=1}$. By proposition 6.3.3.1 we deduce that $\mathrm{Ha}'(G')$ extends on each irreducible components of $X_{\mathrm{par},1}^{\leq 1} \setminus SS$. Moreover, to check that it glues to a section over $X_{\mathrm{par},1}^{\leq 1} \setminus SS$ we need to prove that the values of $\mathrm{Ha}'(G')$ agree on the intersections of the irreducible components. Since this value is zero, this is true. Over $X_{\mathrm{par},1}^{\leq 1} \setminus SS$ we have a chain $G' \to G \to (G')^t \to G'' \to G' \to G$. This chain is constructed as follows. Let $K(\lambda)$ be the kernel of the polarization $G' \to (G')^t$ and $K(\lambda^t)$ the kernel of the polarization $\lambda^t : (G')^t \to G$. Set $H = K(\lambda) \cap \mathrm{Ker}\ F$ and set $H' = K(\lambda^t) \cap \mathrm{Ker}\ F$. These are groups of order p because $K(\lambda)$ and $K(\lambda^t)$ are BT_1 of height 2 and dimension 1. We set $G = G'/H$ and $G'' = (G')^t/H'$. This chain provides an embedding of $X_{\mathrm{par},1}^{\leq 1} \setminus SS$ in the moduli of space of abelian surfaces with a polarization of degree prime-to-p and Iwahori level. More precisely, it identifies $X_{\mathrm{par},1}^{\leq 1} \setminus SS$ with an open subscheme of the union of the closure of the stratum denoted by $X_0^{o,m}$ and $X_0^{et,o}$ in [**62**]. We compute that the closure of $X_0^{o,m}$ corresponds on the local model to the ring quotient

$$\mathbb{F}_p[x,y,a,b,c]/(xy, ax+by+abc) \mapsto \mathbb{F}_p[b,c]$$

given $x=y=a=0$. The closure of $X_0^{et,o}$ corresponds on the local model to the ring quotient

$$\mathbb{F}_p[x,y,a,b,c]/(xy, ax+by+abc) \mapsto \mathbb{F}_p[a,c]$$

given $x=b=0$ and $y \mapsto -ac$. Both rings are quotients of

$$\mathbb{F}_p[x,y,a,b,c]/(xy, ax+by+abc, y+ac, x) \simeq \mathbb{F}_p[a,b,c]$$

given by the respective equations $a=0$ and $b=0$. Finally, the open stratum corresponding to $X^{=1}_{\mathrm{par},1}$ is given by $a \neq 0$ or $b \neq 0$.

□

We define the scheme $X_1^{\leq 0}$ as the vanishing locus of $\mathrm{Ha}'(G)$ and the scheme $X^{\leq 0}_{\mathrm{par},1}$ as the vanishing locus of $\mathrm{Ha}'(G')$.

Remark 6.4.1. — It is possible, using lemma 6.3.3.5, to check that the Siegel modular form $\mathrm{Ha}'(G)$ vanishes at order 2 along the rank 0 locus. When $p \geq 3$, the modular form $\mathrm{Ha}'(G)$ has a square root (a modular form of weight $\frac{p^2-1}{2}$) which vanishes at order 1. When $p=2$, it does not have a square root.

7. The T-operator

The goal of this section is to introduce and study the action of an Hecke operator T on the cohomology of automorphic vector bundles $\mathrm{R}\Gamma(X, \Omega^{(k,r)})$ and $\mathrm{R}\Gamma(X, \Omega^{(k,r)}(-D))$ with $r \geq 2$. The Hecke operator T is related to the classical Hecke operator $T_{p,1} = \mathrm{GSp}_4(\mathbb{Z}_p)\mathrm{diag}(1,p,p,p^2)\mathrm{GSp}_4(\mathbb{Z}_p)$. The naive attempt to directly define $T_{p,1}$ on the integral cohomology of vector bundles does not seem to work because we are unable to properly define and study an integral moduli space associated with $T_{p,1}$: the cocharacter $t \mapsto \mathrm{diag}(t^2,t,t,1)$ is not minuscule. We proceed differently, making use of a factorization in $\mathrm{GL}_4(\mathbb{Q}_p)$: $\mathrm{diag}(p^2,p,p,1) = \mathrm{diag}(p,p,p,1) \times \mathrm{diag}(p,1,1,1)$. This suggests to replace $T_{p,1}$ by a composition (denoted T) of two double cosets :

$$\mathrm{GSp}_4(\mathbb{Z}_p)\mathrm{diag}(p,p,p,1)\mathrm{GSp}'_4(\mathbb{Z}_p) \star \mathrm{GSp}'_4(\mathbb{Z}_p)\mathrm{diag}(p,1,1,1)\mathrm{GSp}_4(\mathbb{Z}_p)$$

where GSp'_4 is the paramodular group. The point is that each double coset

$$\mathrm{GSp}_4(\mathbb{Z}_p)\mathrm{diag}(p,p,p,1)\mathrm{GSp}'_4(\mathbb{Z}_p) \quad \text{and} \quad \mathrm{GSp}'_4(\mathbb{Z}_p)\mathrm{diag}(p,1,1,1)\mathrm{GSp}_4(\mathbb{Z}_p)$$

has a clear moduli interpretation in terms of parahoric level structure.

It is instructive to compare T and $T_{p,1}$. At the level of double cosets, an elementary computation reveals that $T = T_{p,1} + (1+p+p^2+p^3)T_{p,0}$ where $T_{p,0} = p\mathrm{GSp}_4(\mathbb{Z}_p)$ [(8)].

Assume that π_p is a spherical irreducible smooth representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ which contributes to the cohomology $\mathrm{R}\Gamma(X, \Omega^{(k,r)}) \otimes^L \overline{\mathbb{Q}}_p$ or $\mathrm{R}\Gamma(X, \Omega^{(k,r)}(-D)) \otimes^L \overline{\mathbb{Q}}_p$. Let Θ_{π_p} be the corresponding character of the spherical Hecke algebra (valued in $\overline{\mathbb{Q}}_p$). Let us denote by $(\alpha_p, \beta_p, \gamma_p, \delta_p)$ the Hecke parameters of π_p which are the roots of the reciprocal Hecke polynomial evaluated at Θ_{π_p}, ordered to have non-decreasing p-adic valuation and

8. The double coset T parametrizes chains $G \to G/H_1 \to (G/H_1)/H_2$ where $H_1 \subset G[p]$ is an order p^3 group and $H_2 \subset (G/H_1)[p]$ is an order p group contained in the kernel of the polarization of G/H_1. The component $T_{p,1}$ of T corresponds to any choice of H_1 and the choice of $H_2 \neq G[p]/H_1$. The component $T_{p,0}$ of T corresponds to any choice of H_1 and the choice of $H_2 = G[p]/H_1$. It has multiplicity $p^3+p^2+p+1 = \sharp \mathrm{GSp}_4(\mathbb{Z}_p)/Kli(p)$.

such that $\alpha_p\delta_p = \beta_p\gamma_p$. The Newton polygon associated to the Hecke parameters is (at least conjecturally, see remark 5.3.2) above the Hodge polygon with slopes $0, r-2, k+r-1, k+2r-3$, with the same initial and ending point. We assume that this inequality holds in the following discussion. By definition of the Hecke polynomial (see lemma 5.1.5.1), we find that $\Theta_{\pi_p}(T_{p,1}) = p^{-1}(\alpha_p\beta_p+\alpha_p\gamma_p+\alpha_p\delta_p+\beta_p\delta_p+\gamma_p\delta_p)-p^{-3}\alpha_p\delta_p$, and that $\Theta_{\pi_p}(T_{p,0}) = p^{-3}\alpha_p\delta_p$. The Hecke operator T that we use acts like $p^{3-r}(T_{p,1} + (1+p+p^2+p^3)T_{p,0})$ (the normalization factor by p^{3-r} optimizes integrality) and we find that:

$$\Theta_{\pi_p}(T) = p^{2-r}(\alpha_p\beta_p + \alpha_p\gamma_p + \alpha_p\delta_p + \beta_p\delta_p + \gamma_p\delta_p) + p^{1-r}(1+p+p^2)\alpha_p\delta_p.$$

In this work, we mainly focus on the case that $r = 2$, and we observe that in this case, the expression $\Theta_{\pi_p}(T)$ is p-integral for all $k \geq 0$. Moreover, we find that T and $pT_{p,1}$ are congruent modulo p for $k \geq 2$. Recall that our goal is to construct ordinary families when $r = 2$ and k varies. The Hecke parameter $(\alpha_p, \beta_p, \gamma_p, \delta_p)$ is called ordinary if the Newton and Hodge polygon agree. This condition translates into (when $r = 2$) : $\alpha_p\beta_p$ is a p-adic unit. We see that when $k \geq 1$, it further translates into : $\Theta_{\pi_p}(T)$ is a p-adic unit.

7.1. Definition of the T-operator. — Consider the schemes X, $X_{Kli}(p)$ and X_{par} for choices of good polyhedral decompositions Σ, Σ' and Σ'' (see section 6.1). We also assume that Σ' refines both Σ and Σ''. As a result we have maps $p_1 : X_{Kli}(p) \to X$ and $p_2 : X_{Kli}(p) \to X_{\mathrm{par}}$. We recall that G denotes the semi-abelian scheme over X and G' the semi-abelian scheme over X_{par}. Over $X_{Kli}(p)$ we have the chain of isogenies $G \to G' \to G$ where the first isogeny $G \to G'$ has degree p^3, the second isogeny $G' \to G$ has degree p and the composite is multiplication by p. The map p_1 forgets G', the map p_2 forgets G. By theorem 6.1.5.1, the schemes X, $X_{Kli}(p)$ and X_{par} are normal and lci over Spec $\mathbb{Z}_p$. Their non-smooth locus is included in the non-ordinary locus of the special fiber. As a result, it is of codimension 2.

We will apply the formalism developed in section 4 to construct cohomological correspondences. We note that the morphisms p_1 and p_2 are not finite flat, because they are not quasi-finite over the rank 0 loci $X_1^{=0}$ and $X_{\mathrm{par}}^{=0}$ [9]. This is a consequence of the fact that p-rank 0 abelian sufaces may have infinitely many subgroups of order p. This explains why we will need advanced results on coherent duality to construct the cohomological correspondences.

Let $(k,r) \in \mathbb{Z}^2_{\geq 0}$. The differential of the isogeny $G \to G'$ provides a map $p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)}$. Moreover, we have by construction 1 (see section 4.2.1), a fundamental class $p_1^\star\mathscr{O}_X \to p_1^!\mathscr{O}_X$ and $p_1^!\mathscr{O}_X$ is an invertible sheaf. We thus obtain by tensor product with $\Omega^{(k,r)}$ and proposition 4.1.2.1 a map $p_1^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$. Finally, if we compose with the map $p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)}$, we obtain a cohomological correspondence

$$T_1' : p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$$

that we need to normalize.

Lemma 7.1.1. — *The map T_1' factors through $p^{2+r}p_1^!\Omega^{(k,r)}$ if $k+2r \geq 2+r$.*

Proof. It is enough to prove the divisibility over the complement of the non-ordinary locus. This is sufficient because $X_{Kli}(p)$ is normal and the closed subscheme "non-ordinary locus" is of codimension 2. We are thus left to prove the divisibility over the localization of $X_{Kli}(p)$ at each generic point of the ordinary locus. There are two types of components in the ordinary locus. We first consider the components where $G \to G'$ has kernel a group of étale rank two. Over these components, the map $p_2^\star\omega^r \to p_1^\star\omega^r$ factors through $p^r p_1^\star\omega^r$

9. The maps p_1 and p_2 are also not finite flat over the boundary, but this is not a serious issue.

because the multiplicative rank of the kernel of the isogeny $G \to G'$ is exactly 1. As a result, the map $p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)}$ factors through $p^r p_1^\star\Omega^{(k,r)}$. On the other hand, we claim that the map $p_1^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$ factors through $p^2 p_1^!\Omega^{(k,r)}$. Let k be an algebraically closed field of characteristic p and $x : \mathrm{Spec}\ k \to X$ be an ordinary point corresponding to an abelian scheme G. Let T be the Tate module of G. We fix an isomorphism $T \simeq \mathbb{Z}_p^2$. The formal deformation space of this point is $\mathrm{Hom}(\mathrm{Sym}^2 T, \widehat{\mathbb{G}_m})$ by Serre-Tate theory ([**39**]). This space has underlying ring $W(k)[[X,Y,Z]]$ where the Serre-Tate parameter is the map $\mathbb{Z}_p^2 \to \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m}$ given by the symmetric matrix $\begin{pmatrix} X & Z \\ Z & Y \end{pmatrix}$. The components of the fiber of this deformation space under p_1 where $G \to G'$ has kernel a group of étale rank two are a disjoint union (parametrized by $\ker(G \to G') \cap G[p]^m$) of spaces with associated rings

$$W(k)[[X,Y,Z,X',Y',Z']]/((1+X')^p - 1 - X, (1+Z')^p - 1 - Z, Y' - Y),$$

which parametrize the following diagram of Serre-Tate parameters :

$$\begin{tikzcd}
\mathbb{Z}_p^2 \arrow[r, "\begin{pmatrix} X & Z \\ Z & Y \end{pmatrix}"] \arrow[d, "\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}"'] & \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m} \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}"] \\
\mathbb{Z}_p^2 \arrow[r, "\begin{pmatrix} X' & p.Z' \\ Z' & Y' \end{pmatrix}"] & \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m}
\end{tikzcd}$$

The trace

$W(k)[[X,Y,Z,X',Y',Z']]/((1+X')^p - 1 - X, (1+Z')^p - 1 - Z, Y' - Y) \to W(k)[[X,Y,Z]]$ factors through $p^2 W(k)[[X,Y,Z]]$ which implies that the map $p_1^\star\mathscr{O}_X \to p_1^!\mathscr{O}_X$ factors through $p^2 p_1^!\mathscr{O}_X$.

On the components where $G \to G'$ has kernel a group of p-rank two, the map $p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)}$ factors through $p^{(k+2r)} p_1^\star\Omega^{(k,r)}$ and the map $p_1^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$ is an isomorphism. □

Under the assumption $k + 2r \geq 2 + r$ (which holds if $r \geq 2$), we denote by $T_1 = p^{-2-r} T_1' : p_2^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$ the normalized map or the map on cohomology :

$$T_1 : \mathrm{R}\Gamma(X_{\mathrm{par}}, \Omega^{(k,r)}) \to \mathrm{R}\Gamma(X, \Omega^{(k,r)}).$$

We now define a second cohomological correspondence in the other direction (we exchange the roles of p_1 and p_2). We have maps :

$$T_2' : p_1^\star\Omega^{(k,r)} \to p_2^\star\Omega^{(k,r)} \to p_2^!\Omega^{(k,r)},$$

where the first map arises from the differential of the isogeny $G' \to G$ and the second map from the fundamental class.

Lemma 7.1.2. — *The map T_2' factors through $p p_2^!\Omega^{(k,r)}$ if $r \geq 1$.*

Proof. We compute over the localization at generic points in the ordinary locus as in the proof of lemma 7.1.1. There are two types of generic points : the points where the kernel of $G' \to G$ is an étale group scheme and the points where the kernel of $G' \to G$ is a multiplicative group scheme. Over the "étale" points, the map $p_1^\star\Omega^{(k,r)} \to p_2^\star\Omega^{(k,r)}$ is an isomorphism and we claim that the map $p_2^\star\Omega^{(k,r)} \to p_2^!\Omega^{(k,r)}$ factors through $p p_2^!\Omega^{(k,r)}$. This can be checked in the complete local ring, using Serre-Tate parameters. Namely, let k be an algebraically closed field of characteristic p and $x : \mathrm{Spec}\ k \to X_{\mathrm{par}}$ be an ordinary point corresponding to an abelian scheme G'. The formal deformation space at x has underlying ring isomorphic to $W(k)[[X,Y,Z]]$ and parametrizes the Serre-Tate parameter: $\mathbb{Z}_p^2 \to \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m}$ given by the matrix $\begin{pmatrix} X & pZ \\ Z & Y \end{pmatrix}$. The components of the fiber of this deformation space under p_2 where $G' \to G$ is étale has associated ring

$$W(k)[[X,Y,Z,Y']]/((1+Y')^p - 1 - Y)$$

which parametrizes the following diagram of Serre-Tate parameters :

$$\begin{tikzcd} \mathbb{Z}_p^2 \arrow[r, "\begin{pmatrix} X & pZ \\ Z & Y \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}"'] & \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m} \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] \\ \mathbb{Z}_p^2 \arrow[r, "\begin{pmatrix} X & Z \\ Z & Y' \end{pmatrix}"] & \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m} \end{tikzcd}$$

The trace

$$W(k)[[X,Y,Z,Y']]/((1+Y')^p - 1 - Y) \to W(k)[[X,Y,Z]]$$

factors through $pW(k)[[X,Y,Z]]$ which implies that the map $p_2^\star \mathscr{O}_{X_{\mathrm{par}}} \to p_1^! \mathscr{O}_{X_{\mathrm{par}}}$ factors through $pp_2^! \mathscr{O}_{X_{\mathrm{par}}}$ at these points.

At the points where the kernel of $G' \to G$ is a multiplicative group scheme, the map $p_1^\star \Omega^{(k,r)} \to p_2^\star \Omega^{(k,r)}$ factors through $p^r p_2^\star \Omega^{(k,r)}$ and the map $p_2^\star \Omega^{(k,r)} \to p_2^! \Omega^{(k,r)}$ is an isomorphism. $\square$

Under the assumption $r \geq 1$, we denote by T_2 the associated normalized map $p^{-1}T_2'$: $p_1^\star \Omega^{(k,r)} \to p_2^! \Omega^{(k,r)}$ or the map on cohomology :

$$T_2 : \mathrm{R}\Gamma(X, \Omega^{(k,r)}) \to \mathrm{R}\Gamma(X_{\mathrm{par}}, \Omega^{(k,r)}).$$

We let $T = T_1 \circ T_2$. The operator T corresponds to the (normalized) operator attached to the composition of Hecke operators $\mathrm{GSp}_4(\mathbb{Z}_p)\mathrm{diag}(p,p,p,1)\mathrm{GSp}_4'(\mathbb{Z}_p) \star \mathrm{GSp}_4'(\mathbb{Z}_p)\mathrm{diag}(p,1,1,1)\mathrm{GSp}_4(\mathbb{Z}_p)$ as explained in the beginning of this section.

7.2. Independence on the choice of the toroidal compactification. — We justify that the action of our Hecke operators T_1 and T_2 does not depend on special choices of polyhedral cone decompositions. We assume that $r \geq 1$ and $k + 2r \geq r + 2$ throughout this section. Suppose we have a commutative diagram for choices $\Sigma, \Sigma', \Sigma''$ and $\Lambda, \Lambda', \Lambda''$ of good polyhedral cone decompositions :

$$\begin{tikzcd} X_{\mathrm{par},\Lambda''} \arrow[d, "r"] & X_{Kli}(p)_{\Lambda'} \arrow[l, "l_2"'] \arrow[r, "l_1"] \arrow[d, "s"] & X_\Lambda \arrow[d, "t"] \\ X_{\mathrm{par},\Sigma''} & X_{Kli}(p)_{\Sigma'} \arrow[l, "p_2"'] \arrow[r, "p_1"] & X_\Sigma \end{tikzcd}$$

By theorem 6.1.5.1, we have isomorphisms :

$$t^\star : \mathrm{R}\Gamma(X_\Sigma, \Omega^{(k,r)}) \to \mathrm{R}\Gamma(X_\Lambda, t^\star \Omega^{(k,r)}),$$

$$r^\star : \mathrm{R}\Gamma(X_{\mathrm{par},\Sigma''}, \Omega^{(k,r)}) \to \mathrm{R}\Gamma(X_{\mathrm{par},\Lambda''}, r^\star \Omega^{(k,r)}),$$

$$s^\star : \mathrm{R}\Gamma(X_{Kli}(p)_{\Sigma'}, \Omega^{(k,r)}) \to \mathrm{R}\Gamma(X_{Kli}(p)_{\Lambda'}, s^\star \Omega^{(k,r)}),$$

where in this last isomorphisms $\Omega^{(k,r)}$ stands for either $p_1^\star \Omega^{(k,r)}$ or $p_2^\star \Omega^{(k,r)}$.

Proposition 7.2.1. — *The diagrams :*

$$\begin{tikzcd} \mathrm{R}\Gamma(X_{\mathrm{par},\Lambda''},\Omega^{(k,r)}) \arrow[r, "T_{1,\Lambda}"] & \mathrm{R}\Gamma(X_{\Lambda},\Omega^{(k,r)}) \\ \mathrm{R}\Gamma(X_{\mathrm{par},\Sigma''},\Omega^{(k,r)}) \arrow[r, "T_{1,\Sigma}"] \arrow[u, "r^\star"] & \mathrm{R}\Gamma(X_{\Sigma},\Omega^{(k,r)}) \arrow[u, "t^\star"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{R}\Gamma(X_{\Lambda},\Omega^{(k,r)}) \arrow[r, "T_{2,\Lambda}"] & \mathrm{R}\Gamma(X_{\mathrm{par},\Lambda''},\Omega^{(k,r)}) \\ \mathrm{R}\Gamma(X_{\Sigma},\Omega^{(k,r)}) \arrow[r, "T_{2,\Sigma}"] \arrow[u, "t^\star"] & \mathrm{R}\Gamma(X_{\mathrm{par},\Sigma''},\Omega^{(k,r)}) \arrow[u, "r^\star"'] \end{tikzcd}$$

are commutative.

Proof. We only prove the commutativity of the first diagram. The commutativity of the second diagram follows along similar lines. The bottom horizontal map is induced by the cohomological correspondence $T_{1,\Sigma} : p_2^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$ which by adjunction is a map : $R(p_1)_\star p_2^\star\Omega^{(k,r)} \to \Omega^{(k,r)}$. Since $Rs_\star s^\star p_2^\star\Omega^{(k,r)} \simeq p_2^\star\Omega^{(k,r)}$, this map is equivalently a map :

$$T'_{1,\Sigma} : R(p_1)_\star Rs_\star s^\star p_2^\star\Omega^{(k,r)} = Rt_\star R(l_1)_\star l_2^\star r^\star\Omega^{(k,r)} \to \Omega^{(k,r)}.$$

We can obtain another map. We have a second cohomological correspondence $T_{1,\Lambda} : R(l_1)_\star l_2^\star r^\star\Omega^{(k,r)} \to t^\star\Omega^{(k,r)}$. Using the adjunction property and the isomorphism $Rt_\star t^\star\Omega^{(k,r)} \simeq \Omega^{(k,r)}$ we obtain a map that we denote by

$$T'_{1,\Lambda} : Rt_\star R(l_1)_\star l_2^\star r^\star\Omega^{(k,r)} \to \Omega^{(k,r)}.$$

The commutativity of the diagram is equivalent to the equality $T'_{1,\Sigma} = T'_{1,\Lambda}$. By adjunction, both can be seen as maps of locally free shaves $l_2^\star r^\star\Omega^{(k,r)} \to l_1^! t^!\Omega^{(k,r)}$. Both maps coincide over the complement of the boundary. Thus, they coincide everywhere. □

7.3. The operator on cuspidal cohomology. — The boundary of the toroidal compactification X, X_{par} or $X_{Kli}(p)$ is denoted by D_X, $D_{X_{\mathrm{par}}}$ or $D_{X_{Kli}(p)}$. If no confusion will arise, it is simply denoted by D.

Lemma 7.3.1. —

1. *If $k+2r \geq r+2$, the cohomological correspondences $T_1 : p_2^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$ induces a cohomological correspondence $T_1 : p_2^\star\Omega^{(k,r)}(-D_{X_{\mathrm{par}}}) \to p_1^!\Omega^{(k,r)}(-D_X)$.*
2. *If $r \geq 1$, the cohomological correspondences $T_2 : p_1^\star\Omega^{(k,r)} \to p_2^!\Omega^{(k,r)}$ induces a cohomological correspondence $T_2 : p_1^\star\Omega^{(k,r)}(-D_X) \to p_2^!\Omega^{(k,r)}(-D_{X_{\mathrm{par}}})$.*
3. *These cohomological correspondences are functorial with respect to the change of polyhedral cone decomposition, in the sense that the analogue of proposition 7.2.1 holds for cuspidal automorphic sheaves.*

Proof. We only prove point 1 because point 2 is similar and point 3 is proved exactly in the same way as is proposition 7.2.1. We have a map $p_2^\star\Omega^{(k,r)}(-D_{X_{\mathrm{par}}}) \to p_2^\star\Omega^{(k,r)}(-D_{X_{Kli}(p)})$. Twisting the map $p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)}$ we get a map $p_2^\star\Omega^{(k,r)}(-D_{X_{Kli}(p)}) \to p_1^\star\Omega^{(k,r)}(-D_{X_{Kli}(p)})$. By lemma 4.2.4.1, the fundamental class induces a map $\mathscr{O}_{X_{Kli}(p)}(-D_{X_{Kli}(p)}) \to p_1^!\mathscr{O}_X(-D_X)$. Tensoring with $\Omega^{(k,r)}$ and composing everything gives a non-normalized map $p_2^\star\Omega^{(k,r)}(-D_{X_{\mathrm{par}}}) \to p_1^!\Omega^{(k,r)}(-D_X)$. This map factors through $p^{r+2}p_1^!\Omega^{(k,r)} \cap p_1^!\Omega^{(k,r)}(-D_X) = p^{r+2}p_1^!\Omega^{(k,r)}(-D_X)$. $\square$

7.4. Restriction of the correspondence. — In this section, we work over Spec $\mathbb{F}_p$. Let $p_1 : X_{Kli}(p)_1 \to X_1$ and $p_2 : X_{Kli}(p)_1 \to X_{\mathrm{par},1}$ be the reduction modulo p of the maps p_1 and p_2. We keep the notation p_1 and p_2 for the two projections. We will also make use of the following notation : if we have a scheme S, a locally closed subscheme $i : T \hookrightarrow S$, and a coherent sheaf $\mathcal{F}$ on S, we often write $\mathcal{F}|_T$ for $i^\star\mathcal{F}$.

We have (by reduction modulo p and proposition 4.1.2.1), two normalized cohomological correspondences $T_1 : p_2^\star(\Omega^{(k,r)}|_{X_{\mathrm{par},1}}) \to p_1^!(\Omega^{(k,r)}|_{X_1})$ and $T_2 : p_1^\star(\Omega^{(k,r)}|_{X_1}) \to p_2^!(\Omega^{(k,r)}|_{X_{\mathrm{par},1}})$. Again, we keep the notations T_1, T_2 for the reduction of the cohomological correspondences. We deduce maps on cohomology $T_1 \in \mathrm{Hom}(\mathrm{R}\Gamma(X_{\mathrm{par},1}, \Omega^{(k,r)}), \mathrm{R}\Gamma(X_1, \Omega^{(k,r)}))$ and $T_2 \in \mathrm{Hom}(\mathrm{R}\Gamma(X_1, \Omega^{(k,r)}), \mathrm{R}\Gamma(X_{\mathrm{par},1}, \Omega^{(k,r)}))$. We keep writting $T = T_1 \circ T_2$.

7.4.1. Restriction to the non-ordinary locus. — We now study the restriction of the correspondence to the non-ordinary locus.

Proposition 7.4.1.1. — *For $r \geq 2$ and $k + r > 2$, the following diagrams commute :*

$$\begin{tikzcd} p_2^\star\Omega^{(k,r)} \arrow[r, "T_1"] \arrow[d, "p_2^\star\mathrm{Ha}"] & p_1^!\Omega^{(k,r)} \arrow[d, "p_1^\star\mathrm{Ha}"] \\ p_2^\star\Omega^{(k,r+(p-1))} \arrow[r, "T_1"] & p_1^!\Omega^{(k,r+(p-1))} \end{tikzcd}$$

$$\begin{tikzcd} p_1^\star\Omega^{(k,r)} \arrow[r, "T_2"] \arrow[d, "p_1^\star\mathrm{Ha}"] & p_2^!\Omega^{(k,r)} \arrow[d, "p_2^\star\mathrm{Ha}"] \\ p_1^\star\Omega^{(k,r+(p-1))} \arrow[r, "T_2"] & p_2^!\Omega^{(k,r+(p-1))} \end{tikzcd}$$

Proof. It is enough to prove the commutativity over some dense open subscheme since $X_{Kli}(p)_1$ is Cohen-Macaulay. We can thus work over the intersection of the ordinary locus and the complement of the boundary. We consider the first diagram. There are two types of ordinary components. First, the components where the kernel of the isogeny $G \to G'$ is of étale rank 2. Over these components, the diagram can be rewritten as the composition of two diagrams :

$$\begin{tikzcd} p_2^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "p_2^\star\mathrm{Ha}"] & p_1^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "p_1^\star\mathrm{Ha}"] & p_1^!\Omega^{(k,r)} \arrow[d, "p_1^\star\mathrm{Ha}"] \\ p_2^\star\Omega^{(k,r+(p-1))} \arrow[r] & p_1^\star\Omega^{(k,r+(p-1))} \arrow[r] & p_1^!\Omega^{(k,r+(p-1))} \end{tikzcd}$$

The map $p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)}$ is obtained as the tensor product of the natural map $p_2^\star\Omega^{(k,0)} \to p_1^\star\Omega^{(k,0)}$ and a normalized map $p_2^\star\Omega^{(0,r)} \to p_1^\star\Omega^{(0,r)}$. By lemma 6.3.4.1 (observe that the normalization used in that lemma is the same as the normalization used in the definition of the cohomological correspondence), the left square is commutative. The right square diagram is obtained by tensoring a normalized fundamental class $p_1^\star\mathscr{O}_{X_1} \to p_1^!\mathscr{O}_{X_1}$ with the morphism $\Omega^{(k,r)} \overset{p_1^\star\mathrm{Ha}}{\to} \Omega^{(k,r+(p-1))}$ and is obviously commutative. We next deal with the components where the kernel of the isogeny $G \to G'$ is of étale rank 1 and thus of multiplicative rank 2. Going back to the definition (see lemma 7.1.1), we deduce that the map $p_2^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$ vanishes as soon as $k+2r > r+2$. As a result, the commutativity is obvious on these components.

We now deal with the commutativity of the second diagram. First, we consider the components where the isogeny $G' \to G$ has étale kernel. On those components, we can again split the diagram as

$$\begin{tikzcd}
p_1^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "p_1^\star\mathrm{Ha}"] & p_2^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "p_2^\star\mathrm{Ha}"] & p_2^!\Omega^{(k,r)} \arrow[d, "p_2^\star\mathrm{Ha}"] \\
p_1^\star\Omega^{(k,r+(p-1))} \arrow[r] & p_2^\star\Omega^{(k,r+(p-1))} \arrow[r] & p_2^!\Omega^{(k,r+(p-1))}
\end{tikzcd}$$

The left square is commutative because the Hasse invariant commutes with étale isogenies. The right square is commutative because it is obtained by tensoring the normalized fundamental class $p_2^\star\mathscr{O}_{X_1} \to p_2^!\mathscr{O}_{X_1}$ with the morphism $\Omega^{(k,r)} \to \Omega^{(k,r+(p-1))}$.

Finally, we consider components where the kernel of the map $G' \to G$ is multiplicative. Then, as soon as $r > 1$, the map $p_1^\star\Omega^{(k,r)} \to p_2^!\Omega^{(k,r)}$ vanishes and commutativity is obvious. $\square$

We recall that $X^{\leq 1}_{\mathrm{par},1}$ and $X^{\leq 1}_1$ are the vanishing locus of the Hasse invariant in $X_{\mathrm{par},1}$ and X_1.

Lemma 7.4.1.1. — *The sections* $p_2^\star\mathrm{Ha}$ *and* $p_1^\star\mathrm{Ha}$ *are not zero divisors in* $X_{Kli}(p)_1$.

Proof. The scheme $X_{Kli}(p)_1$ is Cohen-Macaulay and the non-ordinary locus has codimension 1. $\square$

By proposition 7.4.1.1 and proposition 4.1.2.1, for all $r \geq 2+p-1$ and $k+r > 2+p-1$, we have cohomological correspondences :

$$T_1 : p_2^\star(\Omega^{(k,r)}|_{X^{\leq 1}_{\mathrm{par},1}}) \to p_1^!(\Omega^{(k,r)}|_{X^{\leq 1}_1})$$

and

$$T_2 : p_1^\star(\Omega^{(k,r)}|_{X^{\leq 1}_1}) \to p_2^!(\Omega^{(k,r)}|_{X^{\leq 1}_{\mathrm{par},1}}).$$

They induce a map $T_1 \in \mathrm{Hom}(\mathrm{R}\Gamma(X^{\leq 1}_{\mathrm{par},1}, \Omega^{(k,r)}), \mathrm{R}\Gamma(X^{\leq 1}_1, \Omega^{(k,r)}))$ and a map $T_2 \in \mathrm{Hom}(\mathrm{R}\Gamma(X^{\leq 1}_1, \Omega^{(k,r)}), \mathrm{R}\Gamma(X^{\leq 1}_{\mathrm{par},1}, \Omega^{(k,r)}))$. We let $T = T_1 \circ T_2$. We obtain maps of exact triangles for all $r \geq 2$ and $k+r > 2$:

$$\begin{tikzcd}
\mathrm{R}(p_1)_\star p_2^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "p_2^\star\mathrm{Ha}"] & \Omega^{(k,r)} \arrow[d, "p_1^\star\mathrm{Ha}"] \\
\mathrm{R}(p_1)_\star p_2^\star\Omega^{(k,r+(p-1))} \arrow[r] \arrow[d] & \Omega^{(k,r+(p-1))} \arrow[d] \\
\mathrm{R}(p_1)_\star (p_2)^\star\Omega^{(k,r+(p-1))}|_{X^{\leq 1}_{\mathrm{par},1}} \arrow[r] \arrow[d, "+1"] & \Omega^{(k,r+(p-1))}|_{X^{\leq 1}_1} \arrow[d, "+1"] \\
{} & {}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{R}(p_2)_\star p_1^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "p_1^\star \mathrm{Ha}"] & \Omega^{(k,r)} \arrow[d, "p_2^\star \mathrm{Ha}"] \\
\mathrm{R}(p_2)_\star p_1^\star\Omega^{(k,r+(p-1))} \arrow[r] \arrow[d] & \Omega^{(k,r+(p-1))} \arrow[d] \\
\mathrm{R}(p_2)_\star (p_1)^\star\Omega^{(k,r+(p-1))}|_{X_1^{\leq 1}} \arrow[r] \arrow[d, "+1"] & \Omega^{(k,r+(p-1))}|_{X^{\leq 1}_{\mathrm{par},1}} \arrow[d, "+1"] \\
{} & {}
\end{tikzcd}$$

For $r \geq 2$ and $k+r > 2$, we deduce that there is a long exact sequence on which T acts equivariantly:

$$\mathrm{H}^\star(X_1, \Omega^{(k,r)}) \stackrel{\times \mathrm{Ha}}{\to} \mathrm{H}^\star(X_1, \Omega^{(k,r+(p-1))}) \to \mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r+(p-1))}) \to .$$

7.4.2. Restriction to the rank zero locus. — For $r \geq 2+p-1$ and $k+r > 2+p-1$, we have cohomological correspondences :

$$T_1 : p_2^\star\Omega^{(k,r)}|_{X^{\leq 1}_{\mathrm{par},1}} \to p_1^!\Omega^{(k,r)}|_{X_1^{\leq 1}}, \quad \text{and} \quad T_2 : p_1^\star\Omega^{(k,r)}|_{X_1^{\leq 1}} \to p_2^!\Omega^{(k,r)}|_{X^{\leq 1}_{\mathrm{par},1}}$$

We are going to decompose these correspondences into pieces.

Lemma 7.4.2.1. — *Let $N \in \mathbb{Z}_{\geq 1}$ and let S be a scheme of characteristic p and G be a truncated Barsotti-Tate group of level N over S. Assume that the étale rank and the multiplicative rank of G is constant over S. Let $H \subset G$ be a subgroup scheme of order p. Then S is the union of three types of open and closed subschemes $S = S^{et} \coprod S^m \coprod S^{oo}$ such that over each geometric point of S^{et}, S^m and S^{oo}, the group H is respectively isomorphic to $\mathbb{Z}/p\mathbb{Z}$, μ_p, α_p.*

Proof. We can assume that S is perfect because a scheme and its perfection have the same underlying topological space and the same geometric points. We have a decomposition: $G = G^m \oplus G^{oo} \oplus G^{et}$ into multiplicative, biconnected and étale groups because the usual multiplicative-connected-étale filtration splits over a perfect scheme (one can use the Verschiebung on G and G^D to produce the splitting, see [**59**], prop. 1.3 for example). The condition that H is of étale, multiplicative or biconnected type is then obviously closed. The condition that H is étale or multiplicative is open. Thus we have open and closed components S^{et} and S^m. Their complement is S^{oo}. □

We will now make use of the following notation : if we have a map of schemes $S \to T$ and $Z \hookrightarrow T$ a locally closed subscheme, we will often write $S|_Z$ for $S \times_T Z$.

Using this lemma we can decompose certain schemes. Consider the chain of isogenies $G \to G' \to G$ over $X_{Kli}(p)$.

Lemma 7.4.2.2. — *The scheme $X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}}$ is the disjoint union of three open and closed subschemes. The étale component $(X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}})^{et}$ where the isogeny $G' \to G$ has multiplicative kernel, the multiplicative component $(X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}})^m$ where the isogeny $G' \to G$ is étale and the bi-infinitesimal component $(X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}})^{oo}$ where the isogeny $G' \to G$ has bi-connected kernel.*

Proof. We first establish the decomposition on $Y_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}}$, the locus where G is an abelian scheme. We can consider the universal order p subgroup H of $G[p]$ and apply the above lemma. This decomposition extends to $X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}}$ by the description of the local charts. □

We deduce that the scheme $X_{Kli}(p)|_{X_1^{=1}}$ (which has the same topological space as $X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}}$) is also the union of three types of components : $(X_{Kli}(p)|_{X_1^{=1}})^{et}$, $(X_{Kli}(p)|_{X_1^{=1}})^{m}$ and $(X_{Kli}(p)|_{X_1^{=1}})^{oo}$.

Lemma 7.4.2.3. — *The scheme $X^{=1}_{\mathrm{par},1}$ is the union of two types of components. The components $X^{=1,oo}_{\mathrm{par},1}$ where the kernel of the quasi-polarization $G'[p^\infty] \to (G')^t[p^\infty]$ is isomorphic to a biconnected group and the components $X^{=1,m-et}_{\mathrm{par},1}$ where the kernel of the polarization contains a multiplicative group.*

Proof. Over $X^{=1,oo}_{\mathrm{par},1}$ we consider $K(\lambda)$ the kernel of the quasi-polarization $G'[p^\infty] \to (G')^t[p^\infty]$. If G' is an abelian scheme, this group is either a connected BT_1 of height 2 and dimension 1 or an extension of an étale by a multiplicative group. We consider the group $\mathrm{Ker}F : K(\lambda) \to K(\lambda)^{(p)}$. This is a rank p group either of multiplicative type or locally isomorphic to α_p. We can apply lemma 7.4.2.1. □

Lemma 7.4.2.4. — *We have :*

$$p_2((X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}})^{oo}) \subset X^{=1,oo}_{\mathrm{par},1}$$

and

$$p_2((X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}})^{m} \cup (X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}})^{et}) \subset X^{=1,m-et}_{\mathrm{par},1}.$$

Proof. The group $\mathrm{Ker}(G' \to G)$ is a closed subgroup of $K(\lambda)$ and therefore it determines its type : it is étale or multiplicative if $K(\lambda)$ contains a mutliplicative group, and it is biconnected if $K(\lambda)$ is. □

The cohomological correspondence $T_1 : p_2^\star \Omega^{(k,r)}|_{X^{=1}_{\mathrm{par},1}} \to p_1^! \Omega^{(k,r)}|_{X_1^{=1}}$ is naturally the sum $T_1^m + T_1^{et} + T_1^{oo}$ of three cohomological correspondences where we denote by T_1^m, T_1^{et} and T_1^{oo} the projection of the cohomological correspondence T_1 respectively on the multiplicative, étale and bi-infinitesimal components.

Similarly, the cohomological correspondence $T_2 : p_1^\star \Omega^{(k,r)}|_{X_1^{=1}} \to p_2^! \Omega^{(k,r)}|_{X^{=1}_{\mathrm{par},1}}$ decomposes into $T_2 = T_2^m + T_2^{et} + T_2^{oo}$, where we denote by T_2^m, T_2^{et} and T_2^{oo} the projection of the cohomological correspondence T_2 respectively on the étale, multiplicative and bi-infinitesimal components (note that the roles of étale and multiplicative components are switched between T_1 and T_2).

We have maps on cohomology :

$$\mathrm{H}^\star(X_1^{=1}, \Omega^{(k,r)}(-D)) \stackrel{(T_2^{oo}, T_2^m + T_2^{et})}{\longrightarrow}$$

$$\mathrm{H}^\star(X^{=1,oo}_{\mathrm{par},1}, \Omega^{(k,r)}(-D)) \oplus \mathrm{H}^\star(X^{=1,m-et}_{\mathrm{par},1}, \Omega^{(k,r)}(-D)) \stackrel{(T_1^{oo}, T_1^{et} + T_1^m)}{\longrightarrow} \mathrm{H}^\star(X_1^{=1}, \Omega^{(k,r)}(-D)).$$

The first important result of this section is :

Proposition 7.4.2.1. — *For $r \geq 2 + (p-1)$ and $k + r > 2(p+1)$, the following diagrams are commutative :*

$$\begin{tikzcd}
p_2^\star\Omega^{(k,r)}|_{X^{\leq 1}_{\mathrm{par},1}} \arrow[r, "T_1"] \arrow[d, "p_2^\star \mathrm{Ha}'"] & p_1^!\Omega^{(k,r)}|_{X_1^{\leq 1}} \arrow[d, "p_1^\star \mathrm{Ha}'"] \\
p_2^\star\Omega^{(k,r+(p^2-1))}|_{X^{\leq 1}_{\mathrm{par},1}} \arrow[r, "T_1"] & p_1^!\Omega^{(k,r+(p^2-1))}|_{X_1^{\leq 1}}
\end{tikzcd}$$

$$\begin{tikzcd}
p_1^\star\Omega^{(k,r)}|_{X_1^{=1}} \arrow[r, "T_2^{et}"] \arrow[d, "p_1^\star \mathrm{Ha}'"] & p_2^!\Omega^{(k,r)}|_{X^{=1}_{\mathrm{par},1}} \arrow[d, "p_2^\star \mathrm{Ha}'"] \\
p_1^\star\Omega^{(k,r+(p^2-1))}|_{X_1^{=1}} \arrow[r, "T_2^{et}"] & p_2^!\Omega^{(k,r+(p^2-1))}|_{X^{=1}_{\mathrm{par},1}}
\end{tikzcd}$$

Moreover, $T_1^m = T_1^{oo} = 0$ *and* $T_2^m = 0$. *Finally, if* $r \geq p+2$, $T_2^{oo} = 0$ *and the diagram:*

$$\begin{tikzcd}
p_1^\star\Omega^{(k,r)}|_{X_1^{\leq 1}} \arrow[r, "T_2"] \arrow[d, "p_1^\star \mathrm{Ha}'"] & p_2^!\Omega^{(k,r)}|_{X^{\leq 1}_{\mathrm{par},1}} \arrow[d, "p_2^\star \mathrm{Ha}'"] \\
p_1^\star\Omega^{(k,r+(p^2-1))}|_{X_1^{\leq 1}} \arrow[r, "T_2"] & p_2^!\Omega^{(k,r+(p^2-1))}|_{X^{\leq 1}_{\mathrm{par},1}}
\end{tikzcd}$$

is commutative.

Proof. We first deal with the operator T_1. We notice that it is enough to prove the claim over $X_{Kli}(p)|_{X_1^{=1}}$ which is dense in the support of the Cohen-Macaulay sheaf $p_1^!\Omega^{(k,r+(p^2-1))}|_{X_1^{\leq 1}}$. We will actually work over the interior of the moduli space $Y_{Kli}(p)|_{X_1^{=1}}$ which is dense. We can treat separately the different connected components. We first deal with the components of étale type. We take some simplifying notations. Let $A = Y^{=1}_{\mathrm{par},1}$ and $\hat{A}$ be the completion of $Y_{\mathrm{par},1}$ along this locally closed subscheme. Let $B = Y_1^{=1}$ and $\hat{B}$ be the completion of Y_1 along B. The ideal of definition of $\hat{A}$ and $\hat{B}$ are $(p, \mathrm{Ha}.\omega^{(1-p)})$. Finally, consider $\hat{C}$, the completion of $X_{Kli}(p)$ along $(X_{Kli}(p)|_{Y^{=1}_{\mathrm{par},1}})^{et} = (p_2^{-1}(A))^{et}$ (or the completion along $(p_1^{-1}(B))^{et}$, it makes no difference). We consider the following restriction of the correspondence (we keep using the same notations for the projections):

$$\begin{tikzcd}
& \hat{C} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\
\hat{A} & & \hat{B}
\end{tikzcd}$$

We observe that the map p_1 is finite flat because $\hat{B}$ is regular, p_1 is finite (because we removed the p-rank 0 locus) and dominant, and $\hat{C}$ is Cohen-Macaulay.

We are now going to give a description of the cohomological correspondence T_1 restricted to $\hat{C}$ (10). Consider the following commutative diagram over $\hat{C}$:

10. Since the map p_1 is finite flat away from the p-rank 0 locus and the boundary, it makes sense to base change the cohomological correspondence to an arbitrary (formal) scheme by section 4.2.5. Also, the reader who wishes to avoid using formal schemes could replace $\hat{C}$ by some open dense affine formal subscheme Spf V and then replace Spf V by Spec V.

$$\begin{tikzcd} G[p^\infty]^m \arrow[r] \arrow[d] & G[p^\infty] \arrow[r] \arrow[d] & G[p^\infty]/G[p^\infty]^m \arrow[d] \\ G'[p^\infty]^m \arrow[r] & G'[p^\infty] \arrow[r] & G'[p^\infty]/G'[p^\infty]^m \end{tikzcd}$$

The middle vertical map is the universal isogeny. The exponent m means the multiplicative part of the BT. The left vertical map is an isomorphism and the right vertical map is multiplication by p composed with an isomorphism. The non-normalized map $p_2^\star\omega \to p_1^\star\omega$ can be normalized by p^{-1} to give an isomorphism. The non-normalized map $p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)}$ can be normalized by p^{-r}. Under the isomorphism $p_2^\star\omega^{(p-1)} \simeq p_1^\star\omega^{(p-1)}$ we have $p_1^\star\mathrm{Ha} = p_2^\star\mathrm{Ha}$ by lemma 6.3.4.2 (applied on the formal scheme $\hat{C}$ which we view as the inductive limit of the schemes defined by the zero locus of increasing powers of the ideal of definition $(p, p_1^\star\mathrm{Ha}.p_1^\star\omega^{(1-p)})$). We now define $C = V(p, p_1^\star\mathrm{Ha}.p_1^\star\omega^{1-p}) \hookrightarrow \hat{C}$ (we could have used instead $p_2^\star\mathrm{Ha}.p_2^\star\omega^{1-p}$). The fundamental class $p_1^\star\mathscr{O}_{\hat{B}} \to p_1^!\mathscr{O}_{\hat{B}}$ is divisible by p^2 as we can check over the ordinary locus as in lemma 7.1.1. We can thus write the cohomological correspondence T_1 over $\hat{C}$ as the composition of a normalized map $p_2^\star\Omega^{(k,r)}|_{\hat{C}} \to p_1^\star\Omega^{(k,r)}|_{\hat{C}}$ and the map which is the tensor product with $p_1^\star\Omega^{(k,r)}$ of a normalized fundamental class. We are using here 4.2.5 to check the compatibility of the fundamental class with base change via the morphism $\hat{B} \to X$.

After this analysis, we can prove the commutativity of the diagram of the proposition over C. We can write the diagram as the composition of two diagrams

$$\begin{tikzcd} p_2^\star\Omega^{(k,r)}|_A \arrow[r] \arrow[d, "p_2^\star\mathrm{Ha}'"] & p_1^\star\Omega^{(k,r)}|_B \arrow[r] \arrow[d, "p_1^\star\mathrm{Ha}'"] & p_1^!\Omega^{(k,r)}|_B \arrow[d, "p_1^\star\mathrm{Ha}'"] \\ p_2^\star\Omega^{(k,r+(p^2-1))}|_A \arrow[r] & p_1^\star\Omega^{(k,r+(p^2-1))}|_B \arrow[r] & p_1^!\Omega^{(k,r+(p^2-1))}|_B \end{tikzcd}$$

The commutativity of the left square follows from lemma 6.3.4.2 and the commutativity of the right square is obvious.

We now deal with the components of $X_{Kli}(p)|_{X^{=1}_{\mathrm{par},1}}$ of multiplicative and bi-infinitesimal type. We have denoted by T_1^{oo} and T_1^m the restriction of the cohomological correspondence to bi-infinitesimal and multiplicative components. Over these components, we will actually prove that the cohomological correspondences T_1^{oo} and T_1^m are zero. The commutativity is thus obvious.

Let Spec $l \to X_1^{=1}$ be a point corresponding to a p-rank 1 principally polarized abelian surface A over an algebraically closed field l of characteristic p. Consider the lift $\tilde{A} \to$ Spec $W(l)$ with associated Barsotti-Tate group $\mu_{p^\infty} \oplus E[p^\infty] \oplus \mathbb{Q}_p/\mathbb{Z}_p$ with $E[p^\infty]$ the Barsotti-Tate group of a supersingular elliptic curve over $W(l)$. Consider the following commutative diagram :

$$\begin{tikzcd} \mathrm{H}^0(X_{Kli}(p) \times_{X,p_1} \mathrm{Spec}\ W(l), p_2^\star\Omega^{(k,r)}) \arrow[r, "T_1^{oo}"] \arrow[d] & \mathrm{H}^0(\mathrm{Spec}\ W(l), \Omega^{(k,r)}) \arrow[d] \\ \mathrm{H}^0(X_{Kli}(p)_1 \times_{X_1,p_1} \mathrm{Spec}\ l, p_2^\star\Omega^{(k,r)}) \arrow[r, "T_1^{oo}"] & \mathrm{H}^0(\mathrm{Spec}\ l, \Omega^{(k,r)}) \end{tikzcd}$$

All vertical maps are surjective because all schemes are affine. Let $f \in \mathrm{H}^0(X_{Kli}(p) \times_{\mathfrak{X},p_1} \operatorname{Spec} \ W(l), p_2^\star \Omega^{(k,r)})$. Then by definition and section 4.2.5,

$$T_1^{00} f(\tilde{A}, \mu) = \frac{1}{p^{2+r}} \sum_{L \subset \tilde{A}[p],\ L^\perp \text{biconnected}} f(\tilde{A}/L, \mu')$$

In this formula, $\mu : W(l)^2 \simeq e^\star \Omega^1_{\tilde{A}}$ is an isomorphism. Let $\mathbb{C}$ be the completion of an algebraic closure of $W(l)[1/p]$. Then

$$\mu' : \mathbb{C}^2 \stackrel{\psi \otimes 1}{\to} e^\star \Omega^1_{\tilde{A}} \otimes \mathbb{C} \stackrel{\mathrm{d}\xi^{-1}}{\to} e^\star \Omega^1_{\tilde{A}/L} \otimes \mathbb{C}$$

where $\xi : \tilde{A} \to \tilde{A}/L$ is the isogeny. We have a non-canonical decomposition over $\mathcal{O}_{\mathbb{C}}$: $L = L^m \oplus L^0 \oplus L^{et}$ where each of these groups is multiplicative/bi-connected/étale of order p. Moreover, it is easy to see that L^0 has degree $\frac{1}{p+1}$ in the sense of [**20**] (see [**62**], example A.2.2). As a result, the map : $e^\star \Omega^1_{\tilde{A}/L} \to e^\star \Omega^1_{\tilde{A}}$ has elementary divisors (p, ϖ) with the p-adic valuation of ϖ (normalized by $v(p) = 1$) equal to $\frac{1}{p+1}$. If $r + k > 2(p+1)$ then $\frac{1}{p^{2+r}} f(\tilde{A}/L, \mu') \in \mathfrak{m}_{\mathcal{O}_{\mathbb{C}}}$ and as a result, $T_1^{oo} f(\tilde{A}, \mu) \mod p = 0$. The proof of the vanishing of T_1^m is similar (actually one sees that T_1^m is zero as soon as $k + 2r > r + 2$ as in the proof of proposition 7.4.1.1.

The commutativity of the second diagram follows easily from the observation that the isogeny $G' \to G$ is étale. The proof of the vanishing of T_2^m or T_2^{oo} (if $r \geq p+2$) is similar to the proof of the vanishing of T_1^{oo}. The commutativity of the last diagram follows. □

Remark 7.4.2.1. —

1. For $r = p+1$, one can prove that the correspondence T_2^{oo} does not commute with Ha' and does not vanish and therefore the operator T_2 does not commute with Ha'.
2. Our vanishing condition for T_1^{oo} is not optimal because we have not used estimates on the fundamental class. It will nevertheless be sufficient for our purpose.

Corollary 7.4.2.1. — *We have $T = T_1 \circ T_2 = T_1^{et} \circ T_2^{et}$ as endomorphisms of $\mathrm{H}^\star(X_1^{=1}, \omega^{(k,r)})$ when $r \geq p+1$ and $k + r > 2(p+1)$.*

Proof. This follows from the vanishing $T_1^m = T_1^{oo} = T_2^m = 0$. □

Lemma 7.4.2.5. — *The section $p_1^\star \mathrm{Ha}'$ is not a zero divisor in $X_{Kli}(p)_1 \times_{X_1} X_1^{\leq 1}$.*

Proof. The scheme $X_{Kli}(p)_1 \times_{X_1} X_1^{\leq 1}$ is Cohen-Macaulay and the p-rank 0 locus has codimension 1 [(11)]. □

By proposition 7.4.2.1 and proposition 4.1.2.1, we have for $r \geq p^2 + p = 2 + p - 1 + p^2 - 1$ and $k + r > 2(p+1) + p^2 - 1$ a cohomological correspondence :

$$T_1 : p_2^\star \Omega^{(k,r)}|_{X^{=0}_{\mathrm{par},1}} \to p_1^! \Omega^{(k,r)}|_{X_1^{=0}}.$$

Moreover, we have for all $r \geq 2 + p - 1$ and $k + r > 2(p+1) + p^2 - 1$ a commutative diagram of long exact sequences :

11. The p-rank 0 locus $X_1^{=0}$ is of dimension 1 and the map p_1 is bijective over the dense open subscheme of $X_1^{=0}$ parametrizing abelian surfaces which are not isomorphic to a product of supersingular elliptic curves. On the other hand, the map p_1 is a $\mathbb{P}^1$-fibration over the finite set of superspecial points SS of $X_1^{=0}$ parametrizing abelian surfaces isomorphic to a product of supersingular elliptic curves.

$$\begin{tikzcd}
\mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r)}) \arrow[r, "\mathrm{Ha}'"] & \mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r+(p^2-1))}) \arrow[r] & \mathrm{H}^\star(X_1^{\leq 0}, \Omega^{(k,r+(p^2-1))}) \arrow[r] & {} \\
\mathrm{H}^\star(X_{\mathrm{par},1}^{\leq 1}, \Omega^{(k,r)}) \arrow[u, "T_1"] \arrow[r, "\mathrm{Ha}'"] & \mathrm{H}^\star(X_{\mathrm{par},1}^{\leq 1}, \Omega^{(k,r+(p^2-1))}) \arrow[u, "T_1"] \arrow[r] & \mathrm{H}^\star(X_{\mathrm{par},1}^{\leq 0}, \Omega^{(k,r+(p^2-1))}) \arrow[u, "T_1"] \arrow[r] & {}
\end{tikzcd}$$

The following proposition is absolutely crucial to the argument of the paper.

Proposition 7.4.2.2. — *There is a constant C which does not depend on the prime-to-p level K^p such that for all $k \geq C$ and all $r \geq p^2+p$, the cohomological correspondence $T_1 : p_2^\star\Omega^{(k,r)}|_{X_{\mathrm{par},1}^{=0}} \to p_1^!\Omega^{(k,r)}|_{X_1^{=0}}$ is zero.*

Proof. Let $\mathcal{I} \subset \mathscr{O}_X$ be the ideal of the closed subscheme $X_1^{=0}$. In a local trivialization of the sheaf ω, the ideal is generated by p and lifts of Ha and Ha$'$. Since $X_1^{=0}$ is a local complete intersection in X, we deduce that $\mathscr{O}_{X_1^{=0}}$ has finite tor dimension as an $\mathscr{O}_X$-module.

The cohomological correspondence $T_1 : p_2^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$ induces a cohomological correspondence

$$p_2^\star\Omega^{(k,r)} \to p_1^!(\Omega^{(k,r)} \otimes \mathscr{O}_{X_1^{=0}})$$

thanks to proposition 4.1.2.1. Moreover, thanks to proposition 7.4.2.1, this cohomological correspondence factors through the map $T_1 : p_2^\star\Omega^{(k,r)}|_{X_{\mathrm{par},1}^{=0}} \to p_1^!\Omega^{(k,r)}|_{X_1^{=0}}$ of the proposition. Thus, in order to prove the proposition it is enough to show that there is a constant C such that for all $k \geq C$, the map $T_1 : p_2^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}$ factors through $T_1 : p_2^\star\Omega^{(k,r)} \to \mathcal{I}p_1^!\Omega^{(k,r)}$.

We now need to analyze one more time the construction of T_1. Let $\Psi : G \to G'$ be the universal isogeny. Its differential is a map $\mathrm{d}\Psi : p_2^\star\Omega^1 \to p_1^\star\Omega^1$. Call $\Psi_{k,r} : p_2^\star\Omega^{(k,r)} \to p_1^\star\Omega^{(k,r)}$ the map obtained by applying the functor $\mathrm{Sym}^k \otimes \det^r$. The determinant $\Psi_{0,1} : p_2^\star\omega^1 \to p_1^\star\omega^1$ factors through $pp_1^\star\omega^1$ (check this over the tube of the ordinary locus).

Secondly, we have a non-normalized fundamental class $\Theta : p_1^\star\mathscr{O}_X \to p_1^!\mathscr{O}_X$. Tensoring with $\Omega^{(k,r)}$ gives a non-normalized map

$$\Theta_{k,r} : p_1^\star\Omega^{(k,r)} \to p_1^!\Omega^{(k,r)}.$$

We have established in lemma 7.1.1 that the composite $\Theta_{k,r} \circ \Psi_{k,r}$ is divisible by p^{2+r} when $r \geq 1$, and the cohomological correspondence T_1 is $p^{-2-r}\Theta_{k,r} \circ \Psi_{k,r}$.

To prove the proposition, it is enough to show that there is a constant C such that

$$\Theta_{k,r} \circ \Psi_{k,r}(p_2^\star\Omega^{(k,r)}) \subset p^{2+r}\mathcal{I}p_1^!\Omega^{(k,r)}$$

for $k \geq C$.

The problem is local. Let Spec A be an open in $X_{Kli}(p)$ and $I = p_1^\star\mathcal{I}(\mathrm{Spec}\ A)$. Set $M_2 = p_2^\star\Omega^1(\mathrm{Spec}\ A)$, $M_3 = p_1^\star\Omega^1(\mathrm{Spec}\ A)$, $M_1 = p_1^!\Omega^1(\mathrm{Spec}\ A)$.

Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be the minimal prime ideals in Spec A/I. One sees that for each i, $\mathrm{d}\Psi(M_2) \subset \mathfrak{p}_iM_3$ as the differential $\mathrm{d}\Psi : \Omega^1_{G'} \to \Omega^1_G$ is 0 modulo $\mathfrak{p}_i$ because the isogeny $\Psi : G \to G'$ factors through the Frobenius map at $\mathfrak{p}_i$ by lemma 7.4.2.6 below.

We deduce that

$$\Theta_{k,r} \circ \Psi_{k,r}(M_2) \subset p^{2+r}M_1 \bigcap (\cap_i p^r\mathfrak{p}_i^k)M_1.$$

By Artin-Rees lemma, there exists $C(A) \geq 0$ such that $p^2A \bigcap \cap_i\mathfrak{p}_i^{C(A)} \subset p^2I$. It follows that for all $k \geq C(A)$, $\Theta_{k,r} \circ \Psi_{k,r}(M_2) \subset p^{2+r}IM_1$. Since $X_{Kli}(p)$ is quasi-compact, it can be covered by finitely many affines as above.

We finally justify that the constant C does not depend on K^p. Say we have $K_1^p \subset K_2^p$ two open compact subgroups of $\mathrm{GSp}_4(\mathbb{A}_f^p)$ and we denote by $K_1 = K_1^p\mathrm{GSp}_4(\mathbb{Z}_p)$ and $K_2 = K_2^p\mathrm{GSp}_4(\mathbb{Z}_p)$. Let C_1 and C_2 be constants that fulfill the conclusion of our proposition for the levels K_1 and K_2. We claim that the constant $C = \inf\{C_1, C_2\}$ works as well for both level. Indeed, there is a map $X_{K_1} \to X_{K_2}$ which correspond to the change of level away from p and which is finite étale away from the boundary (and we have similar maps for the other level structures at p). All our constructions at level K_1 are obtained by base change from level K_2 (they clearly do not depend on the level structure away from p). In particular the map of the lemma for the level K_1 is obtained by base change from the same map at level K_2 under the map $X_{Kli(p)K_1} \to X_{Kli(p)K_2}$ which is finite étale away from the boundary. The map of the lemma is supported on the p-rank 0 locus which does not meet the boundary and therefore C works at level K_1 and K_2.

□

Lemma 7.4.2.6. — *Let $A \to \mathrm{Spec}\ l$ be an abelian surface of p-rank 0 over a field l of characteristic p. Let $L \subset A[p]$ be a group scheme of order p^3. Then $\mathrm{Ker}\ F \subset L$.*

Proof. We have a perfect pairing $A[p] \times A[p]^D \to \mu_p$. The orthogonal of $\mathrm{Ker}\ F \subset A[p]$ is $\mathrm{Ker}\ F \subset A[p]^D$. The group $L^\perp \subset A[p]^D$ is a group of rank p and is necessarily killed by F, since A has p-rank 0. It follows that $L^\perp \subset \mathrm{Ker}\ (F : A[p]^D \to A[p]^D)$ and that $\mathrm{Ker}\ F \subset L$. □

Remark 7.4.2.2. — Finding an explicit bound for the constant C appearing in proposition 7.4.2.2 would be a first important step towards proving an integral classicity theorem for all cohomological degrees improving on theorem 1.1, point 2. This would require new ideas and a deeper analysis of the correspondence.

8. Finiteness of the ordinary cohomology

The purpose of this section is to study the T-ordinary part of the cohomology of automorphic vector bundles over various subsets of the Shimura variety. The results of section 7 provide the necessary background material.

8.1. Finiteness of the ordinary cohomology on $X_1^{=1}$. — We begin with the following lemma.

Lemma 8.1.1. — *For all $r \geq 2 + (p-1)$ and all $k > p+1$, the action of T on $\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D))$ is locally finite.*

Proof. We let $\mathrm{Ha}' \in \mathrm{H}^0(X_1^{\leq 1}, \omega^{p^2-1})$ be the second Hasse invariant. Since $\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D)) = \mathrm{colim}_n \mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r+n(p^2-1))}(-D))$ where the inductive limit is over multiplication by Ha' and $\mathrm{Ha}'T = T\mathrm{Ha}'$ by proposition 7.4.2.1 and corollary 7.4.2.1, the lemma follows. □

Using the result of section 2.3, we can define an ordinary projector e associated to T on $\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D))$ for $k > p+1$, $r \geq p+1$.

Lemma 8.1.2. —

1. *If $r \geq 2 + (p-1)$ and $k > p+1$, we have an equality of morphisms $\mathrm{Ha}'T = T\mathrm{Ha}' : \mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r+(p^2-1))}(-D))$.*
2. *If $r > 2 + (p-1)$ and $k > p+1$, we have an equality of morphisms $\mathrm{Ha}'T = T\mathrm{Ha}'$: $\mathrm{H}^i(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) \to \mathrm{H}^i(X_1^{\leq 1}, \Omega^{(k,r+(p^2-1))}(-D))$ for all i.*

Proof. The first point follows from the fact that we have a T-equivariant embedding $\mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) \hookrightarrow \mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D))$ for $r \geq 2 + (p-1)$ and $k > p+1$ and that the identity holds for the map $\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X_1^{=1}, \Omega^{(k,r+(p^2-1))}(-D))$ by corollary 7.4.2.1. Point 2 follows from proposition 7.4.2.1. $\square$

Remark 8.1.1. — We have not been able to establish that $T\mathrm{Ha}' = \mathrm{Ha}'T$ as morphisms : $\mathrm{H}^i(X_1^{\leq 1}, \Omega^{(k,p+1)}(-D)) \to \mathrm{H}^i(X_1^{\leq 1}, \Omega^{(k,p^2+p))}(-D))$ for k large enough and $i \geq 1$, although we believe this should be true. (12)

Proposition 8.1.1. — *There is a constant C (see prop. 7.4.2.2) which is independent of the level K^p such that for $k \geq C$ and $r \geq p+1$ we have isomorphisms :*

$$e\mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) = e\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D)).$$

If $r \geq p+2$, we moreover have $e\mathrm{H}^i(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) = e\mathrm{H}^i(X_1^{=1}, \Omega^{(k,r)}(-D)) = 0$ for $i = 1, 2$.

Proof. Consider the following exact sequence of sheaves over $X_1^{\leq 1}$ or $X_{\mathrm{par},1}^{\leq 1}$:

$$0 \to \Omega^{(k,r)}(-D) \to \Omega^{(k,r+(p^2-1))}(-D) \to \Omega^{(k,r+(p^2-1))}(-D)/(\mathrm{Ha}') \to 0$$

Applying the functor global sections, we get a commutative diagram of long exact sequences :

$$\begin{tikzcd}
\mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) \arrow[r, "\mathrm{Ha}'"] & \mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r+(p^2-1))}(-D)) \arrow[r] & \mathrm{H}^\star(X_1^{\leq 0}, \Omega^{(k,r+(p^2-1))}(-D)) \arrow[r] & {} \\
\mathrm{H}^\star(X_{\mathrm{par},1}^{\leq 1}, \Omega^{(k,r)}(-D)) \arrow[u, "T_1"] \arrow[r, "\mathrm{Ha}'"] & \mathrm{H}^\star(X_{\mathrm{par},1}^{\leq 1}, \Omega^{(k,r+(p^2-1))}(-D)) \arrow[u, "T_1"] \arrow[r] & \mathrm{H}^\star(X_{\mathrm{par},1}^{\leq 0}, \Omega^{(k,r+(p^2-1))}(-D)) \arrow[u, "T_1"] \arrow[r] & {}
\end{tikzcd}$$

Ideally, we would like to apply the ordinary projectors for T and $T_2 \circ T_1$ to the top and bottom vertical lines of this diagram, but all the maps may not be equivariant by lemma 8.1.2, so some care is necessary.

The map

$$T_1 : \mathrm{H}^\star(X_{\mathrm{par},1}^{=0}, \Omega^{(k,r+(p^2-1))}(-D)) \to \mathrm{H}^\star(X_1^{=0}, \Omega^{(k,r+(p^2-1))}(-D))$$

is the zero map by proposition 7.4.2.2. If $f \in e\mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r+(p^2-1))}(-D))$, we deduce that there exists $f' \in \mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r)}(-D))$ mapping to f. It follows from lemma 8.1.2 that on degree 0 cohomology we have $T\mathrm{Ha}' = \mathrm{Ha}'T$ so that the injective map

$$\mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) \hookrightarrow \mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r+(p^2-1))}(-D))$$

commutes with the projector e. We deduce that the map

$$e\mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) \to e\mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r+(p^2-1))}(-D))$$

is an isomorphism (it is obviously injective, and surjective because ef' maps to f). Passing to the limit over multiplication by $(\mathrm{Ha}')^n$ we get that $e\mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,r)}(-D)) = e\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D))$.

When $r \geq p+2$, we can apply the ordinary projector associated to $T = T_1 \circ T_2$ on $\mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r)}(-D))$ and $\mathrm{H}^\star(X_1^{\leq 1}, \Omega^{(k,r+(p^2-1))}(-D))$ and to $T_2 \circ T_1$ on

12. In some sense, we are paying here the price for our indirect definition of the operator T as a composition of two operators.

$\mathrm{H}^\star(X^{\leq 1}_{\mathrm{par},1}, \Omega^{(k,r)}(-D))$ and $\mathrm{H}^\star(X^{\leq 1}_{\mathrm{par},1}, \Omega^{(k,r+(p^2-1))}(-D))$, because all maps are equivariant by lemma 8.1.2 point 2 (and a slight generalization of it for $T_2 \circ T_1$ instead of T). The map T_1 is an isomorphism between the ordinary parts. On the other hand,

$$T_1 : \mathrm{H}^\star(X^{=0}_{\mathrm{par},1}, \Omega^{(k,r+(p^2-1))}(-D)) \to \mathrm{H}^\star(X^{=0}_1, \Omega^{(k,r+(p^2-1))}(-D))$$

is the zero map by proposition 7.4.2.2. It follows that

$$e\mathrm{H}^\star(X^{\leq 1}_1, \Omega^{k,r}(-D)) = e\mathrm{H}^\star(X^{\leq 1}_1, \Omega^{(k,r+(p^2-1))}(-D)).$$

Passing to the limit over multiplication by $(\mathrm{Ha}')^n$ we get that $e\mathrm{H}^\star(X^{\leq 1}_1, \Omega^{(k,r)}(-D)) = e\mathrm{H}^\star(X^{=1}_1, \Omega^{(k,r)}(-D))$. Finally, for all r, the sheaf $\Omega^{(k,r)}(-D)$ is acyclic relatively to the minimal compactification by thm 6.2.2.1. Moreover, the rank 1 locus $X^{=1}_1$ has affine image in the minimal compactification. As a result $\mathrm{H}^i(X^{=1}_1, \Omega^{(k,r)}(-D)) = 0$ for $i > 0$.

□

Remark 8.1.2. — If we had been able to establish that $T\mathrm{Ha}' = \mathrm{Ha}'T$ as morphisms $\mathrm{H}^i(X^{\leq 1}_1, \Omega^{(k,p+1)}(-D)) \to \mathrm{H}^i(X^{\leq 1}_1, \Omega^{(k,p^2+p))}(-D))$ for all i, we would have deduce that $e\mathrm{H}^i(X^{\leq 1}_1, \Omega^{(k,p+1)}(-D)) = e\mathrm{H}^i(X^{=1}_1, \Omega^{(k,p+1)}(-D))$ for all i.

8.2. Finiteness of the cohomology on $X^{\geq 1}_1$. — We now turn to understand the cohomology of $X^{\geq 1}_1 = X_1 \setminus X^{=0}_1$.

Lemma 8.2.1. — *The action of T on $\mathrm{R}\Gamma(X^{\geq 1}_1, \Omega^{(k,r)}(-D))$ is locally finite for $k > p+1$ and $r \geq 2$.*

Proof. Consider the following resolution over $X^{\geq 1}_1$ of the sheaf $\Omega^{(k,r)}(-D)$:

$$0 \to \Omega^{(k,r)}(-D) \to \mathrm{colim}_{n,\times \mathrm{Ha}} \Omega^{(k,r+(p-1)n)}(-D) \to \mathrm{colim}_n \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n \to 0.$$

All sheaves are acyclic relatively to the minimal compactification by thm 6.2.2.1. Moreover, the support of $\mathrm{colim}_{n,\times \mathrm{Ha}} \Omega^{(k,r+(p^2-1)n)}(-D)$ is the rank 2 locus which is affine in the minimal compactification. The support of $\mathrm{colim}_n \Omega^{(k,r+(p^2-1)n)}(-D)/(\mathrm{Ha})^n$ is the rank 1 locus which is also affine in the minimal compactification. It follows that the above sequence is an acyclic resolution of the sheaf $\Omega^{(k,r)}(-D)$ over $X^{\geq 1}_1$.

The cohomology $\mathrm{R}\Gamma(X^{\geq 1}_1, \Omega^{(k,r)}(-D))$ is thus represented by the following complex :

$$\mathrm{H}^0(X^{=2}_1, \Omega^{(k,r)}(-D)) \to \mathrm{colim}_n \mathrm{H}^0(X^{\geq 1}_1, \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n)$$

We will see that the action of T is locally finite on both terms. Since

$$\mathrm{H}^0(X^{=2}_1, \Omega^{(k,r)}(-D)) = \mathrm{colim}_n \mathrm{H}^0(X_1, \Omega^{(k,r+n(p-1))}(-D))$$

where the transition maps are given by multiplication by Ha and T commutes with multiplication by Ha by proposition 7.4.1.1, the action of T is locally finite on the first term. We now prove that it is locally finite on the second term. It is enough to see that it is locally finite on $\mathrm{H}^0(X^{\geq 1}_1, \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n)$. For $n = 1$, this follows from lemma 8.1.1. For general n, we use induction, lemma 8.1.1, lemma 2.1.1 and the following exact sequence :

$$0 \to \mathrm{H}^0(X^{\geq 1}_1, \Omega^{(k,r+(p-1)(n-1))}(-D)/\mathrm{Ha}^{n-1}) \to \mathrm{H}^0(X^{\geq 1}_1, \Omega^{(k,r+(p-1)n)}(-D)/\mathrm{Ha}^n)$$
$$\to \mathrm{H}^0(X^{\geq 1}_1, \Omega^{(k,r+(p-1)n)}(-D)/\mathrm{Ha}).$$

□

We can now prove the following proposition, which is one of the main technical results of the paper :

Proposition 8.2.1. — *For all $r \geq 2$ and $k \geq C$ (see prop 7.4.2.2), $e\mathrm{R}\Gamma(X_1^{\geq 1}, \Omega^{(k,r)}(-D))$ is a perfect complex of amplitude $[0,1]$ of $\mathbb{F}_p$-vector spaces.*

For all $r \geq 3$ and $k \geq C$, the map $e\mathrm{R}\Gamma(X_1, \Omega^{(k,r)}(-D)) \to e\mathrm{R}\Gamma(X_1^{\geq 1}, \Omega^{(k,r)}(-D))$ is a quasi-isomorphism.

For all $k \geq C$, $e\mathrm{H}^0(X_1^{\geq 1}, \Omega^{(k,2)}(-D)) = e\mathrm{H}^0(X_1, \Omega^{(k,2)}(-D))$ and the map $e\mathrm{H}^1(X_1, \Omega^{(k,2)}(-D)) \to e\mathrm{H}^1(X_1^{\geq 1}, \Omega^{(k,2)}(-D))$ is injective.

Proof. Since $X_1^{=0}$ is of codimension 2 in X_1, and X_1 is smooth, we have unconditionally $\mathrm{H}^0(X_1^{\geq 1}, \Omega^{(k,r)}(-D)) = \mathrm{H}^0(X_1, \Omega^{(k,r)}(-D))$ and in particular $e\mathrm{H}^0(X_1^{\geq 1}, \Omega^{(k,r)}(-D)) = e\mathrm{H}^0(X_1, \Omega^{(k,r)}(-D))$.

We consider the following exact sequence over X_1 :

$$0 \to \Omega^{(k,r)}(-D) \to \mathrm{colim}_{n,\times \mathrm{Ha}} \Omega^{(k,r+(p-1)n)}(-D) \to \mathrm{colim}_n \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n \to 0$$

From the above short exact sequence of sheaves we obtain the following long exact sequences :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(X_1^{\geq 1}, \Omega^{(k,r)}(-D)) \arrow[r] & \mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D)) \arrow[r] & {} \\
0 \arrow[r] & \mathrm{H}^0(X_1, \Omega^{(k,r)}(-D)) \arrow[r] \arrow[u] & \mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D)) \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{colim}\mathrm{H}^0(X_1^{\geq 1}, \Omega^{(k,r+n(p-1))}(-D)/\mathrm{Ha}^n)) \arrow[r] & \mathrm{H}^1(X_1^{\geq 1}, \Omega^{(k,r)}(-D)) \arrow[r] & 0 \\
\mathrm{colim}\mathrm{H}^0(X_1, \Omega^{(k,r+n(p-1))}(-D)/\mathrm{Ha}^n)) \arrow[r] \arrow[u] & \mathrm{H}^1(X_1, \Omega^{(k,r)}(-D)) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and the isomorphisms : $\mathrm{colim}\mathrm{H}^i(X_1, \Omega^{(k,r+n(p-1))}(-D)/\mathrm{Ha}^n)) \simeq \mathrm{H}^{i+1}(X_1, \Omega^{(k,r)}(-D))$ for $i = 1, 2$.

The first two vertical maps in the diagram are isomorphisms. We now check that $e\mathrm{H}^i(X_1, \Omega^{(k,r+n(p-1))}(-D)/\mathrm{Ha}^n)) = 0$ for all $n \geq 1$, $k \geq C$, $r \geq 3$ and $i \in \{1, 2\}$. The case $n = 1$ follows from proposition 8.1.1. For the general case, we take the long exact sequence of cohomology associated to the short exact sequence of sheaves :

$$0 \to \Omega^{(k,r+n(p-1))}(-D)/\mathrm{Ha}^n \overset{\mathrm{Ha}}{\to} \Omega^{(k,r+(n+1)(p-1))}(-D)/\mathrm{Ha}^{n+1} \to$$
$$\Omega^{(k,r+(n+1)(p-1))}(-D)/\mathrm{Ha} \to 0.$$

We now check that $e\mathrm{H}^0(X_1, \Omega^{(k,r+n(p-1))}(-D)/\mathrm{Ha}^n)) \to e\mathrm{H}^0(X_1^{\geq 1}, \Omega^{(k,r+n(p-1))}(-D)/\mathrm{Ha}^n))$ is bijective for all $n \geq 1$, $k \geq C$ and $r \geq 3$. We prove this by induction on n. The case $n = 1$ follows from proposition 8.1.1. The general case follows by taking one more time the long exact sequence of cohomology associated to the following short exact sequence of sheaves (when $r \geq 3$, there is no $e\mathrm{H}^1$ as we just checked) :

$$0 \to \Omega^{(k,r+n(p-1))}(-D)/\mathrm{Ha}^n \overset{\mathrm{Ha}}{\to} \Omega^{(k,r+(n+1)(p-1))}(-D)/\mathrm{Ha}^{n+1} \to$$
$$\Omega^{(k,r+(n+1)(p-1))}(-D)/\mathrm{Ha} \to 0.$$

We finally prove that $e\mathrm{H}^1(X_1, \Omega^{(k,2)}(-D)) \to e\mathrm{H}^1(X_1^{\geq 1}, \Omega^{(k,2)}(-D))$ is an injection of finite dimensional vector spaces when $k \geq C$. We use the long exact sequence associated to

$$0 \to \Omega^{(k,2)}(-D) \overset{\mathrm{Ha}}{\to} \Omega^{(k,p+1)}(-D) \to \Omega^{(k,p+1)}(-D)/\mathrm{Ha} \to 0$$

and the claim follows from the isomorphism

$$e\mathrm{H}^1(X_1, \Omega^{(k,p+1)}(-D)) \to e\mathrm{H}^1(X_1^{\geq 1}, \Omega^{(k,p+1)}(-D))$$

that we just established and the isomorphism of proposition 8.1.1 :

$$e\mathrm{H}^0(X_1^{\leq 1}, \Omega^{(k,p+1)}(-D)) \to e\mathrm{H}^0(X_1^{=1}, \Omega^{(k,p+1)}(-D)).$$

□

Remark 8.2.1. — If we had been able to establish the claims of remark 8.1.2, we could improve the above proposition and show that for all $r \geq 2$ and $k \geq C$, the map $e\mathrm{R}\Gamma(X_1, \Omega^{(k,r)}(-D)) \to e\mathrm{R}\Gamma(X_1^{\geq 1}, \Omega^{(k,r)}(-D))$ is a quasi-isomorphism.

9. Families of sheaves

In this section we give the construction of certain p-adic sheaves, defined over the p-rank at least one locus, which interpolate the classical automorphic sheaves in a one-dimensional direction of the weight space.

9.1. Deep Klingen level structure and Igusa tower. — We introduce certain level structure that will allow us to define p-adic sheaves.

9.1.1. Deep Klingen level structure. — We let $X^{\geq 1}_{Kli}(p^m)_n \to X^{\geq 1}_n$ be the moduli space of subgroups $H_m \subset G[p^m]$ where H_m is étale locally isomorphic to μ_{p^m}. We denote by $X^{ord}_{Kli}(p^m)_n$ or $X^{=2}_{Kli}(p^m)_n$ the ordinary locus of $X^{\geq 1}_{Kli}(p^m)_n$.

Remark 9.1.1.1. — We have previously considered the space $X_{Kli}(p)$ (a toroidal compactification of the Shimura variety with Klingen level at p, let us further assume in this remark that the polyhedral cone decomposition is the same for X and $X_{Kli}(p)$). We warn the reader that $X^{\geq 1}_{Kli}(p)_1$ is a strict open subscheme of $X_{Kli}(p)_1 \times_{X_1} X_1^{\geq 1}$. This is the open subscheme where the universal subgroup H_1 (defined as the orthogonal of the kernel of the degree p^3 isogeny $G \to G'$, or as the kernel of $G \to (G')^t$) is a multiplicative subgroup of order p.

Lemma 9.1.1.1. — *The map $X^{\geq 1}_{Kli}(p^m)_n \to X^{\geq 1}_{Kli}(p^{m-1})_n$ is étale and affine.*

Proof. We first prove that the map is étale. It suffices to show that the map $f : X^{\geq 1}_{Kli}(p^m)_n \to X^{\geq 1}_n$ is étale. We can prove this over the spectrum S of a completed local ring in $X^{\geq 1}_n$. Over S, there is a finite flat subgroup scheme $\tilde{G}[p^m] \subset G[p^m]$ such that the connected component of $G[p^m]$ is contained in $\tilde{G}[p^m]$ [13]. Let $g : T \to X^{\geq 1}_{Kli}(p^m)_n \times_{X^{\geq 1}_n} S$. Let $T \hookrightarrow T'$ be an infinitesimal thickening of T. We suppose that $h = f \circ g$ extends to $h' : T' \to S$ and we want to prove that h' can be lifted to a unique map $g' : T' \to X^{\geq 1}_{Kli}(p^m)_n \times_{X^{\geq 1}_n} S$ such that $f \circ g' = h'$. To the map g is associated a surjective map $\psi_T : \tilde{G}^D[p^m]|_T \to H^D_m|_T$ over T where $H^D_m|_T$ is an étale group scheme, locally isomorphic to $\mathbb{Z}/p^m\mathbb{Z}$. The group scheme $H^D_m|_T$ deforms uniquely to an étale group scheme

13. Away from the boundary, we can of course take $\tilde{G}[p^m] = G[p^m]$. At the boundary we find it easier to work with a finite flat group scheme and we can replace $G[p^m]$ (which is only quasi-finite) by $\tilde{G}[p^m]$ where $\tilde{G}$ is the semi-abelian scheme with constant toric rank that occurs in Mumford's construction (see [**18**], chap. III).

$H_m^D|_{T'}$ over T' and the data of h' provides a deformation $\tilde{G}[p^m]_{T'}$ to T' of $\tilde{G}^D[p^n]|_T$. By Illusie's deformation theory ([**37**], thm VII, 4.2.5) , the map ψ_T admits a unique extension $\psi_{T'} : \tilde{G}^D[p^m]|_{T'} \to H_m^D|_{T'}$.

We are left to prove that the map is affine. It will be enough to prove this for $n = 1$. Let us denote by $Z \to X_{Kli}^{\geq 1}(p^{m-1})_1$ the Grassmannian of subgroups of order p^m inside $G[F^m]$ (the kernel of $F^m : G \to G^{(p^m)}$). We note that $G[F^m]$ is a finite flat group scheme. As a result Z is proper and moreover, it is easy to see that Z is quasi-finite. As a result, Z is finite. We denote by C the universal subgroup. Let us denote by Z' the closed subscheme of Z where $C[p^{m-1}] = H_{m-1}$. The group scheme C/H_{m-1} is connected of order p over Z'. Its co-normal sheaf is $\mathcal{L}$, an invertible sheaf over Z' and the differential of the Verschiebung map $V : (C/H_{m-1})^{(p)} \to C/H_{m-1}$ provides a section $s \in \mathrm{H}^0(Z', \mathcal{L}^{(p-1)})$. The non vanishing locus of this section is the open subscheme $(Z')^m$ of Z where C/H_{m-1} is of multiplicative type. The map $(Z')^m \to X_{Kli}^{\geq 1}(p^{m-1})_1$ is affine as the composite of the affine open immersion $(Z')^m \hookrightarrow Z'$ and the finite map $Z' \to X_{Kli}^{\geq 1}(p^{m-1})_1$. Finally, $X_{Kli}^{\geq 1}(p^m)_1$ is the open and closed subscheme of $(Z')^m$ where C is locally for the étale topology isomorphic to μ_{p^m}. We have thus proved that the map $X_{Kli}^{\geq 1}(p^m)_1 \to X_{Kli}^{\geq 1}(p^{m-1})_1$ is affine. □

Remark 9.1.1.2. — The map $X_{Kli}^{\geq 1}(p^m)_n \to X_{Kli}^{\geq 1}(p^{m-1})_n$ is not finite because it induces an isomorphism over the p-rank 1 locus, and is of rank p (resp. $p+1$) over the p-rank 2 locus if $m \geq 2$ (resp. if $m = 1$).

9.1.2. Igusa tower. — We let $IG(p^m)_n = \mathrm{Isom}_{X_{Kli}^{\geq 1}(p^m)_n}(\mu_{p^m}, H_m)$. This is a $(\mathbb{Z}/p^m\mathbb{Z})^\times$-torsor over $X_{Kli}^{\geq 1}(p^m)_n$. There is an obvious commutative diagram :

$$\begin{tikzcd} X_{Kli}^{\geq 1}(p^m)_{n-1} \arrow[r] \arrow[d] & X_{Kli}^{\geq 1}(p^m)_n \arrow[d] \\ X_{Kli}^{\geq 1}(p^{m-1})_{n-1} \arrow[r] & X_{Kli}^{\geq 1}(p^{m-1})_n \end{tikzcd}$$

The horizontal maps are closed immersions and the vertical maps are étale and affine maps.

Above the last diagram, there is a commutative diagram :

$$\begin{tikzcd} IG(p^m)_{n-1} \arrow[r] \arrow[d] & IG(p^m)_n \arrow[d] \\ IG(p^{m-1})_{n-1} \arrow[r] & IG(p^{m-1})_n \end{tikzcd}$$

9.2. Formal schemes. — In this section we pass to the limit over n, and we are thus led to consider formal schemes. Let $\mathfrak{X} \to \mathrm{Spf}\ \mathbb{Z}_p$ be the p-adic completion of X and we let $\mathfrak{X}^{\geq 1} \hookrightarrow \mathfrak{X}$ be the open formal subscheme where the multiplicative rank of G is at least 1.

Let $\mathfrak{X}_{Kli}^{\geq 1}(p^m) \to \mathfrak{X}$ be the moduli of $H_m \hookrightarrow G[p^m]$ where H_m is locally for the étale topology isomorphic to μ_{p^m}. The map $\mathfrak{X}_{Kli}^{\geq 1}(p^m) \to \mathfrak{X}$ is étale and affine (but not finite !). We let $\mathfrak{X}_{Kli}^{\geq 1}(p^\infty)$ be the formal scheme equal to the inverse limit of $\mathfrak{X}_{Kli}^{\geq 1}(p^m)$ as m varies. It exists because the transition maps are affine. Let $H_\infty \hookrightarrow G[p^\infty]$ be the universal multiplicative Barsotti-Tate group. Above $\mathfrak{X}_{Kli}^{\geq 1}(p^m)$, we set $\mathfrak{IG}(p^m) = \mathrm{Isom}(\mu_{p^m}, H_m)$.

This is a $(\mathbb{Z}/p^m\mathbb{Z})^\times$-torsor. Above $\mathfrak{X}^{\geq 1}_{Kli}(p^\infty)$, we set $\mathfrak{IG}(p^\infty) = \mathrm{Isom}(\mu_{p^\infty}, H_\infty)$. This is a $\mathbb{Z}_p^\times$-torsor.

9.3. p-adic sheaves. — We now define sheaves of p-adic modular forms. Let $\pi : \mathfrak{IG}(p^\infty) \to \mathfrak{X}^{\geq 1}_{Kli}(p)$ be the projection. Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ and $\kappa : \mathbb{Z}_p^\times \to \Lambda^\times$ is the universal character. We can define the sheaf $\mathfrak{F}^\kappa = (\pi_\star \mathscr{O}_{\mathfrak{IG}(p^\infty)} \hat{\otimes}_{\mathbb{Z}_p} \Lambda)^{\mathbb{Z}_p^\times}$ where $\mathbb{Z}_p^\times$ acts diagonally, through its natural action on $\pi_\star \mathscr{O}_{\mathfrak{IG}(p^\infty)}$ and via the universal character $\kappa : \mathbb{Z}_p^\times \to \Lambda^\times$ on Λ. This is an invertible sheaf of $\mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^\infty)} \hat{\otimes}_{\mathbb{Z}_p} \Lambda$-modules over $\mathfrak{X}^{\geq 1}_{Kli}(p)$.

Remark 9.3.1. — We have decided to define our p-adic sheaves over $\mathfrak{X}^{\geq 1}_{Kli}(p)$, although we could also have defined them over $\mathfrak{X}^{\geq 1}_{Kli}(p^\infty)$. The base $\mathfrak{X}^{\geq 1}_{Kli}(p)$ is more directly related to the classical Shimura variety, and since the map $\mathfrak{X}^{\geq 1}_{Kli}(p^\infty) \to \mathfrak{X}^{\geq 1}_{Kli}(p)$ is affine, this does not make any difference on the cohomology.

For any adic complete $\mathbb{Z}_p$-algebra R and any continuous character $\chi : \mathbb{Z}_p^\times \to R^\times$ we let $\mathfrak{F}^\chi := \mathfrak{F}^\kappa \hat{\otimes}_{\Lambda,\chi} R$.

For some arguments, it is useful to consider certain truncated versions of the sheaf $\mathfrak{F}^\kappa$. Let $\Lambda_n = \mathbb{Z}/p^n\mathbb{Z}[(\mathbb{Z}/p^n\mathbb{Z})^\times]$. Let $\pi_{m,n} : IG(p^m)_n \to X^{\geq 1}_{Kli}(p)_n$ be the projection. For $m \geq n$, we let $\kappa_{m,n} : (\mathbb{Z}/p^m\mathbb{Z})^\times \to \Lambda_n^\times$ be the obvious character that factorizes through $(\mathbb{Z}/p^n\mathbb{Z})^\times$. We let $\mathscr{F}^\kappa_{m,n} = (\pi_{m,n})_\star(\mathscr{O}_{IG(p^m)_n} \otimes_{\mathbb{Z}_p} \Lambda_n)[\kappa_{m,n}]$. The sheaf $\mathscr{F}^\kappa_{m,n}$ is a sheaf of $\mathscr{O}_{X^{\geq 1}_{Kli}(p^m)_n} \otimes \Lambda_n$-modules. If $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to R^\times$ is any character with R a $\mathbb{Z}/p^n\mathbb{Z}$-algebra, we denote by $\mathscr{F}^\chi_{m,n}$ the sheaf obtained by base change.

We have the following maps of sheaves (with a slight abuse we think of them as sheaves over $X^{\geq 1}_{Kli}(p^m)_n$) :

$$\begin{array}{ccc} \mathscr{F}^\kappa_{m,n} & \longrightarrow & \mathscr{F}^\kappa_{m,n-1} \\ \uparrow & & \uparrow \\ \mathscr{F}^\kappa_{m-1,n} & \longrightarrow & \mathscr{F}^\kappa_{m-1,n-1} \end{array}$$

where the vertical maps are inclusions and the horizontal maps are induced by reduction modulo the kernel of $\Lambda_n \to \Lambda_{n-1}$. We can set $\mathscr{F}^\kappa_{\infty,n} = \mathrm{colim}_m \mathscr{F}^\kappa_{m,n}$. Then we have surjective maps $\mathscr{F}^\kappa_{\infty,n} \to \mathscr{F}^\kappa_{\infty,n-1}$ and $\mathfrak{F}^\kappa = \lim_n \mathscr{F}^\kappa_{\infty,n}$.

9.4. Comparison map. — Let $f_n : X^{\geq 1}_{Kli}(p^n)_n \to X^{\geq 1}_{Kli}(p)_n$. Over $X^{\geq 1}_{Kli}(p^n)_n$, we have a universal multiplicative subgroup $H_n \hookrightarrow G$. Passing to the conormal sheaves we get a surjective map :

$$\omega_G \to \omega_{H_n}$$

where ω_G is a locally free sheaf of rank 2 and ω_{H_n} is a locally free sheaf of rank 1. Moreover, the Hodge-Tate map provides an isomorphism :

$$\mathrm{HT} : H_n^D \otimes_{\mathbb{Z}_p} \mathscr{O}_{X^{\geq 1}_{Kli}(p^n)_n} \to \omega_{H_n}$$

and it induces an isomorphism $\mathscr{F}^k_{n,n} \to (\omega_{H_n})^k$.

As a consequence, there is a surjective map $\Omega^{(k,0)} \to (\omega_{H_n})^k \simeq \mathscr{F}^k_{n,n}$ of locally free sheaves on $X^{\geq 1}_{Kli}(p^n)_n$. We denote by $K\Omega^{(k,0)}$ the kernel of this map and we set $K\Omega^{(k,r)} = K\Omega^{(k,0)} \otimes \omega^r$.

Remark 9.4.1. — One can think of the map $\Omega^{(k,r)} \to \mathscr{F}^k_{n,n} \otimes \omega^r$ as the projection to the highest weight vector on the representation $\mathrm{Sym}^k\mathrm{St} \otimes \det^r$ of the group GL_2.

9.5. Variant. — All the constructions can be performed over X_{par} instead of X, because the polarization has never been used. We have defined classical sheaves $\Omega^{(k,r)}$ over X_{par} obtained by using the conormal sheaf of $G' \to X_{\mathrm{par}}$.

We let $X^{\geq 1}_{\mathrm{par},n}$ be the open subscheme of $X_{\mathrm{par},n}$ where the p-rank is at least one. We let $X^{\geq 1}_{\mathrm{par},Kli}(p^m)_n \to X^{\geq 1}_{\mathrm{par},n}$ the moduli space of subgroups $H'_m \subset G'$ which are locally isomorphic to μ_{p^m} in the étale topology.

Lemma 9.5.1. — *The map $X^{\geq 1}_{\mathrm{par},Kli}(p^m)_n \to X^{\geq 1}_{\mathrm{par},Kli}(p^{m-1})_n$ is étale and affine.*

Proof. Similar to the proof of lemma 9.1.1.1. □

We let $\mathfrak{X}^{\geq 1}_{\mathrm{par},Kli}(p^m)$ be the formal scheme equal to the limit indexed by n of the schemes $X^{\geq 1}_{\mathrm{par},Kli}(p^m)_n$ and we let $\mathfrak{X}^{\geq 1}_{\mathrm{par},Kli}(p^\infty)$ be the formal scheme equal to the inverse limit over m of the formal schemes $\mathfrak{X}^{\geq 1}_{\mathrm{par},Kli}(p^m)$. We can define a sheaf $\mathfrak{F}^\kappa$ of $\mathscr{O}_{\mathfrak{X}^{\geq 1}_{\mathrm{par},Kli}(p^\infty)} \hat{\otimes}_{\mathbb{Z}_p} \Lambda$-modules over $\mathfrak{X}^{\geq 1}_{\mathrm{par},Kli}(p)$. Similarly, we can define sheaves $\mathscr{F}^\kappa_{m,n}$ of $\mathscr{O}_{X^{\geq 1}_{\mathrm{par},Kli}(p^m)_n} \otimes \Lambda_n$-modules.

10. The U-operator

In this section we introduce the U-operator, which is an operator at Klingen level and is strongly related to the T-operator of section 7 defined at spherical level. This operator U corresponds to the operator $p^{3-r}U_{Kli(p),1}$ of section 5.1.4 on the cohomology in weight (k,r).

10.1. Definition of the correspondence. — The operator U is associated to the matrix $\mathrm{diag}(p^2,p,p,1)$ inside $\mathrm{GSp}_4(\mathbb{Q})$. We start by giving the definition of the moduli space associated to this operator. Let $\mathfrak{Y}^{\geq 1}_{Kli}(p^m) \hookrightarrow \mathfrak{X}^{\geq 1}_{Kli}(p^m)$ be the open subscheme where the semi-abelian scheme is an abelian scheme. Let $\mathfrak{C}_{\mathfrak{Y}}(p^m)$ be the moduli over $\mathfrak{Y}^{\geq 1}_{Kli}(p^m)$ of triples (G, H_m, L) where $L \subset G[p^2]$ is totally isotropic, $L[p]$ is of rank p^3 and $L \cap H_m = \{0\}$. We recall that $H_1 = H_m[p]$. In the following lemma the orthogonal is taken for the Weil pairing inside $G[p]$.

Lemma 10.1.1. — *We have exact sequences : $0 \to L \cap H_1^\perp \to L \to L/(L \cap H_1^\perp) \to 0$ where $L \cap H_1^\perp$ is a truncated Barsotti-Tate group of level 1, height 2 and dimension 1 (the (p,p) part of the correspondence) and $L/(H_1^\perp \cap L)$ is étale locally isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$ (the p^2-part of the correspondence).*

Proof. We start by recalling the following classical fact [14]. Let S be a scheme over which p is nilpotent, and let $M \to S$ be a finite flat group scheme. Then $M \to S$ is a truncated Barsotti-Tate group scheme of level n if and only if M is killed by p^n and for all $s \in S$, $M_s \to s$ is a truncated Barsotti-Tate group scheme of level n. We give the argument when $n = 1$ (a similar argument works for arbitrary n). By definition, we can suppose that S is a scheme over $\mathrm{Spec}\,\mathbb{F}_p$. By assumption, G is killed by p and it follows that the Frobenius map $F : G \to G^{(p)}$ factors into a map $G \to \mathrm{Ker}(V : G^{(p)} \to G)$. We have to prove that the morphism $G \to \mathrm{Ker}(V : G^{(p)} \to G)$ is faithfully flat. By assumption, it is surjective. By the criterion for flatness by fiber it is flat.

It follows that we can check all the assertions of the lemma on geometric points. We now work over a geometric point (we recall that the category of finite flat group schemes

14. We learnt this from Fargues, but we could not find a reference.

over a field is abelian). The assumptions imply that $L[p] \oplus H_1 = G[p]$, from which it follows that $L[p]$ is an extension $1 \to H_1^{\perp} \cap L[p] \to L[p] \to L[p]/H_1^{\perp} \cap L[p] \to 1$ where $H_1^{\perp} \cap L[p]$ is a Barsotti-Tate group of dimension 1 and height 2 (actually isomorphic to $H_1^{\perp}/H_1$) and $L[p]/(H_1^{\perp} \cap L[p])$ is a rank p étale group scheme. The map $L/L[p] \to pL$ is an isomorphism for rank reasons. Moreover, because L is totally isotropic, $pL = L[p]^{\perp}$ maps isomorphically to $L[p]/(H_1^{\perp} \cap L[p])$ which is étale. The lemma is proven. □

We have two projections t_1 and t_2 from $\mathfrak{C}_{\mathfrak{Y}}(p^m)$ to $\mathfrak{Y}^{\geq 1}_{Kli}(p^m)$. They are defined by $t_1 : (G, H_m, L) \mapsto (G, H_m)$ and $t_2 : (G, H_m, L) \mapsto (G/L, H_m + L/L)$.

10.2. Compactification of the correspondence. — As we want to define an action of the correspondence on cohomology groups it is necessary to consider toroidal compactifications. We will actually factor the correspondence as a product of two correspondences and we will compactify both. The advantage of this approach is that it will be easy to compare U and the other correspondence T studied in section 7.

We fix toroidal compactifications X_{Σ}, $X_{Kli}(p)_{\Sigma'}$ and $X_{\mathrm{par},\Sigma''}$ (for good polyhedral cone decompositions such that Σ' refines both Σ and Σ''). We have maps $p_1 : X_{Kli}(p)_{\Sigma'} \to X_{\Sigma}$ and $p_2 : X_{Kli}(p)_{\Sigma'} \to X_{\mathrm{par},\Sigma''}$. We call as usual G the semi-abelian scheme over X_{Σ}, G' the semi-abelian scheme over $X_{\mathrm{par},\Sigma''}$. Over $X_{Kli}(p)_{\Sigma'}$ we have the chain $G \to G' \to G$ where the first isogeny has degree p^3 and the total isogeny is multiplication by p. We drop Σ, Σ' and Σ'' from the notations if no confusion will arise.

Let $\mathfrak{X}_{\mathrm{par}}$ be the formal completion of X_{par}. Let us define $\mathfrak{X}^{m-et}_{\mathrm{par}}$ as the open subscheme of $\mathfrak{X}_{\mathrm{par}}$ where the kernel of the polarization $\lambda' : G' \to (G')^t$ contains a multiplicative group. When G' is an abelian scheme, this group is an extension of an étale by a multiplicative group. We observe that $\mathfrak{X}^{m-et}_{\mathrm{par}}$ is contained in the p-rank at least 1 locus. Let $\mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m) \to \mathfrak{X}^{m-et}_{\mathrm{par}}$ be the moduli space of subgroups $H'_m \subset G'$ locally isomorphic in the étale topology to μ_{p^m} (where G' is the semi-abelian scheme over $\mathfrak{X}_{\mathrm{par}}$).

We let $\mathfrak{C}^1(p^m)$ be the formal subscheme of $\mathfrak{X}_{Kli}(p) \times_{\mathfrak{X}} \mathfrak{X}^{\geq 1}_{Kli}(p^m)$ where the universal triple $(G \to G', H_m)$ satisfies $\mathrm{Ker}(G \to G') \cap H_m = \{0\}$.

Lemma 10.2.1. — *The formal subscheme $\mathfrak{C}^1(p^m)$ of $\mathfrak{X}_{Kli}(p) \times_{\mathfrak{X}} \mathfrak{X}^{\geq 1}_{Kli}(p^m)$ is open and closed.*

Proof. We first check that it is open. Let $J \subset \mathscr{O}_{H_m}$ be the ideal defining $\mathrm{Ker}(G \to G') \cap H_m \subset H_m$. Let $I_e \subset \mathscr{O}_{H_m}$ be the augmentation ideal. The locus where $\mathrm{Ker}(G \to G') \cap H_m = \{0\}$ is the complement of the support of the I_e/J (viewed has a coherent sheaf over $\mathfrak{X}_{Kli}(p) \times_{\mathfrak{X}} \mathfrak{X}^{\geq 1}_{Kli}(p^m)$). It is closed by the rigidity property of multiplicative groups. □

We let $q_1 : \mathfrak{C}^1(p^m) \to \mathfrak{X}^{\geq 1}_{Kli}(p^m)$ be the tautological projection sending $(G \to G', H_m)$ to (G, H_m). We have another projection $\mathfrak{C}^1(p^m) \to \mathfrak{X}_{\mathrm{par}}$ induced from the map p_2. It factors through $\mathfrak{X}^{m-et}_{\mathrm{par}}$ and can moreover be lifted to a map $q_2 : \mathfrak{C}^1(p^m) \to \mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m)$. Indeed, under the isogeny of semi-abelian schemes $G \to G'$ the subgroup $H_m \subset G$ maps isomorphically to its image $H'_m \subset G'$ which provides the required lift. In conclusion, we have $q_2(G \to G', H_m) = (G', H'_m)$.

As a result we have defined a correspondence (observe that the maps q_2 and q_1 are proper as they can be written by construction as a composition of proper maps):

$$\begin{tikzcd} & \mathfrak{C}^1(p^m) \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\ \mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m) & & \mathfrak{X}^{\geq 1}_{Kli}(p^m) \end{tikzcd}$$

We let $\mathfrak{C}^2(p^m)$ be the open and closed formal subscheme of $\mathfrak{X}_{Kli}(p) \times_{\mathfrak{X}_{\mathrm{par}}} \mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m)$ where the universal triple $(G' \to G, H'_m \subset G')$ satisfies $\mathrm{Ker}(G' \to G)$ is not a multiplicative group. By definition $\mathrm{Ker}(G' \to G)$ is a subgroup of the kernel $K(\lambda')$ of the polarization $\lambda' : G' \to (G')^t$ which contains a unique multiplicative subgroup of order p, $K(\lambda')^m$. Therefore the condition defining $\mathfrak{C}^2(p^m)$ is that $\mathrm{Ker}(G' \to G) \cap K(\lambda)^m = \{0\}$. One checks as in lemma 10.2.1 that this condition is closed and open. Observe that over the interior of the moduli space, $\mathrm{Ker}(G' \to G)$ is an étale group scheme. We let $r_1 : \mathfrak{C}^2(p^m) \to \mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m)$ be the tautological projection given by $r_1(G' \to G, H'_m \subset G') = (G', H'_m)$.

There is a second projection $\mathfrak{C}^2(p^m) \to \mathfrak{X}$ induced by the projection p_1. It factors through $\mathfrak{X}^{\geq 1}_{Kli}(p)$ and moreover it can be lifted to a map $r_2 : \mathfrak{C}^2(p^m) \to \mathfrak{X}^{\geq 1}_{Kli}(p^m)$. Indeed, under the isogeny $G' \to G$ the group H'_m is mapped isomorphically to its image $H_m \subset G$. In conclusion, $r_2(G' \to G, H'_m \subset G') = (G, H_m)$.

As a result we have a second correspondence (observe that the maps r_2 and r_1 are proper as they can be written by construction as a composition of proper maps) :

$$\begin{tikzcd} & \mathfrak{C}^2(p^m) \arrow[dl, "r_2"'] \arrow[dr, "r_1"] & \\ \mathfrak{X}^{\geq 1}_{Kli}(p^m) & & \mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m) \end{tikzcd}$$

Lemma 10.2.2. — *The structural morphisms $\mathfrak{C}^i(p^m) \to \mathrm{Spf}\ \mathbb{Z}_p$ for $i \in \{1,2\}$ are local complete intersection morphisms*[15].

Proof. The morphism $\mathfrak{X}_{Kli}(p) \to \mathrm{Spf}\ \ \mathbb{Z}_p$ is a local complete intersection morphism. There are étale morphisms $\mathfrak{C}^i(p^m) \to \mathfrak{X}_{Kli}(p)$ by construction. So the proposition follows. $\square$

We let $\mathfrak{C}(p^m)$ be the composite of these correspondences. Namely, we set

$$\mathfrak{C}(p^m) = \mathfrak{C}^2(p^m) \times_{r_1, \mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m), q_2} \mathfrak{C}^1(p^m)$$

and we obtain the following commutative diagram with cartesian center :

15. We say that a morphism of formal schemes $\mathfrak{S} \to \mathrm{Spf}\ \mathbb{Z}_p$ is a local complete intersection morphism if it is locally topologically of finite type, flat, and its special fiber $S = \mathfrak{S} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spec}\ \mathbb{F}_p \to \mathrm{Spec}\ \mathbb{F}_p$ is a local complete intersection morphism (in the schematic sense, see section 4.1.3).

$$\begin{tikzcd}
 & & \mathfrak{C}(p^m) \arrow[dl, "q_2'"'] \arrow[dr, "r_1'"] & & \\
 & \mathfrak{C}^2(p^m) \arrow[dl, "r_2"'] \arrow[dr, "r_1"] & & \mathfrak{C}^1(p^m) \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\
\mathfrak{X}^{\geq 1}_{Kli}(p^m) & & \mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m) & & \mathfrak{X}^{\geq 1}_{Kli}(p^m)
\end{tikzcd}$$

There are two projections $t_1 = q_1 \circ r_1', t_2 = r_2 \circ q_2' : \mathfrak{C}(p^m) \to \mathfrak{X}^{\geq 1}_{Kli}(p^m)$. The notation t_1, t_2 for these maps is justified by the following proposition :

Proposition 10.2.1. — *The restriction of $\mathfrak{C}(p^m)$ to $\mathfrak{Y}^{\geq 1}_{Kli}(p^m)$ is the correspondence $\mathfrak{C}_{\mathfrak{Y}}(p^m)$.*

Proof. Let (G, H_m, L) be a point of $\mathfrak{C}_{\mathfrak{Y}}(p^m)$. The isogeny $G \to G/L$ factors into $G \to G/(L[p]) \to G/L$ where $L[p]$ is a subgroup of $G[p]$ of order p^3 such that $L[p] \cap H_m = \{0\}$, $G/(L[p])$ carries a polarization whose degree is a prime-to-p multiple of p^2 (it comes from the p^2-multiple of the polarization on G) whose kernel is an extension of an étale by a multiplicative group. The kernel of $G/(L[p]) \to G/L$ is an étale subgroup of order p in the kernel of the polarization on $G/(L[p])$. This gives a map $\mathfrak{C}_{\mathfrak{Y}}(p^m) \to \mathfrak{C}(p^m)$ which identifies $\mathfrak{C}_{\mathfrak{Y}}(p^m)$ with the locus of $\mathfrak{C}(p^m)$ where the semi-abelian schemes are abelian. $\square$

10.3. Trace maps. — We now construct trace maps (or fundamental classes) which will be used later to define the action on the cohomology. We start with the interior of the moduli space.

Lemma 10.3.1. — *The map $t_1 : \mathfrak{C}_{\mathfrak{Y}}(p^m) \to \mathfrak{Y}^{\geq 1}_{Kli}(p^m)$ is finite flat.*

Proof. The map is proper. The quasi-finiteness follows from the fact that an abelian surface over a field of characteristic p and of p-rank at least 1 has only finitely many subgroups of order p. Therefore the map is finite. We prove the flatness. The formal scheme $\mathfrak{Y}^{\geq 1}_{Kli}(p^m)$ is regular and $\mathfrak{C}_{\mathfrak{Y}}(p^m)$ is Cohen-Macaulay by lemma 10.2.2. Flatness follows from [**53**], chap. 23, thm. 2.3.1. $\square$

Lemma 10.3.2. — *There is a normalized trace map $\frac{1}{p^3}\mathrm{Tr}_{t_1} : (t_1)_\star \mathscr{O}_{\mathfrak{C}_{\mathfrak{Y}}(p^m)} \to \mathscr{O}_{\mathfrak{Y}^{\geq 1}_{Kli}(p^m)}$.*

Proof. We have a usual trace map for finite flat morphism $\frac{1}{p^3}\mathrm{Tr}_{t_1} : (t_1)_\star \mathscr{O}_{\mathfrak{C}_{\mathfrak{Y}}(p^m)}[1/p] \to \mathscr{O}_{\mathfrak{Y}^{\geq 1}_{Kli}(p^m)}[1/p]$ and we need to check that lattices match. It is enough to check this over the ordinary locus and away from the boundary. Let $(G, H_m) \in \mathfrak{X}^{=2}_{Kli}(p^m)(\overline{\mathbb{F}}_p)$ be an ordinary point with G an abelian scheme. Let T be the Tate module of this point. Then $T \simeq \mathbb{Z}_p^2$. The deformation space of this point is $\mathrm{Hom}(\mathrm{Sym}^2 T, \widehat{\mathbb{G}_m})$ with ring $W(\overline{\mathbb{F}}_p)[[X, Y, Z]]$ where the Serre-Tate parameter is the map $\mathbb{Z}_p^2 \to \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m}$ given by the symmetric matrix $\begin{pmatrix} X & Z \\ Z & Y \end{pmatrix}$. The fiber of this deformation space under t_1 is a disjoint union[16] of spaces with ring

$$W(\overline{\mathbb{F}}_p)[[X, Y, Z, X', Y', Z']]/((1+X')^p - 1 - X, (1+Z')^{p^2} - 1 - Z, Y' - Y)$$

16. The disjoint union parametrizes the position of $L^m \subset T^\vee \otimes \mu_{p^\infty}$ and $L^{et} \subset T \otimes \mathbb{Q}_p/\mathbb{Z}_p$ for the universal rank p^4 subgroup L and L^m, L^{et} its multiplicative subgroup and étale quotient respectively.

which parametrize the following diagram of Serre-Tate parameters :

$$\begin{tikzcd}
\mathbb{Z}_p^2 \arrow[r, "\begin{pmatrix} X & Z \\ Z & Y \end{pmatrix}"] \arrow[d, "\begin{pmatrix} p^2 & 0 \\ 0 & p \end{pmatrix}"'] & \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m} \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}"] \\
\mathbb{Z}_p^2 \arrow[r, "\begin{pmatrix} X' & Z' \\ Z' & Y' \end{pmatrix}"] & \mathbb{Z}_p^2 \otimes \widehat{\mathbb{G}_m}
\end{tikzcd}$$

It is now clear that division by p^3 preserves the integrality of the Trace map. □

We now extend this normalized trace to the compactification. The next two lemmas are the analogues of lemmas 7.1.1 and 7.1.2. We have to be a little bit careful since we are now dealing with formal schemes.

Lemma 10.3.3. — *There is a normalized Trace map* $\frac{1}{p^2}\mathrm{Tr}_{q_1} : \mathrm{R}(q_1)_\star \mathscr{O}_{\mathfrak{C}^1(p^m)} \to \mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)}$.

Proof. By reduction modulo p^n we have a map of schemes over Spec $\mathbb{Z}/p^n\mathbb{Z}$:

$$q_1 : C^1(p^m)_n \to X^{\geq 1}_{Kli}(p^m)_n.$$

By construction, $C^1(p^m)_n$ and $X^{\geq 1}_{Kli}(p^m)_n$ are local complete intersections over Spec $\mathbb{Z}/p^n\mathbb{Z}$ and the morphism q_1 is projective. The dualizing complex $q_1^! \mathscr{O}_{X^{\geq 1}_{Kli}(p^m)_n}$ is an invertible sheaf and we have canonical isomorphisms $q_1^! \mathscr{O}_{X^{\geq 1}_{Kli}(p^m)_n} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^{n-1}\mathbb{Z} = q_1^! \mathscr{O}_{X^{\geq 1}_{Kli}(p^m)_{n-1}}$. We define $q_1^! \mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)} = \lim_n q_1^! \mathscr{O}_{X^{\geq 1}_{Kli}(p^m)_n}$. Here is an alternative definition (suggested by the referee). The morphism $q_1 : \mathfrak{C}^1(p^m) \to \mathfrak{X}^{\geq 1}_{Kli}(p^m)$ is projective, therefore over each open affine Spf $A \hookrightarrow \mathfrak{X}^{\geq 1}_{Kli}(p^m)$, the fiber $\mathfrak{C}^1(p^m) \times_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)}$ Spf A can be algebraized to a projective scheme over Spec A. The definition of $q_1^! \mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)}$ being local on the base, we can reduce that way to the algebraic situation. We want to produce a fundamental class :

$$\Theta : q_1^\star \mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)} \to q_1^! \mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)}.$$

Away from the boundary, this map is provided by the trace map of the finite flat morphism $q_1 : \mathfrak{C}^1(p^m)|_{\mathfrak{Y}^{\geq 1}_{Kli}(p^m)} \to \mathfrak{Y}^{\geq 1}_{Kli}(p^m)$ (see section 4.2.2). We need to check that the map Θ is well defined at the boundary. Actually, it is enough to see that it is well defined over the entire ordinary locus since the intersection of the boundary and the non-ordinary locus is of codimension 1 in the special fiber and the boundary is flat over Spf $\mathbb{Z}_p$ (in other words, in the spectrum of the local rings Spec $\mathscr{O}_{\mathfrak{C}^1(p^m),x}$ at closed points x of $\mathfrak{C}^1(p^m)$, the intersection of the boundary and the non-ordinary locus is of codimension at least 2).

The formal schemes $\mathfrak{X}^{=2}_{Kli}(p^m)$ and $\mathfrak{C}^1(p^m)|_{\mathfrak{X}^{=2}_{Kli}(p^m)}$ are smooth. The smoothness of $\mathfrak{X}^{=2}_{Kli}(p^m)$ follows from the smoothness of X. The smoothness of $\mathfrak{C}^1(p^m)|_{\mathfrak{X}^{=2}_{Kli}(p^m)}$ away from the boundary follows from the proof of lemma 7.1.1 where we established that the completed local rings are isomorphic to $W(\overline{\mathbb{F}}_p)[[X,Y,Z,X',Y',Z']]/((1+X')^p-1-X,(1+Z')^p-1-Z,Y'-Y)$ using Serre-Tate theory. The smoothness at the boundary follows from the description of the local charts. The main point being the smoothness of the modular curves of level $\Gamma_0(p)$ over the ordinary locus. As a consequence, the fundamental class extends over the ordinary locus : it is given by the determinant of the map on differentials

$$\Omega^1_{\mathfrak{X}^{=2}_{Kli}(p^m)/\mathbb{Z}_p} \to \Omega^1_{\mathfrak{C}^1(p^m)|_{\mathfrak{X}^{=2}_{Kli}(p^m)}/\mathbb{Z}_p}.$$

Moreover, this fundamental class is divisible by p^2 since it is over the complement of the boundary by a variant of lemma 7.1.1. Therefore we get a map $\frac{1}{p^2}\Theta : q_1^\star \mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)} \to q_1^! \mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)}$, and by adjunction (to prove the adjunction we can reduce to the algebraic situation) a map $\mathrm{R}(q_1)_\star \mathscr{O}_{\mathfrak{C}^1(p^m)} \to \mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p^m)}$.

□

Remark 10.3.1. — It is possible to prove that $\mathrm{R}^i(q_1)_\star \mathscr{O}_{\mathfrak{C}^1(p^m)} = 0$ if $i > 0$.

The proof of the next lemma is left to the reader. It is completely analogous to the proof of the previous lemma.

Lemma 10.3.4. — *There is a normalized trace map* $\frac{1}{p}\mathrm{Tr}_{r_1} : \mathrm{R}(r_1)_\star \mathscr{O}_{\mathfrak{C}^2(p^m)} \to \mathscr{O}_{\mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m)}$.

10.4. Action on modular forms. — Over $\mathfrak{C}^1(p^m)$ we have a universal isogeny $G \to G'$ whose differential is a map $\Omega^1_{G'/\mathfrak{C}^1(p^m)} \to \Omega^1_{G/\mathfrak{C}^1(p^m)}$.

Assume for a second we work over $\mathfrak{C}^1(p^\infty)$ (the projective limit of all $\mathfrak{C}^1(p^m)$) or over $C^1(p^m)_n$ (the reduction modulo p^n of $\mathfrak{C}^1(p^m)$) with $m \geq n$. Then there is a commutative diagram of group schemes :

$$\begin{tikzcd} H_m \arrow[r] \arrow[d] & H'_m \arrow[d] \\ G \arrow[r] & G' \end{tikzcd}$$

which induces a commutative diagram of conormal sheaves :

$$\begin{tikzcd} \omega_{G'} \arrow[r] \arrow[d] & \omega_{H'_m} \arrow[r] \arrow[d] & 0 \\ \omega_G \arrow[r] & \omega_{H_m} \arrow[r] & 0 \end{tikzcd}$$

Moreover, there is a Zariski covering of $\mathfrak{C}^1(p^\infty)$ by affine opens Spf R (resp. of $C^1(p^m)_n$ by Spec R) such that the above diagram becomes isomorphic over Spf R (resp. Spec R) to

$$\begin{tikzcd} R \oplus R \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}"'] & R \arrow[r] \arrow[d, "1_R"] & 0 \\ R \oplus R \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & R \arrow[r] & 0 \end{tikzcd} \tag{10.4.A}$$

We drop the hypothesis that $m \geq n$. It follows from the above discussion that we can define a normalized morphism :

$$q_2^\star \Omega^{(k,r)} \to q_1^\star \Omega^{(k,r)}$$

as the tensor product of the natural map $q_2^\star \Omega^k \to q_1^\star \Omega^k$ and a normalized map $\frac{1}{p^r} q_2^\star \omega^r \to q_1^\star \omega^r$.

By composing with the trace map of lemma 10.3.3, we get a map $\mathrm{R}(q_1)_\star q_2^\star \Omega^{(k,r)} \to \Omega^{(k,r)}$ which gives an operator :

$$U_1 \in \mathrm{Hom}\big(\mathrm{R}\Gamma(\mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m), \Omega^{(k,r)}), \mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p^m), \Omega^{(k,r)})\big).$$

We check as usual that the definition of U_1 is independent of the choices of good polyhedral decompositions.

We can proceed in a similar way with the correspondence $\mathfrak{C}^2(p^m)$. The main simplification is that the tautological isogeny $G' \to G$ over $\mathfrak{C}^2(p^m)$ is étale, and induces an isomorphism on differentials. Thus, we obtain a canonical isomorphism

$$r_2^\star \Omega^{(k,r)} \to r_1^\star \Omega^{(k,r)}$$

with no need to take a normalization. Applying the trace map of lemma 10.3.4 produces a cohomological correspondence $\mathrm{R}(r_1)_\star r_2^\star \Omega^{(k,r)} \to \Omega^{(k,r)}$ and as a result an operator

$$U_2 \in \mathrm{Hom}\big(\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p^m), \Omega^{(k,r)}), \mathrm{R}\Gamma(\mathfrak{X}^{m-et}_{\mathrm{par},Kli}(p^m), \Omega^{(k,r)})\big).$$

We denote by $U = U_1 \circ U_2$.

10.5. Action on mod-p forms. — In this section we analyze the action of the U operator in caracteristic p.

10.5.1. reduction modulo p. — By taking $m = 1$ and reducing modulo p, we obtain the following diagram (we still use the same letters to denote the various projections) :

$$\begin{tikzcd}
 & & C(p)_1 \arrow[dl, "q_2'"'] \arrow[dr, "r_1'"] & & \\
 & C^2(p)_1 \arrow[dl, "r_2"'] \arrow[dr, "r_1"] & & C^1(p)_1 \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\
X^{\geq 1}_{Kli}(p)_1 & & X^{m-et}_{\mathrm{par},Kli}(p)_1 & & X^{\geq 1}_{Kli}(p)_1
\end{tikzcd}$$

By reduction modulo p (and proposition 4.1.2.1), we obtain the following two cohomological correspondences $q_2^\star \Omega^{(k,r)}|_{X^{m-et}_{\mathrm{par},Kli}(p)_1} \to q_1^! \Omega^{(k,r)}|_{X^{\geq 1}_{Kli}(p)_1}$ on $C^1(p)_1$ and $r_2^\star \Omega^{(k,r)}|_{X^{\geq 1}_{Kli}(p)_1} \to r_1^! \Omega^{(k,r)}|_{X^{m-et}_{\mathrm{par},Kli}(p)_1}$ on $C^2(p)_1$.

They induce operators (we keep using the same notations as in the previous paragraph)

$$U_1 \in \mathrm{Hom}(\mathrm{R}\Gamma(X^{m-et}_{\mathrm{par},Kli}(p)_1, \Omega^{(k,r)}), \mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r)}))$$

and

$$U_2 \in \mathrm{Hom}(\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r)}), \mathrm{R}\Gamma(X^{m-et}_{\mathrm{par},Kli}(p)_1, \Omega^{(k,r)})).$$

We set $U = U_1 \circ U_2$.

10.5.2. The non-ordinary locus. — We now study the restriction to the non-ordinary locus. The following lemma is the analogue of proposition 7.4.1.1. Notice that everything is simpler in this setting and that there are no restrictions on the weight.

Lemma 10.5.2.1. — *1. Under the isomorphism $q_2^\star \omega^{p-1} = q_1^\star \omega^{p-1}$, we have $q_2^\star \mathrm{Ha} = q_1^\star \mathrm{Ha}$.*

2. Under the isomorphism $r_2^\star \omega^{p-1} = r_1^\star \omega^{p-1}$, we have $r_2^\star \mathrm{Ha} = r_1^\star \mathrm{Ha}$.

3. The following diagrams are commutative :

$$\begin{tikzcd} q_2^\star\Omega^{(k,r)} \arrow[r, "U_1"] \arrow[d, "\mathrm{Ha}"] & q_1^!\Omega^{(k,r)} \arrow[d, "\mathrm{Ha}"] \\ q_2^\star\Omega^{(k,r+(p-1))} \arrow[r, "U_1"] & q_1^!\Omega^{(k,r+(p-1))} \end{tikzcd}$$

$$\begin{tikzcd} r_2^\star\Omega^{(k,r)} \arrow[r, "U_2"] \arrow[d, "\mathrm{Ha}"] & r_1^!\Omega^{(k,r)} \arrow[d, "\mathrm{Ha}"] \\ r_2^\star\Omega^{(k,r+(p-1))} \arrow[r, "U_2"] & r_1^!\Omega^{(k,r+(p-1))} \end{tikzcd}$$

Proof. The correspondence $C^1(p)_1$ and $C^2(p)_1$ are Cohen-Macaulay. It is enough to prove the statements over the interior of the moduli space and the ordinary locus. Then 1 follows from lemma 6.3.4.2. Remark that the way the isomorphism $q_2^\star\omega^{(p-1)} \simeq q_1^\star\omega^{(p-1)}$ is constructed is precisely the canonical map of the lemma.

The point 2 is easier since the isogeny $G' \to G$ over $C^2(p)_1$ is étale and the formation of the Hasse invariant commutes with étale isogeny.

We now prove the commutativity of the diagrams. We can rewrite the first diagram as the composition of two diagrams

$$\begin{tikzcd} q_2^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "\mathrm{Ha}"] & q_1^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "\mathrm{Ha}"] & q_1^!\Omega^{(k,r)} \arrow[d, "\mathrm{Ha}"] \\ q_2^\star\Omega^{(k,r+(p-1))} \arrow[r] & q_1^\star\Omega^{(k,r+(p-1))} \arrow[r] & q_1^!\Omega^{(k,r+(p-1))} \end{tikzcd}$$

The first left square commutes by 1. The second square is the tensor product of the normalized fundamental class $q_1^\star\mathscr{O}_{X_1} \to q_1^!\mathscr{O}_{X_1}$ and the map $\mathrm{Ha} : q_1^\star\Omega^{(k,r)} \to q_1^\star\Omega^{(k,r+(p-1))}$. It is also commutative. One proves the commutativity of the second diagram along similar lines. □

Remark 10.5.2.1. — We can speak of the Hasse invariant on $C^1(p)_1$ and $C^2(p)_1$ without having to worry about which semi-abelian scheme is used to define it.

Lemma 10.5.2.2. — *The Hasse invariant is not a zero divisor in $C^1(p)_1$ and $C^2(p)_1$.*

Proof. Both schemes are Cohen-Macaulay of dimension 3. Since an abelian surface with p-rank at least one has only finitely many subgroups of order p, we deduce that the non-ordinary locus in $C^1(p)_1$ or $C^2(p)_1$ has dimension 2. As a result, the Hasse invariant cannot be a zero divisor. □

We let $X^{=1}_{Kli}(p)_1 \subset X^{\geq 1}_{Kli}(p)_1$ be the vanishing locus of Ha. This scheme is canonically isomorphic to $X_1^{=1}$ under the projection p_1. Taking the non-ordinary locus at all places, we obtain a diagram:

$$\begin{tikzcd}[column sep=small]
 & & C^{=1}(p)_1 \arrow[dl] \arrow[dr] & & \\
 & C^{2,=1}(p)_1 \arrow[dl, "r_2"'] \arrow[dr, "r_1"] & & C^{1,=1}(p)_1 \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\
X_1^{=1} & & X^{m-et,=1}_{\mathrm{par},Kli}(p)_1 & & X_1^{=1}
\end{tikzcd}$$

Using lemma 10.5.2.1, 3. and proposition 4.1.2.1, we obtain cohomological correspondences:

$\mathrm{R}(q_1)_\star(q_2)^\star\Omega^{(k,r)}|_{X^{m-et,=1}_{\mathrm{par},Kli}(p)_1} \to \Omega^{(k,r)}|_{X_1^{=1}}$ and $\mathrm{R}(r_1)_\star(r_2)^\star\Omega^{(k,r)}|_{X_1^{=1}} \to \Omega^{(k,r)}|_{X^{m-et,=1}_{\mathrm{par},Kli}(p)_1}$.
They induce operators (that we still denote by the same way as in the previous paragraph):

$$U_1 \in \mathrm{Hom}\big(\mathrm{R}\Gamma(X^{m-et,=1}_{\mathrm{par},Kli}(p)_1, \Omega^{(k,r)}), \mathrm{R}\Gamma(X_1^{=1}, \Omega^{(k,r)})\big)$$

and

$$U_2 \in \mathrm{Hom}\big(\mathrm{R}\Gamma(X_1^{=1}, \Omega^{(k,r)}), \mathrm{R}\Gamma(X^{m-et,=1}_{\mathrm{par},Kli}(p), \Omega^{(k,r)})\big).$$

We set $U = U_1 \circ U_2$. By lemma 10.5.2.2, we have a map of triangles:

$$\begin{tikzcd}
\mathrm{R}(q_1)_\star q_2^\star\Omega^{(k,r)} \arrow[r] \arrow[d, "\mathrm{Ha}"] & \Omega^{(k,r)} \arrow[d, "\mathrm{Ha}"] \\
\mathrm{R}(q_1)_\star q_2^\star\Omega^{(k,r+(p-1))} \arrow[r] \arrow[d] & \Omega^{(k,r+(p-1))} \arrow[d] \\
\mathrm{R}(q_1)_\star(q_2)^\star\Omega^{(k,r+(p-1))}|_{X^{m-et,=1}_{\mathrm{par},Kli}(p)_1} \arrow[r] \arrow[d, "+1"] & \Omega^{(k,r+(p-1))}|_{X_1^{=1}} \arrow[d, "+1"] \\
 {} & {}
\end{tikzcd}$$

A similar result holds for the other correspondence. It follows that the U-operator acts equivariantly on the long exact sequence

$$\mathrm{H}^\star(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r)}) \overset{\mathrm{Ha}}{\to} \mathrm{H}^\star(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r+(p-1))}) \to \mathrm{H}^\star(X^{=1}_{Kli}(p)_1, \Omega^{(k,r+(p-1))}) \to$$

10.5.3. Invariance under multiplication by Ha′. — The following lemma is the analogue of proposition 7.4.2.1.

Lemma 10.5.3.1. — *1. Under the isomorphism* $(q_2)^\star\omega^{p^2-1} = (q_1)^\star\omega^{p^2-1}$, *we have* $(q_2)^\star\mathrm{Ha}' = (q_1)^\star\mathrm{Ha}'$.

2. Under the isomorphism $(r_2)^\star\omega^{p^2-1} = (r_1)^\star\omega^{p^2-1}$, *we have* $(r_2)^\star\mathrm{Ha}' = (r_1)^\star\mathrm{Ha}'$.

3. The following diagram is commutative :

$$\begin{tikzcd}
\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}) \arrow[r, "U"] \arrow[d, "\mathrm{Ha}'"] & \mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}) \arrow[d, "\mathrm{Ha}'"] \\
\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r+p^2-1)}) \arrow[r, "U"] & \mathrm{H}^0(X_1^{=1}, \Omega^{(k,r+p^2-1)})
\end{tikzcd}$$

Proof. Point 1 follows from lemma 6.3.4.2. Point 2 is easy (the isogeny is étale). Point 3 is an immediate consequence of 1 and 2. □

10.6. Action on p-adic modular forms. — The universal isogeny over $\mathfrak{C}^1(p^\infty)$ or $C^1(p^m)_n$ induces an isomorphism $q_2^\star H_m \to q_1^\star H_m$ and thus a map $q_2^\star \mathscr{F}^\kappa_{m,n} \to q_1^\star \mathscr{F}^\kappa_{m,n}$ for $m \geq n$ and $q_2^\star \mathfrak{F}^\kappa \to q_1^\star \mathfrak{F}^\kappa$. As a result we can define the U_1 operator. The definition of U_2 is highly similar and we let $U = U_1 \circ U_2$. It acts on $\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p^m)_n, \mathscr{F}^\kappa_{m,n} \otimes \omega^r)$ and $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p^\infty), \mathfrak{F}^\kappa \otimes \omega^r)$.

10.7. Comparison map and the U correspondence. — By section 9.4, for all $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ we have an exact sequence of sheaves over $X^{\geq 1}_{Kli}(p^n)_n$:

$$0 \to K\Omega^{(k,r)} \to \Omega^{(k,r)} \to \mathscr{F}^k_{n,n} \otimes \omega^r \to 0.$$

Lemma 10.7.1. — $U \in p\mathrm{End}\big(\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p^n)_n, K\Omega^{(k,r)})\big)$.

Proof. This is obvious on the diagram 10.4.A. □

11. Perfect complexes of p-adic modular forms

In this section we finally consider the cohomology of our interpolation sheaf and apply the ordinary projector U to produce a perfect complex.

11.1. Finiteness of the cohomology on $X^{\geq 1}_{Kli}(p)_1$. — In this section, we will deduce the finiteness of the ordinary cohomology (with respect to U) over $X^{\geq 1}_{Kli}(p)_1$ from the finiteness of the ordinary cohomology (with respect to T) on $X^{\geq 1}_1$ established in section 8. In order to do so, we need to analyze carefully the relation between U and T.

11.1.1. The operators U and T over the ordinary locus. — In this subsection, we will work over the ordinary locus. Since we are only interested in degree 0 cohomology groups, we can work over the complement of the boundary by Koecher's principle. The various Hecke operators we will introduce respect cuspidality. That way, we do not need to worry about compactifications (although taking care of what happens with compactifications would have been possible).

First of all, we claim that we can decompose the Hecke operators $T_1 : \mathrm{H}^0(X^{=2}_{\mathrm{par},1}, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X^{=2}_1, \Omega^{(k,r)}(-D))$ and $T_2 : \mathrm{H}^0(X^{=2}_1, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X^{=2}_{\mathrm{par},1}, \Omega^{(k,r)}(-D))$ into $T_1 = T_1^{et} + T_1^m$ and $T_2 = T_2^{et} + T_2^m$. The operator T_1^{et} accounts for all isogenies $G \to G'$ with kernel a group of étale rank 2 and multiplicative rank one. The operator T_1^m accounts for all isogenies $G \to G'$ with kernel a group of multiplicative rank 2 and étale rank one. Similarly, the operator T_2^{et} accounts for all isogenies $G' \to G$ with kernel an étale group. The operator T_2^m accounts for all isogenies $G' \to G$ with kernel a multiplicative group.

Lemma 11.1.1.1. — *For all $r \geq 2$ and $k \geq 1$, the operators*

$$T_2^m : \mathrm{H}^0(X^{ord}_1, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X^{ord}_{\mathrm{par},1}, \Omega^{(k,r)}(-D)) \quad \textit{and}$$

$$T_1^m : \mathrm{H}^0(X^{ord}_{\mathrm{par},1}, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X^{ord}_1, \Omega^{(k,r)}(-D))$$

are 0.

Proof. This follows from the proof of proposition 7.4.1.1 (see also lemma 7.1.1 and lemma 7.1.2). □

We recall that $\mathfrak{Y} \subset \mathfrak{X}$ is the open formal subscheme where G is an abelian scheme. The ordinary locus of $\mathfrak{Y}$ is denoted by $\mathfrak{Y}^{ord}$. We now introduce a Hecke correspondence $\mathfrak{D}$

over $\mathfrak{Y}^{ord}$. It parametrizes pairs (G,L) where $L \subset G[p^2]$ is a totally isotropic group scheme which is an extension of an étale group scheme locally isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ by a multiplicative group scheme locally isomorphic to μ_p. We have two finite flat projections $g_1, g_2 : \mathfrak{D} \to \mathfrak{Y}ord$ given by $g_1((G,L)) = G$ and $g_2((G,L)) = G/L$. We can associate to this correspondence an Hecke operator T' (normalized by p^{-3-r}) and it is clear that T' acting on $\mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D))$ is the operator $T_1^{et} \circ T_2^{et}$ which is also equal to T by the lemma above if $r \geq 2$ and $k \geq 1$. The second projection $g_2 : \mathfrak{D} \to \mathfrak{Y}^{ord}$ actually lifts to $g_2 : \mathfrak{D} \to \mathfrak{Y}^{ord}_{Kli}(p)$ by mapping (G,L) to $(G/L, G[p]/L)$. If follows that the map $T' \in \mathrm{End}\big(\mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D))\big)$ factors through a map

$$\mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D)) \xrightarrow{i} \mathrm{H}^0(X_{Kli}^{ord}(p)_1, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D))$$

where the first map is the canonical inclusion i. We call $T'' : \mathrm{H}^0(X_{Kli}^{ord}(p)_1, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D))$ the second map. We can compose it again with the natural inclusion i and we obtain that way T''' an endomorphism of $\mathrm{H}^0(X_{Kli}^{ord}(p)_1, \Omega^{(k,r)}(-D))$. The correspondence underlying the operator T''' parametrizes triples (G,H,L) (with $H \subset G[p]$ multiplicative of order p, L as above, we do not require that $L \cap H = \{0\}$). The first projection is $(G,H,L) \mapsto (G,H)$ and the second $(G,H,L) \mapsto (G/L, G[p]/L)$. It is key to observe that the definition of L is independent of H. As a consequence, for $r \geq 2$ and $k \geq 1$, there is a commutative diagram where all vertical maps are the obvious inclusions:

$$\begin{tikzcd}
\mathrm{H}^0(X_{Kli}^{ord}(p)_1, \Omega^{(k,r)}(-D)) \arrow[r, "T'''"] \arrow[dr, "T''"] & \mathrm{H}^0(X_{Kli}^{ord}(p)_1, \Omega^{(k,r)}(-D)) \\
\mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D)) \arrow[u, "i"] \arrow[r, "T=T'"] & \mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D)) \arrow[u, "i"']
\end{tikzcd}$$

Lemma 11.1.1.2. — *The action of T''' is locally finite on $\mathrm{H}^0(X_{Kli}^{ord}(p)_1, \Omega^{(k,r)}(-D))$ if $r \geq 2$ and $k \geq 1$.*

Proof. The action of T is locally finite on $\mathrm{H}^0(X_1^{ord}, \Omega^{(k,r)}(-D))$ by proposition 7.4.1.1 and we have that $\langle (T''')^n f,\ n \in \mathbb{Z}_{\geq 0}\rangle = i\langle (T)^n T'' f,\ n \in \mathbb{Z}_{\geq 0}\rangle + \mathbb{F}_p f$. □

Lemma 11.1.1.3. — *On $\mathrm{H}^0(X_{Kli}^{ord}(p)_1, \Omega^{(k,r)}(-D))$ we have $U \circ T''' = U \circ U$ for $r \geq 2$ and $k \geq 1$.*

Proof. Over $Y_{Kli}^{ord}(p)_1$, we can decompose $T''' = U + F$ where F accounts for all isogenies $G \to G/L$ where L is such that $L \cap H \neq \{0\}$. We are left to prove that $U \circ F = 0$. Let $\mathfrak{H} \to \mathfrak{Y}^{ord}_{Kli}(p)$ be the moduli space of (G,H,L,L') where $(G,H) \in Y_{Kli}^{ord}(p)_1$, $L \subset G[p^2]$ is of type $(1,p,p,p^2)$ (that is, an extension of an étale group scheme locally isomorphic to $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ by a multiplicative group scheme locally isomorphic to μ_p) and $L \cap H = \{0\}$, $L' \subset (G/L)[p^2]$ is of type $(1,p,p,p^2)$ and $L' \cap (G[p]/(L \cap G[p])) \neq \{0\}$. We have two projections $s_1(G,H,L,L') = (G,H)$, $s_2(G,H,L,L') = ((G/L)/L', (G/L[p])/L')$. This correspondence is associated to the operator $U \circ F$. We observe that $G[p]/L \subset L'$. As a result, the map $s_2^\star \Omega^{(1,0)} \to s_1^\star \Omega^{(1,0)}$ factors through $p s_1^\star \Omega^{(1,0)}$. It then follows easily that the non normalized cohomological correspondence $\Theta : s_2^\star \Omega^{(k,r)} \to s_1^! \Omega^{(k,r)}$ factors through $p^{6+2r+k} p_1^! \Omega^{(k,r)}$. The factor p^{2r+k} arises from the map on differential and the factor p^6 from the fundamental class (we get p^3 from U and p^3 from F which is a "direct factor" of T'''). The operator $U \circ F$ arises from the normalized cohomological correspondence $\frac{1}{p^{6+2r}}\Theta$ (we get p^{3+r} from both F and U). When $k \geq 1$, this map reduces to 0 modulo p. □

Corollary 11.1.1.1. — *The action of U on $\mathrm{H}^0(X^{ord}_{Kli}(p)_1, \Omega^{(k,r)}(-D))$ is locally finite for all $r \geq 2$ and $k \geq 1$.*

Proof. Let $f \in \mathrm{H}^0(X^{ord}_{Kli}(p)_1, \Omega^{(k,r)}(-D))$. Then

$$\langle U^n f,\ n \in \mathbb{Z}_{\geq 0}\rangle = U(\langle (T''')^n f,\ n \in \mathbb{Z}_{\geq 0}\rangle) + \mathbb{F}_p f$$

and local finiteness follows from lemma 11.1.1.2. □

We denote by f the ordinary projector associated to U on $\mathrm{H}^0(X^{ord}_{Kli}(p)_1, \Omega^{(k,r)}(-D))$. The morphism $\mathfrak{X}^{ord}_{Kli}(p) \to \mathfrak{X}^{ord}$ is finite étale of rank $p+1$. We let $\mathrm{Tr} : \mathrm{H}^0(\mathfrak{X}^{ord}_{Kli}(p), \Omega^{(k,r)}) \to \mathrm{H}^0(\mathfrak{X}^{ord}, \Omega^{(k,r)})$ be the trace map.

Lemma 11.1.1.4. — *For all $n \geq 0$, we have $\mathrm{Tr} \circ U^n \circ i = pT^n \mod p^k$ as endomorphisms of $\mathrm{H}^0(\mathfrak{X}^{ord}, \Omega^{(k,r)})$.*

Proof. By definition, the Hecke correspondence associated to $\mathrm{Tr} \circ U^n \circ i$ parametrizes over $\mathfrak{Y}^{ord}$ triples (G, H, L_n) where $H \subset G[p]$ is a group of order p and multiplicative type, and $L_n \subset G[p^{2n}]$ is a totally isotropic subgroup which is an extension of an étale group locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^{2n}\mathbb{Z}$ by a multiplicative group which is locally isomorphic to μ_{p^n}, and such that $H \cap L_n = \{0\}$. The projections are given by $(G, H, L_n) \mapsto G$ and $(G, H, L_n) \mapsto (G/L_n)$. We can decompose the operator T^n as $T^{n,et} + T^{n,m}$, where $T^{n,et}$ accounts for all isogenies $G \to G/L_n$ where L_n is a totally isotropic subgroup which is an extension of an étale group locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^{2n}\mathbb{Z}$ by a multiplicative group which is locally isomorphic to μ_{p^n}, and $T^{n,m}$ accounts for all the other isogenies (we have $T^{1,et} = T'$). We now observe that any isogeny $G \to G/L$ occuring in $T^{n,m}$ will factor through multiplication by p on G. So we deduce, reasoning as in the proof of lemma 11.1.1.3, that $T^{n,et} = T^n \mod p^k$. It is clear that $\mathrm{Tr} \circ U^n \circ i = pT^{n,et}$ and the factor p arises from the fact that given L_n, we can find p different subgroups H of order p and multiplicative type such that $H \cap L_n = \{0\}$. □

Corollary 11.1.1.2. — *Assume that $r \geq 2$ and $k \geq 2$. Then the canonical map $f \circ i$: $e\mathrm{H}^0\big(X^{ord}_1, \Omega^{(k,r)}(-D)\big) \to f\mathrm{H}^0\big(X^{ord}_{Kli}(p)_1, \Omega^{(k,r)}(-D)\big)$ is bijective, where e is the ordinary projector for T and f is the ordinary projector for U.*

Proof. We first prove the surjectivity of the map. Let

$$G \in f\mathrm{H}^0\big(X^{ord}_{Kli}(p)_1, \Omega^{(k,r)}(-D)\big).$$

Then $T''U^{-1}G \in \mathrm{H}^0\big(X^{ord}_1, \Omega^{(k,r)}(-D)\big)$ and $eT''U^{-1}G \in e\mathrm{H}^0\big(X^{ord}_1, \Omega^{(k,r)}(-D)\big)$. By lemma 11.1.1.3, $fi(eT''U^{-1}G) = fT'''U^{-1}G = fUU^{-1}G = G$. We now prove injectivity. We consider the map $f \circ i : e\mathrm{H}^0\big(\mathfrak{X}^{ord}, \Omega^{(k,r)}(-D)\big) \to f\mathrm{H}^0\big(\mathfrak{X}^{ord}_{Kli}(p), \Omega^{(k,r)}(-D)\big)$. This is a map of complete flat $\mathbb{Z}_p$-modules. The reduction of this map modulo p is the map of the corollary (by a cohomological vanishing which ensures the surjectivity of the reduction map), and it suffices to prove the injectivity of this new map. This map is surjective by Nakayama's lemma and the first part of the proof. Let $F \in e\mathrm{H}^0\big(\mathfrak{X}^{ord}, \Omega^{(k,r)}(-D)\big)$. We assume that $f \circ i(F) = 0$. Applying lemma 11.1.1.4 for n large enough, we find that $\mathrm{Tr}(f \circ (i))(F) = pF = 0 \mod p^k$. We deduce that $F \in pe\mathrm{H}^0\big(\mathfrak{X}^{ord}, \Omega^{(k,r)}(-D)\big)$. By induction, one proves easily that $F \in \bigcap_n p^n e\mathrm{H}^0\big(\mathfrak{X}^{ord}, \Omega^{(k,r)}(-D)\big) = \{0\}$. □

11.1.2. The operators T and U on $X^{=1}_1$. — In section 7.4.1, we have constructed two cohomological correspondences (for $k + r > 2 + p - 1$ and $r \geq 2 + p - 1$):

$$T_1 : p_2^\star \Omega^{(k,r)}|_{X^{\leq 1}_{\mathrm{par},1}} \to p_1^! \Omega^{(k,r)}|_{X^{\leq 1}_1}$$

and

$$T_2 : p_1^\star \Omega^{(k,r)}|_{X_1^{\leq 1}} \to p_2^! \Omega^{(k,r)}|_{X_{\mathrm{par},1}^{\leq 1}}$$

which we can restrict to the p-rank one locus to get two cohomological correspondences (still denoted in the same way) :

$$T_1 : p_2^\star \Omega^{(k,r)}|_{X_{\mathrm{par},1}^{=1}} \to p_1^! \Omega^{(k,r)}|_{X_1^{=1}}$$

and

$$T_2 : p_1^\star \Omega^{(k,r)}|_{X_1^{=1}} \to p_2^! \Omega^{(k,r)}|_{X_{\mathrm{par},1}^{=1}}$$

and we obtain operators $T_1 : \mathrm{H}^0(X_{\mathrm{par},1}^{=1}, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D))$ and $T_2 : \mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D)) \to \mathrm{H}^0(X_{\mathrm{par},1}^{=1}, \Omega^{(k,r)}(-D))$. We let $T = T_1 \circ T_2$. The operators T_1 and T_2 can be decomposed in this setting into $T_1 = T_1^m + T_1^{et} + T_1^{oo}$ and $T_2 = T_2^m + T_2^{et} + T_2^{oo}$ (see section 7.4.2).

Lemma 11.1.2.1. — *$U = T$ on $\mathrm{H}^0(X_1^{=1}, \Omega^{(k,r)}(-D))$ if $k+r > 2(p+1)$, $r \geq 2+(p-1)$.*

Proof. By definition, $U = T_1^{et} \circ T_2^{et}$. It is enough to prove that $T_1^{oo} = 0$ and $T_1^m = T_2^m = 0$ and this follows from corollary 7.4.2.1. □

11.1.3. Finiteness. — We are now ready to prove the finiteness of the ordinary cohomology on $X_{Kli}^{\geq 1}(p)_1$.

Corollary 11.1.3.1. —

1. *For all $r \geq 2$ and $k > p+1$, the action of U on*
$$\mathrm{R}\Gamma(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,r)}(-D))$$
is locally finite. We denote by f the corresponding projector.
2. *For all $r \geq 2$ and $k > p+1$, the natural map induced by pull back:*
$$e\mathrm{R}\Gamma(X_1^{\geq 1}, \Omega^{(k,r)}(-D)) \to f\mathrm{R}\Gamma(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,r)}(-D))$$
is a quasi-isomorphism.
3. *There is a constant C which does not depend on the prime-to-p level K^p such that for all $k \geq C$ and $r \geq 3$, the map*
$$e\mathrm{R}\Gamma(X_1, \Omega^{(k,r)}(-D)) \to f\mathrm{R}\Gamma(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,r)}(-D))$$
is a quasi-isomorphism.
4. *The map*
$$e\mathrm{H}^i(X_1, \Omega^{(k,2)}(-D)) \to f\mathrm{H}^i(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,2)}(-D))$$
is bijective for $k \geq C$ and $i = 0$ and injective for $k \geq C$ and $i = 1$.
5. *For $r \geq 2$ and $k \geq C$, $f\mathrm{R}\Gamma(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,r)}(-D))$ is a perfect complex of $\mathbb{F}_p$-vector spaces of amplitude $[0,1]$.*

Proof. The cohomology $\mathrm{R}\Gamma(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,r)}(-D))$ is computed by the complex :

$$\mathrm{H}^0(X_{Kli}^{=2}(p)_1, \Omega^{(k,r)}(-D)) \to \mathrm{colim}_n \mathrm{H}^0(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n).$$

By corollary 11.1.1.1, the action is locally finite on the first term when $r \geq 2$ and $k \geq 1$. It is enough to prove that it is locally finite on each $\mathrm{H}^0(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n)$ for $r \geq 2$ and $k > p+1$. The case $n = 1$ follows from lemma 11.1.2.1 and lemma 8.1.1. In general, one argues by induction.

The map

$$e\mathrm{R}\Gamma(X_1^{\geq 1}, \Omega^{(k,r)}(-D)) \to f\mathrm{R}\Gamma(X_{Kli}^{\geq 1}(p)_1, \Omega^{(k,r)}(-D))$$

is represented by the following map of complexes :

$$\begin{tikzcd}
\mathrm{H}^0(X^{=2}_{Kli}(p)_1, \Omega^{(k,r)}(-D)) \arrow[r] & \mathrm{colim}_n \mathrm{H}^0(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n) \\
\mathrm{H}^0(X^{=2}_1, \Omega^{(k,r)}(-D)) \arrow[r] \arrow[u] & \mathrm{colim}_n \mathrm{H}^0(X^{\geq 1}_1, \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n) \arrow[u]
\end{tikzcd}$$

We need to prove that the vertical maps become isomorphisms after applying f on the top and e on the bottom. For the left vertical map, this is corollary 11.1.1.2. We will see that for each n, the map $e\mathrm{H}^0(X_1^{\geq 1}, \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n) \to f\mathrm{H}^0(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r+(p-1)n)}(-D)/(\mathrm{Ha})^n)$ is an isomorphism. For $n = 1$, this follows from lemma 11.1.2.1. The general case follows easily by induction. Points 3, 4 and 5 follow from proposition 8.2.1. $\square$

11.2. Finiteness of the ordinary cohomology over $\mathfrak{X}^{\geq 1}$ and $\mathfrak{X}^{\geq 1}_{Kli}(p)$. — In the following theorem we establish relations between the ordinary cohomology over $\mathfrak{X}^{\geq 1}$ and classical cohomology in weight (k, r) if k is large enough.

Theorem 11.2.1. — *For $k > p + 1$ and $r \geq 2$:*

1. *The Hecke operator U acts locally finitely on $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \Omega^{(k,r)}(-D))$.*
2. *The Hecke operator T acts locally finitely on $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}, \Omega^{(k,r)}(-D))$.*
3. *The complexes $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}, \Omega^{(k,r)}(-D))$ and $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \Omega^{(k,r)}(-D))$ only have cohomology in degree $0, 1$.*
4. *Let us denote by f the ordinary projector associated to U and by e the ordinary projector associated to T. Then the natural map :*
$$e\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}, \Omega^{(k,r)}(-D)) \to f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \Omega^{(k,r)}(-D))$$
is a quasi-isomorphism.
5. *There is a constant C which does not depend on the level K^p such that for $k \geq C$ and $r \geq 3$, the map*
$$e\mathrm{R}\Gamma(\mathfrak{X}, \Omega^{(k,r)}(-D)) \to e\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}, \Omega^{(k,r)}(-D))$$
is a quasi-isomorphism.
6. *For all $k \geq C$,*
$$e\mathrm{H}^i(\mathfrak{X}, \Omega^{(k,2)}(-D)) \to e\mathrm{H}^i(\mathfrak{X}^{\geq 1}, \Omega^{(k,r)}(-D))$$
is bijective for $i = 0$ and injective if $i = 1$.
7. *For all $k \geq C$ and $r \geq 2$, $f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \Omega^{(k,r)}(-D))$ is a perfect complex of $\mathbb{Z}_p$-modules of amplitude $[0, 1]$.*

Proof. Over $X^{\geq 1}_{Kli}(p)_n$ or $X^{\geq 1}_n$, we have the following exact sequence of sheaves :

$$0 \to \Omega^{(k,r)}(-D) \to \mathrm{colim}_l \Omega^{(k,r+lp^{n-1}(p-1))}(-D) \to \mathrm{colim}_l \Omega^{(k,r+lp^{n-1}(p-1))}(-D)/Ha^{lp^{n-1}} \to 0$$

where the limit in the middle is over multiplication by powers of $\mathrm{Ha}^{p^{n-1}}$ which lifts to a section of $\mathrm{H}^0(X_n, \omega^{p^{n-1}(p-1)})$. The middle sheaf is also the restriction of $\Omega^{(k,r)}(-D)$ to the ordinary locus. This is an acyclic resolution of $\Omega^{(k,r)}(-D)$ by flat $\mathbb{Z}/p^n\mathbb{Z}$-sheaves.

Indeed, all sheaves are acyclic relatively to the minimal compactification and the middle and right sheaves are supported over affine subschemes of the minimal compactification. Passing to the limit over n we obtain an acyclic resolution of $\Omega^{(k,r)}(-D)$ over $\mathfrak{X}^{\geq 1}_{Kli}(p)$ or $\mathfrak{X}^{\geq 1}$ by flat, p-adically complete and separated sheaves of $\mathbb{Z}_p$-modules. Let us denote by $M^\bullet$ and $N^\bullet$ the complexes concentrated in degree $[0,1]$ that compute the cohomologies $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1},\Omega^{(k,r)}(-D))$ and $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p),\Omega^{(k,r)}(-D))$ using these resolutions. They are objects of $\mathbf{C}^{flat}(\mathbb{Z}_p)$ by construction. By lemma 8.2.1, corollary 11.1.3.1 and lemma 2.1.2, we deduce that the actions of T and U are locally finite on $M^\bullet$ and $N^\bullet$. The points 4 and 5 follow from corollary 11.1.3.1 using proposition 2.2.2. The point 6 also follows by induction on n from corollary 11.1.3.1. Finally, we deduce 7 by another application of proposition 2.2.2.

□

Corollary 11.2.1. — *For $k > p+1$, the natural map*

$$e\mathrm{H}^0(X,\Omega^{(k,2)}(-D)\otimes\mathbb{Q}_p/\mathbb{Z}_p) \to f\mathrm{H}^0(\mathfrak{X}^{\geq 1}_{Kli}(p),\Omega^{(k,2)}(-D)\otimes\mathbb{Q}_p/\mathbb{Z}_p)$$

is a quasi-isomorphism.

Proof. The map

$$e\mathrm{H}^0(X,\Omega^{(k,2)}(-D)\otimes\mathbb{Q}_p/\mathbb{Z}_p) \to e\mathrm{H}^0(\mathfrak{X}^{\geq 1},\Omega^{(k,2)}(-D)\otimes\mathbb{Q}_p/\mathbb{Z}_p)$$

is an isomorphism since the complement of $\mathfrak{X}^{\geq 1}$ in $\mathfrak{X}$ is of codimension 2. The claim follows from theorem 11.2.1, point 4. □

11.3. The perfect complex. — We can finally construct a perfect complex over Λ and obtain an Hida theory for higher cohomology. We specialize to $r=2$ as this is the case of interest.

Theorem 11.3.1. — *Consider the complex $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p),\mathfrak{F}^\kappa\otimes\omega^2(-D))$.*

1. *The action of U is locally finite. Call f the associated projector.*
2. *The complex $f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p),\mathfrak{F}^\kappa\otimes\omega^2(-D))$ is a perfect complex of Λ-modules concentrated in degree $[0,1]$.*
3. *For all $k\geq 0$, U is locally finite on $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p),\Omega^{(k,2)}(-D))$ and there is a quasi-isomorphism :*

$$f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p),\Omega^{(k,2)}(-D)) \to f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p),\mathfrak{F}^\kappa\otimes\omega^2(-D))\otimes^L_{\Lambda,k}\mathbb{Z}_p.$$

4. *There is a constant C which does not depend of the level K^p such that for all $k\geq C$, the canonical map*

$$e\mathrm{H}^i(\mathfrak{X},\Omega^{(k,2)}(-D)) \to \mathrm{H}^i\big(f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p),\mathfrak{F}^\kappa\otimes\omega^2(-D))\otimes^L_{\Lambda,k}\mathbb{Z}_p\big)$$

is bijective for $i=0$ and injective for $i=1$.

Proof. For all $m\geq n$, we have the following acyclic resolution of the sheaf $\mathscr{F}^\kappa_{m,n}\otimes\omega^2(-D)$ over $X^{\geq 1}_{Kli}(p)_n$:

$$\begin{aligned}0 \to \mathscr{F}^\kappa_{m,n}\otimes\omega^2(-D) &\to \mathrm{colim}_l\mathscr{F}^\kappa_{m,n}(-D)\otimes\omega^{2+lp^{n-1}(p-1)}(-D)\\ &\to \mathrm{colim}_l\mathscr{F}^\kappa_{m,n}(-D)\otimes\omega^{2+lp^{n-1}(p-1)}(-D)/Ha^{lp^{n-1}} \to 0.\end{aligned}$$

Indeed, all theses sheaves are acyclic relatively to the minimal compactification by [**50**], thm. 8.6 and the middle and right sheaves have affine support in the minimal compactification. For all $k \in \mathbb{Z}_{\geq 0}$, we have an exact sequence of sheaves over $X^{\geq 1}_{Kli}(p)_1$: $0 \to K\Omega^{(k,2)}(-D) \to \Omega^{(k,2)}(-D) \to \mathscr{F}^k_{1,1}(-D) \to 0$ (see section 9.4). Using a resolution as above for all sheaves in this exact sequence, we get a commutative diagram :

$$\begin{tikzcd}
0 & 0 \\
\mathrm{H}^0(X^{=2}_{Kli}(p)_1, \mathscr{F}^k_{1,1} \otimes \omega^2(-D)) \arrow[u] \arrow[r] & \mathrm{H}^0(X_{Kli}(p)^{\geq 1}_1, \mathrm{colim}\mathscr{F}^k_{1,1} \otimes \omega^{2+l(p-1)}(-D)/Ha^l) \arrow[u] \\
\mathrm{H}^0(X^{=2}_{Kli}(p)_1, \Omega^{(k,2)}(-D)) \arrow[u] \arrow[r] & \mathrm{H}^0(X^{\geq 1}_{Kli}(p)_1, \mathrm{colim}\Omega^{(k,2+l(p-1))}(-D)/Ha^l) \arrow[u] \\
\mathrm{H}^0(X^{=2}_{Kli}(p)_1, K\Omega^{(k,2)}(-D)) \arrow[u] \arrow[r] & \mathrm{H}^0(X^{\geq 1}_{Kli}(p)_1, \mathrm{colim}K\Omega^{(k,2+l(p-1))}(-D)/Ha^l) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Assume that $k > p+1$. Since U is locally finite on $\mathrm{H}^0(X^{=2}_{Kli}(p)_1, \Omega^{(k,2)}(-D))$ and on

$$\mathrm{H}^0(X^{\geq 1}_{Kli}(p)_1, \mathrm{colim}\Omega^{(k,2+n(p-1))}(-D)/Ha^n),$$

by corollary 11.1.1.1 and corollary 11.1.3.1, it is locally finite on all the modules in the above diagram by lemma 2.1.1. Moreover, by lemma 10.7.1, U acts by zero on the bottom horizontal complex. Applying the projector, we obtain a quasi-isomorphism:

$$f\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r)}(-D)) \to f\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \mathscr{F}^k_{1,1} \otimes \omega^2(-D)).$$

For all m, the operator U^m arises from the correspondence C_m which parametrizes triples (G, H_1, G_m) with $(G, H_1) \in X^{\geq 1}_{Kli}(p)_1$ and $G \to G_m$ is an isogeny whose kernel is a group L_m satisfying $L_m \cap H_1 = \{0\}$ and moreover, if G is abelian L_m is an extension of an étale group scheme locally isomorphic to $\mathbb{Z}/p^{2m}\mathbb{Z}$ by a truncated Barsotti-Tate group of level m, height 2 and dimension 1. We have two projections $z_1 : C_m \to X^{\geq 1}_{Kli}(p)_1$ defined by $z_1(G, H_1, G_m) = (G, H_1)$ and $z_2 : C \to X^{\geq 1}_{Kli}(p)_1$ defined by $z_2(G, H_1, G_m) = (G_m, \mathrm{Im}(H_1))$. Actually, z_2 lifts to a map $z_2 : C_m \to X^{\geq 1}_{Kli}(p^m)_1$ defined by $z_2(G, H_1, G_m) = (G_m, H'_m)$ where H'_m is the image of $G[p^m]$ in G_m.

As a result we have a factorization $(U^m)'$ of U^m in the following diagram :

$$\begin{tikzcd}
\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \mathscr{F}^k_{m,1} \otimes \omega^2(-D)) \arrow[r, "U^m"] \arrow[dr, "(U^m)'"] & \mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \mathscr{F}^k_{m,1} \otimes \omega^2(-D)) \\
\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \mathscr{F}^k_{1,1} \otimes \omega^2(-D)) \arrow[u] \arrow[r, "U^m"] & \mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \mathscr{F}^\kappa_{1,1} \otimes \omega^2(-D)) \arrow[u]
\end{tikzcd}$$

It follows that U is locally finite on $\mathrm{colim}_m\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \mathscr{F}^k_{m,1} \otimes \omega^2(-D))$ and that we have an isomorphism :

$$f\mathrm{colim}_m\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \mathscr{F}^k_{m,1} \otimes \omega^2(-D)) = f\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r)}(-D)).$$

We deduce from lemma 2.1.2 that U is locally finite on $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^\kappa \otimes \omega^2(-D))$.

We now remove the assumption that $k > p+1$, so we take $k \in \mathbb{Z}_{\geq 0}$. Specializing in weight k we deduce that U is locally finite on $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^k \otimes \omega^2(-D))$. Using one more time the exact sequence $0 \to K\Omega^{(k,2)}(-D) \to \Omega^{(k,2)}(-D) \to \mathscr{F}^k_{1,1}(-D) \to 0$ we may now deduce that U is locally finite on $\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r)}(-D))$. Reasoning as before we find that

$$f\mathrm{colim}_m \mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \mathscr{F}^k_{m,1} \otimes \omega^2(-D)) = f\mathrm{R}\Gamma(X^{\geq 1}_{Kli}(p)_1, \Omega^{(k,r)}(-D))$$

is a quasi-isomorphism (for all $k \geq 0$). Moreover, proposition 2.2.2 and theorem 11.2.1 imply directly the points 3 and 4 of the theorem. □

In order to complete the proof of theorem 1.1 of the introduction, we still have to obtain a control theorem for characteristic 0 classes of weight $k \geq 0$. This will be obtained at the end of the next part of this work in theorem 14.8.1.

PART III
HIGHER COLEMAN THEORY

12. Overconvergent cohomology

The goal of this section is to construct an overconvergent version of the cohomologies considered in part II of this work. We first consider analytic Siegel threefolds of deep Klingen level and neighborhoods of the locus where the universal subgroup is multiplicative. We then construct certain Banach sheaves which interpolate the classical automorphic sheaves and we take their cohomology. The most delicate result of this section is a cohomological vanishing for these overconvergent cohomologies (proposition 12.9.1).

12.1. Notation. — We introduce certain notations that are specific to this part of the work. In this section, the base ring for our constructions is $\mathcal{O}$ the ring of integers of $\mathbb{C}_p$ rather than $\mathbb{Z}_p$. The p-adic valuation is normalized by $v(p) = 1$. For any rational number w, we let $p^w \in \mathcal{O}$ be an element of valuation w. If M is an $\mathcal{O}$-module, we denote by $M_w = M/p^w M$. We let **Adm** be the category of admissible $\mathcal{O}$-algebras. We recall that an admissible $\mathcal{O}$-algebra is a flat $\mathcal{O}$-algebra which is the quotient of a convergent power series ring $\mathcal{O}\langle X_1, \cdots, X_s\rangle$ by a finitely generated ideal. We let **NAdm** be the category of normal admissible $\mathcal{O}$-algebras.

12.2. Formal Siegel threefold and the Hodge-Tate period map. —

12.2.1. The Hodge-Tate period map. — We start by introducing several formal and analytic Siegel threefolds as in section 1.2 of [**63**]. Let Σ be a polyhedral decomposition which is Γ-admissible and let $X \to \mathrm{Spec}\ \mathcal{O}$ be a toroidal compactification of the Siegel threefold with spherical level at p and tame level K^p.

Let $\mathcal{X}$ be the associated analytic adic space over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O})$. Let $\mathfrak{X}$ be the formal p-adic completion of X. We let $\mathcal{X}(p^n) \to \mathcal{X}$ be the adic Siegel threefold with full p^n level structure at p. Let $\mathfrak{X}(p^n)$ be the normalization of $\mathfrak{X}$ in $\mathcal{X}(p^n)$.

We denote by $\mathfrak{Y}$ the complement of the boundary in $\mathfrak{X}$ and by $\mathfrak{Y}(p^n)$ the complement of the boundary in $\mathfrak{X}(p^n)$. Over $\mathfrak{Y}(p^n)$ we have a universal map $(\mathbb{Z}/p^n\mathbb{Z})^4 \to G[p^n]$ of group schemes which is a symplectic isomorphism up to a similitude factor on the analytic generic fiber. We also have a Hodge-Tate period map $G[p^n] \to \omega_G/p^n\omega_G$ (we are using the polarization to identify ω_G and ω_{G^t}). We denote by $\mathrm{HT} : (\mathbb{Z}/p^n\mathbb{Z})^4 \to \omega_G/p^n\omega_G$ the composite of the two maps.

In [**63**], prop. 1.2 we show that the Hodge-Tate period map can be extended over $\mathfrak{X}(p^n)$ to a morphism

$$\mathrm{HT} : (\mathbb{Z}/p^n\mathbb{Z})^4 \to \omega_G/p^n.$$

Following [**63**], prop. 1.10, there is a formal scheme $\mathfrak{X}(p^n)^{mod} \to \mathfrak{X}(p^n)$ which is the normalization of a blow up and which carries a rank 2-locally-free modification $\omega_G^{mod} \hookrightarrow \omega_G$ such that

1. $p^{\frac{1}{p-1}}\omega_G \subset \omega_G^{mod} \subset \omega_G$,
2. the Hodge-Tate map factors through $\omega_G^{mod}/p^n\omega_G$ and induces a surjective homomorphism :

$$(\mathbb{Z}/p^n\mathbb{Z})^4 \otimes_{\mathbb{Z}} \mathscr{O}_{\mathfrak{X}(p^n)^{mod}} \to \omega_G^{mod}/p^{n-\frac{1}{p-1}}\omega_G^{mod}.$$

12.2.2. The canonical filtration. — We equip $(\mathbb{Z}/p^n\mathbb{Z})^4$ with the canonical basis (e_1, e_2, e_3, e_4) and the standard symplectic form given by the matrix J (see section 5.1). For all $\epsilon \in [0, n - \frac{1}{p-1}] \cap \mathbb{Q}$, we let $\mathfrak{X}(p^n, \epsilon) \to \mathfrak{X}(p^n)^{mod}$ be the formal scheme where $\mathrm{HT}(e_1) = 0$ in $\omega_G^{mod}/p^\epsilon\omega_G^{mod}$. This is an open subscheme of an admissible blow up of $\mathfrak{X}(p^n)^{mod}$.

Over $\mathfrak{X}(p^n, \epsilon)$ we denote by $\mathrm{Fil}_\epsilon^{can} \subset (\omega_G^{mod})_\epsilon = \omega_G^{mod}/p^\epsilon\omega_G^{mod}$ the coherent subsheaf generated by $\mathrm{HT}(e_2)$ and $\mathrm{HT}(e_3)$.

Lemma 12.2.2.1. — *The sheaf* $\mathrm{Fil}_\epsilon^{can}$ *is a locally free sheaf of rank one of* $\mathscr{O}_{\mathfrak{X}(p^n,\epsilon)}/p^\epsilon$*-modules and locally a direct summand in* $(\omega_G^{mod})_\epsilon$.

Proof. We work locally over some open affine $\mathrm{Spf}\ R$ of $\mathfrak{X}(p^n, \epsilon)$. So we can assume that we have $(\omega_G^{mod})_\epsilon(\mathrm{Spf}\ R) \simeq R_\epsilon^2$ and the matrix of HT is given by

$$\begin{pmatrix} 0 & a & c & e \\ 0 & b & d & f \end{pmatrix}$$

in the canonical basis of $(\mathbb{Z}/p^n\mathbb{Z})^4$. By symplecticity (the kernel of the map $\mathrm{HT} \otimes 1 : R_\epsilon^4 \to R_\epsilon^2$ is a Lagrangian subspace) we get $ad - bc = 0$. The map $\mathrm{HT} \otimes 1$ is surjective and therefore there is (locally on R) a 2×2 minor which is invertible. Let us assume that $cf - de$ is a unit in R_ϵ. Localizing further on R, we can assume that c, f or d, e are units in R_ϵ. Let us assume that c, f are units for example. We deduce that $\mathrm{HT}(e_2) = \frac{a}{c}\mathrm{HT}(e_3)$ and that $\mathrm{Fil}_\epsilon^{can}$ is generated by $\mathrm{HT}(e_2)$, a direct factor is provided by the submodule generated by $\mathrm{HT}(e_4)$. □

The formal scheme $\mathfrak{X}(p^n, \epsilon)$ is covered by the open formal subschemes $\mathfrak{X}(p^n, \epsilon, e_2)$ and $\mathfrak{X}(p^n, \epsilon, e_3)$ which are respectively defined by the conditions $\mathrm{HT}(e_2)$ generates $\mathrm{Fil}_\epsilon^{can}$ and $\mathrm{HT}(e_3)$ generates $\mathrm{Fil}_\epsilon^{can}$.

12.2.3. The canonical quotient. — We denote by

$$\mathrm{Gr}_\epsilon^{can} = \mathrm{coker}(\mathrm{Fil}_\epsilon^{can} \to (\omega_G^{mod})_\epsilon).$$

Passing to the quotient we get a canonical map

$$\mathrm{HT}_4 : (\mathbb{Z}/p^n\mathbb{Z})^4/\langle e_1, e_2, e_3\rangle \simeq \mathbb{Z}/p^n\mathbb{Z} \to \mathrm{Gr}_\epsilon^{can}$$

inducing an isomorphism

$$\mathrm{HT}_4 \otimes 1 : \mathbb{Z}/p^n\mathbb{Z} \otimes (\mathscr{O}_{\mathfrak{X}(p^n,\epsilon)})_\epsilon \to \mathrm{Gr}_\epsilon^{can}.$$

12.3. Flag varieties. — We let $\mathfrak{FL}_n \to \mathfrak{X}(p^n)^{mod}$ be the flag variety parametrizing locally free direct summands of rank one $\mathrm{Fil}\omega_G^{mod} \subset \omega_G^{mod}$. This is a $\mathbb{P}^1$-bundle.

For all rational numbers $0 \leq w \leq \epsilon$, we denote by $\mathfrak{FL}_{n,\epsilon,w} \to \mathfrak{FL}_n \times_{\mathfrak{X}(p^n)^{mod}} \mathfrak{X}(p^n,\epsilon) \to \mathfrak{X}(p^n,\epsilon)$ the admissible formal scheme parametrizing invertible sheaves $\mathrm{Fil}\omega_G^{mod} \subset \omega_G^{mod}$ satisfying

$$(\mathrm{Fil}\omega_G^{mod})_w = \mathrm{Fil}_w^{can}.$$

For all positive rational numbers $w' \leq w$, we also denote by $\mathfrak{FL}^+_{n,\epsilon,w,w'} \to \mathfrak{FL}_{n,\epsilon,w}$ the normal admissible formal scheme which parametrizes basis $\rho : \mathscr{O}_{\mathfrak{FL}^+_{n,\epsilon,w,w'}} \simeq \omega_G^{mod}/\mathrm{Fil}\omega_G^{mod}$ such that $\rho_{w'} = \mathrm{HT}_4 \otimes 1 \mod p^{w'}$.

12.4. Group action. — Denote by $\mathfrak{GSp}_4$ the formal p-adic completion of GSp_4. Let $\mathfrak{Kli} \subset \mathfrak{GSp}_4$ be the Klingen parabolic of upper triangular matrices with blocks of size 1×1, 2×2 and 1×1. There is a well-defined action of $\mathfrak{GSp}_4(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}(p^n)$, trivial over $\mathcal{X}$ and it extends to an action on $\mathfrak{X}(p^n)$ by normality and on $\mathfrak{X}(p^n)^{mod}$ (since $\mathfrak{X}(p^n)^{mod}$ is obtained by blowing up along ideals which are invariant under the group action and by normalization). It is clear that there is an induced action of $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathfrak{X}(p^n,\epsilon)$. We denote by $\mathfrak{X}_{Kli}(p^n,\epsilon)$ the quotient of $\mathfrak{X}(p^n,\epsilon)$ by the finite group $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$.

For all rational numbers $w' \geq 0$, we let $\mathfrak{T}_{w'}$ be the formal group scheme defined by $\mathfrak{T}_{w'}(R) = \mathbb{Z}_p^\times(1+p^{w'}R)$ for all R in **Adm**. We let $\mathfrak{T}^0_{w'}$ be the connected component of the identity in $\mathfrak{T}_{w'}$. For all R in **Adm**, $\mathfrak{T}^0_{w'}(R) = 1 + p^{w'}R$. The group $\mathfrak{T}^0_{w'}$ acts on $\mathfrak{FL}^+_{n,\epsilon,w,w'}$ (it acts on ρ) and the map $\mathfrak{FL}^+_{n,\epsilon,w,w'} \to \mathfrak{FL}_{n,\epsilon,w}$ is a $\mathfrak{T}^0_{w'}$-torsor.

For all integers $n \geq w'$ we let $\mathfrak{T}_{w',n}$ be the fiber product $\mathfrak{T}_{w'} \times_{\mathfrak{T}_{w'}/\mathfrak{T}^0_{w'}} \mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$ where the map $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z}) \to \mathfrak{T}_{w'}/\mathfrak{T}^0_{w'}$ is the composite of $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z}) \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ (given by the last diagonal entry) and the natural projection $(\mathbb{Z}/p^n\mathbb{Z})^\times \to \mathfrak{T}_{w'}/\mathfrak{T}^0_{w'}$ (recall that $w' \leq n$).

Observe that $\mathfrak{T}^0_{w'}$ is naturally a subgroup of $\mathfrak{T}_{w',n}$. The action of $\mathfrak{T}^0_{w'}$ on $\mathfrak{FL}^+_{n,\epsilon,w,w'}$ can be extended to an action of $\mathfrak{T}_{w',n}$ on $\mathfrak{FL}^+_{n,\epsilon,w,w'}$, which induces the action of $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathfrak{X}(p^n,\epsilon)$.

12.5. Local description. — Let $\mathrm{Spf}\ R \hookrightarrow \mathfrak{X}(p^n,\epsilon)$ be a Zariski open subset such that we have $\omega_G^{mod}|_{\mathrm{Spf}\ R} = R\omega_1 \oplus R\omega_2$ where ω_1 lifts a basis of Fil^{can} and ω_2 lifts $\mathrm{HT}(e_4)$.

Over $\mathrm{Spf}\ R$, $\mathfrak{FL}^+_{n,\epsilon,w,w'}$ is identified with the set

$$\begin{pmatrix} 1 & 0 \\ p^w\mathfrak{B}(0,1)_R & 1 \end{pmatrix} \times (1 + p^{w'}\mathfrak{B}(0,1)_R)$$

with $\mathfrak{B}(0,1)_R = \mathrm{Spf}\ R\langle X\rangle$. We associate to the universal matrix

$$\begin{pmatrix} 1 & 0 \\ p^w X & 1 \end{pmatrix} \times (1 + p^{w'}X')$$

the flag $\mathrm{Fil}\omega_G^{mod} = \omega_1 + p^w X\omega_2$ and the trivialization ρ of the quotient $\mathrm{Gr}\omega_G^{mod}$ given by $\rho(1) = (1+p^{w'}X').\omega_2$.

12.6. Banach sheaves. — We construct in this section formal Banach sheaves of locally analytic and overconvergent modular forms.

12.6.1. Formal Banach sheaves. — We recall some definitions taken from [**3**], def. A.1.1.1. We let $\mathfrak{S}$ be an admissible formal scheme. A formal Banach sheaf over $\mathfrak{S}$ is a family $(\mathfrak{F}_n)_{n\geq 0}$ of quasi-coherent sheaves such that :

1. $\mathfrak{F}_n$ is a sheaf of $\mathscr{O}_{\mathfrak{S}}/p^n$-modules,
2. $\mathfrak{F}_n$ is flat over $\mathcal{O}/p^n$,
3. For all $0 \leq m \leq n$, we have compatible isomorphisms $\mathfrak{F}_n \otimes_{\mathcal{O}} \mathcal{O}/p^m \simeq \mathfrak{F}_m$.

We can associate to $(\mathfrak{F}_n)_n$ a sheaf $\mathfrak{F}$ over $\mathfrak{S}$ equal to the inverse limit $\lim_n \mathfrak{F}_n$ (the maps in the inverse limit are those provided by 3) above). The sheaf $\mathfrak{F}$ clearly determines the $(\mathfrak{F}_n)$ and we identify $\mathfrak{F}$ and the family $(\mathfrak{F}_n)$ in the sequel. We say that a Banach sheaf is flat if $\mathfrak{F}_n$ is a flat $\mathscr{O}_{\mathfrak{S}}/p^n$-module for all n.

12.6.2. Formal Banach sheaf of overconvergent modular forms. — Let $\epsilon \in]0, n - \frac{1}{p-1}] \cap \mathbb{Q}$ and $0 < w' \leq w \leq \epsilon$ be rational numbers. Let A be an object of **Nadm**. We assume that we are given a continuous character $\kappa_A : \mathbb{Z}_p^\times \to A^\times$ which is w'-analytic in the sense that it extends to a pairing $\kappa_A : \mathfrak{T}_{w'} \times \mathrm{Spf}\ A \to \mathbb{G}_m$.

We have a series of affine maps

$$\pi : \mathfrak{F}\mathfrak{L}^+_{n,\epsilon,w,w'} \to \mathfrak{F}\mathfrak{L}_{n,\epsilon,w} \to \mathfrak{X}(p^n, \epsilon).$$

Let $\pi_1 : \mathfrak{F}\mathfrak{L}^+_{n,\epsilon,w,w'} \to \mathfrak{F}\mathfrak{L}_{n,\epsilon,w}$. This map is a torsor under $\mathfrak{T}^0_{w'}$. We define an invertible sheaf

$$\mathfrak{L}^{\kappa_A} = \left((\pi_1)_\star \mathscr{O}_{\mathfrak{F}\mathfrak{L}^+_{n,\epsilon,w,w'}} \hat{\otimes}_{\mathcal{O}} A\right)^{\mathfrak{T}^0_{w'}}$$

over $\mathfrak{F}\mathfrak{L}_{n,\epsilon,w} \times \mathrm{Spf}\ A$. The invariants are taken with respect to the diagonal action of $\mathfrak{T}^0_{w'}$.

Remark 12.6.2.1. — The sheaf $\mathfrak{L}^{\kappa_A}$ does not depend on w' for if we choose $w'' \in [w', w]$, we can view κ_A as a character of $\mathfrak{T}^0_{w''}$ and perform a similar construction as above with $\mathfrak{F}\mathfrak{L}_{n,\epsilon,w,w''}$. This will give an invertible sheaf canonically isomorphic to $\mathfrak{L}^{\kappa_A}$. The isomorphism is deduced from the natural map $\mathfrak{F}\mathfrak{L}^+_{n,\epsilon,w,w''} \to \mathfrak{F}\mathfrak{L}^+_{n,\epsilon,w,w'}$, equivariant for the map $\mathfrak{T}^0_{w''} \to \mathfrak{T}^0_{w'}$.

Let $\pi_2 : \mathfrak{F}\mathfrak{L}_{n,\epsilon,w} \to \mathfrak{X}(p^n, \epsilon)$. We define a formal Banach sheaf

$$\mathfrak{G}^{\kappa_A, w} = (\pi_2)_\star \mathfrak{L}^{\kappa_A}$$

over $\mathfrak{X}(p^n, \epsilon) \times \mathrm{Spf} A$.

Lemma 12.6.2.1. — *The formal Banach sheaf $\mathfrak{G}^{\kappa_A, w}$ is flat.*

Proof. Using a covering as in section 12.5, $\mathfrak{F}\mathfrak{L}^+_{n,\epsilon,w,w'}|_{\mathrm{Spf}\ R}$ is identified with the set of matrices

$$\begin{pmatrix} 1 & 0 \\ p^w \mathfrak{B}(0,1)_R & 1 \end{pmatrix} \times (1 + p^{w'} \mathfrak{B}(0,1)_R).$$

The action of $\mathfrak{T}^0_w$ is on the right term. It follows that $\mathfrak{G}^{\kappa_A, w}(\mathrm{Spf}\ R \times \mathrm{Spf}\ A) \simeq R \hat{\otimes} A\langle X \rangle$. □

Lemma 12.6.2.2. — *For $i \in \{2, 3\}$, the restriction of the quasi-coherent sheaf $\mathfrak{G}^{\kappa_A, w}/p^w$ to $\mathfrak{X}(p^n, \epsilon, e_i)$ is an inductive limit of coherent sheaves which are extensions of the sheaf $\mathscr{O}_{\mathfrak{X}(p^n, \epsilon, e_i)}/p^w$.*

Proof. Over $\mathfrak{X}(p^n,\epsilon,e_i)$, the vectors $\mathrm{HT}(e_i)$, $\mathrm{HT}(e_4)$ are a basis of $(\omega_G^{mod})_\epsilon$. We are therefore in a situation similar to [**3**], main construction, section 4.5. The claim follows from corollary 8.1.5.4 and corollary 8.1.6.4 of [**3**]. □

We let $\pi_3 : \mathfrak{X}(p^n,\epsilon) \to \mathfrak{X}_{Kli}(p^n,\epsilon)$ be the finite projection. The sheaf $(\pi_3)_\star \mathfrak{G}^{\kappa_A,w}$ is $\mathfrak{T}_{w,n}$-equivariant. We define a Banach sheaf

$$\mathfrak{F}^{\kappa_A,w} = \big((\pi_3)_\star \mathfrak{G}^{\kappa_A,w}\big)^{\mathfrak{T}_{w,n}} = (\pi_\star \mathscr{O}_{\mathfrak{F}\mathfrak{L}^+_{n,\epsilon,w,w'}} \hat{\otimes} A)^{\mathfrak{T}_{w,n}}$$

over $\mathfrak{X}_{Kli}(p^n,\epsilon) \times \operatorname{Spf} A$.

12.7. Analytic geometry. — The aim of this section is to translate some of our constructions in the setting of analytic adic spaces. One of the improvements in the analytic setting is that the constructions can be performed for Klingen type level structure rather than full level structure. It will be natural to work with Klingen level structure when we consider Hecke operators.

12.7.1. Siegel analytic spaces. — We have an action of $\mathrm{GSp}_4(\mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}(p^n)$. We denote by $\mathcal{X}_{Kli}(p^n)$ the quotient of this space by the group $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z}) \subset \mathrm{GSp}_4(\mathbb{Z}/p^n\mathbb{Z})$ of matrices which are upper triangular with blocks of size 1×1, 2×2 and 1×1.

Let $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})^+$ be the subgroup of elements whose lower diagonal entry is trivial. This is a normal subgroup of $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$ and the quotient $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})/\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})^+$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^\times$. We let $\mathcal{X}_{Kli}(p^n)^+$ be the quotient of $\mathcal{X}(p^n)$ by this group.

For all $\epsilon \in [0, n - \frac{1}{p-1}] \cap \mathbb{Q}$, we denote by $\mathcal{X}(p^n,\epsilon)$ the analytic space associated to $\mathfrak{X}(p^n,\epsilon)$. This is an open subset of $\mathcal{X}(p^n)$ stabilized by the action of the Klingen parabolic $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z}) \subset \mathrm{GSp}_4(\mathbb{Z}/p^n\mathbb{Z})$ on this space. We denote by $\mathcal{X}_{Kli}(p^n,\epsilon) \subset \mathcal{X}_{Kli}(p^n)$ the quotient of $\mathcal{X}(p^n,\epsilon)$ by $\mathfrak{Kli}(\mathbb{Z}/p^n)$ and by $\mathcal{X}_{Kli}(p^n,\epsilon)^+ \subset \mathcal{X}_{Kli}(p^n)^+$ the quotient of $\mathcal{X}(p^n,\epsilon)$ by $\mathfrak{Kli}(\mathbb{Z}/p^n)^+$. We therefore have diagrams for all $n \in \mathbb{Z}_{\geq 1}$:

$$\begin{tikzcd}
\mathcal{X}_{Kli}(p^n,\epsilon) \arrow[r] \arrow[d] & \mathcal{X}_{Kli}(p^n) \arrow[d] \\
\mathcal{X}_{Kli}(p^{n-1},\epsilon) \arrow[r] & \mathcal{X}_{Kli}(p^{n-1})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{X}_{Kli}(p^n,\epsilon)^+ \arrow[r] \arrow[d] & \mathcal{X}_{Kli}(p^n)^+ \arrow[d] \\
\mathcal{X}_{Kli}(p^{n-1},\epsilon)^+ \arrow[r] & \mathcal{X}_{Kli}(p^{n-1})^+
\end{tikzcd}$$

Over $\mathcal{X}$ we define a sheaf $\omega_G^{mod,+}$ of $\mathscr{O}^+_{\mathcal{X}}$-modules for the étale topology. This is the subsheaf of the sheaf ω_G^+ of integral differential forms at the origin of G, generated by the image of the Hodge-Tate period map (compare with section 12.2.1).

Remark 12.7.1.1. — The sheaf $\omega_G^{mod,+}$ is really a sheaf on the étale site and does not come from the analytic site. Nevertheless, its pullback to $\mathcal{X}(p^n)$ for $n \geq 1$ (or $n \geq 2$ for $p = 2$) comes from a sheaf on the Zariski site.

The space $\mathcal{X}_{Kli}(p^n,\epsilon)$ has the following simple modular interpretation. It parametrizes pairs (x, H_n) where x is a point of $\mathcal{X}$ and $H_n \subset G_x[p^n]$ is a finite flat group scheme locally isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, which is locally for the étale topology generated by an element e_1 which satisfies $\mathrm{HT}(e_1) = 0$ in $\omega_{G_x}^{mod,+}/p^\epsilon \omega_{G_x}^{mod,+}$.

We can define sheaves for the étale topology

$$\mathrm{Fil}_\epsilon^{can} = \mathrm{Im}(\mathrm{HT}\otimes 1 : H_n^{\perp}\otimes \mathscr{O}^+_{\mathcal{X}_{Kli}(p^n,\epsilon)} \to (\omega_G^{mod,+})_\epsilon)$$

and

$$\mathrm{Gr}_\epsilon^{can} = \mathrm{coker}(\mathrm{HT}\otimes 1 : H_n^{\perp}\otimes \mathscr{O}^+_{\mathcal{X}_{Kli}(p^n,\epsilon)} \to (\omega_G^{mod,+})_\epsilon).$$

These are locally free sheaves of $\mathscr{O}^+_{\mathcal{X}_{Kli}(p^n,\epsilon)}/p^\epsilon$-modules (compare with section 12.2.2 and section 12.2.3).

The space $\mathcal{X}_{Kli}(p^n,\epsilon)^+ \to \mathcal{X}_{Kli}(p^n,\epsilon)$ is the torsor of trivializations of H_n^D. We let $\psi : \mathbb{Z}/p^n\mathbb{Z} \to H_n^D$ be the universal trivialization.

Over $\mathcal{X}_{Kli}(p^n,\epsilon)^+$ we have a canonical isomorphism

$$\mathrm{HT}_4\otimes 1 : \mathbb{Z}/p^n\mathbb{Z}\otimes (\mathscr{O}^+_{\mathcal{X}_{Kli}(p^n,\epsilon)})_\epsilon \to \mathrm{Gr}_\epsilon^{can}$$

obtained via the map ψ and the Hodge-Tate map for $G[p^n]$ (compare with section 12.2.3).

Remark 12.7.1.2. — We have obtained the analogue of paragraphs 12.2.2 and 12.2.3 in the analytic setting. We observe that in the analytic setting we are able to work at the level of $\mathcal{X}_{Kli}(p^n,\epsilon)$ rather than $\mathcal{X}(p^n,\epsilon)$. The main reason being that the map $\mathcal{X}(p^n,\epsilon)\to\mathcal{X}_{Kli}(p^n,\epsilon)$ is finite flat and étale away from the boundary while this fails for the map $\mathfrak{X}(p^n,\epsilon)\to\mathfrak{X}_{Kli}(p^n,\epsilon)$. It will turn out to be more natural later when we want to define the action of the Hecke operator U to work at "Klingen" level.

12.7.2. Analytic flag varieties. — We let $\mathcal{FL}^+_{n,\epsilon,w,w'} \to \mathcal{FL}_{n,\epsilon,w}\to\mathcal{X}(p^n,\epsilon)$ be the analytic spaces associated to $\mathfrak{FL}_{n,\epsilon,w}$ and $\mathfrak{FL}^+_{n,\epsilon,w,w'}$.

Lemma 12.7.2.1. — *The space $\mathcal{FL}_{n,\epsilon,w}$ descends to an open-subspace of the flag variety $\mathcal{FL}\to\mathcal{X}_{Kli}(p^n,\epsilon)$ of ω_G (17) that we denote by $\mathcal{FL}_{Kli,n,\epsilon,w}$. This is the space of flags $\mathrm{Fil}\omega_G\subset\omega_G$ such that locally for the étale topology $(\mathrm{Fil}\omega_G\cap\omega_G^{mod,+})_w = \mathrm{Fil}_w^{can}$.*

Proof. Consider the map of analytic spaces : $\mathcal{X}(p^n,\epsilon)\times_{\mathcal{X}_{Kli}(p^n,\epsilon)}\mathcal{FL}\to\mathcal{FL}$. This map is finite flat. Moreover, $\mathcal{FL}_{n,\epsilon,w}\hookrightarrow\mathcal{X}(p^n,\epsilon)\times_{\mathcal{X}_{Kli}(p^n,\epsilon)}\mathcal{FL}$ is an admissible open subset. We can therefore apply the descent of admissible opens of [**16**], lem. 4.2.4. □

Let us denote by $\mathcal{FL}^+\to\mathcal{X}_{Kli}(p^n,e_1)^+$ the moduli space of flags (a locally direct summand of rank 1 in this case) $\mathrm{Fil}\omega_G$ of ω_G together with a trivialization $\rho\in\mathrm{Gr}\omega_G = \omega_G/\mathrm{Fil}\omega_G$ (18).

Lemma 12.7.2.2. — *The space $\mathcal{FL}^+_{n,\epsilon,w,w'}$ descends to an open-subspace of $\mathcal{FL}^+\to\mathcal{X}_{Kli}(p^n,e_1)^+$ that we denote by $\mathcal{FL}^+_{Kli,n,\epsilon,w,w'}$. This is the space of flags $\mathrm{Fil}\omega_G\subset\omega_G$ and trivialization $\rho\in\mathrm{Gr}_{\omega_G}$ which satisfy the following conditions :*

— $(\mathrm{Fil}\omega_G\cap\omega_G^{mod,+})_w = \mathrm{Fil}_w^{can}$,

— *The trivialization ρ belongs to $\omega_G^{mod,+}/(\mathrm{Fil}\omega_G\cap\omega_G^{mod,+})$ and reduces to the element $\mathrm{HT}_4(1)$ of $\mathrm{Gr}_{w'}^{can}$.*

Proof. This is another application of [**16**], lem. 4.2.4. □

Let us denote by $\mathcal{T}_{w'}$, $\mathcal{T}^0_{w'}$, $\mathcal{T}_{w',n}$ the analytic fibers of $\mathfrak{T}_{w'}$ and $\mathfrak{T}^0_{w'}$ and $\mathfrak{T}_{w',n}$. We denote by $\mathscr{L}^{\kappa_A}$ the invertible sheaf over $\mathcal{FL}_{n,\epsilon,w}\times\mathrm{Spa}(A[1/p],A)$ associated to $\mathfrak{L}^{\kappa_A}$. We denote by $\mathscr{G}^{\kappa_A,w}$ the Banach sheaf generic fiber of $\mathfrak{G}^{\kappa_A,w}$ over $\mathcal{X}(p^n,\epsilon)\times\mathrm{Spa}(A[1/p],A)$ (see [**3**], def. A.2.1.2 and prop. A.2.2.3). We finally denote by $\mathscr{F}^{\kappa_A,w}$ the Banach sheaf

17. This flag variety is simply a $\mathbb{P}^1$-bundle.

18. ρ is a nowhere vanishing section of the line bundle $\mathrm{Gr}(\omega_G)$.

associated to $\mathfrak{F}^{\kappa_A,w}$ over $\mathcal{X}_{Kli}(p^n,\epsilon)\times \mathrm{Spa}(A[1/p],A)$. A more direct definition of $\mathscr{F}^{\kappa_A,w}$ is the following

$$\mathscr{F}^{\kappa_A,w} = (\pi_\star \mathscr{O}_{\mathcal{FL}^+_{Kli,n,\epsilon,w,w'}}\hat{\otimes} A)^{\mathcal{T}_{w',n}}$$

where $\pi:\mathcal{FL}^+_{Kli,n,\epsilon,w,w'}\to \mathcal{X}_{Kli}(p^n,\epsilon)$ is the projection.

12.8. Overconvergent cohomology. — We are now ready to define overconvergent, locally analytic cohomology.

12.8.1. Definitions. — The (n,ϵ)-overconvergent, w-analytic cohomology of weight (κ_A,r) is the cohomology :

$$C(n,\epsilon,w,\kappa_A,r) := \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r).$$

There is also a cuspidal version :

$$C_{cusp}(n,\epsilon,w,\kappa_A,r) := \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r(-D)).$$

There are obvious maps $C(n,\epsilon,w,\kappa,r)\to C(n_1,\epsilon_1,w_1,\kappa,r)$ for $n_1\geq n$, $\epsilon_1\geq\epsilon$, $w_1\geq w$ (and $w\leq\epsilon$, $w_1\leq\epsilon_1$, $\epsilon\leq n-\frac{1}{p-1}$, $\epsilon_1\leq n_1-\frac{1}{p-1}$).

We may define the overconvergent, locally analytic degree i cohomology of weight (κ_A,r) to be

$$\mathrm{H}^i(\dagger,\kappa_A,r) = \mathrm{colim}_{n,\epsilon,w\to\infty}\mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r)$$

and similarly for the cuspidal version :

$$\mathrm{H}^i_{cusp}(\dagger,\kappa_A,r) = \mathrm{colim}_{n,\epsilon,w\to\infty}\mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r(-D)).$$

12.8.2. Another interpretation. — Here is another way to think about these cohomology groups in terms of coherent cohomology. Thanks to section 12.5, we observe that $\mathcal{FL}_{n,\epsilon,w}$ is locally affine over $\mathcal{X}(p^n,\epsilon)$: this means that there is a covering of $\mathcal{X}(p^n,\epsilon)$ by affinoid spaces such that the fiber of $\mathcal{FL}_{n,\epsilon,w}$ over each such affinoid is affinoid (19). The sheaf $\mathscr{G}^{\kappa_A,w}$ comes from the line bundle $\mathscr{L}^{\kappa_A}$ over $\mathcal{FL}_{n,\epsilon,w}$ by pushforward via the map π_2 : $\mathcal{FL}_{n,\epsilon,w}\to\mathcal{X}(p^n,\epsilon)$. Since $\mathrm{R}^i(\pi_2)_\star\mathscr{L}^{\kappa_A}=0$ for $i>0$, we obtain that

$$\mathrm{R}\Gamma(\mathcal{X}(p^n,\epsilon),\mathscr{G}^{\kappa_A,w}\otimes\omega^r) = \mathrm{R}\Gamma(\mathcal{FL}_{n,\epsilon,w},\mathscr{L}^{\kappa_A}\otimes\omega^r)$$

and

$$\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r) = \mathrm{R}\Gamma(\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z}),\mathrm{R}\Gamma(\mathcal{FL}_{n,\epsilon,w},\mathscr{L}^{\kappa_A}\otimes\omega^r)).$$

Similar statements hold for cuspidal cohomology.

Proposition 12.8.2.1. — *The cohomology complexes $C(n,\epsilon,w,\kappa_A,r)$ and $C_{cusp}(n,\epsilon,w,\kappa_A,r)$ are represented by bounded complexes of projective Banach $A[1/p]$-modules.*

Proof. We only treat the non-cuspidal version. We take a covering $\mathcal{U}$ of $\mathcal{FL}_{n,\epsilon,w}$ by affinoids such that the sheaf $\mathscr{L}^{\kappa_A}$ is isomorphic to $A\hat{\otimes}_{\mathcal{O}}\mathscr{O}_U$ over each $U\in\mathcal{U}$. Refining $\mathcal{U}$ by adding all the $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$-translates of each opens, we can assume that $\mathcal{U}$ is $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$-stable. The $\mathcal{U}$-Čech complex of the sheaf $\mathscr{L}^{\kappa_A}\otimes\omega^r$ is a bounded complex of projective Banach $A[1/p]$-modules which computes $\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{G}^{\kappa_A,w}\otimes\omega^r)$. The group $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$ acts on this complex and the direct factor of invariants computes the cohomology $\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r)$. □

19. This does not imply that the fiber of any affinoid is an affinoid, unlike in the case of schemes.

12.9. Cohomological vanishing. — The main result of this section is a cohomological vanishing.

Proposition 12.9.1. — *The cuspidal overconvergent locally analytic cohomology* $\mathrm{H}^i_{cusp}(\dagger, \kappa_A, r)$ *vanishes for* $i > 1$.

The proof of this proposition follows [**3**] section 8 closely. The strategy is to compute this cohomology on the minimal compactification. The cohomological vanishing results from two facts :

1. that the relative cuspidal cohomology between toroidal and minimal compactification vanishes in higher degree,
2. that the pushforward of our overconvergent sheaves to the minimal compactification are supported on open subsets that can be covered by two affines.

12.9.1. The minimal compactification. — We let $\mathfrak{X}^\star$ be the minimal compactification of $\mathfrak{Y}$. There is a natural map $\mathfrak{X}(p^n) \to \mathfrak{X}^\star$ and we define $\mathfrak{X}(p^n)^\star$ to be the Stein factorization of this map. In [**63**], we proved that the determinant of the Hodge-Tate map :

$$\Lambda^2\mathrm{HT} : \Lambda^2((\mathbb{Z}/p^n\mathbb{Z})^4) \to \det \omega_G/p^n$$

descends from $\mathfrak{X}(p^n)$ to $\mathfrak{X}(p^n)^\star$.

In [**63**] section 1.8 we have introduced a formal scheme $\mathfrak{X}(p^n)^{\star-mod} \to \mathfrak{X}(p^n)^\star$. This space is the normalization of a blow up and it carries a locally free modification $\det \omega_G^{mod} \subset \det \omega_G$ such that :

1. $p^{\frac{2}{p-1}} \det \omega_G \subset \det \omega_G^{mod} \subset \det \omega_G$,
2. The Hodge-Tate map induces a surjective map :
$$\Lambda^2\mathrm{HT} : \Lambda^2((\mathbb{Z}/p^n\mathbb{Z})^4) \otimes \mathscr{O}_{\mathfrak{X}(p^n)^{\star-mod}} \to \det \omega_G^{mod}/p^{n-\frac{2}{p-1}}.$$

By the universal property of blow-up and normalization, there is a map $\mathfrak{X}(p^n)^{mod} \to \mathfrak{X}(p^n)^{\star-mod}$ such that the pull back of $\det \omega_G^{mod}$ is det of ω_G^{mod} and the pull back of the map $\Lambda^2\mathrm{HT} : \Lambda^2((\mathbb{Z}/p^n\mathbb{Z})^4) \to \det \omega_G^{mod}/p^{n-\frac{2}{p-1}}$ agrees with Λ^2 applied to the map $\mathrm{HT} : (\mathbb{Z}/p^n\mathbb{Z})^4 \to \omega_G^{mod}/p^{n-\frac{2}{p-1}}$.

Let $\epsilon \in [0, n-\frac{2}{p-1}] \cap \mathbb{Q}$. We let $\mathfrak{X}(p^n, \epsilon)^\star$ be the formal scheme where $\mathrm{HT}(e_1) \wedge \mathrm{HT}(e_2) = \mathrm{HT}(e_1) \wedge \mathrm{HT}(e_3) = \mathrm{HT}(e_1) \wedge \mathrm{HT}(e_4) = 0 \mod p^\epsilon$.

Lemma 12.9.1.1. — *There is a cartesian diagram :*

$$\begin{tikzcd} \mathfrak{X}(p^n,\epsilon) \arrow[r] \arrow[d] & \mathfrak{X}(p^n)^{mod} \arrow[d] \\ \mathfrak{X}(p^n,\epsilon)^\star \arrow[r] & \mathfrak{X}(p^n)^{\star-mod} \end{tikzcd}$$

Proof. It suffices to prove that the condition $\mathrm{HT}(e_1) \wedge \mathrm{HT}(e_2) = \mathrm{HT}(e_1) \wedge \mathrm{HT}(e_3) = \mathrm{HT}(e_1) \wedge \mathrm{HT}(e_4) = 0 \mod p^\epsilon$ is equivalent to the condition $\mathrm{HT}(e_1) = 0 \mod p^\epsilon$. The reverse implication is obvious so let us prove the direct implication. Under the natural perfect pairing $(\omega_G^{mod})_\epsilon \times (\omega_G^{mod})_\epsilon \to (\det \omega_G^{mod})_\epsilon$, we have by assumption that $\mathrm{HT}(e_1)$ is orthogonal to the entire $(\omega_G^{mod})_\epsilon$ (which is generated by $\mathrm{HT}(e_i)$, $1 \leq i \leq 4$) so $\mathrm{HT}(e_1) = 0 \mod p^\epsilon$.

□

We denote $\mathfrak{X}(p^n,\epsilon,e_2)^\star$ and $\mathfrak{X}(p^n,\epsilon,e_3)^\star$ the open formal subschemes of $\mathfrak{X}(p^n,\epsilon)^\star$ where the sheaf $(\det\omega_G^{mod})_\epsilon$ is generated by $\mathrm{HT}(e_4)\wedge\mathrm{HT}(e_2)$ and $\mathrm{HT}(e_4)\wedge\mathrm{HT}(e_3)$ respectively.

Lemma 12.9.1.2. — *We have cartesian diagrams :*

$$\begin{tikzcd} \mathfrak{X}(p^n,\epsilon,e_i) \arrow[r] \arrow[d] & \mathfrak{X}(p^n)^{mod} \arrow[d] \\ \mathfrak{X}(p^n,\epsilon,e_i)^\star \arrow[r] & \mathfrak{X}(p^n)^{\star-mod} \end{tikzcd}$$

for $i\in\{2,3\}$.

Proof. This follows from the fact that $(\omega_G^{mod})_\epsilon$ is generated by $\mathrm{HT}(e_i)$ and $\mathrm{HT}(e_4)$ if and only if $\mathrm{HT}(e_4)\wedge\mathrm{HT}(e_i)$ generates $\det(\omega_G^{mod})_\epsilon$. □

By [**71**], p. 72 (see also [**63**], thm. 1.16), there is an integer N such that for all $n\geq N$ there is a formal scheme $\mathfrak{X}(p^n)^{\star-HT}$ and a projective map $\mathfrak{X}(p^n)^{\star-mod}\to\mathfrak{X}(p^n)^{\star-HT}$ such that :

1. $\mathfrak{X}(p^n)^{\star-HT}$ is a normal admissible formal scheme with generic analytic adic fiber $\mathcal{X}(p^n)^\star$,
2. The invertible sheaf $\det\omega_G^{mod}$ descends to an ample invertible sheaf $\det\omega_G^{mod}$ over $\mathfrak{X}(p^n)^{\star-HT}$,
3. For all $\epsilon>0$, there is $n(\epsilon)\geq N$ such that for all $n\geq n(\epsilon)$ we have sections $s_{i,j}\in\mathrm{H}^0(\mathfrak{X}(p^n)^{\star-HT},\det\omega_G^{mod})$ for $1\leq i,j\leq 4$ such that $s_{i,j}=\mathrm{HT}(e_i)\wedge\mathrm{HT}(e_j)\in\det\omega_G^{mod}/p^\epsilon$.

Let $\epsilon>0$ and let $n\geq n(\epsilon)$. Let us define $\mathfrak{X}(p^n,\epsilon,e_i)^{\star-HT}\to\mathfrak{X}(p^n)^{\star-HT}$ by the conditions :
— $s_{i,4}\neq 0$,
— $s_{1,j}\in p^\epsilon\det\omega_G^{mod},\ \forall 1\leq j\leq 4$.

Lemma 12.9.1.3. — *The formal scheme $\mathfrak{X}(p^n,\epsilon,e_i)^{\star-HT}$ is affine and the map*

$$\mathfrak{X}(p^n,\epsilon,e_i)^{\star-mod}\to\mathfrak{X}(p^n,\epsilon,e_i)^{\star-HT}$$

is a projective map and is an isomorphism on the generic fiber.

Proof. The open formal subscheme of $\mathfrak{X}(p^n)^{\star-HT}$ defined by $s_{i,4}\neq 0$ is affine since $\det\omega_G^{mod}$ is ample. Let us denote by A its ring of functions. Observe that $\det\omega_G^{mod}$ is trivial over $\mathrm{Spf}\ A$, generated by $s_{i,4}$. The formal scheme defined by the equation $s_{1,j}\in p^\epsilon\det\omega_G^{mod}$ is

$$\mathrm{Spf}\ A\langle\frac{s_{1,j}}{s_{i,4}p^\epsilon},\ 1\leq j\leq 4\rangle$$

and is again affine. The final claim follows from the obvious equality

$$\mathfrak{X}(p^n,\epsilon,e_i)^{\star-mod}=\mathfrak{X}(p^n)^{\star-mod}\times_{\mathfrak{X}(p^n)^{\star-HT}}\mathfrak{X}(p^n,\epsilon,e_i)^{\star-HT}.$$

□

12.9.2. Vanishing. — A formal Banach sheaf $\mathfrak{F}$ over an admissible formal scheme $\mathfrak{S}$ is small if $\mathfrak{F}_1$ can be written has an inductive limit of coherent sheaves $\mathrm{colim}_{j\in\mathbb{N}}\mathfrak{F}_{1,j}$ with injective transition maps, and there exists a coherent sheaf $\mathscr{G}$ over $\mathfrak{S}$ such that the quotients $\mathfrak{F}_{1,j+1}/\mathfrak{F}_{1,j}$ are direct summands of $\mathscr{G}$. We now recall a vanishing result of [**3**], thm. A.1.2.2 :

Theorem 12.9.2.1. — *Let $\mathfrak{S}$ be an admissible formal scheme. Assume that $\mathfrak{S}$ admits a projective map $\mathfrak{S} \to \mathfrak{S}'$ to an affine admissible formal scheme which is an isomorphism of the associated analytic adic spaces over* $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O})$*. Let $\mathfrak{F}$ be a small Banach sheaf over $\mathfrak{S}$. Let $\mathfrak{U}$ be an affine cover of $\mathfrak{S}$. Then the Čech complex*

$$\check{C}ech(\mathfrak{U}, \mathfrak{F}) \otimes_{\mathcal{O}} \mathbb{C}_p$$

is acyclic in positive degree.

We denote by $\pi : \mathfrak{X}(p^n, \epsilon) \to \mathfrak{X}(p^n, \epsilon)^\star$ the projection. The following proposition is the analogue of [**3**], proposition 8.2.2.4 (see also [**51**]) :

Proposition 12.9.2.1. — *We have the vanishing* $\mathrm{R}^i\pi_\star\mathscr{O}_{\mathfrak{X}(p^n,\epsilon)}(-D)$ *for all $i \geq 1$.*

Proof. The formal scheme $\mathfrak{X}(p^n, \epsilon)$ carries a stratification indexed by a subset of the set of all Lagrangian locally direct factors W of $V = \mathbb{Z}^4$. We are going to describe briefly this stratification, based on the analogous description of the stratification of $\mathfrak{X}(p^n)^{mod}$ given in proposition 4.9 of [**63**]. The case $W = \{0\}$ corresponds to the open and dense stratum with complement the boundary D. This stratum maps isomorphically to its image in $\mathfrak{X}(p^n, \epsilon)^\star$. We now deal with the case W is one-dimensional. First of all there is a one-dimensional affine formal scheme $\mathfrak{X}_W(p^n, \epsilon)$ constructed as follows. We start with the formal affine modular curve $\mathfrak{X}_W$ of some prime-to-p level determined by W and the tame level K^p. Then we can construct a normal formal scheme $\mathfrak{X}_W(p^n)$ and a finite map $\mathfrak{X}_W(p^n) \to \mathfrak{X}_W$ by adding a full level structure of level p^n. We then perform a blow up and a normalization to define $\mathfrak{X}_W(p^n)^{mod}$ which carries a locally free modification of the conormal sheaf of the universal elliptic curve. We finally consider a formal scheme $\mathfrak{X}_W(p^n, \epsilon) \to \mathfrak{X}_W(p^n)^{mod}$ which is an open subscheme of a blow up defined by a condition on the Hodge-Tate period map.

Over $\mathfrak{X}_W(p^n, \epsilon)$ we have an elliptic curve $\mathfrak{B}_W(p^n, \epsilon) \to \mathfrak{X}_W(p^n, \epsilon)$, isogenous to the universal elliptic curve. There is a $\mathbb{G}_m$-torsor $\mathfrak{M}_W(p^n, \epsilon) \to \mathfrak{B}_W(p^n, \epsilon)$ isogenous to the torsor of trivializations of $\mathscr{O}_{\mathfrak{B}_W(p^n,\epsilon)}(-2O)$ (where O is the identity section of the elliptic curve) and a relative toroidal embedding $\mathfrak{M}_W(p^n, \epsilon) \hookrightarrow \overline{\mathfrak{M}}_W(p^n, \epsilon)$ (obtained by adjoining to the $\mathbb{G}_m$-torsor the 0 element). The complement of $\mathfrak{M}_W(p^n, \epsilon) \hookrightarrow \overline{\mathfrak{M}}_W(p^n, \epsilon)$ maps isomorphically to $\mathfrak{B}_W(p^n, \epsilon)$. The W-stratum in $\mathfrak{X}(p^n, \epsilon)$ is $\mathfrak{B}_W(p^n, \epsilon)$ and the completion of $\mathfrak{X}(p^n, \epsilon)$ along $\mathfrak{B}_W(p^n, \epsilon)$ is isomorphic to the completion of $\overline{\mathfrak{M}}_W(p^n, \epsilon)$ along $\mathfrak{B}_W(p^n, \epsilon)$.

The morphism π restricts to a morphism $\mathfrak{B}_W(p^n, \epsilon) \to \mathfrak{X}(p^n, \epsilon)^\star$ and factors through $\mathfrak{B}_W(p^n, \epsilon) \to \mathfrak{X}_W(p^n, \epsilon) \to \mathfrak{X}(p^n, \epsilon)^\star$ where $\mathfrak{X}_W(p^n, \epsilon) \to \mathfrak{X}(p^n, \epsilon)^\star$ is finite (compare with [**63**], lem. 4.4 and thm. 4.7).

In the case W is two dimensional, the boundary component is included in the ordinary locus and the maps $\mathfrak{X}(p^n, \epsilon) \to \mathfrak{X}(p^n)^{mod} \to \mathfrak{X}(p^n)$ restrict on the ordinary locus respectively to an open immersion and an isomorphism. The description of the boundary component is given in [**63**], thm 4.1. We recall that there is a formal torus T_W isogenous to the p-adic completion of $\mathrm{Hom}(\mathrm{Sym}^2 V/W^\perp, \mathbb{G}_m)$, a T_W-torsor $\mathfrak{M}_W(p^n, \epsilon)$, a relative toroidal embedding $\mathfrak{M}_W(p^n, \epsilon) \hookrightarrow \overline{\mathfrak{M}}_W(p^n, \epsilon)$, a closed codimension 1 formal subscheme $\mathfrak{Z}_W(p^n, \epsilon) \hookrightarrow \overline{\mathfrak{M}}_W(p^n, \epsilon)$ in the complement of $\mathfrak{M}_W(p^n, \epsilon)$ and an arithmetic subgroup Γ_W of $\mathrm{GL}(W)$ such that the closed W-stratum is isomorphic to $\mathfrak{Z}_W(p^n, \epsilon)/\Gamma_W$ and the completion of $\mathfrak{X}(p^n, \epsilon)$ along $\mathfrak{Z}_W(p^n, \epsilon)/\Gamma_W$ is isomorphic to the completion of $\overline{\mathfrak{M}}_W(p^n, \epsilon)/\Gamma_W$

along $\mathfrak{Z}_W(p^n,\epsilon)/\Gamma_W$. Lastly, the image of $\mathfrak{Z}_W(p^n,\epsilon)/\Gamma_W$ in $\mathfrak{X}(p^n,\epsilon)^\star$ is a closed formal subscheme, finite over Spf $\mathcal{O}$.

By the theorem on formal functions, the vanishing theorem is equivalent to :

1. $\mathrm{H}^i(\overline{\mathfrak{M}}_W(p^n,\epsilon)/\Gamma_W, \mathscr{O}_{\overline{\mathfrak{M}}_W(p^n,\epsilon)/\Gamma_W}(-\mathfrak{Z}_W(p^n,\epsilon)/\Gamma_W)) = 0$ for all $i>0$ and W two dimensional,
2. $\mathrm{H}^i(\widehat{\overline{\mathfrak{M}}_W(p^n,\epsilon)}^x, \mathscr{O}_{\overline{\mathfrak{M}}_W(p^n,\epsilon)}(-\mathfrak{B}_W(p^n,\epsilon))$ for all $i>0$, W one-dimensional, $x\in \mathfrak{X}_W(p^n,\epsilon)$ a closed point. We have denoted by $\widehat{\overline{\mathfrak{M}}_W(p^n,\epsilon)}^x$ the completion of $\overline{\mathfrak{M}}_W(p^n,\epsilon)$ along the fiber of the map $\mathfrak{B}_W(p^n,\epsilon)\to\mathfrak{X}_W(p^n,\epsilon)$ at x.

We are therefore in a similar situation to [**3**], proposition 8.2.2.4, or to [**51**], sect. 4. One can conclude by repeating the arguments of these papers.

□

Lemma 12.9.2.1. — *Let* $\epsilon>0$. *There exists* $n(\epsilon)$ *such that for all* $n\geq n(\epsilon)$, $\mathrm{R}\Gamma(\mathcal{X}(p^n,\epsilon),\mathscr{G}^{\kappa_A,w}\otimes\omega^r(-D))$ *is concentrated in degree* 0 *and* 1.

Proof. We let $\pi:\mathfrak{X}(p^n,\epsilon)\to\mathfrak{X}(p^n,\epsilon)^\star$ denote the usual projection. By lemma 12.6.2.2, proposition 12.9.2.1 and proposition A.1.3.1 of [**3**], $\pi_\star\mathfrak{G}^{\kappa_A,w}\otimes\omega^r(-D)$ is a small formal Banach sheaf over $\mathfrak{X}(p^n,\epsilon)^\star$ and $\mathrm{R}^i\pi_\star\mathfrak{G}^{\kappa_A,w}\otimes\omega^r(-D)=0$ for all $i>0$.

Let us take an affine covering $\mathfrak{V}_i$ of $\mathfrak{X}(p^n,\epsilon,e_i)^\star$ and an affine covering $\mathfrak{U}_i$ of $\mathfrak{FL}_{n,\epsilon,w}|_{\mathfrak{X}(p^n,\epsilon,e_i)^\star}$ which refines the inverse image of $\mathfrak{V}_i$ in $\mathfrak{FL}_{n,\epsilon,w}|_{\mathfrak{X}(p^n,\epsilon,e_i)^\star}$ for $i\in\{2,3\}$.

Since $\mathrm{R}^i\pi_\star\mathfrak{G}^{\kappa_A,w}\otimes\omega^r(-D)=0$ for all $i>0$ we deduce that the map

$$\check{\mathrm{C}}\mathrm{ech}(\mathfrak{V}_i,\pi_\star\mathfrak{G}^{\kappa_A,w}\otimes\omega^r(-D))\to\check{\mathrm{C}}\mathrm{ech}(\mathfrak{U}_i,\mathfrak{L}^{\kappa_A}\otimes\omega^r(-D))$$

is a quasi-isomorphism.

We deduce from thm 12.9.2.1 that $\check{\mathrm{C}}\mathrm{ech}(\mathfrak{U}_i,\mathfrak{L}^{\kappa_A}\otimes\omega^r(-D))[1/p]$ is concentrated in degree 0. We now consider the Čech complex associated to the covering $\mathfrak{U}=\mathfrak{U}_2\cup\mathfrak{U}_3$ of $\mathfrak{FL}_{n,\epsilon,w}$ for the sheaf $\mathfrak{L}^{\kappa_A}(-D))$. Then $\check{\mathrm{C}}\mathrm{ech}(\mathfrak{U},\mathfrak{L}^{\kappa_A}\otimes\omega^r(-D))[1/p]$ computes $\mathrm{R}\Gamma(\mathcal{FL}_{n,\epsilon,w},\mathscr{L}^{\kappa_A}\otimes\omega^r(-D))$. But this Čech complex is quasi-isomorphic to the complex:

$$\mathrm{H}^0(\mathcal{X}(p^n,\epsilon,e_2)^\star,\pi_\star\mathscr{G}^{\kappa_A}\otimes\omega^r(-D))\oplus\mathrm{H}^0(\mathcal{X}(p^n,\epsilon,e_3)^\star,\pi_\star\mathscr{G}^{\kappa_A}\otimes\omega^r(-D))$$
$$\longrightarrow\mathrm{H}^0(\mathcal{X}(p^n,\epsilon,e_2)^\star\cap\mathcal{X}(p^n,\epsilon,e_3)^\star,\pi_\star\mathscr{G}^{\kappa_A}\otimes\omega^r(-D))$$

and has therefore cohomology in degree 0 and 1.

□

Corollary 12.9.2.1. — *Let* $\epsilon>0$. *There exists* $n(\epsilon)$ *such that for all* $n\geq n(\epsilon)$, $\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r(-D))$ *is concentrated in degree* 0 *and* 1.

Proof. This follows from the formula

$$\mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r(-D))=\mathrm{H}^0(\mathfrak{Kli}(\mathbb{Z}/p^n),\mathrm{H}^i(\mathcal{X}(p^n,\epsilon),\mathscr{G}^{\kappa_A,w}\otimes\omega^r(-D))).$$

□

13. Finite slope families

In this section we will apply Coleman's spectral theory to our overconvergent cohomology in order to construct finite slope families.

13.1. Review of spectral theory. — We quickly review the notion of slope decomposition and the construction of spectral varieties.

13.1.1. Slope decomposition. — The valuation on $\mathbb{Q}_p$ is normalized by $v(p) = 1$ as usual. Let k be a complete non-archimedean field extension of $\mathbb{Q}_p$ for a valuation extending the p-adic valuation. Let M be a vector space over k and let U be an endomorphism of the vector space M. Let $h \in \mathbb{Q}$. A h-slope decomposition of M with respect to U is a direct sum decomposition of k-vector spaces $M = M^{\leq h} \oplus M^{>h}$ such that:

1. $M^{\leq h}$ and $M^{>h}$ are stable under the action of U.
2. $M^{\leq h}$ is finite dimensional over k.
3. All the eigenvalues (in an algebraic closure of k) of U acting on $M^{\leq h}$ are of valuation less or equal to h.
4. For any polynomial Q with roots of valuation strictly greater than h, the restriction of $Q^\star(U)$ to $M^{>h}$ is an invertible endomorphism. Here $Q^\star$ is the reciprocal of Q.

By [**81**], coro. 2.3.3, if such a slope decomposition exists, it is unique. If M has h-slope decomposition for all $h \in \mathbb{Q}$, we simply say that M has slope decomposition. In this situation we can obviously define submodules $M^{=h}$ and $M^{<h}$ of M for all $h \in \mathbb{Q}$.

13.1.2. Spectral varieties. — Let A be a Tate algebra over k. We let $\mathbf{Ban}(A)$ be the category of Banach modules over A. A Banach module is called projective if it is a direct factor of an orthonormalizable Banach module. We let $\mathbf{K}^{proj}(A)$ be the category whose objects are bounded complexes of projective Banach modules over A and morphisms are homotopy classes of morphisms of complexes. Let $M^\bullet \in \mathbf{K}^{proj}(A)$. An element $U \in \mathrm{End}_{\mathbf{K}^{proj}(A)}(M^\bullet)$ is compact if it has a representative $\tilde{U} \in \mathrm{End}_A(M^\bullet)$ whose restriction to each M^k is compact.

Given a compact representative $\tilde{U}$, we can define by [**14**], Part A, (or [**9**]) the characteristic series $\tilde{P}(X) = \det(1 - X\tilde{U}|M^\bullet) = \prod_k \det(1 - X\tilde{U}|M^k)$. This characteristic series is entire: it defines a function on $\mathbb{A}^1 \times \mathrm{Spa}(A, A^+)$. We denote by $\tilde{\mathcal{Z}} \hookrightarrow \mathbb{A}^1 \times \mathrm{Spa}(A, A^+)$ the spectral variety which is the closed subspace defined by $\tilde{P}(X)$. It depends on $\tilde{U}$. Over $\tilde{\mathcal{Z}}$ we have a complex of coherent sheaves $\mathcal{M}^\bullet$. It is the universal eigenspace of $M^\bullet$ for the action of $\tilde{U}$. There is a covering of $\tilde{\mathcal{Z}}$ by opens $\mathcal{U}$ which are finite over their image $\mathcal{V}$ in $\mathrm{Spa}(A, A^+)$ and such that $\mathcal{M}^\bullet|_{\mathcal{U}}$ is a perfect complex of $\mathscr{O}_{\mathcal{V}}$-module.

The cohomology groups $H^\bullet(\mathcal{M}^\bullet)$ are coherent sheaves over $\tilde{\mathcal{Z}}$. Let $\mathscr{I} \subset \mathscr{O}_{\tilde{\mathcal{Z}}}$ be the annihilator of this graded module. We let $\mathcal{Z} = V(\mathscr{I}) \subset \tilde{\mathcal{Z}}$ be the spectral variety associated to U and $M^\bullet$. It does not depend on the choice of $\tilde{U}$. It comes equipped with a graded coherent sheaf $H^\bullet(\mathcal{M}^\bullet)$.

13.1.3. Euler characteristic. — Let $M^\bullet$ be a complex of Banach modules and U be a compact operator as above. If $x : \mathrm{Spa}(K, \mathcal{O}_K) \to \mathrm{Spa}(A, A^+)$ is a rank one point, it follows from [**72**] that the space $H^i(M^\bullet_x)$ has a slope decomposition (the valuation v_x corresponding to x is normalized by $v_x(p) = 1$). We have :

Proposition 13.1.3.1. — *For all $h \in \mathbb{Q}$, the Euler-characteristic function*

$$x \mapsto \sum_i (-1)^i \dim H^i(M^\bullet_x)^{=h}$$

is a locally constant function of the rank one points of $\mathrm{Spa}(A, A^+)$ [20].

20. This means that for any rank one point x, there exists a neighborhood $\mathcal{U}_x$ of x in $\mathrm{Spa}(A, A^+)$, such that for all rank one point $y \in \mathcal{U}_x$, we have $\sum_i (-1)^i \dim H^i(M^\bullet_x)^{=h} = \sum_i (-1)^i \dim H^i(M^\bullet_y)^{=h}$. Be careful that $\{\mathcal{U}_x\}_{x \in \mathrm{Spa}(A,A^+),\ \mathrm{rk}(x)=1}$ is not a covering $\mathrm{Spa}(A, A^+)$ in general.

Proof. This follows from the equality

$$\sum_i (-1)^i \dim H^i(M_x^\bullet)^{=h} = \sum_i (-1)^i \dim (M_x^i)^{=h}$$

and the local constancy of $\dim(M_x^i)^{=h}$ (see [**14**], Part A). $\square$

13.2. The U-operator on overconvergent cohomology. — We construct the U-operator in the setting of overconvergent cohomology. The construction is parallel to section 10.

13.2.1. The cohomological correspondence C. — Let $\mathcal{Y}_{Kli}(p^n)$ be the open subspace of $\mathcal{X}_{Kli}(p^n)$ where the semi-abelian scheme is an abelian scheme. There is a Hecke correspondence $t_1 : C|_{\mathcal{Y}_{Kli}(p^n)} \to \mathcal{Y}_{Kli}(p^n)$ where $C|_{\mathcal{Y}_{Kli}(p^n)}$ is the moduli space of (G, H_n, L) where (G, H_n) is a point of $\mathcal{Y}_{Kli}(p^n)$ and $L \subset G[p^2]$ is a totally isotropic subgroup which is locally for the étale topology isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2 \oplus \mathbb{Z}/p^2\mathbb{Z}$ and $L \cap H_n = \{0\}$. The map t_1 sends (G, H_n, L) to (G, H). There is a map $t_2 : C_n|_{\mathcal{Y}_{Kli}(p^n)} \to \mathcal{Y}_{Kli}(p^{n+1})$ defined by mapping (G, H_n, L) to $(G/L, p^{-1}H_n + L/L)$.

By the theory of toroidal compactification (see [**48**] for instance), there exist a polyhedral cone decompositions Σ' and toroidal compactifications of $C|_{\mathcal{Y}_{Kli}(p^n)}$ which we denote by $C_{\Sigma'}$ or simply C and maps $t_1 : C_{\Sigma'} \to \mathcal{X}_{Kli}(p^n)_\Sigma$ and $t_2 : C_{\Sigma'} \to \mathcal{X}_{Kli}(p^{n+1})_\Sigma$ which extend the maps t_1 and t_2 previously defined. We drop Σ and Σ' from the notations if not necessary. We also recall that the map $(t_1)_\star \mathscr{O}_C \to \mathrm{R}(t_1)_\star \mathscr{O}_C$ is a quasi-isomorphism.

Lemma 13.2.1.1. — *Let $C_\epsilon = C \times_{t_1, \mathcal{X}_{Kli}(p^n)} \mathcal{X}_{Kli}(p^n, \epsilon)$. Then C_ϵ factorizes to a correspondence*

$$\begin{tikzcd} & C_\epsilon \arrow[dl, "t_2"'] \arrow[dr, "t_1"] & \\ \mathcal{X}_{Kli}(p^{n+1}, \epsilon+1) & & \mathcal{X}_{Kli}(p^n, \epsilon) \end{tikzcd}$$

Proof. All adic spaces are topologically of finite type, so it is enough to check that the map t_2 has the expected factorization for rank one points. Let $(K, \mathcal{O}_K)$ be a rank one point of C_n corresponding to an isogeny $\xi : G \to G_1$. Let $\hat{\bar{K}}$ be the completion of an algebraic closure of K. Over $\mathcal{O}_{\hat{\bar{K}}}$, we have a commutative diagram (where T_p is the Tate module and HT is the Hodge-Tate map) :

$$\begin{tikzcd} T_p(G) \arrow[r, "\xi"] \arrow[d, "\mathrm{HT}"] & T_p(G_1) \arrow[d, "\mathrm{HT}"] \\ \omega_G^{mod,+} \arrow[r, "\xi^D"] & \omega_{G_1}^{mod,+} \end{tikzcd}$$

In case G and G_1 are semi-abelian scheme, one can interpret $T_p(G)$ and $T_p(G_1)$ as the Tate modules of the corresponding 1-motives. We take a basis of $T_p(G) \simeq \mathbb{Z}_p^4$ and $T_p(G_1) \simeq \mathbb{Z}_p^4$ lifting the basis of $G[p^n]$ and $G_1[p^n]$ provided by the moduli problems. For suitable basis of ω_G and ω_{G_1} respecting the canonical filtration, this diagram is isomorphic to

$$\begin{array}{ccc} \mathbb{Z}_p^4 & \xrightarrow{[1,p,p,p^2]} & \mathbb{Z}_p^4 \\ \downarrow p_1 & & \downarrow p_2 \\ \mathcal{O}_{\hat{\bar{K}}}^2 & \xrightarrow{[p,p^2]} & \mathcal{O}_{\hat{\bar{K}}}^2 \end{array}$$

where $[1,p,p,p^2]$ and $[p,p^2]$ represent diagonal matrices. Moreover, by definition $p_1(e_1) \in p^{\epsilon}\mathcal{O}_{\hat{\bar{K}}}^2$. We deduce at once that the group generated by the image $[1,p,p,p^2](e_1)$ in $G_1[p^{n+1}]$ is independent of choices and that $p_2([1,p,p,p^2](e_1)) \in p^{\epsilon}p\mathcal{O}_{\hat{\bar{K}}}^2$. Therefore, at the level of points, we have proved that $t_2(C_\epsilon)$ factors through $\mathcal{X}_{Kli}(p^{n+1}, \epsilon+1)$.

□

13.2.2. Action on the sheaf. — In this section we prove that for all positive rational $w \leq \epsilon$ we can define over the correspondence C_ϵ a natural map:

$$t_2^\star \mathscr{F}^{\kappa_A, w+1} \to t_1^\star \mathscr{F}^{\kappa_A, w}.$$

Over the correspondence C_ϵ we consider the universal isogeny $\xi : G \to G_1$ and its differential $\xi^\star : \omega_{G_1} \to \omega_G$. Therefore we get a map $t_1^\star \mathcal{FL} \to t_2^\star \mathcal{FL}$ obtained by $\mathrm{Fil}\omega_G \mapsto (\xi^\star)^{-1}\mathrm{Fil}\omega_G$.

Lemma 13.2.2.1. — *The map $t_1^\star \mathcal{FL} \to t_2^\star \mathcal{FL}$ restricts to a map*

$$t_1^\star \mathcal{FL}_{Kli,n,\epsilon,\omega} \to t_2^\star \mathcal{FL}_{Kli,n+1,\epsilon+1,\omega+1}.$$

Proof. It is enough to check this on rank one points. Let $(K, \mathcal{O}_K)$ be a rank one point of C_ϵ corresponding to an isogeny $\xi : G \to G_1$. As in the proof of lemma 13.2.1.1, we obtain over $\mathcal{O}_{\hat{\bar{K}}}$ a commutative diagram :

$$\begin{array}{ccc} T_p(G) & \xrightarrow{\xi} & T_p(G_1) \\ \downarrow \mathrm{HT} & & \downarrow \mathrm{HT} \\ \omega_G^{mod,+} & \xrightarrow{\xi^D} & \omega_{G_1}^{mod,+} \end{array}$$

isomorphic to

$$\begin{array}{ccc} \mathbb{Z}_p^4 & \xrightarrow{[1,p,p,p^2]} & \mathbb{Z}_p^4 \\ \downarrow p_1 & & \downarrow p_2 \\ \mathcal{O}_{\hat{\bar{K}}}^2 & \xrightarrow{[p,p^2]} & \mathcal{O}_{\hat{\bar{K}}}^2 \end{array}$$

Let $\mathrm{Fil}\omega_G^{mod}$ be a flag. We may assume that it is generated by a vector $\mathrm{HT}(e_2)+\alpha p^w \mathrm{HT}(e_4)$ with $\alpha \in \mathcal{O}_{\hat{\bar{K}}}$ (up to changing e_2 and e_3). Its image via ξ^D is the line generated by $p\mathrm{HT}(e_2) + \alpha p^w p^2 \mathrm{HT}(e_4)$ or equivalently $\mathrm{HT}(e_2) + \alpha p^{w+1}\mathrm{HT}(e_4)$. □

Corollary 13.2.2.1. — *We have a map $\xi^\star : t_2^\star \mathscr{F}^{\kappa_A, w+1} \to t_1^\star \mathscr{F}^{\kappa_A, w}$.*

Proof. Let $\mathrm{Spa}(R, R^+) \to C_\epsilon$ be a point. Let $\xi : G \to G_1$ be the associated isogeny. To $(\mathrm{Fil}\omega_G, \rho_G : R \simeq \mathrm{Gr}(\omega_G) = \omega_G/\mathrm{Fil}\omega_G) \in \mathcal{FL}^+_{Kli,n,\epsilon,w,w'}$ we associate $(\xi^\star)^{-1}\mathrm{Fil}\omega_G$ and a trivialization $(\xi^\star)^{-1}\rho_G : R \simeq \mathrm{Gr}(\omega_G) \simeq \mathrm{Gr}(\omega_{G_1})$. This defines a point on $\mathcal{FL}^+_{Kli,n+1,\epsilon+1,w+1,w'}$. Given a section $s \in t_2^\star \mathscr{F}^{\kappa_A, w+1}$, we set $\xi^\star s(\mathrm{Fil}\omega_G, \rho_G) = s((\xi^\star)^{-1}\mathrm{Fil}\omega_G, (\xi^\star)^{-1}\rho_G)$. □

13.2.3. The action of U on overconvergent cohomology. — We now get an operator U as the composite

$$\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r)\to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^{n+1},\epsilon+1),\mathscr{F}^{\kappa_A,w+1}\otimes\omega^r)\to \mathrm{R}\Gamma(C_\epsilon,t_2^\star\mathscr{F}^{\kappa_A,w+1}\otimes\omega^r)$$

$$\stackrel{\frac{1}{p^r}\xi^\star}{\to} \mathrm{R}\Gamma(C_\epsilon,t_1^\star\mathscr{F}^{\kappa_A,w}\otimes\omega^r)\to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),(t_1)_\star t_1^\star\mathscr{F}^{\kappa_A,w}\otimes\omega^r)\stackrel{\frac{1}{p^3}\mathrm{Tr}}{\to} \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r)$$

and similarly on cuspidal cohomology. The map $\xi^\star$ is the tensor product of the map of corollary 13.2.2.1 and the obvious map $t_2^\star\omega^r\to t_1^\star\omega^r$.

Remark 13.2.3.1. — Note the normalization of the map $\xi^\star$ and of the Trace map.

13.2.4. Compactness. — We prove the compactness of the operator U.

Lemma 13.2.4.1. — *The natural map*

$$\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r)\to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^{n+1},\epsilon+1),\mathscr{F}^{\kappa_A,w+1}\otimes\omega^r)$$

is compact. A similar statement holds for cuspidal cohomology.

Proof. We have an obvious injective map $\mathcal{FL}_{n+1,\epsilon+1,w+1}\to\mathcal{X}(p^{n+1})\times_{\mathcal{X}(p^n)}\mathcal{FL}_{n,\epsilon,w}$. All these spaces are open subspaces of the the proper analytic spaces $\mathcal{FL}$ which parametrizes flags in ω_G over $\mathcal{X}(p^{n+1})$. It follows from the definitions that the closure of $\mathcal{FL}_{n+1,\epsilon+1,w+1}$ is contained in $\mathcal{X}(p^{n+1})\times_{\mathcal{X}(p^n)}\mathcal{FL}_{n,\epsilon,w}$.

Let $\mathcal{U}=\{\mathcal{U}_i\}_{i\in I}$ be an affinoid covering of $\mathcal{FL}_{n+1,\epsilon+1,w+1}$. We may assume that this covering is stable under the action of $\mathfrak{Kli}(\mathbb{Z}/p^{n+1}\mathbb{Z})$. By [**52**], thm. 5.1, for each $\mathcal{U}_i\in\mathcal{U}$ we can find an affinoid open $\mathcal{U}_i'\subset\mathcal{X}_{Kli}(p^{n+1})\times_{\mathcal{X}_{Kli}(p^n)}\mathcal{FL}_{n,\epsilon,w}$ such that $\overline{\mathcal{U}_i}\subset\mathcal{U}_i'$. We may refine $\{\mathcal{U}_i'\}$ by adding all translates under the action of $\mathfrak{Kli}(\mathbb{Z}/p^{n+1}\mathbb{Z})$ so we can suppose that $\mathcal{U}'=\{\mathcal{U}_i'\}$ is stable under the action of $\mathfrak{Kli}(\mathbb{Z}/p^n\mathbb{Z})$. We let $\mathcal{T}=\cup_i{\mathcal{U}'}_i$.

The cohomology $\mathrm{R}\Gamma(\mathcal{T},\mathscr{L}^{\kappa_A}\otimes\omega^r)$ is represented by the Čech complex $\check{\mathrm{C}}\mathrm{ech}(\mathcal{U}',\mathscr{L}^{\kappa_A}\otimes\omega^r)$. Similarly, the cohomology $\mathrm{R}\Gamma(\mathcal{FL}_{n+1,\epsilon+1,w+1},\mathscr{L}^{\kappa_A}\otimes\omega^r)$ is represented by the Čech complex $\check{\mathrm{C}}\mathrm{ech}(\mathcal{U},\mathscr{L}^{\kappa_A}\otimes\omega^r)$. The map $\check{\mathrm{C}}\mathrm{ech}(\mathcal{U}',\mathscr{L}^{\kappa_A}\otimes\omega^r)\to\check{\mathrm{C}}\mathrm{ech}(\mathcal{U},\mathscr{L}^{\kappa_A}\otimes\omega^r)$ is compact. It follows that the map of the proposition is compact as it can be factored into :

$$\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w}\otimes\omega^r)\to\left(\check{\mathrm{C}}\mathrm{ech}(\mathcal{U}',\mathscr{L}^{\kappa_A}\otimes\omega^r)\right)^{\mathfrak{Kli}(\mathbb{Z}/p^{n+1}\mathbb{Z})}$$
$$\to\left(\check{\mathrm{C}}\mathrm{ech}(\mathcal{U},\mathscr{L}^{\kappa_A}\otimes\omega^r)\right)^{\mathfrak{Kli}(\mathbb{Z}/p^{n+1}\mathbb{Z})}=\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^{n+1},\epsilon+1),\mathscr{F}^{\kappa_A,w+1}\otimes\omega^r).$$

□

Corollary 13.2.4.1. — *The operator U is compact.*

Proof. It is the composition of several continuous maps and one of the maps is compact. □

Corollary 13.2.4.2. — *The restriction maps $C(n,\epsilon,w,\kappa_A,r)\to C(n',\epsilon',w',\kappa_A,r)$ for $n'\geq n$, $\epsilon'\geq\epsilon$, $w'\geq w$ induces an isomorphism on the finite slope part for U. A similar statement holds for cuspidal cohomology.*

Proof. Without loss of generality, we can assume that $n'\leq n+1$, $w'\leq w+1$, $\epsilon'\leq\epsilon+1$. The map $U:\mathrm{H}^i(C(n',\epsilon',w',\kappa_A,r))\to\mathrm{H}^i(C(n',\epsilon',w',\kappa_A,r))$ factors canonically into

$$\mathrm{H}^i(C(n',\epsilon',w',\kappa_A,r))\stackrel{\tilde{U}}{\to}\mathrm{H}^i(C(n,\epsilon,w,\kappa_A,r))\stackrel{res}{\to}\mathrm{H}^i(C(n',\epsilon',w',\kappa_A,r)),$$

where the second map is the obvious restriction map. Given a finite slope class $f\in\mathrm{H}^i(C(n',\epsilon',w',\kappa_A,r))$, there is by definition (locally on A) a non-zero polynomial $P(X)\in A[X]$ with $P(0)=0$ such that $f=P(U)f$. We define the extension of f to $\mathrm{H}^i(C(n,\epsilon,w,\kappa_A,r))$ to be $P(\tilde{U})f$. This provides a map $ext:\mathrm{H}^i(C(n',\epsilon',w',\kappa_A,r))^{fs}\to$

$\mathrm{H}^i(C(n,\epsilon,w,\kappa_A,r))$ on finite slope classes. It is clear that $ext \circ res = Id$ and $res \circ ext = Id$ on finite slope classes. $\square$

Remark 13.2.4.1. — This corollary allows us to identify finite slope cohomology classes in $\mathrm{H}^i(\dagger,\kappa_A,r)$ with classes of prescribed radius of convergence and analyticity.

Remark 13.2.4.2. — One proves in a similar way that U acts compactly on $\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)})$ and $\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)}(-D))$.

13.3. Classicity at the level of the sheaf. — Let $(k,r) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. There is a natural map going from (n,ϵ)-overconvergent cohomology of the classical sheaf to (n,ϵ)-overconvergent, w-analytic cohomology :

$$\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)}) \to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{(k,w)} \otimes \omega^r),$$

and similarly for cuspidal cohomology. The goal of this section is to prove that on the small slope part, this map is a quasi-isomorphism.

13.3.1. Slopes. — The aim of this paragraph is to bound the possible slopes for U on overconvergent cohomology.

Proposition 13.3.1.1. — *Let $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}^\times$ be a w-analytic character. The operator U has slopes ≥ -3 on $\mathrm{H}^i(\dagger,\kappa,r)$ or $\mathrm{H}^i_{cusp}(\dagger,\kappa,r)$ for all i. Moreover, on degree 0 cohomology, it has slopes ≥ 0.*

Proof. The Banach sheaf $\mathscr{F}^{\kappa,w}$ is a subsheaf of the structural sheaf $\mathscr{O}_{\mathcal{FL}_{Kli,n,\epsilon,w,w'}}$ and we let $\mathscr{F}^{\kappa,w,++}$ be the sheaf $\mathscr{F}^{\kappa,w} \cap \mathscr{O}^{++}_{\mathcal{FL}_{Kli,n,\epsilon,w,w'}}$ (we recall that the superscript ++ stands for topologically nilpotent sections).

The map

$$t_2^\star \mathscr{F}^{\kappa,w+1} \to t_1^\star \mathscr{F}^{\kappa,w}$$

arises from a map of spaces

$$t_1^\star \mathcal{FL}_{Kli,n,\epsilon,w} \to t_2^\star \mathcal{FL}_{Kli,n+1,\epsilon+1,w+1}$$

therefore, it respects the integral structure and induces a map :

$$t_2^\star \mathscr{F}^{\kappa,w+1,++} \to t_1^\star \mathscr{F}^{\kappa,w,++}.$$

Next, the differential of the universal isogeny induces $\xi^\star : t_2^\star\omega^r \to t_1^\star\omega^r$ and factors through $\xi^\star : t_2^\star(\omega^{++})^r \to p^r t_1^\star(\omega^{++})^r$ by lemma 14.3.1, 2 [(21)]. By proposition 14.4.1.1 [(22)] we have that $\mathrm{R}^i(t_1)_\star\mathscr{O}^{++}_{C_\epsilon} = (t_1)_\star\mathscr{O}^{++}_{C_\epsilon} = 0$ for all $i > 0$. Finally, the trace map $\mathrm{Tr} : (t_1)_\star\mathscr{O}_{C_\epsilon} \to \mathscr{O}_{\mathcal{X}_{Kli}(p^n,\epsilon)}$ restricts to $\mathrm{Tr} : (t_1)_\star\mathscr{O}^{++}_{C_\epsilon} \to \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p^n,\epsilon)}$ Therefore there is a map $p^3U : \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w,++} \otimes (\omega^{++})^r) \to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa_A,w,++} \otimes (\omega^{++})^r)$ fitting in the commutative diagram :

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa,w} \otimes \omega^r) \arrow[r, "p^3U"] & \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa,w} \otimes \omega^r) \\
\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa,w,++} \otimes (\omega^{++})^r) \arrow[r, "p^3U"] \arrow[u] & \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa,w,++} \otimes (\omega^{++})^r) \arrow[u]
\end{tikzcd}$$

21. We use a result that is only proved in the next section. The reader can check that the proof of lemma 14.3.1 is completely self-contained and independent of any other result of this paper.

22. We again use a result that is only proved in the next section, the proof of this proposition depends only on results obtained in section 3.4, so there is no circularity in our arguments.

We now consider an affinoid covering $\mathcal{U}$ of $\mathcal{X}_{Kli}(p^n,\epsilon)$ (chosen such that for all $U \in \mathcal{U}$, one has $\mathcal{FL}_{Kli,n,\epsilon,w}$ is affinoid). The Čech complex $C^\bullet$ associated to $\mathcal{U}$ of the sheaf $\mathscr{F}^{\kappa,w}\otimes\omega^r$ computes the cohomology $\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa,w}\otimes\omega^r)$. This is a bounded complex of Banach spaces and we can lift the U operator to a compact endomorphism $\tilde{U}$ of $C^\bullet$ by lemma 13.2.4.1. Let a be rational number and let $(C^\bullet)^{=a}$ be the associated direct factor of $C^\bullet$ computing the slope a cohomology. This is a perfect complex of $\mathbb{C}_p$ vector spaces and the projection $C^\bullet \to (C^\bullet)^{=a}$ is continuous. We now consider the Čech complex $C^{\bullet,++}$ of the sheaf $\mathscr{F}^{\kappa,w,++}\otimes(\omega^{++})^r$ for the covering $\mathcal{U}$. This is a subcomplex of $C^\bullet$ of open and bounded $\mathcal{O}$-modules. Its image $(C^{\bullet,++})^{=a}$ under the continuous projection $C^\bullet \to (C^\bullet)^{=a}$ is again open and bounded. Therefore, the image of $\mathrm{H}^i(C^{\bullet,++})$ in $\mathrm{H}^i(C^\bullet)^{=a}$ is bounded.

We consider the compositions of maps :

$$\mathrm{H}^i(C^{\bullet,++}) \to \mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa,w,++}\otimes(\omega^{++})^r) \to \mathrm{H}^i(C^\bullet) \to \mathrm{H}^i(C^\bullet)^{=a},$$

where the first map is the map from Čech cohomology with respect to the covering $\mathcal{U}$ to cohomology, the second map is the functorial map between cohomology groups associated to the map of sheaves $\mathscr{F}^{\kappa,w,++}\otimes(\omega^{++})^r \to \mathscr{F}^{\kappa,w}\otimes\omega^r$, and the last map is the continuous projection to the slope a cohomology.

We now deduce from lemma 3.2.2 that the map

$$\mathrm{H}^i(C^{\bullet,++}) \to \mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa,w,++}\otimes(\omega^{++})^r)$$

has kernel and co-kernel of bounded p-torsion. It follows that the image of

$$\mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{\kappa,w,++}\otimes(\omega^{++})^r)$$

in $\mathrm{H}^i(C^\bullet)^{=a}$ is open and bounded. It follows that in $\mathrm{H}^i(C^\bullet)^{=a}$, the operator p^3U stabilizes an open and bounded submodule. Therefore, we deduce that $a+3\geq 0$.

On degree 0 cohomology we can argue a bit differently and improve on the result. It follows from the construction that the cohomology $\mathrm{H}^0(\dagger,\kappa,r)$ embeds in the module of p-adic modular forms of weight (κ,r) tensored with $\mathbb{C}$ (see definition 4.3 in [**60**]). The claim follows from the fact that our U-operator stabilizes the integral structure on p-adic modular forms. In more down to earth terms, we have a q-expansion map for p-adic modular forms, and the U-operator preserves integrality on q-expansions (see [**34**]). □

Remark 13.3.1.1. — Although we believe only non-negative slopes can occur in all cohomological degree, it is difficult to improve the above argument. The reason is that the trace map is normalized by a factor p^{-3}. This normalization does not preserve integrality in general.

13.3.2. Classicity for the sheaf. — For all $(k,r)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}$ we have a classical sheaf $\Omega^{(k,r)}$.

Lemma 13.3.2.1. — *There is a canonical map of sheaves over $\mathcal{X}_{Kli}(p^n,\epsilon)$:*

$$\Omega^{(k,r)} \to \mathscr{F}^{k,w}\otimes\omega^r.$$

Proof. Remark that $\Omega^{(k,r)} = \Omega^{(k,0)}\otimes\omega^r$. It suffices to construct the map for $r=0$. Let $\mathcal{FL}\to\mathcal{X}_{Kli}(p^n,\epsilon)$ be the analytic flag variety parametrizing flags $\mathrm{Fil}\omega_G\subset\omega_G$. Let $\mathcal{FL}^+\to\mathcal{FL}$ be the $\mathbb{G}_m$-torsor parametrizing trivializations of $\mathrm{Gr}(\omega_G)$. We denote by $f:\mathcal{FL}^+\to\mathcal{X}_{Kli}(p^n,\epsilon)$ the structural map. Then by definition $\Omega^{(k,0)} = f_\star\mathscr{O}_{\mathcal{FL}^+}[-k]$ where $[-k]$ means the subsheaf of $f_\star\mathscr{O}_{\mathcal{FL}^+}$ where $\mathbb{G}_m$ acts via the character $-k$. There is an obvious map $i:\mathcal{FL}^+_{n,\epsilon,w,w'}\to\mathcal{FL}^+$, equivariant for the action of $\mathcal{T}_{w',n}$ on the left and $\mathbb{G}_m$ on the right (under the map $\mathcal{T}_{w',n}\to\mathbb{G}_m$). Taking the $-k$ invariants part of $i^\star:\mathscr{O}_{\mathcal{FL}^+}\to\mathscr{O}_{\mathcal{FL}^+_{Kli,n,\epsilon,w,w'}}$ provides a map

$$\Omega^{(k,0)}\hookrightarrow\mathscr{F}^{k,w}.$$

□

For the next proposition, we shall denote $\mathscr{F}^{k,w^-}$ the inductive limit $\mathrm{colim}_{w'<w}\mathscr{F}^{k,w'}$.

Proposition 13.3.2.1. — *Let (k,r) be an algebraic weight. Then we have an exact sequence over $\mathcal{X}_{Kli}(p^n,\epsilon)$:*

$$0 \to \Omega^{(k,r)} \xrightarrow{d_0} \mathscr{F}^{k,w^-} \otimes \omega^r \xrightarrow{d_1} \mathscr{F}^{-2-k,w^-} \otimes \omega^{k+r+1} \to 0$$

Proof. See [**3**], prop. 7.2.1. This is a relative version of the locally analytic BGG resolution. We note that in [**3**] there is the minor error that the proposition is given for a fixed w, without taking the colimit $\mathrm{colim}_{w'<w}$. The colimit is necessary because integration changes the radius of convergence. □

We let $C(n,\epsilon,w^-,k,r) = \mathrm{colim}_{w'<w}C(n,\epsilon,w,k,r)$.

Corollary 13.3.2.1. — *There is an exact triangle :*

$$\mathrm{R}\Gamma(X_{Kli}(p^n,\epsilon),\Omega^{(k,r)}) \to C(n,\epsilon,w^-,k,r) \to C(n,\epsilon,w^-,-2-k,k+r+1) \xrightarrow{+1}$$

A similar statement holds for cuspidal cohomology.

13.3.3. Equivariance of the BGG resolution. — We will now prove that certain (n,ϵ)-overconvergent and w-analytic cohomology classes are in fact (n,ϵ)-overconvergent cohomology classes of a classical sheaf.

Proposition 13.3.3.1. — *The following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{k,w^-}\otimes\omega^r) \arrow[r, "U"] \arrow[d, "d_1"] & \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{k,w^-}\otimes\omega^r) \arrow[d, "d_1"] \\
\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{-2-k,w^-}\otimes\omega^{k+r+1}) \arrow[r, "p^{-k-1}U"] & \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{-2-k,w^-}\otimes\omega^{k+r+1})
\end{tikzcd}$$

Proof. See [**3**], prop. 7.2.3. □

Corollary 13.3.3.1. — *1. The maps*

$$\mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)})^{<k-2} \to \mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{k,w}\otimes\omega^r)^{<k-2}$$

and

$$\mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)}(-D))^{<k-2} \to \mathrm{H}^i(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{k,w}\otimes\omega^r(-D))^{<k-2}$$

are isomorphisms.

2. The maps

$$\mathrm{H}^0(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)})^{<k+1} \to \mathrm{H}^0(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{k,w}\otimes\omega^r)^{<k+1}$$

and

$$\mathrm{H}^0(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)}(-D))^{<k+1} \to \mathrm{H}^0(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{k,w}\otimes\omega^r(-D))^{<k+1}$$

are isomorphisms.

3. The maps

$$\mathrm{H}^1(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)})^{<k+1} \to \mathrm{H}^1(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{k,w}\otimes\omega^r)^{<k+1}$$

and

$$\mathrm{H}^1(\mathcal{X}_{Kli}(p^n,\epsilon),\Omega^{(k,r)}(-D))^{<k+1} \to \mathrm{H}^1(\mathcal{X}_{Kli}(p^n,\epsilon),\mathscr{F}^{k,w}\otimes\omega^r(-D))^{<k+1}$$

are injective.

Proof. This follows from proposition 13.3.1.1, proposition 13.3.3.1, and corollary 13.3.2.1. □

13.4. The spectral variety. — Let $\mathcal{W} = \mathrm{Spa}(\Lambda, \Lambda) \times \mathrm{Spa}(\mathbb{C}_p, \mathcal{O})$ be the analytic weight space in characteristic zero where we recall that $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is the one-dimensional Iwasawa algebra. We can write $\mathcal{W}$ as an increasing union of affinoids $\mathrm{Spa}(A_l[1/p], A_l)$. We let $\kappa_{A_l} : \mathbb{Z}_p^\times \to A_l^\times$ be the universal character. We can apply the formalism of section 13.1.2 to the cohomology $C_{cusp}(n, \epsilon, w, \kappa_{A_l}, 2)$ (for, n, ϵ, w large enough) and the compact U-operator acting on it. We obtain a complex $C_{cusp}(A_l)$ over $\mathrm{Spa}(A_l[1/p], A_l) \times \mathbb{G}_m$ of finite slope cuspidal overconvergent cohomology of weight $(\kappa_{A_l}, 2)$ which is concentrated in degree 0 and 1. We observe that $C_{cusp}(A_l)$ is independant of n, ϵ, w as the operator U improves convergence and analyticity (see corollary 13.2.4.2 and the remark below the proof).

Moreover, for all $\kappa : \mathrm{Spa}(\mathbb{C}_p, \mathcal{O}) \to \mathrm{Spa}(A_l[1/p], A_l)$ and $\alpha^{-1} \in \mathbb{C}_p^\times$ providing a point $(\kappa, \alpha^{-1}) : \mathrm{Spa}(\mathbb{C}_p, \mathcal{O}) \to \mathrm{Spa}(A_l[1/p], A_l) \times \mathbb{G}_m$, we have isomorphisms :

$$\mathrm{H}^i((\kappa, \alpha^{-1})^\star C_{cusp}(A_l)) = \mathrm{H}^i_{cusp}(\kappa, r)[U = \alpha].$$

The annihilator of $\mathrm{H}^\bullet(C_{cusp}(A_l))$ is a coherent ideal $\mathscr{I}_l \subset \mathscr{O}_{\mathrm{Spa}(A_l[1/p], A_l) \times \mathbb{G}_m}$ and the associated closed subspace is the spectral variety $\mathcal{Z}_l$. The map $\mathcal{Z}_l \to \mathrm{Spa}(A_l[1/p], A_l)$ is quasi-finite and locally finite.

For all l, the spectral varieties $\mathcal{Z}_l$ glue to $\mathcal{Z} \to \mathcal{W}$ and there is a universal graded coherent module $\mathrm{H}^\bullet(C_{cusp})$ over $\mathcal{Z}$ supported in degree 0 and 1.

We deduce the following proposition:

Proposition 13.4.1. — *The function defined on $\mathbb{Z}_{\geq 0} \hookrightarrow \mathcal{W}$ [23]:*

$$k \mapsto \dim_{\mathbb{C}} \mathrm{H}^1_{cusp}(\dagger, k, 2)^{=0} - \dim_{\mathbb{C}} \mathrm{H}^0_{cusp}(\dagger, k, 2)^{=0}$$

is locally constant.

Proof. This is a corollary of the discussion above and proposition 13.1.3.1. □

14. Small slope cohomology classes are classical

The goal of this section is to prove a generalization of Coleman's classicity criterion that small slope overconvergent modular forms are classical. We shall here work mainly with a classical sheaf of coefficients $\Omega^{(k,r)}$ or $\Omega^{(k,r)}(-D)$ and prove that restriction from small slope cohomology classes on $\mathcal{X}_{Kli}(p)$ to small slope overconvergent cohomology classes over $\mathcal{X}^{\geq 1}_{Kli}(p)$ is an isomorphism. That way, we will be able to complete the proof of theorem 1.1.

14.1. Neighborhoods of the ordinary locus in $\mathcal{X}_{Kli}(p)$. — We recall that $\mathcal{X}_{Kli}(p)$ is the analytic Siegel threefold of Klingen level at p. There is a universal chain of isogenies $G \to G' \to G$ where $G \to G'$ is a degree p^3 isogeny and the composition of the two isogenies is multiplication by p. We let H be the group scheme $\mathrm{Ker}(G \to G')^\perp$ (the orthogonal is for the Weil pairing). When G is an abelian scheme, H is a finite flat group scheme of order p. We let $G'' = G/H = (G')^t$. We denote by ω_G^+ the invertible sheaf of $\mathscr{O}^+_{\mathcal{X}_{Kli}(p)}$ modules of integral differential form at the unit section on G (a similar notation applies to G''). Let $\delta_H \in \det\ \omega_G^+ \otimes \det^{-1} \omega_{G''}^+$ be the determinant of the map $\omega_{G''}^+ \to \omega_G^+$ induced

23. The set $\mathbb{Z}_{\geq 0}$ carries the subspace topology, which is the p-adic topology on each of the residue classes modulo $p-1$ (or modulo 2 if $p = 2$).

by the isogeny $G \to G''$. We recall that for all rank 1 point $x : \mathrm{Spa}(K, \mathcal{O}_K) \to \mathcal{X}_{Kli}(p)$ with associated valuation v_x normalized by $v_x(p) = 1$, we have $v_x(\delta_H) = \deg H_x \in [0,1]$ in the sense of [**20**] whenever H_x is a finite flat group scheme whose schematic closure is a finite flat subgroup scheme of G over Spf $\mathcal{O}_K$ (this holds when G has good reduction at x for example).

We let $\mathcal{X}_{Kli}(p)_\epsilon \subset \mathcal{X}_{Kli}(p)$ be the locus where $|\delta_H| \leq |p^\epsilon|$. This is another way to measure the distance to the p-rank one locus that is more adapted to the arguments of this part of the work. Before proceeding, we make a comparison with the spaces $\mathcal{X}_{Kli}(p^n, \epsilon)$ introduced in section 12.7.1.

Lemma 14.1.1. — *The natural map $\mathcal{X}_{Kli}(p^n, \epsilon) \to \mathcal{X}_{Kli}(p)$ factorizes through*

$$\mathcal{X}_{Kli}(p)_{\max\{0, 1 - \frac{2}{n}(n - \epsilon + \frac{1}{p-1})\}}.$$

Proof. It is enough to do the proof for all rank 1 points $\mathrm{Spa}(K, \mathcal{O}_K) \to \mathcal{X}_{Kli}(p^n, \epsilon)$. Let $G \to \mathrm{Spec}\ \mathcal{O}_K$ be the corresponding semi-abelian surface. Let $H_n \subset G[p^n]$ be the group generated by e_1. There is a commutative diagram :

$$\begin{array}{ccccc} 0 & \longrightarrow & H_n & \longrightarrow & G[p^n] \\ & & \downarrow{\scriptstyle \mathrm{HT}_{H_n}} & & \downarrow{\scriptstyle \mathrm{HT}} \\ 0 & \longrightarrow & \omega_{H_n^D} & \longrightarrow & \omega_G / p^n \omega_G \end{array}$$

The group $\omega_{H_n^D}$ is generated by two elements as an $\mathcal{O}_K$-module (because H_n^D can be embedded in a two dimensional p-divisible group) and the cokernel of $\mathrm{HT}_{H_n} \otimes 1 : H_n \otimes \mathcal{O}_K \to \omega_{H_n^D}$ is killed by $p^{\frac{1}{p-1}}$ by [**20**], thm. 7. Since the map $\mathrm{HT} : H_n \to \omega_G^{mod}/p^\epsilon$ is zero by assumption, we deduce that $\mathrm{HT}_{H_n}(H_n)$ is killed by $p^{n-\epsilon}$ so that $\omega_{H_n^D}$ is killed by $p^{\frac{1}{p-1}+n-\epsilon}$. Since $\omega_{H_n^D}$ is generated by 2 elements, we deduce that $\deg H_n^D \leq 2(n - \epsilon + \frac{1}{p-1})$.

The group H_n has degree at least $n - 2(n - \epsilon + \frac{1}{p-1})$. Morover the maps $p^{k-1} : H_n[p^k]/H_n[p^{k-1}] \to H_n[p^{n-1}] = H_1$ are morphisms which are isomorphisms on the generic fiber. Therefore, using [**20**], coro. 3 on p. 13, we deduce that $\deg H_1 \geq \frac{1}{n} \deg H_n \geq 1 - \frac{2}{n}(n - \epsilon + \frac{1}{p-1})$.

□

Remark 14.1.1. — So in particular, if $\epsilon = n - \frac{1}{p-1}$ and $n \to +\infty$, $1 - \frac{2}{n}(n - \epsilon + \frac{1}{p-1}) \to 1$.

Lemma 14.1.2. — *We have $\mathcal{X}_{Kli}(p)_\epsilon \subset \mathcal{X}_{Kli}(p, 1 - \frac{1}{p-1})$ for all $\epsilon \geq 1 - \frac{1}{p}$.*

Proof. This is an easy computation using Oort-Tate theory [**58**]. □

14.2. The correspondences C_n. — Let $\mathcal{Y}_{Kli}(p)$ be the open subspace of $\mathcal{X}_{Kli}(p)$ where the semi-abelian scheme is an abelian scheme. For all $n \in \mathbb{N}$, there is a Hecke correspondence $t_{n,1}, t_{n,2} : C_n|_{\mathcal{Y}_{Kli}(p)} \to \mathcal{Y}_{Kli}(p)$ where $C_n|_{\mathcal{Y}_{Kli}(p)}$ is the moduli space of (G, H, L_n) where $(G, H) \in \mathcal{Y}_{Kli}(p)$ and $L_n \subset G[p^n]$ is a totally isotropic subgroup which is locally for the étale topology isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^2 \oplus \mathbb{Z}/p^{2n}\mathbb{Z}$ and $L_n \cap H = \{0\}$. The map $t_{n,1}$ sends (G, H, L_n) to (G, H). The map $t_{n,2}$ sends (G, H, L_n) to $(G/L_n, H + L_n/L_n)$. We remark that $C_n|_{\mathcal{Y}_{Kli}(p)}$ is simply obtained by iterating n times the correspondence $C_1|_{\mathcal{Y}_{Kli}(p)}$ (which is the correspondence $C|_{\mathcal{Y}_{Kli}(p)}$ considered in section 13.2.1).

There exist smooth polyhedral cone decompositions Σ and Σ' and toroidal compactifications of $C_n|_{\mathcal{Y}_{Kli}(p)}$ which we denote by $C_{n,\Sigma'}$ or simply C_n, of $\mathcal{Y}_{Kli}(p)$ which we denote by $\mathcal{X}_{Kli}(p)_\Sigma$ or simply by $\mathcal{X}_{Kli}(p)$, and maps $t_{n,1} : C_{n,\Sigma'} \to \mathcal{X}_{Kli}(p)_\Sigma$ and $t_{n,2} : C_{n,\Sigma'} \to \mathcal{X}_{Kli}(p)_\Sigma$ which extend the maps $t_{n,1}$ and $t_{n,2}$ previously defined.

14.3. Variation of the degree. — Over C_n we have an isogeny $G \to G_n$ with kernel L_n. The differential of this isogeny provides a map $(\Omega^1_{G_n/C_n})^+ \to (\Omega^1_{G/C_n})^+$ where $(\Omega^1_{G_n/C_n})^+ \subset \Omega^1_{G_n/C_n}$ is the locally free $\mathscr{O}^+_{C_n}$ module of integral differentials. Taking the determinant yields a section $\delta_{L_n} \in \det(\Omega^1_{G/C_n})^+ \otimes \det^{-1}(\Omega^1_{G_n/C_n})^+$.

When we have a rank one point $x : \mathrm{Spa}(K, \mathcal{O}_K) \to C_n$, with associated valuation v_x normalized by $v_x(p) = 1$, we can define the degree $\deg L_n|_x = v_x(\delta_{L_n})$ where $v_x(\delta_{L_n})$ means the valuation of $\delta_{L_n}(x)$ computed in any local trivialization of the sheaf $\det(\Omega^1_{G/C_n})^+ \otimes \det^{-1}(\Omega^1_{G_n/C_n})^+$. When $G|_x$ is an abelian scheme and extends to an abelian scheme $\mathfrak{G}$ over $\mathrm{Spf}\ \mathcal{O}_K$, this is also the degree of the schematic closure of $L_n|_x$ in $\mathfrak{G}$ defined in [**20**]. In general, $G|_x$ can be uniformized as the quotient of a semi-abelian scheme G^0 by a lattice. The semi-abelian scheme G^0 extends to a semi-abelian scheme $\mathfrak{G}^0$ over $\mathrm{Spf}\ \mathcal{O}_K$. In this case, $\deg L_n|_x = \deg L_n|_x \cap G^0$.

Lemma 14.3.1. — *Let $x : \mathrm{Spa}(K, \mathcal{O}_K) \to C_1$ be a rank 1 point corresponding to a triple $(G, H, L = L_1)$. Then we have :*

1. $\deg H + \deg L[p] \leq 2$,
2. $\deg L[p]/pL = 1$,
3. $\deg L/L[p] \leq \deg pL$,
4. $\deg(G[p] + L)/L = 1 - \deg L/L[p]$,
5. $\deg(G[p] + L)/L \geq \deg H$. *In case of equality, H is either of multiplicative or étale type.*

Proof. It is enough to prove all the points when G is an abelian scheme, by Zariski density. The first point follows from the fact that there is a morphism which is an isomorphism on the generic fiber : $H \times L[p] \to G[p]$ and properties of the degree [**20**], coro. 3 on p. 13.

Using the lemma below the proof, we deduce that the perfect Weil pairing on $G[p]$ induces a perfect pairing between $L[p]$ and $G[p]/pL$ which restricts to a perfect pairing on $L[p]/pL$. As a result $L[p]/pL \simeq (L[p]/pL)^D$. We deduce from [**20**], lem. 4 on p. 12 that we have $\deg L[p]/pL + \deg L[p]/pL = 2$ and it follows that $\deg L[p]/pL = 1$.

The map given by multiplication by $p : L/L[p] \to pL$ is a generic isomorphism. It follows from [**20**], coro. 3 on p. 13 that $\deg L/L[p] \leq \deg pL$.

As before, the perfect Weil pairing on $G[p^2]$ induces a pairing between L and $G[p^2]/L$ which restricts to a pairing between $(G[p] + L)/L$ and $L/L[p]$. It follows that $\deg(G[p] + L)/L + \deg L/L[p] = 1$.

The map $H \to (G[p]+L)/L$ is a generic isomorphism. As a result, $\deg H \leq \deg(G[p]+L)/L$. In case of equality, we deduce that $H \to (G[p] + L)/L$ is an isomorphism, that $H \to G[p]/L[p]$ is also an isomorphism (because we have a factorization $H \to G[p]/L[p] \to (G[p]+L)/L$), and therefore that the map $H \oplus L[p] \to G[p]$ is an isomorphism. The group H is a direct factor of a truncated Barsotti-Tate group of level 1, therefore it is a truncated Barsotti-Tate group of level 1. Since it is of order p, we deduce that H is either of étale or multiplicative type. □

In the course of the proof of the above lemma, we have used the following easy lemma whose proof is left to the reader :

Lemma 14.3.2. — *Let J be a finite flat group scheme over $\mathcal{O}_K$. Let $M_K \subset J_K$ be a subgroup and let M be the schematic closure of M_K. Let $M_K^{\perp}$ be the orthogonal of M_K in J_K^D. Let $M^{\perp}$ be the schematic closure of $M_K^{\perp}$. Then $J^D/M^{\perp} = M^D$.*

Corollary 14.3.1. — *Let $n \geq 1$. Let $x : \mathrm{Spa}(K, \mathcal{O}_K) \to C_n$ be a rank 1 point corresponding to a triple (G, H, L_n), let $\epsilon \in \mathbb{R}$ and assume that $\deg L_n \leq n(3 - 2\epsilon)$. Then $\deg(G[p] + L_n)/L_n \geq \epsilon$.*

Proof. We first give the proof for $n = 1$ and write $L = L_1$. Note that $\deg L = \deg pL + \deg L[p]/pL + \deg L/L[p]$, so that $\deg L \geq 1 + 2\deg L/L[p]$ (by lemma 14.3.1, point 2 and 3). We deduce that $\deg L/L[p] \leq 1 - \epsilon$ and the claim follows from the formula $\deg(G[p] + L)/L = 1 - \deg L/L[p]$ (lemma 14.3.1, point 4).

We now give the proof for a general n. There is a filtration $L_1 \subset L_2 \subset \cdots \subset L_n$ with L_i locally isomorphic to $\mathbb{Z}/p^{2i} \oplus (\mathbb{Z}/p^i)^2$ and totally isotropic in $G[p^{2i}]$. By elementary properties of the degree map, there is an index i such that $\deg(L_i/L_{i-1}) \leq 3 - 2\epsilon$. Since $(G[p]+L_i)/L_i = ((G/L_{i-1})[p]+L_i/L_{i-1})/L_i/L_{i-1}$ we deduce by application of the corollary for $n = 1$ that $\deg(G[p] + L_i)/L_i \geq \epsilon$. Since the map $(G[p] + L_i)/L_i \to (G[p] + L_n)/L_n$ is an isomorphism over K, the corollary follows. ☐

We can deduce the following result on the dynamic of the Hecke correspondence C_1.

Corollary 14.3.2. — *Let $[a, b] \subset]0, 1[$. There exists $r(a, b) > 0$ such that for all $\epsilon \in [a, b]$ we have $t_{1,2}(t_{1,1}^{-1}(\mathcal{X}_{Kli}(p)_\epsilon)) \subset \mathcal{X}_{Kli}(p)_{\epsilon + r(a,b)}$.*

Proof. See [**62**], prop. 2.3.6. For the reader's convenience, let us mention that this is an application of lemma 14.3.1, point 5., together with the maximal principle applied over suitable quasi-compact subsets of C_1. ☐

14.4. Cohomological correspondences in the analytic setting. —

14.4.1. Basic vanishing. — In this section we establish a vanishing result for coherent cohomology with respect to the change of polyhedral cone decomposition and also a vanishing result for higher direct images of the correspondence. These results will allow us to consider safely the action of Hecke operators on cohomology.

Proposition 14.4.1.1. —
1. *Let Σ and Σ' be smooth polyhedral cone decompositions. Consider the map $\pi_{\Sigma', \Sigma} : \mathcal{X}_{Kli}(p)_{\Sigma'} \to \mathcal{X}_{Kli}(p)_\Sigma$. We have $\mathrm{R}(\pi_{\Sigma',\Sigma})_\star \mathscr{O}_{\mathcal{X}_{Kli}(p)_{\Sigma'}} = \mathscr{O}_{\mathcal{X}_{Kli}(p)_\Sigma}$ and $\mathrm{R}(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}} = \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_\Sigma}$.*
2. *Let $t_{n,1} : C_n \to \mathcal{X}_{Kli}(p)$. Then we have $\mathrm{R}(t_{n,1})_\star \mathscr{O}_{C_n} = (t_{n,1})_\star \mathscr{O}_{C_n}$ and $\mathrm{R}(t_{n,1})_\star \mathscr{O}^{++}_{C_n} = (t_{n,1})_\star \mathscr{O}^{++}_{C_n}$.*

Proof. The points 1 and 2 for the structural sheaves (not the ++ version) follow from standard computations and the comparison theorem stated in [**69**], thm. 9.1. We now proceed to deduce 1 and 2 for the "++" sheaves. Let $\sigma \subset \Sigma$ be a cone. Then, $\sigma \cap \Sigma'$ is a refinement of σ. Associated to σ is a boundary component $\mathcal{Z}_\sigma \hookrightarrow \mathcal{X}_{Kli}(p)_\Sigma$. Its inverse image in $\mathcal{X}_{Kli}(p)_{\Sigma'}$ is a union of boundary stratum $\mathcal{Z}_{\sigma\cap\Sigma'}$.

We have local charts

$$\begin{array}{ccc} \mathcal{M}_{\sigma\cap\Sigma'} & \xrightarrow{\pi} & \mathcal{M}_\sigma \\ \uparrow & & \uparrow \\ \mathcal{Z}_{\sigma\cap\Sigma'} & \longrightarrow & \mathcal{Z}_\sigma \end{array}$$

and there is an isomorphism :

$$\begin{tikzcd} \widehat{\mathcal{M}_{\sigma\cap\Sigma'}}^{\mathcal{Z}_{\sigma\cap\Sigma'}} \arrow[r, "\pi"] & \widehat{\mathcal{M}_{\sigma}}^{\mathcal{Z}_{\sigma}} \\ \widehat{\mathcal{X}_{Kli}(p)_{\Sigma'}}^{\mathcal{Z}_{\sigma\cap\Sigma'}} \arrow[u] \arrow[r, "\pi_{\Sigma',\Sigma}"] & \widehat{\mathcal{X}_{Kli}(p)_{\Sigma}}^{\mathcal{Z}_{\sigma}} \arrow[u] \end{tikzcd}$$

There is a Kuga-Sato variety $\mathcal{B}$, a split torus T and a natural map $\mathcal{M}_\sigma \to \mathcal{B}$ such that $\mathcal{M}_{\sigma\cap\Sigma} \to \mathcal{M}_\sigma$ is locally isomorphic over $\mathcal{B}$ to $T_{\Sigma'} \times \mathcal{B} \to T_\sigma \times \mathcal{B}$. By proposition 3.4.1, we deduce that $\mathrm{R}\pi_\star \mathscr{O}^{++}_{\mathcal{M}_{\sigma\cap\Sigma}} = \mathscr{O}^{++}_{\mathcal{M}_\sigma}$.

By proposition 3.3.1, this implies that $\mathrm{R}\pi_\star \mathscr{O}^{++}_{\mathcal{Z}_{\sigma\cap\Sigma}}/p^n = \mathscr{O}^{++}_{\mathcal{Z}_\sigma}/p^n$. This implies in turn that

$$\mathrm{R}(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_\Sigma}/p^n = \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}}/p^n.$$

We have a long exact sequence :

$$\cdots \to \mathrm{R}^i(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}} \xrightarrow{p} \mathrm{R}^i(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}} \to \mathrm{R}^i(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}}/p \to \cdots$$

We look at the sequence for $i = 0$. Since $(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}}/p = \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_\Sigma}/p$ and $\mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_\Sigma} \hookrightarrow (\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}}$, we deduce that the map $(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}} \to (\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}}/p$ is surjective.

This implies that for all $i > 0$, multiplication by p is an isomorphism on $\mathrm{R}^i(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}}$. As a result, $\mathrm{R}^i(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}} = \mathrm{R}^i(\pi_{\Sigma',\Sigma})_\star \mathscr{O}_{\mathcal{X}_{Kli}(p)_{\Sigma'}}$. The latter vanishes. We also deduce easily that $(\pi_{\Sigma',\Sigma})_\star \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_{\Sigma'}} = \mathscr{O}^{++}_{\mathcal{X}_{Kli}(p)_\Sigma}$.

We next deal with point 2. We have $C_n = C_{n,\Sigma'}$ and $\mathcal{X}_{Kli}(p) = \mathcal{X}_{Kli}(p)_\Sigma$ for two smooth polyhedral decompositions Σ and Σ' (for different integral structures). Actually we can use Σ to produce a toroidal compactification $C_{n,\Sigma}$ which is not going to be smooth (because of the change of integral structure). We then have a factorization of $t_{n,1}$ into $C_{n,\Sigma'} \xrightarrow{f} C_{n,\Sigma} \xrightarrow{g} \mathcal{X}_{Kli}(p)_\Sigma$. As in point 1, we show that $\mathrm{R}f_\star \mathscr{O}^{++}_{C_{n,\Sigma'}} = \mathscr{O}^{++}_{C_{n,\Sigma}}$ (notice that the smoothness of Σ was not used in the proof of 1). On the other hand, the morphism g is finite and has no higher cohomology. □

14.4.2. Cohomological correspondences for classical sheaves. — Let $\mathscr{F}$ be any of $\Omega^{(k,r)}$ or $\Omega^{(k,r)}(-D)$. We can define an unnormalized analytic cohomological correspondence $(t_{n,1})_\star t^\star_{n,2}\mathscr{F} \to \mathscr{F}$ by taking (for instance) the analytification of the algebraic cohomological correspondence. We normalize this map by dividing by the factor $p^{n(3+r)}$ and call it U^n. This normalization is consistent with section 10.4. Restricting this map to $\mathscr{F}^{++}$ provides a map $U^n : (t_{n,1})_\star t^\star_{n,2}\mathscr{F}^{++} \to p^{-3n}\,\mathscr{F}^{++}$. The reason the map lands in $p^{-3n}\mathscr{F}^{++}$ instead of $p^{-3n-nr}\mathscr{F}^{++}$ is that the kernel L_n of the isogeny $G \to G_n$ has degree at least one by lemma 14.3.1, 2.

Remark 14.4.2.1. — When we work on the analytic space, we cannot expect the cohomological correspondence to have a better integral property than the integral property stated above. The cohomological correspondence has a better integral property on the formal scheme ordinary locus (see sect. 10.4).

We denote by $U^n : \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p), \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p), \mathscr{F})$ and $U^n : \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p), \mathscr{F}^{++}) \to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p), p^{-3n}\mathscr{F}^{++})$ the corresponding maps on cohomology. Obviously, U^n is the n-th iterate of $U = U^1$.

14.5. Analytic continuation. — Let ϵ' and ϵ be such that $t_{n,2}t_{n,1}^{-1}(\mathcal{X}_{Kli}(p)_{\epsilon'}) \subset \mathcal{X}_{Kli}(p)_{\epsilon}$. Then we get a map :

$$U^n_{\epsilon,\epsilon'} : \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F}).$$

On the other hand, if $\epsilon' \geq \epsilon$, we have a restriction map

$$res_{\epsilon,\epsilon'} : \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})$$

induced by the inclusions $\mathcal{X}_{Kli}(p)_{\epsilon} \hookrightarrow \mathcal{X}_{Kli}(p)_{\epsilon'}$. When it makes sense, we have $U^n_{\epsilon,\epsilon'} \circ res_{\epsilon'',\epsilon} = U^n_{\epsilon'',\epsilon'}$ and $res_{\epsilon',\epsilon''} \circ U^n_{\epsilon,\epsilon'} = U^n_{\epsilon,\epsilon''}$. We often write U^n instead of $U^n_{\epsilon,\epsilon'}$ and res instead of $res_{\epsilon,\epsilon'}$ if the context is clear.

Proposition 14.5.1. — *Let $f \in \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F})$ with $\epsilon < 1$. We assume that $Uf = af$ with $a \neq 0$. Then for all $\epsilon > \epsilon' > 0$, there is a unique section $g \in \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})$ such that $Ug = ag$ and $res_{\epsilon',\epsilon} g = f$*

Proof. Let $[c,d] \subset]0,1[$ such that $\epsilon, \epsilon' \in [c,d]$ and choose n such that $nr(c,d) + \epsilon' \geq \epsilon$ (see coro 14.3.2). We consider the operator $a^{-n}U^n : \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F}) \to \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})$ and we set $g = a^{-n}U^n f$.

The following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F}) \arrow[r, "U^n"] \arrow[d] & \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F}) \arrow[r, "U"] & \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F}) \arrow[d] \\
\mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F}) \arrow[r, "U"] & \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F}) \arrow[r, "U^n"] & \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})
\end{tikzcd}$$

and we deduce that $Ug = ag$. Moreover, since we can factor $a^{-n}U^n : \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F}) \to \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})$ into

$$\mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F}) \overset{res}{\to} \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F}) \overset{a^{-n}U^n}{\to} \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})$$

we deduce that g is unique.

□

We can slightly improve the last proposition, in the spirit of [**38**].

Proposition 14.5.2. — *Let $f \in \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon}, \mathscr{F})$ with $\epsilon < 1$. Let $P = X^m + a_{m-1}X^{m-1} + \cdots + a_0 \in \mathcal{O}[X]$ be a polynomial of degree m with $a_0 \neq 0$. We assume that $P(U)f = 0$. Then for all $\epsilon > \epsilon' > 0$, there is a unique section $g \in \mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})$ such that $P(U)g = 0$ and $res_{\epsilon',\epsilon} g = f$.*

Proof. Let $Q = -a_0^{-1}(X^m + a_{m-1}X^{m-1} + \cdots + a_1 X)$. Then $Q(U)f = f$ and $g = Q(U)^n f$ for n large enough. □

Remark 14.5.1. — Using lemmas 14.1.1, 14.1.2, corollary 13.2.4.2 and the above proposition we deduce that we can think of finite slope sections on $\mathrm{H}^i(\mathcal{X}_{Kli}(p^n, \epsilon), \mathscr{F})$ for any $\epsilon > 0$ and n as sections of $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})$ for any $\epsilon' > 0$ and similarly for cuspidal cohomology.

14.6. More analytic continuation. — We show that we can improve the last proposition if we work with torsion coefficients. Let $j : \mathcal{X}_{Kli}(p)_\epsilon \hookrightarrow \mathcal{X}_{Kli}(p)$ be the open inclusion. For any sheaf $\mathscr{G}$ over $\mathcal{X}_{Kli}(p)$ we will abusively write in this paragraph $\mathscr{G}|_{\mathcal{X}_{Kli}(p)_\epsilon}$ for $j_\star j^\star \mathscr{G}$ in order to simplify the notations.

Proposition 14.6.1. — *Let $0 < \epsilon < \epsilon'$. There is a map $U^n_{\epsilon,0}$ fitting in the following commutative diagram of normalized cohomological correspondences :*

$$\begin{tikzcd}
(t_{n,1})_\star(t_{n,2})^\star(\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}) \arrow[r, "U^n_{\epsilon,\epsilon}"] \arrow[dr, "U^n_{\epsilon,0}"] & \mathscr{F}/p^{n(2r+k-3-2\epsilon'(r+k))}\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon} \\
(t_{n,1})_\star(t_{n,2})^\star(\mathscr{F}^{++}) \arrow[r, "U^n"] \arrow[u] & \mathscr{F}/p^{n(2r+k-3-2\epsilon'(r+k))}\mathscr{F}^{++} \arrow[u]
\end{tikzcd}$$

Before giving the proof we need the following lemma.

Lemma 14.6.1. — *Let $x : \mathrm{Spa}(K, \mathcal{O}_K) \to C_n$ be a point. Assume that $|\delta_{L_n}|_x \leq |p^{3n-\alpha}|_x$. The map $\Omega^+_{G/L_n}|_x \to \Omega^+_G|_x$ factorizes through $p^{n-\alpha}\Omega^+_G|_x$. The map*

$$\mathrm{Sym}^k\Omega^+_{G/L_n} \otimes \det{}^r\Omega^+_{G/L_n}|_x \to \mathrm{Sym}^k\Omega^+_G \otimes \det{}^r\Omega^+_G|_x$$

factorizes through $p^{k(n-\alpha)+r(3n-\alpha)}\mathrm{Sym}^k\Omega^+_G \otimes \det{}^r\Omega^+_G|_x$.

Proof. We fix an isomorphism between $\Omega^+_{G/L_n}|_x \to \Omega^+_G|_x$ and $\mathcal{O}_K^2 \xrightarrow{M} \mathcal{O}_K^2$ with M a diagonal matrix with coefficients m_1, m_2. We have $|m_1 m_2|_x \leq |p^{3n-\alpha}|_x$. But on the other hand, $|m_i|_x \geq |p^{2n}|_x$ since $L_n \subset G[p^{2n}]$. We deduce that $|m_i|_x \leq |p^{n-\alpha}|_x$. □

Proof.[Proof of proposition 14.6.1] Let $x \in \mathcal{X}_{Kli}(p)$. We have to find a neighborhood U of x in $\mathcal{X}_{Kli}(p)$ and to construct a canonical map :

$$t^\star_{n,2}\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}(t^{-1}_{n,1}U) \to \mathscr{F}/p^{n(2r+k-3-2\epsilon'(r+k))}\mathscr{F}^{++}(U).$$

Pick $\epsilon'' \in]\epsilon, \epsilon'[$ such that for all $y = (G, H, L_n) \in t^{-1}_{n,1}(x)$ we have $|\delta_{L_n}|_y \neq |p^{n(3-2\epsilon'')}|_y$. This is possible since the fiber of $t_{n,1}$ is finite away from the boundary. At the boundary, it is easy to see that there are only finitely many possibilities for $|\delta_{L_n}|_y$.

It follows that there exists a neighborhood U of x and a disjoint decomposition of $t^{-1}_{n,1}(U) = V \coprod W$ where for all $(G, H, L_n) \in W$, we have $|\delta_{L_n}| > |p^{n(3-2\epsilon'')}|$ and for all $(G, H, L_n) \in V$, we have $|\delta_{L_n}| < |p^{n(3-2\epsilon'')}|$.

We have a map $U^n : t^\star_{n,2}\mathscr{F}^{++}(V) \oplus t^\star_{n,2}\mathscr{F}^{++}(W) \to \mathscr{F}(U)$. The image of $t^\star_{n,2}\mathscr{F}^{++}(V)$ in $\mathscr{F}(U)$ lands in $p^{n(2r+k-3-2\epsilon''(r+k))}\mathscr{F}^{++}(U)$ by the above lemma 14.6.1. We deduce a factorization

$$U^n : (t_{n,1})_\star t^\star_{n,2}\mathscr{F}^{++}(U) \to t^\star_{n,2}\mathscr{F}^{++}(W) \to \mathscr{F}(U)/p^{n(2r+k-3-2\epsilon''(r+k))}\mathscr{F}^{++}(U).$$

Moreover $t_{n,2}(W) \subset \mathcal{X}_{Kli}(p)_\epsilon$ by corollary 14.3.1, so that $t^\star_{n,2}\mathscr{F}^{++}(W) = t^\star_{n,2}\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}(W)$. We can construct the expected map as the composition :

$$t^\star_{n,2}\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}(t^{-1}_{n,1}U) \to t^\star_{n,2}\mathscr{F}^{++}(W) \to \mathscr{F}/p^{n(2r+k-3-2\epsilon'(r+k))}\mathscr{F}^{++}(U).$$

It clearly does not depend on the choice of ϵ''. □

Corollary 14.6.1. — *Let $\epsilon > 0$. Let $f \in \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})$ be a form satisfying $Uf = af$. Assume $v(a) < 2r + k - 3$. There is a projective system*

$$(f_n) \in \lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^n\mathscr{F}^{++})$$

which satisfies $U(f_n) = a(f_n)$ and such that $res_{0,\epsilon}(f_n)$ is the image of f in

$$\lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}/p^n\mathscr{F}^{++}).$$

Remark 14.6.1. — The U operator induces maps

$$\mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^n\mathscr{F}^{++}) \to \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{n-3}\mathscr{F}^{++}).$$

It follows that it acts on $\lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^n\mathscr{F}^{++})$.

Proof. Let $\epsilon' > 0$ be such that $\alpha = 2r+k-3-2\epsilon'(r+k)-v(a) > 0$. We can assume that $0 < \epsilon < \epsilon'$ and that $f \in \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})$ satisfies $Uf = af$ by proposition 14.5.1. The map $\mathcal{X}_{Kli}(p)_\epsilon \hookrightarrow \mathcal{X}_{Kli}(p)$ is affine (there is a covering of $\mathcal{X}_{Kli}(p)$ by affinoids, such that the fiber over these affinoids is affinoid). It follows that $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}) = \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}|_{\mathcal{X}_{Kli}(p)_\epsilon})$.

After rescaling f we may assume that f comes from a section (still denoted f) in $\mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon})$ and that $Uf \in \mathrm{H}^i(\mathcal{X}_{Kli}(p), p^{-3}\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon})$ is the image of af in $\mathrm{H}^i(\mathcal{X}_{Kli}(p), p^{-3}\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon})$. We define the sections $f_n = a^{-n}U^n_{\epsilon,0}f \in \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{n\alpha}\mathscr{F}^{++})$.

Consider the following commutative diagram :

$$\begin{tikzcd}
\mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}) \arrow[r, "a^{-n}U^n_{\epsilon,0}"] \arrow[d, "a^{-1}U"] & \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{n\alpha}\mathscr{F}^{++}) \arrow[d] \\
\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, p^{-3-v(a)}\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}) \arrow[r, "a^{-n-1}U^{n-1}_{\epsilon,0}"] & \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{(n-1)\alpha-3-v(a)}\mathscr{F}^{++}) \\
\mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}) \arrow[r, "a^{-n-1}U^{n-1}_{\epsilon,0}"] \arrow[u] & \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{(n-1)\alpha}\mathscr{F}^{++}) \arrow[u]
\end{tikzcd}$$

where the vertical maps going from the bottom to the middle line are the obvious ones. Since the image of $f \in \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon})$ is the same via any of the two left vertical maps, we deduce that $f_n = f_{n-1}$ in $\mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{(n-1)\alpha-3-v(a)}\mathscr{F}^{++})$. Consider the following commutative diagram :

$$\begin{tikzcd}
& \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{n\alpha}\mathscr{F}^{++}) \arrow[dr, "U"] & \\
\mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}) \arrow[ur, "a^{-n}U^n"] \arrow[dr, "U"'] & & \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{n\alpha-3}\mathscr{F}^{++}) \\
& \mathrm{H}^i(\mathcal{X}_{Kli}(p), p^{-3}\mathscr{F}^{++}|_{\mathcal{X}_{Kli}(p)_\epsilon}) \arrow[ur, "a^{-n}U^n"'] &
\end{tikzcd}$$

It follows that $Uf_n = af_n$ in $\mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{n\alpha-3}\mathscr{F}^{++})$. As a conclusion, we obtain a projective system

$$(f_n) \in \lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^{n\alpha-3-v(a)}\mathscr{F}^{++}) = \lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^n\mathscr{F}^{++})$$

which satisfies $U(f_n) = a(f_n)$. By construction, $res_{0,\epsilon}(f_n)$ is the image of f in $\lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}/p^n\mathscr{F}^{++})$.

$\square$

We can again slightly improve the above corollary :

Corollary 14.6.2. — *Let $f \in \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})$. Let $P = X^m + a_{m-1}X^{m-1} + \cdots + a_0 \in \mathcal{O}[X]$ be a polynomial of degree m. We assume that $P(U)f = 0$ and that for all the roots a of P in $\mathbb{C}$, we have $v(a) < 2r + k - 3$. There is a projective system*

$$(f_n) \in \lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p), \mathscr{F}/p^n\mathscr{F}^{++})$$

which satisfies $P(U)(f_n) = 0$ and such that $res_{0,\epsilon}(f_n)$ is the image of f in

$$\lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}/p^n\mathscr{F}^{++}).$$

Proof. We let $Q = -a_0^{-1}(X^m + a_{m-1}X^{m-1} + \cdots + X)$. Then $Q(U)f = f$ and we let $f_n = Q(U)^n f$ as in the proof of corollary 14.6.1. □

14.7. Classicity of overconvergent cohomology. — We are now ready to state our main result on the classicity of small slope cohomology classes.

Lemma 14.7.1. — *For any finite slope $h \in \mathbb{Q}$, the map $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})^{\leq h} \to \lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}/p^n\mathscr{F}^+)$ is injective.*

Proof. Denote by V the image of $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}^+)$ in $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})$. We have to prove that $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})^{\leq h} \cap V$ is bounded. Let I be a finite set and $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{U}' = \{U'_i\}_{i\in I}$ be two finite affinoid coverings of $\mathcal{X}_{Kli}(p)$. We assume that $\overline{U'_i} \subset U_i$. Such a covering exists because $\mathcal{X}_{Kli}(p)$ is proper. Let $\mathcal{U}_\epsilon = \{U_{i,\epsilon}\}$ be the finite affinoid covering $\mathcal{U} \cap \mathcal{X}_{Kli}(p)_\epsilon$. Let $\epsilon < \epsilon'$ be such that $U(\mathcal{X}_{Kli}(p)_{\epsilon'}) \subset \mathcal{X}_{Kli}(p)_\epsilon$. Let $\mathcal{U}_{\epsilon'} = \{U_{i,\epsilon'}\}$ be the covering $\mathcal{U}' \cap \mathcal{X}_{Kli}(p)_{\epsilon'}$. For all $i \in I$, we have $\overline{U_{i,\epsilon'}} \subset U_{i,\epsilon}$.The U operator is defined as the composite

$$\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}) \stackrel{res}{\to} \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F}) \stackrel{U_{\epsilon,\epsilon'}}{\to} \mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}).$$

We can represent $\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})$ by the Čech complex $M^\bullet = \check{\mathrm{C}}\mathrm{ech}(\mathcal{U}_\epsilon, \mathscr{F})$ and $\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_{\epsilon'}, \mathscr{F})$ by $N^\bullet = \check{\mathrm{C}}\mathrm{ech}(\mathcal{U}_{\epsilon'}, \mathscr{F})$. The map U can be represented by

$$\tilde{U} : M^\bullet \stackrel{res}{\to} N^\bullet \stackrel{\tilde{U}_{\epsilon',\epsilon}}{\to} M^\bullet$$

which is compact. We have a direct summand $(M^\bullet)^{\leq h}$ which is a complex of finite dimensional vector spaces and $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})^{\leq h} = \mathrm{H}^i((M^\bullet)^{\leq h})$. Since the natural map $\check{\mathrm{H}}^i_{\mathcal{U}_\epsilon}(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}^+) \to \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}^+)$ has cokernel of bounded torsion by lemma 3.2.2, we can replace V by V' the image of $\check{\mathrm{H}}^i_{\mathcal{U}_\epsilon}(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F}^+)$ in $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})$. Let $\mathcal{Z}^i((M^\bullet)^{\leq h}) \subset M^i$ be the cocycles of slope less than h. This is a finite dimensional vector space. We denote by $M^{+\bullet}$ the Čech complex $\check{\mathrm{C}}\mathrm{ech}(\mathcal{U}_\epsilon, \mathscr{F}^+)$. Then M^{+i} is bounded in M^i. It follows that $M^{+i} \cap \mathcal{Z}^i((M^\bullet)^{\leq h})$ is bounded and thus a lattice. As a result, its image in $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})^{\leq h}$ (which is $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \mathscr{F})^{\leq h} \cap V'$) is bounded. □

Theorem 14.7.1. — *For any $\epsilon \in [0,1[\cap\mathbb{Q}$, the restriction map*

$$\mathrm{H}^i(\mathcal{X}_{Kli}(p), \Omega^{(k,r)})^{<k+2r-3} \to \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,r)})^{<k+2r-3}$$

is bijective. A similar statement holds for cuspidal cohomology

Proof. Denote by res the map of the corollary. We first exhibit a map ext : $\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,r)})^{<k+2r-3} \to \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,r)})^{<k+2r-3}$ in the other direction. Given $f \in \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,r)})^{<k+2r-3}$, we obtain $(f_n) \in \lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p), \Omega^{(k,r)}/p^n(\Omega^{(k,r)})^+)$ by corollary 14.6.2. Since

$$\lim_n \mathrm{H}^i(\mathcal{X}_{Kli}(p), \Omega^{(k,r)}/p^n(\Omega^{(k,r)})^+) = \mathrm{H}^i(\mathcal{X}_{Kli}(p), \Omega^{(k,r)})$$

by proposition 3.2.1, this defines the map ext. Using lemma 14.7.1, we deduce that $res \circ ext = \mathrm{id}$. Unravelling the construction of ext, we deduce that $ext \circ res = \mathrm{id}$. □

Corollary 14.7.1. — *1. The map*

$$\mathrm{H}^i(\mathcal{X}_{Kli}(p), \Omega^{(k,r)})^{<\min\{k+2r-3,k-2\}} \to \mathrm{H}^i(\dagger, k, r)^{<\min\{k+2r-3,k-2\}}$$

is an isomorphism. A similar statement holds for cuspidal cohomology.

2. The map

$$\mathrm{H}^0(\mathcal{X}_{Kli}(p), \Omega^{(k,r)})^{<\min\{k+2r-3,k+1\}} \to \mathrm{H}^0(\dagger, k, r)^{<\min\{k+2r-3,k+1\}}$$

is an isomorphism and a similar statement holds for cuspidal cohomology.

3. The map

$$\mathrm{H}^1(\mathcal{X}_{Kli}(p), \Omega^{(k,r)})^{<\min\{k+2r-3,k+1\}} \to \mathrm{H}^1(\dagger, k, r)^{<\min\{k+2r-3,k+1\}}$$

is injective and a similar statement holds for cuspidal cohomology.

Proof. This is a combination of theorem 14.7.1 and corollary 13.3.3.1 (see also remark 14.5.1). □

14.8. Application to ordinary cohomology. — We are now able to deduce a classicity theorem for ordinary classes in ordinary cohomology. We recall that f is the ordinary projector attached to U.

Theorem 14.8.1. — *The map*

$$f\mathrm{R}\Gamma(X_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p \to f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism for all $k \geq 0$.

The proof of this theorem will be split into several lemmas. We denote by $\mathcal{X}^{\geq 1}_{Kli}(p)$ the adic space over $\mathrm{Spa}(\mathbb{C}, \mathcal{O})$ attached to $\mathfrak{X}^{\geq 1}_{Kli}(p)$. By definition $\mathcal{X}^{\geq 1}_{Kli}(p) = \mathcal{X}_{Kli}(p)_1$, but we prefer to use the notation $\mathcal{X}^{\geq 1}_{Kli}(p)$ for this space.

Lemma 14.8.1. — *We have quasi-isomorphisms :*

1. $f\mathrm{R}\Gamma(X_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes^L_{\mathbb{Z}_p} \mathbb{C} \simeq f\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p), \Omega^{(k,2)}(-D))$,
2. $f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes^L_{\mathbb{Z}_p} \mathbb{C} \simeq f\mathrm{R}\Gamma(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$.

Proof. The first point follows from the GAGA theorem stated in [**69**], thm. 9.1. The second point follows classically from the fact that $\mathcal{X}^{\geq 1}_{Kli}(p)$ is the adic space over $\mathrm{Spa}(\mathbb{C}, \mathcal{O})$ attached to $\mathfrak{X}^{\geq 1}_{Kli}(p)$. □

Remark 14.8.1. — The $\mathbb{C}$-vector space $\mathrm{H}^i(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ admits a 0-slope decomposition in the sense of section 13.1.1 : $f\mathrm{H}^i(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D)) = \mathrm{H}^i(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))^{=0}$ is the slope 0 part and $(1-f)\mathrm{H}^i(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D)) = \mathrm{H}^i(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))^{>0}$ is the slope strictly greater than 0 part.

To prove the theorem, it suffices to show that the restriction map

$$f\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D)) \to f\mathrm{R}\Gamma(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$$

is a quasi-isomorphism for $k \geq 0$ and any $\epsilon \in [0, 1[$, by theorem 14.7.1.

Lemma 14.8.2. — *For any $k \geq 0$, and any $\epsilon \in]0,1[\cap \mathbb{Q}$, the cohomology complexes $f\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D))$ and $f\mathrm{R}\Gamma(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ are concentrated in degree 0 and 1. Moreover, the map $f\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D)) \to f\mathrm{H}^i(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ is injective if $i = 0$ and surjective if $i = 1$.*

Proof. We prove that these cohomology complexes are concentrated in degree 0 and 1 by exhibiting a special covering computing the cohomology that will be useful to establish the other claims of the lemma (24). Let us denote by $X^\star \to \mathrm{Spec}\ \mathbb{Z}_p$ the minimal compactification of the Siegel threefold with spherical level at p, and by $X^\star_{Kli}(p)$ the minimal compactification with Klingen level at p. We denote by $\mathfrak{X}^\star$ and $\mathfrak{X}^\star_{Kli}(p)$ the associated formal schemes. The p-rank at least one locus $\mathfrak{X}^{\star,\geq 1} \hookrightarrow \mathfrak{X}^\star$ is covered by two affines : sufficiently high powers $\mathrm{Ha}(G)^{n(p+1)}$ and $\mathrm{Ha}'(G)^n$ of the first and second Hasse invariant (defined respectively on $V(p)$ and $V(p, \mathrm{Ha}(G))$) lift (non-canonically) to sections s_1 and s_2 of the ample sheaf $(\det \omega_G)^{n(p^2-1)}$ over $\mathfrak{X}^\star$, and their non-zero locus is $\mathfrak{X}^{\star,\geq 1}$. Therefore $\mathfrak{X}^{\star,\geq 1} = D(s_1) \cup D(s_2)$ is covered by two affines.

The map $\mathfrak{X}^\star_{Kli}(p) \times_{\mathfrak{X}^\star} \mathfrak{X}^{\star,\geq 1} \to \mathfrak{X}^{\star,\geq 1}$ is proper and quasi-finite, therefore it is affine. We deduce that $\mathfrak{X}^\star_{Kli}(p) \times_{\mathfrak{X}^\star} \mathfrak{X}^{\star,\geq 1}$ is also covered by two affines. Over the toroidal compactification $\mathfrak{X}_{Kli}(p)$ we have the canonical chain of isogenies $G \to G' \to G$. The non-zero locus of the map on differentials $\det \omega_G \to \det \omega_{G'}$ is by definition $\mathfrak{X}^{\geq 1}_{Kli}(p) \hookrightarrow \mathfrak{X}_{Kli}(p)$. The map $\det \omega_G \to \det \omega_{G'}$ descends to a map of invertible sheaves over $\mathfrak{X}^\star_{Kli}(p)$ and its non-zero locus defines the open formal subscheme $\mathfrak{X}^{\star,\geq 1}_{Kli}(p) \hookrightarrow \mathfrak{X}^\star_{Kli}(p)$, whose inverse image in $\mathfrak{X}_{Kli}(p)$ is $\mathfrak{X}^{\geq 1}_{Kli}(p)$. The map $\mathfrak{X}^{\star,\geq 1}_{Kli}(p) \hookrightarrow \mathfrak{X}^\star_{Kli}(p)$ is affine, and moreover it factors through $\mathfrak{X}^\star_{Kli}(p) \times_{\mathfrak{X}^\star} \mathfrak{X}^{\star,\geq 1} \hookrightarrow \mathfrak{X}^\star_{Kli}(p)$. We deduce that $\mathfrak{X}^{\star,\geq 1}_{Kli}(p)$ is covered by two affines, say $\mathfrak{V}_1$ and $\mathfrak{V}_2$. Let $\mathcal{X}^\star_{Kli}(p)$ be the analytic adic space over $\mathrm{Spa}(\mathbb{C}, \mathcal{O})$ attached to $\mathfrak{X}^\star_{Kli}(p)$, let V_1 and V_2 denote the inverse images of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ in $\mathcal{X}^\star_{Kli}(p)$ and set $\mathcal{X}^{\star,\geq 1}_{Kli}(p) = V_1 \cup V_2$. Let $\pi : \mathcal{X}_{Kli}(p) \to \mathcal{X}^\star_{Kli}(p)$ be the projection. Then $\mathrm{R}\pi_\star \Omega^{(k,2)}(-D) = \pi_\star \Omega^{(k,2)}(-D)$ (by [**50**], thm. 8.9). Moreover, the image by π of $\mathcal{X}^{\geq 1}_{Kli}(p)$ is $\mathcal{X}^{\star,\geq 1}_{Kli}(p)$. Let $U_i = \pi^{-1} V_i$ for $i \in \{1,2\}$. We deduce that $\mathrm{R}\Gamma(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ is represented by the complex :

$$\mathrm{H}^0(U_1, \Omega^{(k,2)}(-D)) \oplus \mathrm{H}^0(U_2, \Omega^{(k,2)}(-D)) \to \mathrm{H}^0(U_1 \cap U_2, \Omega^{(k,2)}(-D))$$

and thus that $f\mathrm{R}\Gamma(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ is concentrated in degree 0 and 1. This complex contains a subcomplex of overconvergent sections :

$$\mathrm{H}^0(U_1, \Omega^{(k,2),\dagger}(-D)) \oplus \mathrm{H}^0(U_2, \Omega^{(k,2),\dagger}(-D)) \to \mathrm{H}^0(U_1 \cap U_2, \Omega^{(k,2),\dagger}(-D)).$$

whose i-th cohomology group computes $\mathrm{colim}_{\epsilon<1} \mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D))$. Since $f\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D)) = f\mathrm{H}^i(\mathcal{X}_{Kli}(p)_{\epsilon'}, \Omega^{(k,2)}(-D))$ for any $\epsilon, \epsilon' \in]0,1[\cap \mathbb{Q}$ by proposition 14.5.2, we deduce that $f\mathrm{R}\Gamma(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D))$ is concentrated in degree 0 and 1.

It is clear that restriction induces an injection $\mathrm{colim}_{\epsilon<1} \mathrm{H}^0(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D)) \to \mathrm{H}^0(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$.

The cohomology group $\mathrm{H}^1(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ carries the natural quotient topology from the surjection $\mathrm{H}^0(U_1 \cap U_2, \Omega^{(k,2)}(-D)) \to \mathrm{H}^1(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ (25) and we deduce

24. That these cohomologies are concentrated in degree 0 and 1 has already been established by slightly different methods, see theorem 11.3.1, point 2 and proposition 12.9.1.

25. We can describe the topology on the $\mathbb{C}$-vector space $\mathrm{H}^i(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ in a more intrinsic way as follows : an open and bounded submodule is given by the image of

$$\mathrm{H}^i(\mathfrak{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D) \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}),$$

that the map $\mathrm{colim}_{\epsilon\to 1}\mathrm{H}^1(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D)) \to \mathrm{H}^1(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ has dense image. If we apply the ordinary projector on both sides (the ordinary projector is compatible because 0-slope decomposition is functorial), we get a surjection since the ordinary part is finite dimensional. □

Let us denote by

$$d_i(k) = \dim_{\mathbb{C}} f\mathrm{H}^i(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$$

and by

$$d^\dagger_i(k) = \dim_{\mathbb{C}} f\mathrm{H}^i(\mathcal{X}_{Kli}(p)_\epsilon, \Omega^{(k,2)}(-D))$$

for any $\epsilon \in [0,1[\cap\mathbb{Q}$.

Lemma 14.8.3. — *For all $k \geq 0$, we have $d_0(k) \geq d^\dagger_0(k)$ and $d^\dagger_1(k) \geq d_1(k)$. If k is large enough, we have $d_i(k) = d^\dagger_i(k)$.*

Proof. We have $d_0(k) \geq d^\dagger_0(k)$ and $d^\dagger_1(k) \geq d_1(k)$ by lemma 14.8.2. Moreover, if k is large enough, we have an isomorphism $e\mathrm{H}^0(\mathcal{X}, \Omega^{(k,2)}(-D)) \to f\mathrm{H}^0(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ and an injection $e\mathrm{H}^1(\mathcal{X}, \Omega^{(k,2)}(-D)) \to f\mathrm{H}^1(\mathcal{X}^{\geq 1}_{Kli}(p), \Omega^{(k,2)}(-D))$ (by theorem 11.3.1). The lemma follows from the claim that the maps : $e\mathrm{H}^i(\mathcal{X}, \Omega^{(k,2)}(-D)) \to f\mathrm{H}^i(\mathcal{X}_{Kli}(p), \Omega^{(k,2)}(-D))$ are isomorphisms for $i \in \{0,1\}$. The Hecke parameters $(\alpha,\beta,\gamma,\delta)$ of an irreducible smooth admissible representation π_p of $\mathrm{GSp}_4(\mathbb{Q}_p)$ contributing to either $e\mathrm{H}^i(\mathcal{X}, \Omega^{(k,2)}(-D))$ or $f\mathrm{H}^i(\mathcal{X}_{Kli}(p), \Omega^{(k,2)}(-D))$ have p-adic valuations (in a suitable order) $0, 0, k+1, k+1$ by corollary 14.9.1. The claim follows from proposition 5.1.5.2 and lemma 5.1.5.2. □

Lemma 14.8.4. — *For all $k \geq 0$, we have $d_i(k) = d^\dagger_i(k)$.*

Proof. Let us denote by $d^{\dagger,'}_i(k) = \dim_{\mathbb{C}} \mathrm{H}^i_{cusp}(\dagger, k, 2)$. We have $d^{\dagger,'}_0(k) = d^\dagger_0(k)$ for all $k \geq 0$, and $d^{\dagger,'}_1(k) \geq d^\dagger_1(k)$ for all $k \geq 0$, with equality if $k \geq 3$ by corollary 13.3.3.1. The Euler characteristics $d_1(k) - d_0(k)$ and $d^{\dagger,'}_1(k) - d^{\dagger,'}_0(k)$ are locally constant functions of $k \in \mathbb{Z}_{\geq 0}$ by theorem 11.3.1 and proposition 13.4.1. We deduce that $d_1(k) - d_0(k) = d^{\dagger,'}_1(k) - d^{\dagger,'}_0(k)$ for all $k \geq 0$ by lemma 14.8.3. It follows that for all $k \geq 0$, $d_1(k) - d^{\dagger,'}_1(k) = d_0(k) - d^{\dagger,'}_0(k)$, but since the first difference is non-positive and the second difference is non-negative by lemma 14.8.3, we deduce that $d^{\dagger,'}_i(k) = d_i(k)$ for all $k \geq 0$. Since $d^{\dagger,'}_0(k) = d^\dagger_0(k)$ and $d^{\dagger,'}_1(k) \geq d^\dagger_1(k) \geq d_1(k)$ for all $k \geq 0$, the lemma and the theorem are proven. □

14.9. Estimates on Satake parameters. — We have Hecke operators $T_{p,2}$ and $U_{Kli(p),2}$ acting on $\mathrm{H}^i(X, \Omega^{(k,2)}(-D)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathrm{H}^i(X_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The goal of this section is to establish the following result :

Proposition 14.9.1. — *For all $k \geq 0$, the Hecke operators $T_{p,2}$ and $U_{Kli(p),2}$ acting on $\mathrm{H}^i(X, \Omega^{(k,2)}(-D)) \otimes_{\mathbb{Z}_p} \mathbb{C}$ and $\mathrm{H}^i(X_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes_{\mathbb{Z}_p} \mathbb{C}$ respectively only have eigenvalues of positive p-adic valuation.*

We deduce the following corollary :

Corollary 14.9.1. — *Assume that $k \geq 1$. The Hecke parameters $(\alpha,\beta,\gamma,\delta)$ of an irreducible smooth admissible representation π_p of $\mathrm{GSp}_4(\mathbb{Q}_p)$ contributing to either $e\mathrm{H}^i(X, \Omega^{(k,2)}(-D)) \otimes \mathbb{C}$ or $f\mathrm{H}^i(X_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes \mathbb{C}$ have p-adic valuations (in a suitable order) $(0, 0, k+1, k+1)$.*

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endowed with the p-adic topology. Be careful that this last space is complete for the p-adic topology, but not necessarily separated.

Proof. This is an elementary Newton polygon computation in the spherical case and obvious in the Klingen case (in which case the corollary holds for $k \geq 0$). □

Over $X_{Kli}(p)$, we consider the chain $G \to G' \to G$. We have the differential $\det \omega_G \to \det \omega_{G'}$ of the second isogeny and we denote by $\mathcal{I} = \det \omega_G \otimes \det \omega_{G'}^{-1}$. This is an invertible sheaf of ideals in $\mathscr{O}_{\mathfrak{X}_{Kli}(p)}$. It defines a Cartier divisor supported in the special fiber of $X_{Kli}(p)$, whose complement in the special fiber is $X_{Kli}^{\geq 1}(p)_1$. We denote as usual $\mathfrak{X}^{\geq 1}$ the open of p-rank at least one of the formal completion of X. We also let $(\mathfrak{X}_{Kli}(p))^{\geq 1}$ be the open of p-rank at least one of the formal completion of $X_{Kli}(p)$ (it contains strictly $\mathfrak{X}_{Kli}^{\geq 1}(p)$ which is the locus where the universal rank p group scheme is multiplicative). Our key lemma is:

Lemma 14.9.1. — *The Hecke operator $T_{p,2}$ acts on $\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}, \Omega^{(k,2)}(-D))$ and the Hecke operator $U_{Kli(p),2}$ acts on $\mathrm{R}\Gamma((\mathfrak{X}_{Kli}(p))^{\geq 1}, \Omega^{(k,2)}(-D) \otimes \mathcal{I}^{-3})$.*

Remark 14.9.1. — Using the techniques developed in section 4, we could easily construct an action of $T_{p,2}$ on $\mathrm{R}\Gamma(X, \Omega^{(k,2)}(-D))$.

Remark 14.9.2. — The sheaf $\Omega^{(k,2)}(-D)\otimes\mathcal{I}^{-3}$ is an integral structure on $\Omega^{(k,2)}(-D)\otimes_{\mathbb{Z}_p} \otimes\mathbb{Q}_p$. The reason for which we need to modify the obvious integral structure $\Omega^{(k,2)}(-D)$ will become clear during the proof.

14.9.1. proof of proposition 14.9.1 assuming lemma 14.9.1. — We will only give a full proof of the proposition for classes at Klingen level. The spherical case is identical. We start by the following lemma:

Lemma 14.9.2. — *For any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ (26), the maps*

$$\mathrm{H}^i(X, \Omega^{(k,2)}(-D)/p^n) \to \mathrm{H}^i(\mathfrak{X}^{\geq 1}, \Omega^{(k,2)}(-D)/p^n)$$

and

$$\mathrm{H}^i(X_{Kli}(p), \Omega^{(k,2)}(-D) \otimes \mathcal{I}^{-3}/p^n) \to \mathrm{H}^i((\mathfrak{X}_{Kli}(p))^{\geq 1}, \Omega^{(k,2)}(-D) \otimes \mathcal{I}^{-3}/p^n)$$

are isomorphisms for $i = 0$ and injective for $i = 1$.

Proof. This follows from [**SGA**], exposé III, section 3. The point is that $\mathfrak{X}$ and $\mathfrak{X}_{Kli}(p)$ are Cohen-Macaulay and $\mathfrak{X}^{\geq 1}$ and $(\mathfrak{X}_{Kli}(p))^{\geq 1}$ are opens of codimension 2. □

Let $i \in \{0, 1\}$. Let $f \in \mathrm{H}^i(X_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes \mathbb{C}$ be an eigenclass. We assume that $U_{Kli(p),2} f = \alpha f$ with $v(\alpha) < 0$, so that $\alpha^{-1} U_{Kli(p),2} f = f$. We want to deduce that $f = 0$. We have a commutative diagram where the horizontal maps are injective :

$$\begin{tikzcd}
\mathrm{H}^i(X_{Kli}(p), \Omega^{(k,2)}(-D)) \otimes \mathbb{C} \arrow[r] & \mathrm{H}^i((\mathfrak{X}_{Kli}(p))^{\geq 1}, \Omega^{(k,2)}(-D)) \otimes \mathbb{C} \\
\mathrm{H}^i(X_{Kli}(p), \Omega^{(k,2)}(-D) \otimes \mathcal{I}^{-3}) \otimes \mathcal{O} \arrow[r] \arrow[u] & \mathrm{H}^i((\mathfrak{X}_{Kli}(p))^{\geq 1}, \Omega^{(k,2)}(-D) \otimes \mathcal{I}^{-3}) \otimes \mathcal{O} \arrow[u]
\end{tikzcd}$$

After rescaling f, we may assume that f comes from a class $g \in \mathrm{H}^i(X_{Kli}(p), \Omega^{(k,2)}(-D) \otimes \mathcal{I}^{-3}) \otimes \mathcal{O}$. Moreover, the image g' of g in $\mathrm{H}^i((\mathfrak{X}_{Kli}(p))^{\geq 1}, \Omega^{(k,2)}(-D) \otimes \mathcal{I}^{-3}) \otimes \mathcal{O}$ satisfies $\alpha^{-1} U_{Kli(p),2} g' = g' + h$ where $h \in \mathrm{H}^i((\mathfrak{X}_{Kli}(p))^{\geq 1}, \Omega^{(k,2)}(-D) \otimes \mathcal{I}^{-3}) \otimes \mathcal{O}$ is a torsion class. Rescaling g further, we may assume that $h = 0$. For any $n \geq 0$, there exists n' such that $p^n \in \alpha^{-n'} \mathcal{O}$ and we deduce that $g' = \alpha^{-n} U^n_{Kli(p),2} g'$ is zero in $\mathrm{H}^i((\mathfrak{X}_{Kli}(p))^{\geq 1}, \Omega^{(k,2)}(-D) \otimes$

26. With the convention that $p^\infty = 0$.

$\mathcal{I}^{-3}/p^n)\otimes\mathcal{O}$, and therefore g maps to zero in $\mathrm{H}^i(X_{Kli}(p),\Omega^{(k,2)}(-D)\otimes\mathcal{I}^{-3}/p^n)\otimes\mathcal{O}$. Since $\lim_n \mathrm{H}^i(X_{Kli}(p),\Omega^{(k,2)}(-D)\otimes\mathcal{I}^{-3}/p^n) = \mathrm{H}^i(X_{Kli}(p),\Omega^{(k,2)}(-D)\otimes\mathcal{I}^{-3})$, we conclude that $g=0$ and that $f=0$.

14.9.2. proof of lemma 14.9.1. — We can define a Hecke correspondences attached to the double coset $T_{p,2}$ (see [**18**], p. 253) for suitable choices of polyhedral cone decompositions Σ,Σ' and Σ'':

$$\begin{tikzcd} & C_{p,2,\Sigma''} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ X_{\Sigma'} & & X_{\Sigma} \end{tikzcd}$$

It will convenient for us to take $\Sigma=\Sigma''$. We drop the subscript corresponding to the choices of polyhedral cone decompositions. Recall that $C_{p,2}$ parametrizes isogenies $p_1^\star G\to p_2^\star G$ whose kernel is (away from the boundary) an isotropic rank p^2 subgroup of $p_1^\star G[p]$.

Denote by $\mathfrak{C}_{p,2}$ the formal completion of $C_{p,2}$ and by $(\mathfrak{C}_{p,2})^{\geq 1}$ its restriction to the p-rank at least one locus. The map $p_1:(\mathfrak{C}_{p,2})^{\geq 1}\to\mathfrak{X}^{\geq 1}$ is finite, generically étale. Therefore, we have a trace morphism $\mathrm{Tr}_{p_1}:(p_1)_\star\mathscr{O}_{(\mathfrak{C}_{p,2})^{\geq 1}}\to\mathscr{O}_{\mathfrak{X}^{\geq 1}}$ (since $\mathfrak{X}^{\geq 1}$ is smooth, hence normal). We also have a morphism $p_2^\star\Omega^{(k,2)}\to p_1^\star\Omega^{(k,2)}$ coming from the differential of the isogeny. Using these, we get a map $T'_{p,2}:(p_1)_\star p_2^\star\Omega^{(k,2)}(-D)\to\Omega^{(k,2)}(-D)$ and we let $T_{p,2}=p^{-3}T'_{p,2}$. We claim that $T'_{p,2}:(p_1)_\star p_2^\star\Omega^{(k,2)}(-D)\to p^3\Omega^{(k,2)}(-D)$ so that the map $T_{p,2}$ is well-defined. It is enough to check the claim over the ordinary locus by normality. Over the formal neighborhood of an ordinary point $x\in\mathfrak{X}^{\geq 1}$, the correspondence $(\mathfrak{C}_{p,2})^{\geq 1}$ splits into several components : the locus where the isogeny $p_1^\star G\to p_2^\star G$ has multiplicative kernel, has kernel an extension of an étale by a multiplicative group, and has kernel an étale group. In the first case, the map $p_2^\star\Omega^{(k,2)}\to p_1^\star\Omega^{(k,2)}$ factors through $p^{k+4}p_1^\star\Omega^{(k,2)}$, in the second case it factors through $p^2p_1^\star\Omega^{(k,2)}$ and in the last case it is an isomorphism. On the other hand, the restriction of the trace map Tr_{p_1} is an isomorphism in the first case, factors through $p\mathscr{O}_{\mathfrak{X}^{\geq 1}}$ in the second case, and through $p^3\mathscr{O}_{\mathfrak{X}^{\geq 1}}$ in the last case (computations using Serre-Tate). The lemma is thus proven over $\mathfrak{X}^{\geq 1}$.

We now consider the situation over $(\mathfrak{X}_{Kli}(p))^{\geq 1}$. Our first task is to produce a model for the correspondence attached to $U_{Kli(p),2}$. We first consider $(\mathfrak{D}_{p,2})^{\geq 1}=(\mathfrak{C}_{p,2})^{\geq 1}\times_{\mathfrak{X}^{\geq 1},p_1}(\mathfrak{X}_{Kli}(p))^{\geq 1}$ and we denote by $(\mathcal{D}_{p,2})^{\geq 1}$ the associated analytic space over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$. This is not quiet the correspondence corresponding to $U_{Kli(p),2}$: we need to select certain irreducible components. Over $(\mathfrak{D}_{p,2})^{\geq 1}$, we have an isogeny $p_1^\star G\to p_2^\star G$ has well as the universal chain $G\to(G')^t\to G'\to G$ (where the first map has degree p, the second p^2, the third p and the total map is multiplication by p). We let $H=\mathrm{Ker}(G\to(G')^t)$. We let $(\mathcal{E}_{p,2})^{\geq 1}$ be the union of components of $(\mathcal{D}_{p,2})^{\geq 1}$ where the kernel of the universal isogeny $p_1^\star G\to p_2^\star G$ has generically (that is away from the boundary) a trivial intersection with the universal subgroup H. We let $(\mathfrak{E}_{p,2})^{\geq 1}\to(\mathfrak{D}_{p,2})^{\geq 1}$ be the normalization of the closure of $(\mathcal{E}_{p,2})^{\geq 1}$ in $(\mathfrak{D}_{p,2})^{\geq 1}$. The map $p_2:(\mathfrak{E}_{p,2})^{\geq 1}\to\mathfrak{X}^{\geq 1}$ lifts on the generic fiber to a map $(\mathcal{E}_{p,2})^{\geq 1}\to(\mathcal{X}_{Kli}(p))^{\geq 1}$ (we equip $p_2^\star G$ with the image of H via $p_1^\star G\to p_2^\star G$), and thus by normality it extends to a map of formal schemes $p_2:(\mathfrak{E}_{p,2})^{\geq 1}\to(\mathfrak{X}_{Kli}(p))^{\geq 1}$. We have thus produced a model for the correspondence.

We have a trace morphism $\mathrm{Tr}_{p_1}:(p_1)_\star\mathscr{O}_{(\mathfrak{E}_{p,2})^{\geq 1}}\to\mathscr{O}_{(\mathfrak{X}_{Kli}(p))^{\geq 1}}$ (because p_1 is finite, generically finite étale and $(\mathfrak{X}_{Kli}(p))^{\geq 1}$ is normal). We have also have morphism $p_2^\star\Omega^{(k,2)}\to p_1^\star\Omega^{(k,2)}$.

Lemma 14.9.2.1. — *There is a natural morphism $p_2^\star\mathcal{I}^{-1} \to p_1^\star\mathcal{I}^{-1}$.*

Proof. The isogeny induces a generic isomorphism $p_1^\star H \to p_2^\star H$ and by Cartier duality a map : $p_2^\star H^D \to p_1^\star H^D$. We thus have a map $p_1^\star\mathscr{O}_{H^D} \to p_2^\star\mathscr{O}_{H^D}$ of finite flat, generically étale $\mathscr{O}_{\mathfrak{C}_{p,2}}$-algebras (27). We deduce that there is a map

$$p_2^\star D_{\mathscr{O}_{H^D}/\mathscr{O}_{(\mathfrak{X}_{Kli(p)})^{\geq 1}}} \to p_1^\star D_{\mathscr{O}_{H^D}/\mathscr{O}_{(\mathfrak{X}_{Kli(p)})^{\geq 1}}}$$

between the inverse different of both algebras. By [**20**], section 1.3, 1.4 and definition 3, the tensor product

$$D_{\mathscr{O}_{H^D}/\mathscr{O}_{(\mathfrak{X}_{Kli(p)})^{\geq 1}}} \otimes_{\mathscr{O}_{H^D},e} \mathscr{O}_{(\mathfrak{X}_{Kli(p)})^{\geq 1}}$$

for the unit section $e : \mathscr{O}_{H^D} \to \mathscr{O}_{(\mathfrak{X}_{Kli(p)})^{\geq 1}}$ is canonically isomorphic to $\mathcal{I}^{-1}$. □

Remark 14.9.3. — If we take an ordinary point $x \in (\mathfrak{C}_{p,2})^{\geq 1}$, there are three possibilities for the map $p_1^\star H \to p_2^\star H$ at x :

1. $p_1^\star H$ and $p_2^\star H$ are multiplicative groups and the map is an isomorphism. In that case, the map $p_2^\star\mathcal{I}^{-1} \to p_1^\star\mathcal{I}^{-1}$ is an isomorphism at x.
2. $p_1^\star H$ and $p_2^\star H$ are étale groups and the map is an isomorphism. In that case, the map $p_2^\star\mathcal{I}^{-1} \to p_1^\star\mathcal{I}^{-1}$ is an isomorphism at x.
3. $p_1^\star H$ is an étale group and $p_2^\star H$ is a multiplicative group, and the map is zero at the point x. In that case, the map $p_2^\star\mathcal{I}^{-1} \to p_1^\star\mathcal{I}^{-1}$ factors through $pp_1^\star\mathcal{I}^{-1}$ over the local ring at x (see [**20**], prop. 2 on page 11).

All together, we can use this to produce a map $U'_{Kli(p),2} : (p_1)_\star p_2^\star\Omega^{(k,2)}(-D)\otimes\mathcal{I}^{-3} \to \Omega^{(k,2)}(-D)\otimes\mathcal{I}^{-3}$. We claim that this map factors through $p^3\Omega^{(k,2)}(-D)\otimes\mathcal{I}^{-3}$. We take an ordinary point $x \in (\mathfrak{X}_{Kli(p)})^{\geq 1}$ and work in the formal neighborhood of this point. We first consider the case where H is a multiplicative group at x. The correspondence splits into several components over the formal neighborhood of x : the locus where the isogeny $p_1^\star G \to p_2^\star G$ has kernel an extension of an étale by a multiplicative group which intersects trivially with H, and the locus where it has kernel an étale group. In the first case, the map $p_2^\star\Omega^{(k,2)} \to p_1^\star\Omega^{(k,2)}$ factors through $p^2 p_1^\star\Omega^{(k,2)}$ and in the last case it is an isomorphism. On the other hand, the restriction of the trace map Tr_{p_1} factors through $p\mathscr{O}_{\mathfrak{X}^{\geq 1}}$ in the first case, and through $p^3\mathscr{O}_{\mathfrak{X}^{\geq 1}}$ in the second case. Finally, the map $p_2^\star\mathcal{I}^{-1} \to p_1^\star\mathcal{I}^{-1}$ is an isomorphism. We now consider the case where H is étale. The correspondence splits into several components over x again :

1. the locus where the isogeny $p_1^\star G \to p_2^\star G$ has a multiplicative kernel,
2. the locus where the isogeny $p_1^\star G \to p_2^\star G$ has kernel an extension of an étale by a multiplicative group which intersects trivially (at x) with H,
3. the locus where the isogeny $p_1^\star G \to p_2^\star G$ has kernel an extension of an étale by a multiplicative group which intersects non-trivially (at x) with H, so that $p_2^\star H$ is multiplicative on this component,
4. the locus where $p_1^\star G \to p_2^\star G$, has kernel an étale group, so that $p_2^\star H$ is again multiplicative on this component.

We now list the divisibility we get in each of these cases :

1. In the first case, the map $p_2^\star\Omega^{(k,2)} \to p_1^\star\Omega^{(k,2)}$ factors through $p^{4+k}p_1^\star\Omega^{(k,2)}$,
2. In the second case, the map $p_2^\star\Omega^{(k,2)} \to p_1^\star\Omega^{(k,2)}$ factors through $p^2 p_1^\star\Omega^{(k,2)}$, and the trace map Tr_{p_1} factors through $p\mathscr{O}_{(\mathfrak{X}_{Kli(p)})^{\geq 1}}$,

27. By [**75**], lem. 2.4.3, the group H can be extended to a finite flat group scheme at the boundary.

3. In the third case, the map $p_2^\star\Omega^{(k,2)} \to p_1^\star\Omega^{(k,2)}$ factors through $p^2p_1^\star\Omega^{(k,2)}$, and the map $p_2^\star\mathcal{I}^{-1} \to p_1^\star\mathcal{I}^{-1}$ factors through $pp_1^\star\mathcal{I}^{-1}$,
4. In the fourth case, the map $p_2^\star\mathcal{I}^{-1} \to p_1^\star\mathcal{I}^{-1}$ factors through $pp_1^\star\mathcal{I}^{-1}$.

This finishes the proof of lemma 14.9.1.

PART IV
EULER CHARACTERISTIC

15. Vanishing of Euler characteristic

In this last section, we use automorphic methods to compute the Euler characteristic of a non-Eisenstein localization of the complex of theorem 1.1 and prove theorem 1.2.

15.1. Action of the Hecke algebra. — We construct an action of the prime-to-p Hecke algebra on the cohomology of our p-adic sheaves. This is a routine construction. Let ℓ be a prime. We have introduced the spherical Hecke algebra $\mathcal{H}_\ell = \mathbb{Z}[T_{\ell,0}, T_{\ell,0}^{-1}, T_{\ell,1}, T_{\ell,2}]$ in section 5.1.3. Let $K = \prod_\ell K_\ell \subset \mathrm{GSp}_4(\mathbb{A}_f)$ be a compact open subgroup. We assume as usual that $K_p = \mathrm{GSp}_4(\mathbb{Z}_p)$.

Proposition 15.1.1. — *Let $\ell \neq p$ be a prime such that $K_\ell = \mathrm{GSp}_4(\mathbb{Z}_\ell)$. We have an action of $\mathcal{H}_\ell$ on $\mathrm{R}\Gamma(\mathfrak{X}_{Kli}(p)_K^{\geq 1}, \mathfrak{F}^\kappa \otimes \omega^2(-D))$.*

Proof. We suppress the subscript K from the notations in this proof. For certain choices of polyhedral cone decompositions that we suppress from the notation, we can define Hecke correspondences attached to the double coset $T_{\ell,i}$ (see [**18**], p. 253) :

$$\begin{tikzcd} & C_{\ell,i} \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ X & & X \end{tikzcd}$$

Denote by $\mathfrak{C}_{\ell,i}$ the formal completion of $C_{\ell,i}$. We can form the fiber product $\mathfrak{D}_{\ell,i} = \mathfrak{C}_{\ell,i} \times_{p_1,\mathfrak{X}} \mathfrak{X}^{\geq 1}_{Kli}(p)$. The second projection $p_2 : \mathfrak{D}_{\ell,i} \to \mathfrak{X}$ can be lifted naturally to $p_2 : \mathfrak{D}_{\ell,i} \to \mathfrak{X}^{\geq 1}_{Kli}(p)$. Since the universal isogeny associated to the double coset $T_{\ell,i}$ is étale, we have a canonical isomorphism :

$$p_2^\star\mathfrak{F}^\kappa \otimes \omega^2(-D) \to p_1^\star\mathfrak{F}^\kappa \otimes \omega^2(-D).$$

The formal schemes $\mathfrak{X}^{\geq 1}_{Kli}(p)$ and $\mathfrak{D}_{\ell,1}$ are smooth, and as a result there is a fundamental class $p_1^\star\mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p)} \to p_1^!\mathscr{O}_{\mathfrak{X}^{\geq 1}_{Kli}(p)}$. We can thus form an unnormalized cohomological correspondence $T'_{\ell,i} : p_2^\star\mathfrak{F}^\kappa \otimes \omega^2(-D) \to p_1^!\mathfrak{F}^\kappa \otimes \omega^2(-D)$. We shall set $T_{\ell,2} = \ell^{-3}T'_{\ell,2}$ and $T_{\ell,i} = \ell^{-6}T'_{\ell,i}$ for $i = 0, 1$ [28].

□

28. see remark 5.3.1 for a justification of this normalization.

15.2. Euler characteristic. — Let $K = \prod_\ell K_\ell \subset \mathrm{GSp}_4(\mathbb{A}_f)$ be a compact open subgroup. We assume that $K_p = \mathrm{GSp}_4(\mathbb{Z}_p)$. Let N be the product of primes ℓ such that $K_\ell \neq \mathrm{GSp}_4(\mathbb{Z}_\ell)$. Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{F}}_p)$ be a Galois representation, unramified away from the primes ℓ dividing pN. We assume that $\overline{\rho}$ is absolutely irreducible. We let $\mathfrak{m}$ be the associated maximal ideal of the abstract Hecke algebra $\mathcal{H}^{Np}$ and $\Theta_{\mathfrak{m}} : \mathcal{H}^{Np} \to \overline{\mathbb{F}}_p$ the corresponding morphism. The map $\Theta_{\mathfrak{m}}$ is thus defined by the rule $\Theta_{\mathfrak{m}}(Q_\ell(X)) = \det\big(1 - X\overline{\rho}(\mathrm{Fob}_\ell)\big)$.

The algebra $\mathcal{H}^{Np}$ acts on the perfect complex $f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^\kappa \otimes \omega^2(-D))$. The Λ-subalgebra of $\mathrm{End}\big(f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^\kappa \otimes \omega^2(-D))\big)$ generated by $\mathcal{H}^{Np}$ is a finite Λ-algebra. In particular it is semi-local. We can define a direct factor (which may be trivial if $\bar{\rho}$ does not occur in our cohomology) of $f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^\kappa \otimes \omega^2(-D))$ associated to the maximal ideal $\mathfrak{m}$ (see [**41**], lemma 2.12) :

$$f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^\kappa \otimes \omega^2(-D))_{\mathfrak{m}}.$$

Theorem 15.2.1. — *The Euler characteristic of the perfect complex*

$$f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^\kappa \otimes \omega^2(-D))_{\mathfrak{m}}$$

is equal to 0*.*

Remark 15.2.1. — We conjecture that the support over Λ of $\oplus_{i=0}^1 f\mathrm{H}^i(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^\kappa \otimes \omega^2(-D))_{\mathfrak{m}}$ has Krull dimension less or equal to 1 if the representation $\bar{\rho}$ is not induced from a real quadratic extension of $\mathbb{Q}$ (in that case, one should be able to construct positive dimensional families using inductions of families of Hilbert modular forms). Compare with conjecture 7.2 in [**41**].

The proof of this theorem will be given in section 15.2.5 below. Before giving the proof we need to collect a certain number of results concerning automorphic forms.

15.2.1. Limits of discrete series. — Given $\lambda = (\lambda_1, \lambda_2; c) \in X^\star(\mathrm{T}) + (2, 1; 0) \subset X^\star(\mathrm{T})_{\mathbb{C}}$ which satisfies $-\lambda_1 \geq \lambda_2 > -\lambda_1$ and a Weyl chamber C positive for λ we have a (limit of) discrete series $\pi(\lambda, C)$ (see [**28**], 3.3).

Let $\mathfrak{Z}$ be the center of the enveloping algebra $U(\mathfrak{g})$. By Harris-Chandra isomorphism, $\mathfrak{Z} \simeq \mathbb{C}[X_\star(\mathrm{T})]^W$ where W is the Weyl group. The infinitesimal character of $\pi(\lambda, C)$ is the Weyl group orbit of λ.

Si $\lambda_2 \neq 0$ and $\lambda_2 \neq -\lambda_1$, λ determines uniquely C and $\pi(\lambda, C)$ is a discrete series. The case of interest to us is $\lambda_2 = 0$ and $0 > \lambda_1$. We now make these hypothesis. Under these assumptions, there are two choices for C. The natural choice (C is the chamber corresponding to our choice of positive roots) provides a limit of discrete series that we denote by $\pi(\lambda)^h$ (it contains the holomorphic and anti-holomorphic limits of discrete series of the derived group). The other choice of C provides another limit of discrete series that we denote by $\pi(\lambda)^g$.

15.2.2. Cohomology of limits of discrete series. — For $\lambda = (\lambda_1, \lambda_2; c)$ with $\lambda_2 = 0$, $0 > \lambda_1$, consider the character $(-\lambda_1 + 1, 2; -c) \in X^\star(\mathrm{T})$. This character is dominant for the Levi $M_{Si} \simeq \mathrm{GL}_2 \times \mathbb{G}_m$ of the Siegel parabolic $P_{Si} \subset \mathrm{GSp}_4$ which stabilizes the space $\langle e_1, e_2\rangle$. Associated to this character is a complex irreducible representation of P_{Si} of highest weight $(-\lambda_1 + 1, 2; -c)$ that we denote by $V_{(\lambda_1+1,2;-c)}$.

Recall that we have a map $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \to \mathrm{GSp}_4|_{\mathbb{R}}$ given by $h(a + ib) = a1_2 + bJ$ and that $K_\infty \subset \mathrm{GSp}_4(\mathbb{R})$ is the centralizer of the image of h. We let $\mathfrak{g}$ be the complex Lie algebra of GSp_4. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since $\mathfrak{k}$ is also the complex lie algebra of M_{Si}, the representation $V_{(-\lambda_1+1,2;c)}$ can also be viewed as a representation of $\mathfrak{k}$ and K_∞. Let W be a $(\mathfrak{g}, K_\infty)$-module. Then one can define the $(\mathfrak{p}, K_\infty)$-cohomology of W, denoted by $\mathrm{H}^\bullet(\mathfrak{p}, K_\infty; W)$ (see [**30**], sect. 4.1.1).

Theorem 15.2.2.1 **([4], thm. 3.2.1, sect. 4.2).** — *1. We have :*

— $\mathrm{H}^i(\mathfrak{p}, K_\infty; \pi(\lambda)^h \otimes V_{(-\lambda_1+1,2;-c)}) = \mathbb{C}$ *if* $i = 0$ *and* $\mathrm{H}^i(\mathfrak{p}, K_\infty; \pi(\lambda)^h \otimes V_{(-\lambda_1+1,2;-c)}) = 0$ *otherwise,*

— $\mathrm{H}^i(\mathfrak{p}, K_\infty; \pi(\lambda)^g \otimes V_{(-\lambda_1+1,2;-c)}) = \mathbb{C}$ *if* $i = 1$ *and* $\mathrm{H}^i(\mathfrak{p}, K_\infty; \pi(\lambda)^g \otimes V_{(-\lambda_1+1,2;-c)}) = 0$ *otherwise.*

2. There is a constant R such that if $\lambda_1 \geq R$ and π_∞ in an irreducible, essentially unitary representation of $\mathrm{GSp}_4(\mathbb{R})$ *and :*

— *if* $\mathrm{H}^0(\mathfrak{p}, K_\infty; \pi_\infty \otimes V_{(-\lambda_1+1,2;-c)}) \neq 0$ *then* $\pi_\infty \simeq \pi(\lambda)^h$,

— *if* $\mathrm{H}^1(\mathfrak{p}, K_\infty; \pi_\infty \otimes V_{(-\lambda_1+1,2;-c)}) \neq 0$ *then* $\pi_\infty \simeq \pi(\lambda)^g$.

15.2.3. Representing cohomology classes by automorphic forms. — We let S_K be the Siegel threefold of level K over $\mathbb{C}$. We fix a toroidal compactification $S^{tor}_{K,\Sigma}$ of S_K. Recall that $\lambda = (\lambda_1, 0; c) \in X^\star(\mathrm{T}) + (2, 1; 0)$. We set $k = -\lambda_1 - 1$. We also fix the central character c to be $-\lambda_1 + 3$. This the "correct" normalization. We denote by $\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma}, \Omega^{(k,2)})$ the image of $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \Omega^{(k,2)}(-D))$ in $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \Omega^{(k,2)})$, this is called the interior coherent cohomology.

Theorem 15.2.3.1 **([30], coro. 5.3.2).** — *For every integer $k \geq R - 1$ (see thm. 15.2.2.1, 2.), we have*

$$\overline{\mathrm{H}}^0(S^{tor}_{K,\Sigma}, \Omega^{(k,2)}) = \oplus_{\pi_f} (\pi_f^K)^{m^h(\pi_f)}$$

where π_f runs over all irreducible admissible representations of $\mathrm{GSp}_4(\mathbb{A}_f)$ *such that $\pi_f \otimes \pi(\lambda)^h$ is cuspidal automorphic and $m^h(\pi_f)$ is the multiplicity of $\pi_f \otimes \pi(\lambda)^h$.*

Similarly,

$$\overline{\mathrm{H}}^1(S^{tor}_{K,\Sigma}, \Omega^{(k,2)}) = \oplus_{\pi_f} (\pi_f^K)^{m^g(\pi_f)}$$

where π_f runs over all irreducible admissible representations of $\mathrm{GSp}_4(\mathbb{A}_f)$ *such that $\pi_f \otimes \pi(\lambda)^g$ is cuspidal automorphic and $m^g(\pi_f)$ is the multiplicity of $\pi_f \otimes \pi(\lambda)^g$.*

We fix an isomorphism $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$. Thanks to this isomorphism, we can make sense of the localized cohomology groups $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \Omega^{(k,2)}(-D))_{\mathfrak{m}}$.

Corollary 15.2.3.1. — *For $k \geq R - 1$, we have*

$$\mathrm{H}^0(S^{tor}_{K,\Sigma}, \Omega^{(k,2)}(-D))_{\mathfrak{m}} = \oplus_{\pi_f} (\pi_f^K)^{m^h(\pi_f)}$$

where π_f runs over all irreducible admissible representations of $\mathrm{GSp}_4(\mathbb{A}_f)$ *such that $\pi_f \otimes \pi(\lambda)^h$ is cuspidal automorphic and $m^h(\pi_f)$ is the multiplicity of $\pi_f \otimes \pi(\lambda)^h$ and the character $\Theta_{\pi_f} : \mathcal{H}^{Np} \to \mathbb{C}$ is congruent to $\Theta_{\mathfrak{m}}$.*

Similarly,

$$\mathrm{H}^1(S^{tor}_{K,\Sigma}, \Omega^{(k,2)}(-D)) = \oplus_{\pi_f} (\pi_f^K)^{m^g(\pi_f)}$$

where π_f runs over all irreducible admissible representations of $\mathrm{GSp}_4(\mathbb{A}_f)$ *such that $\pi_f \otimes \pi(\lambda)^g$ is cuspidal automorphic and $m^g(\pi_f)$ is the multiplicity of $\pi_f \otimes \pi(\lambda)^g$ and the character $\Theta_{\pi_f} : \mathcal{H}^{Np} \to \mathbb{C}$ is congruent to $\Theta_{\mathfrak{m}}$.*

Proof. In order to deduce the corollary from theorem 15.2.3.1, we need to prove that the natural map $\mathrm{H}^1(S^{tor}_{K,\Sigma}, \Omega^{(k,2)}(-D))_{\mathfrak{m}} \to \mathrm{H}^1(S^{tor}_{K,\Sigma}, \Omega^{(k,2)})$ is injective. We have a short exact sequence :

$$\mathrm{H}^0(S^{tor}_{K,\Sigma}, \Omega^{(k,2)}) \to \mathrm{H}^0(S^{tor}_{K,\Sigma}, \Omega^{(k,2)} \otimes \mathscr{O}_D) \to \mathrm{H}^1(S^{tor}_{K}, \Omega^{(k,2)}(-D)).$$

We shall prove that the cohomology group $\mathrm{H}^0(S^{tor}_{K,\Sigma}, \Omega^{(k,2)} \otimes \mathscr{O}_D)_{\mathfrak{m}}$ is zero. Let $S^\star_K$ be the minimal compactification. Recall that there is a stratification

$$S^\star_K = S_K \coprod S^{(1)}_K \coprod S^{(0)}_K.$$

where $S^{(1),\star}_K = S^{(1)}_K \coprod S^{(0)}_K$ is a union of compactified modular curves. Let $\pi : S^{tor}_{K,\Sigma} \to S^\star_K$ be the projection. There is an induced projection $D \to S^{(1),\star}_K$. One computes that $\pi_\star \Omega^{(k,2)}|_D = \omega^{k+2}(-cusp)$ if $k \neq 0$ and ω^2 when $k = 0$, where ω^{k+2} is the usual sheaf of modular forms of weight $k+2$ on the modular curve.

Let ℓ be a prime that is prime to the level K. We let $T_{\ell,2}$ be the corresponding Hecke operator. We let T_ℓ be the usual Hecke operator on modular forms for the group $\mathrm{GL}_2/\mathbb{Q}$. On $\mathrm{H}^0(S^{tor}_{K,\Sigma}, \Omega^{(k,2)} \otimes \mathscr{O}_D) \simeq \mathrm{H}^0(S^{(1),\star}_K, \omega^{k+2}(-cusp))$ (resp. $\simeq \mathrm{H}^0(S^{(1),\star}_K, \omega^2)$ if $k = 2$), we have the formula $T_{\ell,2} = 2T_\ell$ by [**22**], IV, satz 4.4. Let f be an eigenform in $\mathrm{H}^0(S^{(1),\star}_K, \omega^{k+2})$, with associated Galois representation $\rho_f : \mathbb{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$. Then, associated to the character $\Theta_f : \mathcal{H}^{Np} \to \overline{\mathbb{Q}_p}$, we have the reducible 4-dimensional Galois representation $\rho_f \oplus \rho_f$ which is not congruent to $\overline{\rho}$.

□

15.2.4. An application of Arthur's results. — We use here Arthur's classification for GSp_4 as announced in [**1**].

Proposition 15.2.4.1. — *Let π_f be an admissible irreducible representation of $G(\mathbb{A}_f)$ which is unramified at primes ℓ not dividing Np. Let $\Theta_{\pi_f} : \mathcal{H}^{Np} \to \overline{\mathbb{Q}}_p$ be the associated character of the Hecke algebra. Assume that Θ_{π_f} is congruent to $\Theta_{\mathfrak{m}}$. Let $\lambda = (\lambda_1, 0; c) \in X^\star(\mathrm{T}) + (2,1;0)$ with $\lambda_1 > 0$.*

Then $\pi_f \otimes \pi(\lambda)^h$ is automorphic if and only if $\pi_f \otimes \pi(\lambda)^g$ is automorphic and moreover, $m^h(\pi_f) = m^g(\pi_f) = 1$.

Proof. Assume that $\pi_f \otimes \pi(\lambda)^h$ is automorphic (the argument would be the same if we assumed that $\pi_f \otimes \pi(\lambda)^g$ is automorphic). Let Π be the associated global A-packet. We claim that Π is of generic type in the sense of [**1**], classification theorem on p. 78. Hence Π is stable and tempered. It follows that Π_∞ is an L-packet, and this is $\{\pi(\lambda)^g, \pi(\lambda)^h\}$ (see [**4**], prop. 5.3.7). The conclusion follows. In order to see that Π is of generic type, we first observe that since $\pi(\lambda)^h$ is a limit of discrete series, Π can either be of generic, Yoshida or Saito-Kurokawa type (compare [**68**], sect. 1.1 and 1.2 with the description of the parameters attached to $\pi(\lambda)^h$ in [**67**], p.11). In the last two cases, the associated Galois representation is reducible, while $\overline{\rho}$ is irreducible.

□

15.2.5. Proof of theorem 15.2.1. — In order to prove the theorem, we can specialize at a very large weight k. Then $f\mathrm{R}\Gamma(\mathfrak{X}^{\geq 1}_{Kli}(p), \mathfrak{F}^\kappa \otimes \omega^2(-D))_{\mathfrak{m}} \otimes_{\Lambda,k} \mathbb{Q}_p = f\mathrm{R}\Gamma(X_{Kli}(p), \Omega^{(k,2)}(-D))_{\mathfrak{m}}$ by theorem 11.3.1. The cohomology is concentrated in degree 0 and 1. Extending the scalars to $\overline{\mathbb{Q}}_p$ we can express the cohomology in automorphic terms using corollary 15.2.3.1 and proposition prop 15.2.4.1 :

$$f\mathrm{H}^0(X_{Kli}(p), \Omega^{(k,2)}(-D))_{\mathfrak{m}} \otimes \overline{\mathbb{Q}}_p = \oplus_{\pi_f} f(\pi_f^{K^pKli(p)}) = f\mathrm{H}^1(X_{Kli}(p), \Omega^{(k,2)}(-D))_{\mathfrak{m}} \otimes \overline{\mathbb{Q}}_p$$

where π_f runs over all irreducible admissible representations of $\mathrm{GSp}_4(\mathbb{A}_f)$ such that $\pi_f \otimes \pi(\lambda)^h$ is cuspidal automorphic, the character $\Theta_{\pi_f} : \mathcal{H}^{Np} \to \mathbb{C}$ is congruent to $\Theta_{\mathfrak{m}}$. The projector f acts on $\pi_p^{Kli(p)}$.

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