MODULARITY THEOREMS FOR ABELIAN SURFACES

GEORGE BOXER, FRANK CALEGARI, TOBY GEE, AND VINCENT PILLONI

Abstract. We prove the modularity of a positive proportion of abelian surfaces over $\mathbf{Q}$. More precisely, we prove the modularity of abelian surfaces which are ordinary at 3 and are 3-distinguished, subject to some assumptions on the 3-torsion representation (a "big image" hypothesis, and a technical hypothesis on the action of a decomposition group at 2). We employ a 2–3 switch and a new classicality theorem (in the style of Lue Pan) for ordinary p-adic Siegel modular forms.

Contents

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1. Introduction

1.1. **The main theorems.** Our main theorem is as follows (see §9.5).

Theorem A. *Let $A/\mathbf{Q}$ be an abelian surface with a polarization of degree prime to 3. Suppose the following holds:*

(1) The mod 3 representation:

$$\overline{\rho}_{A,3} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{F}_3)$$

is surjective.

(2) $\overline{\rho}_{A,3}|_{G_{\mathbf{Q}_2}}$ *is unramified, and the characteristic polynomial of* $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ *is not* $(x^2 \pm x + 2)^2$.

(3) A has good ordinary reduction at 3 *and the characteristic polynomial of Frobenius at* 3 *does not have repeated roots.*

Then A is modular. More precisely, there exists a cuspidal automorphic representation π *for* $\mathrm{GL}_4 / \mathbf{Q}$ *(the transfer of a cuspidal automorphic representation of* $\mathrm{GSp}_4 / \mathbf{Q}$ *of weight* 2*) such that* $L(s, H^1(A)) = L(s, \pi)$*, and hence* $L(s, H^1(A))$ *has a holomorphic continuation to* $\mathbf{C}$ *satisfying the expected functional equation.*

In our previous paper [BCGP21], we proved the *potential* modularity of all abelian surfaces over totally real fields. (We refer the reader to the introduction to [BCGP21] for a history of the modularity conjecture for abelian surfaces.) As a consequence, the main result of [BCGP21] implies that $L(s, H^1(A))$ has a *meromorphic* continuation to all of $\mathbf{C}$, but it does not suffice to prove the conjectured holomorphicity, for essentially the same reason that Brauer [Bra47] was able to prove the meromorphic continuation of Artin L-functions but their holomorphicity remains conjectural. The results of [BCGP21] also allowed one to establish the *modularity* of an abelian surface under extremely restrictive conditions, and in particular to produce [BCGP21, Thm 10.2.6] infinite (thin) sets of modular abelian surfaces $A/\mathbf{Q}$ (up to twist) with $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$. These sets, however, account for 0% of all abelian surfaces over $\mathbf{Q}$ counted in any reasonable way. Indeed, even producing *any* explicit examples where our modularity theorems applied was somewhat of a challenge [CCG20]. In contrast, we expect that Theorem A applies to a positive proportion of all abelian surfaces over $\mathbf{Q}$ counted in any reasonable way.[1] For example, conditions (1)–(3) can be guaranteed by imposing congruence conditions at finitely many primes (including 2 and 3). See Section 10.1 for some more precise heuristics and examples; in particular we show that Theorem A applies to the Jacobians of precisely 11743 of the 66158 genus two curves in [LMF24, BSS[+]16].

The hypothesis (1) (which comes from the Taylor–Wiles method) on the residual image can be weakened; the allowable subgroups are precisely those listed in Lemma 6.4.3 (they are all absolutely irreducible).

Although there is some scope for marginal improvement on the local conditions (2) and (3) (as a direct consequence of the modularity theorems proved in this paper), our expectation is that the best way to relax the local assumptions is to make use of base change by generalizing our main results to totally real fields, which we hope to return in the future. While some of our arguments will generalize straightforwardly, the proof of the main classicality theorem will require new ideas. In [BCGP21], we were able to work over totally real fields F in which p splits completely, and additionally there was considerable freedom to choose the prime p. In contrast, in the current paper, we are often forced to take $p = 3$ or $p = 2$, and in order to relax (2) and (3) it will be necessary to allow these primes to behave arbitrarily in the totally real field F.

We also note the following easy to formulate corollary of Theorem A (again, see §9.5).

Theorem B. *Let* $X : y^2 = f(x)$ *with* $\deg(f(x)) = 5$ *be a genus two curve over* $\mathbf{Q}$*. Suppose that:*

(1) $\overline{\rho}_{X,3} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{F}_3)$ *is surjective.*

[1]It is very hard to say anything rigorous (even for elliptic curves) if one orders by conductor.

(2) X has good ordinary reduction at 2.
(3) X has good ordinary reduction at 3.

Then X is modular.

1.2. **The 2-3 switch.** The starting point of this paper is an analogue of the 3-5 switch used by Wiles [Wil95] to prove residual modularity, which exploited the rationality of certain twists of the modular curve $X(5)/\mathbf{Q}$. In our case, we use a rational moduli space of abelian surfaces to carry out a 2-3 switch. This space was defined in [BCGP21] as follows: given an abelian surface A with a prime to 3 polarization, one may consider the moduli space $P(A[3])$ of genus two curves X equipped with a symplectic isomorphism $\mathrm{Jac}(X)[3] \simeq A[3]$ and a fixed rational Weierstrass point. By forgetting the Weierstrass point, the variety $P(A[3])$ admits a degree 6 map to a twist of the Siegel 3-fold of full level three, which is rational over $\mathbf{C}$ but almost never over $\mathbf{Q}$ [CC22]. However, $P(A[3])$ is always rational by [BCGP21, Thm 10.2.1]. In particular, we may find another abelian surface $B/\mathbf{Q}$ with $B[3] \simeq A[3]$ and such that the map

$$\overline{\rho}_{B,2} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_2) \simeq S_6$$

has image isomorphic to S_5 (because of the rational Weierstrass point). Condition (2) of Theorem A ensures that we can find such a B with good ordinary reduction at 2. After restricting to a quadratic extension $F^+/\mathbf{Q}$ (which we can arrange to be totally real), we may assume that $\overline{\rho}_{B,2}|_{G_{F^+}}$ is absolutely irreducible with image A_5 in $\mathrm{GSp}_4(\mathbf{F}_2)$. Known cases of the Artin conjecture in dimension two [PS16b, Sas19] allow us to identify this representation with the mod 2 reduction of the symmetric cube of the 2-adic Galois representation associated to a Hilbert modular form of parallel weight 2, and thus (via known functorialities) to the mod 2 representation associated to a Hilbert–Siegel eigenform (see also [TY22, Thm. 4.7]). The goal is now to use modularity lifting theorems to go from the modularity of $B[2]$ to the modularity of B and thus to the modularity of $A[3] \simeq B[3]$, and finally to the modularity of A.

One difficulty that we encounter is that we need to prove modularity lifting theorems which apply when $p = 2$ and the residual image is rather small. However by far the most serious difficulty compared to our previous work is that the argument above gives modularity of the residual representation $\overline{\rho}_{B,2}$ in regular weight, but the representation $\rho_{B,2}$ has irregular weight. The main innovation in our earlier work [BCGP21] was to prove a modularity lifting theorem in irregular weight; however, this theorem crucially depended on having residual modularity in irregular weight as an input.

Deducing residual modularity in irregular weight from regular weight would be a higher dimensional analogue of showing (for modular forms) that a residual modular representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which is unramified at p arises from a Katz modular form of weight one [Gro90, CV92], and we do not know how to do this. Instead we use modularity lifting theorems to prove the existence of a p-adic Siegel modular form associated to the p-adic Tate module of our abelian surface, and we then prove a classicality theorem for ordinary p-adic Siegel forms in irregular weight. Such p-adic modular forms are not necessarily classical; indeed their associated Galois representations need not be de Rham. However, we prove (under mild technical hypotheses, see Theorem 4.12.4) that if the Galois representation is (ordinary and) de Rham, then the form is indeed classical. (Condition (3) in Theorem A guarantees that we can apply Theorem 4.12.4 to $\rho_{A,3}$.)

1.3. **Classicality for ordinary p-adic Siegel modular forms.** Our classicality theorem follows the strategy introduced by Lue Pan in his paper [Pan22a], as reinterpreted in [Pil24]. Both of these papers (and Pan's sequel [Pan22b]) are in the setting of the modular curve. While Pan gives a complete treatment of arbitrary de Rham representations, we restrict to the ordinary setting, which is considerably simpler; but there are still many additional complications for higher-dimensional Shimura varieties. We prove our classicality theorem in the case of the Siegel threefold (the Shimura variety for $\mathrm{GSp}_4/\mathbf{Q}$) and it asserts that if an ordinary p-adic modular eigenform of weight 2 has an associated Galois representation which is irreducible and de Rham (and satisfies a few more technical hypothesis), then it is classical (see Theorem 4.12.4). The strategy for proving this theorem is to realize the Galois representation in the completed cohomology of the Siegel threefold (it does not contribute to the classical étale cohomology because its Hodge–Tate weights are singular) and to relate the Sen operator of this Galois representation to a Cousin map which measures the obstruction for a p-adic modular form to be a classical modular form. In our ordinary case, the de Rhamness is equivalent to the semi-simplicity of the Sen operator which translates into the vanishing of the Cousin map and therefore implies the classicality of the p-adic modular form.

We now give a more precise account of our strategy. Let $\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}$ be a toroidal compactification of the Siegel threefold of level K^pK_p over $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$, and let $\mathrm{Sh}^{\mathrm{tor}}_{K^p} = \lim_{K_p} \mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}$ be the perfectoid Siegel threefold over $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$, of prime-to-$p$ level K^p. Let $\omega^2_{K_p}$ be the sheaf of weight 2 Siegel modular forms over $\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}$ and let $\omega^{2,\mathrm{sm}} = \mathrm{colim}_{K_p}\,\omega^2_{K_p}$, viewed as a sheaf over $\mathrm{Sh}^{\mathrm{tor}}_{K^p}$, whose cohomology is $\mathrm{colim}_{K_p}\,\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^2_{K_p})$. Thus, an element of the degree 0 cohomology of $\omega^{2,\mathrm{sm}}$ is a weight 2 Siegel modular form of level K^pK_p for some K_p.

The ordinary part $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{2,\mathrm{sm}})^{\mathrm{ord}}$ is computed by the following complex (more precisely, this is only true for cuspidal cohomology but we ignore this subtlety in the introduction) in degrees 0 and 1:

$$[H^0_{Id}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{2,\mathrm{sm}})^{\mathrm{ord}} \stackrel{Cous}{\to} H^1_{1_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{2,\mathrm{sm}})^{\mathrm{ord}}] \tag{1.3.1}$$

where the module in degree 0 in the complex is the space of ordinary p-adic modular forms of weight 2, and the module in degree 1 is a space of ordinary higher p-adic modular forms (studied in higher Coleman theory). The differential is the Cousin map.

Let $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)$ denote the complex of completed cohomology. We prove (under technical assumptions, Theorem 4.9.9) that the ordinary part of the $\mathfrak{b}$-cohomology

$$\mathrm{R}\,\mathrm{Hom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}})^{\mathrm{ord}} \tag{1.3.2}$$

of locally analytic vectors in completed cohomology is concentrated in degree 3. Here λ is a non-dominant character of the torus of GSp_4 where we expect to see (by interpolation from what happens for dominant characters) the Galois representation of weight 2 modular forms.

After we tensor (1.3.2) with $\mathbf{C}_p$, by the p-adic Eichler–Shimura theory developed in this paper, the cohomology admits a 4 step filtration and the graded pieces are given by the various relevant higher Coleman theories. If we denote by V this degree 3 cohomology group, we prove that $V_{\mathbf{C}_p}$ has a decreasing filtration with $\mathrm{Gr}^0 V_{\mathbf{C}_p} = H^0_{Id}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{2,\mathrm{sm}})^{\mathrm{ord}}$ and $\mathrm{Gr}^1 V_{\mathbf{C}_p} = H^1_{1_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{2,\mathrm{sm}})^{\mathrm{ord}}$. We use this to verify that the Galois representation of our p-adic modular eigenform of weight 2 is realized in completed cohomology.

The Sen operator respects the filtration and acts by the scalars $1, 1, 0, 0$ on the respective graded pieces, but possibly acts non semi-simply on $V_{\mathbf{C}_p}$. Looking at the generalized 0 eigenspace for the Sen operator, we get an induced map

$$\mathrm{Sen} : \mathrm{Gr}^0 V_{\mathbf{C}_p} = H^0_{Id}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{2,\mathrm{sm}})^{\mathrm{ord}} \to H^1_{1_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{2,\mathrm{sm}})^{\mathrm{ord}} = \mathrm{Gr}^1 V_{\mathbf{C}_p} \quad (1.3.3)$$

measuring the failure of semi-simplicity of the Sen operator.

The main result from which we deduce our classicality theorem is the property that the Cousin map (1.3.1) and the Sen map (1.3.3) agree up to a non-zero scalar (Theorem 4.10.12). The key idea behind the proof is that the Sen operator which acts on the cohomology arises from an operator defined on the complex of sheaves $\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ on the perfectoid Shimura varieties whose cohomology is the $\mathfrak{b}$-cohomology of locally analytic vectors in completed cohomology. The complex $\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ is closely related to twisted D-modules on the flag variety (a form of the Beilinson–Bernstein localization of the Verma module of weight λ), and the Sen operator to a certain horizontal Cartan action. It turns out that the relation between the Sen and Cousin maps can already be studied and established at this more "explicit" geometric representation level (Theorem 3.6.9).

1.4. **p-adic Eichler–Shimura theory.** A substantial part of this paper is dedicated to p-adic Eichler–Shimura theory. In this part of the work, we are able to work in the generality of Hodge type Shimura varieties. Let us recall first the classical Hodge–Tate decomposition of modular curves. Let $G = \mathrm{GL}_2$, and let $K \subseteq G(\mathbf{A}_f)$ be a compact open subgroup. Let Sh_K be the modular curve over $\mathbf{C}_p$ of level K, with compactification $\mathrm{Sh}^{\mathrm{tor}}_K$. Let $E \to \mathrm{Sh}_K$ be the universal elliptic curve. In [Fal87], Faltings proved the following Hodge–Tate decomposition for the étale cohomology of modular curves:

$$H^1(\mathrm{Sh}_K, \mathrm{Sym}^k T_p E) \otimes_{\mathbf{Z}_p} \mathbf{C}_p = H^0(\mathrm{Sh}^{\mathrm{tor}}_K, \omega^{k+2})(-k-1) \oplus H^1(\mathrm{Sh}^{\mathrm{tor}}_K, \omega^{-k}) \quad (1.4.1)$$

This isomorphism is equivariant for the Hecke action and the local Galois action, and $(-k-1)$ indicates a Tate twist. This kind of statement has been generalized to all Shimura varieties (see for example [LLZ23]).

Both sides of (1.4.1) are classical instances of bigger p-adic objects. On the left hand side we may consider completed cohomology, and on the right hand side we can consider higher Coleman theory [BP21]. The main goal of p-adic Eichler–Shimura theory (as taken up in §4) is to express some relation between both big p-adic spaces, generalizing Faltings's theory to non-classical cohomologies. We note that p-adic Eichler–Shimura theory was initiated in [AIS15] and completely transformed after the work of Pan [Pan22a] followed by that of Rodríguez Camargo [RC22, RC23]. In order to state our main results, we need to introduce a certain amount of notation as well as recall a number of facts from higher Coleman theory. For this reason, we defer any further discussion to the more technical introduction given in §4.1.

1.5. **An outline of the paper.** Here is a brief synopsis of the sections in our paper; see also the introductions to the individual sections for more details.

§2 is concerned with Lie algebras. We consider the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ and its Fréchet completion $\hat{U}(\mathfrak{g})$, as well as modules over them. Our main result (of independent interest) is Theorem 2.3.32 which compares the Lie algebra cohomology of a unipotent radical of a parabolic of $\mathfrak{g}$ of certain algebraic $U(\mathfrak{g})$-modules and of their completions.

§3 is about equivariant twisted D-modules on flag varieties. We develop a somewhat ad hoc language to describe them. One of the main difficulties is to keep track of the various topologies and finiteness conditions we want to impose. We use the language of condensed mathematics to deal with functional analysis. We introduce a version of Beilinson–Bernstein localization and describe it using the results of Section 2.

§4 contains our main classicality result. We also give some complements on higher Coleman theory and establish the p-adic Eichler–Shimura theory.

§5 proves an $R = \mathbf{T}$ theorem in regular weight when $p = 2$ under a suitable oddness hypothesis, following [Tho17]. For technical reasons (due to the small residual image of our representations), we need to work with unitary groups rather than symplectic groups.

§6 proves an $R = \mathbf{T}$ theorem in regular weight for $p > 2$ for symplectic representations. Curiously enough, when $p = 3$ and the image of $\overline{\rho}$ is $\mathrm{GSp}_4(\mathbf{F}_3)$ (the main case of interest), technical reasons now mandate that we work with symplectic groups rather than unitary groups; see Remark 6.4.5.

§7 proves a multiplicity one result for certain Hida families, which (once again) for technical reasons is necessary for our classicality argument. This is where the main modularity theorems Proposition 7.5.10 and Theorem 7.5.11 are proved, using the classicality result Theorem 4.12.4.

§8 beings by recalling the basic theory of 2-torsion points on an abelian surface, and then establishes some basic but necessary facts concerning the modular representation theory of A_5. This section also addresses the residual modularity of mod-2 representations with image A_5 using known cases of the Artin conjecture for $n = 2$.

§9 studies the representations $\overline{\rho} : G_{\mathbf{Q}_p} \to \mathrm{GSp}_4(\mathbf{F}_3)$ such that $\overline{\rho}^\vee \simeq \mathrm{Jac}(X)[3]$ for a genus two curve $X/\mathbf{Q}_p$ with good ordinary reduction and a rational Weierstrass point when $p = 2$ or 3. We also study the related question of when $\overline{\rho}^\vee \simeq A[3]$ where $A/\mathbf{Q}_p$ is an abelian surface with good ordinary reduction and a rational odd theta characteristic, as well as variants in which ordinary semistable reduction is allowed — note that even when $A = \mathrm{Jac}(X)$, it is possible that A has good reduction even when X does not. This analysis is then used in §9.4 to carry out the 2-3 switch and then in §9.5 to complete the proofs of our main modularity theorems (Theorem A and B).

§10 gives some examples and complements to our main theorem, proving a residual modularity theorem for mod 2 representations with image A_6 or S_6, and proving the automorphy of any abelian surface $A/\mathbf{Q}$ which neither has $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$ nor satisfies $\mathrm{End}(A_K) \otimes \mathbf{R} = \mathrm{End}(A_{\overline{\mathbf{Q}}}) \otimes \mathbf{R} = \mathbf{R} \oplus \mathbf{R}$ for some quadratic field K (this excluded case includes the restriction of scalars of a general elliptic curve over K). We also explain why the full modularity theorem for all abelian surfaces over $\mathbf{Q}$ would follow from a version of Serre's conjecture for $\mathrm{GSp}_4(\mathbf{F}_p)$ in *regular* weight.

1.6. **The work of Arthur.** It should be noted that this paper, as with the paper [BCGP21] (see [BCGP21, 1.4.1]), relies on results stated by Arthur in [Art04] which ultimately rely on references [A24], [A25], [A26], and [A27] which have not (still) yet appeared, as well as cases of the twisted weighed fundamental lemma announced in [CL10]. However, the situation has improved remarkably in recent times. As a result of the recent preprint [AGI$^+$] of Atobe, Gan, Ichino, Kaletha, Mínguez, and Shin, a complete proof of all the missing ingredients from Arthur's papers is now available, and thus the only result we use for which a proof is not yet available is the twisted weighted fundamental lemma.

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1.8. **Notation and conventions.**

1.8.1. *Assorted notation.* We write $\mathbf{Z}^n_+ \subset \mathbf{Z}^n$ for the subset of tuples $(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If $L/\mathbf{Q}_p$ is a finite extension, we write $L_{\mathrm{cycl}} := L(\zeta_{p^\infty})$ for the cyclotomic extension.

1.8.2. *Coefficients.* We let E be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k. We will always assume that E is chosen to be large enough such that all irreducible components of all deformation rings that we consider, and all irreducible components of their special fibres, are geometrically irreducible. (We are always free to enlarge E in all of the arguments that we make, so this is not a serious assumption.) Given a complete Noetherian local $\mathcal{O}$-algebra Λ with residue field k, we let CNL_Λ denote the category of complete Noetherian local Λ-algebras with residue field k. We refer to an object in CNL_Λ as a CNL_Λ-algebra. If G is a group functor on CNL_Λ then we write $\widehat{G}$ for the group functor on CNL_Λ given by $\widehat{G}(R) := \ker(\widehat{G}(R) \to \widehat{G}(k))$.

1.8.3. *Galois representations and p-adic Hodge theory.* We assume without further comment that all Galois representations are continuous with respect to the natural topologies. We normalize Hodge–Tate weights so that the cyclotomic character has Hodge–Tate weight -1, and the Sen operator acts via 1 on the Sen module of $\mathbf{Q}_p(1)$, so that the (generalized) Hodge–Tate weights are the negatives of the eigenvalues of the Sen operator. We write ε for the p-adic cyclotomic character.

Let $K/\mathbf{Q}_l$ be a finite extension for some l (possibly equal to p). As in [BCGP21, §2.8] we say that a representation $G_K \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is pure if the corresponding Weil–Deligne representation is pure; in the case $l = p$, this presupposes that the representation is de Rham. We say that a representation $G_K \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ is pure if the corresponding representation $G_K \to \mathrm{GL}_4(\overline{\mathbf{Q}}_p)$ is pure. If F is a number field then we say that a representation $G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ (or $G_F \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$) if it is pure at all finite places of F.

1.8.4. *Notation for reductive groups.* We consider a split reductive group G over Spec E, with Borel B and torus T. We denote by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$ their Lie algebras. We write $\bar{B}$ for the opposite Borel of B and write $\bar{\mathfrak{b}}$ for its Lie algebra. We let Φ be the set of roots of G, with positive roots Φ^+ and negative roots $\Phi^- = -\Phi^+$. For $\alpha \in \Phi$ choose standard basis elements X_α so that $(X_\alpha, X_{-\alpha}, H_\alpha)$ is an $\mathfrak{sl}_2$-triple, where $H_\alpha := [X_\alpha, X_{-\alpha}]$. We write Δ for the set of simple roots. We let W be the Weyl group of G, with length function $\ell : W \to \mathbf{Z}_{\geq 0}$, and write w_0 for the longest element of W. The Weyl group acts on the left on the character group $X^*(T)$ via $(w\lambda)(t) := \lambda(w^{-1}tw)$. It also acts on the left on $X_*(T)$, and the natural pairing $\langle , \rangle$ between $X^*(T)$ and $X_*(T)$ is W-equivariant.

Let $P \supseteq B$ be a standard parabolic with Levi quotient M. We let $\mathfrak{p}$ be its Lie algebra, with unipotent radical $\mathfrak{u}_\mathfrak{p}$ and Levi $\mathfrak{m}$, and we write $\mathfrak{z}_\mathfrak{m}$ for the centre of $\mathfrak{m}$. We have a Borel $\mathfrak{b}_\mathfrak{m} = \mathfrak{m} \cap \mathfrak{b} \subseteq \mathfrak{m}$. We write $\mathfrak{u}$ for the unipotent radical of $\mathfrak{b}$ and $\mathfrak{u}_\mathfrak{p}$ for the unipotent radical of $\mathfrak{p}$. Let Φ^+_M be the subset of Φ^+ which lie in the Lie algebra of M, and set $\Phi^{+,M} := \Phi^+ \smallsetminus \Phi^+_M$; and write $\Phi^-_M := -\Phi^+_M$, $\Phi^{-,M} := -\Phi^{+,M}$. We let W_M be the Weyl group of M, with longest element $w_{0,M}$, and we let ${}^M W \subseteq W$ be the set of Kostant representatives of $W_M \backslash W$ (i.e. those $w \in W$ with $\Phi^+_M \subseteq w\Phi^+$; this is a set of coset representatives of minimal length). There is an involution of ${}^M W$ given by $w \mapsto w_{0,M} w w_0$, and we have $l(w_{0,M} w w_0) + l(w) = |\Phi^{+,M}|$. In particular the Kostant representative of maximal length is $w_0^M := w_{0,M} w_0$.

We let ρ be half the sum of the positive roots, and write $\rho = \rho^M + \rho_M$ where ρ^M is half the sum of the roots in $\Phi^{+,M}$ and ρ_M is half the sum of the roots in Φ^+_M.

We define the "dot action" of W on $X^*(T)$ by $w \cdot \lambda := w(\lambda + \rho) - \rho$. We say that $\lambda \in X^*(T)$ is *regular* if the stabilizer of λ for the dot action is trivial, and otherwise we say that λ is *singular* or *irregular.*

Let $w \in W$. Then we let $P_w := w^{-1} P w$, with Lie algebra $\mathfrak{p}_w := w^{-1} \mathfrak{p} w$, and similarly we define $\mathfrak{u}_{\mathfrak{p}_w}$, $\mathfrak{m}_w$, and so on.

1.8.5. *Notations for a p-adic torus.* Let $T \to$ Spec $\mathbf{Q}_p$ be a torus. We let T^d be its maximal split subtorus. The group $T(\mathbf{Q}_p)$ has a unique maximal compact subgroup that we denote (abusing notation) by $T(\mathbf{Z}_p)$. We have an exact sequence

$$0 \to \bar{\mathbf{Z}}_p^\times \to \overline{\mathbf{Q}}_p^\times \xrightarrow{v} \mathbf{Q} \to 0$$

given by the p-adic valuation, normalized by $v(p) = 1$. Tensoring this sequence by $X_*(T)$ and taking invariants under the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ yields an exact sequence

$$0 \to T(\mathbf{Z}_p) \to T(\mathbf{Q}_p) \xrightarrow{v} X_*(T^d) \otimes \mathbf{Q} \tag{1.8.6}$$

where the image of v is a lattice.

Let $\chi : T(\mathbf{Q}_p) \to \overline{\mathbf{Q}}_p^\times$ be a character. We can compose it with the p-adic valuation and get a map $v(\chi) : T(\mathbf{Q}_p) \to \mathbf{Q}$, which factors as $T(\mathbf{Q}_p) \xrightarrow{v} X_*(T^d) \otimes \mathbf{Q} \to \mathbf{Q}$. We can therefore think of $v(\chi)$ as an element of $X^*(T^d)_\mathbf{Q}$.

If T is a maximal torus contained in a Borel of a quasi-split reductive group G defined over $\mathbf{Q}_p$ and if $\Phi = \Phi^+ \cup \Phi^-$ is the set of absolute roots, we let $T^+(\mathbf{Q}_p) = \{t \in T(\mathbf{Q}_p), v(\alpha(t)) \geq 0 \ \forall \alpha \in \Phi^+\}$ and $T^{++}(\mathbf{Q}_p) = \{t \in T(\mathbf{Q}_p), v(\alpha(t)) > 0 \ \forall \alpha \in \Phi^+\}$.

1.8.7. *Notations in the symplectic case.* We will often consider the case where $G = \mathrm{GSp}_{2g}$ and P is the Siegel parabolic. In this case we make some more explicit choices. The group G has a natural model over Spec $\mathbf{Z}$, namely we realize G as the subgroup of GL_{2g} acting on the free $\mathbf{Z}$-modules of rank $2g$, with basis $e_1, \cdots, e_{2g}$ and preserving up to a similitude factor the symplectic form with matrix

$$J = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}$$

where S is the $g \times g$ anti-diagonal matrix with only 1's on the anti-diagonal. We denote by $\nu : \mathrm{GSp}_{2g} \to \mathbf{G}_m$ the similitude factor.

We let P be the stabilizer of $\langle e_{g+1}, \cdots, e_{2g}\rangle$. We choose $B \subseteq P$ to be upper triangular on each diagonal block. We let T be the diagonal torus. An element of T is labelled $t = \mathrm{diag}(zt_1, \cdots, zt_g, zt_g^{-1}, \cdots, zt_1^{-1})$. Characters $X^*(T)$ of T are tuples: $\kappa = (k_1, \cdots, k_g; w) \in \mathbf{Z}^g \times \mathbf{Z}$ with $w = \sum k_i \pmod 2$, and $\kappa(t) = z^w \prod_{i=1}^g t_i^{k_i}$. A character is M-dominant if $k_1 \geq \cdots \geq k_g$. The set of M-dominant characters is denoted by $X^*(T)^{M,+}$. A character is G-dominant if $0 \geq k_1 \geq \cdots \geq k_g$. The set of G-dominant characters is denoted by $X^*(T)^{G,+}$.

1.8.8. GSp_4. We now specialize further to the case $G = \mathrm{GSp}_4$. We continue to take P to be the ("block lower-triangular") Siegel parabolic stabilizing e_3, e_4, and B the Borel inside it which is upper-triangular in each of the diagonal 2×2 blocks. We let Q be the Klingen parabolic containing B (this is the other maximal parabolic in GSp_4) with Levi M_Q.

Let $\kappa = (k_1, k_2; w) \in X^*(T)^+$ be a dominant weight for GSp_4, so that $0 \geq k_1 \geq k_2$. Given our choice of Borel, the positive roots Φ^+ are $\alpha = (1, -1; 0), \beta = (-2, 0; 0), \gamma = \alpha + \beta = (-1, -1; 0), \delta = 2\alpha + \beta = (0, -2; 0)$. We have $\rho = (-1, -2; 0)$.

The Weyl group W is generated by s_α and s_β where $s_\alpha(k_1, k_2; w) = (k_2, k_1; w)$ and $s_\beta(k_1, k_2; w) = (-k_1, k_2; w)$, so that $w_0 = s_\alpha s_\beta s_\alpha s_\beta$, and $w_0(k_1, k_2; w) = (-k_1, -k_2; w)$. We have $W_M = \{\mathrm{Id}, s_\alpha\}$ and $W_{M_Q} = \{\mathrm{Id}, s_\beta\}$. The elements of ${}^M W$ are $\mathrm{Id}, s_\beta, s_\beta s_\alpha, s_\beta s_\alpha s_\beta$. We label them ${}^0w, {}^1w, {}^2w, {}^3w$; they respectively have length $0, 1, 2, 3$. In particular, ${}^3w = w_0^M$ is the length three element. We use the pairing between characters and cocharacters coming from the standard pairing on $\mathbf{Q}^3$. Thus, we label cocharacters $X_*(T)$ as triples $(a, b; c) \in \frac{1}{2}\mathbf{Z}^3$, with $a + c, b + c \in \mathbf{Z}$. To $(a, b; c)$ we attach the cocharacter $t \mapsto \mathrm{diag}(t^{a+c}, t^{b+c}, t^{-b+c}, t^{-a+c})$. We let $\mu = (-1/2, -1/2; 1/2) \in X_*(T)$. We sometimes view μ as an element of $\mathfrak{z}_\mathfrak{m}$ (the centre of the Lie algebra $\mathfrak{m}$).

We let $\widehat{\mathrm{GSp}_4}$ be the dual group of GSp_4. Our choice of Borel B and torus T in GSp_4 gives a Borel $\widehat{B}$ and torus $\widehat{T}$ in $\widehat{\mathrm{GSp}_4}$. We use the spin representation to identify $\widehat{\mathrm{GSp}_4}$, the Borel $\widehat{B}$ and torus $\widehat{T}$ with the group GSp_4, its usual upper triangular Borel and diagonal torus. In particular, this fixes an isomorphism $X^*(T) = X_*(\widehat{T}) \simeq X_*(T)$, given by

$$(\lambda_1, \lambda_2; w) \mapsto [t \mapsto \mathrm{diag}(t^{\frac{-\lambda_1-\lambda_2+w}{2}}, t^{\frac{\lambda_1-\lambda_2+w}{2}}, t^{\frac{-\lambda_1+\lambda_2+w}{2}}, t^{\frac{\lambda_1+\lambda_2+w}{2}})]$$

Dually, there is an isomorphism $X_*(T) = X^*(\widehat{T}) \simeq X^*(T)$ for which μ corresponds to the dominant character $(1, 0; 1)$ of $X^*(\widehat{T})$. When we work on the dual side (typically when we consider Galois representations), we will also denote by B the upper triangular Borel in $\mathrm{GSp}_4 \simeq \widehat{\mathrm{GSp}_4}$. This should not cause any confusion.

1.8.9. *Ordinary Galois representations.*

Definition 1.8.10. Let $K/\mathbf{Q}_p$ be a finite extension, and let $\rho : G_K \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ be a representation with similitude factor ε^{-1}. We say that ρ is *ordinary* if there are characters $\chi_1, \chi_2 : G_K \to \overline{\mathbf{Q}}_p^\times$ with

$$\rho \cong \begin{pmatrix} \chi_1 & * & * & * \\ 0 & \chi_2 & * & * \\ 0 & 0 & \varepsilon^{-1}\chi_2^{-1} & * \\ 0 & 0 & 0 & \varepsilon^{-1}\chi_1^{-1} \end{pmatrix}.$$

We say that the ordered pair (χ_1, χ_2) is a *p-stabilization* of ρ. We say that ρ is *p-distinguished* if the 4 characters $\chi_1, \chi_2, \varepsilon^{-1}\chi_2^{-1}, \varepsilon^{-1}\chi_1^{-1}$ are pairwise distinct. We say that ρ is *semistable of weight 2* if the subrepresentation

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

is unramified. (Such a representation is automatically semistable in the usual sense.) In this case we will sometimes denote the p-stabilization (χ_1, χ_2) by (α, β) with $\alpha = \chi_1(\mathrm{Frob}_K)$, $\beta = \chi_2(\mathrm{Frob}_K)$.

Similarly, we say that a representation $\overline{\rho} : G_K \to \mathrm{GSp}_4(\overline{\mathbf{F}}_p)$ with similitude factor $\overline{\varepsilon}$ is *ordinary* if there are characters $\overline{\chi}_1, \overline{\chi}_2 : G_K \to \overline{\mathbf{F}}_p^\times$ with

$$\overline{\rho} \cong \begin{pmatrix} \overline{\chi}_1 & * & * & * \\ 0 & \overline{\chi}_2 & * & * \\ 0 & 0 & \varepsilon^{-1}\overline{\chi}_2^{-1} & * \\ 0 & 0 & 0 & \varepsilon^{-1}\overline{\chi}_1^{-1} \end{pmatrix}.$$

We say that the ordered pair $(\overline{\chi}_1, \overline{\chi}_2)$ is a *p-stabilization* of $\overline{\rho}$. We say that ρ is *residually p-distinguished* if the 4 characters $\overline{\chi}_1, \overline{\chi}_2, \varepsilon^{-1}\overline{\chi}_2^{-1}, \varepsilon^{-1}\overline{\chi}_1^{-1}$ are pairwise distinct. We say that $\overline{\rho}$ is *of weight 2* if the subrepresentation

$$\begin{pmatrix} \overline{\chi}_1 & * \\ 0 & \overline{\chi}_2 \end{pmatrix}$$

is unramified; in particular the characters $\overline{\chi}_1, \overline{\chi}_2$ are unramified. (Conversely, if $\overline{\chi}_1, \overline{\chi}_2$ are distinct and unramified, then $\overline{\rho}$ is of weight 2.) If $\overline{\rho}$ is of weight 2, then we will usually denote the p-stabilization $(\overline{\chi}_1, \overline{\chi}_2)$ by $(\overline{\alpha}, \overline{\beta})$, where $\overline{\alpha} = \overline{\chi}_1(\mathrm{Frob}_K)$, $\overline{\beta} = \overline{\chi}_2(\mathrm{Frob}_K)$.

We make the same definitions for integral representations $\rho : G_K \to \mathrm{GSp}_4(\overline{\mathbf{Z}}_p)$, in which case a p-stabilization (χ_1, χ_2) induces a p-stabilization $(\overline{\chi}_1, \overline{\chi}_2)$ of the mod p representation $\overline{\rho} : G_K \to \mathrm{GSp}_4(\overline{\mathbf{F}}_p)$. Note that if ρ is semistable of weight 2, then $\overline{\rho}$ is of weight 2. If we regard ρ as a lift of $\overline{\rho}$, then we say that (χ_1, χ_2) is *compatible* with $(\overline{\chi}_1, \overline{\chi}_2)$. Given two ordinary lifts ρ_1, ρ_2 of $\overline{\rho}$, we say that p-stabilizations of ρ_1 and ρ_2 respectively are compatible if they induce the same p-stabilization of $\overline{\rho}$.

Remark 1.8.11. We again (see [BCGP21, Rem. 7.3.2]) apologize for the terminology "of weight 2"; these definitions are convenient later in the paper when we wish to appeal to results from [BCGP21]. In particular we caution the reader that if ρ is of weight 2 and pure, then it is pure of weight 1 in the usual sense. Since we will never use the terminology "pure of weight 1" (or "pure of weight 2", for that matter), we hope that this will not lead to any confusion.

1.8.12. *Galois representations associated to automorphic representations.* We for the most part follow the conventions of our earlier paper [BCGP21], to which we refer for further details. We begin with some brief recollections from [BCGP21, §2.3]. If $K/\mathbf{Q}_l$ is a finite extension for some l, then we let rec_K be the local Langlands correspondence of [HT01], which assigns to an irreducible complex admissible representation π of $\mathrm{GL}_n(K)$ a Frobenius semi-simple Weil–Deligne complex representation $\mathrm{rec}_K(\pi)$ of the Weil group W_K. We will write rec for rec_K when the choice of K is clear. In the case $n = 1$, rec_K is obtained from the Artin map $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$, which we normalize to send uniformizers to geometric Frobenius elements. Similarly, we denote the local Langlands correspondence of [GT11] by $\mathrm{rec}_{\mathrm{GT}}$; this assigns a GSp_4-conjugacy classes of $\mathrm{GSp}_4(\mathbf{C})$-valued Weil–Deligne representation of W_K to each irreducible smooth complex representation of $\mathrm{GSp}_4(K)$. If (r, N) is a Weil–Deligne representation of W_K we will write $(r, N)^{F-\mathrm{ss}}$ for its Frobenius semi-simplification.

We fix once and for all for each prime p an isomorphism $\imath = \imath_p : \mathbf{C} \cong \overline{\mathbf{Q}}_p$. We will sometimes omit these isomorphisms from our notation, in order to avoid clutter. In particular, we will frequently use that $\imath$ determines a square root of p in $\overline{\mathbf{Q}}_p$ (corresponding to the positive square root of p in $\mathbf{C}$). We will often regard automorphic representations as being defined over $\overline{\mathbf{Q}}_p$, rather than $\mathbf{C}$, by means of the fixed isomorphism $\imath : \mathbf{C} \cong \overline{\mathbf{Q}}_p$. We write rec_p and $\mathrm{rec}_{\mathrm{GT},p}$ for the local Langlands correspondences for $\overline{\mathbf{Q}}_p$-representations given by conjugating by $\imath$.

Suppose that F^+ is a totally real field and that π is a cuspidal automorphic representation of GSp_4 / F^+. We will always assume that such a π has central character $|\cdot|^2$. (We apologize for this assumption, which seemed helpful at some points when writing [BCGP21], and suffices for applications to abelian surfaces.) We say that π is *algebraic* if it is C-algebraic, and we say that it is *regular algebraic* if π_∞ is an (essentially) discrete series representation. Suppose that π is algebraic. We say that it has weight $(\lambda_v)_{v|\infty}$ where $\lambda_v \in (X^*(T)^+_{\mathbf{Q}} - \rho) \cap X^*(T)$, if π_v has infinitesimal character $-\lambda_v - \rho$. If π is regular algebraic then $\lambda_v \in X^*(T)^+$, and we know that $\pi \otimes \bigotimes_{v|\infty} V_{\lambda_v}$ has non-trivial $(\mathfrak{g}, K_\infty)$-cohomology where V_{λ_v} is the highest weight λ_v-representation.

We now come to the definition of ordinarity. Assume furthermore that p splits completely in F^+ (this is sufficient to us). Our fixed isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}}_p$ identifies $\{w \mid p\}$ and $\{v \mid \infty\}$. Suppose that $w \mid p$. We say that π_w is *finite slope* if it has non-trivial Jacquet module. The Jacquet module of π_w is then a direct sum of characters $\chi_w : T(\mathbf{Q}_p) \to \bar{\mathbf{Q}}_p^\times$. We say that π_w is *ordinary* if there is a character χ_w occurring in the Jacquet module such that $v(\chi_w) = -\lambda_w$ (see Section 1.8.5 for the definition of $v(\chi_w)$). We refer to a choice of such a character as an (ordinary) *p-stabilization* of π_w. We say that π is ordinary if π_w is ordinary for all $w \mid p$. If π is ordinary and regular algebraic, then each π_w has a unique (ordinary) p-stabilization.

Theorem 1.8.13. *Suppose that F^+ is totally real and that p splits completely in F^+. If π is regular algebraic of weight $\lambda = ((k_v, l_v; 2))_{v|\infty}$, then for each prime p there is (see e.g.* [BCGP21, Thms. 2.7.1, 2.7.2]*) a semi-simple representation $\rho_{\pi,p} : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ satisfying the following properties.*

- $\nu \circ \rho_{\pi,p} = \varepsilon^{-1}$.

- *For each finite place $v \nmid p$, we have*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{F_v^+}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GT},p}(\pi_v \otimes |\nu|^{-3/2})^{\mathrm{ss}}.$$

- *If $\rho_{\pi,p}$ is irreducible, then for each finite place v of F, $\rho_{\pi,p}|_{G_{F_v^+}}$ is pure and*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{F_v^+}})^{F-\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GT},p}(\pi_v \otimes |\nu|^{-3/2}).$$

- *For each $v|p$, $\rho_{\pi,p}|_{G_{F_v^+}}$ is de Rham with Hodge–Tate weights $((k_v+l_v)/2 - 1, -(k_v-l_v)/2, (k_v-l_v)/2+1, 2-(k_v+l_v)/2)$.*
- *If p splits completely in F, $v \mid p$, and π_v is ordinary, then there are potentially unramified characters α, β such that:*

$$\rho_{\pi,p}|_{G_{F_v^+}} \cong \begin{pmatrix} \alpha\varepsilon^{1-(k_v+l_v)/2} & * & * & * \\ 0 & \beta\varepsilon^{(k_v-l_v)/2} & * & * \\ 0 & 0 & \beta^{-1}\varepsilon^{-1-(k_v-l_v)/2} & * \\ 0 & 0 & 0 & \alpha^{-1}\varepsilon^{(k_v+l_v)/2-2} \end{pmatrix}. \tag{1.8.14}$$

Remark 1.8.15. We can spell out more precisely the characters on the diagonal in (1.8.19). Let $\chi_v : T(\mathbf{Q}_p) \to \bar{\mathbf{Q}}_p^\times$ be an ordinary p-stabilization of π_v. This induces a p-stabilization of $\rho_{\pi,p}|_{G_{F_v^+}}$ in the sense of Definition 1.8.10 as follows. Let

$$\widetilde{\chi}_v = \chi_v \lambda_v \rho \nu^{-\frac{3}{2}} |\rho\nu^{-\frac{3}{2}}|^{-1} : T(\mathbf{Q}_p) \to \bar{\mathbf{Z}}_p^\times.$$

This character is valued in $\bar{\mathbf{Z}}_p^\times$ by the ordinarity assumption. We can identify $\widetilde{\chi}_v$ with an homomorphism $\mathbf{Q}_p^\times \to \widehat{T}(\bar{\mathbf{Z}}_p)$, where $\widehat{T}$ is the dual torus, which we identify with T by using the isomorphism $\mathrm{GSp}_4 \simeq \widehat{\mathrm{GSp}_4}$ of Section 1.8.8. Then by class field theory, we interpret $\widetilde{\chi}_v : G_{\mathbf{Q}_p} \to T(\bar{\mathbf{Z}}_p)$. This is the character on the diagonal of $\rho_{\pi,p}|_{G_{F_v^+}}$.

We will also need to use the Galois representations associated to certain irregular weight algebraic cuspidal automorphic representations for GSp_4 / F^+.

Definition 1.8.16. We say that π has weight 2 if it is algebraic of weight $\lambda = (1,1;2)_{v|\infty}$ (remember that by our convention, this λ is not G-dominant) and π_∞ is a non-degenerate limit of discrete series.

The following theorem is well known. We provide a sketch of proof since we couldn't find a precise reference in the literature.

Theorem 1.8.17. *Suppose that F^+ is totally real and that p splits completely in F^+. Let π be an ordinary weight 2 automorphic representation for GSp_4 / F^+. There is a semi-simple representation $\rho_{\pi,p} : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ satisfying the following properties.*

- $\nu \circ \rho_{\pi,p} = \varepsilon^{-1}$.
- *For each finite place $v \nmid p$, we have*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{F_v^+}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GT},p}(\pi_v \otimes |\nu|^{-3/2})^{\mathrm{ss}}. \tag{1.8.18}$$

- *There are potentially unramified characters α, β such that:*

$$\rho_{\pi,p}|_{G_{F_v^+}} \cong \begin{pmatrix} \alpha & * & * & * \\ 0 & \beta & * & * \\ 0 & 0 & \beta^{-1}\varepsilon^{-1} & * \\ 0 & 0 & 0 & \alpha^{-1}\varepsilon^{-1} \end{pmatrix}. \tag{1.8.19}$$

In fact, the character on the diagonal is described by the recipe explained in Remark 1.8.15.

Proof. The representation π_f will realize in the interior coherent cohomology of the Hilbert–Siegel Shimura variety by [Har90, Thm. 2.7, Thm. 3.6.2]. By [BP21, Thm. 1.4.3 (1) (4)], π_f defines a point x on an equidimensional eigenvariety which dominates weight space. Let $\mathrm{Spa}(A, A^+)$ be an affinoid open subset of the eigenvariety containing x. By [BP21, Thm. 1.4.3 (2)], there is a Zariski dense set of classical points in $\mathrm{Spa}(A, A^+)$, with regular algebraic weight. Let X be the space of GSp_4-valued pseudorepresentations of G_{F^+} (in the sense of Lafforgue, see [Qua23]). Then by interpolation of the representations in Theorem 1.8.13 there is a map $\mathrm{Spec}\, A \to X$. Specializing at x produces the semi-simple representation $\rho_{\pi,p}$. By interpolation the representation $\wedge^2 \rho_{\pi,p}$ contains the character ε^{-1}, so $\rho_{\pi,p}$ admits a symplectic pairing with multiplier ε^{-1}. The statement regarding local-global compatibility away from p follows by a standard argument from p-adic interpolation (note that the Weil–Deligne representations are only considered up to semi-simplification). If we assume that π is ordinary, then we can assume that $\mathrm{Spa}(A, A^+)$ is an ordinary component of the eigenvariety, and by interpolation our local-global compatibility statement at p follows (see also [BP21, Thm. 1.4.8], for a more general statement in the finite slope case). ☐

Remark 1.8.20. In the situation of Theorem A, we can upgrade the semi-simplified local-global compatibility (1.8.18) in Theorem 1.8.17 to full local-global compatibility. More precisely, if π is of general type and $\rho_{\pi,p}$ in is pure, then

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{F_v^+}})^{F-\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GT},p}(\pi_v \otimes |\nu|^{-3/2})$$

for all v; that is, in addition to (1.8.18), the monodromy operators N on each side agree. To see this, note firstly that since π is of general type, and cuspidal automorphic representations of GL_n are generic, the L-packet containing $\pi_v \otimes |\nu|^{-3/2}$ is generic; so by part vii of the main theorem of [GT11], the adjoint L-factor $L(s, \mathrm{ad}(\mathrm{rec}_{\mathrm{GT},p}(\pi_v \otimes |\nu|^{-3/2})))$ is holomorphic at $s = 1$. Equivalently,

$$(\mathrm{ad}(\mathrm{rec}_{\mathrm{GT},p}(\pi_v \otimes |\nu|^{-3/2}))(1))^{\varphi=1, N=0} = 0. \tag{1.8.21}$$

On the other hand, since $\rho_{\pi,p}|_{G_{F_v^+}}$ is pure, so is $\mathrm{WD}(\rho_{\pi,p}|_{G_{F_v^+}})^{F-\mathrm{ss}}$ (by [TY07, Lem. 1.4(1)]). By [TY07, Lem. 1.4(4)] and its proof, this means that $\mathrm{WD}(\rho_{\pi,p}|_{G_{F_v^+}})^{F-\mathrm{ss}}$ is equipped with the unique choice of N satisfying (1.8.21), as required.

Finally, we will need to use the Galois representations associated to certain automorphic representations of GL_n, which we now very briefly recall. Let F be an imaginary CM field. Recall that an automorphic representation π of GL_n / F is RACSDC if it is regular algebraic, conjugate self-dual, (i.e. $\pi^c \cong \pi^\vee$), and cuspidal. (See e.g. [BLGGT14, §2]) for more details.) Associated to a RACSDC automorphic representation π is a continuous semi-simple representation $r_{\pi,p} : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ such that $r_{\pi,p}|_{G_{F_v}}$ is de Rham for all $v|p$, and for each finite place v of F we have

$$\imath\,\mathrm{WD}(r_{\pi,p}|_{G_{F_v}})^{F-\mathrm{ss}} \cong \mathrm{rec}(\pi_v \otimes |\det|_v^{(1-n)/2})$$

(see e.g. [BLGGT14, Thm. 2.1.1] and [Car14, Thm. 1.1]). In particular, we have $r^c_{\pi,p} \cong r^\vee_{\pi,p}\varepsilon^{1-n}$.

1.8.22. *Transfer between* GL_4 *and* GSp_4. We firstly very briefly recall some results on Arthur's classification of discrete automorphic representations of GSp_4; see [BCGP21, §2.9] for a slightly longer treatment with precise references to the literature. Suppose that F is a number field, that Π is a cuspidal automorphic representation of GL_4/F, and that $\chi : \mathbf{A}_F^\times/F^\times \to \mathbf{C}^\times$ is unitary. Then we say that Π is χ-self dual if $\Pi \cong \Pi^\vee \otimes \chi \circ \det$, in which case the pair (Π, χ) is of *symplectic type* if the partial L-function $L^S(s, \Pi, \bigwedge^2 \otimes \chi^{-1})$ has a pole at $s = 1$ (where S is any finite set of places of F) or of *orthogonal type* if $L^S(s, \Pi, \mathrm{Sym}^2 \otimes \chi^{-1})$ has a pole at $s = 1$. Exactly one of these alternatives holds, and if (Π, χ) is of symplectic (resp. orthogonal) type then it descends to a discrete automorphic representation π of GSp_4/F (resp. $\mathrm{GSpin}_4^\alpha/F$ for some inner form GSpin_4^α of GSpin_4) with central character $\omega_\pi = \chi$. (See for example [GT19, Prop. 6.1.7].) We say that a discrete automorphic representation π of GSp_4/F is *of general type* if it arises in this way for some (Π, χ), in which case we say that Π is the *transfer* of π, and that π is a *descent* of Π. For each place v of F, the L-parameter obtained from $\mathrm{rec}_{\mathrm{GT}}(\pi_v)$ by composing with the usual embedding $\mathrm{GSp}_4 \hookrightarrow \mathrm{GL}_4$ is $\mathrm{rec}(\Pi_v)$. In this case π is necessarily cuspidal, and it is stable. In fact if $\pi' := \otimes'\pi'_v$ with π_v, π'_v in the same L-packet for all v, then π' is automorphic, and occurs with multiplicity one in the discrete spectrum. If π is (regular) algebraic then Π is also (regular) algebraic.

If F is totally real, and π is regular algebraic and *not* of general type, then the Galois representations $\rho_{\pi,p}$ associated to π are reducible by [BCGP21, Lem. 2.9.1]. Since we will always be in a situation where our Galois representations are irreducible (even irreducible modulo p), we will only need to consider π of general type in this paper.

1.8.23. *Galois representations associated to abelian surfaces.* Let F be a number field, and let A/F be an abelian surface. For each prime p, we may write $\rho_{A,p}$ for the Galois representation associated to $H^1(A_{\overline{F}}, \mathbf{Z}_p)$. We often think of $\rho_{A,p}$ as a representation

$$\rho_{A,p} : G_F \to \mathrm{GSp}_4(\mathbf{Q}_p)$$

with multiplier given by the inverse cyclotomic character ε^{-1} (compare [BCGP21, Defn. 2.8.2]). We also let $\overline{\rho}_{A,p}$ denote the Galois representation associated to $H^1(A_{\overline{F}}, \mathbf{F}_p)$. If A admits a principal polarization of degree prime to p, then we can and do think of $\overline{\rho}_{A,p}$ as a representation

$$\overline{\rho}_{A,p} : G_F \to \mathrm{GSp}_4(\overline{\mathbf{F}}_p).$$

We take the coefficient field of $\rho_{A,p}$ (respectively, of $\overline{\rho}_{A,p}$) to be $\mathbf{Q}_p$ or $\overline{\mathbf{Q}}_p$ (resp. $\mathbf{F}_p$ or $\overline{\mathbf{F}}_p$) depending on what is most convenient. If $T_p(A)$ denotes the p-adic Tate module of A, then (in our conventions) the Galois representations associated to $T_p(A)$ and $A[p]$ are the dual representations $\rho^\vee_{A,p} \simeq \rho_{A,p} \otimes \varepsilon$ and $\overline{\rho}^\vee_{A,p} \simeq \overline{\rho}_{A,p} \otimes \varepsilon$ respectively. The representation $\rho_{A,p}$ is unramified at all but finitely many places v of F, and if $v|p$ then $\rho_{A,p}|_{G_{F_v}}$ is de Rham with Hodge–Tate weights 0,0,1,1 for every choice of embedding $F \to \overline{\mathbf{Q}}_p$. Furthermore $\rho_{A,p}|_{G_{F_v}}$ is pure at all finite places v (see e.g. [BCGP21, Prop. 2.8.1]). If A/F_v has good ordinary reduction for some $v|p$, then $\rho_{A,p}|_{G_{F_v}}$ is crystalline and ordinary of weight 2.

1.8.24. *Notions of modularity.* Let F be a number field.

Definition 1.8.25. An abelian surface A/F is *modular*, or equivalently, *automorphic* if there exist C-algebraic cuspidal automorphic representations π_i for GL_{n_i}/F with $4 = \sum n_i$ such that

$$L(s, H^1(A)) = \prod L(s, \pi_i \otimes |\det|^{(1-n_i)/2}).$$

A genus two curve X/F is modular if $A = \mathrm{Jac}(X)/F$ is modular.

If A is modular, then $L(s, H^i(A)) = L(s, \wedge^i H^1(A)) = L(s, \Pi_i)$ for some automorphic representation Π_i. This follows from known functorialities in small degrees, most notably [Kim03, Hen09] (cf. the proof of [BCGP21, Thm 9.3.1]).

Remark 1.8.26 (Warning)**.** In [BCGP21], particularly [BCGP21, Defn. 9.1.8], we reserved the term *modular* to specifically refer to the stronger statement that F was totally real and that A was associated to a cuspidal automorphic representation of GSp_4/F with certain properties. With such a restriction, there are abelian surfaces and genus two curves which fail to be modular, for example, when $A/\mathbf{Q} = \mathrm{Jac}(X)/\mathbf{Q}$ is isogenous to a product of two elliptic curves or an abelian surface of GL_2-type. In retrospect, we feel that this distinction is unhelpful. In the main theorems of this paper, we (under certain hypotheses) establish the modularity of $A/\mathbf{Q}$ by proving the modularity of $\rho_{A,p}$ for some p. More precisely, we assume that $\rho_{A,p}$ is absolutely irreducible, and show that $\rho_{A,p} \cong \rho_{\pi,p}$ for some weight 2 cuspidal automorphic representation π for $\mathrm{GSp}_4/\mathbf{Q}$. This π will be of general type, and thus transfers to a C-algebraic cuspidal automorphic representation of GL_4.

1.8.27. *The Eichler–Shimura relation.* We let ℓ be a prime. We let

$$\mathbf{T}_\ell = \mathbf{Z}[\mathrm{GSp}_4(\mathbf{Q}_\ell)/\!/\mathrm{GSp}_4(\mathbf{Z}_\ell)]$$

be the spherical Hecke algebra for $\mathrm{GSp}_4(\mathbf{Q}_\ell)$ with $\mathbf{Z}$-coefficients. As a $\mathbf{Z}$-module, it has a basis consisting of the characteristic functions of the double cosets $T_\lambda = [\mathrm{GSp}_4(\mathbf{Z}_\ell)\lambda(\ell)\,\mathrm{GSp}_4(\mathbf{Z}_\ell)]$ where $\lambda \in X_*(T)^+$. In particular, we define $T_{\ell,i} = [\mathrm{GSp}_4(\mathbf{Z}_l)\beta_{\ell,i}\,\mathrm{GSp}_4(\mathbf{Z}_l)]$, where

$$\beta_{\ell,0} = \mathrm{diag}(\ell,\ell,\ell,\ell),$$
$$\beta_{\ell,1} = \mathrm{diag}(\ell,\ell,1,1),$$
$$\beta_{\ell,2} = \mathrm{diag}(\ell^2,\ell,\ell,1).$$

We write $Q_\ell(X) \in \mathbf{T}[X]$ for the polynomial

$$X^4 - T_{\ell,1}X^3 + (\ell T_{\ell,2} + (\ell^3 + l)T_{\ell,0})X^2 - \ell^3 T_{\ell,0}T_{\ell,1}X + \ell^6 T_{\ell,0}^2.$$

We have the Satake isomorphism

$$S : \mathbf{Q}(\sqrt{\ell})[X^*(\widehat{T})]^W \xrightarrow{\sim} \mathbf{Q}(\sqrt{\ell}) \otimes_{\mathbf{Z}} \mathbf{T}_\ell.$$

For each representation V of $\widehat{\mathrm{GSp}_4}$ we let $[V]$ be the character of $\widehat{T}$ on V. This defines an element of $\mathbf{Q}(\sqrt{\ell})[X^*(\widehat{T})]^W$. To each $\lambda \in X_*(T)$ we can associate a representation V_λ of $\widehat{\mathrm{GSp}_4}$ with highest weight λ. The $[V_\lambda]$ form a basis of

$\mathbf{Q}(\sqrt{\ell})[X^*(\widehat{T})]^W$. We consider in particular $\lambda = (-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}) \in X_*(T)$, corresponding to the Spin representation $\widehat{\mathrm{GSp}}_4 \to \mathrm{GL}_4$ (which, as explained above, we use to identify $\widehat{\mathrm{GSp}}_4$ and GSp_4, so that via the isomorphism $X_*(T) = X^*(\hat{T}) = X^*(T)$, $(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2})$ goes to $(1, 0; 1)$). We also consider the dual of the Spin representation, corresponding to $\lambda = (-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2})$. We write $P_{\ell,\lambda}(X) = X^4 - [V_\lambda]X^3 + [\Lambda^2 V_\lambda]X^2 - [\Lambda^3 V_\lambda]X + [\Lambda^4 V_\lambda]$ for the characteristic polynomial of the representation V_λ in either of these cases. We write $Q_\ell(X) = \ell^6 P_{\ell,(-\frac{1}{2},-\frac{1}{2};\frac{1}{2})}(\ell^{-\frac{3}{2}}X)$. Then Q_ℓ is the usual Hecke polynomial in $\mathbf{T}_\ell[X]$ whose definition was recalled above. The coefficient of X^3 is $-T_{(-\frac{1}{2},-\frac{1}{2};\frac{1}{2})}$. We also let

$$P_\ell(X) := \ell^6 P_{\ell,(-\frac{1}{2},-\frac{1}{2};-\frac{1}{2})}(\ell^{-\frac{3}{2}}X). \tag{1.8.27}$$

The coefficient of X^3 in P_ℓ is $-T_{(-\frac{1}{2},-\frac{1}{2};-\frac{1}{2})}$.

Let π be a C-algebraic automorphic representation of $\mathrm{GSp}_4/\mathbf{Q}$ whose component at infinity is a non-degenerate limit of discrete series. Let S be the set of finite places at which π is not spherical. Let $\mathbf{T}^S := \otimes_{\ell \notin S} \mathbf{T}_\ell$ be the spherical Hecke algebra away from S. We let $\Theta_\pi : \mathbf{T}^S \to \mathbf{C}$ be the character describing the action of $\mathbf{T}^S$ on the one dimensional $\mathbf{C}$-vector space of spherical vectors of $\otimes_{v \notin S} \pi_v$. Then by the definition of rec, the Galois representation $\rho_{\pi,p} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ has the property for all primes $\ell \notin S \cup \{p\}$, $\imath(\Theta_\pi(Q_\ell))(X)$ is the characteristic polynomial of $\rho_{\pi,p}(\mathrm{Frob}_\ell)$ (here Frob_ℓ denotes a geometric Frobenius element).

Lemma 1.8.28. *For all primes $\ell \notin S \cup \{p\}$, $\imath(\Theta_\pi(P_\ell))(X)$ is the characteristic polynomial of $(\rho_{\pi,p}^\vee \otimes \varepsilon^{-3})(\mathrm{Frob}_\ell)$.*

Proof. Unraveling the definitions, we find that $\ell^{-6}\imath(\Theta_\pi(P_{\ell,(-\frac{1}{2},-\frac{1}{2};-\frac{1}{2})}))(\ell^{\frac{3}{2}}X)$ is the characteristic polynomial of $\rho_{\pi,p}^\vee(\mathrm{Frob}_\ell)$. The characteristic polynomial of $\ell^3\rho_{\pi,p}^\vee(\mathrm{Frob}_\ell)$ is therefore $\ell^6\imath(\Theta_\pi(P_{\ell,(-\frac{1}{2},-\frac{1}{2};-\frac{1}{2})}))(\ell^{-\frac{3}{2}}X)$. □

For any neat compact open subgroup $K = \prod_\ell K_\ell \subseteq \mathrm{GSp}_4(\mathbf{A}_f)$, let $\mathrm{Sh}_K^{alg} \to \mathrm{Spec}\ \mathbf{Q}$ denote the Siegel threefold of level K. We will make use of the following Eichler–Shimura relation.

Theorem 1.8.29. *On $\mathrm{R}\Gamma(\mathrm{Sh}^{alg}_{K,\bar{\mathbf{Q}}}, \mathbf{Z}/p^n\mathbf{Z})$ and $\mathrm{R}\Gamma_c(\mathrm{Sh}^{alg}_{K,\bar{\mathbf{Q}}}, \mathbf{Z}/p^n\mathbf{Z})$, for each place $\ell \neq p$ at which K_ℓ is hyperspecial, the local Galois representation of $G_{\mathbf{Q}_\ell}$ is unramified at ℓ and $P_\ell(\mathrm{Frob}_\ell) = 0$.*

Proof. Let $\ell \neq p$ be a place at which K_ℓ is hyperspecial. We have a natural smooth integral model $\mathrm{Sh}^{alg}_{K,\mathbf{Z}_\ell} \to \mathrm{Spec}\ \mathbf{Z}_\ell$. We first claim that $\mathrm{R}\Gamma(\mathrm{Sh}^{alg}_{K,\bar{\mathbf{Q}}_\ell}, \mathbf{Z}/p^n\mathbf{Z}) = \mathrm{R}\Gamma(\mathrm{Sh}^{alg}_{K,\bar{\mathbf{F}}_\ell}, \mathbf{Z}/p^n\mathbf{Z})$. By [LS18, Coro. 5.20][2],

$$\mathrm{R}\Gamma(\mathrm{Sh}^{alg}_{K,\bar{\mathbf{Q}}_\ell}, \mathbf{Z}/p^n\mathbf{Z}) = \mathrm{R}\Gamma(\mathrm{Sh}^{alg}_{K,\bar{\mathbf{F}}_\ell}, \mathrm{R}\Psi\mathbf{Z}/p^n\mathbf{Z}).$$

Since $\mathrm{Sh}^{alg}_{K,\mathbf{Z}_\ell} \to \mathrm{Spec}\ \mathbf{Z}_\ell$ is smooth, the map $\mathbf{Z}/p^n\mathbf{Z} \to \mathrm{R}\Psi\mathbf{Z}/p^n\mathbf{Z}$ is an isomorphism. By Poincaré duality, we deduce that

$$\mathrm{R}\Gamma_c(\mathrm{Sh}^{alg}_{K,\bar{\mathbf{Q}}_\ell}, \mathbf{Z}/p^n\mathbf{Z}) = \mathrm{R}\Gamma_c(\mathrm{Sh}^{alg}_{K,\bar{\mathbf{F}}_\ell}, \mathbf{Z}/p^n\mathbf{Z}).$$

[2] whose proof considerably simplifies in our case, due to the existence of smooth toroidal compactifications, with normal crossing boundary divisor.

We now use the Eichler–Shimura relation of [FC90, VII, Thm 4.2], to deduce that $P_\ell(\mathrm{Frob}_\ell) = 0$. It only remains to explain why it is the polynomial P_ℓ and not Q_ℓ that we need to use. This all boils down to understanding how we attach to a characteristic function of a double coset in the Hecke algebra, a Hecke correspondence. Using our conventions (which we think are standard, but are the transpose of that of [FC90]), to the double coset T_λ is associated the Hecke correspondence:

$$\begin{array}{ccc} \mathrm{Sh}^{alg}_{\lambda(\ell)^{-1}K\lambda(\ell)\cap K} & \xleftarrow[\lambda(\ell)]{} & \mathrm{Sh}^{alg}_{\lambda(\ell)K\lambda(\ell)^{-1}\cap K} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} \\ \mathrm{Sh}^{alg}_K & & \mathrm{Sh}^{alg}_K \end{array}$$

For example, for $\lambda = (-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2})$, this is the moduli space parametrizing abelian surfaces p_1^*A and p_2^*A, with certain prime-to-ℓ level structure and prime-to-ℓ polarization, together with an isogeny (compatible with level structure and polarization) $p_1^*A \to p_2^*A$ whose kernel is a maximal isotropic subgroup of $p_1^*A[\ell]$. The reduction of the natural integral model of this correspondence modulo ℓ contains the Frobenius correspondence. □

2. Lie algebra homology

2.1. **Introduction.** Let $\mathfrak{g}$ be a reductive Lie algebra over E, let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi $\mathfrak{m}$ and unipotent radical $\mathfrak{u}_\mathfrak{p}$, and let $\mathfrak{b}$ be a Borel of $\mathfrak{g}$ containing a Cartan $\mathfrak{h}$, which we assume is also contained in $\mathfrak{m}$. In this section we study the $\mathfrak{u}_\mathfrak{p}$-cohomology of objects of category $\mathcal{O}$ and of category $\hat{\mathcal{O}}$, a p-adic analytic version of the BGG category $\mathcal{O}$. The categories $\mathcal{O}$ and $\hat{\mathcal{O}}$ are equivalent, via base change from the universal enveloping algebra $U(\mathfrak{g})$ to its completion, the Fréchet–Stein algebra $\hat{U}(\mathfrak{g})$. We establish in particular the key Theorem 2.3.32, which shows that in a fixed p-adically non-Liouville weight, the operation of taking $\mathfrak{u}_\mathfrak{p}$-cohomology is compatible with completion, i.e. with passage from category $\mathcal{O}$ to category $\hat{\mathcal{O}}$.

We use the language of condensed mathematics throughout, and we begin in Section 2.2.1 with an overview of the results that we need (mostly from [RJRC22]) on solid E-vector spaces, together with a summary of some results from [Sch13a] on category $\hat{\mathcal{O}}$.

2.2. **Solid functional analysis and representations.**

2.2.1. *Solid E-vector spaces.* Rather than use the classical theory of topological vector spaces, we work throughout with the condensed mathematics of Clausen–Scholze [CS]; for the convenience of the reader, here and below we recall some of the comparisons to the classical definitions. Let E be a finite extension of $\mathbf{Q}_p$. By [CS, lecture 7], the non-archimedean field E can be viewed as a solid abelian group. It follows that E can be equipped with a structure of an analytic ring, where for any profinite set S, $E_\blacksquare[S] = E \otimes_{\mathbf{Z}} \mathbf{Z}_\blacksquare[S]$. We let $\mathrm{Mod}(E)$ be the abelian category of solid E-vector spaces; this has a tensor product, which we denote by $\otimes$, and an internal Hom, which we denote by $\underline{\mathrm{Hom}}(-,-)$. We refer to [RJRC22, §3], for a complete treatment of non-archimedean functional analysis from the condensed perspective. We simply recall what is strictly necessary for us.[3]

We have a functor $V \mapsto \underline{V}$ from topological spaces to condensed sets, where $\underline{V}$ is the condensed set defined by $\underline{V}(S) = \mathcal{C}^0(S, V)$ for any profinite set S. This functor has a left adjoint $X \mapsto X(*)_{\mathrm{top}}$ from condensed sets to topological spaces, given by evaluating a condensed set X on the point $*$ and endowing $X(*)$ with the quotient topology of the map $\coprod_{S, x \in X(S)} S \to X(*)$, where S runs through all profinite sets. The restriction of the functor $V \mapsto \underline{V}$ to the category of compactly generated topological spaces is fully faithful, and if V is compactly generated then $V = \underline{V}(*)_{\mathrm{top}}$ (more precisely, the counit $\underline{V}(*)_{\mathrm{top}} \to V$ of the adjunction restricts to the identity functor on compactly generated topological spaces, see [CS, Prop. 1.7]).

By [RJRC22, Proposition 3.7], the functor $V \mapsto \underline{V}$ restricts to a functor from the category of complete locally convex E-vector spaces to the category of solid E-vector spaces. All the complete locally convex E-vector spaces that we will encounter will be considered as solid E-vector spaces unless explicitly specified otherwise.

We introduce certain full subcategories of $\mathrm{Mod}(E)$.

Definition 2.2.2.

(1) A Banach space is a solid E-module of the form $(\lim_n(\oplus_I \mathcal{O}_E/p^n\mathcal{O}_E))[1/p]$ for some set I.
(2) A Smith space is a solid E-module which has the form $(\prod_I \mathcal{O}_E)[1/p]$ for some set I.

We let $B(E)$ be the category of Banach spaces and $S(E)$ be the category of Smith spaces.

Remark 2.2.3. The categories of solid and classical Banach spaces (resp. Smith spaces) are equivalent via the functors $V \mapsto V(*)_{\mathrm{top}}$ and $V \mapsto \underline{V}$. The essential surjectivity follows from the explicit description of the objects. The full faithfulness is a consequence of the fact that classical Banach spaces and Smith spaces are compactly generated. (See for example [RJRC22, Prop. 3.5].)

Proposition 2.2.4. [RJRC22, Lem. 3.10] *There is an anti-equivalence of categories between Smith and Banach spaces given by* $V \mapsto V^\vee := \underline{\mathrm{Hom}}(V, E)$. *Moreover,* $(V^\vee)^\vee = V$.

Remark 2.2.5. The functor $V \mapsto V^\vee$ is exact in the sense that it sends short exact sequences of Banach spaces (resp. Smith spaces) to short exact sequence of Smith spaces (resp. Banach spaces). In fact, any short exact sequence is split.

Remark 2.2.6. If V is in $B(E)$, then $V^\vee(*)_{\mathrm{top}}$ is the classical Smith space equal to the continuous dual $\mathrm{Hom}(V(*)_{\mathrm{top}}, E)$ equipped with the compact open topology.

Definition 2.2.7.

(1) A Fréchet space is a solid E-module which can be written as a sequential limit of Banach spaces.
(2) An LS-space is a solid E-module which can be written as a sequential colimit of Smith spaces with injective transition maps.

[3]In order to fix set-theoretical issues, we choose a strongly inaccessible cardinal κ and we only consider κ-small profinite sets. See [CS, Lecture 1, rem. 1.3]

(3) An LB-space is a solid E-module which can be written as a sequential colimit of Banach spaces with injective transition maps.

We let $F(E)$ be the category of Fréchet spaces, we let $LS(E)$ be the category of LS-spaces, and we let $LB(E)$ be the category of LB-spaces.

Remark 2.2.8. The categories of solid and classical Fréchet spaces are equivalent under the functors $V \mapsto V(*)_{\text{top}}$ and $V \mapsto \underline{V}$, [RJRC22, Lem. 3.24(1)]. To see this, we claim that it suffices to show that $V \mapsto \underline{V}$ is an essentially surjective functor from classical to solid Fréchet spaces. Indeed, since the counit $\underline{V}(*)_{\text{top}} \to V$ is an isomorphism (because Fréchet spaces are in particular compactly generated), we will then know that $V \mapsto \underline{V}$ is fully faithful and essentially surjective, and thus an equivalence; it follows formally from this that the unit $V \mapsto \underline{V(*)_{\text{top}}}$ of the adjunction is also an isomorphism of Fréchet spaces, as required.

Now, if $V = \lim_r V_r$ is a classical Fréchet space (where the V_r are classical Banach spaces), then by Remark 2.2.3, $\underline{V} = \lim_r \underline{V_r}$ is a solid Fréchet space (note that $V \mapsto \underline{V}$ commutes with limits, being a right adjoint). Conversely, since by definition a solid Fréchet space can be written as $V = \lim_r V_r$ where the V_r are Banach spaces, we have

$$V = \lim_r V_r \xrightarrow{\sim} \lim_r \underline{V_r(*)_{\text{top}}},$$

which gives the essential surjectivity.

Note in particular that as a consequence of this equivalence, any (solid) Fréchet space admits a presentation where $V = \lim_r V_r$ with $V_{r+1}(*)_{\text{top}} \to V_r(*)_{\text{top}}$ has dense image.

Proposition 2.2.9. [RJRC22, Thm. 3.40] *We have an anti-equivalence of categories $V \mapsto V^\vee := \underline{\text{Hom}}(V, E)$ between $F(E)$ and $LS(E)$ extending the biduality between $B(E)$ and $S(E)$. Moreover, $(V^\vee)^\vee = V$. The functor $V \mapsto V^\vee$ is exact.*

Definition 2.2.10.

(1) A map $f : V \to W$ of Smith spaces is *trace class* if there exists a map $g : E \to V^\vee \otimes W$ such that f is the composite $V \overset{\text{Id}_V \otimes g}{\to} V \otimes V^\vee \otimes W \overset{\text{ev}\otimes\text{Id}_W}{\to} W$.

(2) A map $f : V \to W$ of Banach spaces is *compact* if its dual is trace class.

Example 2.2.11. Let I be a set and let $(a_i)_{i\in I} \in E^I := \prod_{i\in I} E$ be a family converging to zero with respect to the net of the complements of finite subsets of I. Let $f : \mathcal{O}_E^I[1/p] \to \mathcal{O}_E^I[1/p]$ be the map sending $(x_i)_{i\in I}$ to $(a_i x_i)_{i\in I}$. Then one sees that f is trace class, represented by the tensor $\sum_i a_i e_i^\vee \otimes e_i$ in $(\lim_n(\oplus_I \mathcal{O}_E/p^n\mathcal{O}_E))[1/p] \otimes \mathcal{O}_E^I[1/p]$ (where e_i is i-th basis vector of $\mathcal{O}_E^I[1/p]$).

Definition 2.2.12.

(1) An object V of $LS(E)$ is of compact type if it has a presentation $V = \operatorname{colim}_n V_n$ where the maps $V_n \to V_{n+1}$ are trace class.

(2) An object V of $F(E)$ is of compact type if it has a presentation $V = \lim_n V_n$ where the maps $V_n \to V_{n-1}$ are compact.

(3) An object V of $LB(E)$ is of compact type if it has a presentation $V = \operatorname{colim}_n V_n$ where the maps $V_n \to V_{n+1}$ are compact.

Proposition 2.2.13. [RJRC22, Cor. 3.38] *A solid E-module is an LB-space of compact type if and only if it is a LS-space of compact type.*

We will use the following lemma in Remark 2.3.8.

Lemma 2.2.14. *A Smith, Banach, LB or LS-space is flat. A Fréchet space of compact type is flat over E.*

Proof. The flatness of Smith, Banach, LB or LS-spaces is [RJRC22, Lem. 3.21]. Let V be a Fréchet space of compact type. By [RJRC22, Cor. 3.38(1)], we can write $V = \lim_n V_n$ as an inverse limit of Smith spaces. Following the proof of [RJRC22, Lem. 3.21], it suffices to show that if $W' \to W$ is an injection of Smith spaces, then $W' \otimes V \to W \otimes V$ is injective.

Since we have an injection $\lim_n V_n \hookrightarrow \prod_n V_n$, and since Smith spaces are flat over E (by [RJRC22, Prop. 3.20, Lem. 3.21]), it suffices to show that $(\prod_n V_n) \otimes_E W' \to (\prod_n V_n) \otimes_E W$ is injective. For any Smith space X, we have (see [RJRC22, Prop. 3.12])

$$(\prod_n V_n) \otimes_E X = \prod_n (V_n \otimes_E X),$$

so it suffices in turn to show that $\prod_n (V_n \otimes_E W') \to \prod_n (V_n \otimes_E W)$ is injective. Since the Smith spaces V_n are flat, each morphism $V_n \otimes_E W' \to V_n \otimes_E W$ is injective, and we are done. ☐

2.2.15. *Representations of algebraic groups.* In this section we recall the classical notion of representation of an algebraic group, before moving to representations of analytic groups. We let $\mathrm{Mod}^{\delta}(E)$ be the usual category of E-vector spaces (the superscript δ stands for discrete). Let $G = \operatorname{Spec} \mathcal{O}_G$ be an affine group scheme over $\operatorname{Spec} E$. The algebra $\mathcal{O}_G$ is a Hopf algebra with comultiplication $\mu : \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$ and augmentation $e : \mathcal{O}_G \to E$. We let $\mathrm{Mod}^{\delta}_G(E)$ be the category of algebraic representations of G. Its objects are vector spaces V over E, equipped with a co-action map $c : V \to V \otimes \mathcal{O}_G$ such that:

(1) (associativity) The maps $(c \otimes \mathrm{Id}) \circ c$ and $(\mathrm{Id} \otimes \mu) \circ c$: $V \to V \otimes \mathcal{O}_G \to V \otimes \mathcal{O}_G \otimes \mathcal{O}_G$ agree.
(2) (neutral element) The map $(\mathrm{Id} \otimes e) \circ c$: $V \to V \otimes \mathcal{O}_G \to V$ is the identity.

2.2.16. *Representations of analytic groups.* We recall the following standard definition.

Definition 2.2.17. An adic space $\mathcal{X}$ is called quasi-Stein if it has an open cover given by an increasing countable union of affinoid spaces of finite type $\mathcal{X} = \cup_n \mathcal{X}_n$ where $H^0(\mathcal{X}_{n+1}, \mathcal{O}_{\mathcal{X}_{n+1}}) \to H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ has dense image. A quasi-Stein space is Stein if it admits a covering as before having the property that $\mathcal{X}_n$ is relatively compact in $\mathcal{X}_{n+1}$ ([L9̈0, 2.4]); equivalently, if the closure $\overline{\mathcal{X}_n}$ of $\mathcal{X}_n$ in $\mathcal{X}_{n+1}$ is proper over $\mathrm{Spa}(E, \mathcal{O}_E)$.

We now let G be a Stein analytic group over $\mathrm{Spa}(E, \mathcal{O}_E)$.

Remark 2.2.18. We have two cases in mind: either G is the analytification of an affine group scheme over $\operatorname{Spec} E$, or G is a quasi-compact affinoid open subgroup of such an analytification.

We let $\mathcal{O}_G$ be the algebra of functions on G, which is an object of $\mathrm{Mod}(E)$ (it is a Fréchet space). It has a structure of a Hopf algebra. We define the category $\mathrm{Mod}_G(E)$ of representations of G on solid E-vector spaces. Its objects are solid vector spaces equipped with a co-action map $c : V \to V \otimes \mathcal{O}_G$ satisfying the same conditions as before. Similarly, we let $B_G(E)$ be the category of representations of

G on Banach modules. We let $LB_G(E)$ be the category of representations of G on LB-spaces.

We let $D(G) = \underline{\mathrm{Hom}}(\mathcal{O}_G, E) = \mathcal{O}_G^\vee$ be the distribution algebra of G. The dual of the comultiplication $\mu : \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$ induces the algebra structure on $D(G)$. If V is an object of $\mathrm{Mod}_G(E)$, then it is naturally a $D(G)$-module (via $V \otimes D(G) \xrightarrow{c} V \otimes \mathcal{O}_G \otimes D(G) \to V$). We therefore have a natural functor $\mathrm{Mod}_G(E) \to \mathrm{Mod}(D(G))$ from the category of solid G-representations to the category of solid $D(G)$-modules.

Remark 2.2.19. In some cases, one can go backwards. For example if G is quasi-compact and V is a Banach space, we have that $\underline{\mathrm{Hom}}(D(G), V) = \mathcal{O}_G \otimes V$ by [RJRC22, Cor. 3.17] so that any $D(G)$-module structure on V can be turned into an action of G on V. If we denote by $B(D(G))$ the category of $D(G)$-modules which are Banach spaces, then the categories $B_G(E)$ and $B(D(G))$ are equivalent.

2.2.20. *Representations of locally profinite groups.* We now let M be a locally profinite group. We view M as a condensed group. We let $E[M]$ be the associated condensed ring and we let $E_{\blacksquare}[M]$ be its solidification. If M is compact, then $E_{\blacksquare}[M] = (\lim_N \mathcal{O}_E[M/N][1/p]$ where N runs through the compact open subgroups of M. In general, if $M_0 \subseteq M$ is a compact open subgroup, then we have the formula $E_{\blacksquare}[M] = E[M] \otimes_{E[M_0]} E_{\blacksquare}[M_0]$.

Definition 2.2.21. A representation of M over a solid E-vector space is a solid $E_{\blacksquare}[M]$-module. The category of M-representations is denoted by $\mathrm{Mod}_M(E)$.

Remark 2.2.22. Equivalently, a representation of M is a solid E-vector space V and an action map $M \times V \to V$ of condensed sets satisfying the usual group action axioms.

2.2.23. *Smooth representations.* Let $V \in \mathrm{Mod}_M(E)$. We let $V^{\mathrm{sm}} = \mathrm{colim}_{N \subseteq M} V^N$ where N runs through all compact open subgroups of M. We say that V is smooth if the natural map $V^{\mathrm{sm}} \to V$ is an isomorphism. We let $\mathrm{Mod}_M^{\mathrm{sm}}(E)$ be the category of smooth representations.

We let M_{disc} be the group M equipped with the discrete topology. There is a natural map $M_{\mathrm{disc}} \to M$ of condensed sets. One can consider the category $\mathrm{Mod}_{M_{\mathrm{disc}}}(E)$ of $E[M_{\mathrm{disc}}]$-modules. We can define the subcategory $\mathrm{Mod}^{\mathrm{sm}}_{M_{\mathrm{disc}}}(E)$ of smooth representations of M_{disc}. Its objects are representations V of M_{disc} such that $V = \mathrm{colim}\, V^{N_{\mathrm{disc}}}$ where N goes through all compact open subgroups of M.

Lemma 2.2.24. *The categories* $\mathrm{Mod}_M^{\mathrm{sm}}(E)$ *and* $\mathrm{Mod}^{\mathrm{sm}}_{M_{\mathrm{disc}}}(E)$ *are equivalent.*

Proof. We have a natural functor $\mathrm{Mod}_M(E) \to \mathrm{Mod}_{M_{\mathrm{disc}}}(E)$, induced by the map $M_{\mathrm{disc}} \to M$. This induces a functor $\mathrm{Mod}_M^{\mathrm{sm}}(E) \to \mathrm{Mod}^{\mathrm{sm}}_{M_{\mathrm{disc}}}(E)$. We now construct a functor $\mathrm{Mod}^{\mathrm{sm}}_{M_{\mathrm{disc}}}(E) \to \mathrm{Mod}_M^{\mathrm{sm}}(E)$. Let $V \in \mathrm{Mod}^{\mathrm{sm}}_{M_{\mathrm{disc}}}(E)$. Let M_0 be a compact open subgroup of M and let M_n be a system of normal compact open subgroups of M_0. We see that V^{M_n} is an $E[(M_0)_{\mathrm{disc}}/(M_n)_{\mathrm{disc}}]$-module. Since $(M_0)_{\mathrm{disc}}/(M_n)_{\mathrm{disc}} = M_0/M_n$, we deduce that the $(M_0)_{\mathrm{disc}}$-module structure on V^{M_n} extends uniquely to an M_0-module structure. Passing to the colimit over n, we deduce that V is an M_0-module. Since $E[M] = E[M_{\mathrm{disc}}] \otimes_{E[(M_0)_{\mathrm{disc}}]} E[M_0]$, we are done. □

Recall that an abelian category is an *Grothendieck abelian category* if it has arbitrary colimits, it has a generator, and filtered colimits are exact (AB5). By [Sta13, Tag 079H], any Grothendieck abelian category has enough injectives. We will now show that $\mathrm{Mod}_M^{\mathrm{sm}}(E)$ is a Grothendieck abelian category; note that this relies on our set-theoretic assumption that we only consider κ-small profinite sets for some fixed κ.

Lemma 2.2.25. *The category* $\mathrm{Mod}_M^{\mathrm{sm}}(E)$ *is a Grothendieck abelian category, and in particular it has enough injectives.*

Proof. This is obvious, except for the existence of a generator. For a totally disconnected S and compact open subgroup $N \subseteq M$, we consider $V_{S,N} = E_{\blacksquare}[M] \otimes_{E_{\blacksquare}[N]} E_{\blacksquare}[S]$ with N acting trivially on $E_{\blacksquare}[S]$. We claim that $\oplus_{S,N} V_{S,N}$ is a generator. This follows from the property that for any $V \in \mathrm{Mod}_M^{\mathrm{sm}}(E)$, $\mathrm{Hom}_{\mathrm{Mod}_M^{\mathrm{sm}}(E)}(V_{S,N}, V) = V^N(S)$. □

Here is a slight generalization of the concept of smooth. Let $\lambda : M \to E^\times$ be a character, and write $E(-\lambda)$ for the corresponding representation of M (with underlying vector space E).

Definition 2.2.26. We say that V is λ-smooth if $V \otimes E(-\lambda)$ is smooth. We let $\mathrm{Mod}_M^{\lambda-\mathrm{sm}}(E)$ be the category of λ-smooth M-modules.

Note that if λ, λ' are two characters such that $\lambda \otimes (\lambda')^{-1}$ is smooth, then the categories $\mathrm{Mod}_M^{\lambda-\mathrm{sm}}(E)$ and $\mathrm{Mod}_M^{\lambda'-\mathrm{sm}}(E)$ are canonically equivalent.

2.2.27. *Locally analytic representations.* We now assume that M arises as the set of $\mathbf{Q}_p$-points of an analytic group G over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and we also assume that we have a fundamental system of quasi-compact open subgroups $\{G_r\}_{r \geq 0}$ of G, where G_r is a polydisc. We let $G_r(\mathbf{Q}_p) = M_r$. The $\{M_r\}$ form a fundamental system of compact open subgroups in M. We now define the locally analytic vectors of $V \in \mathrm{Mod}_M(E)$. Note that $\mathcal{O}_{G_r} \otimes_E V$ has three commuting left actions of M_r: $*_l$, $*_r$ and $*_V$ (induced respectively by left translation on the group, right translation on the group, and the original action on V). The action $*_V$ comes from an action of M. Moreover, the group M acts by conjugations $*_{l,r}$ on its system of neighborhoods of identity $\{M_r\}$. We set $V^{M_r-\mathrm{an}} = H^0(M_r, (\mathcal{O}_{G_r} \otimes V))$, where the invariants are taken for the action $*_l \otimes *_V$. The space $V^{M_r-\mathrm{an}}$ still carries a $*_{l,r} \otimes *_V$-action of M_r. The evaluation map at e, $\mathcal{O}_{G_{r'}} \otimes V \to V$ induces an injective map $V^{M_r-\mathrm{an}} \to V$. We let $V^{\mathrm{la}} = \mathrm{colim}_r\, V^{M_r-\mathrm{an}}$. This is an M-representation. We thus have inclusions $V^{\mathrm{sm}} \to V^{\mathrm{la}} \to V$.

Remark 2.2.28. The functor $V \mapsto V^{\mathrm{la}}$ can naturally be derived into a functor $V \mapsto V^{\mathrm{Rla}}$. See [RJRC22, sect. 4.4].

2.2.29. *The algebra* $\hat{U}(\mathfrak{g})$. Let us assume now that we have an algebraic group $G^{\mathrm{alg}} \to \mathrm{Spec}\, \mathcal{O}_E$. Its analytification defines a quasi-compact affinoid analytic group $G = G_0 \to \mathrm{Spa}(E, \mathcal{O}_E)$. For any $r \in \mathbf{Q}_{\geq 0}$, we let G_r be the quasi-compact analytic subgroup of G_0 of elements reducing to the identity e modulo p^r. We have that $\mathcal{O}_{G,e} = \mathrm{colim}\, \mathcal{O}_{G_r}$ is an LB-space of compact type. We let $\mathfrak{g}$ be the Lie-algebra of G. We define $\hat{U}(\mathfrak{g}) = \mathcal{O}_{G,e}^\vee$. This is a Fréchet space of compact type.

Since the categories of solid and classical Fréchet spaces are equivalent, we will freely write $\hat{U}(\mathfrak{g})$ for the underlying classical E-algebra $\hat{U}(\mathfrak{g})(*)_{\mathrm{top}}$ of the solid E-algebra $\hat{U}(\mathfrak{g})$. We have a natural map $\mathfrak{g} \to \hat{U}(\mathfrak{g})$ given by $X \mapsto [f(g) \mapsto f'(g\exp(-tX))|_{t=0}]$, which extends to a map from the enveloping algebra $U(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$ with dense image.

One can describe $\hat{U}(\mathfrak{g})$ as a completion of $U(\mathfrak{g})$ as follows (following [ST02, lemma 2.4] and [Sch13a, sect. 3.2]). If we fix a basis $x_1, \cdots, x_n$ of $\mathfrak{g}$, then $U(\mathfrak{g}) = \oplus_{(n_i)} E \prod x_i^{n_i}$ by the PBW theorem. For each $r \in \mathbf{R}_{>0}$, we define a norm $| \, |_r$ on $U(\mathfrak{g})$ by putting $|\sum a_{\underline{n}} x^{\underline{n}}|_r = \sup_{\underline{n}} |a_{\underline{n}}| r^{\sum n_i}$. We let $U(\mathfrak{g})_r$ be the completion of $U(\mathfrak{g})$ for $| - |_r$ and we have $\hat{U}(\mathfrak{g}) = \lim_{r \geq 1} U(\mathfrak{g})_r$.

Remark 2.2.30. For any r, there exists r' such that $U(\mathfrak{g})_{r'} \to D(G_r)$ and $D(G_{r'}) \to U(\mathfrak{g})_r$. We thus have two presentations of $\hat{U}(\mathfrak{g}) = \lim_r U(\mathfrak{g})_r = \lim_r D(G_r)$, as an inverse limit of Banach spaces with compact transition maps and as an inverse limit of Smith spaces with trace class transition maps.

Since $\hat{U}(\mathfrak{g})$ is a Fréchet–Stein algebra (see [ST03, sect. 3]), there is an associated abelian category of coadmissible modules $\mathrm{Mod}^{\mathrm{coad}}(\hat{U}(\mathfrak{g}))$, which is defined as follows.

Definition 2.2.31. A (left) $\hat{U}(\mathfrak{g})$-module M is coadmissible if it has a presentation $M = \lim M_r$ where M_r is a finitely generated $\hat{U}(\mathfrak{g})_r$-module and $M_{r+1} \otimes_{\hat{U}(\mathfrak{g})_{r+1}} \hat{U}(\mathfrak{g})_r = M_r$.

We let $\mathrm{Mod}(\hat{U}(\mathfrak{g}))$ be the category of solid $\hat{U}(\mathfrak{g})$-modules.

Theorem 2.2.32. *We have a fully faithful exact functor* $\mathrm{Mod}^{\mathrm{coad}}(\hat{U}(\mathfrak{g})) \to \mathrm{Mod}(\hat{U}(\mathfrak{g}))$.

Proof. By for example [Sch13a, Prop. 3.1.1], any coadmissible module is canonically an object of $F(E)$ of compact type. □

Definition 2.2.33. An *admissible module* is the dual of a coadmissible module. Admissible modules are objects of $LB(E)$ of compact type.

2.2.34. *Categories of $\hat{U}(\mathfrak{g})$ and $U(\mathfrak{g})$-modules.* We recall the the maps (of classical rings) $U(\mathfrak{g}) \to U(\mathfrak{g})_r$ and $U(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$ are flat (see for example [Sch13a, Thm. 4.3.3]). Let $\mathrm{Mod}^{\mathrm{fg}}(U(\mathfrak{g}))$ be the category of finitely generated left $U(\mathfrak{g})$-modules.

Proposition 2.2.35. *We have an exact functor:*

$$\begin{array}{rcl} \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} - : \mathrm{Mod}^{\mathrm{fg}}(U(\mathfrak{g})) & \to & \mathrm{Mod}^{\mathrm{coad}}(\hat{U}(\mathfrak{g})) \\ M & \mapsto & \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M \end{array}$$

Proof. This follows from the flatness of $U(\mathfrak{g}) \to \hat{U}(\mathfrak{g})$. □

Corollary 2.2.36. *We have an exact functor:*

$$\begin{array}{rcl} \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} - : \mathrm{Mod}^{\mathrm{fg}}(U(\mathfrak{g})) & \to & \mathrm{Mod}(\hat{U}(\mathfrak{g})) \\ M & \mapsto & \widehat{M} := \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M \end{array}$$

Proof. By combining Theorem 2.2.32 and Proposition 2.2.35, we obtain an exact functor $\mathrm{Mod}^{\mathrm{fg}}(U(\mathfrak{g})) \to \mathrm{Mod}(\hat{U}(\mathfrak{g}))$. □

2.2.37. *Category $\mathcal{O}$ and category $\hat{\mathcal{O}}$.* We assume that $\mathfrak{g}$ is a reductive Lie algebra with Borel $\mathfrak{b}$ and Cartan $\mathfrak{h}$, and as usual we write Φ^+ (resp. Φ^-) for the positive (resp. negative) roots determined by our fixed Borel subgroup $\mathfrak{b}$. We can consider the abelian category $\mathcal{O}(\mathfrak{g}, \mathfrak{b})$ (simply denoted $\mathcal{O}$ if $\mathfrak{g}$ and $\mathfrak{b}$ are clear from the context) whose objects are finitely generated left $U(\mathfrak{g})$-modules for which the $\mathfrak{b}$-action is locally finite and the $\mathfrak{h}$-action is semi-simple (see [Hum08], chapter 1).

Definition 2.2.38. Following [Sch13a, Defn. 3.6.2], we let $\hat{\mathcal{O}}$ be the category whose objects are coadmissible $\hat{U}(\mathfrak{g})$-modules M for which the action of $\hat{U}(\mathfrak{h})$ is diagonalizable and the following properties hold:

1. (1) All weights of M are contained in finitely many subsets of the form $\lambda + \mathbf{N}[\Phi^-]$, and
2. (2) all weight spaces of M are finite dimensional.

Theorem 2.2.39 ([Sch13a], Thm. 4.3.1)**.** *We have an equivalence of categories:*

$$\begin{array}{rcl} \mathcal{O} & \to & \hat{\mathcal{O}} \\ M & \mapsto & \hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M. \end{array}$$

A quasi-inverse to this functor is given by the functor: $M \mapsto M^{\mathrm{ss}}$, which takes M to the direct sum of its weight spaces.

2.3. **Lie algebra cohomology and homology.**

2.3.1. *Definitions.* Recall that $\mathrm{Mod}^{\delta}(E)$ is the usual category of E-vector spaces, and $\mathrm{Mod}(E)$ is the category of solid E-vector spaces. We let $D(\mathrm{Mod}^{\delta}(E))$ be the derived category of $\mathrm{Mod}^{\delta}(E)$, and we let $D(\mathrm{Mod}(E))$ be the derived category of $\mathrm{Mod}(E)$. Let $\mathfrak{g}$ be a Lie algebra (not necessarily reductive) with enveloping Lie algebra $U(\mathfrak{g})$. We let $\mathrm{Mod}(U(\mathfrak{g}))$ be the category of (discrete) $U(\mathfrak{g})$-modules, and let $D(\mathrm{Mod}(U(\mathfrak{g}))$ be its derived category.

We have a functor "homology of $\mathfrak{g}$":

$$\begin{array}{rcl} E \otimes^L_{U(\mathfrak{g})} - : D(\mathrm{Mod}(U(\mathfrak{g}))) & \to & D(\mathrm{Mod}^{\delta}(E)) \\ M & \mapsto & E \otimes^L_{U(\mathfrak{g})} M \end{array}$$

We let $H_i(\mathfrak{g}, M) := H^{-i}(E \otimes^L_{U(\mathfrak{g})} M)$.

We also have a functor "cohomology of $\mathfrak{g}$":

$$\begin{array}{rcl} \mathrm{R}\Gamma(\mathfrak{g}, -) : D(\mathrm{Mod}(U(\mathfrak{g}))) & \to & D(\mathrm{Mod}^{\delta}(E)) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{g}}(E, M) \end{array}$$

These functors can be computed by taking the Chevalley–Eilenberg resolution $CE(E)$ of E, in cohomological degrees $[-d, 0]$ with $d = \dim(\mathfrak{g})$ (see [Wei94, sect. 7]):

$$0 \to U(\mathfrak{g}) \otimes \Lambda^d \mathfrak{g} \to \cdots \to U(\mathfrak{g}) \to 0.$$

We can also define functors:

$$\begin{array}{rcl} E \otimes^L_{U(\mathfrak{g})} - : D(\mathrm{Mod}(\hat{U}(\mathfrak{g}))) & \to & D(\mathrm{Mod}(E)) \\ M & \mapsto & E \otimes^L_{U(\mathfrak{g})} M \end{array}$$

$$\begin{array}{rcl} \mathrm{R}\Gamma(\mathfrak{g}, -) : D(\mathrm{Mod}(\hat{U}(\mathfrak{g}))) & \to & D(\mathrm{Mod}(E)) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{g}}(E, M) \end{array}$$

These functors can also be computed by taking the solid Chevalley–Eilenberg resolution $\hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} CE(E)$ of E (which remains a resolution of E by Corollary 2.2.36):

$$0 \to \hat{U}(\mathfrak{g}) \otimes \Lambda^d \mathfrak{g} \to \cdots \to \hat{U}(\mathfrak{g}) \to 0.$$

We sometimes write respectively $\mathrm{RHom}_{U(\mathfrak{g})}$ or $\mathrm{RHom}_{\hat{U}(\mathfrak{g})}$ in place of $\mathrm{RHom}_{\mathfrak{g}}$.

Remark 2.3.2. We have the following trivial relation between homology and cohomology: $\underline{\mathrm{RHom}}(E \otimes^L_{U(\mathfrak{g})} M, E) = \mathrm{RHom}_{U(\mathfrak{g})}(E, \underline{\mathrm{RHom}}(M, E))$. We also have the following relation between homology and cohomology ([Haz70]):

$$E \otimes^L_{U(\mathfrak{g})} (M[-d] \otimes_E \Lambda^d \mathfrak{g}^\vee) = \mathrm{RHom}_{U(\mathfrak{g})}(E, M). \tag{2.3.3}$$

We have the following well-known lemma.

Lemma 2.3.4. *$\mathcal{O}_{G,e}$ is an acyclic $\hat{U}(\mathfrak{g})$-module.*

Proof. This is a simple consequence of the standard relationship between the Chevalley–Eilenberg resolution and the de Rham complex, *cf.* [RJRC22, Prop. 5.12]. □

2.3.5. *Homology and cohomology of $\mathfrak{g}$ and $\mathfrak{m}$.* For the rest of this section we put ourselves in the situation of Section 1.8.4, so that in particular $\mathfrak{g}$ is a reductive Lie algebra with Cartan and Borel $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$, and $\mathfrak{p} \supseteq \mathfrak{b}$ is a standard parabolic with Levi $\mathfrak{m} \supseteq \mathfrak{h}$. We let $d = \dim \mathfrak{u}_{\mathfrak{p}}$. From now on until the end of Section 2.3, we fix a $w \in {}^M W$ and consider the conjugates $\mathfrak{p}_w, \mathfrak{m}_w, \mathfrak{u}_{\mathfrak{p}_w}, \mathfrak{b}_{\mathfrak{m}_w}$. We note that because $w \in {}^M W$, $\mathfrak{b}_{\mathfrak{m}_w} = \mathfrak{b} \cap \mathfrak{m}_w$.

We can define the homology functor of $\mathfrak{u}_{\mathfrak{p}_w}$:

$$\begin{array}{rcl} E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} - : D^-(\mathrm{Mod}(U(\mathfrak{g}))) & \to & D^-(\mathrm{Mod}(U(\mathfrak{m}_w))) \\ M & \mapsto & E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M \end{array}$$

Remark 2.3.6. This functor is defined by taking a projective resolution of M as a $U(\mathfrak{p}_w)$-module. By the PBW theorem, $U(\mathfrak{p}_w)$ is free over $U(\mathfrak{u}_{\mathfrak{p}_w})$, and so this is also a projective resolution of M as a $U(\mathfrak{u}_{\mathfrak{p}_w})$-module. We also have a natural functor $D(\mathrm{Mod}(U(\mathfrak{m}_w))) \to D(\mathrm{Mod}(U(\mathfrak{p}_w)))$. If we resolve E via the Chevalley–Eilenberg resolution

$$U(\mathfrak{u}_{\mathfrak{p}_w}) \otimes \Lambda^i \mathfrak{u}_{\mathfrak{p}_w} \to \cdots \to U(\mathfrak{u}_{\mathfrak{p}_w}) \to E$$

then we get a complex which computes $E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M$ in $D(\mathrm{Mod}(U(\mathfrak{p}_w)))$ (but on the cohomology groups, the action of $\mathfrak{p}_w$ factors through an action of $\mathfrak{m}_w$).

We similarly have a cohomology functor of $\mathfrak{u}_{\mathfrak{p}_w}$:

$$\begin{array}{rcl} \mathrm{R}\Gamma(\mathfrak{u}_{\mathfrak{p}_w}, -) : D^+(\mathrm{Mod}(U(\mathfrak{g}))) & \to & D^+(\mathrm{Mod}(U(\mathfrak{m}_w))) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{u}_{\mathfrak{p}_w}}(E, M) \end{array}$$

Remark 2.3.7. Similarly to Remark 2.3.6, this functor is obtained by taking an injective resolution of M as a $U(\mathfrak{p}_w)$-module. If one uses the Chevalley–Eilenberg resolution of E instead, then we obtain a complex which computes the composition of this functor with the natural functor $D(\mathrm{Mod}(U(\mathfrak{m}_w))) \to D(\mathrm{Mod}(U(\mathfrak{p}_w)))$.

We can also define functors:

$$\begin{array}{rcl} E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} - : D^-(\mathrm{Mod}(\hat{U}(\mathfrak{g}))) & \to & D^-(\mathrm{Mod}(\hat{U}(\mathfrak{m}_w))) \\ M & \mapsto & E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M \end{array}$$

$$\begin{array}{rcl} \mathrm{R}\Gamma(\mathfrak{u}_{\mathfrak{p}_w}, -) : D^+(\mathrm{Mod}(\hat{U}(\mathfrak{g}))) & \to & D^+(\mathrm{Mod}(\hat{U}(\mathfrak{m}_w))) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{u}_{\mathfrak{p}_w}}(E, M) \end{array}$$

Remark 2.3.8. Following Remarks 2.3.6 and 2.3.7, these functors are well defined because $\hat{U}(\mathfrak{p}_w)$ is flat over $\hat{U}(\mathfrak{u}_{\mathfrak{p}_w})$ so that $\hat{U}(\mathfrak{m}_w) = \hat{U}(\mathfrak{p}_w) \otimes^L_{\hat{U}(\mathfrak{u}_{\mathfrak{p}_w})} E$. In order to see the flatness, by the PBW theorem and the description of $\hat{U}(\mathfrak{p}_w)$ given in section 2.2.29, we find that $\hat{U}(\mathfrak{p}_w) = \hat{U}(\mathfrak{u}_{\mathfrak{p}_w}) \otimes_E \hat{U}(\mathfrak{m}_w)$ and so it remains to note that the Fréchet space of compact type $\hat{U}(\mathfrak{m}_w)$ is flat over E by Lemma 2.2.14.

2.3.9. *Finiteness of the algebraic cohomology.* Let $Z(\mathfrak{g})$ and $Z(\mathfrak{m}_w)$ denote the centres of $U(\mathfrak{g})$ and $U(\mathfrak{m}_w)$ respectively, and let W and $W_{\mathfrak{m}_w}$ be the Weyl groups of $\mathfrak{g}$ and $\mathfrak{m}_w$. If M is a $U(\mathfrak{g})$-module on which $Z(\mathfrak{g})$ acts via a character χ, then we say that χ is the *infinitesimal character* of M. Recall the Harish-Chandra isomorphism $HC_{\mathfrak{g}} : Z(\mathfrak{g}) \to U(\mathfrak{h})^{W,\cdot}$ (where the target is the invariants for the dotted action of W), determined by the property that $z \otimes 1 = 1 \otimes HC_{\mathfrak{g}}(z)$ in $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} U(\mathfrak{h})$. Using this isomorphism, any character $\lambda : Z(\mathfrak{g}) \to E$ is identified with an element of $\mathfrak{h}^\vee$, well defined up to the dotted action of W.

Remark 2.3.10. Let $\iota : Z(\mathfrak{g}) \to Z(\mathfrak{g})$ be the map induced by the inverse map on $\mathfrak{g}$, $X \mapsto -X$. We have $HC \circ \iota = -w_0 \circ HC$. If $\lambda \in \mathfrak{h}^\vee$ represents a character of $Z(\mathfrak{g})$, $-w_0\lambda$ represents the character $\lambda \circ \iota$.

We similarly have a natural Harish-Chandra isomorphism $HC_{\mathfrak{m}_w} : Z(\mathfrak{m}_w) \to U(\mathfrak{h})^{W_{\mathfrak{m}_w},\cdot}$, and we deduce that there is a natural Harish-Chandra map

$$HC : Z(\mathfrak{g}) \to Z(\mathfrak{m}_w). \tag{2.3.11}$$

This map is characterized by the property that in $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_w)} U(\mathfrak{m}_w)$, we have $z \otimes 1 = 1 \otimes HC(z)$.

We also have a map from $Z(\mathfrak{g})$ (resp. $Z(\mathfrak{m}_w)$) to the centre of the derived categories $D(\mathrm{Mod}(U(\mathfrak{g})))$ (resp. $D(U(\mathfrak{m}_w))$) (i.e. the t-centre in the sense of [Mil14]). The following is known as the Casselman–Osborne theorem.

Theorem 2.3.12 ([CO75], [Mil14])**.** *The functor* $\mathrm{R}\Gamma(\mathfrak{u}_{\mathfrak{p}_w}, -) : D^+(\mathrm{Mod}(U(\mathfrak{g}))) \to D^+(\mathrm{Mod}(U(\mathfrak{m}_w)))$ *is* $Z(\mathfrak{g})$*-homogeneous, in the sense that for* $z \in Z(\mathfrak{g})$*, we have*

$$\mathrm{R}\Gamma(\mathfrak{u}_{\mathfrak{p}_w}, z) = HC(z).$$

In particular, if M *is a* $\mathfrak{g}$*-module with infinitesimal character* $\lambda \in \mathfrak{h}^\vee$*, then* $H^i(\mathfrak{u}_{\mathfrak{p}_w}, M)$ *is a* $Z(\mathfrak{m}_w) \otimes_{Z(\mathfrak{g})} \lambda$*-module.*

Theorem 2.3.13. *Let* $M \in \mathcal{O}(\mathfrak{g}, \mathfrak{b})$*. Then* $H^i(\mathfrak{u}_{\mathfrak{p}_w}, M) \in \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$*.*

Proof. Using the Chevalley–Eilenberg resolution, the cohomology is computed by the complex $0 \to M \to M \otimes \mathfrak{u}_{\mathfrak{p}_w}^\vee \to \cdots$. We see that all modules occurring in this complex have locally nilpotent action of $\mathfrak{u} \cap \mathfrak{m}_w$ (the unipotent radical of $\mathfrak{b}_{\mathfrak{m}_w}$) and semi-simple $\mathfrak{h}$-action, and furthermore each $\mathfrak{h}$-eigenspace has finite dimension.

This also holds for the cohomology groups. It follows that the cohomology groups admit a (possibly infinite) increasing filtration where each graded is a simple object in category $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$. Indeed, if a cohomology group is non-zero, we can find a highest weight vector since $\mathfrak{u} \cap \mathfrak{m}_w$ acts locally nilpotently, so we get a map from a Verma module. We can repeat the process with the quotient. Since all simple objects of category $\mathcal{O}$ have a generalized infinitesimal character, it follows from the Casselman–Osborne Theorem 2.3.12 that there are only a finite number of possible infinitesimal characters of the simple subquotients, and therefore only finitely many possible highest weight vectors of all irreducible subquotients. Since $\mathfrak{h}$ acts semi-simply with finite-dimensional eigenspaces, we deduce that the filtration is finite and that the cohomology groups belong to $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$, as required. □

We write $D^{\mathrm{perf}}(U(\mathfrak{m}_w))$ for the category of perfect complexes of $U(\mathfrak{m}_w)$-modules. Since $U(\mathfrak{m}_w)$ is Noetherian and has finite global dimension (see e.g. [Wei94, Ex. 7.7.2]), these are equivalently the complexes whose cohomologies are finitely generated $U(\mathfrak{m}_w)$-modules, and are nonzero in only finitely many degrees.

Corollary 2.3.14. *If $M \in \mathcal{O}(\mathfrak{g}, \mathfrak{b})$, then $\mathrm{R}\Gamma(\mathfrak{u}_{\mathfrak{p}_w}, M)$ and $E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M$ belong to $D^{\mathrm{perf}}(U(\mathfrak{m}_w))$, and their cohomologies belong to $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$.*

Proof. This is immediate from Theorem 2.3.13 and Remark 2.3.2. □

2.3.15. *Cohomology of Verma modules.*

Definition 2.3.16. For any $\lambda \in \mathfrak{h}^\vee$, we write $M_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$ for the corresponding Verma module for $\mathfrak{g}$, and $M(\mathfrak{m}_w)_\lambda := U(\mathfrak{m}_w) \otimes_{U(\mathfrak{b}_{\mathfrak{m}_w})} \lambda$ for the Verma module for $\mathfrak{m}_w$. We write $L(\mathfrak{m}_w)_\lambda$ for the simple object in $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$ with highest weight λ.

Definition 2.3.17. Let $\lambda \in \mathfrak{h}^\vee$. We let $W_{<\lambda}$ be the subset of W consisting of elements w' which satisfy: $w' \cdot \lambda = \lambda - \sum_{\alpha \in \Phi^+} n_\alpha \alpha$, where $n_\alpha \in \mathbf{Z}_{\geq 0}$, and $n_\alpha > 0$ for some α.

Remark 2.3.18. If $\lambda \in \mathfrak{h}^\vee$ is such that $w' \cdot \lambda - \lambda \notin \mathbf{Z}\Phi$ for $w' \neq 1$ (a generic condition on λ), then $W_{<\lambda} = \emptyset$.

Theorem 2.3.19. *Let $\lambda \in \mathfrak{h}^\vee$, and let $w \in {}^M W$. Assume that $\mathfrak{u}_{\mathfrak{p}}$ is abelian.*

1. *The groups $H_i(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda)$ belong to the category $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$.*
2. *These homology groups vanish if $i > d - \ell(w)$.*
3. *There is an injective "highest weight" map*
$$M(\mathfrak{m}_w)_{w^{-1}(w \cdot \lambda + 2\rho^M)} \hookrightarrow H_{d-\ell(w)}(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda).$$
4. *The cokernel of the map $M(\mathfrak{m}_w)_{w^{-1}(w \cdot \lambda + 2\rho^M)} \hookrightarrow H_{d-\ell(w)}(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda)$ and the homology groups $H_i(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda)$ for $i < d - \ell(w)$ have Jordan– Hölder factors among the $L(\mathfrak{m}_w)_{w^{-1}(w' \cdot \lambda + 2\rho^M)}$ with $w' \in wW_{<\lambda}$.*

Remark 2.3.20. In particular, if $W_{<\lambda} = \emptyset$, then $H_i(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda)$ is concentrated in degree $d - \ell(w)$ and $H_{d-\ell(w)}(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda) = M(\mathfrak{m}_w)_{w^{-1}(w \cdot \lambda + 2\rho^M)}$.

Proof of Theorem 2.3.19. By Corollary 2.3.14 , the homologies belong to the category $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$. Since M_λ is free as a $U(\bar{\mathfrak{u}})$-module and so also as a $U(\mathfrak{u}_{\mathfrak{p}_w} \cap \bar{\mathfrak{u}})$-module, there is an isomorphism

$$H_*(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda) = H_*(\mathfrak{u}_{\mathfrak{p}_w} \cap \mathfrak{u}, H_0(\mathfrak{u}_{\mathfrak{p}_w} \cap \bar{\mathfrak{u}}, M_\lambda)). \tag{2.3.21}$$

(Here we used that $\mathfrak{u}_{\mathfrak{p}_w}$ is abelian, since $\mathfrak{u}$ is abelian by assumption.) Write $N := H_0(\mathfrak{u}_{\mathfrak{p}_w} \cap \bar{\mathfrak{u}}, M_\lambda)$. The homology (2.3.21) is computed by the Chevalley–Eilenberg complex:

$$0 \to \wedge^{d-\ell(w)}(\mathfrak{u}_{\mathfrak{p}_w} \cap \mathfrak{u}) \otimes N \to \cdots \to N \to 0$$

and so in particular vanishes in degree bigger than $d-\ell(w) = \dim(\mathfrak{u}_{\mathfrak{p}_w} \cap \mathfrak{u})$. Moreover the highest weight occurring in this complex is

$$\lambda + w^{-1}w_{0,M}\rho + \rho = w^{-1}(w \cdot \lambda + 2\rho^M) \tag{2.3.22}$$

(the equality holding because we have assumed that $w \in {}^M W$), which occurs exactly once in $\wedge^{d-\ell(w)}(\mathfrak{u}_{P_w} \cap \mathfrak{b}) \otimes N$. It follows that there is a natural map:

$$M(\mathfrak{m}_w)_{w^{-1}(w\cdot\lambda+2\rho^M)} \to \wedge^{d-\ell(w)}(\mathfrak{u}_{\mathfrak{p}_w} \cap \mathfrak{u}) \otimes N$$

which induces the (necessarily injective) map of part (3)

$$M(\mathfrak{m}_w)_{w^{-1}(w\cdot\lambda+2\rho^M)} \hookrightarrow H_{d-\ell(w)}(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda).$$

It remains to prove (4). By the Casselman–Osborne Theorem 2.3.12, together with (2.3.3), we see that the possible infinitesimal characters of simple subquotients in the homology belong to the set $\{w^{-1}(w' \cdot \lambda + 2\rho^M), w' \in {}^M W\}$. Thus the simples which can occur are the $L(\mathfrak{m}_w)_{w^{-1}(w'\cdot\lambda+2\rho^M)}$. Moreover in order for $L(\mathfrak{m}_w)_{w^{-1}(w'\cdot\lambda+2\rho^M)}$ to occur as a subquotient of $H_i(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda)$ for $i < d - \ell(w)$ or of $H_{d-\ell(w)}(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda)/M(\mathfrak{m}_w)_{w^{-1}(w\cdot\lambda+2\rho^M)}$ we must have that $w^{-1}(w' \cdot \lambda + 2\rho^M)$ is of the form $w^{-1}(w \cdot \lambda + 2\rho^M) - \sum n_\alpha \alpha$ with $n_\alpha \geq 0$ and some $n_\alpha \neq 0$. Therefore $w' \in wW_{<\lambda}$. ☐

Remark 2.3.23. One can show that the highest weight map computes the Euler characteristic of $H_*(\mathfrak{u}_{\mathfrak{p}_w}, M_\lambda)$ in the Grothendieck group, see Proposition 2.4.11.

2.3.24. *Strictness.* We now state our main theorem on the comparison of algebraic and solid cohomology of Lie algebras (Theorem 2.3.32).

Definition 2.3.25. We say $x \in E$ is *p-adically non-Liouville*, or simply *non-Liouville*, if $\liminf_{r\in\mathbf{Z}_{>0}} |x-r|^{1/r} \neq 0$.

Remark 2.3.26 (Inconsistencies in the literature concerning the definition of non-Liouville)**.** There are a number of conflicting definitions in the literature of what it means for $\alpha \in \overline{\mathbf{Q}}_p$ to be p-adically non-Liouville. The original definition in [Cla66, Def 1] is equivalent to the existence of a real number d such that

$$|\alpha - r| \gg |r|^{-d}, \quad r \in \mathbf{Z},\ r \to \infty. \tag{2.3.27}$$

This is the most direct analogue of the definition over $\mathbf{R}$ — Liouville's original argument shows that any $\alpha \in \overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p$ satisfies (2.3.27) with $d = [\mathbf{Q}(\alpha) : \mathbf{Q}]$. There is a weaker definition of non-Liouville given in [Ado76], which is equivalent to

$$\forall B \in (0,1), \quad |\alpha - r| \gg B^{|r|}, \quad r \in \mathbf{Z},\ r \to \infty. \tag{2.3.28}$$

Definition 2.3.25, following Pan [Pan22a, Rem 5.2.11] (see also [MA13, Def. 1]) is weaker still, and is equivalent to

$$\exists B \in (0,1), \quad |\alpha - r| \gg B^{|r|}, \quad r \in \mathbf{Z},\ r \to \infty. \tag{2.3.29}$$

Despite these differences, both [Ado76] and [Pan22a] attribute their respective definitions to [Cla66]. The definition given in Kedlaya's book [Ked10, §13] is equivalent to the one in [Ado76]. The reason we use the definition in [Pan22a] is because (for a number of arguments) our results are true *if and only if* λ is non-Liouville in the sense of Definition 2.3.25, although we do not stress this point. As a practical matter, however, the reader should feel free to take any definition they like, since:

(1) In applications, the strongest assumption from [Cla66] is satisfied.
(2) In proofs, only the weakest assumption from [Pan22a] will be assumed.

Definition 2.3.30. We say that a weight $\lambda \in \mathfrak{h}^\vee$ is p-adically non-Liouville if $\langle \lambda+\rho, \alpha^\vee\rangle$ is p-adically non-Liouville for all $\alpha \in \Phi^+$. Again, we will often abbreviate "p-adically non-Liouville" to "non-Liouville".

Remark 2.3.31. Any integer is p-adically non-Liouville, and if $(\mathfrak{g}, \mathfrak{h}, \mathfrak{b})$ arises as the Lie algebras of (G, T, B) with G a reductive group, then any algebraic weight $\lambda \in X^*(T) \subseteq \mathfrak{h}^\vee = X^*(T)_E$ is p-adically non-Liouville.

We let $\mathcal{O}_{nL}(\mathfrak{g}, \mathfrak{b}) \subseteq \mathcal{O}(\mathfrak{g}, \mathfrak{b})$ be the direct factor abelian subcategory consisting of objects whose weights are p-adically non-Liouville.

Theorem 2.3.32. *Let $M \in \mathcal{O}_{nL}(\mathfrak{g}, \mathfrak{b})$. The canonical map*

$$\hat{U}(\mathfrak{m}_w)\otimes_{U(\mathfrak{m}_w)}(E\otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})}M) = \hat{U}(\mathfrak{p}_w)\otimes_{U(\mathfrak{p}_w)}(E\otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})}M) \to E\otimes^L_{\hat{U}(\mathfrak{u}_{\mathfrak{p}_w})}(\hat{U}(\mathfrak{g})\otimes_{U(\mathfrak{g})}M)$$

is a quasi-isomorphism.

Proof. This is proved below as Theorem 2.8.2. □

Corollary 2.3.33. *For all i, $H^i(\mathfrak{u}_{\mathfrak{p}_w}, \widehat{M}^\vee) = \widehat{H_i(\mathfrak{u}_{\mathfrak{p}_w}, M)}^\vee$ is an admissible $\hat{U}(\mathfrak{m}_w)$-module (where as usual we write $\widehat{M} := M \otimes_{U(\mathfrak{g})} \hat{U}(\mathfrak{g})$).*

Proof. The Chevalley–Eilenberg complex $CE(E)\otimes_E \widehat{M}$ which computes $E\otimes^L_{\hat{U}(\mathfrak{u}_{\mathfrak{p}_w})} \widehat{M}$ has the shape: $0 \to \Lambda^d \otimes \widehat{M}\mathfrak{u}_{\mathfrak{p}_w} \to \cdots \to \widehat{M} \to 0$. This is a complex of Fréchet spaces, and its cohomology groups are also Fréchet spaces by Theorem 2.3.32. Moreover, we have that $H^{-i}(CE(E)\otimes_E \widehat{M}) = \widehat{H_i(\mathfrak{u}_{\mathfrak{p}_w}, M)}$. We recall from Remark 2.2.5 that $\underline{\mathrm{Hom}}(-, E)$ is an exact functor between Fréchet and LB-spaces. We see in the first place that $\underline{\mathrm{Hom}}(CE(E)\otimes_E \widehat{M}, E) = \underline{\mathrm{Hom}}(CE(E), \widehat{M}^\vee)$ computes $\mathrm{R}\Gamma(\mathfrak{u}_{\mathfrak{p}_w}, \widehat{M}^\vee)$ and that $H^i(\underline{\mathrm{Hom}}(CE(E)\otimes_E \widehat{M}, E)) = \underline{\mathrm{Hom}}(H^{-i}(CE(E)\otimes_E \widehat{M}), E)$. Since $\underline{\mathrm{Hom}}(H^{-i}(CE(E)\otimes_E \widehat{M}), E) = \widehat{H_i(\mathfrak{u}_{\mathfrak{p}_w}, M)}^\vee$ by Theorem 2.3.32, we are done by Theorem 2.3.13. □

2.4. **Algebraic local cohomology and twisted Verma modules.**

Now we fix a split reductive group G/E with Lie algebra $\mathfrak{g}$. We work over the field E, viewed as a discrete field (we ignore its natural p-adic topology for the moment). We want to introduce twisted Verma modules as local cohomology on the flag variety.

Remark 2.4.1. We will make a small variation on the classical presentation since we will use the six-functor formalism in coherent cohomology of Clausen and Scholze [CS], and endow the Bruhat cells with the structure of analytic stacks. We feel that this perspective clarifies the discussion. We note that most of our statements are classical (see for example [AL03]), and our proofs can easily be translated into more classical language.

To any affine scheme Spec A, we can attach an analytic stack $\mathrm{AnSpec}(A, \mathrm{Mod}(A))$ where $\mathrm{Mod}(A)$ is the category of condensed A-modules which are solid $\mathbf{Z}$-modules.

This procedure glues to define a functor from the category of schemes to the category analytic stacks, which we denote by $X \mapsto \tilde{X}$.

For any Zariski open subset Spec $A[1/f]$, the corresponding map

$$i : \mathrm{AnSpec}(A[1/f], \mathrm{Mod}(A[1/f])) \to \mathrm{AnSpec}(A, \mathrm{Mod}(A))$$

is proper (!) and can be regarded as a closed immersion. The inclusion i has an open complement $j : U \to \mathrm{AnSpec}(A, \mathrm{Mod}(A))$. We can describe U as the ind-scheme equal to the formal completion of A along the ideal (f). Via the morphism j_*, the (derived) category of quasi-coherent sheaves on U identifies with the subcategory of $D(\mathrm{Mod}(A))$ of modules which are derived (f)-complete (i.e. modules M which satisfy $\lim_{\times f} M = 0$).

We now let $FL := B\backslash G$. We consider the classical Bruhat stratification $FL = \coprod_{w\in W} C_w$ where $C_w = B\backslash BwB$. We let $X_w = \overline{C_w}$ be the Schubert variety. We now equip each X_w with the structure of an analytic stack $\tilde{X}'_w$ which admits a map to $\tilde{X}_w$. Let Y_w be the open complement of X_w in FL. Then Y_w defines an analytic stack $\tilde{Y}_w$, the map $\tilde{Y}_w \to \tilde{FL}$ is proper, and we let $\tilde{X}'_w$ be its open complement. Its structure sheaf is the completion $\widehat{\mathcal{O}_{FL}}^{I_{X_w}}$ where I_{X_w} is the ideal of X_w in $\mathcal{O}_{FL}$. The corresponding category of modules are the $\mathcal{O}_{FL}$-modules which are $\mathbf{Z}$-solid and derived complete modules for the I_{X_w}-adic topology. In other words, we are considering the formal scheme equal to the formal completion of FL along X_w, which we view naturally as an object of the category of analytic stacks. From that perspective, the map $\tilde{X}'_w \to \tilde{FL}$ is an open immersion.

This induces a structure of analytic stack $\tilde{C}'_w$ on each Schubert cell C_w. Indeed, we have classically $C_w = X_w \smallsetminus \cup_{w' \leq w} X_{w'}$ and we let $\tilde{C}'_w = \tilde{X}'_w \smallsetminus \cup_{w'\leq w} \tilde{X}'_{w'}$. Note that $\tilde{C}'_w$ is naturally closed in $\tilde{X}'_w$. We now simplify our notations, and denote by FL, C_w, X_w the analytic stacks we just defined.

Example 2.4.2. We can illustrate how this works for SL_2. In this case, we have $FL = \mathbf{P}^1$. We have that C_{Id} has structure sheaf $E[\![T^{-1}]\!]$ and category of modules the solid $E[\![T^{-1}]\!]$-modules which are derived complete for the T^{-1}-adic topology. We have that C_{w_0} has the structure sheaf $E[T]$ and modules are the $E[T]$-modules which are solid $\mathbf{Z}$-modules.

Example 2.4.3. We can also describe C_w in general. Write $E[T_\alpha]$ for the underlying ring of the root group U_α. Then C_w has structure sheaf $E[T_\alpha, \alpha \in w^{-1}\Phi^- \cap \Phi^+][\![T_\alpha, \alpha \in w^{-1}\Phi^- \cap \Phi^-]\!]$ and category of modules the solid $\mathbf{Z}$-modules which are $E[T_\alpha, \alpha \in w^{-1}\Phi^- \cap \Phi^+][\![T_\alpha, \alpha \in w^{-1}\Phi^- \cap \Phi^-]\!]$-modules and are $(T_\alpha, \alpha \in w^{-1}\Phi^- \cap \Phi^-)$-derived complete.

We can in fact consider the action of an analytic stack in groups $\hat{G} \rtimes B$ on FL (via the the product $\hat{G} \rtimes B \to G$ and the obvious G-action on FL), which is such that the C_w (with their analytic structures) are the $\hat{G} \rtimes B$-orbits. We begin with some definitions.

Whenever we have a classical affine algebraic group H we view it as an analytic stack using the functor $H \mapsto \tilde{H}$. In other words, it is equipped with the structure sheaf $\mathcal{O}_H$ and category of modules the condensed $\mathcal{O}_H$-modules which are solid $\mathbf{Z}$-modules. Similarly, we define $\hat{H}$ (the completion at identity) with structure sheaf

$\widehat{\mathcal{O}_{H,e}}$ (the completed structure sheaf at the identity) and modules the solid $\mathbf{Z}$-modules which are $\widehat{\mathcal{O}_{H,e}}$-modules and are derived complete modules for the $\mathfrak{m}_{\mathcal{O}_{H,e}}$-adic topology. Note that $\hat{H}$ is the complement of $H \smallsetminus \{e\}$ (where again $H \smallsetminus \{e\}$ is equipped with its structure sheaf $\mathcal{O}_{H \smallsetminus \{e\}}$ and category of modules all the solid $\mathbf{Z}$-modules which are $\mathcal{O}_{H \smallsetminus \{e\}}$-modules). Note also that $\hat{H}$ is open in H and $H \smallsetminus \{e\}$ is closed.

The relevance of the group $\hat{G}$ is clarified by the following lemma.

Lemma 2.4.4. *The category of representations of $\hat{G}$ is naturally equivalent to the category of $U(\mathfrak{g})$-modules on solid E-vector spaces.*

Proof. Write $\pi : \hat{G} \to \mathrm{Spec}\ E$, so that a representation of $\hat{G}$ is a solid E-module M, together with a comodule map $M \to \pi^* M$ satisfying the usual cocycle condition. We have $\pi^* M = \mathrm{R}\lim_n M \otimes_E \mathcal{O}_G/\mathfrak{m}^n_{\mathcal{O}_{G,e}}$, so that the map $M \to \pi^* M$ is equivalent to the data of compatible maps $M \to M \otimes_E \mathcal{O}_G/\mathfrak{m}^n_{\mathcal{O}_{G,e}}$, which dually corresponds to a map $U(\mathfrak{g}) \otimes_E M \to M$. □

The following computation will be used repeatedly.

Lemma 2.4.5. *Let U_α be a root group, with underlying ring $E[T_\alpha]$. Then we have $\mathrm{R}\Gamma_c(U_\alpha, \mathcal{O}_{U_\alpha}) = E[T_\alpha]$ and $\mathrm{R}\Gamma_c(\hat{U}_\alpha, \mathcal{O}_{U_\alpha}) = E[T_\alpha, T_\alpha^{-1}]/E[T_\alpha][-1]$. Moreover, $E \otimes^L_{U(\mathfrak{u}_\alpha)} \mathrm{R}\Gamma_c(U_\alpha, \mathcal{O}_{U_\alpha}) = E(\alpha)[1]$ and $E \otimes^L_{U(\mathfrak{u}_\alpha)} \mathrm{R}\Gamma_c(\hat{U}_\alpha, \mathcal{O}_{U_\alpha}) = E(-\alpha)[-1]$.*

Proof. When regarded as above as an analytic stack, U_α is proper over $\mathrm{Spec}\ E$ so that $\mathrm{R}\Gamma_c(U_\alpha, \mathcal{O}_{U_\alpha}) = E[T_\alpha]$. We can then compute $\mathrm{R}\Gamma_c(\hat{U}_\alpha, \mathcal{O}_{U_\alpha}) = E[T_\alpha, T_\alpha^{-1}]/E[T_\alpha][-1]$ by using the triangle:

$$\mathrm{R}\Gamma_c(\hat{U}_\alpha, \mathcal{O}_{U_\alpha}) \to \mathrm{R}\Gamma(U_\alpha, \mathcal{O}_{U_\alpha}) \to \mathrm{R}\Gamma(\hat{U}_\alpha \smallsetminus \{e\}, \mathcal{O}_{U_\alpha}) \xrightarrow{+1} .$$

Next, $E \otimes^L_{U(\mathfrak{u}_\alpha)} \mathrm{R}\Gamma_c(U_\alpha, \mathcal{O}_{U_\alpha})$ is computed by the (Chevalley–Eilenberg) complex in degrees -1 and 0: $E[T_\alpha] \otimes \mathfrak{u}_\alpha \to E[T_\alpha]$ with basis vector u_α of $\mathfrak{u}_\alpha$ acting by the derivation ∂_{T_α}. It is thus $E(\alpha)[1]$. Similarly, $E \otimes^L_{U(\mathfrak{u}_\alpha)} \mathrm{R}\Gamma_c(\hat{U}_\alpha, \mathcal{O}_{U_\alpha})$ is computed by the complex in degrees 0 and 1: $E[T_\alpha, T_\alpha^{-1}]/E[T_\alpha] \otimes \mathfrak{u}_\alpha \to E[T_\alpha, T_\alpha^{-1}]/E[T_\alpha]$. It is thus $E(-\alpha)[-1]$. □

We check that the semi-direct product $\hat{G} \rtimes B$ is well defined (i.e. there is an action of B on $\hat{G}$). First, there is an action of B on G by conjugation (with G equipped with its structure sheaf $\mathcal{O}_G$ and category of modules the solid $\mathbf{Z}$-modules which are also $\mathcal{O}_G$-modules). We observe next that $\hat{G}$ is the complement of $G \smallsetminus \{e\}$. It is clear that B preserves $G \smallsetminus \{e\}$ and thus it also acts on its open complement $\hat{G}$.

We see that each C_w is a $\hat{G} \rtimes B$-orbit in FL. Therefore, we have an equivalence of categories between $\hat{G} \rtimes B$-equivariant sheaves on C_w and representations of the stabilizer $Stab(w)$ of w, given by the fiber functor $\mathscr{F} \mapsto \mathscr{F}|_w$. An inverse of this functor is given by $V \mapsto \pi_*(\mathcal{O}_{\hat{G} \rtimes B} \otimes V)^{Stab(w)}$ where $\pi : \hat{G} \rtimes B \to C_w$ is the uniformization map. One can describe the stabilizer $Stab(w)$ of the point w under this map.

Lemma 2.4.6. *We have $Stab(w) = \widehat{B \cap B_w} \backslash [(\hat{B}_w \times \hat{B}) \rtimes B \cap B_w]$ where the map $(\hat{B}_w \times \hat{B}) \rtimes B \cap B_w \to \hat{G} \rtimes B$ is given by $(b, b', b'') \mapsto (b(b')^{-1}, b'b'')$ and the map $\widehat{B \cap B_w} \to (\hat{B}_w \times \hat{B}) \rtimes B \cap B_w$ is given by $b \mapsto (b^{-1}, b^{-1}, b)$.*

Proof. This is straightforward; see Lemma 3.3.6 for the proof of a very similar statement. ☐

Lemma 2.4.7. *Let $\mathscr{F}$ be a $\hat{G} \rtimes B$ equivariant sheaf on C_w. Then $\mathscr{F}$ is isomorphic as a $\hat{B}_{w_0w}$-equivariant sheaf to $\mathscr{F}|_w \otimes_E \mathcal{O}_{C_w}$ (where the $\hat{B}_{w_0w}$-equivariant sheaf structure on $\mathscr{F}$ arises from regarding $\hat{B}_{w_0w}$ as a subgroup of $\widehat{G}$; and $\hat{B}_{w_0w}$ acts on $\mathscr{F}|_w$ through $\hat{T}$ and via its natural action on $\mathcal{O}_{C_w}$).*

Proof. By Example 2.4.3 $\widehat{B_{w_0} \cap U}_{w_0w} \times B \cap U_{w_0w}$ (viewed as a substack – but not a subgroup – of $\hat{G} \rtimes B$) maps isomorphically to C_w via the uniformization map $x \mapsto wx$. It follows that the product map gives an isomorphism of analytic stacks (not of groups):

$$Stab(w) \times (\widehat{B_{w_0} \cap U}_{w_0w} \times B \cap U_{w_0w}) \to \hat{G} \rtimes B.$$

This isomorphism is equivariant for the $\hat{U}_{w_0w}$-action by translation on the right on $\widehat{B_{w_0} \cap U}_{w_0w} \times B \cap U_{w_0w}$ and $\hat{G} \rtimes B$. It is also equivariant for the $\hat{T}$-action (the one by right translation on $\hat{G} \rtimes B$, and by right translation on $Stab(w)$ and conjugation on $\widehat{B_{w_0} \cap U}_{w_0w} \times B \cap U_{w_0w}$). We construct a map $\mathscr{F}|_w \to \mathscr{F}$, by sending $v \in \mathscr{F}|_w$ to $(ss' \mapsto sv)$ viewed as an element of $\pi_*(\mathcal{O}_{\hat{G} \rtimes B} \otimes \mathscr{F}|_w)^{Stab(w)} = \mathscr{F}$ for $s \in Stab(w)$ and $s' \in \widehat{B_{w_0} \cap U}_{w_0w} \times B \cap U_{w_0w}$. This induces an isomorphism $\mathscr{F}|_w \otimes_E \mathcal{O}_{C_w} \to \mathscr{F}$ which satisfies the expected properties. ☐

Let $(\kappa, \nu) \in X^*(T)^2_E$ with the property that $\kappa + \nu \in X^*(T)$. We define a character of $Stab(w)$ as follows: we let $\hat{B}_w$ act via κ, we let $\hat{B}$ act via ν, and we let $B \cap B_w$ act via $\kappa + \nu$. This defines a $\hat{G} \rtimes B$ equivariant sheaf $\mathcal{L}_\kappa(\nu)$ over C_w. We sometimes drop ν from the notation since we are mostly interested in the $\hat{G}$-equivariant action (the B-action rigidifies the construction and will be used in the construction of intertwining maps). We let $j_w : C_w \to FL$ be the inclusion. Let d_{FL} be the dimension of FL. We can now define the twisted Verma modules:

Definition 2.4.8. We define the twisted Verma module $M(\mathfrak{g})^w_\lambda = H^{d_{FL}-\ell(w)}(FL, (j_w)_! \mathcal{L}_{\lambda + w^{-1}\rho + \rho}(\nu))$.

This is a representation of $\hat{G} \rtimes B$. By Lemma 2.4.4, the $\hat{G}$-action amounts to a $\mathfrak{g}$-module structure. We will usually write M^w_λ for $M(\mathfrak{g})^w_\lambda$.

Proposition 2.4.9. *The $\mathfrak{g}$-module M^w_λ belongs to $\mathcal{O}(\mathfrak{g}, \mathfrak{b})$. It has the following properties:*

1. *Its highest weight is λ.*
2. *It is isomorphic to the direct sum*
$$\oplus_{k_\alpha \geq 0, \alpha \in w^{-1}\Phi^- \cap \Phi^+, k_\alpha < 0, \alpha \in w^{-1}\Phi^- \cap \Phi^-} E(\lambda + w^{-1}\rho + \rho) \prod T_\alpha^{k_\alpha}$$
and in particular the action of $\mathfrak{b}_{w_0w}$ is completely explicit.
3. *M^{Id}_λ is the Verma module of highest weight λ.*
4. *$M^{w_0}_\lambda$ is the dual Verma module of highest weight λ.*
5. *The elements $[M^w_\lambda]$ of the Grothendieck group are independent of w.*

Proof. This is proven in the course of the proof of [BP21, Lem. 3.2.2]. Let us give some details. Given Lemma 2.4.7, and the projection formula, the key computation is that $\mathrm{R}\Gamma_c(C_w, \mathcal{O}_{C_w}) = \oplus_{k_\alpha \geq 0, \alpha \in w^{-1}\Phi^- \cap \Phi^+, k_\alpha < 0, \alpha \in w^{-1}\Phi^- \cap \Phi^-} E \prod T_\alpha^{k_\alpha}$. This follows from Lemma 2.4.5. We deduce that M^w_λ is a finitely generated $U(\mathfrak{g})$-module and that the action of $\mathfrak{h}$ is semi-simple with the same character as that of the Verma module M_λ^{Id}. This implies that M_λ^w belongs to category $\mathcal{O}$ and that $[M_\lambda^w]$ is independent of w (see [Hum08, 1.15]). □

One can compute easily the homology of $\mathfrak{u}_{\mathfrak{p}_w}$ on $M_\lambda^{w_0 w}$ as follows. We continue to write $d = \dim_E(\mathfrak{u}_\mathfrak{p})$.

Proposition 2.4.10. *We have*

$$E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M_\lambda^{w_0 w} = M(\mathfrak{m}_w)^{(w^M)^{-1} w_{0,M} w}_{(w^M)^{-1}(w^M \cdot \lambda + 2\rho^M)}[d - \ell(w^M)]$$

where $w = w_M w^M$ for $w_M \in W_M$ and $w^M \in {}^M W$.

Proof. We have a map

$$\pi : \prod_{\alpha \in w^{-1}\Phi^+ \cap \Phi^+} U_\alpha \prod_{\alpha \in w^{-1}\Phi^+ \cap \Phi^-} \hat{U}_\alpha \to \prod_{\alpha \in w^{-1}\Phi^+_M \cap \Phi^+} U_\alpha \prod_{\alpha \in w^{-1}\Phi^+_M \cap \Phi^-} \hat{U}_\alpha$$

of analytic stacks with fiber

$$F := \prod_{\alpha \in w^{-1}\Phi^{+,M} \cap \Phi^+} U_\alpha \prod_{\alpha \in w^{-1}\Phi^{+,M} \cap \Phi^-} \hat{U}_\alpha.$$

This map is $\hat{P}_w \rtimes B \cap P_w$-equivariant (the action of $\hat{P}_w \rtimes B \cap P_w$ factors through an action $\hat{M}_w \rtimes B_{M_w}$ on the target). The space $\prod_{\alpha \in w^{-1}\Phi^+ \cap \Phi^+} U_\alpha \prod_{\alpha \in w^{-1}\Phi^+ \cap \Phi^-} \hat{U}_\alpha$ is the Bruhat cell $C_{w_0 w}$ in $B \backslash G$. The space $\prod_{\alpha \in w^{-1}\Phi^+_M \cap \Phi^+} U_\alpha \prod_{\alpha \in w^{-1}\Phi^+_M \cap \Phi^-} \hat{U}_\alpha$ is the Bruhat cell $C'_{(w^M)^{-1} w_{0,M} w_M w^M}$ in $B_{M_w} \backslash M_w$. We deduce that

$$E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M_\lambda^{w_0 w} = \mathrm{R}\Gamma_c(C'_{(w^M)^{-1} w_{0,M} w_M w^M}, \pi_! \mathcal{L}_{\lambda + w^{-1} w_0^{-1} \rho + \rho} \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} E).$$

It therefore suffices to compute the sheaf $\pi_! \mathcal{L}_{\lambda + w^{-1} w_0^{-1} \rho + \rho} \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} E$. This is a $\hat{M}_w \rtimes B_{M_w}$-equivariant sheaf, so it is determined by its fiber at w.

It follows from the basic computations of Lemma 2.4.5 that $\mathrm{R}\Gamma_c(F, \mathcal{O}_F)$ is concentrated in degree $\ell(w^M)$ and equals

$$\bigoplus_{k_\alpha \geq 0, \alpha \in w^{-1}\Phi^{+,M} \cap \Phi^+, k_\alpha < 0, \alpha \in w^{-1}\Phi^{+,M} \cap \Phi^-} E \prod_\alpha T_\alpha^{k_\alpha}.$$

We then compute that

$$\begin{aligned} E \otimes^L_{\mathfrak{u}_{P_w}} \mathrm{R}\Gamma_c(F, \mathcal{O}_F) &= H_0(\mathfrak{u}_{P_w} \cap \bar{\mathfrak{b}}, H_{d-\ell(w^M)}(\mathfrak{u}_{P_w} \cap \mathfrak{b}, H_c^{\ell(w)}(F, \mathcal{O}_F))[d - 2\ell(w^M)] \\ &= E(\sum_{w^{-1}\Phi^{+,M} \cap \Phi^+} \alpha - \sum_{w^{-1}\Phi^{+,M} \cap \Phi^-} \alpha)[d - 2\ell(w^M)]. \end{aligned}$$

It follows that $\pi_! \mathcal{L}_{\lambda + w^{-1} w_0^{-1} \rho + \rho} \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} E$ is an invertible sheaf in degree $2\ell(w^M) - d$ of weight

$$\lambda + w^{-1} w_0^{-1} \rho + \rho + \sum_{w^{-1}\Phi^{+,M} \cap \Phi^+} \alpha - \sum_{w^{-1}\Phi^{+,M} \cap \Phi^-} \alpha.$$

It follows that $M_\lambda^{w_0 w} \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} E$ is concentrated in degree $2\ell(w^M) - d + \ell(w^M) - \ell(w) = \ell(w^M) - d$, and is the twisted Verma of weight

$$\lambda + w^{-1} w_0^{-1} \rho + \rho + \sum_{w^{-1}\Phi^{+,M} \cap \Phi^+} \alpha - \sum_{w^{-1}\Phi^{+,M} \cap \Phi^-} \alpha - \sum_{w^{-1}\Phi^+_M \cap \Phi^-} \alpha$$

$$= \lambda + \sum_{w^{-1}\Phi^{+,M}\cap\Phi^+} \alpha = (w^M)^{-1}(w^M \cdot (\lambda+\rho) + 2\rho^M). \qquad \square$$

Proposition 2.4.11. *Let $w \in {}^M W$. In the Grothendieck group of category $\mathcal{O}$ for $\mathfrak{m}_w$, we have*

$$\big[E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M_\lambda^{Id}\big] = (-1)^{d-\ell(w)}\big[M(\mathfrak{m}_w)^{Id}_{w^{-1}(w\cdot\lambda+2\rho^M)}\big].$$

Proof. In the Grothendieck group of $\mathcal{O}(\mathfrak{g},\mathfrak{b})$ we have $[M_\lambda^{Id}] = [M_\lambda^{w_0 w_{0,M} w}]$ and by Proposition 2.4.10 we have $E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M_\lambda^{w_0 w_{0,M} w} = M(\mathfrak{m}_w)^{Id}_{w^{-1}(w\cdot\lambda+2\rho^M)}[d - \ell(w)]$. $\square$

2.5. **Some SL_2-computations.** We now make some explicit calculations in the case $G = \mathrm{SL}_2$. We let $H, X, \bar{X}$ be the standard basis of $\mathfrak{g}$ with $E\cdot H \oplus E \cdot X = \mathfrak{b}$ and $[X,\bar{X}] = H$. Let $\lambda : \mathfrak{h} \to E$ be a character (identified with its value $\lambda(H) \in E$), and write $E(\lambda)$ for the underlying representation. The Verma module $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} E(\lambda)$ has basis the $\{\bar{X}^n\}_{n\geq 0}$ (or more precisely, $\bar{X}^n \otimes 1$ where $1 \in E(\lambda)$ is a basis vector).

We let $M_\lambda^\vee$ be the dual Verma module (the dual in category $\mathcal{O}$). Concretely, $M_\lambda^\vee$ is the subspace of the algebraic dual $\mathrm{Hom}_E(M_\lambda, E)$ which has basis the vectors $\{(\bar{X}^n)^*\}_{n\geq 0}$ where $(\bar{X}^n)^*(\bar{X}^m) = 1$ if $m = n$ and $(\bar{X}^n)^*(\bar{X}^m) = 0$ if $m \neq n$. For $g \in \mathfrak{g}$ and $f \in M_\lambda^\vee$, we have that $gf = f({}^t g -)$.

Lemma 2.5.1. *We have that $\bar{X}\cdot(\bar{X}^n)^* = (n+1)(\lambda-n)(\bar{X}^{n+1})^*$ and $X \cdot (\bar{X}^n)^* = (\bar{X}^{n-1})^*$.*

Proof. These follow from the corresponding formulas in M_λ: $X\cdot\bar{X}^n = n(\lambda-n+1)\bar{X}^{n-1}$ and $\bar{X}\cdot\bar{X}^n = \bar{X}^{n+1}$. $\square$

Corollary 2.5.2.

(1) There is a unique map of $U(\mathfrak{g})$-modules, $I : M_\lambda \to M_\lambda^\vee$ which sends $\bar{X}^n$ to $n!\lambda(\lambda-1)\cdots(\lambda-(n-1))(\bar{X}^n)^*$. Any other map of $U(\mathfrak{g})$-modules is a E-multiple of this map.

(2) If $\lambda \notin \mathbf{Z}_{\geq 0}$, the map $M_\lambda \to M_\lambda^\vee$ is an isomorphism. If $\lambda \in \mathbf{Z}_{\geq 0}$, we have a long exact sequence:

$$0 \to M_{-2-\lambda} \to M_\lambda \to M_\lambda^\vee \to M_{-2-\lambda}^\vee \to 0.$$

The map $M_{-2-\lambda} \to M_\lambda$ sends the basis vector $\bar{X}^n$ of $M_{-2-\lambda}$ to $\bar{X}^{n+\lambda+2}$ in M_λ. The map $M_\lambda^\vee \to M_{-2-\lambda}^\vee$ is dual to the map $M_{-2-\lambda} \to M_\lambda$.

Proof. Giving a map $M_\lambda \to M_\lambda^\vee$ of $U(\mathfrak{g})$-modules amounts to giving a map of $\mathfrak{b}$-modules, $E(\lambda) \to M_\lambda^\vee$. Since $M_\lambda^\vee$ has a unique vector of weight λ, namely $(\bar{X}^0)^*$, the space of maps is one dimensional, generated by the map $\bar{X}^0 \mapsto (\bar{X}^0)^*$. Then we see by Lemma 2.5.1 that $\bar{X}^n \mapsto \bar{X}^n.(\bar{X}^0)^* = n!\lambda(\lambda-1)\cdots(\lambda-(n-1))(\bar{X}^n)^*$. If $\lambda \notin \mathbf{Z}_{\geq 0}$, this map is an isomorphism. Otherwise, let L_λ be the finite dimensional irreducible representation of highest weight λ (and dimension $\lambda+1$). There is a surjective map $M_\lambda \to L_\lambda \to 0$, fitting in an exact sequence:

$$0 \to M_{-2-\lambda} \to M_\lambda \to L_\lambda \to 0$$

The dual in category $\mathcal{O}$ of this exact sequence gives:

$$0 \to L_\lambda \to M_\lambda^\vee \to M_{-2-\lambda}^\vee \to 0$$

and combining these exact sequences concludes the proof. $\square$

2.6. **Intertwining maps.** Let β be a simple root. Let us consider the corresponding parabolic P_β, and the partial flag variety $FL_\beta := P_\beta\backslash G$. We have a map $\pi_\beta : FL \to FL_\beta$ which is a $\mathbf{P}^1$-fibration.

For each $w \in W$, we let $D_w = P_\beta \backslash P_\beta w B$ be the corresponding Bruhat cell. As in Example 2.4.3, it is equipped with the following analytic stack structure. Its structure ring is $E \otimes_{\mathbf{Z}} \mathbf{Z}[T_\alpha, \alpha \in w^{-1}\Phi^{-,M_\beta} \cap \Phi^+][\![T_\alpha, \alpha \in w^{-1}\Phi^{-,M_\beta} \cap \Phi^-]\!]$. Its modules are solid $\mathbf{Z}$-modules which are $E \otimes_{\mathbf{Z}} \mathbf{Z}[T_\alpha, \alpha \in w^{-1}\Phi^{-,M_\beta} \cap \Phi^+][\![T_\alpha, \alpha \in w^{-1}\Phi^{-,M_\beta} \cap \Phi^-]\!]$-modules and are $(T_\alpha, \alpha \in w^{-1}\Phi^{-,M_\beta} \cap \Phi^-)$-derived complete. Assume from now on that $\ell(s_\beta w) = \ell(w)+1$; then we have $\pi_\beta^{-1}(D_w) = C_w \cup C_{s_\beta w}$. Each D_w is a $\hat{G} \rtimes B$-orbit. We let $Stab(w)_\beta$ be the stabilizer of w. We again have an equivalence between $Stab(w)_\beta$-representations and $\hat{G} \rtimes B$-equivariant sheaves on D_w.

Lemma 2.6.1. $Stab(w)_\beta = \widehat{B \cap P_{s_\beta w}} \backslash [(\hat{P}_{s_\beta w} \times \hat{B}) \rtimes B \cap P_{s_\beta w}]$.

Proof. The same as Lemma 2.4.6. □

Lemma 2.6.2. *Let $\mathscr{F}$ be an $\hat{G} \rtimes B$-equivariant sheaf on D_w. There is a $\hat{T}$-equivariant isomorphism $\mathscr{F}|_w \otimes \mathcal{O}_{D_w} \to \mathscr{F}$.*

Proof. This is the same as Lemma 2.4.7. □

Given any pair of characters $(\lambda, \nu) \in X^*(T)^2_E$ with $\lambda + \nu \in X^*(T)$, one can construct representations $M_\lambda(\nu)$ and $M^\vee_\lambda(\nu)$ of $Stab(w)_\beta$ as follows: the underlying representation of $\hat{P}_{s_\beta w}$ factors through $\hat{M}_{s_\beta w}$, and is respectively given by $M(\mathfrak{m}_{s_\beta w})_\lambda$ or $M(\mathfrak{m}_{s_\beta w})^\vee_\lambda$; and we let $\hat{B}$ act via ν. The product of these actions integrates to an action of $B \cap P_{s_\beta w}$.

By Corollary 2.5.2, we see that we have intertwining maps: $I : M_\lambda(\nu) \to M^\vee_\lambda(\nu)$.

Lemma 2.6.3. *We have*

$$\mathrm{R}\Gamma_c(D_w, \mathcal{O}_{D_w}) = \oplus_{k_\alpha \geq 0, \alpha \in w^{-1}\Phi^{-,M_\beta} \cap \Phi^+, k_\alpha < 0, \alpha \in w^{-1}\Phi^{-,M_\beta} \cap \Phi^-} E \prod T_\alpha^{k_\alpha}.$$

Proof. This follows from Lemma 2.4.5. □

Proposition 2.6.4. *Assume that $\ell(s_\beta w) = \ell(w)+1$. There is an intertwining map of $\mathfrak{g}$-modules:*

$$M^w_\lambda \to M^{s_\beta w}_\lambda.$$

This map is given (as $\mathfrak{h}$-modules) by the map

$$M_{\lambda + w^{-1}s_\beta\rho + \rho} \otimes_E \mathrm{R}\Gamma_c(D_w, \mathcal{O}_{D_w})$$
$$\stackrel{I \otimes Id}{\to} M^\vee_{\lambda + w^{-1}s_\beta\rho + \rho} \otimes_E \mathrm{R}\Gamma_c(D_w, \mathcal{O}_{D_w}).$$

(1) *If $\langle \lambda + \rho, w^{-1}\beta^\vee \rangle \in \mathbf{Z}_{>0}$, we have a long exact sequence:*

$$0 \to M^w_{s_{w^{-1}\beta} \cdot \lambda} \to M^w_\lambda \to M^{s_\beta w}_\lambda \to M^{s_\beta w}_{s_{w^{-1}\beta} \cdot \lambda} \to 0$$

which is the tensor product of the long exact sequence of Corollary 2.5.2 with $\mathrm{R}\Gamma_c(D_w, \mathcal{O}_{D_w})$ (as $\mathfrak{h}$-modules). Furthermore $M^w_{s_{w^{-1}\beta} \cdot \lambda} \simeq M^{s_\beta w}_{s_{w^{-1}\beta} \cdot \lambda}$.

(2) *Otherwise, the intertwining map is an isomorphism $M^w_\lambda \simeq M^{s_\beta w}_\lambda$.*

Proof. Consider the map $\pi : FL \to FL_\beta$ and the maps $\pi_w : C_w \to D_w$ and $\pi_{s_\beta w} : C_{s_\beta w} \to D_w$. We construct a map: $\pi_{w,!}\mathcal{L}_{\lambda+w^{-1}\rho+\rho}(\nu) \to \pi_{s_\beta w,!}\mathcal{L}_{\lambda+w^{-1}s_\beta\rho+\rho}(\nu)$. For this, we observe that both are $\hat{G} \rtimes B$-equivariant sheaves on D_w. We compute the corresponding $Stab(w)_\beta$-representations. For this we can work over the fiber at w by proper base change. By Proposition 2.4.9, (3) and (4) we deduce that the fiber $\pi_{w,!}\mathcal{L}_{\lambda+w^{-1}\rho+\rho}|_w$ corresponds to the representation $M_{\lambda+w^{-1}s_\beta\rho+\rho}(\nu)$, and the fiber $\pi_{s_\beta w,!}\mathcal{L}_{\lambda+w^{-1}s_\beta\rho+\rho}|_w$ corresponds to the representation $M^\vee_{\lambda+w^{-1}s_\beta\rho+\rho}(\nu)$. As noted above, we get a map between these representations by using the intertwining map I defined in Corollary 2.5.2. Moreover, by Lemma 2.6.2, both sheaves are trivial, and are respectively $\widehat{T}$-equivariantly isomorphic to $M_{\lambda+w^{-1}s_\beta\rho+\rho} \otimes_E \mathcal{O}_{D_w}$ and $M^\vee_{\lambda+w^{-1}s_\beta\rho+\rho} \otimes_E \mathcal{O}_{D_w}$. We now take cohomology with compact support so that the cohomologies $\mathrm{R}\Gamma_c(D_w, \pi_{w,!}\mathcal{L}_{\lambda+w^{-1}\rho+\rho})$ and $\mathrm{R}\Gamma_c(D_w, \pi_{s_\beta w,!}\mathcal{L}_{\lambda+w^{-1}s_\beta\rho+\rho})$ are indeed given by the claimed formulas (use the projection formulas).

To see (1) and (2), observe that $\langle \lambda+\rho, w^{-1}\beta^\vee\rangle \in \mathbf{Z}_{>0}$ then again by Corollary 2.5.2 (noting that λ there is $\langle \lambda + w^{-1}s_\beta\rho+\rho, w^{-1}\beta^\vee\rangle = \langle \lambda+\rho, w^{-1}\beta^\vee\rangle - 1$), we actually have a long exact sequence of sheaves:

$$0 \to \pi_{w,!}\mathcal{L}_{s_{w^{-1}\beta}\cdot\lambda+w^{-1}\rho+\rho}(\nu) \to \pi_{w,!}\mathcal{L}_{\lambda+w^{-1}\rho+\rho}(\nu) \to$$
$$\pi_{s_\beta w,!}\mathcal{L}_{\lambda+w^{-1}s_\beta\rho+\rho}(\nu) \to \pi_{s_\beta w,!}\mathcal{L}_{s_{w^{-1}\beta}\cdot\lambda+w^{-1}s_\beta\rho+\rho}(\nu) \to 0$$

inducing the expected long exact sequence on cohomology. Otherwise, the intertwining map of sheaves is an isomorphism, inducing an isomorphism on cohomology. □

2.7. **Topology.** In this section we consider E-vector spaces V equipped with a weight space decomposition $V = \oplus_{\nu\in X^*(T)}V_\nu$ where each V_ν is finite dimensional and equipped with a norm $|-|_\nu$. We can define the norms $|-|_\nu$ by choosing a basis for V_ν, and decreeing the basis vectors to have norm 1.

Fix a basis $\{e_i\}$ of $X^*(T)$; then we have a function $|-| : X^*(T) \to \mathbf{N}$ measuring the size of ν as follows: any ν can be written as $\sum n_i e_i$ and we put $|\nu| = 1 + \sum |n_i|$ (where in contrast to the rest of this section, $|n_i|$ is the archimedean norm of n_i).

Let $r \in \mathbf{R}_{>0}$. We define a norm $|-|_r$ on V by letting $|\sum v_\nu|_r = \sup_\nu |v_\nu|_\nu r^{|\nu|}$. We write $\hat{V}_r$ for the Banach space completion; concretely,

$$\hat{V}_r = \{(v_\nu) \in \prod V_\nu \mid \limsup_\nu |v_\nu|_\nu r^{|\nu|} = 0\}.$$

We let $\mathcal{T}_{nat}$ be the natural topology on V defined by the family of norms $\{|-|_r\}_{r\geq 1}$, making V a locally convex E-vector space. We let $\hat{V}_{nat}$ be $\lim_{r\to\infty}\hat{V}_r$, the completion of V for $\mathcal{T}_{nat}$. This is a Fréchet space.

This applies in particular to $V = U(\mathfrak{g})$. Fixing a PBW basis gives a decomposition of $U(\mathfrak{g})$ into weight spaces and defines the natural topology. We have $\hat{U}(\mathfrak{g}) = \hat{U}(\mathfrak{g})_{nat}$. It follows that any object M of $\mathcal{O}$, being a finitely generated $U(\mathfrak{g})$-module, inherits a canonical topology $\mathcal{T}_{can}$ which is a locally convex topology. For any such M, its completion is $\hat{M} = M \otimes_{U(\mathfrak{g})} \hat{U}(\mathfrak{g})$.

Lemma 2.7.1. *Any map $M \to N$ in category $\mathcal{O}$ is strict for the canonical topology.*

Proof. See for example [Sch13a, Prop. 3.1.1]. □

We can consider the twisted Verma module M^w_λ, which admits the basis

$$\oplus_{k_\alpha\geq 0,\alpha\in w^{-1}\Phi^-\cap\Phi^+,k_\alpha<0,\alpha\in w^{-1}\Phi^-\cap\Phi^-}E\prod T_\alpha^{k_\alpha}.$$

We use this basis to define the natural topology $\mathcal{T}_{nat}$ as above. It follows that twisted Verma modules have two topologies $\mathcal{T}_{nat}$ and $\mathcal{T}_{can}$. It is immediate that in case $w = Id$, the canonical topology and the natural topology coincide. We will next show that they coincide more generally if λ is p-adically non-Liouville (see Definitions 2.3.25 and 2.3.30).

Lemma 2.7.2. *Suppose $x \in E$ is p-adically non-Liouville and $x \notin \mathbf{Z}_{>0}$. Then there exists a constant $C > 0$ such that $\prod_{r=1}^n |x - r| \geq C^n$ for all n.*

Proof. If $x \notin \mathbf{Z}_p$, there is a constant $C > 0$ such that $|x - r| \geq C$ for all r (e.g. take $C = \min_{r \in \mathbf{Z}_p} |x - r|$) so the result is clear in this case. So assume $x \in \mathbf{Z}_p$. By assumption, there is a constant $B > 0$ such that $|x - r| \geq B^r$ for all $r \in \mathbf{Z}_{>0}$ (see (2.3.29)). For a given n, let $p^{m-1} \leq n < p^m$ and choose $0 < r_0 \leq p^m$ with $r_0 \equiv x \pmod{p^m}$. Then, for $0 < r \leq n$ with $r \neq r_0$, $|x - r| = |r_0 - r|$, and so we may estimate

$$\begin{aligned}\prod_{r=1}^{n} |x - r| &= |x - r_0| \cdot \prod_{r=1, r \neq r_0}^{n} |r_0 - r| \\ &\geq B^n \cdot |(p^m)!| \geq C^n\end{aligned}$$

for some $C > 0$. □

Lemma 2.7.3. *If λ is a non-Liouville number, the maps of Corollary 2.5.2 are strict for the natural topology.*

Proof. When $\lambda \in \mathbf{Z}_{\geq 0}$, the statement is obvious. We now assume that $\lambda \notin \mathbf{Z}_{\geq 0}$, so we need to show that the isomorphism $I : M_\lambda \to M_\lambda^\vee$ which sends $\bar{X}^n$ to $n!\lambda(\lambda - 1) \cdots (\lambda - (n-1))(\bar{X}^n)^*$ is strict. By Lemma 2.7.2 (and the trivial bound $|n!| \geq p^{-n}$), we see that for $n > 0$, we have $|n!\lambda(\lambda - 1) \cdots (\lambda - (n-1))| \geq C^n$ for a positive constant C. This easily implies strictness. □

Lemma 2.7.4. *If λ is a non-Liouville weight, the sequence of Proposition 2.6.4 (1)*

$$0 \to M^w_{s_{w^{-1}\beta} \cdot \lambda} \to M^w_\lambda \to M^{s_\beta w}_\lambda \to M^{s_\beta w}_{s_{w^{-1}\beta} \cdot \lambda} \to 0$$

or the isomorphism $M^w_\lambda \to M^{s_\beta w}_\lambda$ of Proposition 2.6.4 (2) are strict for the natural topology.

Proof. This follows from Lemma 2.7.3. □

Lemma 2.7.5. *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of E-vector spaces. Assume that V_2 has two locally convex topologies $\mathcal{T}$ and $\mathcal{T}'$. Assume that the induced topologies $\mathcal{T}_1$ and $\mathcal{T}'_1$ on V_1, as well as the induced topologies $\mathcal{T}_2$ and $\mathcal{T}'_3$ on V_3 coincide. Then $\mathcal{T}$ and $\mathcal{T}'$ coincide.*

Proof. The topologies $\mathcal{T}$ and $\mathcal{T}'$ are given by families of lattices $\{L_i\}_{i \in I}$ and $\{L_{i'}\}_{i' \in I'}$ subject to certain conditions (in particular, for any $i, j \in I$, there is a $k \in I$ such that $L_k \subseteq L_i \cap L_j$). By symmetry, it suffices to prove that for any $i' \in I'$, there is $i \in I$ such that $L_i \subseteq L_{i'}$. Any lattice L in V_2 sits in an exact sequence $0 \to L_1 \to L \to L_3 \to 0$, with $L_1 = L \cap V_1$. By assumption, there is $i_1 \in I$ such that $L_{i_1,1} \subseteq L_{i',1}$ and $i_3 \in I$ such that $L_{i_3,3} \subseteq L_{i',3}$. Picking $i \in I$ such that $L_i \subseteq L_{i_1} \cap L_{i_3}$, we find that $L_i \subseteq L_{i'}$. □

Proposition 2.7.6. *If λ is a non-Liouville weight, the canonical and natural topologies on M^w_λ coincide.*

Proof. We use induction on the length of w. We know this is true if $w = Id$. We assume that this is true for w and all λ', and want to prove it for $M_\lambda^{s_\beta w}$, where $\ell(s_\beta w) = \ell(w) + 1$. If the intertwining map $M_\lambda^w \to M_\lambda^{s_\beta w}$ is an isomorphism, it is strict for both the canonical and natural topologies (by Lemmas 2.7.1 and 2.7.4), so we are done.

Otherwise, we have a long exact sequence:

$$0 \to M^w_{s_{w^{-1}\beta}\cdot\lambda} \to M_\lambda^w \to M_\lambda^{s_\beta w} \to M^{s_\beta w}_{s_{w^{-1}\beta}\cdot\lambda} \to 0.$$

This long exact sequence is again strict for the natural and canonical topologies. Since the natural and canonical topologies agree on $M^w_{s_{w^{-1}\beta}\cdot\lambda}$ and M_λ^w, and also on $M^{s_\beta w}_{s_{w^{-1}\beta}\cdot\lambda} \simeq M^w_{s_{w^{-1}\beta}\cdot\lambda}$, we are done by Lemma 2.7.5. □

2.8. **Proof of Theorem 2.3.32.**

Lemma 2.8.1. *Let α be a root. Consider the complexes $C_1 : E[T_\alpha] \otimes \mathfrak{u}_\alpha \to E[T_\alpha]$ and $C_2 : E[T_\alpha, T_\alpha^{-1}]/E[T_\alpha] \otimes \mathfrak{u}_\alpha \to E[T_\alpha, T_\alpha^{-1}]/E[T_\alpha]$. Let $\hat{C}_{1nat}$ and $\hat{C}_{2nat}$ be the completions of these complexes for the natural topology. For $i = 1, 2$, the natural map $C_i \to \hat{C}_{inat}$ is a quasi-isomorphism. In particular, the differentials in $\hat{C}_{inat}$ are strict.*

Proof. This is a standard computation; for instance, $\hat{C}_{1nat}$ computes the de Rham cohomology of the analytic affine line, but in any case it is a simple explicit calculation as in Lemma 2.4.5. □

We now restate Theorem 2.3.32, for the reader's convenience.

Theorem 2.8.2 (Theorem 2.3.32)**.** *Let $M \in \mathcal{O}_{nL}(\mathfrak{g}, \mathfrak{b})$. The canonical map*

$$\hat{U}(\mathfrak{m}_w) \otimes_{U(\mathfrak{m}_w)} (E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M) \to E \otimes^L_{\hat{U}(\mathfrak{u}_{\mathfrak{p}_w})} (\hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M)$$

is a quasi-isomorphism.

Proof. A standard argument using the five lemma shows that we may replace M by a resolution, and thus we reduce to the case that $M = M_\lambda$ for some non-Liouville λ. In fact, it is more convenient to handle all of the twisted Verma modules $M = M_\lambda^{w'}$ by induction on $\ell(w'w^{-1}w_0)$.

We begin with the base case $w' = w_0 w$. Lemma 2.8.1 implies easily that the Chevalley–Eilenberg complex computing $E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M_\lambda^{w_0 w}$ is a strict complex for the natural topology, so that the formula of Proposition 2.4.10

$$E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M_\lambda^{w_0 w} = M(\mathfrak{m}_w)^{(w^M)^{-1}w_{0,M}w}_{(w^M)^{-1}(w^M\cdot\lambda + 2\rho^M)}[d - \ell(w^M)]$$

passes to completions for the natural topology. The result now follows since the canonical and natural topologies coincide (on $M_\lambda^{w_0 w}$, and on $M(\mathfrak{m}_w)^{(w^M)^{-1}w_{0,M}w}_{(w^M)^{-1}(w^M\cdot\lambda+2\rho^M)}$) by Proposition 2.7.6.

For the inductive step, we can suppose that there is some β such that $\ell(s_\beta w'w^{-1}w_0) = \ell(w'w^{-1}w_0) - 1$. Then $\ell(s_\beta w') = \ell(w') \pm 1$. Let us assume that it is $\ell(w') + 1$, so that we have the intertwining map $M_\lambda^{w'} \to M_\lambda^{s_\beta w'}$ (the other case is almost identical, using the intertwining map $M_\lambda^{s_\beta w'} \to M_\lambda^{w'}$, and we leave it to the reader).

If the intertwining map is an isomorphism, we are done. Otherwise, we have a long exact sequence: $0 \to M^{w'}_{s_{(w')^{-1}\beta}\cdot\lambda} \to M_\lambda^{w'} \to M_\lambda^{s_\beta w'} \to M^{s_\beta w'}_{s_{(w')^{-1}\beta}\cdot\lambda} \to 0.$

By induction, the theorem holds for $M_\lambda^{s_\beta w'}$ and $M^{s_\beta w'}_{s_{(w')^{-1}\beta}\cdot\lambda}$. Since $M^{w'}_{s_{(w')^{-1}\beta}\cdot\lambda} \to M^{s_\beta w'}_{s_{(w')^{-1}\beta}\cdot\lambda}$ is an isomorphism, it also holds for $M^{w'}_{s_{(w')^{-1}\beta}\cdot\lambda}$, and thus (again by the five lemma) for $M_\lambda^{w'}$, as required. □

We will use the following result in Section 3.6.

Proposition 2.8.3. *Assume β is a simple root with $w = w_0^M s_\beta \in {}^M W$ and $\langle \lambda, \beta^\vee \rangle \notin \mathbf{Z}_{\geq 0}$. Then*

$$E \otimes^L_{U(\mathfrak{u}_{\mathfrak{p}_w})} M_\lambda^{Id} = M(\mathfrak{m}_w)^{Id}_{w^{-1}(w\cdot\lambda + 2\rho^M)}[1]$$

Proof. By the hypothesis that $\langle \lambda, \beta^\vee \rangle \notin \mathbf{Z}_{\geq 0}$, there is an isomorphism $M_\lambda^{Id} \xrightarrow{\sim} M_\lambda^{s_\beta}$ by Proposition 2.6.4 (2). The result then follows from Proposition 2.4.10 (taking w there to be $w_{0,M} w$). □

Remark 2.8.4. We note that the condition that $w_0^M s_\beta \in {}^M W$ is equivalent to β not being a root of $\mathfrak{m}_{w_0^M}$.

3. Equivariant sheaves on the flag variety and localization

3.1. **Introduction.** This entire section is concerned with geometric representation theory. We fix a split reductive group G, a Parabolic P with Levi M, and a Borel $B \subseteq P$. We consider the partial flag variety $P\backslash G$ and its Bruhat decomposition $P\backslash G = \coprod_{w \in {}^M W} P\backslash PwB$ into B-orbits, indexed by the subset ${}^M W$ of Kostant representatives of the Weyl group W of G.

In Section 3.2 and Section 3.3 we consider equivariant sheaves on the partial flag variety as well as (dagger neighbourhoods of) Bruhat cells, for the action of G, its Lie algebra $\mathfrak{g}$, or a Borel subgroup B, depending on the context. We also establish the connection between these equivariant sheaves and twisted D-modules and introduce the horizontal Levi action. We begin with some generalities on equivariant sheaves on adic and dagger spaces, before turning to the specific cases that we need. We repeatedly make use of the standard equivalence (given by passage to the fibre at a point $x \in X$) between H-equivariant sheaves on a space X on which the group H acts transitively, and the representations of the stabilizer group $\mathrm{Stab}_H(x)$; however, since we are working with topological (or rather solid) sheaves, we have to go to some lengths to make precise the categories that we are working with, and their interactions with these equivalences. (The particular categories that we work with are ultimately dictated by the use of geometric Sen theory in Section 4.)

Remark 3.1.1. All the sheaves we consider will be sheaves on topological spaces, valued in the category of solid E-vector spaces (where E is a finite extension of $\mathbf{Q}_p$). These form an abelian category. Our topological spaces will usually be adic spaces or dagger spaces, and our sheaves will also be "quasi-coherent" and often be twisted D-modules. This means that the objects we manipulate would naturally fit in the formalism of quasi-coherent sheaves on adic spaces of [And21], and the formalism of analytic geometry and the de Rham stack of [RC24]. The much simpler perspective we adopt is sufficient for our purposes.

This preliminary material is used in Section 3.4 to produce, for any $w \in {}^M W$, a functor HCS (for "Higher Coleman sheaf") from category $\mathcal{O}$ for $\mathfrak{m}_w$ (the Lie algebra of $w^{-1}Mw$), to the category of $(\mathfrak{g}, B)$-equivariant sheaves on the dagger neighborhood of the Bruhat cell $P\backslash PwB$. In section 4 we will use these sheaves to produce sheaves on (open subsets of) Shimura varieties whose cohomology with support is Higher Coleman theory of [BP21].

In Section 3.5 we define our localization functor on the partial flag variety. This functor goes from category $\mathcal{O}$ for $\mathfrak{g}$ to twisted D-modules on the flag variety. In Theorem 3.5.11 we describe the localization in terms of Higher Coleman sheaves. Namely, in p-adically non-Liouville weight, the restrictions to the Bruhat cells of the cohomology sheaves of the localization of a Verma module M of $\mathfrak{g}$ are given by the Higher Coleman sheaves associated to the $\mathfrak{u}_{P_w}$-homology of M. (It is here that we use Theorem 2.3.32.) Furthermore we give an explicit filtration on these sheaves in Corollary 3.5.20.

Finally in Section 3.6 we specialize to the case $G = \mathrm{GSp}_4$ and prove the crucial Theorem 3.6.9, which describes the cohomology of the horizontal Cartan action on the localization in a special case of interest to us.

3.2. **Equivariant sheaves on partial flag varieties.** In this section we discuss several kind of equivariant sheaves.

3.2.1. *Equivariant sheaves over adic spaces.* Let C be a rank one field extension of E. In applications, C is either E or $\mathbf{C}_p$. Let X be an adic space which is locally of finite type over $\mathrm{Spa}(C, \mathcal{O}_C)$. Its structure sheaf $\mathcal{O}_X$ is naturally a topological sheaf, whose value on a quasi-compact open subset is a Banach space. It follows that we can think of $\mathcal{O}_X$ as taking values in the category $\mathrm{Mod}(E)$. All the sheaves we will encounter will be sheaves of solid E-vector spaces. By a solid $\mathcal{O}_X$-module we mean a sheaf valued in the category $\mathrm{Mod}(E)$ equipped with an $\mathcal{O}_X$-module structure. We emphasize that we do not impose any kind of quasi-coherence condition in the definition of solid $\mathcal{O}_X$-modules.

Definition 3.2.2.

(1) A sheaf $\mathscr{F}$ of solid $\mathcal{O}_X$-modules is an *orthonormalizable Banach sheaf* if there exists a Banach space V over E such that $\mathscr{F} = \mathcal{O}_X \otimes_E V$.
(2) A sheaf $\mathscr{F}$ of solid $\mathcal{O}_X$-modules is a *summand of orthonormalizable Banach sheaf* if it is a direct summand of an orthonormalizable Banach sheaf.
(3) A sheaf $\mathscr{F}$ of solid $\mathcal{O}_X$-modules is a *Banach sheaf* if there is a covering $X = \cup_i \mathrm{Spa}(A_i, A_i^+)$ and a Banach space V_i over E such that $\mathscr{F}|_{\mathrm{Spa}(A_i,A_i^+)}$ is a direct summand of the sheaf $\mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)} \otimes_E V_i$.
(4) A sheaf $\mathscr{F}$ of solid $\mathcal{O}_X$-modules is an LB-sheaf if there is a covering $X = \cup_i \mathrm{Spa}(A_i, A_i^+)$ and LB-spaces V_i over E such that $\mathscr{F}|_{\mathrm{Spa}(A_i,A_i^+)}$ is a direct summand of the sheaf $\mathcal{O}_{\mathrm{Spa}(A_i,A_i^+)} \otimes_E V_i$.

Banach sheaves define a category $B(X)$ and LB-sheaves define a category $LB(X)$.

Let G be an analytic group acting on X. We have two maps $\mathrm{act}, p : G \times X \to X$, which are respectively the action and projection maps. We let $B_G(X)$ be the category of G-equivariant Banach sheaves, whose objects are objects $\mathscr{F}$ of $B(X)$ together with an isomorphism $\mathrm{act}^*\mathscr{F} \to p^*\mathscr{F}$ (in the category $B(G \times X)$) satisfying the usual cocycle condition. We let $\{G_n\}_{n \in \mathbf{Z}_{\geq 0}}$ be a system of neighborhoods of the identity e in G, given by quasi-compact open subgroups.

Definition 3.2.3. The category $LB_G(X)$ is the category whose objects are objects $\mathscr{F}$ of $LB(X)$ together with an isomorphism $\mathrm{act}^*\mathscr{F} \to p^*\mathscr{F}$ (in the category $LB(G\times X)$) satisfying the usual cocycle condition, and further satisfying the following finiteness condition:

(1) There exists a covering $X = \cup_j U_j$ such that $\mathscr{F}|_{U_j} = \mathrm{colim}_{r\geq 0}\mathscr{F}_{j,r}$ is a filtered countable inductive limit of orthonormalizable Banach sheaves with injective transition maps.
(2) For all j, there exists a quasi-compact open subgroup $G_{r(j)} \subseteq G$ which stabilizes U_j and we can upgrade $\mathscr{F}_{j,r}$ to an object of $B_{G_{r(j)}}(U_j)$, in such a way that the inductive system $\{\mathscr{F}_{j,r}\}$ is an inductive system in $B_{G_{r(j)}}(U_j)$.
(3) The two $G_{r(j)}$-actions on $\mathscr{F}|_{U_j}$ (the one induced by the inclusion $G_{r(j)} \hookrightarrow G$, and the one obtained by taking the colimit of the $\mathscr{F}_{j,r}$) are the same.

We let $\mathfrak{g}$ be the Lie-algebra of G. The action of G on X induces an action of $\mathfrak{g}$ by derivations on $\mathcal{O}_X$. We let $\mathrm{Mod}'_{\mathfrak{g}}(X)$ be the following category: its objects are solid $\mathcal{O}_X$-modules $\mathscr{F}$ together with a map $\mathfrak{g}\otimes_E \mathscr{F} \to \mathscr{F}$ inducing an action of $\mathfrak{g}$ on $\mathscr{F}$ by derivations in the following sense:

(1) For any $X, Y \in \mathfrak{g}$, we have $[X,Y] = XY - YX$ in $\mathrm{End}(\mathscr{F})$.
(2) For any $(X, a, f) \in \mathfrak{g}\times\mathcal{O}_X\times\mathscr{F}$, we have $X(af) = X(a)f + aX(f)$.

Definition 3.2.4. We let $\mathrm{Mod}_{\mathfrak{g}}(X)$ be the subcategory of $\mathrm{Mod}'_{\mathfrak{g}}(X)$ generated under colimits by objects $\mathscr{F}$ which have the following property: for any quasi-compact open subset U of X, there exists r such that the action $\mathfrak{g}\otimes\mathscr{F}(U) \to \mathscr{F}(U)$ can be integrated to an action $D(G_r)\otimes\mathscr{F}(U) \to \mathscr{F}(U)$.

Lemma 3.2.5. *The categories* $\mathrm{Mod}_{\mathfrak{g}}(X)$ *and* $\mathrm{Mod}'_{\mathfrak{g}}(X)$ *are Grothendieck abelian category, and in particular have enough injectives.*

Proof. We begin with the case of $\mathrm{Mod}'_{\mathfrak{g}}(X)$, where the only non-obvious point is the existence of a set of generators. For this we may take the sheaves $j_!(U(\mathfrak{g})\otimes R\otimes\mathcal{O}_U)$ for R a generator of the category of solid E-modules, $j : U \hookrightarrow X$ a quasi-compact open subset and $r \in \mathbf{Q}_{\geq 0}$. We now turn to $\mathrm{Mod}_{\mathfrak{g}}(X)$, where we first make a comment on the condition that the action $\mathfrak{g}\otimes\mathscr{F}(U) \to \mathscr{F}(U)$ can be integrated to an action $D(G_r)\otimes\mathscr{F}(U) \to \mathscr{F}(U)$ for some r. Let $D(G_{r^+}) = \lim_{r'>r} D(G_{r'})$. Then $D(G_{r^+})\otimes_{U(\mathfrak{g})} D(G_{r^+}) = D(G_{r^+})$ by [RJRC22, Lem. 5.13]. As a result, the extension of the $\mathfrak{g}$-action to an action of $D(G_{r^+})$ for some r is a property of $\mathscr{F}(U)$ and not some extra data: it means that $\mathscr{F}(U)\otimes_{U(\mathfrak{g})} D(G_{r^+}) = \mathscr{F}(U)$ for some r. By construction $\mathrm{Mod}_{\mathfrak{g}}(X)$ is an abelian subcategory of $\mathrm{Mod}'_{\mathfrak{g}}(X)$ stable under colimits, and filtered colimits are exact since they are exact in $\mathrm{Mod}'_{\mathfrak{g}}(X)$. Then a set of generators is given by the sheaves $j_!(D(G_r)\otimes R\otimes\mathcal{O}_U)$ for R, U as above. □

We define the subcategory $LB_{\mathfrak{g}}(X)$ of $\mathrm{Mod}_{\mathfrak{g}}(X)$ as follows.

Definition 3.2.6. The objects of $LB_{\mathfrak{g}}(X)$ are LB-sheaves $\mathscr{F}$ on X together with a map $\mathfrak{g}\otimes_E\mathscr{F} \to \mathscr{F}$ inducing an action of $\mathfrak{g}$ on $\mathscr{F}$ by derivations. We furthermore impose that the $\mathfrak{g}$-action can locally be integrated to a locally analytic action. Here is the precise condition:

(1) There exists a covering $X = \cup_j U_j$ such that $\mathscr{F}|_{U_j} = \mathrm{colim}_{r\geq 0}\mathscr{F}_{j,r}$ is an inductive limit of orthonormalizable Banach sheaves with injective transition maps.

(2) For r large enough, we can upgrade $\mathscr{F}_{j,r}$ to an object of $B_{G_r}(U_j)$, in such a way that the transition maps $\mathscr{F}_{j,r} \to \mathscr{F}_{j,r'}$ are equivariant for the maps $G_{r'} \to G_r$.
(3) The two $\mathfrak{g}$-actions on $\mathscr{F}|_{U_j}$ (the one induced by differentiating the action of G_r and passing to the colimit, and the one which is part of the original data) are the same.

3.2.7. *Equivariant sheaves over topological and ringed spaces.* We also need to consider the situation where a locally profinite group M acts continuously on a locally spectral topological space X.

Lemma 3.2.8. *Let V be a quasi-compact open subset of X. There is a compact open subgroup N of M such that $N \cdot V = V$.*

Proof. The map act : $M \times X \to X$ is continuous. It follows that $act^{-1}(V)$ is open, so that for any $x \in V$, there exists a compact open subgroup N_x of M and an open neighborhood $V_x \subseteq V$ of x such that $N_x \cdot V_x \subseteq V$. Since $V = \cup_{x \in V} V_x$, and V is quasi-compact, there is a finite collection of elements $\{x_i\}_{i \in I}$ such that $V = \cup_i V_{x_i}$. We deduce that $N = \cap_i N_{x_i}$ works. □

Definition 3.2.9. We let $\mathrm{Mod}_M(X)$ be the category consisting of:
(1) A $\mathrm{Mod}(E)$-valued sheaf $\mathscr{F}$ on X.
(2) An abstract action of M on $\mathscr{F}$ (that is for any $m \in M$, an isomorphism $m^{-1}\mathscr{F} \to \mathscr{F}$ satisfying compatibility conditions for various m).
(3) For any quasi-compact open subset $V \subseteq X$, for one (equivalently for any) compact open subgroup N_V of M stabilizing V, the abstract action of N_V on $\mathscr{F}(V)$ extends to a $E_\blacksquare[N_V]$-module structure.
(4) For any $U \subseteq V$, and for one (equivalently any) quasi-compact open subgroup $N_{V,U}$ stabilizing U and V, the restriction maps $\mathscr{F}(V) \to \mathscr{F}(U)$ are $E_\blacksquare[N_{V,U}]$-equivariant.

If in (3), the action of N_V is smooth, we say that $\mathscr{F}$ is smooth. We thus have a subcategory category $\mathrm{Mod}^{\mathrm{sm}}_M(X)$ of $\mathrm{Mod}_M(X)$.

Lemma 3.2.10. *The categories $\mathrm{Mod}^{\mathrm{sm}}_M(X)$ and $\mathrm{Mod}_M(X)$ are Grothendieck abelian categories, and in particular they have enough injectives.*

Proof. All claims are obvious, except for the existence of generators. Let us first prove the existence of a generator in $\mathrm{Mod}_M(X)$. Let U be a quasi-compact open in X and let N be a compact open subgroup stabilizing U. Let R be a generator of the category $\mathrm{Mod}_N(E)$. We consider the sheaf $L(U) = \oplus_{m \in M/N} j_{mU,!}R$ where $j_{mU} : mU \to X$ is the open immersion. It is endowed with the obvious M-equivariant action. Let $\mathscr{F}$ be an object of $\mathrm{Mod}_M(X)$. A map $L(U) \to \mathscr{F}$ amounts to a map $R \to \mathscr{F}(U)$ in the category $\mathrm{Mod}_N(E)$. It follows that $\oplus_U L(U)$ is a generator of $\mathrm{Mod}_M(X)$. We construct similarly a generator of $\mathrm{Mod}^{\mathrm{sm}}_M(X)$ by the same construction, but replacing R by a generator of $\mathrm{Mod}^{\mathrm{sm}}_N(E)$. □

Let us briefly indicate some possible variations. If X is equipped with a sheaf of algebras in solid E-vector spaces $\mathcal{O}_X$, which belongs to $\mathrm{Mod}_M(X)$, one can consider the category $\mathrm{Mod}_M(\mathcal{O}_X)$ of M-equivariant $\mathcal{O}_X$-modules. If the M-equivariant sheaf $\mathcal{O}_X$ is smooth, we also have a category $\mathrm{Mod}^{\mathrm{sm}}_M(\mathcal{O}_X)$. In this case, we can also introduce a twist by a character $\lambda : M \to E^\times$. We say that an object $\mathscr{F}$ of $\mathrm{Mod}_M(\mathcal{O}_X)$ is λ-smooth if $\mathscr{F} \otimes E(-\lambda)$ is smooth. The category of λ-smooth objects is denoted by $\mathrm{Mod}_M^{\lambda-\mathrm{sm}}(X)$. The categories $\mathrm{Mod}_M(\mathcal{O}_X)$, $\mathrm{Mod}_M^{\lambda-\mathrm{sm}}(\mathcal{O}_X)$ are again Grothendieck abelian categories, and in particular they have enough injectives.

3.2.11. *Dagger spaces.* We need to enlarge the category of adic spaces and also consider certain limits of adic spaces (for example dagger spaces in the sense of [GK00]). Let Z be a locally closed subset of X. We let $Z^{\dagger,X}$ be the locally ringed space $\lim_{Z\subseteq U} U$ where U runs through the open subsets of X containing Z. As a topological space, $Z^{\dagger,X} = Z$. It carries the structure sheaf $\mathcal{O}_{Z^{\dagger,X}} = i_Z^{-1}\mathcal{O}_X$ where $i_Z : Z \to X$ is the inclusion. This sheaf also takes values in the category $\mathrm{Mod}(E)$.

Remark 3.2.12. The locally ringed space $Z^{\dagger,X}$ depends on Z and on the embedding $Z \hookrightarrow X$. If X is clear from the context, we simply denote $Z^{\dagger,X}$ by $Z^\dagger$.

3.2.13. *Notations for the flag variety.* We let G be a connected split reductive group over E and we let P be a parabolic subgroup. We let $\mathcal{FL} = P\backslash G$ be the partial flag variety (an analytic adic space).

Remark 3.2.14. In section 4, we will use the notation $\mathcal{FL}^{rat}$ for this analytic space, and we will let $\mathcal{FL}$ be its base change to Spa $(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$.

We let U_P be the unipotent radical of P, with Levi quotient $M = P/U_P$. We let $B \subseteq P$ be a Borel with maximal torus T and unipotent radical U_B. We let B_M be the induced Borel on M and U_M be the unipotent radical of B_M. We use gothic letters for the Lie algebras of all groups introduced so far, so that for example $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{b}$ is the Lie algebra of B; the one exception is that following standard conventions, the Lie algebra of T is denoted by $\mathfrak{h}$. For any $x \in \mathcal{FL}$, we let P_x be $x^{-1}Px$, and let U_{P_x} be its unipotent radical. We adopt similar notation for other groups or Lie algebras, and in addition for x replaced by an element w of a Weyl group; so for example for $w \in W$ (the Weyl group of G) we have P_w, U_{P_w} and so on.

From now on G is viewed as an analytic group over $\mathrm{Spa}(E, \mathcal{O}_E)$ (this is a Stein space and is not quasi-compact unless $G = \{1\}$).

We fix a reductive model for G over $\mathcal{O}_E$. Its analytification defines a quasi-compact open subgroup $G_0 \subseteq G$. For any $r \in \mathbf{Q}_{\geq 0}$, we let G_r be the quasi-compact analytic subgroup of G_0 of elements reducing to the identity e modulo p^r. Here are some slightly non-standard conventions and constructions:

- If H is an analytic subgroup of G, we let $H_r = G_r \cap H$.
- If H is an analytic subgroup of G, we let $H_e := \lim_{r\geq 0} H_r$ where the limit is taken in the category of locally ringed spaces. Thus $H_e = \{e\}^{\dagger,H}$. As a space, H_e has only one point (the identity e of G), but it carries the structure sheaf $\mathcal{O}_{H,e}$ whose dual is the distribution algebra $\hat{U}(\mathrm{Lie}(H))$.

3.2.15. *G-equivariant sheaves.* In section 3.2.1, we have introduced the categories $B_G(\mathcal{FL})$ and $LB_G(\mathcal{FL})$. We also have the categories of representations $B_{P_w}(E) = B_{P_w}(\mathrm{Spa}(E, \mathcal{O}_E))$ and $LB_{P_w}(E) = LB_{P_w}(\mathrm{Spa}(E, \mathcal{O}_E))$ for each $w \in W$.

Proposition 3.2.16. *Taking the fiber at $w = P\backslash Pw \in \mathcal{FL}$ gives equivalences of categories between $B_G(\mathcal{FL})$ and $B_{P_w}(E)$ and between $LB_G(\mathcal{FL})$ and $LB_{P_w}(E)$.*

Proof. We have an isomorphism $P\backslash G \xrightarrow{\sim} P_w\backslash G$, $x \mapsto w^{-1}\cdot x$, which takes $w \in \mathcal{FL}$ to $e \in P_w\backslash G$; so we can and do reduce to the case $w = e$. Given a G-equivariant sheaf $\mathcal{V}$, we take its fiber at $e = P\backslash P \in \mathcal{FL}$, which is a representation of P.

Conversely, let $\pi : G \to \mathcal{FL}$, $g \mapsto eg$ be the uniformization map. The sheaf $\pi_*\mathcal{O}_G$ is G-equivariant (via the action of G by right translation on itself) and carries a P-action (via the action of P by left translation on G). Given an object V of $LB_P(E)$, we consider $(\pi_*\mathcal{O}_G \otimes V)^P$. These two functors define the equivalences of categories of the proposition (and in particular match the various finiteness conditions); we leave the details to the reader. □

Example 3.2.17. We have a filtration $\mathfrak{u}_P \subseteq \mathfrak{p} \subseteq \mathfrak{g}$ of finite dimensional P-representations. Via our equivalence of categories, this corresponds to a filtration of G-equivariant coherent sheaves: $\mathfrak{u}_P^0 \subseteq \mathfrak{p}^0 \subseteq \mathfrak{g}^0 = \mathcal{O}_{\mathcal{FL}} \otimes \mathfrak{g}$. The fibers of this filtration at a point $x \in \mathcal{FL}$ are $\mathfrak{u}_{P_x} = x^{-1}\mathfrak{u}_P x \subseteq \mathfrak{p}_x = x^{-1}\mathfrak{p}x \subseteq E(x) \otimes \mathfrak{g}$. Moreover, we have an isomorphism $\mathfrak{g}^0/\mathfrak{p}^0 = T_{\mathcal{FL}}$.

Example 3.2.18. Let $\lambda \in X^*(T)^{M,+}$. There is an associated highest weight representation of M and via our equivalence of categories, this corresponds to a G-equivariant coherent sheaf $\mathcal{L}_\lambda$. Here is an equivalent geometric construction of this sheaf. Let $\pi : U_P\backslash G \to \mathcal{FL}$. This is a G-equivariant M-torsor. Then $\mathcal{L}_\lambda = (\pi_*\mathcal{O}_{U_P\backslash G})[B_M = -w_{0,M}\lambda]$, with the right translation action of G.

Remark 3.2.19. In the Siegel case, the tautological exact sequence over $\mathcal{FL}$ is (for $St = E^{2g}$ the standard $2g$-dimensional representation of G):

$$0 \to \mathcal{L}_{(0,\cdots,0,-1;1)} \to \mathcal{O}_{\mathcal{FL}} \otimes St \to \mathcal{L}_{(1,0,\cdots,0;1)} \to 0.$$

3.2.20. *$\mathfrak{g}$-equivariant sheaves and the horizontal action.* Let U be an open subset of $\mathcal{FL}$. By Definition 3.2.6, we have a category of $\mathfrak{g}$-equivariant sheaves $LB_{\mathfrak{g}}(U)$.

On any object of $LB_{\mathfrak{g}}(U)$, the $\mathfrak{g}$-action extends linearly to an $\mathcal{O}_U \otimes \mathfrak{g}$-action. We recall that we have the moving parabolic Lie-algebra $\mathfrak{p}^0 \subseteq \mathcal{O}_U \otimes \mathfrak{g}$.

Lemma 3.2.21. *We have that $\mathfrak{p}^0 \subseteq \mathcal{O}_U \otimes \mathfrak{g}$ acts $\mathcal{O}_U$-linearly and G-equivariantly on any object of $LB_{\mathfrak{g}}(U)$.*

Proof. We have that $\mathfrak{p}^0$ is a G-equivariant subsheaf of $\mathcal{O}_{\mathcal{FL}} \otimes \mathfrak{g}$. Moreover, it acts trivially on $\mathcal{O}_{\mathcal{FL}}$ since $T_{\mathcal{FL}} = \mathfrak{g}^0/\mathfrak{p}^0$. □

Definition 3.2.22. We let $LB_{\mathfrak{g}}(U)^{\mathfrak{u}_P^0}$ be the full subcategory of $LB_{\mathfrak{g}}(U)$ of objects which are annihilated by $\mathfrak{u}_P^0$.

For any $\mathscr{F} \in LB_{\mathfrak{g}}(U)^{\mathfrak{u}_P^0}$, we have a G-equivariant map $\mathfrak{m}^0 = \mathfrak{p}^0/\mathfrak{u}_P^0 \to \underline{\mathrm{End}}_{\mathcal{O}_U}(\mathscr{F})$ which can be extended to an algebra map:

$$U(\mathfrak{m}^0) \to \underline{\mathrm{End}}_{\mathcal{O}_U}(\mathscr{F}). \tag{3.2.23}$$

Let $Z(\mathfrak{m})$ be the centre of $U(\mathfrak{m})$.

Lemma 3.2.24. *We have an injective algebra homomorphism $Z(\mathfrak{m}) \hookrightarrow H^0(\mathcal{FL}, U(\mathfrak{m}^0))$.*

Proof. The G-equivariant sheaf $U(\mathfrak{m}^0)$ is associated via Proposition 3.2.16 to the P-representation $U(\mathfrak{m})$ (the fiber at e). We have a natural inclusion $Z(\mathfrak{m}) \hookrightarrow U(\mathfrak{m})$, and $Z(\mathfrak{m})$ identifies with the P-invariant subspace of $U(\mathfrak{m})$. It follows that we get an injective map of sheaves $\mathcal{O}_{\mathcal{FL}} \otimes Z(\mathfrak{m}) \to U(\mathfrak{m}^0)$, inducing the expected map on global sections. □

Definition 3.2.25. We define the horizontal action as the map $\Theta_{\mathrm{hor}} : Z(\mathfrak{m}) \to \mathrm{End}_{\mathcal{O}_U}(\mathscr{F})$ obtained by composing the map of Lemma 3.2.24 and the map

$$H^0(U, U(\mathfrak{m}^0)) \to \mathrm{End}_{\mathcal{O}_U}(\mathscr{F})$$

obtained from (3.2.23).

3.2.26. $(\mathfrak{g}, G)$-*equivariant sheaves.* We now consider $(\mathfrak{g}, G)$-equivariant sheaves. We sometimes find it helpful to interpret these as $G_e \rtimes G$-equivariant sheaves, where the action of G on G_e is via conjugation, see Remark 3.2.28 below. We remark there is a group homomorphism $G_e \rtimes G \to G$, $(g_e, g) \mapsto g_e g$.

Definition 3.2.27. The category $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ has objects consisting of a G-equivariant sheaf $\mathscr{F} \in LB_G(\mathcal{FL})$ together with a map $\mathfrak{g} \otimes \mathscr{F} \to \mathscr{F}$ of G-equivariant sheaves (where $\mathfrak{g} \otimes \mathscr{F}$ carries the diagonal G-action), giving a Lie algebra action on $\mathscr{F}$:

(1) For any $X, Y \in \mathfrak{g}$, we have $[X, Y] = XY - YX$ in $\mathrm{End}(\mathscr{F})$.
(2) For any $(X, a, f) \in \mathfrak{g} \times \mathcal{O}_{\mathcal{FL}} \times \mathscr{F}$, we have $X(af) = X(a)f + aX(f)$[4].

We furthermore impose that the $\mathfrak{g}$-action can locally be integrated to a locally analytic action. Here is the precise condition:

(1) There exists a covering $\mathcal{FL} = \cup_j U_j$ such that $\mathscr{F}|_{U_j} = \mathrm{colim}_{r \geq 0} \mathscr{F}_{j,r}$ is an inductive limit of Banach sheaves with injective transition maps.
(2) For all j, there exists a quasi-compact open subgroup $G_{r(j)} \subseteq G$ which stabilizes U_j.
(3) For r large enough, we can upgrade $\mathscr{F}_{j,r}$ to an object of $B_{G_r \rtimes G_{r(j)}}(U_j)$, in such a way that the inductive system $\{\mathscr{F}_{j,r}\}$ is now in $B_{G_r \rtimes G_{r(j)}}(U_j)$.
(4) The two $G_{r(j)}$-actions on $\mathscr{F}|_{U_j}$ (the one induced by the inclusion $G_{r(j)} \hookrightarrow G$, and the one obtained by taking the colimit of the $\mathscr{F}_{j,r}$) are the same.
(5) The two $\mathfrak{g}$-actions on $\mathscr{F}|_{U_j}$ (the one induced by differentiating the action of G_r and passing to the colimit, and the one which is part of the original data) are the same.

Remark 3.2.28. In particular, the $\mathfrak{g}$-action on an object $\mathcal{F}$ of $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ can be upgraded to a $\hat{U}(\mathfrak{g})$-action on $\mathscr{F}$. We can thus think of an object $\mathscr{F}$ of $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ as a $G_e \rtimes G$-equivariant sheaf satisfying certain finiteness conditions.

Remark 3.2.29. In section 4, we will also consider the category $LB_{(\mathfrak{g},G)}(\mathcal{FL}_{\mathbf{C}_p})$ for the flag variety over $\mathbf{C}_p$, whose definition is the obvious variant of the definition 3.2.27. There is a base change map $LB_{(\mathfrak{g},G)}(\mathcal{FL}) \to LB_{(\mathfrak{g},G)}(\mathcal{FL}_{\mathbf{C}_p})$.

Definition 3.2.30. We define $LB_{(\mathfrak{g},P_w)}(E)$ to be the category whose objects consist of an object V of $LB_{P_w}(E)$, together with a P_w-equivariant morphism $\mathfrak{g} \otimes V \to V$ in the category $LB_{P_w}(E)$, inducing a Lie algebra action of $\mathfrak{g}$ on V: for any $X, Y \in \mathfrak{g}$, we have $[X, Y] = XY - YX$ in $\mathrm{End}(V)$. We further impose the following finiteness condition:

(1) $V = \mathrm{colim}_r V_r$ is an inductive limit of Banach spaces with injective transition maps.
(2) There exists s such that for all r large enough, V_r can be upgraded to an object of $B_{G_r \rtimes P_{w,s}}(E)$ and the maps $V_r \to V_{r'}$ are equivariant for the map $G_{r'} \rtimes P_{w,s} \to G_r \rtimes P_{w,s}$.
(3) The action of $P_{w,s}$ on V obtained on the limit is the one induced by restriction from P_w to $P_{w,s}$.
(4) The action of G_r on V_r induces an action of $\mathfrak{g}$, and the action of $\mathfrak{g}$ on V coincides with the original action of $\mathfrak{g}$.

[4]We have a map $\mathfrak{g} \to T_{\mathcal{FL}}$ and thus an action of $\mathfrak{g}$ by derivations on $\mathcal{O}_{\mathcal{FL}}$.

Similarly to Proposition 3.2.16, we have the following equivalence of categories. As usual, this equivalence is obtained by passage to a fiber; note in particular that we are not simply restricting the $\mathfrak{g}$ and G-actions, and indeed the action of $\mathfrak{g}$ is obtained as the difference between the given $\mathfrak{g}$-action and the derivative of the G-action.

Proposition 3.2.31. *Taking the fiber at w induces an equivalence of categories between $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ and $LB_{(\mathfrak{g},P_w)}(E)$.*

Proof. As in the proof of Proposition 3.2.16, we can without loss of generality take $w = e$. We consider the uniformization map $m : G_e \rtimes G \to \mathcal{FL}$, $(g_e, g) \mapsto eg_eg$. The stabilizer $\mathrm{Stab}(e)$ of e is the subgroup of elements $(g_e, g) \in G_e \rtimes G$, such that $g_eg \in P$. This is also the semi-direct product $G_e \rtimes P$ with $G_e = \{(g_e^{-1}, g_e)\}$ and $P = \{(1, p)\}$. A $(\mathfrak{g}, G)$-equivariant sheaf $\mathcal{V}$ gives a $(\mathfrak{g}, P)$-module by taking the fiber at e. Conversely, given a $\mathrm{Stab}(e)$-representation V, we consider the sheaf $\mathcal{V} = (m_*(\mathcal{O}_{G_e\rtimes G})\otimes V)^{G_e\rtimes P}$. It has an action of G_e given by $g_ef(g'_e, g') = f(g'_eg'g_e(g')^{-1}, g')$ and an action of G given by $g.f(g'_e, g') = f(g'_e, g'g)$. We check that this induces an equivalence between $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ and $LB_{(\mathfrak{g},P)}(E)$. For example assume that V is an object of $LB_{(\mathfrak{g},P)}(E)$, thus $V = \mathrm{colim}_r V_r$ where each V_r carries an action of $G_r \rtimes P_s$ for s and r large enough. We can consider the map $G_r \rtimes G_s \to \mathcal{FL}$, $(g, g') \mapsto eg'g$. The image is a neighborhood U of e and the stabilizer of e is $G_r \rtimes P_s$. We deduce that $\mathcal{V}|_U = \mathrm{colim}_r(m_*(\mathcal{O}_{G_r\rtimes G_s}) \otimes V_r)^{G_r\rtimes P_s}$ which proves that $\mathcal{V}$ is indeed an object of $LB_{(\mathfrak{g},G)}(\mathcal{FL})$. The reverse computation is left to the reader. ☐

Remark 3.2.32. We can use the group homomorphism $G_e \rtimes G \to G$, $(g_e, g) \mapsto g_eg$ to turn a G-equivariant sheaf into a $G_e \rtimes G$-equivariant sheaf. This defines a natural fully faithful functor $LB_G(\mathcal{FL}) \to LB_{(\mathfrak{g},G)}(\mathcal{FL})$. We can also interpret this functor as saying that a G-equivariant sheaf is naturally a $(\mathfrak{g}, G)$-equivariant sheaf by differentiating the G-action. There is also a forgetful functor $LB_{(\mathfrak{g},G)}(\mathcal{FL}) \to LB_G(\mathcal{FL})$.

Via the equivalences of categories of Propositions 3.2.16 and 3.2.31, the functor from G-equivariant sheaves to $(\mathfrak{g}, G)$-equivariant sheaves amounts to associating to a P-representation the $(\mathfrak{g}, P)$-representation with trivial $\mathfrak{g}$-action. Indeed, the $(\mathfrak{g}, G)$-equivariant sheaf corresponding to a $(\mathfrak{g}, P)$-representation V is $\mathscr{F} = (m_*\mathcal{O}_{G_e\rtimes G} \otimes V)^{G_e\rtimes P}$. If V has trivial $\mathfrak{g}$-action, then $f(g_eg'_e, (g'_e)^{-1}g) = f(g_e, g)$, and we deduce that $f(g_e, gg'_e) = f(g_eg(g_e)'g^{-1}, g)$, which means that the two $\mathfrak{g}$-actions (the obvious one and the one coming from the G-action) coincide. The converse implication is similar. The forgetful functor from $(\mathfrak{g}, G)$-equivariant sheaves to G-equivariant sheaves corresponds to the forgetful functor associating to a $(\mathfrak{g}, P)$-representation the underlying P-representation.

Definition 3.2.33. We let $LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}_P^0}$ be the full subcategory of $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ of objects which are annihilated by $\mathfrak{u}_P^0$.

We let $LB_{(\mathfrak{g},P)}(E)^{\mathfrak{u}_P}$ be the full subcategory of $LB_{(\mathfrak{g},P)}(E)$ whose objects have the property that the $\mathfrak{u}_P$-action coming from differentiating the P-action coincides with the $\mathfrak{u}_P$-action coming from the $\mathfrak{g}$-action. (We will shortly see that this is equivalent to the category $LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}_P^0}$.)

Any object of $LB_{(\mathfrak{g},P)}(E)^{\mathfrak{u}_P}$ carries an action Θ of $Z(\mathfrak{m})$ defined as follows. The differentiation of the P-action gives an action $d\rho_P$ of $\mathfrak{p}$. On the other hand the action $\rho_{\mathfrak{g}}$ of $\mathfrak{g}$ restricts to an action of $\mathfrak{p}$. We then let $z \in \mathfrak{m}$ act via the formula

$$z \cdot v = d\rho_P(\tilde{z}) \cdot v - \rho_{\mathfrak{g}}(\tilde{z}) \cdot v$$

for any lift $\tilde{z}$ of z in $\mathfrak{p}$ (this is independent of the lift). This induces an action of $U(\mathfrak{m})$ and restricts to an action of $Z(\mathfrak{m})$. This action commutes with the $(\mathfrak{g}, P)$-action.

Proposition 3.2.34. *The equivalence of categories between $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ and $LB_{(\mathfrak{g},P)}(E)$ of Proposition 3.2.31 induces an equivalence between $LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}_P^0}$ and $LB_{(\mathfrak{g},P)}(E)^{\mathfrak{u}_P}$. The actions Θ_{hor} and Θ of $Z(\mathfrak{m})$ correspond to each other.*

Proof. Let $V \in LB_{(\mathfrak{g},P)}(E)^{\mathfrak{u}_P}$. On the sheaf $(m_*\mathcal{O}_{G_e \rtimes G} \hat{\otimes} V)^{G_e \rtimes P}$, we want to see that the action of $\mathfrak{u}_P^0$ is trivial. We have $((g_e g)^{-1} u (g_e g)).f(g_e, g) = f(ug_e, g)$ for $u \in (U_P)_e$. On the other hand, our assumption implies that the action of $(u, 1) \in P_e \rtimes G \subseteq G_e \rtimes G$ is trivial on V (since the actions of (u, u^{-1}) and $(1, u^{-1})$ coincide). This tells us that $f(ug_e, g) = f(g_e, g)$, as required. The converse implication follows similarly. The actions of Θ_{hor} and Θ correspond by construction. □

Remark 3.2.35. Let $\mathrm{Rep}(M)$ be the category of algebraic representations of M. Then the natural functor $\mathrm{Rep}(M) \to LB_{(\mathfrak{g},P)}(E)$ (induced by inflation from M to P, and letting $\mathfrak{g}$ act trivially, see Remark 3.2.32) factors through $LB_{(\mathfrak{g},P)}(E)^{\mathfrak{u}_P}$, and the obvious action of $Z(\mathfrak{m})$ on $\mathrm{Rep}(M)$ induces the action Θ.

Remark 3.2.36. We have a natural functor

$$H^0(\mathfrak{u}_P^0, -) : LB_{(\mathfrak{g},G)}(\mathcal{FL}) \to LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}_P^0},$$

which can be defined as follows. Via our equivalence of categories, it corresponds to the natural functor $H^0(\mathfrak{u}_P, -) : LB_{(\mathfrak{g},P)}(E) \to LB_{(\mathfrak{g},P)}(E)^{\mathfrak{u}_P}$. In this last formula, the $\mathfrak{u}_P$ action is the diagonal one. More precisely, on any object V of $LB_{(\mathfrak{g},P)}(E)$, we can differentiate the P-action to obtain a $\mathfrak{p}$-action. We therefore have an action of $\mathfrak{g} \rtimes \mathfrak{p}$ and $\mathfrak{u}_P$ embeds diagonally via $u \mapsto (-u, u)$ as a normal sub-Lie algebra.

3.2.37. *Twisted differential operators and the sheaf $\mathcal{C}^{\mathrm{la}}$.*

Definition 3.2.38. Let $\tilde{\mathcal{D}}^{\mathrm{la}} = \mathcal{O}_{\mathcal{FL}} \hat{\otimes} \hat{U}(\mathfrak{g}) / \mathfrak{u}_P^0 \mathcal{O}_{\mathcal{FL}} \otimes \hat{U}(\mathfrak{g})$ be the ring of universal twisted differential operators.

Remark 3.2.39. We have that $\mathcal{D}^{\mathrm{la}} = \mathcal{O}_{\mathcal{FL}} \hat{\otimes} \hat{U}(\mathfrak{g}) / \mathfrak{p}^0 \mathcal{O}_{\mathcal{FL}} \otimes \hat{U}(\mathfrak{g})$ is the usual ring of differential operators.

Remark 3.2.40. One also has an "algebraic" version of $\tilde{\mathcal{D}}^{\mathrm{la}}$. Namely, we let $\tilde{\mathcal{D}}^{\mathrm{alg}} = \mathcal{O}_{\mathcal{FL}^{\mathrm{alg}}} \otimes U(\mathfrak{g}) / \mathfrak{u}_P^{0,\mathrm{alg}} \mathcal{O}_{\mathcal{FL}^{\mathrm{alg}}} \otimes U(\mathfrak{g})$ be the ring of (algebraic) universal twisted differential operator on the E-scheme $\mathcal{FL}^{\mathrm{alg}}$.

We have three commuting actions of G on $\mathcal{O}_G \otimes \mathcal{O}_{\mathcal{FL}}$:

(1) $h *_1 f(g, x) = f(h^{-1}g, x)$,
(2) $h *_2 f(g, x) = f(gh, x)$,
(3) $h *_3 f(g, x) = f(g, xh)$.

We write $*_{1,3}$ for the composition of the $*_1$ and $*_3$ action, and similarly for $*_{1,2,3}$ and so on.

Definition 3.2.41. We let $\mathcal{C}^{\mathrm{la}} = (\mathcal{O}_{G,e} \otimes \mathcal{O}_{\mathcal{FL}})^{\mathfrak{u}_P^0}$ where the invariants are for the $*_{1,3}$-action.

Elements of $\mathcal{C}^{\mathrm{la}}$ are functions $f(g,x)$ with $g \in G_e$, $x \in \mathcal{FL}$, satisfying $f(u_x g, x) = f(g,x)$ for $u_x \in U_{P_x,e}$.

In particular $\mathcal{C}^{\mathrm{la}}$ has a $*_{1,3}$-action of $\mathfrak{g}$ and a $*_{1,2,3}$ action of G, and it is easy to check that this gives it the structure of an object of $LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}_P^0}$. It has an extra linear $*_2$-action of $\mathfrak{g}$. Its fiber at e is the module $\mathcal{C}^{\mathrm{la}}_e = \mathcal{O}_{U_P\backslash G,e}$. Under the equivalence of 3.2.31, the $(\mathfrak{g},P)$-module structure is the conjugation action of P and the right translation action of $\mathfrak{g}$. The linear $*_2$-action of $\mathfrak{g}$ also induces the right translation action on the fiber.

Remark 3.2.42. The subsheaf $(\mathcal{C}^{\mathrm{la}})^{\mathfrak{p}^0} \subseteq (\mathcal{O}_{G,e} \otimes \mathcal{O}_{\mathcal{FL}})^{\mathfrak{u}_P^0}$ is some kind of infinite jet bundle over $\mathcal{O}_{\mathcal{FL}}$.

We have a map $\Theta_{\mathrm{hor}} : Z(\mathfrak{m}) \to \mathrm{End}_{\mathcal{O}_{\mathcal{FL}}}(\mathcal{C}^{\mathrm{la}})$. We also have a map $*_2 : Z(\mathfrak{g}) \to \mathrm{End}_{\mathcal{O}_{\mathcal{FL}}}(\mathcal{C}^{\mathrm{la}})$ induced by the $*_2$ action of $\mathfrak{g}$. These maps are related as follows:

Lemma 3.2.43. *We let $\iota : Z(\mathfrak{g}) \to Z(\mathfrak{g})$ be the map induced by the inverse on G. For any $z \in Z(\mathfrak{g})$, we have $*_2(\iota z) = \Theta_{\mathrm{hor}}(HC(z))$, where HC is the map* (2.3.11).

Proof. The endomorphisms of $\mathcal{C}^{\mathrm{la}}$, $*_2(\iota z)$ and $\Theta_{\mathrm{hor}}(HC(z))$ are G-equivariant and $\mathcal{O}_{\mathcal{FL}}$-linear. Therefore it suffices to understand what happens on the fiber at e, namely $\mathcal{O}_{U_P\backslash G,e}$. We first observe that on $\mathcal{O}_{G,e}$ the two actions $*_1$ and $*_2$ of G induce two actions of $Z(\mathfrak{g})$ and these actions are related by $*_2(z) = *_1(\iota z)$. For the second point, by the definition of the map HC, for any $z \in Z(\mathfrak{g})$, we have that $HC(z) = z + z'$ where $z' \in U(\mathfrak{g})\mathfrak{u}_P$. It follows that $*_1(z)$ and $HC(z)$ act in the same way on $\mathcal{O}_{U_P\backslash G,e} = H^0(\mathfrak{u}_P, \mathcal{O}_{G,e})$. $\square$

We have a left action of $\mathfrak{g}$ on $\mathcal{O}_{G,e}$, defined by $g.f(\cdot) = f'(\exp(-tg)\cdot)|_{t=0}$. This induces a natural pairing $\hat{U}(\mathfrak{g}) \otimes \mathcal{O}_{G,e} \to E$, $(\mathfrak{g}, f) \mapsto (g.f)(e)$. This pairing induces a pairing $(\mathcal{O}_{G,e} \hat{\otimes} \mathcal{O}_{\mathcal{FL}}) \otimes (\hat{U}(\mathfrak{g}) \hat{\otimes} \mathcal{O}_{\mathcal{FL}}) \to \mathcal{O}_{\mathcal{FL}}$. It passes to a pairing on the quotient:

$$\tilde{\mathcal{D}}^{\mathrm{la}} \otimes \mathcal{C}^{\mathrm{la}} \to \mathcal{O}_{\mathcal{FL}}.$$

Proposition 3.2.44. *We have that $\underline{\mathrm{RHom}}_{\mathcal{O}_{\mathcal{FL}}}(\mathcal{C}^{\mathrm{la}}, \mathcal{O}_{\mathcal{FL}}) = \tilde{\mathcal{D}}^{\mathrm{la}}$.*

Proof. We will prove that $\underline{\mathrm{RHom}}_{\mathcal{O}_{\mathcal{FL}}}((\mathcal{O}_{G,e} \hat{\otimes} \mathcal{O}_{\mathcal{FL}}), \mathcal{O}_{\mathcal{FL}}) = \hat{U}(\mathfrak{g}) \hat{\otimes} \mathcal{O}_{\mathcal{FL}}$. Since $\mathcal{C}^{\mathrm{la}}$ is locally a direct summand in $(\mathcal{O}_{G,e} \hat{\otimes} \mathcal{O}_{\mathcal{FL}})$ and $\tilde{\mathcal{D}}^{\mathrm{la}}$ is locally a direct summand in $\hat{U}(\mathfrak{g}) \hat{\otimes} \mathcal{O}_{\mathcal{FL}}$, this implies the claim. We take a presentation $\mathcal{O}_{G,e} = \mathrm{colim}_r V_r$ where the V_r are Smith spaces. We deduce that

$$\begin{aligned} \underline{\mathrm{RHom}}_{\mathcal{O}_{\mathcal{FL}}}(\mathcal{O}_{G,e} \otimes \mathcal{O}_{\mathcal{FL}}, \mathcal{O}_{\mathcal{FL}}) &= \mathrm{Rlim}_r \underline{\mathrm{RHom}}_E(V_r, \mathcal{O}_{\mathcal{FL}}) \\ &= \mathrm{Rlim}_r (V_r^\vee \otimes \mathcal{O}_{\mathcal{FL}}) \\ &= \lim_r (V_r^\vee \otimes \mathcal{O}_{\mathcal{FL}}) \\ &= \hat{U}(\mathfrak{g}) \otimes \mathcal{O}_{\mathcal{FL}} \end{aligned}$$

Here, the first equality is formal, the second equality is a consequence of the nuclearity of Banach spaces [RJRC22, Cor. 3.7], the third equality follows from Mittag-Leffler [RJRC22, Lem. 3.27] and the last equality follows from [RJRC22, Lem. 3.28]. $\square$

Remark 3.2.45. Using [RJRC22, Lem. 3.10], one can prove conversely that $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{FL}}}(\tilde{\mathcal{D}}^{\mathrm{la}}, \mathcal{O}_{\mathcal{FL}}) = \mathcal{C}^{\mathrm{la}}$. Conjecture 3.41 of [RJRC22] would imply that furthermore $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{FL}}}(\tilde{\mathcal{D}}^{\mathrm{la}}, \mathcal{O}_{\mathcal{FL}}) = \underline{\mathrm{RHom}}_{\mathcal{O}_{\mathcal{FL}}}(\tilde{\mathcal{D}}^{\mathrm{la}}, \mathcal{O}_{\mathcal{FL}})$.

3.2.46. *The admissible objects.* We let $\mathrm{Adm}_{(\mathfrak{g},P)}(E)$ be the subcategory of $LB_{(\mathfrak{g},P)}(E)$ whose objects are admissible $\hat{U}(\mathfrak{g})$-modules. This is an abelian category. We let $\mathrm{Adm}_{(\mathfrak{g},G)}(\mathcal{FL})$ be the subcategory of $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ which corresponds to $\mathrm{Adm}_{(\mathfrak{g},P)}(E)$.

Example 3.2.47. We see that $\mathcal{C}^{\mathrm{la}}$ is an object of $\mathrm{Adm}_{(\mathfrak{g},G)}(\mathcal{FL})$.

3.3. **Equivariant sheaves on Bruhat cells.** We will now consider the stratification of $\mathcal{FL}$ into its B-orbits (i.e. Bruhat cells).

3.3.1. $(\mathfrak{g}, Q)$-*equivariant sheaves.* Recall that ${}^{M}W$ denotes the Kostant representatives of $W_M\backslash W$, and for $w \in {}^{M}W$ we let $C_w = P\backslash PwB$ be the Bruhat cell. More generally, let Q be a standard parabolic, i.e. $B \subseteq Q$, and write M_Q for the Levi quotient of Q. We write $C_{w,Q} := P\backslash PwQ \stackrel{j}{\hookrightarrow} \mathcal{FL}$ for the corresponding Q-orbit in $\mathcal{FL}$. We write $C^{\dagger}_{w,Q} = \lim_{C_{w,Q}\subseteq U} U$ where U runs through the neighborhoods of $C_{w,Q}$ in $\mathcal{FL}$. By definition $C^{\dagger}_{w,Q}$ is the space $C_{w,Q}$ equipped with the sheaf $\mathcal{O}_{C^{\dagger}_{w,Q}} = j^{-1}\mathcal{O}_{\mathcal{FL}}$.

We consider the semi-direct product $G_e \rtimes Q$. We have a product map $G_e \rtimes Q \to G$, $(g, q) \mapsto gq$. The group $G_e \rtimes Q$ acts on $C^{\dagger}_{w,Q}$.

Definition 3.3.2. We let $LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q})$ be the category whose objects consist of the following list of data:

(1) An LB-sheaf $\mathscr{F}$ over $C^{\dagger}_{w,Q}$: this is a sheaf of $\mathcal{O}_{C^{\dagger}_{w,Q}}$-modules such that there is a covering $C_{w,Q} = \cup_i U_i$ by quasi-compact opens, and for each i a family $\{U_{i,j}\}_j$ of quasi-compact opens of $\mathcal{FL}$ with $U_{i,j} \cap C_{w,Q} = U_i$ and $\cap_j U_{i,j} = U_i$, and Banach sheaves $\mathscr{F}_{i,j}$ over $U_{i,j}$ such that $\{\mathscr{F}_{i,j}|_{U_i}\}_j$ form an inductive system and $\mathscr{F}|_{U_i} = \mathrm{colim}_j \mathscr{F}_{i,j}|_{U_i}$.
(2) We have a Q-equivariant map $\mathfrak{g} \otimes \mathscr{F} \to \mathscr{F}$ of LB-sheaves providing a Lie algebra action of $\mathfrak{g}$ on $\mathscr{F}$.
(3) For each i, there exists $s(i)$ such that each $U_{i,j}$ is stable under the action of $G_j \rtimes Q_{s(i)}$ and $\mathscr{F}_{i,j}$ is an object of $B_{G_j \rtimes Q_{s(i)}}(U_{i,j})$ and the maps $\mathscr{F}_{i,j} \to \mathscr{F}_{i,j'}$ are equivariant for the maps $G_{j'} \rtimes Q_{s(i)} \to G_j \rtimes Q_{s(i)}$.
(4) The induced action of $G_e \rtimes Q_{s(i)}$ on $\mathscr{F}|_{U_i}$ coincides with the restriction of the $(\mathfrak{g}, Q)$-action.

Remark 3.3.3. Similar to remark 3.2.29, we also can define a category $LB_{(\mathfrak{g},Q)}((C^{\dagger}_{w,Q})_{\mathbf{C}_p})$ with a base change functor $LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q}) \to LB_{(\mathfrak{g},Q)}((C^{\dagger}_{w,Q})_{\mathbf{C}_p})$.

3.3.4. *An equivalence of categories.* We consider the uniformization:

$$\begin{array}{rcl} m : G_e \rtimes Q & \to & C^{\dagger}_{w,Q} \\ (g, q) & \mapsto & wgq. \end{array}$$

We let $\mathrm{Stab}_Q(w)$ be the stabilizer of w for this action, so that

$$\mathrm{Stab}_Q(w) = \{(g, q) \in G_e \rtimes Q, gq \in P_w\}.$$

We have an injective homomorphism

$$\mathrm{Stab}_Q(w) \hookrightarrow P_w \times Q$$

given by $(g, q) \mapsto (gq, q)$, which induces an isomorphism

$$\mathrm{Stab}_Q(w)_e \xrightarrow{\sim} P_{w,e} \times Q_e. \tag{3.3.5}$$

From now on we will frequently identify $P_{w,e} \times Q_e$ with $\mathrm{Stab}_Q(w)_e$ via (3.3.5), and in particular we will frequently regard Q_e as a subgroup of $G_e \rtimes Q$ via (3.3.5) (and the inclusion $\mathrm{Stab}_Q(w) \subset G_e \rtimes Q$), i.e. as the subgroup of elements (q^{-1}, q) with $q \in Q_e$.

Lemma 3.3.6.

(1) *The group* $\mathrm{Stab}_Q(w)$ *is generated by its subgroups* $\mathrm{Stab}_Q(w)_e = P_{w,e} \times Q_e$ *and* $P_w \cap Q$.

(2) *There is an isomorphism* $(P_w \cap Q)_e \backslash \big((P_{w,e} \times Q_e) \rtimes (P_w \cap Q)\big) \xrightarrow{\sim} \mathrm{Stab}_Q(w)$.

Proof. Consider an element $(g, q) \in \mathrm{Stab}_Q(w) \subseteq G_e \rtimes Q$, so that $gq \in P_w$. Then $g \in G_e \cap P_w Q = P_{w,e} Q_e$, so we can write $g = pq'$ with $p \in P_{w,e}, q' \in Q_e$. Then $q'q \in Q$, and since $gq \in P_w$ and $p \in P_w$, we in fact have $q'q \in P_w \cap Q$. Thus we can write

$$(g, q) = (p, 1)(q', q'^{-1})(1, q'q) \in G_e \rtimes Q$$

with $(p, 1) \in P_{w,e}$, $(q', q'^{-1}) \in Q_e$ and $(1, q'q) \in P_w \cap Q$, completing the proof of the first part.

It follows from the first part that we have a surjective homomorphism

$$\mathrm{Stab}_Q(w)_e \rtimes (P_w \cap Q) \to \mathrm{Stab}_Q(w)$$

given by $(g, q) \mapsto gq$. The kernel of this homomorphism is $\mathrm{Stab}_Q(w)_e \cap (P_w \cap Q) = (P_w \cap Q)_e$, and the second part is immediate. □

Corollary 3.3.7. *A representation* (V, ρ) *of* $\mathrm{Stab}_Q(w)$ *is the data of a representation* (V, ρ_1) *of* $P_{w,e}$, *a representation* (V, ρ_2) *of* Q_e *and a representation* (V, ρ_3) *of* $P_w \cap Q$, *satisfying:*

(1) ρ_1 *and* ρ_2 *commute.*

(2) $\mathrm{Ad}\rho_3(a)(\rho_1(b)\rho_2(c)) = \rho_1(aba^{-1})\rho_2(aca^{-1})$.

(3) $\rho_3 = \rho_1\rho_2$ *on* $(P_w \cap Q)_e$.

Proof. This is immediate from Lemma 3.3.6. □

Example 3.3.8. We consider the sheaf $\mathcal{C}^{\mathrm{la}}|_{C^{\dagger}_{w,Q}}$. We see that the fiber $\mathcal{C}^{\mathrm{la}}_w$ is $\mathcal{O}_{U_{P_w} \backslash G, e}$, and the action of $\mathrm{Stab}_Q(w)$ is given by $(g, q)f(g') = f(q^{-1}g^{-1}g'q)$. In other words, it has a $P_{w,e}$-action by left translation, a Q_e-action by right translation, and a $P_w \cap Q$-action by conjugation.

If $r, s \geq 1$, we can consider the semi-direct product $G_r \rtimes Q_s$. We let $\mathrm{Stab}_Q(w)_{r,s}$ be the stabilizer of w for the action of $G_r \rtimes Q_s$; again, this is the subgroup of $G_r \rtimes Q_s$ of elements (g, q) such that $gq \in P_w$.

Lemma 3.3.9. *If* $r \geq s$, *we have an isomorphism:*

$$(P_w \cap Q)_r \backslash \big((P_{w,r} \times Q_r) \rtimes (P_w \cap Q_s)\big) \xrightarrow{\sim} \mathrm{Stab}_Q(w)_{r,s}.$$

Proof. This follows exactly as in the proof of Lemma 3.3.6. □

We now define a category $LB_{\mathrm{Stab}_Q(w)}(E)$ as follows.

Definition 3.3.10. The category $LB_{\mathrm{Stab}_Q(w)}(E)$ has objects consisting of the following list of data:

(1) An LB-space $V = \mathrm{colim}_r V_r$ over E,

(2) An action of $\mathrm{Stab}_Q(w)$ on V.

(3) For each r, there exists s such that $V_r \in B_{\mathrm{Stab}_Q(w)_{r,s}}(E)$ and the maps $V_r \to V_{r'}$ are equivariant with respect to the maps $\mathrm{Stab}_Q(w)_{r',s} \to \mathrm{Stab}_Q(w)_{r,s}$.
(4) This induces on V an action of $\mathrm{Stab}_Q(w)_e$ and $P_w \cap Q_s$ which coincides with the restriction of the $\mathrm{Stab}_Q(w)$-action of V.

Proposition 3.3.11. *Taking the fiber at w gives an equivalence between the categories $LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q})$ and $LB_{\mathrm{Stab}_Q(w)}(E)$.*

Proof. As usual, by taking the fiber at w, we obtain a $\mathrm{Stab}_Q(w)$-module. Conversely, we attach to an object V of $LB_{\mathrm{Stab}_Q(w)}(E)$ the sheaf $(m_*(\mathcal{O}_{G_e \rtimes Q})\hat{\otimes}V)^{\mathrm{Stab}_Q(w)}$. □

Remark 3.3.12. We have a restriction map $LB_{(\mathfrak{g},G)}(\mathcal{FL}) \to LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q})$. The category $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ is equivalent to $LB_{(\mathfrak{g},P_w)}(E)$ by Proposition 3.2.31, and this restriction map corresponds to the map $LB_{(\mathfrak{g},P_w)}(E) \to LB_{\mathrm{Stab}_Q(w)}(E)$ which is induced by the inclusion $\mathrm{Stab}_Q(w) \hookrightarrow G_e \rtimes P_w$ where $q \in Q_e \mapsto (q,1)$, $p \in P_{w,e} \mapsto (p^{-1},p)$, $r \in P_w \cap Q \mapsto (1,r)$.

Remark 3.3.13. For any object $\mathscr{F}$ of $LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q})$, one can differentiate the Q-equivariant structure and thus obtain a Q-equivariant map $\mathrm{act}_{\mathfrak{q}} : \mathfrak{q} \otimes \mathscr{F} \to \mathscr{F}$. One may want to compare this map with the map $\mathrm{act}_{\mathfrak{g}} : \mathfrak{g} \otimes \mathscr{F} \to \mathscr{F}$ which is given by the $\mathfrak{g}$-equivariant action. The difference $\mathrm{act}_{\mathfrak{q}} - \mathrm{act}_{\mathfrak{g}}|_{\mathfrak{q}} : \mathfrak{q} \otimes \mathscr{F} \to \mathscr{F}$ is a Q-equivariant linear map. It is therefore entirely determined by its fiber at w. If $\mathscr{F}$ corresponds to a $\mathrm{Stab}_Q(w)$-representation V via the equivalence of Proposition 3.3.11, then the $\mathfrak{q}$-action $\mathrm{act}_{\mathfrak{q}} - \mathrm{act}_{\mathfrak{g}}|_{\mathfrak{q}}$ is induced by the Q_e-action on V.

We let $LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q})^{\mathfrak{u}^0_P}$ be the subcategory of objects which are killed by $\mathfrak{u}^0_P$. We let $LB_{\mathrm{Stab}_Q(w)}(E)^{\mathfrak{u}_{P_w}}$ be the full subcategory of objects with trivial action of the subgroup $(U_{P_w})_e \hookrightarrow P_{w,e}$. The objects of $LB_{\mathrm{Stab}_Q(w)}(E)^{\mathfrak{u}_{P_w}}$ carry an action Θ of $Z(\mathfrak{m}_w)$.

Proposition 3.3.14. *The equivalence of categories between the categories $LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q})$ and $LB_{\mathrm{Stab}_Q(w)}(E)$ induces an equivalence between $LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q})^{\mathfrak{u}^0_P}$ and $LB_{\mathrm{Stab}_Q(w)}(E)^{\mathfrak{u}_{P_w}}$. Via this equivalence, the action of Θ_{hor} of $Z(\mathfrak{m})$ corresponds to the action Θ of $Z(\mathfrak{m}_w)$ via conjugation by w^{-1}.*

Proof. This follows from Proposition 3.3.11, exactly as in the proof of Proposition 3.2.34. □

Remark 3.3.15. Let $\mu \in X^*(M_Q^{\mathrm{ab}})$ be an algebraic character. There is a functor $LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q}) \to LB_{(\mathfrak{g},Q)}(C^{\dagger}_{w,Q})$, $\mathscr{F} \mapsto \mathscr{F} \otimes E(\mu)$, corresponding to twisting the Q-action by μ. There is a map $\mathrm{Stab}_Q(w) \hookrightarrow G_e \rtimes Q \to Q \to M_Q^{\mathrm{ab}}$, so that any character $\mu \in X^*(M_Q^{\mathrm{ab}})$ induces a character of $\mathrm{Stab}_Q(w)$. The operation of twisting the Q-action by μ corresponds to the operation of twisting a $\mathrm{Stab}_Q(w)$-representation by μ.

3.4. **Algebraic and locally analytic representations.** In this section we will explain how to define a functor from a subcategory of the algebraic category $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$ to representations of $\mathrm{Stab}_Q(w)$. We begin with some more general considerations.

3.4.1. *Completion of category $\mathcal{O}$.* In this subsection we consider a reductive group G with Borel B and maximal torus T. Its Lie algebra is $\mathfrak{g}$ with Borel $\mathfrak{b}$ and Cartan $\mathfrak{h}$. We recall that $\mathcal{O}(\mathfrak{g},\mathfrak{b})$ is the corresponding BGG subcategory of $U(\mathfrak{g})$-modules. Let $B \subseteq Q \subseteq G$ be a parabolic with Levi M. Let $\mathfrak{q} \subseteq \mathfrak{g}$ be its Lie algebra, with Levi $\mathfrak{m}$. We let $\mathcal{O}(\mathfrak{g},\mathfrak{q})$ be the parabolic BGG category, which is the subcategory of $\mathcal{O}(\mathfrak{g},\mathfrak{b})$ of objects whose restriction to $\mathfrak{m}$ is a direct sum of finite dimensional representations.

We start with the following definition which is nothing but Definition 3.2.6 specialized to $X = \mathrm{Spa}(E, \mathcal{O}_E)$.

Definition 3.4.2. We let $LB_{\mathfrak{g}}(E)$ be the category of $\mathfrak{g}$-representations on LB-spaces. More precisely, its objects are LB spaces over E, V which are $\hat{U}(\mathfrak{g})$-modules and satisfy the following conditions:

1. We have $V = \mathrm{colim}_r V_r$ and for r large enough $V_r \in B_{G_r}(E)$. Moreover, the transition maps $V_r \to V_{r'}$ for $r \leq r'$ are equivariant for the map $G_r \to G_{r'}$.
2. The actions of G_r on V_r induce the action of $\mathfrak{g}$ on the limit.

Proposition 3.4.3. *There is an exact contravariant functor:*

$$\begin{array}{rcl} \mathcal{O}(\mathfrak{g},\mathfrak{b}) & \to & LB_{\mathfrak{g}}(E) \\ M & \mapsto & [M \otimes_{U(\mathfrak{g})} \hat{U}(\mathfrak{g})]^\vee = \hat{M}^\vee \end{array}$$

Proof. This follows from Theorem 2.2.39. Indeed, our functor is the composition of the functor $\mathcal{O} \to \hat{\mathcal{O}}$ which is an equivalence of abelian categories, and then of the duality functor (which is an exact anti-equivalence of categories, and turns a coadmissible $\hat{U}(\mathfrak{g})$-module into an admissible $\hat{U}(\mathfrak{g})$-module), and finally the forgetful functor to the category of LB-spaces equipped with a $\mathfrak{g}$-action. It remains to justify that $[M \otimes_{U(\mathfrak{g})} \hat{U}(\mathfrak{g})]^\vee$ belongs to $LB_{\mathfrak{g}}(E)$. To see this, note that $\hat{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M = \lim_r M \otimes_{U(\mathfrak{g})} D(G_r)$, so that $\hat{M}^\vee = \mathrm{colim}_r V_r$ where $V_r = (M \otimes_{U(\mathfrak{g})} D(G_r))^\vee$ is a Banach space. Moreover, there is an action of G_r on V_r. □

Remark 3.4.4. We can explicate what this functor is doing on Verma modules. Let $\lambda \in X^*(T)_E$ be a character of $\mathfrak{b}$. Let $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$. We see that $\hat{M}^\vee$ is the submodule of $\mathcal{O}_{G,e}$ of functions f which satisfy $f(gb) = \lambda(b) f(g)$ for $(g,b) \in G_e \times B_e$, with the action of $\mathfrak{g}$ being that given by the action of G_e as $(gf)(x) = f(g^{-1}x)$. More generally, let V be a finite dimensional representation of $\mathfrak{b}$ with dual $V^\vee$. Let $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V$. We see that $\hat{M}^\vee$ is the submodule of the space of functions $f : G_e \to V^\vee$ which satisfy $bf(gb) = f(g)$.

We let $X^*(M)$ be the character space of M.

Definition 3.4.5. Let $\lambda \in X^*(M)_E$. We let $\mathcal{O}(\mathfrak{g},\mathfrak{q})_{\lambda-\mathrm{alg}}$ be the full subcategory of $\mathcal{O}(\mathfrak{g},\mathfrak{q})$, whose objects are those V which have the property that in the weight decomposition $V = \oplus_{\nu \in X^*(T)_E} V[\nu]$ for the action of $\mathfrak{h}$[5], we have $\nu - \lambda \in X^*(T)$. This is an abelian category.

Lemma 3.4.6. *If $V \in \mathcal{O}(\mathfrak{g},\mathfrak{q})_{\lambda-\mathrm{alg}}$, then the $\mathfrak{q}$-action on the twisted module $V(-\lambda)$ integrates to an action of Q.*

[5] By definition, $V[\nu] = \{m \in V, h.m = \nu(h)m\}$. In the direct sum, we suppose $V[\nu] \neq 0$.

Proof. We observe that $V(-\lambda)$ is a union of finite dimensional representations of $\mathfrak{q}$. We claim that on any finite dimensional representation $(W, d\rho)$ of $\mathfrak{q}$, the action integrates to an action (W, ρ) of Q as long as the action of $\mathfrak{h}$ integrates to an action of T (which is the reason why we are introducing a twist). Indeed, the Lie algebra action gives a map $U(\mathfrak{q}) \otimes W \to W$ and since W is finite dimensional, we can dualize this map to a map $W \to W \otimes \widehat{\mathcal{O}_{Q,e}}$ where $\widehat{\mathcal{O}_{Q,e}}$ is the completion of the local ring at e. We claim that this map factorizes through $W \otimes \mathcal{O}_Q$ and gives the coaction map; this establishes the lemma. This claim must be well known but we could not find a reference so we sketch the argument. Using the Levi decomposition $Q = M \ltimes U_Q$ it suffices to treat the case of M and U_Q separately.

To see that the action of the unipotent radical U_Q of Q integrates we can just consider a root groups $U_\beta \simeq \mathbf{G}_a$ inside U_Q with Lie algebra $\mathfrak{u}_\beta$ generated by u_β, and the rule $V \to V \otimes E[T]$, $v \mapsto \exp(Tu_\beta)v$ defines an action of U_β (we use here that the action of u_β is locally nilpotent). On the other hand, the action of $\mathfrak{h}$ on $V(-\lambda)$ integrates to an action of T. Similarly, the action of M^{der} integrates (from the action of $\mathfrak{m}$). Indeed, for a semi-simple group, the categories of finite dimensional representations of the group and of its Lie algebra are equivalent. This actions of M^{der} and T combine together to an action of M. □

3.4.7. *A particular class of representations of* $\mathrm{Stab}_Q(w)$. We now go back to our original setting. Let $Q_{M_w} = P_w \cap Q / U_{P_w \cap Q}$.

Lemma 3.4.8. *If $w \in {}^M W$ then Q_{M_w} is a parabolic subgroup of M_w containing B_{M_w}.*

Proof. Clearly, Q_{M_w} contains $P_w \cap B / U_{P_w \cap Q}$ which is a Borel subgroup of M_w. Any closed subgroup of a reductive group containing a Borel is a parabolic subgroup. □

Let $\lambda \in X^*(M_Q)_E$.

Definition 3.4.9. We let $\mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda$ be the following category. Its objects are admissible $\hat{U}(\mathfrak{m}_w)$-modules $V = \mathrm{colim}_r V_r$ which admit an action of Q_{M_w}, compatible with the action of Q_{M_w} by conjugation on $D(M_{w,e})$. We further demand the following conditions :

(1) For $q \in Q_{M_w}$, $g \in \mathfrak{m}_w$, and $m \in V$, we have $\mathrm{Ad}(q)(g).m = q.g.q^{-1}.m$.
(2) there exists $s \in \mathbf{Z}_{>0}$ and an action of $Q_{M_w,s}$ on each V_r, inducing an action of $V = \mathrm{colim}_r V_r$ which coincides with the restriction of the action of Q_{M_w} to $Q_{M_w,s}$.
(3) Let us denote by $\rho_{Q_{M_w}}$ the action of Q_{M_w} on V. This action differentiates to an action $\mathrm{d}\rho_{Q_{M_w}}$ of $\mathfrak{q}_{M_w}$. We let $\rho_{\mathfrak{m}_w}$ be the action of $\mathfrak{m}_w$. Then we ask that $\mathrm{d}\rho_{Q_{M_w}} = \lambda + \rho_{\mathfrak{m}_w}|_{\mathfrak{q}_{M_w}}$.

Remark 3.4.10. The category $\mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda$ is an abelian category by general results on coadmissible and admissible modules over Fréchet–Stein algebras (see [ST03], sect. 3).

The category $\mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda$ is a full subcategory of $LB_{\mathfrak{m}_w}(E)$. In particular, since Q_{M_w} is connected, the action of Q_{M_w} in the third condition is uniquely determined by the $\mathfrak{m}_w$-action and λ.

Remark 3.4.11. By Proposition 3.2.31 (applied with G replaced by M_w), we see that objects of $\mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda$ define M_w-equivariant D_λ-modules on the flag variety $X = Q_{M_w}\backslash M_w$, where $D_\lambda := \mathcal{O}_X \otimes \hat{U}(\mathfrak{m}_w) \otimes_{\mathfrak{q}^0_{M_w}} \lambda$ is a ring of twisted differential operators.

Proposition 3.4.12. *There is a natural fully faithful functor:* $\mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda \to LB_{\mathrm{Stab}_Q(w)}(E)^{\mathfrak{u}_{\mathfrak{p}_w}}$.

Proof. It suffices to exhibit an equivalence of categories between $\mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda$ and a full subcategory of $LB_{\mathrm{Stab}_Q(w)}(E)^{\mathfrak{u}_{\mathfrak{p}_w}}$. To this end, recall that by Lemma 3.3.6, $\mathrm{Stab}_Q(w) = (P_w \cap Q)_e\backslash\big((P_{w,e} \times Q_e) \rtimes (P_w \cap Q)\big)$. Accordingly, we may consider the full subcategory of representations of $\mathrm{Stab}_Q(w)$ which factor through $(Q_{M_w})_e\backslash\big(((M_w)_e \times M^{\mathrm{ab}}_{Q,e}\big) \rtimes Q_{M_w}$ and have the property that $M^{\mathrm{ab}}_{Q,e}$ acts through the character λ. In other words, we consider the full subcategory of $LB_{\mathrm{Stab}_Q(w)}(E)$ whose objects V satisfy the following properties:

1. (1) The action of $P_{w,e}$ factors through an action ρ_1 of $M_{w,e}$. Moreover V, viewed as an object of $LB_{\mathfrak{m}_w}(E)$, is admissible.
2. (2) The action of Q_e factors through $M^{\mathrm{ab}}_{Q,e}$ acting via λ.
3. (3) The action of $Q \cap P_w$ factors through an action ρ_3 of Q_{M_w}.

Clearly, this is equivalent to $\mathrm{Adm}_{(\mathfrak{m}_w, B_{M_w})}(E)_\lambda$. □

We now consider the parabolic BGG category $\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})$ for $\mathfrak{m}_w$ and the parabolic $\mathfrak{q} \cap \mathfrak{m}_w = \mathfrak{q}_{M_w}$. We think of $\mathfrak{h} \hookrightarrow \mathfrak{b}$ as giving the Cartan of $\mathfrak{m}_w$. We now apply the material of Section 3.4.1 to $\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})$.

By Proposition 3.4.3 (applied to $\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})$), we have a completion functor $\mathcal{O} \to LB_{\mathfrak{m}_w}(E)$, which restricts to a functor $\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}} \to LB_{\mathfrak{m}_w}(E)$.

Proposition 3.4.13. *We can uniquely upgrade the completion functor*

$$\begin{array}{rcl} \mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}} & \to & LB_{\mathfrak{m}_w}(E) \\ V & \mapsto & \hat{V}^\vee \end{array}$$

to a fully faithful functor

$$\begin{array}{rcl} \mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}} & \to & \mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda \\ V & \mapsto & \hat{V}^\vee(\lambda). \end{array}$$

Its essential image is the subcategory of $\mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda$ *of objects which are in the image of the functor* $\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}} \to LB_{\mathfrak{m}_w}(E)$ *when viewed as* $\mathfrak{m}_w$*-representations.*

Remark 3.4.14. As the notation $\hat{V}^\vee(\lambda)$ suggests, we make a twist of the action of $\mathfrak{q}_{M_w}$ on $\hat{V}^\vee$ by λ so that it extends to an action of Q_{M_w}. (See also Lemma 3.4.6.)

Proof of Proposition 3.4.13. Let $V \in \mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}}$. We consider $\hat{V}^\vee$. This space carries an action of $P_{w,e}$ (factoring through $M_{w,e}$). We can also define an action of $(M_Q)_e$ on $\hat{V}^\vee$ via scalar multiplication by the character λ. Clearly these two actions commute. The product of the two actions defines an action of $(M_Q)_e \cap P_{w,e}$ factoring through $Q_{M_w,e}$. We claim that we can extend it to an action of $Q \cap P_w$ factoring through Q_{M_w}. It follows from Lemma 3.4.6 that we have an action on $V(-\lambda)$. We thus get an action on $\hat{V}(-\lambda) = V(-\lambda) \otimes_{U(\mathfrak{m}_w)} \hat{U}(\mathfrak{m}_w)$, since M_w acts on $\hat{U}(\mathfrak{m}_w)$ and $U(\mathfrak{m}_w)$ via the adjoint representation. □

Remark 3.4.15. Clearly the categories $\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}}$ and $\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda+\mu-\mathrm{alg}}$ are equivalent if $\mu \in X^*(M_Q)$. However, the functor $\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}} \to \mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda$ depends on the choice of λ (as clearly the target category depends on λ). We have a functor $\mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda \to \mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_{\lambda+\mu}$, $V \mapsto V \otimes E(\mu)$. We have the following commutative diagram of functors (telling us that $\hat{V}^\vee(\lambda) \otimes E(\mu) = \hat{V}^\vee(\lambda+\mu)$):

$$\begin{array}{ccc} \mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}} & \longrightarrow & \mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda+\mu-\mathrm{alg}} & \longrightarrow & \mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_{\lambda+\mu} \end{array}$$

3.4.16. *Higher Coleman sheaves.*

Definition 3.4.17. We now define a contravariant exact functor

$$HCS_{Q,w,\lambda} : \mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}} \to LB_{(\mathfrak{g},Q)}(C^\dagger_{w,Q})^{\mathfrak{u}^0_{\mathfrak{p}}}$$

(where "HCS" stands for "higher Coleman sheaf") as as the composite

$$\mathcal{O}(\mathfrak{m}_w, \mathfrak{q}_{M_w})_{\lambda-\mathrm{alg}} \to \mathrm{Adm}_{(\mathfrak{m}_w, Q_{M_w})}(E)_\lambda \to LB_{\mathrm{Stab}_Q(w)}(E)^{\mathfrak{u}_{\mathfrak{p}_w}} \to LB_{(\mathfrak{g},Q)}(C^\dagger_{w,Q})^{\mathfrak{u}^0_{\mathfrak{p}}},$$

where the first functor is the one defined in Proposition 3.4.13, the second is the fully faithful functor of Proposition 3.4.12, and the third is the equivalence of Proposition 3.3.14.

In the case $Q = B$ we write $HCS_{w,\lambda}$ for $HCS_{B,w,\lambda}$.

Proposition 3.4.18. *Let* $M \in \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda-\mathrm{alg}}$. *Let* $\mu \in X^*(T)$. *We have* $HCS_{w,\lambda}(M) \otimes E(\mu) = HCS_{w,\lambda+\mu}(M)$.

Proof. This follows from Remark 3.4.15. □

We have an action of $Z(\mathfrak{m}_w)$ on $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda-\mathrm{alg}}$. We also have an action $Z(\mathfrak{m})$ on $LB_{(\mathfrak{g},B)}(C^\dagger_w)^{\mathfrak{u}^0_{\mathfrak{p}}}$ via Θ_{hor}. These two action are related by the following lemma. We let $\iota : Z(\mathfrak{m}) \to Z(\mathfrak{m})$ be the map induced by the inverse map on M, and let $w : \mathfrak{m}_w \to \mathfrak{m}$ be conjugation by w.

Proposition 3.4.19. *Let us consider the map* $w : Z(\mathfrak{m}_w) \to Z(\mathfrak{m})$. *Then we have that* $HCS_{w,\lambda}(z) = \Theta_{\mathrm{hor}}(\iota w z)$ *for any* $z \in Z(\mathfrak{m}_w)$.

Proof. This follows directly from the construction, bearing in mind Proposition 3.3.14. □

3.5. **Localization on the partial flag variety.**

3.5.1. *Statement of the localization problem.* Recall that in Section 3.2.37 we defined an object $\mathcal{C}^{\mathrm{la}} \in LB_{(\mathfrak{g},G)}(\mathcal{FL})$. This is a $\tilde{\mathcal{D}}^{\mathrm{la}}$-module and it carries an action $*_2$ of $\mathfrak{g}$ which commutes with the $\tilde{\mathcal{D}}^{\mathrm{la}}$-module structure.

We define a localization functor:

$$\begin{array}{rcl} \mathrm{Loc} : D^-(U(\mathfrak{g})) & \to & D(\tilde{\mathcal{D}}^{\mathrm{la}}) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{g},*_2}(M, \mathcal{C}^{\mathrm{la}}) \end{array}$$

where $D(\tilde{\mathcal{D}}^{\mathrm{la}})$ is the derived category of solid $\tilde{\mathcal{D}}^{\mathrm{la}}$-modules. We will sometimes drop the subscript $*_2$ from the notation, and simply write $\mathrm{RHom}_{\mathfrak{g}}(M, \mathcal{C}^{\mathrm{la}})$.

Recall that if M is an object of $\mathrm{Mod}^{\mathrm{fg}}(U(\mathfrak{g}))$ then we let $\hat{M} = M \otimes_{U(\mathfrak{g})} \hat{U}(\mathfrak{g})$ and $\hat{M}^\vee = \underline{\mathrm{Hom}}(\hat{M}, E)$. The following lemma gives another description of $\mathrm{Loc}(M)$ for $M \in \mathrm{Mod}^{\mathrm{fg}}(U(\mathfrak{g}))$.

Lemma 3.5.2. *Assume that $M \in \mathrm{Mod}^{\mathrm{fg}}(U(\mathfrak{g}))$. Then we have:*

$$\begin{aligned} \mathrm{RHom}_{\mathfrak{g},*_2}(M, \mathcal{C}^{\mathrm{la}}) &= \mathrm{RHom}_{\mathfrak{g},*_2}(\hat{M}, \mathcal{C}^{\mathrm{la}}) \\ &= \mathrm{RHom}_{\mathfrak{g},*_2}(E, \hat{M}^\vee \otimes \mathcal{C}^{\mathrm{la}}). \end{aligned}$$

Proof. For the first equality, we use that $\mathcal{C}^{\mathrm{la}}$ is a $\hat{U}(\mathfrak{g})$-module so that $\mathrm{RHom}_{U(\mathfrak{g}),*_2}(M, \mathcal{C}^{\mathrm{la}}) = \mathrm{RHom}_{\hat{U}(\mathfrak{g}),*_2}(M \otimes^L_{U(\mathfrak{g})} \hat{U}(\mathfrak{g}), \mathcal{C}^{\mathrm{la}})$. It follows from Corollary 2.2.36 that $M \otimes^L_{U(\mathfrak{g})} \hat{U}(\mathfrak{g}) = \hat{M}[0]$.

For the second equality, we have an obvious map

$$\mathrm{RHom}_{\mathfrak{g},*_2}(E, \hat{M}^\vee \otimes \mathcal{C}^{\mathrm{la}}) \to \mathrm{RHom}_{\mathfrak{g},*_2}(\hat{M}, \mathcal{C}^{\mathrm{la}}) = \mathrm{RHom}_{\mathfrak{g},*_2}(M, \mathcal{C}^{\mathrm{la}}).$$

By resolving M by free modules it suffices to check that this map is a quasi-isomorphism for $M = U(\mathfrak{g})$. But then we have

$$\mathrm{RHom}_{\mathfrak{g},*_2}(U(\mathfrak{g}), \mathcal{C}^{\mathrm{la}}) = \mathcal{C}^{\mathrm{la}}[0] = \mathrm{RHom}_{\mathfrak{g},*_2}(E, \mathcal{O}_{G,e} \otimes \mathcal{C}^{\mathrm{la}})$$

where the first equality is obvious. For the other equality, we think of $\mathcal{O}_{G,e} \otimes \mathcal{C}^{\mathrm{la}}$ as a submodule of $\mathcal{O}_{G,e} \otimes \mathcal{O}_{G,e} \otimes \mathcal{O}_{\mathcal{FL}}$ which is the germs of functions $f(g', g, x)$ at (e, e) in $G \times G \times \mathcal{FL}$. The $\mathfrak{g}$-action is induced from $g''.f(g', g, x) = \lambda^{-1}(b) f((g'')^{-1} g', gg'', x)$ for $g'' \in G_e$. We consider the automorphism of $\mathcal{O}_{G,e} \otimes \mathcal{C}^{\mathrm{la}}$ given by the map $f(g', g, x) \mapsto [(g', g) \mapsto f(g', gg', x)]$. Via this automorphism, the $\mathfrak{g}$-action becomes $g''.f(g, g', x) = f((g'')^{-1} g', g, x)$ for $g'' \in G_e$, and is therefore only on the first factor. We can now use the flatness of $\mathcal{C}^{\mathrm{la}}$ over E and Lemma 2.3.4 to conclude. □

We recall that $Z(\mathfrak{g})$ lies in the centre of $D^-(U(\mathfrak{g}))$. We also have defined a map $\Theta_{\mathrm{hor}} : Z(\mathfrak{m}) \to \mathrm{End}_{\mathcal{O}_{\mathcal{FL}}}(\mathcal{C}^{\mathrm{la}})$ in section 3.2.37.

Lemma 3.5.3. *For any $z \in Z(\mathfrak{g})$, we have $\mathrm{Loc}(z) = \Theta_{\mathrm{hor}}(HC(\iota z))$.*

Proof. This follows from Lemma 3.2.43. □

Corollary 3.5.4. *Let $M \in \mathrm{Mod}(U(\mathfrak{g}))$ be a module with infinitesimal character $\lambda \in X^*(T)_E$ (modulo dotted W-action). Then on $\mathrm{Loc}(M)$, the horizontal action of $Z(\mathfrak{m})$ factors through an action of $Z(\mathfrak{m}) \otimes_{HC, Z(\mathfrak{g})} (-w_0\lambda)$.*

Proof. We recall that for the map $HC_{\mathfrak{g}} : Z(\mathfrak{g}) \to U(\mathfrak{h})$, we have $HC_{\mathfrak{g}} \circ \iota = -w_0 HC_{\mathfrak{g}}$. The rest follows from Lemma 3.5.3. □

Remark 3.5.5. Let us describe Spec $Z(\mathfrak{m}) \otimes_{HC, Z(\mathfrak{g})} (-w_0\lambda)$, or equivalently the idempotents in this finite E-algebra (note however that in singular weight $Z(\mathfrak{m}) \otimes_{HC, Z(\mathfrak{g})} (-w_0\lambda)$ is not reduced). The possible characters of $Z(\mathfrak{m}) \otimes_{HC, Z(\mathfrak{g})} (-w_0\lambda)$ range through the set $\{w \cdot (-w_0\lambda), w \in {}^M W\}$. Since $w \cdot (-w_0\lambda) = -w_{0,M}(w_{0,M} w w_0 \cdot \lambda + 2\rho^M)$ we deduce that

$$\mathrm{Spec}\ Z(\mathfrak{m}) \otimes_{HC, Z(\mathfrak{g})} (-w_0\lambda) = \{-w_{0,M}(w \cdot \lambda + 2\rho^M), w \in {}^M W\}.$$

It follows that if M has infinitesimal character λ, then

$$\mathrm{Loc}(M) = \oplus_{w \in {}^M W} \mathrm{Loc}(M)_{-w_{0,M}(w \cdot \lambda + 2\rho^M)}$$

where $\mathrm{Loc}(M)_{-w_{0,M}(w \cdot \lambda + 2\rho^M)}$ is the direct factor which corresponds to the idempotent in $Z(\mathfrak{m}) \otimes_{HC, Z(\mathfrak{g})} (-w_0\lambda)$ given by $-w_{0,M}(w \cdot \lambda + 2\rho^M)$.

We conclude our generalities on our localization problem by showing that it is pre-dual to an obvious variant of the classical localization problem as in [BB83] (which is of course formulated in the algebraic context, and involves a fixed choice of (generalized) infinitesimal character). In order to do so, we recall the Chevalley–Eilenberg resolution of E (with $d = \dim \mathfrak{g}$):

$$0 \to U(\mathfrak{g}) \otimes \Lambda^d \mathfrak{g} \to \cdots \to U(\mathfrak{g}) \to E \to 0.$$

By Lemma 3.5.2, for a finitely generated $U(\mathfrak{g})$-module M, $\mathrm{Loc}(M) = \mathrm{RHom}_{\mathfrak{g},*_2}(M, \mathcal{C}^{\mathrm{la}})$ is computed by the following complex of LB-sheaves (in degree $[0, d]$):

$$0 \to \mathcal{C}^{\mathrm{la}} \otimes \hat{M}^\vee \to \cdots \to \mathcal{C}^{\mathrm{la}} \otimes \hat{M}^\vee \otimes \Lambda^d \mathfrak{g}^\vee \to 0. \tag{3.5.6}$$

Proposition 3.5.7. *For $M \in \mathrm{Mod}^{\mathrm{fg}}(U(\mathfrak{g}))$, we have $\underline{\mathrm{RHom}}_{\mathcal{O}_{\mathcal{FL}}}(\mathrm{RHom}_{\mathfrak{g},*_2}(M, \mathcal{C}^{\mathrm{la}}), \mathcal{O}_{\mathcal{FL}}) = \tilde{\mathcal{D}}^{\mathrm{la}} \otimes^L_{U(\mathfrak{g})} M$.*

Proof. The same computation as in Proposition 3.2.44 shows that the derived $\mathcal{O}_{\mathcal{FL}}$-dual of the complex:

$$0 \to \mathcal{C}^{\mathrm{la}} \otimes \hat{M}^\vee \to \cdots \to \mathcal{C}^{\mathrm{la}} \otimes \hat{M}^\vee \otimes \Lambda^d \mathfrak{g}^\vee \to 0$$

is the complex:

$$0 \to \tilde{\mathcal{D}}^{\mathrm{la}} \otimes \hat{M} \otimes \Lambda^d \mathfrak{g} \to \cdots \to \tilde{\mathcal{D}}^{\mathrm{la}} \otimes \hat{M} \to 0.$$ □

3.5.8. *B-action.* We can also exploit the B-equivariant structure on $\mathrm{Loc}(M)$. To this end, suppose that $M \in \mathcal{O}(\mathfrak{g}, \mathfrak{b})_{\lambda\text{-alg}}$, so that $M(-\lambda)$ has an action of B (see Remark 3.4.6). Then by Proposition 3.4.3, $\hat{M}^\vee(\lambda) \in LB_{(\mathfrak{g},B)}(E)$. We deduce that the complex (3.5.6) computing $\mathrm{Loc}((M(-\lambda)) = \mathrm{RHom}_{\mathfrak{g},*_2}(M(-\lambda), \mathcal{C}^{\mathrm{la}})$ is a complex in $LB_{(\mathfrak{g},B)}(\mathcal{FL})$. More precisely, $\mathcal{C}^{\mathrm{la}} \otimes \hat{M}^\vee \otimes \Lambda^i \mathfrak{g}^\vee$ carries the induced $\mathfrak{g}$-action from the $*_{1,3}$-action on $\mathcal{C}^{\mathrm{la}}$ and the B-action which is the tensor product of the B-action on $\mathcal{C}^{\mathrm{la}}$ and the B-action on $\hat{M}^\vee \otimes \Lambda^i \mathfrak{g}^\vee$. There is another $\mathfrak{g}$-action which is the tensor product of the $*_2$ action on $\mathcal{C}^{\mathrm{la}}$ and the $\mathfrak{g}$-action on $\hat{M}$ (and which is used to construct the differentials in the complex).

3.5.9. *Main theorem.* For $M \in \mathcal{O}(\mathfrak{g}, \mathfrak{b})_{\lambda\text{-alg}}$, the cohomology sheaves of $\mathrm{Loc}(M(-\lambda))$ are $(\mathfrak{g}, B)$-equivariant sheaves that we want to describe. As a first step we intend to describe their restrictions to $C_w^\dagger$ for each $w \in {}^M W$.

Remark 3.5.10. In principle the cohomology sheaves could be sheaves of solid E-vector spaces which need not arise from nice sheaves of topological spaces (in more classical language, the cohomology could be non-separated). However, under the assumption that λ is non-Liouville, we see as a consequence of the following theorem that they are actually separated objects and again belong to the category $LB_{(\mathfrak{g},B)}(C_w^\dagger)$.

Theorem 3.5.11. *Let $M \in \mathcal{O}(\mathfrak{g}, \mathfrak{b})_{\lambda-\mathrm{alg}}$ and assume that λ is non-Liouville. Then we have:*

$$\underline{H}^i(\mathrm{Loc}(M(-\lambda)))|_{C_w^\dagger} = HCS_{w,\lambda}(H_i(\mathfrak{u}_{P_w}, M)).$$

Proof. By the definition of the functor HCS the sheaf $HCS_{w,\lambda}(H_i(\mathfrak{u}_{P_w}, M))$ corresponds via the equivalence of categories of Proposition 3.3.11 to the $\mathrm{Stab}(w)$-representation $\widehat{H_i(\mathfrak{u}_{\mathfrak{p}}, M)}^\vee(\lambda)$. By Corollary 2.3.33, we can identify this with

$\mathrm{Ext}^i_{\mathfrak{u}_{P_w}}(E,\hat{M}^\vee(\lambda))$, and by Proposition 3.5.14 below, this can in turn be identified with

$$\mathrm{Ext}^i_{\mathfrak{g},*_2}(\hat{M}(-\lambda),\mathcal{O}_{U_{P_w}\backslash G,e}). \tag{3.5.12}$$

By definition we have

$$\underline{H}^i(\mathrm{Loc}(M(-\lambda)))|_{C_w^\dagger} = \mathrm{Ext}^i_{\mathfrak{g},*_2}(M(-\lambda),\mathcal{C}^{\mathrm{la}}|_{C_w^\dagger}). \tag{3.5.13}$$

Morally, it remains to show that passage to the fiber at w identifies the right hand side of (3.5.13) with 3.5.12. However, we have to be a little careful with this comparison, because we have not developed a theory which allows us to consider arbitrary sheaves of solid E-vector spaces.

To this end, we consider the Chevalley–Eilenberg complex computing $\mathrm{Ext}^i_{\mathfrak{g},*_2}(M(-\lambda),\mathcal{C}^{\mathrm{la}}|_{C_w^\dagger})$. Under the equivalence of categories of Proposition 3.3.11 (given by taking the fiber at w), this complex corresponds to the following complex of $\mathrm{Stab}(w)$-representations,

$$0\to\mathcal{O}_{U_{P_w}\backslash G,e}\otimes\hat{M}^\vee(\lambda)\to\mathcal{O}_{U_{P_w}\backslash G,e}\otimes\hat{M}^\vee(\lambda)\otimes\mathfrak{g}^\vee\to\mathcal{O}_{U_{P_w}\backslash G,e}\otimes\hat{M}^\vee(\lambda)\otimes\Lambda^2\mathfrak{g}^\vee\to\dots$$

which computes $\mathrm{RHom}_{\mathfrak{g},*_2}(\hat{M}(-\lambda),\mathcal{O}_{U_{P_w}\backslash G,e})$, as required.

(This cohomology is computed in the category of solid E-vector spaces. Again, the cohomology groups could be very pathological (from the classical perspective). For clarity, we can make explicit the action of $\mathrm{Stab}(w)$ on $\mathcal{O}_{U_{P_w}\backslash G,e}\otimes\hat{M}^\vee(\lambda)\otimes\Lambda^i\mathfrak{g}^\vee$. This action consists of:

- An action of $(P_w)_e$ induced by the action on $\mathcal{O}_{U_{P_w}\backslash G,e}$ via $p.f(-)=f(p^{-1}-)$.
- An action of B_e, which is the tensor product of the action on $\mathcal{O}_{U_{P_w}\backslash G,e}$ via $b.f(-)=f(-b)$ and of the restriction to B_e of the B-action on $\hat{M}^\vee(\lambda)\otimes\Lambda^i\mathfrak{g}^\vee$.
- An action of $B\cap P_w$ which is the tensor product of the action on $\mathcal{O}_{U_{P_w}\backslash G,e}$ via $b.f(-)=f(b^{-1}-b)$ and the action of $B\cap P_w$ on $\hat{M}^\vee(\lambda)\otimes\Lambda^i\mathfrak{g}^\vee$.

The differentials in the complex involve the $\mathfrak{g}$-action which is the tensor product of the $*_2$ action on $\mathcal{O}_{U_{P_w}\backslash G,e}$ and the $\mathfrak{g}$-action on $\hat{M}^\vee(\lambda)\otimes\mathfrak{g}^\vee$.) □

Proposition 3.5.14. *If $M\in\mathcal{O}(\mathfrak{g},\mathfrak{b})_{\lambda\text{-alg}}$, then we have a quasi-isomorphism* $\mathrm{RHom}_{\mathfrak{g},*_2}(M(-\lambda),\mathcal{O}_{U_{P_w}\backslash G,e})=\mathrm{RHom}_{\mathfrak{u}_{P_w}}(E,\hat{M}^\vee(\lambda))$.

Proof. Indeed we have

$$\begin{aligned}\mathrm{RHom}_{\mathfrak{g},*_2}(M(-\lambda),\mathcal{O}_{U_{P_w}\backslash G,e}) &= \mathrm{RHom}_{\mathfrak{g},*_2}(E,\hat{M}^\vee(\lambda)\otimes\mathcal{O}_{U_{P_w}\backslash G,e})\\ &= \mathrm{RHom}_{\mathfrak{g}\oplus\mathfrak{u}_{P_w}}(E,\hat{M}^\vee(\lambda)\otimes\mathcal{O}_{G,e})\\ &= \mathrm{RHom}_{\mathfrak{u}_{P_w}}(E,\hat{M}^\vee(\lambda)).\end{aligned}$$

Here the first equality follows from the same argument as in Lemma 3.5.2, the second equality uses that $\mathrm{RHom}_{\mathfrak{u}_{P_w}}(E,\mathcal{O}_{G,e})=\mathcal{O}_{U_{P_w}\backslash G,e}[0]$ and the flatness of $\hat{M}^\vee(\lambda)$ (as it is a colimit of Smith spaces), and the last equality follows from Lemma 2.3.4. □

3.5.15. *Localization of finite dimensional representations.* Let $\lambda\in X^*(T)^+$. Let V_λ be the irreducible finite dimensional representation of G of highest weight λ viewed as an object of $\mathcal{O}(\mathfrak{g},\mathfrak{b})_{0-alg}$. Let $\lambda\in X^*(T)^{+,M}$. We let L_λ be the irreducible finite dimensional representation of M of highest weight λ. We also let $d=\dim(\mathfrak{u}_P)$. We recall the following theorem of Kostant:

Theorem 3.5.16. *We have that* $H_i(\mathfrak{u}_P, V_\lambda) = \oplus_{w\in {}^MW, \ell(w)=d-i} L_{w\cdot\lambda+2\rho^M}$.

Proof. See for example [oGVAG09, Thm. 4.2.1] (together with (2.3.3) to pass from cohomology to homology). □

Proposition 3.5.17. *We have* $\mathrm{Loc}(V_\lambda) = \bigoplus_{w\in {}^MW} \mathrm{Loc}(V_\lambda)_{-w_{0,M}(w\cdot\lambda+2\rho^M)}$ *and* $\mathrm{Loc}(V_\lambda)_{-w_{0,M}(w\cdot\lambda+2\rho^M)} = \mathcal{L}_{-w_{0,M}(w\cdot\lambda+2\rho^M)}[-d+\ell(w)]$.

Proof. Since there is a G-action on V_λ, we see that $\mathrm{Loc}(V_\lambda)$ is in fact computed by a complex in $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ (and not just in $LB_{(\mathfrak{g},B)}(\mathcal{FL})$). Via the equivalence of categories of Proposition 3.2.31, this complex corresponds to the complex $\mathrm{RHom}_{\mathfrak{g},*_2}(V_\lambda, \mathcal{O}_{U_{P_w}\backslash G,e})$ in $L_{(\mathfrak{g},P)}(E)$, and as in Proposition 3.5.14, we find that this is quasi-isomorphic to $\mathrm{RHom}_{\mathfrak{u}_{P_w}}(E, {V_\lambda}^\vee)$. It follows that the cohomology groups are simply given by the representations of M: $\mathrm{Ext}^i_{\mathfrak{u}_{P_w}}(E, {V_\lambda}^\vee) = H_i(\mathfrak{u}_P, V_\lambda)^\vee$. By Kostant's Theorem 3.5.16, these correspond to $\oplus_{w,\ell(w)=d-i}\mathcal{L}_{-w_{0,M}(w\cdot\lambda+2\rho^M)}$, as required. □

Remark 3.5.18. Proposition 3.5.17 is clearly compatible with Theorem 3.5.11, since

$$\begin{aligned}\mathrm{Loc}(V_\lambda)|_{C_w^\dagger} &= \oplus_{w'\in {}^MW}\mathcal{L}_{-w_{0,M}(w'\cdot\lambda+2\rho^M)}[\ell(w')-d]|_{C_w^\dagger} \\ &= \oplus_{i=0}^d HCS_{w,\lambda}(H_i(\mathfrak{u}_{P_w}, V_\lambda))\end{aligned}$$

but it also gives more information as it describes all the extensions between the sheaves on the Bruhat strata.

3.5.19. *Localization of Verma modules in the non-Liouville case.* Let $\lambda \in X^*(T)_E$ be non-Liouville. We view the Verma module M_λ of weight λ as an object of $\mathcal{O}(\mathfrak{g},\mathfrak{b})_{\lambda-\mathrm{alg}}$. Thanks to Theorem 3.5.11, we see that understanding $\mathrm{Loc}(M_\lambda(-\lambda))|_{C_w^\dagger}$ boils down to understanding the cohomology of some Verma modules, as in Theorem 2.3.19. We also assume for simplicity that $\mathfrak{u}_P$ is abelian (this assumption holds in our applications to Shimura varieties).

Corollary 3.5.20. *Assume that* $\lambda \in X^*(T)_E$ *is non-Liouville and that* $\mathfrak{u}_{P_w}$ *is abelian. Then the following hold:*

(1) *All the cohomology groups* $\underline{H}^i(\mathrm{Loc}(M_\lambda(-\lambda)))|_{C_w^\dagger}$ *belong to the image of the functor* $HCS_{w,\lambda} : \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda\text{-alg}} \to LB_{(\mathfrak{g},B)}(C_w^\dagger)$.
(2) *The cohomology groups are zero if* $i > d-\ell(w)$.
(3) *There is a surjective "highest weight" map:*
$$\underline{H}^{d-\ell(w)}(\mathrm{Loc}(M_\lambda(-\lambda)))|_{C_w^\dagger} \to HCS_{w,\lambda}(M(\mathfrak{m}_w)_{\lambda+w^{-1}w_{0,M}\rho+\rho}).$$
(4) *The kernel of the highest weight map, and the cohomology groups* $\underline{H}^i(\mathrm{Loc}(M_\lambda(-\lambda)))|_{C_w^\dagger}$ *for* $i < d-\ell(w)$, *admit finite filtrations with sub-quotients ranging among the sheaves* $\mathcal{L}_{w,-w_{0,M}(w'\cdot\lambda+2\rho^M)} \otimes E(\lambda - w^{-1}w'\cdot\lambda - w^{-1}w_{0,M}\rho - \rho)$, *where* $w' \in wW_{<\lambda}$.

Proof. This is immediate from Theorem 3.5.11 and Theorem 2.3.19, bearing in mind (2.3.22) and Proposition 3.4.18. □

Remark 3.5.21. In particular, if λ is non-Liouville and antidominant (i.e. that $W_{<\lambda} = \emptyset$), we see that the cohomology is concentrated in degree $d-\ell(w)$ and that the highest weight map is an isomorphism on this cohomology.

3.5.22. *Localization of Verma modules in general.* Let $\lambda \in X^*(T)_E$. For sake of completeness, in this section we give the following general result (without any non-Liouville assumption on λ) which is a weaker form of Corollary 3.5.20. We still assume for simplicity that $\mathfrak{u}_P$ is abelian. This result was obtained by Juan Esteban Rodriguez-Camargo in his PhD thesis.

Theorem 3.5.23 (Rodriguez-Camargo)**.** *Let $M_\lambda \in \mathcal{O}(\mathfrak{g}, \mathfrak{b})_{\lambda-\mathrm{alg}}$ be the Verma module of weight λ. Let $w \in {}^MW$. The following is true:*

(1) $\underline{H}^i(\mathrm{Loc}(M_\lambda(-\lambda)))|_{C_w^\dagger}$ vanishes unless $i \in [0, d-\ell(w)]$.
(2) We have a surjective "highest weight" map

$$\underline{H}^{d-\ell(w)}(\mathrm{Loc}(M_\lambda(-\lambda)))|_{C_w^\dagger} \to HCS_{w,\lambda}(M(\mathfrak{m}_w)_{\lambda+w^{-1}w_{0,M}\rho+\rho}).$$

(3) If $w = w_0^M$, the above map is an isomorphism.

Proof. We remark that

$$\mathrm{Loc}(M_\lambda) = \mathrm{RHom}_{\mathfrak{g}}(M_\lambda, \mathcal{C}^{\mathrm{la}}) = \mathrm{RHom}_{\mathfrak{b}}(E(\lambda), \mathcal{C}^{\mathrm{la}}). \tag{3.5.24}$$

Thus, the fiber of $\mathrm{Loc}(M_\lambda)$ at w is the $Stab(w)$-representation $\mathrm{Ext}_{\mathfrak{b},*_2}(E(\lambda), \mathcal{O}_{U_{P_w}\backslash G,e})$, and the result is immediate from Proposition 3.5.25 below. $\square$

Proposition 3.5.25. *The cohomology groups $\mathrm{Ext}^i_{\mathfrak{b},*_2}(E(\lambda), \mathcal{O}_{U_{P_w}\backslash G,e})$ vanish outside degrees $[0, d-\ell(w)]$. Moreover, there is a canonical surjective map*

$$\mathrm{Ext}^{d-\ell(w)}_{\mathfrak{b},*_2}(E(\lambda), \mathcal{O}_{U_{P_w}\backslash G,e}) \to M(\mathfrak{m}_w)^\vee_{\lambda+w^{-1}w_{0,M}\rho+\rho}.$$

If $w = w_0^M$, this map is an isomorphism.

Proof. We recall that by Proposition 3.5.14 (and its proof), we have:

$$\begin{aligned} \mathrm{RHom}_{\mathfrak{b},*_2}(E(\lambda), \mathcal{O}_{U_{P_w}\backslash G,e}) &= \mathrm{RHom}_{\mathfrak{u}_{P_w}\oplus\mathfrak{b}}(E, \mathcal{O}_{G,e}(-\lambda)) \\ &= \mathrm{RHom}_{\mathfrak{u}_{P_w}}(E, \hat{M}_\lambda^\vee). \end{aligned}$$

Now, the cohomology $\mathrm{R}\Gamma(\mathfrak{u}_{P_w} \cap \bar{\mathfrak{b}}, \hat{M}_\lambda^\vee)$ is concentrated in degree 0. (Indeed, recall from Remark 3.4.4 that $\hat{M}_\lambda^\vee = (\mathcal{O}_{G/U,e}(-\lambda))^{\mathfrak{h}}$. As a $\mathfrak{u}_{P_w} \cap \bar{\mathfrak{b}}$-module, this module can be written in the form $\mathcal{O}_{U_{P_w}\cap\bar{B},e} \otimes V$ where V is an LB-space of compact type with trivial action. We observe that $H^i(\mathfrak{u}_{P_w} \cap \bar{\mathfrak{b}}, \mathcal{O}_{U_{P_w}\cap\bar{B},e}) = 0$ if $i > 0$.)

We therefore have:

$$\begin{aligned} \mathrm{R}\Gamma(\mathfrak{u}_{P_w}, \hat{M}_\lambda^\vee) &= \mathrm{R}\Gamma(\mathfrak{u}_{P_w} \cap \mathfrak{b}, \mathrm{R}\Gamma(\mathfrak{u}_{P_w} \cap \bar{\mathfrak{b}}, \hat{M}_\lambda^\vee)) \\ &= \mathrm{R}\Gamma(\mathfrak{u}_{P_w} \cap \mathfrak{b}, H^0(\mathfrak{u}_{P_w} \cap \bar{\mathfrak{b}}, \hat{M}_\lambda^\vee)), \end{aligned}$$

and we see in particular that the cohomology vanishes above degree $d - \ell(w) = \dim \mathfrak{u}_{P_w} \cap \mathfrak{b}$.

We now consider the surjective "restriction" map $\mathcal{O}_{G,e} \to \mathcal{O}_{P_w(B\cap\bar{U}_{P_w}),e}$ induced by the inclusion $P_w(B \cap \bar{U}_{P_w}) \hookrightarrow G$. We deduce a map:

$$\mathrm{RHom}_{\mathfrak{u}_{P_w}\oplus\mathfrak{b}}(E(\lambda), \mathcal{O}_{G,e}) \to \mathrm{RHom}_{\mathfrak{u}_{P_w}\oplus\mathfrak{b}}(E(\lambda), \mathcal{O}_{P_w(B\cap\bar{U}_{P_w}),e}).$$

We claim that this this map is surjective in degree $d - \ell(w)$, and that it is an isomorphism if $w = w_0^M$. To see this, we first take $\mathfrak{b}$ and $\mathfrak{u}_{P_w} \cap \mathfrak{b}$-cohomology which gives a surjective map (the cohomology is still in degree 0):

$$(\mathcal{O}_{U_{P_w}\cap\bar{B}\backslash G/B,e}(-\lambda))^{\mathfrak{h}} \to (\mathcal{O}_{U_{P_w\cap\bar{B}}\backslash P_w/B\cap P_w,e}(-\lambda))^{\mathfrak{h}}.$$

Taking the cohomology of $\mathfrak{u}_{P_w} \cap \mathfrak{b}$, we see that the above surjective map induces a surjective map in top degree cohomology (which is an isomorphism if $w = w_0^M$).

It remains to identify the target of the surjection with $M(\mathfrak{m}_w)^\vee_{\lambda+w^{-1}w_{0,M}\rho+\rho}$. We cannot immediately deduce this, because the computation above is not $\mathfrak{m}_w$-equivariant (because the decomposition of $\mathfrak{u}_{P_w} = \mathfrak{u}_{P_w} \cap \mathfrak{b} \oplus \mathfrak{u}_{P_w} \cap \bar{\mathfrak{b}}$ is not $\mathfrak{m}_w$-invariant). In order to identify the $\mathfrak{m}_w$-module structure of $\mathrm{RHom}_{\mathfrak{u}_{P_w}\oplus\mathfrak{b}}(E(\lambda), \mathcal{O}_{P_w(B\cap\bar{U}_{P_w}),e})$ we compute the cohomology in a different way, by first considering $\mathfrak{u}_{P_w}$-cohomology, and then $\mathfrak{b}$-cohomology.

Certainly

$$\mathrm{RHom}_{\mathfrak{u}_{P_w}}(E, \mathcal{O}_{P_w(B\cap\bar{U}_{P_w}),e}) = \mathcal{O}_{U_{P_w}\backslash P_w(B\cap\bar{U}_{P_w}),e}[0],$$

so it remains to show that

$$\mathrm{Ext}^{d-\ell(w)}_{\mathfrak{b},*_2}(E(\lambda), \mathcal{O}_{U_{P_w}\backslash P_w(B\cap\bar{U}_{P_w}),e}) = (\mathcal{O}_{M_w/U_{M_w},e}(-\lambda - w^{-1}w_{0,M}\rho - \rho))^{\mathfrak{h}}.$$

We will then be done, because the right hand side is $\hat{M}(\mathfrak{m}_w)^\vee_{\lambda+w^{-1}w_{0,M}\rho+\rho}$ by Remark 3.4.4 and the proof of Proposition 3.4.13.

We now show the claim. Note that $\mathfrak{b} = \mathfrak{b} \cap \mathfrak{p}_w \oplus \mathfrak{b} \cap \bar{\mathfrak{u}}_{P_w}$. We first compute

$$\mathrm{R}\Gamma_{*_2}(\mathfrak{b} \cap \bar{\mathfrak{u}}_{P_w}, \mathcal{O}_{U_{P_w}\backslash P_w(B\cap\bar{U}_{P_w}),e}) = \mathcal{O}_{U_{P_w}\backslash P_w,e}[0] = \mathcal{O}_{M_w,e}[0].$$

We now observe that $\mathcal{O}_{M_w,e}$ is a $\mathfrak{b}\cap\mathfrak{p}_w$-representation, with $\mathfrak{b}\cap\mathfrak{u}_{P_w}$ acting trivially. We deduce that

$$\begin{aligned}\mathrm{Ext}^{d-\ell(w)}_{\mathfrak{b}\cap\mathfrak{p}_w,*_2}(E(\lambda), \mathcal{O}_{M_w,e}) &= \mathrm{Hom}_{\mathfrak{b}\cap\mathfrak{m}_w,*_2}(E(\lambda), \mathcal{O}_{M_w,e} \otimes \wedge^{d-\ell(w)}(\mathfrak{u}_{P_w} \cap \mathfrak{b})^\vee) \\ &= (\mathcal{O}_{M_w/U_{M_w},e}(-\lambda - w^{-1}w_{0,M}\rho - \rho))^{\mathfrak{h}}. \qquad \square\end{aligned}$$

3.6. **Localization and higher Coleman sheaves at singular weight.** In this section we study localization at a singular weight for $G = \mathrm{GSp}_4$ and P is the Siegel parabolic associated to the cocharacter $\mu = (-1/2, -1/2; 1/2) \in X_*(T)_E$. We freely use our notation for GSp_4 (see 1.8.8). We consider the Klingen parabolic $Q \supseteq B$ attached to the simple root β. We denote by M_Q the associated Levi which is a group of semi-simple rank 1. It is important for us that $w_0^M s_\beta \in {}^M W$; this implies that the stratum $C_{w_0^M,Q}$ is the union of two B-orbits $C_{w_0^M}$ and $C_{w_0^M s_\beta}$. We wish to study the localization $\mathrm{Loc}(M(\mathfrak{g})_\lambda(-\lambda))|_{C_{w_0^M,Q}}$ for $\lambda = (1, 1; w)$. We notice that in this singular weight the horizontal action is not semi-simple. We are going to study this action and describe the semi-simple part.

3.6.1. *Geometry of the strata.* We consider the Q-orbit $C_{w_0^M,Q}$ on $\mathcal{FL}$, which is the union of the two Bruhat strata $C_{w_0^M}$ and $C_{w_0^M s_\beta}$.

Note that the map $Q \to C_{w_0^M,Q}$, $q \mapsto w_0^M q$ induces an isomorphism $P_{w_0^M} \cap Q\backslash Q \to C_{w_0^M,Q}$. The projection $Q \to M_Q$ induces a map $(P_{w_0^M} \cap Q)\backslash Q \to (P_{w_0^M} \cap M_Q)\backslash M_Q$. Since M_Q has semi-simple rank one and $(P_{w_0^M} \cap M_Q)$ is a Borel subgroup, we can identify $(P_{w_0^M} \cap M_Q)\backslash M_Q$ with $\mathbf{P}^1$, with ∞ the image of $w_0^M s_\beta$ and 0 the image of w_0^M. We therefore have a natural map $\pi : C_{w_0^M,Q} \to \mathbf{P}^1$, with $\pi^{-1}(\{\infty\}) = C_{w_0^M s_\beta}$ and $\pi^{-1}(\mathbf{A}^1) = C_{w_0^M}$; moreover, $B \cap s_\beta B s_\beta^{-1}$ acts transitively on $\pi^{-1}(\mathbf{G}_m) = C_{w_0^M} \cap C_{w_0^M} s_\beta$.

3.6.2. *Sheaves on the union of strata.* The inclusion $G_e \rtimes B \hookrightarrow G_e \rtimes Q$ induces an inclusion $\mathrm{Stab}_B(w_0^M) \hookrightarrow \mathrm{Stab}_Q(w_0^M)$, which induces the restriction map $LB_{(\mathfrak{g},Q)}(C^\dagger_{w_0^M,Q}) \to LB_{(\mathfrak{g},B)}(C^\dagger_{w_0^M})$. Similarly, conjugation by s_β gives an inclusion $\mathrm{Stab}_B(w_0^M s_\beta) \hookrightarrow \mathrm{Stab}_Q(w_0^M)$, which induces the restriction map $LB_{(\mathfrak{g},Q)}(C^\dagger_{w_0^M,Q}) \to LB_{(\mathfrak{g},B)}(C^\dagger_{w_0^M s_\beta})$.

We also note that the image of $Q \cap P_{w_0^M}$ in $M_{w_0^M}$ is the Borel $B_{M_{w_0^M}}$ (and the image of $Q \cap P_{w_0^M s_\beta}$ in $M_{w_0^M s_\beta}$ is the Borel $B_{M_{w_0^M s_\beta}}$). Therefore, the source category $\mathcal{O}(\mathfrak{m}_{w_0^M}, \mathfrak{q}_{M_{w_0^M}})$ for producing sheaves on $C_{w_0^M,Q}$ (see Definition 3.4.17) is tautologically equal to $\mathcal{O}(\mathfrak{m}_{w_0^M}, \mathfrak{b}_{M_{w_0^M}})$.

Proposition 3.6.3. *Let $\nu \in X^*(M_Q)_E$.*

1. *Conjugation by s_β induces an equivalence of categories $s_\beta : \mathcal{O}(\mathfrak{m}_{w_0^M}, \mathfrak{b}_{M_{w_0^M}})_{\nu\text{-alg}} \to \mathcal{O}(\mathfrak{m}_{w_0^M s_\beta}, \mathfrak{b}_{M_{w_0^M s_\beta}})_{\nu\text{-alg}}$.*
2. *For any $M \in \mathcal{O}(\mathfrak{m}_{w_0^M}, \mathfrak{b}_{M_{w_0^M}})_{\nu\text{-alg}}$, we have*

$$HCS_{Q,w_0^M,\nu}(M)|_{C^\dagger_{w_0^M}} = HCS_{w_0^M,\nu}(M),$$

$$HCS_{Q,w_0^M,\nu}(M)|_{C^\dagger_{w_0^M s_\beta}} = HCS_{w_0^M s_\beta,\nu}(s_\beta M).$$

Proof. The first point is obvious, and the second is immediate from the definition of the functors HCS (i.e. from the construction in the proof of Proposition 3.4.12). $\square$

3.6.4. *Singular localization.* Write $j_{w_0^M} : C^\dagger_{w_0^M} \hookrightarrow C^\dagger_{w_0^M,Q}$, $j_{w_0^M s_\beta} : C^\dagger_{w_0^M s_\beta} \hookrightarrow C^\dagger_{w_0^M,Q}$ for the inclusions. We take $\lambda = (1,1;w) \in X^*(T)$. Recall that $\rho = (-1,-2;0)$, so that $\lambda+\rho = (0,-1;w)$ is invariant under s_β. However, this character is not integral. Let us define $\eta = (0,-1;1)$ which differs from $\lambda+\rho$ by a character of the centre, and is still invariant under s_β.

Applying Proposition 3.6.3 with $\nu = \eta$, we have a short exact sequence of $(\mathfrak{g}, Q)$-equivariant sheaves:

$$0 \to (j_{w_0^M})_! HCS_{w_0^M,\eta}(M(\mathfrak{m}_{w_0^M})_\lambda) \to HCS_{Q,w_0^M,\eta}(M(\mathfrak{m}_{w_0^M})_\lambda) \to HCS_{w_0^M s_\beta,\eta}(M(\mathfrak{m}_{w_0^M s_\beta})_{s_\beta\lambda}) \to 0 \quad (3.6.5)$$

We want to study $\mathrm{Loc}(M(\mathfrak{g})_\lambda(-\lambda))|_{C^\dagger_{w_0^M,Q}} = \mathrm{RHom}_{\mathfrak{b},*_2}(\lambda, \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M,Q}})$.

Proposition 3.6.6. *We have an exact triangle:*

$$(j_{w_0^M})_! HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \to Loc(M(\mathfrak{g})_\lambda(-\lambda))|_{C^\dagger_{w_0^M,Q}} \to HCS_{w_0^M s_\beta,\lambda}(M(\mathfrak{m}_{w_0^M s_\beta})_{s_\beta\lambda})[-1] \xrightarrow{+1}$$

Proof. This is immediate from consequence of Theorem 3.5.11 together with Theorem 2.3.19 and Proposition 2.8.3. $\square$

There is a horizontal action Θ_{hor} of $Z(\mathfrak{m})$ on $\mathrm{Loc}(M_\lambda(-\lambda))|_{C^\dagger_{w_0^M,Q}}$. By Proposition 3.4.19 (see also Remark 4.6.8 below), this action is via $\mu_0 := \langle -w_0\lambda, \mu\rangle$ on both $HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda)$ and $HCS_{w_0^M s_\beta,\lambda}(M(\mathfrak{m}_{w_0^M})_{s_\beta\lambda})$ (and in particular the action of μ doesn't split the triangle). Taking the derived invariants for $\mu - \mu_0$ yields a triangle:

$$\begin{aligned}&(j_{w_0^M})_! HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \oplus (j_{w_0^M})_! HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda)[-1] \to \\ &\mathrm{RHom}_{\mathfrak{b},*_2;\mu}((\lambda,\mu_0),\mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M,Q}}) \to \\ &HCS_{w_0^M s_\beta,\lambda}(M(\mathfrak{m}_{w_0^M})_{s_\beta\lambda})[-1] \oplus HCS_{w_0^M s_\beta,\lambda}(M(\mathfrak{m}_{w_0^M})_{s_\beta\lambda})[-2] \stackrel{+1}{\to}\end{aligned} \tag{3.6.7}$$

Taking the H^1 yields the following short exact sequence of $(\mathfrak{g},B)$-equivariant sheaves:

$$0 \to (j_{w_0^M})_! HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \to \mathrm{Ext}^1_{\mathfrak{b},*_2;\mu}((\lambda,\mu_0),\mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M,Q}}) \to HCS_{w_0^M s_\beta,\lambda}(M(\mathfrak{m}_{w_0^M})_{s_\beta\lambda}) \to 0 \tag{3.6.8}$$

The main result of this section is the following:

Theorem 3.6.9. *The two extensions of $(\mathfrak{g},B)$-equivariant sheaves* (3.6.8) *and* (3.6.5) *differ by a twist of the B-action by the character $\lambda-\eta$ and multiplication by a scalar in $E^\times$.*

Remark 3.6.10. This arguments in the remainder of this section admit a simpler analogue in the GL_2-context; see [Pil24, §6].

3.6.11. *Preparations for the proof of Theorem 3.6.9.* We have commuting actions of $\mathfrak{b}$ (via $*_2$) and μ (via the horizontal action) on $\mathcal{C}^{\mathrm{la}}$. We begin by isolating a certain sub-Lie algebra of $\mathfrak{b}\oplus E\mu$ whose cohomology on $\mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M,Q}}$ is in degree 0.

We use the usual standard basis elements $X_\beta, X_{-\beta}, H_\beta$ with $H_\beta=[X_\beta,X_{-\beta}]$. We set $\mathfrak{h}':=\ker(\beta)$, so that $\mathfrak{b}$-cohomology can be obtained by first taking $\mathfrak{h}'\oplus\mathfrak{u}_Q$-cohomology, and then taking $EX_\beta\oplus EH_\beta$-cohomology. We write $\lambda'=\lambda|\mathfrak{h}'$.

Lemma 3.6.12. $\mathcal{C}^{\mathrm{la},\lambda',\mu_0} := \mathrm{Ext}_{\mathfrak{h}'\oplus\mathfrak{u}_Q,\mu}((\lambda';\mu_0),\mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M,Q}})$ *is concentrated in degree* 0.

Proof. We can do the computation separately on each of the strata $C^\dagger_{w_0^M}$ and $C^\dagger_{w_0^M s_\beta}$, so we reduce to showing that the cohomology on the fibers at w_0^M and $w_0^M s_\beta$ of $\mathcal{C}^{\mathrm{la}}$ vanishes in positive degrees. These fibers are respectively $\mathcal{O}_{U_{P_{w_0^M}}\backslash G,e}$ and $\mathcal{O}_{U_{P_{w_0^M s_\beta}}\backslash G,e}$, and the required vanishing follows from a consideration of the actions of $\bar{\mathfrak{p}}_{w_0^M}\cap\mathfrak{b}=\mathfrak{b}$ and $\bar{\mathfrak{p}}_{w_0^M s_\beta}\cap\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{u}_Q$ respectively. (Note that $\mu=\mathrm{diag}(0,0,1,1)$, so that $w_0^M\mu=\mathrm{diag}(1,1,0,0)$ and $s_\beta w_0^M\mu=\mathrm{diag}(0,1,0,1)$, and therefore $\mathfrak{h}'\oplus E\cdot w_0^M\mu=\mathfrak{h}'\oplus E\cdot s_\beta w_0^M\mu=\mathfrak{h}$.) ∎

We thus see that

$$\mathrm{RHom}_{\mathfrak{b},*_2;\mu}((\lambda,\mu_0),\mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M,Q}}) = \mathrm{RHom}_{EX_\beta\oplus EH_\beta}(\lambda,\mathcal{C}^{\mathrm{la},\lambda',\mu_0})$$

(where the restriction of λ to $EX_\beta\oplus EH_\beta$ takes $X_\beta\mapsto 0$). This cohomology is represented by the following Chevalley–Eilenberg complex $K^\bullet$ (in degrees $0,1$ and 2):

$$\mathcal{C}^{\mathrm{la},\lambda',\mu_0} \xrightarrow{\begin{pmatrix}X_\beta & H_\beta-\lambda(H_\beta)\end{pmatrix}} \mathcal{C}^{\mathrm{la},\lambda',\mu_0}\otimes E(-\beta)\oplus\mathcal{C}^{\mathrm{la},\lambda',\mu_0} \xrightarrow{\begin{pmatrix}H_\beta-\lambda(H_\beta)\\ -X_\beta\end{pmatrix}} \mathcal{C}^{\mathrm{la},\lambda',\mu_0}\otimes E(-\beta).$$

Since
$$HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) = \mathrm{Ext}^0_{\mathfrak{b}}(\lambda, \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M}}),$$
we deduce that $s_\beta^* HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \otimes E(-\beta) = \mathrm{Ext}^0_{s_\beta\mathfrak{b}}(s_\beta\lambda, \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M s_\beta}}) \otimes E(-\beta)$. On this sheaf, $\mathfrak{h}$ acts via $s_\beta\lambda - \beta = \lambda$ (this is a crucial place where we use that the weight is singular), and μ still acts via μ_0. Thus, we can consider the map
$$\begin{aligned} \mathrm{Ext}^0_{s_\beta\mathfrak{b}}(s_\beta\lambda, \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M s_\beta}}) \otimes E(-\beta) &\to K^1|_{C^\dagger_{w_0^M s_\beta}} = \mathcal{C}^{\mathrm{la},\lambda',\mu_0}|_{C^\dagger_{w_0^M s_\beta}} \otimes E(-\beta) \oplus \mathcal{C}^{\mathrm{la},\lambda',\mu_0}|_{C^\dagger_{w_0^M s_\beta}} \\ s &\mapsto (s,0), \end{aligned}$$
which induces a map
$$\mathrm{Ext}^0_{s_\beta\mathfrak{b}}(s_\beta\lambda, \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M s_\beta}}) \otimes E(-\beta) \to \mathrm{Ext}^1_{\mathfrak{b},*_2;\mu}((\lambda,\mu_0), \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M s_\beta}}). \tag{3.6.13}$$

We will show below that (3.6.13) is an isomorphism. We begin by studying its restrictions to $C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta$ and to $C^\dagger_{w_0^M s_\beta}$; it is immediate from the definitions that the former restriction is a map in $LB_{(\mathfrak{g},B\cap B_{s_\beta})}(C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta)$, and the latter is a map in $LB_{(\mathfrak{g},B\cap B_{s_\beta})}(C^\dagger_{w_0^M s_\beta})$.

Lemma 3.6.14.

1. *(1) The restrictions of the left and right hand sides of* (3.6.13) *to* $C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta$ *are isomorphic, and their endomorphism algebras in* $LB_{(\mathfrak{g},B\cap B_{s_\beta})}(C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta)$ *are the scalars* E.
2. *(2) The restrictions of the left and right hand sides of* (3.6.13) *to* $C^\dagger_{w_0^M s_\beta}$ *are isomorphic, and their endomorphisms in* $LB_{(\mathfrak{g},B\cap B_{s_\beta})}(C^\dagger_{w_0^M s_\beta})$ *are the scalars* E.

Proof. To begin, we note that it follows from the Q-equivariance of $HCS_{Q,w_0^M,\eta}(M(\mathfrak{m}_{w_0^M})_\lambda)$ that
$$s_\beta^*(HCS_{Q,w_0^M,\eta}(M(\mathfrak{m}_{w_0^M})_\lambda) \otimes E(\lambda-\eta+\beta)) = HCS_{Q,w_0^M,\eta}(M(\mathfrak{m}_{w_0^M})_\lambda) \otimes E(\lambda-\eta).$$
It follows that
$$\begin{aligned} \mathrm{Ext}^0_{s_\beta\mathfrak{b}}(s_\beta\lambda, \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M s_\beta}})|_{C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta} &= s_\beta^* HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \otimes E(-\beta) \mid_{C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta} \\ &= HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \mid_{C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta}. \end{aligned}$$
On the other hand
$$\mathrm{Ext}^1_{\mathfrak{b},*_2;\mu}((\lambda,\mu_0), \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M s_\beta}})|_{C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta} = HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \mid_{C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta}$$
(by Proposition 3.6.3 and the proof of Proposition 3.6.6.)

We now check that the endomorphisms of the sheaf $HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \mid_{C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta}$ in the category $LB_{(\mathfrak{g},B\cap B_{s_\beta})}(C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta)$ are scalars.

Since $B\cap B_{s_\beta}$ acts transitively on $C_{w_0^M}\cap C_{w_0^M}s_\beta$ by section 3.6.1, an endomorphism is determined by its behavior at any fiber. More precisely, let $x\in C_{w_0^M}\cap C_{w_0^M}s_\beta$. We consider the uniformization map

$$\begin{aligned} G_e\rtimes(B\cap B_{s_\beta}) &\rightarrow C^\dagger_{w_0^M}\cap C^\dagger_{w_0^M}s_\beta\\ (g,b)&\mapsto xgb. \end{aligned}$$

Let $\mathrm{Stab}(x)$ be the stabilizer of x for this action. We have $\mathrm{Stab}(x)=\{(g,b),gb\in P_x\}$. Exactly as in the proof of Lemma 3.3.6, this group is

$$(P_x\cap B\cap B_{s_\beta})_e\backslash[((P_x)_e\times(B\cap B_{s_\beta})_e)\rtimes(P_x\cap B\cap B_{s_\beta})].$$

The fiber we consider is the completion of a dual Verma module for $\mathfrak{m}_x$ (by Definitions 3.4.17 and 4.6.6), and the first part of the lemma follows from the property that the endomorphisms of a Verma module are the scalars (together with Theorem 2.2.39).

The second part is proved in the same way, as follows. We first observe that

$$\begin{aligned} \mathrm{Ext}^0_{s_\beta\mathfrak{b}}(s_\beta\lambda,\mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M s_\beta}})\otimes E(-\beta)|_{C^\dagger_{w_0^M s_\beta}} &= HCS_{w_0^M s_\beta,\lambda}(M(\mathfrak{m}_{w_0^M s_\beta})_{s_\beta\lambda})\\ &= \mathrm{Ext}^1_{\mathfrak{b},*_2;\mu}((\lambda,\mu_0),\mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M s_\beta}}) \end{aligned}$$

We reduce to showing that the endomorphisms of $HCS_{w_0^M s_\beta,\lambda}(M(\mathfrak{m}_{w_0^M s_\beta})_{s_\beta\lambda})$ in the category $LB_{(\mathfrak{g},B\cap B_{s_\beta})}(C^\dagger_{w_0^M s_\beta})$ are scalars which again follows from the property that endomorphisms of Verma modules are scalar. □

Proposition 3.6.15. *The map* (3.6.13) *is an isomorphism.*

Proof. By Lemma 3.6.14 it suffices to show that (3.6.13) is nonzero on the fiber at one point of $C_{w_0^M s_\beta}$ and at one point of $C_{w_0^M}\cap C_{w_0^M}s_\beta$. It suffices in turn to prove that the map

$$\mathcal{C}^{\mathrm{la},\lambda',\mu_0}\xrightarrow{H_\beta-\lambda(H_\beta)}\mathcal{C}^{\mathrm{la},\lambda',\mu_0}$$

induces an injective map on the fibers at $w_0^M s_\beta$ and at one point of $C_{w_0^M}\cap C_{w_0^M}s_\beta$.

By definition, the kernel of this map on the fiber at a point x is

$$\mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{u}_Q,*_2;\mu}((\lambda,\mu_0),\mathcal{C}^{\mathrm{la}}_x).$$

We first consider the fiber at $x=w_0^M s_\beta$, where $\mathcal{C}^{\mathrm{la}}_{w_0^M s_\beta}=\mathcal{O}_{U_{P_{2w}}\backslash G,e}$. We have

$$\mathfrak{u}_{P_{w_0^M s_\beta}}=EX_\beta\oplus EX_{-\alpha}\oplus EX_{-\delta},\ \mathfrak{u}_Q=EX_\alpha\oplus X_\delta\oplus X_\gamma.$$

For any $s\in\Phi$ we write U_s for the corresponding $1-$parameter subgroup. We pick a coordinate x_s on U_s with the property that the corresponding vector field is X_s. Then elements of $\mathrm{Hom}_{\mathfrak{h}\oplus\mathfrak{u}_Q,*_2}(\lambda,\mathcal{C}^{\mathrm{la}}_x)$ are germs of analytic functions on $U_{-\beta}U_{-\gamma}T$, which can be written as

$$f(x_{-\beta},x_{-\gamma},t)=\sum_{k_{-\beta},k_{-\gamma}\in\mathbf{Z}_{\geq0}}x_{-\beta}^{k_{-\beta}}x_{-\gamma}^{k_{-\gamma}}\lambda(t).$$

We need to show that if $(\Theta_{\mathrm{hor}}(\mu)-\mu_0)f=0$, then $f=0$. By definition, $\Theta_{\mathrm{hor}}(\mu)$ acts on the left via the action of $-(w_0^M s_\beta)^{-1}\mu\in\mathfrak{h}$. It follows that

$$\begin{aligned} \Theta_{\mathrm{hor}}(\mu) &= \langle(w_0^M s_\beta)^{-1}\mu,\beta\rangle x_{-\beta}\partial_{x_{-\beta}}+\langle(w_0^M s_\beta)^{-1}\mu,\gamma\rangle x_{-\gamma}\partial_{x_{-\gamma}}-\langle(w_0^M s_\beta)^{-1}\mu,\lambda\rangle\\ &= x_{-\beta}\partial_{x_{-\beta}}-\frac{w}{2}. \end{aligned}$$

(We are using here that $(w_0^M s_\beta)^{-1}\mu = (-\frac{1}{2}, \frac{1}{2}; \frac{1}{2})$, $\beta = (-2, 0; 0)$, $\gamma = (-1, -1; 0)$, $\lambda = (1, 1; w)$.) Thus $\Theta_{\mathrm{hor}}(\mu)$ acts by $k_{-\beta} - \frac{w}{2}$ on $x_{-\beta}^{k_{-\beta}} x_{-\gamma}^{k_{-\gamma}} \lambda(t)$. Since $\mu_0 = -1 - \frac{w}{2}$, we deduce that $\Theta_{\mathrm{hor}}(\mu) - \mu_0$ is injective.

We now consider the point $w_0^M s_\beta g_{-\beta} \in C_{w_0^M} \cap C_{w_0^M} s_\beta$, where $g_{-\beta} = \exp(X_{-\beta})$. We see that

$$(w_0^M s_\beta g_{-\beta})^{-1}\mu = (w_0^M s_\beta)^{-1}\mu + \langle (w_0^M s_\beta)^{-1}\mu, \beta\rangle X_{-\beta}.$$

An element of $\mathcal{C}^{\mathrm{la},\lambda',\mu_0}_{2_{w g_{-\beta}}}$ can still be expressed as a germ of an analytic function on $U_{-\beta} U_{-\gamma} T$, and can thus be written as

$$f(x_{-\beta}, x_{-\gamma}, t) = \sum_{k_{-\beta}, k_{-\gamma} \in \mathbf{Z}_{\geq 0}} x_{-\beta}^{k_{-\beta}} x_{-\gamma}^{k_{-\gamma}} \lambda(t).$$

We now find that $\Theta_{\mathrm{hor}}(\mu) = x_{-\beta}\partial_{x_{-\beta}} - \frac{w}{2} + \partial_{x_{-\beta}}$, so that

$$\Theta_{\mathrm{hor}}(\mu) x_{-\beta}^{k_{-\beta}} x_{-\gamma}^{k_{-\gamma}} \lambda(t) = (k_{-\beta} - \frac{w}{2}) x_{-\beta}^{k_{-\beta}} x_{-\gamma}^{k_{-\gamma}} \lambda(t) + k_{-\beta} x_{-\beta}^{k_{-\beta}-1} x_{-\gamma}^{k_{-\gamma}} \lambda(t),$$

and we again deduce that $\Theta_{\mathrm{hor}}(\mu) - \mu_0$ is injective. □

Proof of Theorem 3.6.9. By Proposition 3.6.15, the sheaf $\mathrm{Ext}^1_{\mathfrak{b},*_2;\mu}((\lambda, \mu_0), \mathcal{C}^{\mathrm{la}}|_{C^\dagger_{w_0^M, Q}})$ is obtained by gluing $HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda)$ and $s_\beta^* HCS_{w_0^M,\lambda}(M(\mathfrak{m}_{w_0^M})_\lambda) \otimes E(-\beta)$ along $C^\dagger_{w_0^M} \cap C^\dagger_{w_0^M} s_\beta$. The same is true of $HCS_{Q,w_0^M,\eta}(M(\mathfrak{m}_{w_0^M})_\lambda) \otimes E(\lambda - \eta)$ by construction. The gluing data is that of an isomorphism of $(\mathfrak{g}, B \cap B_{s_\beta})$-equivariant sheaves, and by Lemma 3.6.14 the space of such isomorphisms identifies with $E^\times$, so there is (up to isomorphism) a unique way to glue, as required. □

4. p-adic Eichler–Shimura theory

4.1. **Introduction.** The main goal of this section (as mentioned in §1.4) is to relate higher Coleman theory to completed cohomology, so that (ultimately) we can connect the Galois-theoretic properties of a p-adic ordinary (overconvergent) modular form (in terms of the action of the Sen operator) to its classicality.

Before proceeding, we introduce some notation. We fix a Hodge type Shimura datum (G, X). We assume that $G_{\mathbf{Q}_p}$ is quasi-split. Let P be the parabolic corresponding to μ with Levi M. Let $B \subseteq G_{\mathbf{Q}_p}$ be a Borel subgroup. We pick a maximal torus $T \subseteq B$. The relevant flag variety is $FL = P\backslash G$ and we have the decomposition $FL = \coprod_{w \in {}^M W} P\backslash PwB$ where ${}^M W \subseteq W$ is the set of Kostant representatives in the absolute Weyl group. We also fix a coefficient field E which is a finite extension of $\mathbf{Q}_p$ and admits a map from the reflex field of the Shimura datum.

4.1.1. *Higher Coleman theory.* Higher Coleman theory [BP21] is a theory of (higher) overconvergent modular forms. The different higher Coleman theories are parameterized by two parameters: an element $w \in {}^M W$, and a weight.

Remark 4.1.2. We note that ${}^M W$ parametrizes chambers in the weight space which are M-dominant, and the w-theory will interpolate those classical cohomologies whose weights belong to the w-chamber.

To describe the weight parameter, we fix $w \in {}^M W$. Let $\mathfrak{m} = \mathrm{Lie}(M)$, and let $\mathfrak{m}_w = w^{-1}\mathfrak{m}w$. Let $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$ be the BGG category $\mathcal{O}$ of $U(\mathfrak{m}_w)$-modules for the Borel $\mathfrak{b}_{\mathfrak{m}_w} = \mathrm{Lie}(B) \cap \mathfrak{m}_w$. In the same way that weights of modular forms are finite dimensional representations of M, weights of (higher) overconvergent modular forms (parameterized by $w \in {}^M W$) are objects of $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$. For any $\lambda \in X^*(T)_E$, we let $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})_{\lambda-\mathrm{alg}}$ be the subcategory of category $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})$ with weights in $\lambda + X^*(T)$.

We have higher Coleman functors (see Definition 4.6.35):

$$HC_{w,\lambda}, HC_{\mathrm{cusp},w,\lambda} : \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})^{\mathrm{op}}_{\lambda-\mathrm{alg}} \to D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)) \tag{4.1.3}$$

where $\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)$ is the category of locally analytic representations of $B(\mathbf{Q}_p)$ with $\mathfrak{b}$ acting like λ. For example, if $\lambda = 0$, this is just the category of smooth $B(\mathbf{Q}_p)$-representations.

Remark 4.1.4. The parameter λ is there to specify the $B(\mathbf{Q}_p)$-action. If $\eta \in X^*(T)$, the categories $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})_{\lambda-\mathrm{alg}}$ and $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})_{\lambda+\eta-\mathrm{alg}}$ are canonically the same and one has $HC_{w,\lambda+\eta}(M) = HC_{w,\lambda}(M) \otimes E(\eta)$ where $- \otimes E(\eta)$ means a twist of the $B(\mathbf{Q}_p)$-action by η.

The functors (4.1.3) are defined by first attaching to every object M of $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})_{\lambda-\mathrm{alg}}$ a "quasi-coherent" $B(\mathbf{Q}_p)$-equivariant sheaf over the pullback of the Bruhat stratum $P \backslash PwB$ via the Hodge–Tate period map and taking its cohomology with suitable support condition.

We also have a finite slope part functor $D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)) \to D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{T(\mathbf{Q}_p)}(E))$ and we can speak of the finite slope part of higher Coleman functors (Section 4.6.54):

$$HC^{fs}_{w,\lambda}, HC^{fs}_{\mathrm{cusp},w,\lambda} : \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})^{\mathrm{op}}_{\lambda-\mathrm{alg}} \to D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{T(\mathbf{Q}_p)}(E)).$$

Remark 4.1.5. Let $\kappa \in X^*(T)^{+,M}$. Let $L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}$ be the finite dimensional representation of $\mathfrak{m}_w$ of of highest weight $-w^{-1}w_{0,M}\kappa$. We show that

$$HC^{fs}_{w,0}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa})$$

is the direct sum of the $\mathrm{R}\Gamma_w(K^p, \kappa, \chi)^{+,fs}$ of [BP21] (Theorem 4.6.56). These are higher Coleman theories with value in the classical sheaf of weight κ. On the other hand, if $M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}$ denotes the Verma of highest weight $-w^{-1}w_{0,M}\kappa$, then

$$HC^{fs}_{w,0}(M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa})$$

corresponds to higher Coleman theory with value in the big "induction" sheaf (Theorem 4.6.57). The surjective map $M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa} \to L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}$ induces a map

$$HC_{w,0}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}) \to HC_{w,\lambda}(M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa})$$

and similarly on the finite slope part. In summary, the improvements on [BP21] are the following:

- We extend the definitions to the infinite slope part (in [BP21], only the finite slope part was canonically defined).
- We introduce a more functorial perspective on the weights. In [BP21] we allowed weights to be either finite dimensional representations or Verma modules, which of course generate the BGG category.

Our main results on higher Coleman theory can be summarized as follows.

Theorem 4.1.6 (Theorems 4.6.45, 4.6.58, and 4.6.60)**.**

(1) $HC_{w,\lambda}$, $HC^{fs}_{w,\lambda}$ *have cohomological amplitude* $[\ell(w), d]$ *and* $HC_{\mathrm{cusp},w,\lambda}$, $HC^{fs}_{\mathrm{cusp},w,\lambda}$ *have cohomological amplitude* $[0, \ell(w)]$.
(2) *Let* $M \in \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w})_{\lambda-\mathrm{alg}}$ *be a module generated by a highest weight vector of weight* ν. *Assume that the Shimura variety is proper or that we are in the Siegel case. The slopes appearing in* $HC^{fs}_{\mathrm{cusp},w,\lambda}(M)$ *and* $HC^{fs}_{w,\lambda}(M)$ *are* $\geq \lambda - \nu + w^{-1}w_{0,M}\rho + \rho$.

Remark 4.1.7. In the proper case, the functors $HC_{w,\lambda}$ and $HC^{fs}_{w,\lambda}$ are exact.

4.1.8. *Completed cohomology.* We let $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)$ be completed cohomology and $\mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)$ denote completed cohomology with compact support.

We let $\mathcal{O}(\mathfrak{g}, \mathfrak{b})$ be the BGG category $\mathcal{O}$ for $\mathfrak{g}$ and $\mathfrak{b}$. We define functors (Section 4.7):

$$\begin{array}{rcl} CC_\lambda : \mathcal{O}(\mathfrak{g}, \mathfrak{b})_{\lambda-\mathrm{alg}} & \to & D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{g}}(M, \mathrm{R}\Gamma(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}) \end{array}$$

$$\begin{array}{rcl} CC_{\mathrm{cusp},\lambda} : \mathcal{O}(\mathfrak{g}, \mathfrak{b})_{\lambda-\mathrm{alg}} & \to & D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{g}}(M, \mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}) \end{array}$$

Remark 4.1.9. The first natural example is to apply these functors to M a finite dimensional representation of G, in which case we recover classical étale cohomology with weight $M^\vee$ (it is natural to take $\lambda = 0$). For a non-classical example, we can take $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$ to be a Verma module of weight λ so that we are computing $\mathfrak{b}$-cohomology.

4.1.10. *p-adic Eichler Shimura.* We are now ready to state our main result comparing completed cohomology and higher Coleman theory. It holds under a non-Liouville condition on λ (see Definition 2.3.25). We observe that if $\lambda \in X^*(T)$ is algebraic, it is non-Liouville.

Theorem 4.1.11 (Theorem 4.7.1)**.** *Assume* λ *is non-Liouville and* M *is an object of* $\mathcal{O}(\mathfrak{g}, \mathfrak{b})_{\lambda-\mathrm{alg}}$. *We have a spectral sequence*

$$E_1^{p,q} = \oplus_{w \in {}^MW, \ell(w)=p} H^{p+q}(HC_{w,\lambda}(M \otimes^L_{\mathfrak{u}_{\mathfrak{p}_w}} E))$$

converging to $H^{p+q}(CC_\lambda(M)) \otimes \mathbf{C}_p$. *Moreover, the Sen operator is given by* $w\mu \in Z(\mathfrak{m}_w)$ *acting on* $H_*(\mathfrak{u}_{\mathfrak{p}_w}, M)$.

Remark 4.1.12. The functor $- \otimes_{U(\mathfrak{u}_{\mathfrak{p}_w})} E : D(\mathcal{O}(\mathfrak{g}, \mathfrak{b})) \to D(\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{\mathfrak{m}_w}))$ is the Lie algebra homology of the unipotent radical $\mathfrak{u}_{\mathfrak{p}_w}$ of $\mathfrak{p}_w = w^{-1}\mathrm{Lie}(P)w$. It is computed by the Koszul complex (in degree $-d$ to 0):

$$0 \to M \otimes \Lambda^d \mathfrak{u}_{\mathfrak{p}_w} \to \cdots \to M \to 0$$

Remark 4.1.13. In Section 3.5 we attached to $M \in \mathcal{O}(\mathfrak{g}, \mathfrak{b})$ a certain twisted D-module $\mathrm{Loc}(M)$ on the flag variety. This is a version of Beilinson–Bernstein localization. This twisted D-module completely encodes the p-adic Eichler–Shimura theory (see Theorem 4.7.1 for a precise statement). We observe that $\mathrm{Loc}(M)$ is "constant" on each Bruhat stratum C_w and its restriction to each C_w is determined (in the non-Liouville case) by the Lie algebra homology $M \otimes^L_{\mathfrak{u}_{\mathfrak{p}_w}} E$.

Under favorable "genericity" assumptions, the spectral sequence simplifies a lot. Let us denote by $M(\mathfrak{g})_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda$ the Verma module of weight λ. We adopt a similar notation to denote Verma modules for other reductive Lie algebras.

Corollary 4.1.14 (Corollary 4.7.3)**.** *Assume that λ is non-Liouville and antidominant in the sense of Remark 3.5.21, and that the Shimura variety is proper. Then $CC_\lambda(M(\mathfrak{g})_\lambda)$ is concentrated in the middle degree d and moreover, it has a decreasing filtration $\mathrm{Fil}^i\mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda))$ with*

- $\mathrm{Fil}^{d+1}\mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda)) = 0$,
- $\mathrm{Fil}^0\mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda)) = \mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda))$,
- $\mathrm{Gr}^p\mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda)) = \oplus_{w \in {}^MW, \ell(w)=p} H^p(HC_{w,\lambda}(M(\mathfrak{m}_w)_{\lambda + w^{-1}w_{0,M}\rho + \rho}))$.

We also refer to Theorem 4.7.5 for a similar result in the ordinary case.

4.2. **Perfectoid Shimura varieties.** We consider a Hodge-type Shimura datum (G, X). We also fix a map from the reflex field of the Shimura datum to E. We let $\mathrm{Sh}^{rat}_{K_pK^p}$ be the analytic space over $\mathrm{Spa}(E, \mathcal{O}_E)$ attached to the Shimura variety of level K_pK^p over Spec E. We let $\mathrm{Sh}_{K_pK^p} = \mathrm{Sh}^{rat}_{K_pK^p} \times_{\mathrm{Spa}(E,\mathcal{O}_E)} \mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. We let $\mathrm{Sh}^{rat,\mathrm{tor}}_{K_pK^p,\Sigma}$ be a toroidal compactification of $\mathrm{Sh}^{rat}_{K_pK^p}$ over $\mathrm{Spa}(E, \mathcal{O}_E)$ for a specific choice of cone decomposition Σ (see [FC90], [Lan13]). For $K'_p \subseteq K_p$, we have a natural map $\mathrm{Sh}^{rat,\mathrm{tor}}_{K'_pK^p,\Sigma} \to \mathrm{Sh}^{rat,\mathrm{tor}}_{K_pK^p,\Sigma}$. We let $\mathrm{Sh}^{rat,\mathrm{tor}}_{K^p,\Sigma} = \lim_{K'_p} \mathrm{Sh}^{rat,\mathrm{tor}}_{K'_pK^p,\Sigma}$. By [Sch15] (and [PS16a], [Lan22] for the extension to the compactification), this is a perfectoid space. We note that the same cone decomposition Σ is used at each stage of the limit and that there is some restriction on the choice of cone decomposition if we are not in the Siegel case. Concretely, the underlying topological space of $\mathrm{Sh}^{rat,\mathrm{tor}}_{K^p,\Sigma}$ is the inverse limit of the topological spaces of the $\mathrm{Sh}^{rat,\mathrm{tor}}_{K'_pK^p,\Sigma}$, and there is a basis of affinoid opens $\mathrm{Spa}(A, A^+)$ of $\mathrm{Sh}^{rat,\mathrm{tor}}_{K^p,\Sigma}$, which are pull backs of affinoid opens $\mathrm{Spa}(A_{K'_p}, A^+_{K'_p})$ in $\mathrm{Sh}^{rat,\mathrm{tor}}_{K'_pK^p,\Sigma}$ for small enough K'_p, and such that A^+ is the p-adic completion of $\mathrm{colim}_{K'_p} A^+_{K'_p}$.

We let $\mathrm{Sh}^{rat}_{K^p} \hookrightarrow \mathrm{Sh}^{rat,\mathrm{tor}}_{K^p,\Sigma}$ be the open subspace $\lim_{K'_p} \mathrm{Sh}^{rat}_{K'_pK^p}$. We let $\pi^{rat}_{HT,\Sigma} : \mathrm{Sh}^{rat,\mathrm{tor}}_{K^p,\Sigma} \to \mathcal{FL}^{rat}$ be the Hodge–Tate period map, where $\mathcal{FL}^{rat}$ is the Flag variety over $\mathrm{Spa}(E, \mathcal{O}_E)$.

We let $\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p,\Sigma}$ be the base change to $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$ of $\mathrm{Sh}^{rat,\mathrm{tor}}_{K_pK^p,\Sigma}$ (which therefore carries an action of $\mathrm{Gal}(\bar{E}/E)$). We similarly define $\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}$, Sh_{K^p} and $\mathcal{FL}$, and the period map $\pi_{HT,\Sigma} : \mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma} \to \mathcal{FL}$ which is $\mathrm{Gal}(\bar{E}/E)$-equivariant.

The period map is also K_p-equivariant. The action of $G(\mathbf{Q}_p)$ on $\mathrm{Sh}_{K^p,\Sigma}$ does not extend to an action on $\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}$ but for a general $g \in G(\mathbf{Q}_p)$ we still have diagrams:

$$\begin{array}{ccc} \mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma} & \xrightarrow{g} & \mathrm{Sh}^{\mathrm{tor}}_{K^p,g\Sigma} \\ \downarrow & & \downarrow \\ \mathcal{FL} & \xrightarrow{g} & \mathcal{FL} \end{array}$$

Remark 4.2.1. The choice of a specific Σ does not usually play any role. If no confusion is likely to arise, we fix a Σ and drop it from the notation. We will eventually allow ourselves to change Σ. It is also important to note that all the cohomologies we will consider (coherent cohomology, completed cohomology) do not depend on Σ.

4.3. **Smooth and locally analytic vectors of the structure sheaf.** We let $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ be the structure sheaf of the perfectoid space. If $U \subseteq \mathrm{Sh}^{\mathrm{tor}}_{K^p}$ is a quasi-compact open, then U is stabilized by an open subgroup K_p of $G(\mathbf{Q}_p)$ and we get a continuous action of K_p on $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}(U)$. One can speak of the smooth and locally analytic vectors of $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}(U)$. We thus obtain subsheaves of smooth and locally analytic vectors:

$$\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \subseteq \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \subseteq \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}.$$

Proposition 4.3.1. *For any compact open subgroup $K'_p \subseteq G(\mathbf{Q}_p)$, let $\pi_{K'_p}$: $\mathrm{Sh}^{\mathrm{tor}}_{K^p} \to \mathrm{Sh}^{\mathrm{tor}}_{K^pK'_p}$ be the natural map. The pullback map $\mathrm{colim}_{K'_p}\, \pi^{-1}_{K'_p}\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K'_pK^p}} \to \mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ is an isomorphism.*

Proof. We consider the map of sites $\nu : (\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p})_{\mathrm{prokt}} \to (\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p})_{\mathrm{ket}}$ from the pro-Kummer-étale site to the Kummer-étale site. It follows from [Sch13b, Coro. 6.19] (which is easily extended to the Kummer-étale case via the machinery of [DLLZ23]) that $\nu_*\hat{\mathcal{O}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}} = \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}}$. We are going to see that the proposition follows directly from this statement. We consider the pro-Kummer étale cover π_{K_p} : $\mathrm{Sh}^{\mathrm{tor}}_{K^p} \to \mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}$. Since $\mathrm{Sh}^{\mathrm{tor}}_{K^p}$ is perfectoid, it follows from [DLLZ23, Thm. 5.4.3] that for any open affine $U \subseteq \mathrm{Sh}^{\mathrm{tor}}_{K^p}$, $\hat{\mathcal{O}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}}(U) = \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}(U)$. Since $\mathrm{Sh}^{\mathrm{tor}}_{K^p} \times_{\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}} \mathrm{Sh}^{\mathrm{tor}}_{K^p} = K_p \times \mathrm{Sh}^{\mathrm{tor}}_{K^p}$, we deduce that $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}} = \nu_*\hat{\mathcal{O}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}} = H^0(K_p, \pi_{K_p,*}\mathcal{O}^{\mathrm{tor}}_{\mathrm{Sh}_{K^p}})$. The proposition follows by taking the colimit over K_p. $\square$

We let $\mathscr{I}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}}$ be the ideal of the (reduced) boundary D_{K_p} in $\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}$. This is an invertible ideal in $\mathscr{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}}$ that we also denote by $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}}(-D_{K_p})$. We let $\mathscr{I}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} = \mathrm{colim}_{K'_p}\, \mathscr{I}_{\mathrm{Sh}^{\mathrm{tor}}_{K'_pK^p}}$.

We let $\mathscr{I}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ be the ideal of the boundary in the structure sheaf $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$. Its subsheaf of locally analytic vectors is denoted by $\mathscr{I}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ and turns out to be equal to $\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \otimes_{\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}} \mathscr{I}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ by the following lemma.

Lemma 4.3.2. *The natural map: $\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \otimes_{\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}} \mathscr{I}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \to \mathscr{I}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ is an isomorphism.*

Proof. This is a consequence of [RC23, Thm. 3.4.1.]. $\square$

4.4. **Completed cohomology.** Let $\mathrm{Sh}^{alg}_{K^pK_p}$ denote the Shimura variety, viewed as a scheme, defined over its reflex field $E(G, X)$. Let $\mathrm{Sh}^{alg}_{K^p} = \lim_{K_p} \mathrm{Sh}^{alg}_{K^pK_p}$. The limit exists as a scheme since the transition maps are affine. We define completed cohomology with $\mathbf{Q}_p$ coefficients to be $\mathrm{R}\Gamma_{\mathrm{proet}}(\mathrm{Sh}^{alg}_{K^p,\bar{\mathbf{Q}}}, \mathbf{Q}_p)$. The cohomology groups of $\mathrm{R}\Gamma_{\mathrm{proet}}(\mathrm{Sh}^{alg}_{K^p,\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ identify with the usual completed cohomology groups with $\mathbf{Q}_p$-coefficients (as defined for example in [CE12]). This cohomology has a $G(\mathbf{Q}_p)$-action, an action of the Hecke algebra away from p, and an action of $G_{E(G,X)}$, the absolute Galois group of $E(G, X)$.

Using comparison theorems in [Hub96, page 30] and [RC22, Cor. 6.1.7], completed cohomology with $\mathbf{Q}_p$-coefficients identifies with $\mathrm{R}\Gamma_{\mathrm{prokt}}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)$. Here the subscript prokt is is short for "pro-Kummer-étale", in the sense of [DLLZ23]; this modification of the pro-étale site is needed because our Shimura varieties are not compact. We usually omit this subscript from now on. Similarly, we identify $\mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)$ with the completed cohomology with compact support.

We let $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}$ be the (derived) locally analytic vectors in completed cohomology. Similarly, we let $\mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}$ be the (derived) locally analytic vectors in completed cohomology with compact support. We remark that both $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)$ and $\mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)$ have admissible cohomology groups, and that the passage to locally analytic vectors is an exact functor on admissible representations by [ST03], Thm. 7.1 (see also [RJRC22, Prop. 4.48]). Therefore, the cohomology groups of $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}$ and $\mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}$ are the locally analytic vectors in the completed cohomology groups.

Theorem 4.4.1. *We have*

$$\begin{aligned}\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{C}_p &= \mathrm{R}\Gamma_{an}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})\\ \mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathbf{C}_p &= \mathrm{R}\Gamma_{an}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{I}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})\end{aligned}$$

We have

$$\begin{aligned}\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p &= \mathrm{R}\Gamma_{an}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})\\ \mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p &= \mathrm{R}\Gamma_{an}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{I}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})\end{aligned}$$

Proof. The first part is immediate from [DLLZ23, Thm. 6.2.1], by passing to the limit as in the proof of [Sch15, Thm. 4.2.1]. The second part is [RC22, Thm. 6.2.6]. □

Remark 4.4.2. The main ideas in the proof of Theorem 4.4.1 are due to Scholze and Pan. More precisely, the statements regarding completed cohomology (before taking locally analytic vectors) are a consequence of Scholze's primitive comparison theorem; see e.g. [Sch15, Thm. 4.2.1]. For the locally analytic vectors, in the case of usual (i.e. not compactly supported) cohomology it is a consequence of the fact that $\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} = \mathcal{O}^{\mathrm{Rla}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ where Rla are the derived locally analytic vectors (in the sense of [RJRC22]). This was proved for modular curves in [Pan22a], Thm. 4.4.6. In *loc. cit.* it is also proved for modular curves that $\mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}) = \mathrm{R}\Gamma_{\mathrm{an}}(\mathcal{FL}, (\pi_{HT})_*\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$. We do not need (and have not proved) this fact more generally.

4.5. **The functor VB.** We now introduce a functor which turns equivariant sheaves on the flag variety into sheaves on the perfectoid Shimura variety.

4.5.1. *Definition of the functor and main properties.* Let us briefly reintroduce Σ to the notation (see Remark 4.2.1). Let $U^{rat}_{\mathcal{FL}}$ be a quasi-compact open subset of $\mathcal{FL}^{rat}$ and let $U_{\mathcal{FL}}$ be its base change to $\mathrm{Spa}(\mathbf{C}_p, \mathcal{O}_{\mathbf{C}_p})$. We write $U_{K^p,\Sigma} := \pi^{-1}_{HT,\Sigma}(U_{\mathcal{FL}}) = \lim_{K'_p} U_{K'_pK^p,\Sigma}$ where $U_{K'_pK^p,\Sigma}$ is a quasi-compact open subset of $\mathrm{Sh}_{K'_pK^p,\Sigma}$ for K'_p small enough. In Definition 3.2.6 (see also Remark 3.2.29) we have defined the categories $LB_{\mathfrak{g}}(U^{rat}_{\mathcal{FL}})$ and $LB_{\mathfrak{g}}(U_{\mathcal{FL}})$ and there is a base change functor $LB_{\mathfrak{g}}(U^{rat}_{\mathcal{FL}}) \to LB_{\mathfrak{g}}(U_{\mathcal{FL}})$. We define a functor

$$\begin{aligned}VB^0_\Sigma : LB_{\mathfrak{g}}(U_{\mathcal{FL}}) &\to \mathrm{Mod}(\mathcal{O}^{\mathrm{sm}}_{U_{K^p,\Sigma}})\\ \mathscr{F} &\mapsto \mathrm{colim}_{K'_p}((\pi^{-1}_{HT,\Sigma}\mathscr{F})\otimes_{\pi^{-1}_{HT,\Sigma}\mathcal{O}_{\mathcal{FL}}}\mathcal{O}_{U_{K^p,\Sigma}})^{K'_p}.\end{aligned}$$

Remark 4.5.2. Note that this functor is (well-) defined, because for any quasi-compact open subset $V_{\mathcal{FL}} \subseteq U_{\mathcal{FL}}$, we have $\mathscr{F}(V_{\mathcal{FL}}) = \mathrm{colim}_{K'_p} \mathscr{F}(V_{\mathcal{FL}})_{K'_p}$ where $\mathscr{F}(V_{\mathcal{FL}})_{K'_p}$ is a submodule of $\mathscr{F}(V_{\mathcal{FL}})$ where the action of $\mathfrak{g}$ integrates to an action of K'_p (compatibly for the transition maps).

We also need a derived version of this functor. We recall that in Definition 3.2.4 we introduced abelian categories $\mathrm{Mod}_{\mathfrak{g}}(U^{rat}_{\mathcal{FL}})$ and $\mathrm{Mod}_{\mathfrak{g}}(U_{\mathcal{FL}})$ which respectively contain $LB_{\mathfrak{g}}(U^{rat}_{\mathcal{FL}})$ and $LB_{\mathfrak{g}}(U_{\mathcal{FL}})$. Moreover there is a base change functor $\mathrm{Mod}_{\mathfrak{g}}(U^{rat}_{\mathcal{FL}}) \to \mathrm{Mod}_{\mathfrak{g}}(U_{\mathcal{FL}})$. We define a functor

$$\begin{aligned} VB_\Sigma : D(\mathrm{Mod}_{\mathfrak{g}}(U_{\mathcal{FL}})) &\to D(\mathrm{Mod}(\mathcal{O}^{\mathrm{sm}}_{U_{K^p,\Sigma}})) \\ \mathscr{F} &\mapsto \mathrm{colim}_{K'_p} \mathrm{R}\Gamma(K'_p, (\pi^{-1}_{HT,\Sigma}\mathscr{F}) \otimes^L_{\pi^{-1}_{HT,\Sigma}\mathcal{O}_{\mathcal{FL}}} \mathcal{O}_{U_{K^p,\Sigma}}). \end{aligned}$$

Remark 4.5.3. The pullback functor

$$\mathcal{F} \mapsto (\pi^{-1}_{HT,\Sigma}\mathscr{F}) \otimes^L_{\pi^{-1}_{HT,\Sigma}\mathcal{O}_{\mathcal{FL}}} \mathcal{O}_{U_{K^p,\Sigma}}$$

in the definition of VB_Σ is exact on the category $LB_{\mathfrak{g}}(U_{\mathcal{FL}})$ (essentially by the definition of LB-sheaves). Thus on $LB_{\mathfrak{g}}(U_{\mathcal{FL}})$, VB^0_Σ is left exact, and we may think of VB_Σ as its right derived functor.

It might be more natural to denote this pullback by $\pi^*_{\mathrm{HT},\Sigma}$, but we reserve this notation below for the underived pullback

$$\mathcal{F} \mapsto (\pi^{-1}_{HT,\Sigma}\mathscr{F}) \otimes_{\pi^{-1}_{HT,\Sigma}\mathcal{O}_{\mathcal{FL}}} \mathcal{O}_{U_{K^p,\Sigma}}.$$

Since the Hodge–Tate period map is $\mathrm{Gal}(\bar{E}/E)$-equivariant, if $\mathscr{F}$ is an object of $\mathrm{Mod}_{\mathfrak{g}}(U_{\mathcal{FL}})$ which comes from $\mathrm{Mod}_{\mathfrak{g}}(U^{rat}_{\mathcal{FL}})$ by base change, then $VB_\Sigma(\mathscr{F})$ carries a semi-linear action of $\mathrm{Gal}(\bar{E}/E)$. One can study this action as follows. We can consider the category $\mathrm{Mod}_{\mathrm{Gal}(\bar{E}/E)}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ of sheaves of $\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$-modules, carrying a semi-linear continuous Galois action. We let $\mathcal{O}^{\mathrm{sm},\mathrm{Gal}(\bar{E}/E)-\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ be the subsheaf of smooth vectors for the action of $\mathrm{Gal}(\bar{E}/E)$. We remark that $\mathcal{O}^{\mathrm{sm},\mathrm{Gal}(\bar{E}/E)-\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} = \mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{rat,\mathrm{tor}}_{K^p}} \otimes_E \bar{E}$.

We define an arithmetic Sen functor:

$$\begin{aligned} S^{\mathrm{arit}} : \mathrm{Mod}_{\mathrm{Gal}(\bar{E}/E)}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}) &\to \mathrm{Mod}(\mathcal{O}^{\mathrm{sm},\mathrm{Gal}(\bar{E}/E)-\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}) \\ \mathscr{F} &\to \mathrm{colim}_{E'} \mathscr{F}^{\mathrm{Gal}(\bar{E}/E'_{\mathrm{cycl}}),\mathrm{Gal}(E'_{\mathrm{cycl}}/E')-an} \end{aligned}$$

where the colimit goes over all finite extensions E'/E and the superscript

$$(-)^{\mathrm{Gal}(\bar{E}/E'_{\mathrm{cycl}}),\mathrm{Gal}(E'_{\mathrm{cycl}}/E')-an}$$

means the $\mathrm{Gal}(\bar{E}/E'_{\mathrm{cycl}})$-fixed and $\mathrm{Gal}(E'_{\mathrm{cycl}}/E')$-analytic vectors (where $\mathrm{Gal}(E'_{\mathrm{cycl}}/E')$ is viewed as a subgroup of $\mathbf{Z}_p^\times$ via the cyclotomic character). We observe that $S^{\mathrm{arit}}(\mathscr{F})$ carries an $\mathcal{O}^{\mathrm{sm},\mathrm{Gal}(\bar{E}/E)-\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$-linear arithmetic Sen operator, obtained by differentiating the $\mathrm{Gal}(E'_{\mathrm{cycl}}/E')$-action on $(\mathscr{F})^{\mathrm{Gal}(\bar{E}/E'_{\mathrm{cycl}}),\mathrm{Gal}(E'_{\mathrm{cycl}}/E')-an}$ and passing to the colimit.

The following theorem is implicit in [Pan22a] in the modular curve case, see [Pil24] for a formulation in this spirit. In higher dimension it is essentially a direct consequence of the results [RC22, RC23], as we will see in the course of the proof.

Theorem 4.5.4.

(1) *For any* $\mathscr{F} \in LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}_P^0}$*, there exists a covering by open affinoids* $U_{K^p,\Sigma} = \cup_i V_i$ *with* $V_i = \lim_{K_p} V_{i,K_p}$ *(for* K_p *small enough; here the* V_{i,K_p} *are open affinoid) a sequence of compact open subgroups* $\{K_{p,r}\}_{r\in\mathbf{Z}_{\geq 0}}$ *and summand of orthonormalizable Banach sheaves* $VB^0_{\Sigma,K_{p,r},V_i}(\mathscr{F})$ *over* $V_{i,K_{p,r}}$ *such that we have:*

$$VB^0_{\Sigma}(\mathscr{F})|_{V_i} = \mathrm{colim}_r\, \mathcal{O}^{\mathrm{sm}}_{V_i} \otimes_{\mathcal{O}_{V_{i,K_{p,r}}}} VB^0_{\Sigma,K_{p,r},V_i}(\mathscr{F}).$$

Moreover, we have

$$VB^0_{\Sigma}(\mathscr{F}) \otimes_{\mathcal{O}^{\mathrm{sm}}_{U_{K^p,\Sigma}}} \mathcal{O}_{U_{K^p,\Sigma}} = (\pi^{-1}_{HT,\Sigma}\mathscr{F}) \otimes_{\pi^{-1}_{HT,\Sigma}(\mathcal{O}_{U_{\mathcal{FL}}})} \mathcal{O}_{U_{K^p,\Sigma}}. \tag{4.5.5}$$

(2) *The restriction of the functor* VB^0_{Σ} *to the category* $LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}_P^0}$ *is an exact functor, and for an object* $\mathscr{F}$ *of* $LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}_P^0}$ *we have*

$$VB_{\Sigma}(\mathscr{F}) = VB^0(\mathscr{F}) \otimes_{\mathcal{O}^{\mathrm{sm}}_{U_{K^p,\Sigma}}} (\mathrm{colim}_{K_p} \oplus_{i=0}^d \Omega^i_{U_{K_pK^p,\Sigma}}(\log(D_{K_p,\Sigma}))[-i]).$$

(3) *Let* $\mathscr{F} \in LB_{\mathfrak{g}}(U_{\mathcal{FL}})$*. Assume that for all* i*, we have* $H^i(\mathfrak{u}_P^0, \mathscr{F}) \in LB_{\mathfrak{g}}(U_{\mathcal{FL}})$*. Then we have isomorphisms:* $VB^0_{\Sigma}(H^i(\mathfrak{u}^0_P, \mathscr{F})) \xrightarrow{\sim} H^i(VB_{\Sigma}(\mathscr{F}))$*.*

(4) *Let* $\mathscr{F} \in LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}_P^0}$*. Assume that* $\mathscr{F}$ *arises from an object of* $LB_{\mathfrak{g}}(U^{rat}_{\mathcal{FL}})$*. Then we have*

$$S^{\mathrm{arit}}(VB^0_{\Sigma}(\mathscr{F})) \otimes_{\mathcal{O}^{\mathrm{sm},\mathrm{Gal}(\bar{E}/E)-\mathrm{sm}}_{U_{K^p,\Sigma}}} \mathcal{O}^{sm}_{U_{K^p,\Sigma}} = VB^0_{\Sigma}(\mathscr{F})$$

and the action of μ *via* Θ_{hor} *is an arithmetic Sen operator on* $S^{\mathrm{arit}}(VB^0_{\Sigma}(\mathscr{F}))$*.*

More precisely, in the notation of (1)*, we can suppose that the covering* $U_{K^p,\Sigma} = \cup_i V_i$ *comes from a covering* $U^{rat}_{K^p,\Sigma} = \cup_i V_i^{rat}$*, and there exists a finite extension* $E_{i,n}$ *of* E *such that*

$$VB^0_{\Sigma,K_{p,n},V_i}(\mathscr{F}) = VB^0_{\Sigma,K_{p,n},V_i}(\mathscr{F})^{\mathrm{Gal}(\bar{E}/E_{i,n,cycl}),\mathrm{Gal}(E_{i,n,cycl}/E_{i,n})-an} \otimes_{E_{i,n}} \mathbf{C}_p.$$

Proof. We explain why the theorem follows from the results of [RC22] and [RC23]. The statement is local so we can assume that $\mathscr{F} = \mathrm{colim}\,\mathscr{F}_r$ is a colimit of orthonormalizable Banach sheaves with injective transition maps and that G_r acts on each $\mathscr{F}_r$. We can assume that $U_{\mathcal{FL}} = \mathrm{Spa}(C, C^+)$ is affinoid. We can also consider $V_{K^p,\Sigma} = \mathrm{Spa}(B, B^+)$ an open affinoid subset of $U_{K^p,\Sigma}$. The open subset $V_{K^p,\Sigma}$ descends to $V_{K^pK_p,\Sigma} = \mathrm{Spa}(B_{K_p}, B^+_{K_p})$ for K_p small enough.

For K_p small enough (such that $K_p \subseteq G_r(\mathbf{Q}_p)$), the pull back $\pi^*_{HT,\Sigma}\mathscr{F}_r$ to $\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}$ carries a K_p-action Let us put $F_r := \mathscr{F}_r(U_{\mathcal{FL}})$. We pick a C^+ lattice F_r^+ (that is, F_r^+ is the completion of a free C^+-module and $F_r^+ \otimes_{C^+} C = F_r$). The action of G_r amounts to a co-action map $c : F_r \to F_r \otimes_C \mathcal{O}_{G_r}$. By continuity, there exists n such that $c(F_r^+) \subseteq p^{-n}F_r^+ \otimes \mathcal{O}^+_{G_r}$. We claim that for $r' = r + n + 1$ the restriction of the co-action map $c' : F_r \to F_r \otimes_C \mathcal{O}_{G_{r'}}$ induces a map $F_r^+ \to F_r^+ \otimes_C \mathcal{O}^+_{G_{r'}}$ and moreover, this co-action map is trivial modulo p. To see this, we may write (for example by using the exponential map) $\mathcal{O}_{G_r} = \mathbf{C}_p\langle X_1, \cdots, X_t\rangle$ so that $\mathcal{O}_{G_{r'}} = \mathbf{C}_p\langle p^{-n-1}X_1, \cdots, p^{-n-1}X_t\rangle$. For any $f \in F_r^+$, we write $c(f) = \sum_{\underline{i}} f_{\underline{i}} X^{\underline{i}}$, where $f_{\underline{0}} = f$ and $f_{\underline{i}} \in p^{-n}F_r^+$ is tending to 0. Our claim is thus clear. By shrinking K_p, we can assume that $K_p \subseteq G_{r'}(\mathbf{Q}_p)$. We remark that we have in particular checked that the action of K_p is "locally analytic" in the sense of [RC23, Defn. 1.0.1]; more precisely, the pro-Kummer-étale $\widehat{\mathcal{O}}_{V_{K^pK_p,\Sigma}}$-module corresponding to $\pi^*_{HT,\Sigma}\mathscr{F}_r$ is relatively analytic ON Banach in the sense of [RC23, Defn. 1.0.1]. Note that this is the familiar smallness condition in p-adic Simpson theory (see [RC23, Rem. 1.0.1]).

We let:

- $\mathbf{T} = \mathrm{Spa}(\mathbf{C}_p\langle T, T^{-1}\rangle, \mathbf{C}_p^+\langle T, T^{-1}\rangle)$,
- $\mathbf{T}_n = \mathrm{Spa}(\mathbf{C}_p\langle T^{p^{-n}}, T^{-p^{-n}}\rangle, \mathbf{C}_p^+\langle T^{p^{-n}}, T^{-p^{-n}}\rangle)$ for any $n \geq 1$.
- $\mathbf{D} = \mathrm{Spa}(\mathbf{C}_p\langle T\rangle, \mathbf{C}_p^+\langle T\rangle)$,
- $\mathbf{D}_n = \mathrm{Spa}(\mathbf{C}_p\langle T^{p^{-n}}\rangle, \mathbf{C}_p^+\langle T^{p^{-n}}\rangle)$ for any $n \geq 1$.

We can also assume (after shrinking $V_{K^p,\Sigma}$ and taking K_p small enough) that we have a toric chart (in the sense of [DLLZ23, Prop. 3.1.10]) $V_{K_pK^p,\Sigma} \to \mathbf{T}^e \times \mathbf{D}^{d-e}$. We let $\mathrm{Spa}(B_{K_p,n}, B^+_{K_p,n}) = U_{K_pK^p,\Sigma} \times_{\mathbf{T}^e\times\mathbf{D}^{d-e}} \mathbf{T}_n^e \times \mathbf{D}_n^{d-e}$. We let $\mathrm{Spa}(B_{K_p,\infty}, B^+_{K_p,\infty}) = \lim_n \mathrm{Spa}(B_{K_p,n}, B^+_{K_p,n})$. We let $\mathrm{Spa}(B_n, B_n^+) = U_{K^p,\Sigma} \times_{\mathbf{T}^e\times\mathbf{D}^{d-e}} \mathbf{T}_n^e \times \mathbf{D}_n^{d-e}$. We let $\mathrm{Spa}(B_\infty, B_\infty^+) = \lim_n \mathrm{Spa}(B_n, B_n^+)$. After making a choice of compatible p-power roots of unity, We let $\Gamma = \mathbf{Z}_p^d$, acting on $\mathbf{T}_n^e \times \mathbf{D}_n^{d-e}$. We thus have an action of $K_p \times \Gamma$ on B_∞. By [RC23, Prop. 3.2.3], the triple $(B_\infty, K_p \times \Gamma, pr_2 : K_p \times \Gamma \to \Gamma)$ is a strongly decomposable Sen theory in the sense of [RC23, Defn. 2.2.6].

We consider the semi-linear representation of K_p, $B \otimes_C F_r$. Our goal is to compute $\mathrm{colim}_{K_p} H^i(K_p, B \otimes_C F_r)$ using Sen theory. By almost purity, we have

$$\mathrm{R\Gamma}(K_p, B\otimes_C F_r) = \mathrm{R\Gamma}(K_p\times\Gamma, B_\infty\otimes_C F_r) = \mathrm{R\Gamma}(\Gamma, H^0(K_p, B_\infty\otimes_C F_r)). \quad (4.5.6)$$

By [RC23, Thm. 2.4.3], we have (after possibly shrinking K_p and for all n large enough) the Sen module

$$S_{K_p,n}(F_r) := (B\otimes_C F_r)^{K_p, p^n\Gamma-an} \quad (4.5.7)$$

which is obtained by taking the K_p-invariants and the $p^n\Gamma$-analytic vectors. This is an orthonormalizable Banach $B_{K_p,n}$-module with a locally analytic action of Γ, and it satisfies

$$B_\infty \otimes_{B_{K_p,n}} S_{K_p,n}(F_r) = B_\infty \otimes_C F_r. \quad (4.5.8)$$

In addition by (4.5.6) we have

$$H^0(\Gamma, S_{K_p,n}(F_r)) = H^0(K_p, B\otimes_C F_r). \quad (4.5.9)$$

We have an action of $\mathrm{Lie}(\Gamma)$ on $S_{K_p,n}(F_r)$, which are the "geometric" Sen operators, and by [RJRC22, Thm. 1.7] we have

$$\mathrm{R\Gamma}(\Gamma, H^0(K_p, B_\infty\otimes_C F_r)) = H^0(\Gamma, \mathrm{R\Gamma}(\mathrm{Lie}(\Gamma), S_{K_p,n}(F_r))) \quad (4.5.10)$$

where $\mathrm{R\Gamma}(\mathrm{Lie}(\Gamma), S_{K_p,n}(F_r))$ is a complex of smooth Γ-modules and $H^0(\Gamma, -)$ is the exact functor of Γ-invariants on smooth Γ-modules. It is a consequence of [RC22, Thm. 1.1.5] that these Sen operators are induced by functoriality from the map $\mathfrak{u}_P^0 \otimes \mathscr{F}_r \to \mathscr{F}_r$, using the identification $\pi_{HT}^*\mathfrak{u}_P^0 \simeq \mathrm{Lie}(\Gamma)\otimes\mathcal{O}_{V_{K^p,\Sigma}}$.

Since the transition maps in the colimit $\mathcal{F} = \mathrm{colim}\,\mathcal{F}_r$ are injective, it follows in particular that $\mathscr{F} \in LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}_P^0}$ if and only if the geometric Sen operator of each $\mathscr{F}_r$ is trivial. Let us assume that this is the case. Then the action of Γ on $S_{K_p,n}(F_r)$ is smooth, and this action factors through $\Gamma/p^n\Gamma$. By finite étale descent, we find that

$$B_{K_p,n} \otimes_{B_{K_p}} S_{K_p,n}(F_r)^\Gamma = S_{K_p,n}(F_r) \quad (4.5.11)$$

and that $S_{K_p,n}(F_r)^\Gamma$ is a direct summand of the orthonormalizable Banach B_{K_p}-module $S_{K_p,n}(F_r)$. Taking $p^n\Gamma$-invariants in (4.5.8), we obtain

$$B_n \otimes_{B_{K_p,n}} S_{K_p,n}(F_r) = B_n \otimes_C F_r$$

and thus (using (4.5.11))

$$B_n \otimes_{B_{K_p}} S_{K_p,n}(F_r)^\Gamma = B_n \otimes_C F_r.$$

Taking Γ-invariants and using (4.5.9), we deduce that

$$B \otimes_{B_{K_p}} H^0(K_p, B \otimes_C F_r) = B \otimes_C F_r. \tag{4.5.12}$$

Passing to the colimit over r and K_p, we obtain (4.5.5). This completes the proof of (1) (taking $VB^0_{\Sigma,K_{p,r},V_i}(\mathscr{F})$ to be the sheaf associated to $H^0(K_p, B \otimes_C F_r)$, and $K_{p,r}$ to be K_p).

We now prove (2). We have

$$\mathrm{R}\Gamma(\mathrm{Lie}(\Gamma), S_{K_p,n}(F_r)) = \oplus S_{K_p,n}(F_r) \otimes \Lambda^i(\mathrm{Lie}(\Gamma))^\vee[-i]$$

from which we deduce that

$$VB_\Sigma(\mathscr{F}) = VB^0(\mathscr{F}) \otimes_{\mathcal{O}^{\mathrm{sm}}_{U_{K^p,\Sigma}}} (\mathrm{colim}_{K_p} \oplus_{i=0}^d \Omega^i_{U_{K_pK^p,\Sigma}}(\log(D_{K_p,\Sigma}))[-i]). \tag{4.5.13}$$

Let $0 \to \mathscr{F} \to \mathscr{L} \to \mathscr{H} \to 0$ be an exact sequence in $LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}^0_P}$. Applying VB_Σ yields exact sequences:

$$0 \to VB^0_\Sigma(\mathscr{F}) \to VB^0_\Sigma(\mathscr{L}) \to VB^0_\Sigma(\mathscr{H})$$

and

$$VB^0_\Sigma(\mathscr{F}) \otimes_{\mathcal{O}^{\mathrm{sm}}_U} VB^0_\Sigma(\Lambda^d(\mathfrak{u}^0_P)^\vee) \to VB^0_\Sigma(\mathscr{L}) \otimes_{\mathcal{O}^{\mathrm{sm}}_U} VB^0_\Sigma(\Lambda^d(\mathfrak{u}^0_P)^\vee) \to VB^0_\Sigma(\mathscr{H}) \otimes_{\mathcal{O}^{\mathrm{sm}}_U} VB^0_\Sigma(\Lambda^d(\mathfrak{u}^0_P)^\vee) \to 0.$$

Since $VB^0_\Sigma(\Lambda^d(\mathfrak{u}^0_P)^\vee)$ is an invertible sheaf, we conclude that $0 \to VB^0_\Sigma(\mathscr{F}) \to VB^0_\Sigma(\mathscr{L}) \to VB^0_\Sigma(\mathscr{H}) \to 0$ is exact, as required.

We now turn to (3), so we no longer assume that $\mathscr{F}$ is killed by $\mathfrak{u}^0_P$. We let $S_{K_p}(F_r) = \mathrm{colim}_n S_{K_p,n}(F_r)$, we let $S(F_r) = \mathrm{colim}_{K_p} S_{K_p}(F_r)$ and finally we let $S(F) = \mathrm{colim}_r S(F_r)$. We claim that

$$H^i(\mathrm{Lie}(\Gamma), S(F)) = S(H^i(\mathfrak{u}^0_P, F)). \tag{4.5.14}$$

Granting (4.5.14), we claim that taking Γ-invariants gives

$$\mathrm{colim}_{r,K_p} H^i(K_p, B \otimes_C F_r) = \mathrm{colim}_{r,K_p} H^0(K_p, B \otimes_C H^i(\mathfrak{u}^0_P, F_r)), \tag{4.5.15}$$

which immediately gives (3). Indeed, by 4.5.10 and (4.5.6) we have

$$H^0(\Gamma, H^i(\mathrm{Lie}(\Gamma), S_{K_p,n}(F_r))) = H^i(\Gamma, H^0(K_p, B_\infty \otimes_C F_r)) = H^i(K_p, B \otimes_C F_r).$$

On the other hand, passing to colimits in (4.5.9) we see that

$$\mathrm{colim}_{K_p} H^0(K_p, B \otimes_C F_r) = H^0(\Gamma, S(F_r)), \tag{4.5.16}$$

so that (replacing F_r by $H^i(\mathfrak{u}^0_P, F_r)$)

$$H^0(\Gamma, S(H^i(\mathfrak{u}^0_P, F))) = \mathrm{colim}_{r,K_p} H^0(K_p, B \otimes_C H^i(\mathfrak{u}^0_P, F_r)),$$

as required.

We now establish (4.5.14). Firstly, we claim that $B_{K_p,\infty}$ is an orthonormalizable $B_{K_p,n}$-module. Indeed the algebra of $\mathbf{T}_\infty = \lim_n \mathbf{T}_n$ has a topological basis $\{T^i\}_{i \in \mathbf{Q}_p/\mathbf{Z}_p}$ over $\mathbf{C}_p\langle T, T^{-1}\rangle$, and similarly the algebra of $\mathbf{D}_\infty \lim_n \mathbf{D}_n$ has a topological basis $\{T^i\}_{i \in \mathbf{Q}_p/\mathbf{Z}_p}$ over $\mathbf{C}_p\langle T\rangle$. We next claim that B_∞ is a direct summand of an orthonormalizable $B_{K_p,\infty}$-module. To see this, let us fix a decreasing sequence of compact open subgroups $\{K_{p,r}\}_{r\geq 0}$ tending to $\{e\}$ with $K_{p,0} = K_p$. Since $B^+_{K_{p,r},\infty} \to B^+_{K_{p,r+1},\infty}$ is almost étale, there exist finite $B^+_{K_{p,r},\infty}$ modules $X_{K_{p,r+1}}$ and $Y_{K_{p,r+1}}$ together with:

- an injective map $B^+_{K_p,r,\infty} \oplus X_{K_p,r+1} \to B^+_{K_p,r+1,\infty}$ whose cokernel is annihilated by p, and
- an integer $n_r \in \mathbf{Z}_{\geq 0}$ and an injective map $X_{K_p,r+1} \oplus Y_{K_p,r+1} \to (B^+_{K_p,r,\infty})^{n_r}$ whose cokernel is annihilated by p.

We deduce that $B_\infty \bigoplus (\widehat{\oplus_r Y_{K_p,r+1}}[1/p])$ is orthonormalizable over $B_{K_p,\infty}$, as required.

In particular, we have shown that B_∞ is flat over $B_{K_p,n}$. We deduce that $H^i(\mathrm{Lie}(\Gamma), B_\infty \otimes_{B_{K_p,n}} S_{K_p,n}(F_r)) = H^i(\mathrm{Lie}(\Gamma), B_\infty \otimes_{B_{K_p,n}} S_{K_p,n}(F_r))$, and passing to the colimit over K_p, n, r we obtain that (for B_∞^{sm} the subring of smooth vectors for the $K_p \times \Gamma$-action):

$$H^i(\mathrm{Lie}(\Gamma), B_\infty \otimes_{B_\infty^{\mathrm{sm}}} S(F)) = B_\infty \otimes_{B_\infty^{\mathrm{sm}}} H^i(\mathrm{Lie}(\Gamma), S(F)).$$

On the other hand, $H^i(\mathfrak{u}_P^0, B_\infty \otimes_C F) = B_\infty \otimes_C H^i(\mathfrak{u}_P^0, F)$ since all the $H^j(\mathfrak{u}_P^0, F)$ are flat over C (here we use our assumption that $H^i(\mathfrak{u}_P^0, \mathscr{F}) \in LB_{\mathfrak{g}}(U_{\mathcal{FL}})$). Recalling (4.5.8), we deduce that we have $K_p \times \Gamma$- equivariant isomorphisms:

$$B_\infty \otimes_{B_\infty^{\mathrm{sm}}} H^i(\mathrm{Lie}(\Gamma), S(F)) = B_\infty \otimes_C H^i(\mathfrak{u}_P^0, F) = B_\infty \otimes_{B_\infty^{\mathrm{sm}}} S(H^i(\mathfrak{u}_P^0, F))$$

Taking $K_p \times \Gamma$ smooth vectors yields $H^i(\mathrm{Lie}(\Gamma), S(F)) = S(H^i(\mathfrak{u}_P^0, F))$, as required.

We now prove the last point. We take $\mathscr{F} \in LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}_P^0}$, arising from an object of $LB_{\mathfrak{g}}(U_{\mathcal{FL}}^{rat})$. We can therefore choose the $\mathcal{F}_r$ to be defined over E, so that there is a semi-linear $\mathrm{Gal}(\bar{E}/E)$-action on $B \otimes_C F_r$. Moreover, taking a topological basis $\{v_i\}_{i \in I}$ of F_r defined over E, we see that the Galois action is trivial in this basis. Assuming that K_p acts trivially modulo p^2 on F_r^+ (which we can always arrange after shrinking K_p), we deduce that $S_{K_p,n}(F_r)$ has a topological basis $\{v_i'\}_{i \in I}$ where the change of basis matrix (for the isomorphism (4.5.8)) from $\{v_i\}$ to $\{v_i'\}$ is congruent to 1 modulo p (see [RC23, Thm. 2.4.3, (1), (b)]). As a result, the matrix of the Galois action on $S_{K_p,n}(F_r)$ is congruent to 1 modulo p. One can therefore apply Sen theory to the extension $B^{rat}_{K_p,n} \to B_{K_p,n}$ where $B^{rat}_{K_p,n} = (B_{K_p,n})^{\mathrm{Gal} \bar{E}/E(\zeta_{p^n})}$. We let $S^{\mathrm{arit},s}(S_{K_p,n}(F_r)) = S_{K_p,n}(F_r)^{G_{L^{\mathrm{cycl}}}, G_{L(\zeta_{p^s})}-an}$. For s large enough, $S^{\mathrm{arit},s}(S_{K_p,n}(F_r)) \otimes_{B^{rat}_{K_p,n}(\zeta_{p^s})} B_{K_p,n} = S_{K_p,n}(F_r)$ and the derivative of the $\mathrm{Gal}(E_{\mathrm{cycl}}/E(\zeta_{p^s}))$ action provides an arithmetic Sen operator.

In order to prove (4), it only remains to identify this Sen operator with the operator μ coming from the horizontal action. The orbit map provides an embedding $F_r \hookrightarrow \mathcal{C}^{an}(K_p, F_r)$, $f \mapsto [k \mapsto k.f]$. It intertwines the action of K_p on F_r with the action of K_p on functions $h(-) \in \mathcal{C}^{an}(K_p, F_r)$ via $k *_2 h(-) = h(-k)$ (therefore the $*_2$ action does not depend on the K_p-action on F_r). Note that $\mathcal{C}^{an}(K_p, F_r)$ is a C-module as there is a orbit map $C \to \mathcal{C}^{an}(K_p, C)$ and $\mathcal{C}^{an}(K_p, C)$ acts on $\mathcal{C}^{an}(K_p, F_r)$ naturally. Moreover, the embedding $F_r \hookrightarrow \mathcal{C}^{an}(K_p, F_r)$ factors through $\mathcal{C}^{an}(K_p, F_r)^{\mathfrak{u}_P^0}$. It therefore suffices to identify the arithmetic Sen operator of $\mathcal{C}^{an}(K_p, F_r)^{\mathfrak{u}_P^0}$. Since F_r has a topological basis over C we can reduce to understanding the Sen operator of $\mathcal{C}^{an}(K_p, C)^{\mathfrak{u}_P^0}$. The orbit map $C \mapsto \mathcal{C}^{an}(K_p, C)$ induces an isomorphism $\mathcal{C}^{an}(K_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C \to \mathcal{C}^{an}(K_p, C)$. Moreover, the subspace of algebraic functions $\mathcal{C}^{alg}(K_p, \mathbf{Q}_p) \hookrightarrow \mathcal{C}^{an}(K_p, \mathbf{Q}_p)$ is dense, and it induces a dense map $(\mathcal{C}^{alg}(K_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C)^{\mathfrak{u}_P^0} \hookrightarrow (\mathcal{C}^{an}(K_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C)^{\mathfrak{u}_P^0}$. Viewing the arithmetic Sen operator as an endomorphism of

$$B_\infty \otimes_{B_{K_p,n}} S_{K_p,n}(\mathcal{C}^{an}(K_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C)^{\mathfrak{u}_P^0}) = B_\infty \otimes_C (\mathcal{C}^{an}(K_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C)^{\mathfrak{u}_P^0}$$

we deduce that it suffices to prove it coincides with μ on $B_\infty \otimes_C (\mathcal{C}^{alg}(K_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} C)^{\mathfrak{u}_P^0}$. Since $\mathcal{C}^{alg}(K_p, \mathbf{Q}_p) = \oplus_{\kappa \in X^*(T)^+} V_\kappa \otimes V_\kappa^\vee$ and $(V_\kappa \otimes V_\kappa^\vee \otimes_{\mathbf{Q}_p} \mathcal{O}_{\mathcal{FL}})^{\mathfrak{u}_P^0} = \mathcal{L}_\kappa \otimes_{\mathbf{Q}_p} V_\kappa^\vee$ we deduce that it suffices to understand the Sen operator of the classical automorphic vector bundles. It therefore suffices to show that $VB^0_\Sigma(\mathcal{L}_\kappa) = \omega^{\kappa, sm}(\kappa(\mu))$, where $\kappa(\mu)$ is the Tate twist and $\omega^{\kappa,\mathrm{sm}} = \mathrm{colim}_{K_p} \omega^\kappa_{K_p}$ is the colimit of the automorphic vector bundles defined over $\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}$ (equipped with their rational structure). This follows from an inspection of the rationality properties of the Hodge–Tate map and of the universal M-torsor, see [RC22, Thm. 4.2.1]. □

Observe that by (4.5.13)

$$VB_\Sigma(\mathcal{O}_{U_{\mathcal{FL}}}) = \mathrm{colim}_{K_p} \oplus_{i=0}^d \Omega^i_{U_{K_pK^p,\Sigma}}(\log(D_{K_p,\Sigma}))[-i]$$

is a DG algebra and admits an augmentation map to $\mathcal{O}^{\mathrm{sm}}_{U_{K^p,\Sigma}}[0]$. We can therefore make the following definition.

Definition 4.5.17. We let $VB^{\mathrm{red}}_\Sigma : D(\mathrm{Mod}_{\mathfrak{g}}(U_{\mathcal{FL}})) \to D(\mathrm{Mod}(\mathcal{O}^{\mathrm{sm}}_{U_{K^p,\Sigma}}))$ be the functor given by

$$VB^{\mathrm{red}}_\Sigma(\mathscr{F}) := VB_\Sigma(\mathscr{F}) \otimes^L_{VB_\Sigma(\mathcal{O}_{U_{\mathcal{FL}}})} \mathcal{O}^{\mathrm{sm}}_U[0].$$

Remark 4.5.18. By Theorem 4.5.4 (2), if $\mathscr{F} \in LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}_P^0}$, then $VB^{\mathrm{red}}_\Sigma(\mathscr{F}) = VB^0_\Sigma(\mathscr{F})[0]$. Consequently given a complex of objects in $LB_{\mathfrak{g}}(U_{\mathcal{FL}})^{\mathfrak{u}_P^0}$, we can evaluate VB^{red}_Σ by applying VB^0_Σ termwise.

4.5.19. *Variants.* We now introduce variants of the above functor carrying extra structure. Let us define $\mathrm{Mod}_{G(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ to be the category of sheaves $(\mathscr{F}_\Sigma)_\Sigma$ of $\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}}$-modules with the properties:

(1) For any refinement Σ' of Σ, inducing a map $\pi_{\Sigma',\Sigma} : \mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma'} \to \mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}$, we have an isomorphism $\pi^*_{\Sigma',\Sigma}\mathscr{F}_\Sigma \to \mathscr{F}_{\Sigma'}$ of $\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma'}}$-modules (and these isomorphisms are compatible).
(2) For any $g \in G(\mathbf{Q}_p)$, inducing an isomorphism $g : \mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma} \to \mathrm{Sh}^{\mathrm{tor}}_{K^p,g\Sigma}$, there is an isomorphism $g^*\mathscr{F}_{g\Sigma} \to \mathscr{F}_\Sigma$ of $\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K_p,\Sigma}}$-modules (and they satisfy the usual cocycle condition).

Then we have a functor

$$VB^0 : LB_{(\mathfrak{g},G)}(\mathcal{FL}) \to \mathrm{Mod}_{G(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}),$$

constructed as follows. Composing the functor VB^0_Σ with the forgetful functor $LB_{(\mathfrak{g},G)}(\mathcal{FL}) \to LB_{\mathfrak{g}}(\mathcal{FL})$ gives a functor $VB^0_\Sigma : LB_{(\mathfrak{g},G)}(\mathcal{FL}) \to \mathrm{Mod}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$. For each $g \in G(\mathbf{Q}_p)$, there is a map $g : \mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma} \to \mathrm{Sh}^{\mathrm{tor}}_{K^p,g\Sigma}$ and a map $g^*VB^0_{g\Sigma}(\mathscr{F}) \to VB^0_\Sigma(\mathscr{F})$ satisfying the usual cocycle condition. The various VB^0_Σ thus define a functor $VB^0 : LB_{(\mathfrak{g},G)}(\mathcal{FL}) \to \mathrm{Mod}_{G(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ as claimed. Note that in practice, we fix some Σ and really work with the functor VB^0_Σ (but see Remark 4.2.1 for our notational convention).

We recall the stratification into B-orbits $\mathcal{FL} = \coprod_{w \in {}^MW} C_w$, with $C_w = P\backslash PwB$. We let $j_w : C_w \hookrightarrow \mathcal{FL}$ be the locally closed immersion. It induces $j_{w,\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}} : \pi^{-1}_{HT,\Sigma}(C_w) \to \mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}$. Instead of working on the whole Shimura variety, we can also work over $\pi^{-1}_{HT,\Sigma}(C_w^\dagger)$ for any $w \in {}^MW$. We recall that this is a ringed space, whose underlying topological space is $\pi_{HT,\Sigma}^{-1}(C_w)$ and whose structure sheaf is $j_{w,\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}}^{-1}\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}}$. In this case, we can consider functors:

$$VB^0_\Sigma : LB_{(\mathfrak{g},B)}(C_w^\dagger) \to \mathrm{Mod}(\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT,\Sigma}(C_w^\dagger)}).$$

If $b \in B(\mathbf{Q}_p)$, there is a map $b : \pi^{-1}_{HT,\Sigma}C_w^\dagger \to \pi^{-1}_{HT,b\Sigma}C_w^\dagger$ and a map $b^*VB^0_{b\Sigma}(\mathscr{F}) \to VB^0_\Sigma(\mathscr{F})$ satisfying the usual cocycle condition. This leads us to consider the category $\mathrm{Mod}_{B(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT}C_w^\dagger})$ whose objects are collections of sheaves $(\mathscr{F}_\Sigma)_\Sigma$ of $\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT,\Sigma}C_w^\dagger}$-modules, such that for any refinement Σ' of Σ, inducing a map $\pi_{\Sigma',\Sigma}$: $\mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma'} \to \mathrm{Sh}^{\mathrm{tor}}_{K^p,\Sigma}$, we have an isomorphism $\pi^*_{\Sigma',\Sigma}\mathscr{F}_\Sigma \to \mathscr{F}_{\Sigma'}$ of $\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT,\Sigma'}C_w^\dagger}$-modules (and these isomorphisms are compatible), and such that for any $b \in B(\mathbf{Q}_p)$, there is a map $b^*\mathscr{F}_{b\Sigma} \to \mathscr{F}_\Sigma$ satisfying the usual cocycle condition. The various VB^0_Σ thus define a functor $VB^0 : LB_{(\mathfrak{g},B)}(C_w^\dagger)^{\mathfrak{u}_P^0} \to \mathrm{Mod}_{B(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT}C_w^\dagger})$. Again, in practice, we fix some Σ and really work with the functor VB^0_Σ (but drop Σ from the notation).

We also remark that we have E-rational structures $\mathcal{FL}^{rat}$ on $\mathcal{FL}$ and C_w^{rat} on C_w, and we can consider the categories $LB_{(\mathfrak{g},G)}(\mathcal{FL}^{rat})$ and $LB_{(\mathfrak{g},B)}(C_w^{rat,\dagger})$ which admit base change functors to the categories $LB_{(\mathfrak{g},G)}(\mathcal{FL})$ and $LB_{(\mathfrak{g},B)}(C_w^\dagger)$.

Theorem 4.5.20.

(1) *The functor*

$$VB^0 : LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}_P^0} \to \mathrm{Mod}_{G(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$$

is an exact functor.

(2) *For any $\mathscr{F} \in LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}_P^0}$, we have an analytic covering $\mathrm{Sh}^{\mathrm{tor}}_{K^p} = \cup V_i$, a sequence of compact open subgroups $K_{p,n}$ and summand of orthonormalizable Banach sheaves $VB^0_{K_{p,n},V_i}(\mathscr{F})$ over $V_{i,K_{p,n}}$ such that*

$$VB^0(\mathscr{F})|_{V_i} = \mathrm{colim}_n\, VB^0_{K_{p,n},V_i}(\mathscr{F}) \otimes_{\mathcal{O}_{V_{i,K_{p,n}}}} \mathcal{O}^{\mathrm{sm}}_{V_i}.$$

Moreover, there is a compact open subgroup K_p fixing V_i such that all sheaves $VB^0_{K_{p,n},V_i}(\mathscr{F}) \otimes_{\mathcal{O}_{V_{i,K_{p,n}}}} \mathcal{O}^{\mathrm{sm}}_{V_i}$ are K_p-equivariant (compatibly with n) and this induces in the limit the K_p-equivariant structure on $VB^0(\mathscr{F})|_{V_i}$.

(3) *For any $\mathscr{F} \in LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}_P^0}$, we have*

$$VB^0(\mathscr{F}) \otimes_{\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}} \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} = \pi^{-1}_{HT}\mathscr{F} \otimes_{\pi^{-1}_{HT}\mathcal{O}_{\mathcal{FL}}} \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}.$$

(4) *We have that $VB^0(\mathcal{C}^{\mathrm{la}}) = \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$.*

(5) *The functor*

$$VB^0 : LB_{(\mathfrak{g},B)}(C_w^\dagger)^{\mathfrak{u}_P^0} \to \mathrm{Mod}_{B(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT}C_w^\dagger})$$

is an exact functor.

(6) *For any $\mathscr{F} \in LB_{(\mathfrak{g},B)}(C_w^\dagger)^{\mathfrak{u}_P^0}$, there exists an analytic covering by quasi-compact subsets $\pi^{-1}_{HT}(C_w) = \cup_i V_i$, a cofinal decreasing family of quasi-compact strict neighborhoods of V_i: $V_{i,n} = \lim_{K_p} V_{i,n,K_p}$, compact open subgroups $K_{p,n}$, and summand of orthonormalizable Banach sheaves $VB^0_{K_{p,n},V_{i,n}}(\mathscr{F})$*

over $V_{i,n,K_{p,n}}$ such that

$$VB^0(\mathscr{F})|_{V_i} = \mathrm{colim}_n\, VB^0_{K_{p,n},V_{i,n}}(\mathscr{F}) \otimes_{\mathcal{O}_{V_{i,n,K_{p,n}}}} \mathcal{O}_{V^{\mathrm{sm}}_{i,n}}$$

Moreover, for each i, there is a compact open subgroup $K_B \subseteq B(\mathbf{Q}_p)$ stabilizing V_i and all $V_{i,n}$ and such that the sheaves $VB^0_{K_{p,n},V_{i,n}}(\mathscr{F}) \otimes_{\mathcal{O}_{V_{i,n,K_{p,n}}}} \mathcal{O}_{V^{\mathrm{sm}}_{i,n}}$ are K_B-equivariant, compatibly in n, and this induces in the limit the K_B-equivariant structure on $VB^0(\mathscr{F})|_{V_i}$.

(7) *For any $\mathscr{F} \in LB_{(\mathfrak{g},B)}(C^\dagger_w)^{\mathfrak{u}^0_P}$, we have that*

$$VB^0(\mathscr{F}) \otimes_{\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT}C^\dagger_w}} \mathcal{O}_{\pi^{-1}_{HT}C^\dagger_w} = \pi^{-1}_{\mathrm{HT}}\mathscr{F} \otimes_{\pi^{-1}_{\mathrm{HT}}\mathcal{O}_{C^\dagger_w}} \mathcal{O}_{\pi^{-1}_{HT}C^\dagger_w}.$$

(8) *The following diagram of functors is commutative (where the horizontal functors are given by VB^0, and the vertical functors are the natural restriction functors):*

$$\begin{array}{ccc} LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}^0_P} & \longrightarrow & \mathrm{Mod}_{G(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}) \\ \downarrow & & \downarrow \\ LB_{(\mathfrak{g},B)}(C^\dagger_w)^{\mathfrak{u}^0_P} & \longrightarrow & \mathrm{Mod}_{B(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT}C^\dagger_w}) \end{array}$$

(9) *If $\mathscr{F}$ arises from $LB_{(\mathfrak{g},G)}(\mathcal{FL}^{rat})^{\mathfrak{u}^0_P}$ or $LB_{(\mathfrak{g},B)}(C^{rat,\dagger}_w)^{\mathfrak{u}^0_P}$, the action of μ via Θ_{hor} is an arithmetic Sen operator on $VB^0(\mathscr{F})$.*

Proof. Everything but part (4) is an immediate consequence of Theorem 4.5.4. To see (4), note firstly that since $\mathrm{R}\Gamma(\mathfrak{u}_P, \mathcal{O}_{G,e}) = \mathcal{O}_{U_{P_w}\backslash G,e}[0]$ (*cf.* Lemma 2.3.4), it follows from Theorem 4.5.4 (3) that $VB(\mathcal{O}_{G,e} \otimes \mathcal{O}_{\mathcal{FL}}) = VB^0(\mathcal{C}^{\mathrm{la}})[0]$. On the other hand, bearing in mind Remark 4.5.2, we see that

$$VB(\mathcal{O}_{G,e} \otimes \mathcal{O}_{\mathcal{FL}}) = \mathcal{O}^{\mathrm{Rla}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}},$$

where Rla is the functor of derived locally analytic vectors defined in [RJRC22, Defn. 4.40]. The result follows immediately. □

Remark 4.5.21. In particular the proof of Theorem 4.5.20 showed that $\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} = \mathcal{O}^{\mathrm{Rla}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$, confirming Remark 4.4.2.

Proposition 4.5.22.

(1) *Assume that $\mathscr{F} \in LB_{(\mathfrak{g},G)}(\mathcal{FL})^{\mathfrak{u}^0_P}$ is such that the $\mathfrak{g}$-action is the derivative of the G-action. Then $VB^0(\mathscr{F}) \in \mathrm{Mod}^{\mathrm{sm}}_{G(\mathbf{Q}_p)}(\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$.*

(2) *Assume that $\mathscr{F} \in LB_{(\mathfrak{g},B)}(C^\dagger_w)^{\mathfrak{u}^0_P}$ is such that the restriction to $\mathfrak{b}$ of the $\mathfrak{g}$-action is the derivative of the B-action. Then $VB^0(\mathscr{F}) \in \mathrm{Mod}^{\mathrm{sm}}_{B(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\pi^{-1}_{HT}C^\dagger_w})$.*

Proof. This is immediate from the definition of VB_Σ (and of the $G(\mathbf{Q}_p)$-action). □

We now address the existence of an arithmetic Sen operator on the locally analytic vectors in completed cohomology. One can consider the category $\mathrm{Mod}_{\mathrm{Gal}(\bar{E}/E)}(\mathbf{C}_p)$ of semi-linear $\mathbf{C}_p$-representations and define a Sen module functor

$$\begin{array}{rcl} S^{\mathrm{arit}} : \mathrm{Mod}_{\mathrm{Gal}(\bar{E}/E)}(\mathbf{C}_p) & \to & \mathrm{Mod}(\bar{E}) \\ V & \to & \mathrm{colim}_{E'}(V)^{\mathrm{Gal}(\bar{E}/E'_{\mathrm{cycl}}),\mathrm{Gal}(E'_{\mathrm{cycl}}/E')-an} \end{array}$$

We still denote by S^{arit} the right derived functor. The following is [RC22, Cor. 6.3.6].

Theorem 4.5.23.

(1) We have that

$$\begin{aligned}\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p &= S^{\mathrm{arit}}(\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p)\otimes_{\bar{E}}\mathbf{C}_p,\\ \mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p},\mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p &= S^{\mathrm{arit}}(\mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p},\mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p)\otimes_{\bar{E}}\mathbf{C}_p.\end{aligned}$$

(2) The action of μ via Θ_{hor} on $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p=\mathrm{R}\Gamma_{an}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ and $\mathrm{R}\Gamma_c(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p=\mathrm{R}\Gamma_{an}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\mathscr{I}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ is the arithmetic Sen operator for the semilinear action of $\mathrm{Gal}(\bar{E}/E)$ (whose existence was guaranteed by (1)).

Proof. We can take an affinoid covering $\mathcal{V}=\{V_i\}$ of $\mathrm{Sh}^{\mathrm{tor}}_{K^p}$ with the property that $\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{V_i}$ is a colimit of acyclic sheaves $VB^0_{K_{p,n},V_i}(\mathcal{C}^{\mathrm{la}})$. It follows that the Čech complex $\mathcal{C}(\mathcal{V},\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ represents $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p$ and carries a semi-linear $\mathrm{Gal}(\bar{E}/E)$-action. It follows from Theorem 4.5.4 (4), that we have

$$\mathcal{C}(\mathcal{U},\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})=\mathcal{C}(\mathcal{U},S^{\mathrm{arit}}(\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}))\otimes_{\bar{E}}\mathbf{C}_p,$$

where $\mathcal{C}(\mathcal{U},S^{\mathrm{arit}}(\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}))$ is therefore the sub-complex of $\mathrm{Gal}(\bar{E}/E^{\mathrm{cycl}})$-smooth and $\mathrm{Gal}(\bar{E}/E)$-locally analytic vectors. We deduce that $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\mathbf{Q}_p)^{\mathrm{la}}\otimes_{\mathbf{Q}_p}\mathbf{C}_p$ admits a Sen operator and it is given by $\Theta_{\mathrm{hor}}(\mu)$ by Theorem 4.5.20, (9). □

4.6. **Higher Coleman theory.**

4.6.1. *Automorphic vector bundles.* One can apply the functor VB^0 to the G-equivariant locally free sheaves of finite rank which are parameterized by finite dimensional representations of M. Let $\kappa\in X^*(T)^{M,+}$. We let $\omega^{\kappa,\mathrm{sm}}=VB^0(\mathcal{L}_\kappa)$ (where $\mathcal{L}_\kappa$ is constructed in Example 3.2.18). The sheaf $\omega^{\kappa,\mathrm{sm}}$ descends to a sheaf $\omega^\kappa_{K_p}$ on the Shimura variety $\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}$ (the usual sheaf of modular forms of weight κ). By construction $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\kappa,\mathrm{sm}})$ is a complex of smooth admissible $G(\mathbf{Q}_p)$-representations, equal to $\mathrm{colim}_{K_p}\,\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p},\omega^\kappa_{K_p})$. Recall that we have denoted by D_{K_p} the divisor of the boundary in $\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}$. We then consider the cuspidal subsheaf $\omega^\kappa_{K_p}(-D_{K_p})$. Passing to the limit, we get

$$\omega^{\kappa,\mathrm{sm}}(-D)=\mathrm{colim}\,\pi^{-1}_{K_p}\omega^\kappa_{K_p}(-D_{K_p})=\omega^{\kappa,\mathrm{sm}}\otimes_{\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}}\mathscr{I}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}.$$

Similarly, $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\kappa,\mathrm{sm}}(-D))$ is a complex of smooth admissible $G(\mathbf{Q}_p)$-representations, equal to $\mathrm{colim}_{K_p}\,\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p},\omega^\kappa_{K_p}(-D_{K_p}))$.

Remark 4.6.2. For $G=\mathrm{GSp}_4$, the tautological exact sequence over $\mathcal{FL}$ is

$$0\to\mathcal{L}_{(0,-1;1)}\to St\otimes\mathcal{O}_{\mathcal{FL}}\to\mathcal{L}_{(1,0;1)}\to 0$$

which pulls back to

$$0\to Lie(A)_{K_p}(1)\otimes_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}}}\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}\to T_pA\otimes_{\mathbf{Z}_p}\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}\to(\omega_{A^t})_{K_p}\otimes_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}}}\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}\to 0$$

for any level K_p. We deduce that $\omega^{(0,-1;1),\mathrm{sm}}=Lie(A)_{K_p}(1)\otimes\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$ and thus that

$$\omega^{(1,0;-1),\mathrm{sm}}=(\omega_A)_{K_p}(-1)\otimes_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}}}\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$$

by duality; so our normalization of the weights of Siegel modular forms is the standard one.

4.6.3. *Higher Coleman sheaves.* We now fix $w \in {}^MW$. Let $\lambda \in X^*(T)_E$. We consider the exact functor:

$$VB^0 \circ HCS_{w,\lambda} : \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda-\mathrm{alg}} \to LB_{(\mathfrak{g},B)}(C_w^\dagger) \to \mathrm{Mod}_{B(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\pi_{HT}^{-1}C_w^\dagger}).$$

Lemma 4.6.4. *The functor $VB^0 \circ HCS_{w,\lambda}$ factors through the category $\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\pi_{HT}^{-1}C_w^\dagger})$.*

Proof. This is a combination of Proposition 4.5.22 and of Remark 3.3.13. □

Lemma 4.6.5. *Let $\lambda \in X^*(T)^{M,+}$. Let $L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\lambda} \in \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{0-\mathrm{alg}}$ be the finite dimensional irreducible representation of highest weight $-w^{-1}w_{0,M}\lambda$. Then $VB^0 \circ HCS_{w,0}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\lambda}) = \omega^{\lambda,sm}|_{\pi_{HT}^{-1}C_w^\dagger}$.*

Proof. . Let V_λ be the highest weight λ-representation of M. Then V_λ identifies with $L(\mathfrak{m}_w)_{w^{-1}\lambda}$ if we conjugate $\mathfrak{m}_w$ to $\mathfrak{m}$. It follows from the definitions that $\mathcal{L}_\lambda|_{C_w^\dagger} = HCS_{w,0}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\lambda})$ and the conclusion follows from applying VB^0. □

As in Definition 2.3.16, we have the Verma module $M(\mathfrak{m}_w)_\lambda$ of weight λ.

Definition 4.6.6. We define the following object of $\mathrm{Mod}^{-w^{-1}w_{0,M}\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(\mathcal{O}^{\mathrm{sm}}_{\pi_{HT}^{-1}C_w^\dagger})$:

$$\omega_w^{\dagger,\lambda} := VB^0 \circ HCS_{w,-w^{-1}w_{0,M}\lambda}(M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\lambda}).$$

Remark 4.6.7. This definition compares with [BP21, §6.3] as follows. In that reference we (GB+VP) defined Banach sheaves $\mathcal{V}_\nu^{n-an}$ for characters $\nu : T(\mathbf{Z}_p) \to \mathbf{C}_p^\times$, n large enough, over certain quasi-compact open subspaces $\pi_{HT}^{-1}(]C_{w,k}[_{n,n}K_p)$ of $\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}$ (for K_p small enough), where $]C_{w,k}[_{n,n}K_p$ is a quasi-compact open subset of C_w. The colimit over K_p, over all ν with $\mathrm{d}\nu = \lambda$ and over all n of the $\mathcal{V}_\nu^{n-an}$ identifies canonically with the germ of $\omega_w^{\dagger,\lambda}$ at $\pi_{HT}^{-1}(\{w\})$. A slight change of perspective from [BP21] is therefore this passage to the limit, and the fact that we define the sheaf $\omega_w^{\dagger,\lambda}$ over the entire $\pi_{HT}^{-1}(C_w)$. For the definition and computation of the finite slope part of higher Coleman theory, the sheaves $\mathcal{V}_\nu^{n-an}$ are however sufficient (and seemed to us easier to define in the first place). See Section 4.6.55 for further details.

One reason for the twist from λ to $-w^{-1}w_{0,M}\lambda$ is in order to obtain Remark 4.6.8 and Proposition 4.6.9; more conceptually, the multiplication by w^{-1} is justified by the change of base point in the flag variety, and the appearance of $-w_{0,M}$ is due to the usual duality involution on highest weights which comes from the contravariance of $HCS_{w,\lambda}$.

Remark 4.6.8. We see from Proposition 3.4.19 that the Θ_{hor}-action on

$$HCS_{w,-w^{-1}w_{0,M}\lambda}(M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\lambda})$$

is via λ. (We recall from Remark 2.3.10 that $HC_\mathfrak{m}\iota = -w_{0,M}HC_\mathfrak{m}$.) Thus, the arithmetic Sen operator acts via $\langle \mu, \lambda \rangle$ on $\omega_w^{\dagger,\lambda}$.

Proposition 4.6.9. *Assume that $\lambda \in X^*(T)^{M,+}$. We have an injective, $B(\mathbf{Q}_p)$-equivariant map: $\omega^{\lambda,\mathrm{sm}}|_{\pi_{HT}^{-1}(C_w^\dagger)} \otimes E(-w^{-1}w_{0,M}\lambda) \to \omega_w^{\dagger,\lambda}$.*

Proof. By Lemma 4.6.5, $\omega^{\lambda,\mathrm{sm}}|_{\pi_{HT}^{-1}(C_w^\dagger)} = VB^0 \circ HCS_{w,0}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\lambda})$. The map of the proposition is thus given by applying $VB^0 \circ HCS_{w,-w^{-1}w_{0,M}\lambda}$ to the surjection $M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\lambda} \to L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\lambda}$, using Proposition 3.4.18. □

4.6.10. *The Bruhat stratification and a Cousin spectral sequence.* We recall the stratification into B-orbits $\mathcal{FL} = \coprod_{w\in {}^MW} C_w$, with $C_w = P\backslash PwB$. We let $X_w = \cup_{w'\leq w} C_{w'}$ be the Schubert variety. We let $j_w : C_w \hookrightarrow \mathcal{FL}$ be the locally closed immersion. It induces $j_{w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}} : \pi_{HT}^{-1}(C_w) \to \mathrm{Sh}^{\mathrm{tor}}_{K^p}$.

Definition 4.6.11. If $\mathscr{F}$ is any sheaf of solid E-modules on $\pi_{HT}^{-1}(C_w)$, we let

$$\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}) = \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, (j_{w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!\mathscr{F}).$$

In this definition, $(j_{w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!$ is the extension by zero functor on abelian sheaves of solid abelian groups. By abuse of notation, if $\mathscr{F}$ is defined over any subset of $\mathrm{Sh}^{\mathrm{tor}}_{K^p}$ containing $\pi_{HT}^{-1}(C_w)$, we let $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}) = \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, (j_{w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!\mathscr{F}|_{\pi_{HT}^{-1}(C_w)})$.

We now explain that if $\mathscr{F}$ carries a $B(\mathbf{Q}_p)$-equivariant structure, the cohomology is a $B(\mathbf{Q}_p)$-representation.

Lemma 4.6.12. *The functor*

$$\begin{array}{rcl} \mathrm{Mod}(\pi_{HT}^{-1}(C_w)) & \to & D(\mathrm{Mod}(E)) \\ \mathscr{F} & \mapsto & \mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}) \end{array}$$

can be upgraded to functors:

$$\begin{array}{rcl} \mathrm{Mod}_{B(\mathbf{Q}_p)}(\pi_{HT}^{-1}(C_w)) & \to & D(\mathrm{Mod}_{B(\mathbf{Q}_p)}(E)). \\ \mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(\pi_{HT}^{-1}(C_w)) & \to & D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)). \end{array}$$

Proof. The existence of $\mathrm{Mod}_{B(\mathbf{Q}_p)}(\pi_{HT}^{-1}(C_w)) \to D(\mathrm{Mod}_{B(\mathbf{Q}_p)}(E))$ follows from the fact that $\mathrm{Mod}_{B(\mathbf{Q}_p)}(\pi_{HT}^{-1}(C_w))$ has enough injectives (see Lemma 3.2.10).

We verify that this induces a functor $\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(\pi_{HT}^{-1}(C_w)) \to D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E))$. We can reduce to the case that $\lambda = 0$. Then $(j_{w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!\mathscr{F}$ is a smooth $B(\mathbf{Q}_p)$-equivariant sheaf on $\mathrm{Sh}^{\mathrm{tor}}_{K^p}$, which is quasi-compact, so by definition the global sections $H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}) \in \mathrm{Mod}^{\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)$ are a smooth $B(\mathbf{Q}_p)$-representation. The result follows by applying this to an injective resolution. □

Proposition 4.6.13. *Let $\mathscr{F}$ be a solid abelian sheaf defined over $\mathrm{Sh}^{\mathrm{tor}}_{K^p}$. We have a spectral sequence (Cousin spectral sequence):*

$$E_1^{p,q} = \oplus_{w\in {}^MW, \ell(w)=p} H^{p+q}_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}) \Rightarrow H^{p+q}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}).$$

Proof. See e.g. [BP21, §2.3]. □

Remark 4.6.14. If $\mathcal{F}$ in Proposition 4.6.13 is $B(\mathbf{Q}_p)$-equivariant, then so (by construction) is the Cousin spectral sequence.

4.6.15. *Tools for computing the cohomology.* In this section, we give a few basic tools for computing cohomology. All adic spaces are locally of finite type over $\mathrm{Spa}(E, \mathcal{O}_E)$ and are separated unless specifically mentioned (thus they correspond to separated rigid analytic spaces in the sense of Tate). We will often consider the case of Stein or quasi-Stein spaces, which we recalled in Definition 2.2.17.

Let $\mathcal{X}$ be an adic space over $\mathrm{Spa}(E, \mathcal{O}_E)$. Let $\mathscr{F}$ be a coherent sheaf over $\mathcal{X}$. We can think of $\mathscr{F}$ as an abelian sheaf of solid E-modules and compute $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F})$ in the derived category of solid E-modules, $D(\mathrm{Mod}(E))$.

Theorem 4.6.16 (Tate acyclicity)**.** *Let $\mathcal{X}$ be an affinoid adic space. Let $\mathscr{F}$ be a coherent sheaf over $\mathcal{X}$. The cohomology $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F}) \in D(\mathrm{Mod}(E))$ is concentrated in degree 0.*

Proof. Let $\{\mathcal{U}_i\}_{i\in I}$ be a finite affinoid covering of $\mathcal{X}$. We know by [Hub94, Thm. 2.5] that the augmented Čech complex of $\{\mathcal{U}_i\}_{i\in I}$ is a long exact sequence of classical Banach spaces. Therefore, it is also an exact sequence of solid Banach spaces (see Remark 2.2.3). One concludes by [Gro57, 3.8, coro. 4]. □

Corollary 4.6.17. *Let $\mathcal{X}$ be a finite type, separated adic space over $\mathrm{Spa}(E, \mathcal{O}_E)$. Let $\mathscr{F}$ be a coherent sheaf over $\mathcal{X}$. The cohomology of $\mathscr{F}$ agrees with Čech cohomology.*

Proof. This is an application of [Gro57, 3.8, coro. 4] and Theorem 4.6.16. □

Corollary 4.6.18. *Let $\mathcal{X}$ be a quasi-Stein affinoid adic space. Let $\mathscr{F}$ be a coherent sheaf over $\mathcal{X}$. The cohomology $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F}) \in D(\mathrm{Mod}(E))$ is concentrated in degree 0.*

Proof. By Corollary 4.6.17, the cohomology is given by $\mathrm{R}\lim_n \mathrm{H}^0(\mathcal{X}_n, \mathscr{F})$. One knows by [Kie67] that the maps $\mathrm{H}^0(\mathcal{X}_{n+1}, \mathscr{F}) \to \mathrm{H}^0(\mathcal{X}_n, \mathscr{F})$ of classical Banach spaces have dense image. Then the topological Mittag-Leffler [RJRC22, Lem. 3.27] shows that $\mathrm{R}\lim_n \mathrm{H}^0(\mathcal{X}_n, \mathscr{F}) = \lim_n \mathrm{H}^0(\mathcal{X}_n, \mathscr{F})$, as required. □

By [RJRC22, Lem. 3.21] a Banach or Smith space over E is flat.

Corollary 4.6.19. *Let $\mathcal{X}$ be a finite type, separated adic space over $\mathrm{Spa}(E, \mathcal{O}_E)$, and let $\mathscr{F}$ be a coherent sheaf. Let V be a Banach or Smith space over E. Then $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F} \otimes_E V) = \mathrm{R}\Gamma(\mathcal{X}, \mathscr{F}) \otimes_E V$.*

Proof. In the affinoid case, we check that $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F} \otimes_E V) = H^0(\mathcal{X}, \mathscr{F}) \otimes_E V[0]$. To see this, we simply check it on the Čech cohomology of arbitrary finite affinoid covers, and this follows from the flatness of V. For a general $\mathcal{X}$, we deduce (again using [Gro57, 3.8, coro. 4]) that the cohomology of $\mathscr{F} \otimes V$ is computed by the Čech cohomology of a finite affinoid cover, and we use one more time that V is flat. □

Corollary 4.6.20. *Let $\mathscr{F}$ be a coherent sheaf over $\mathcal{X}$, and let V be a Banach or Smith space over L. Suppose that $\mathcal{X}$ is covered by finitely many quasi-Stein spaces. Then $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F} \otimes_E V) = \mathrm{R}\Gamma(\mathcal{X}, \mathscr{F}) \otimes_E V$, and if $\mathcal{X}$ is itself a quasi-Stein space, then we have $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F} \otimes_E V) = H^0(\mathcal{X}, \mathscr{F}) \otimes_E V[0]$.*

Proof. If $\mathcal{X}$ is quasi-Stein, then by Corollary 4.6.19, the cohomology is given by $\mathrm{R}\lim_n(\mathrm{H}^0(\mathcal{X}_n, \mathscr{F}) \otimes_E V)$. By the topological Mittag-Leffler and [RJRC22, Lem. 3.28], this limit is simply $(\lim_n \mathrm{H}^0(\mathcal{X}_n, \mathscr{F})) \otimes_E V$. From this, we deduce that if $\mathcal{X}$ is covered by finitely many quasi-Stein spaces, then the cohomology of $\mathscr{F} \otimes_E V$ is computed by the Čech cohomology of any finite quasi-Stein cover, and the claim follows again from flatness of V. □

We also consider duality and cohomology with compact support. Let $\mathcal{X}$ be an adic space. Let $\mathscr{F}$ be a solid sheaf of abelian groups. Following [Hub96, 5.2], we let $H^0_c(\mathcal{X}, \mathscr{F}) = \mathrm{colim}_{\mathcal{Z}}\, H^0_{\mathcal{Z}}(\mathscr{F})$ be the space of sections with compact support where

$\mathcal{Z}$ runs through the poset of closed subsets of $\mathcal{X}$ which are proper over $\mathrm{Spa}(E, \mathcal{O}_E)$. We let $\mathrm{R}\Gamma_c(\mathcal{X}, \mathscr{F}) = \mathrm{colim}_{\mathcal{Z}} \mathrm{R}\Gamma_{\mathcal{Z}}(\mathcal{X}, \mathscr{F})$ be the cohomology with compact support. (Here $\mathrm{R}\Gamma_{\mathcal{Z}}(\mathcal{X}, \mathscr{F})$ is by definition equal to $\mathrm{R}\Gamma(\mathcal{X}, (i_{\mathcal{Z}})_* i^!_{\mathcal{Z}} \mathscr{F})$, where $i_{\mathcal{Z}} : \mathcal{Z} \hookrightarrow \mathcal{Z}$.)

Remark 4.6.21. For example, $\mathcal{Z}$ is proper if it arises as the inverse image of a proper subset of a formal model of a quasi-compact open subset of $\mathcal{X}$.

Consider a quasi-compact and separated adic space of finite type $\mathcal{X}$. Let $j : \mathcal{U} \hookrightarrow \mathcal{X}$ be an open adic subspace of $\mathcal{X}$. We assume that $\mathcal{U}$ admits an increasing covering by quasi-compact opens $\mathcal{U} = \cup_n \mathcal{U}_n$ with the property that $\mathcal{U}_n$ is relatively compact in $\mathcal{U}_{n+1}$.

Example 4.6.22. One can take $\mathcal{X} = \mathbf{P}^1, \mathcal{U} = \mathbf{A}^{1,an}$. More generally, one can take $\mathcal{X}$ a proper finite type analytic space and $\mathcal{U}$ be the complement of a Zariski closed subset of $\mathcal{X}$ (see [L9̈0, 5.9]). One can also take $\mathcal{X} = \mathrm{Spa}(L\langle T\rangle, \mathcal{O}_L\langle T\rangle)$ (the closed unit ball of radius 1), with $\mathcal{U}$ the open unit ball of radius 1 inside $\mathcal{X}$.

Lemma 4.6.23. *Let $\mathscr{F}$ be a solid abelian sheaf over $\mathcal{U}$ as above. Then* $\mathrm{R}\Gamma(\mathcal{X}, j_! \mathscr{F}) = \mathrm{R}\Gamma_c(\mathcal{U}, \mathscr{F}) = \mathrm{colim}\, \mathrm{R}\Gamma_{\mathcal{Z}}(\mathcal{U}, \mathscr{F})$ *where $\mathcal{Z}$ runs through all closed subsets of $\mathcal{X}$ contained in $\mathcal{U}$. Moreover, there exists an increasing family of closed subspaces $\mathcal{U}_n \subseteq \mathcal{Z}_n \subseteq \mathcal{U}_{n+1}$, each with quasi-compact complement in $\mathcal{X}$, such that* $\mathrm{R}\Gamma_c(\mathcal{U}, \mathscr{F}) = \mathrm{colim}_n \mathrm{R}\Gamma_{\mathcal{Z}_n}(\mathcal{U}, \mathscr{F})$.

Proof. Firstly, we check that a subset $\mathcal{Z} \subset \mathcal{U}$ is closed in $\mathcal{X}$ if and only if $\mathcal{Z}$ is proper over $\mathrm{Spa}(E, \mathcal{O}_E)$. Suppose that $\mathcal{Z}$ is a closed subset of $\mathcal{X}$ contained in $\mathcal{U}$. Then $\mathcal{Z}$ is quasi-compact (as it is a closed subset of $\mathcal{X}$). Let $\mathcal{U} = \cup_n \mathcal{U}_n$. By assumption, $\overline{\mathcal{U}_n}$ (the closure of $\mathcal{U}_n$ in $\mathcal{U}$) is proper over $\mathrm{Spa}(E, \mathcal{O}_E)$. Since $\mathcal{Z}$ is quasi-compact, $\mathcal{Z} \subseteq \mathcal{U}_n$ for n large enough. Hence, $\mathcal{Z}$ is closed in $\overline{\mathcal{U}_n}$, thus partially proper. Conversely, if $\mathcal{Z}$ is a subset of $\mathcal{U}$, proper over $\mathrm{Spa}(E, \mathcal{O}_E)$, then $\mathcal{Z} \to \mathcal{X}$ is proper, as claimed.

For any closed subset $\mathcal{Z} \subseteq \mathcal{U}$, the counit of adjunction for $i_{\mathcal{Z}} : \mathcal{Z} \hookrightarrow \mathcal{X}$ gives a map $(i_{\mathcal{Z}})_* i^!_{\mathcal{Z}} j_! \mathscr{F} \to j_! \mathscr{F}$ which induces a map $\mathrm{colim}_{\mathcal{Z}} (i_{\mathcal{Z}})_* i^!_{\mathcal{Z}} j_! \mathscr{F} \to j_! \mathscr{F}$. This map is injective (both sheaves are subsheaves of $j_* \mathscr{F}$), and both sheaves restrict to $\mathscr{F}$ on $\mathcal{U}$ and have zero stalk at points of $\mathcal{X} \smallsetminus \mathcal{U}$; thus the map is an isomorphism. Since $\mathcal{X}$ is quasi-compact and separated, this implies that $\mathrm{R}\Gamma(\mathcal{X}, j_! \mathscr{F}) = \mathrm{colim}_Z \mathrm{R}\Gamma_{\mathcal{Z}}(\mathcal{X}, j_! \mathscr{F})$. Moreover, $\mathrm{colim}_{\mathcal{Z}} \mathrm{R}\Gamma_{\mathcal{Z}}(\mathcal{X}, j_! \mathscr{F}) = \mathrm{R}\Gamma_c(\mathcal{U}, \mathscr{F})$ by definition. We finally claim that for each n there exists a closed subspace $\mathcal{U}_n \subseteq \mathcal{Z}_n \subseteq \mathcal{U}_{n+1}$ with quasi-compact complement in $\mathcal{X}$. We let $\cup_{i \in I} V_i$ be a covering by quasi-compact opens of $\overline{\mathcal{U}_n}^c$. We claim that there exists a finite subset I' of I such that $\cup_{i \in I'} V_i$ contains $\mathcal{U}^c_{n+1}$. This follows from endowing $\mathcal{X}$ with the constructible topology and noticing that $\mathcal{U}^c_{n+1}$ is compact in this topology. We can take $\mathcal{Z}_n = (\cup_{i \in I'} V_i)^c$. The $\mathcal{Z}_n$ are clearly cofinal among all $\mathcal{Z}$'s. ◻

The basic duality statement is the following.

Theorem 4.6.24 ([Chi90])**.** *Let $\mathscr{F}$ be a locally free coherent sheaf defined over a smooth Stein space $\mathcal{X}$ of dimension d. The cohomology $\mathrm{R}\Gamma_c(\mathcal{X}, \mathscr{F})$ is concentrated in degree d and is an LB-space of compact type. Moreover,*

$$H^d_c(\mathcal{X}, \mathscr{F}) = \mathrm{Hom}_E(H^0(\mathcal{X}, D(\mathscr{F})), E)$$

where $D(\mathscr{F}) = \underline{\mathrm{Hom}}(\mathscr{F}, \Omega^d_{\mathcal{X}})$.

Remark 4.6.25. Although we use the formalism of compactly supported cohomology for abelian sheaves, we get the correct "coherent duality". This is a favorable property of Stein spaces.

We now extend this result to orthonormalizable Banach sheaves.

Corollary 4.6.26. *Let $\mathscr{F}$ be a locally free coherent sheaf defined over a Stein space $\mathcal{X}$ of dimension d. Let V be a Smith space over E. We have that $\mathrm{R}\Gamma_c(\mathcal{X}, \mathscr{F} \otimes_E V) = \mathrm{R}\Gamma_c(\mathcal{X}, \mathscr{F}) \otimes_E V$ is concentrated in degree d and moreover,*

$$H^d_c(\mathcal{X}, \mathscr{F} \otimes V) = \mathrm{Hom}_E(H^0(\mathcal{X}, D(\mathscr{F}) \otimes V^\vee), E).$$

Proof. We let $\mathcal{X} = \cup_n \mathcal{X}_n$ be an open cover by affinoids, with $\mathcal{X}_n$ relatively compact in $\mathcal{X}_{n+1}$. We also let $\mathcal{X}_n \subseteq \mathcal{Z}_n \subseteq \mathcal{X}_{n+1}$ be closed subspaces with quasi-compact complement as in the statement of Lemma 4.6.23 (with $\mathcal{U}_n = \mathcal{X}_n$). It follows from Lemma 4.6.23 that $\mathrm{R}\Gamma_c(\mathcal{X}, \mathscr{F} \otimes_E V) = \mathrm{colim}_n \mathrm{R}\Gamma_{\mathcal{Z}_n}(\mathcal{X}_{n+1}, \mathscr{F} \otimes_E V)$. We observe that $\mathrm{R}\Gamma_{\mathcal{Z}_n}(\mathcal{X}_{n+1}, \mathscr{F} \otimes V) = \mathrm{R}\Gamma_{\mathcal{Z}_n}(\mathcal{X}_{n+1}, \mathscr{F}) \otimes V$ by an easy reduction to Corollary 4.6.19. We deduce that $\mathrm{R}\Gamma_c(\mathcal{X}, \mathscr{F} \otimes_E V) = \mathrm{R}\Gamma_c(\mathcal{X}, \mathscr{F}) \otimes_E V$. Next, we see that

$$\begin{aligned}
\mathrm{Hom}_E(H^0(\mathcal{X}, D(\mathscr{F}) \otimes V^\vee), E) &= \mathrm{Hom}_E(H^0(\mathcal{X}, D(\mathscr{F})) \otimes V^\vee, E) \text{ by Corollary 4.6.20} \\
&= \mathrm{Hom}_E(H^0(\mathcal{X}, D(\mathscr{F})), V) \text{ by adjunction and Proposition 2.2.4} \\
&= \mathrm{Hom}_E(H^0(\mathcal{X}, D(\mathscr{F})), E) \otimes_E V \text{ by [RJRC22, Thm. 3.40]} \\
&= H^d_c(\mathcal{X}, \mathscr{F}) \otimes V \text{ by Theorem 4.6.24} \\
&= H^d_c(\mathcal{X}, \mathscr{F} \otimes V) \text{ by the first point.}
\end{aligned}$$

□

4.6.27. *On the computation of the local cohomology.* We want to give some formulas for computing $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F})$.

The Bruhat cell C_w is an affine space of dimension $\ell(w)$ and its closure X_w is a compactification of this affine space. We recall (see [BP21, Lemma 3.1.3] for example) that $C_w = w \prod_{\alpha \in \Phi^+ \cap w^{-1}\Phi^{-,M}} U_\alpha$ where U_α is the α-root space, isomorphic to $\mathbf{A}^{1,an}$. We also have a neighborhood of C_w, $U_w = w \prod_{\alpha \in w^{-1}\Phi^{-,M}} U_\alpha$. Let us pick a coordinate u_α on each U_α.

Definition 4.6.28. We now define certain subsets of $Z_n, Z_{n,m}$, U_n and $U_{n,m}$ of C_w, for $n, m \in \mathbf{Z}_{\geq 0}$. Let $n_0 \geq 0$ be fixed. We take:

- $Z_n = \{x \in C_w, \forall \alpha \in \Phi^+ \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x < |p|_x^{n_0 - n}\}$.
- $U_n = \{x \in C_w, \forall \alpha \in \Phi^+ \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x \leq |p|_x^{n_0 - n - 1}\}$.
- $Z_{n,m} = \{x \in U_w, \forall \alpha \in \Phi^+ \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x < |p|_x^{n_0 - n}, \forall \alpha \in \Phi^- \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x < |p|_x^m\}$,
- $U_{n,m} = \{x \in U_w, \forall \alpha \in \Phi^+ \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x \leq |p|_x^{n_0 - n - 1}, \forall \alpha \in \Phi^- \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x \leq |p|_x^m\}$.

Here are some obvious properties of these sets.

- $\cup_n Z_n = \cup_n U_n = C_w$,
- The complement of Z_n in X_w is a quasi-compact open subset,
- U_n is a quasi-compact open subset of X_w,
- $U_n = \cap_m U_{n,m}$, and $U_{n,m} \cap X_w = U_n$,
- $U_{n,m}$ is a quasi-compact open of $\mathcal{FL}$,
- $Z_{n,m}$ is a closed subset of $U_{n,m}$ with quasi-compact complement,
- $Z_n = U_n \cap Z_{n,m}$.

We let $F_n = \pi_{HT}^{-1}(Z_n)$, $V_n = \pi_{HT}^{-1}(U_n)$, $F_{n,m} = \pi_{HT}^{-1}(Z_{n,m})$ and $V_{n,m} = \pi_{HT}^{-1}(U_{n,m})$.

Lemma 4.6.29. *Let $\mathscr{F}$ be a solid abelian sheaf over $\pi_{HT}^{-1}(C_w)$. We have* $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}) = \mathrm{colim}_n \mathrm{R}\Gamma_{F_n}(V_n, \mathscr{F})$.

Proof. This is a consequence of Lemma 4.6.23 (since the V_n give a cofinal system of closed subsets of Z_w).

□

Lemma 4.6.30. *Let $m_0 \geq 0$ and let $\mathscr{F}'$ be a sheaf on V_{n,m_0} and let $\mathscr{F} = i^{-1}\mathscr{F}'$ on V_n for $i : V_n \to V_{n,m_0}$. Then we have:* $\mathrm{R}\Gamma_{F_n}(V_n, \mathscr{F}) = \mathrm{colim}_{m \geq m_0} \mathrm{R}\Gamma_{F_{n,m}}(V_{n,m}, \mathscr{F}')$.

Proof. We consider the triangle:

$$\mathrm{R}\Gamma_{F_{n,m}}(V_{n,m}, \mathscr{F}') \to \mathrm{R}\Gamma(V_{n,m}, \mathscr{F}') \to \mathrm{R}\Gamma(V_{n,m} \smallsetminus F_{n,m}), \mathscr{F}') \stackrel{+1}{\to}$$

Passing to the colimit over m, we need to see that $\mathrm{colim}_m \mathrm{R}\Gamma(V_{n,m}, \mathscr{F}') = \mathrm{R}\Gamma(V_n, \mathscr{F})$ and $\mathrm{colim}_m \mathrm{R}\Gamma(V_{n,m} \smallsetminus F_{n,m}, \mathscr{F}') = \mathrm{R}\Gamma(V_n \smallsetminus F_n, \mathscr{F})$. This follows from the fact that all spaces involved are quasi-compact. □

The combination of lemmas 4.6.29 and 4.6.30 allows us to compute the cohomology as a colimit of cohomologies of the shape $\mathrm{R}\Gamma_{F_{n,m}}(V_{n,m}, \mathscr{F}')$ where $\mathrm{R}\Gamma_{F_{n,m}}(V_{n,m}, \mathscr{F}')$ fits in a triangle:

$$\mathrm{R}\Gamma_{F_{n,m}}(V_{n,m}, \mathscr{F}') \to \mathrm{R}\Gamma(V_{n,m}, \mathscr{F}') \to \mathrm{R}\Gamma(V_{n,m} \smallsetminus F_{n,m}, \mathscr{F}') \stackrel{+1}{\to}$$

and both $V_{n,m}$ and $V_{n,m} \smallsetminus F_{n,m}$ are quasi-compact opens.

We now give another similar presentation of the cohomology which puts emphasis on the cohomology with compact support. We will be using Stein spaces because of remark 4.6.25.

Definition 4.6.31. We define sets X_n, $X_{n,m}$ and $T_{n,m,r}$ for $n, m, r \in \mathbf{Z}_{\geq 0}$.

- For any $\epsilon \in \mathbf{Q}_{>0}$, we put $X_{n,\epsilon} = \{x \in C_w, \forall \alpha \in \Phi^+ \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x \leq |p|_x^{n_0-n+\epsilon}\}$ and set $X_n = \cup_{\epsilon>0} X_{n,\epsilon}$.
- $X_{n,m,\epsilon} = \{x \in U_w, \forall \alpha \in \Phi^+ \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x \leq |p|_x^{n_0-n+\epsilon}, \forall \alpha \in \Phi^- \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x \leq |p|_x^{m+\epsilon}\}$, and set $X_{n,m} = \cup_{\epsilon>0} X_{n,m,\epsilon}$.
- We also put $T_{n,m,r,\eta} = \{x \in X_{n,m}, \exists \alpha \in \Phi^- \cap w^{-1}\Phi^{-,M}, |u_\alpha|_x \geq |p|^{r-\eta}\}$. We let $T_{n,m,r} = \cup_{\eta>0} T_{n,m,r,\eta}$.

We see that X_n is an increasing sequence of Stein subsets of C_w with the property that $C_w = \cup_n X_n$ and that each X_n is included in a quasi-compact open subset Z_n of C_w. We see that $X_{n,m}$ is a decreasing (in m) family of Stein open subsets of $\mathcal{FL}$ with $X_n = X_{n,m} \cap X_w$ and $\cap_m X_{n,m} = X_n$. We have $\cup_{r>0} T_{n,m,r} = X_{n,m} \smallsetminus X_n$.

Let $Y_n = \pi_{HT}^{-1}(X_n)$. We let $Y_{n,m} = \pi_{HT}^{-1}(X_{n,m})$ and $W_{n,m,r} = \pi_{HT}^{-1}(T_{m,n,r})$.

Lemma 4.6.32. *Let $\mathscr{F}$ be a solid abelian sheaf over $\pi_{HT}^{-1}(C_w)$. We have* $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}) = \mathrm{colim}_n \mathrm{R}\Gamma_c(Y_n, \mathscr{F})$.

Proof. This follows from Lemma 4.6.29. Indeed the two inductive systems are equivalent as $Y_n \subseteq F_n \subseteq Y_{n+1}$, and we have a series of natural maps $\mathrm{R}\Gamma_{F_n}(V_n, \mathscr{F}) \to \mathrm{R}\Gamma_c(Y_n, \mathscr{F}) \to \mathrm{R}\Gamma_{F_{n+1}}(V_{n+1}, \mathscr{F})$. □

Lemma 4.6.33. *Let $\mathscr{F}'$ be a sheaf on $Y_{n,m}$ and let $\mathscr{F} = i^{-1}\mathscr{F}'$ on Y_n. Then we have a triangle:*

$$\mathrm{R}\Gamma_c(Y_{n,m} \smallsetminus Y_n, \mathscr{F}') \to \mathrm{R}\Gamma_c(Y_{n,m}, \mathscr{F}') \to \mathrm{R}\Gamma_c(Y_n, \mathscr{F}) \stackrel{+1}{\to}$$

Moreover, $\mathrm{R}\Gamma_c(Y_{n,m} \smallsetminus Y_n, \mathscr{F}') = \mathrm{colim}_r \mathrm{R}\Gamma_c(W_{n,m,r}, \mathscr{F}')$.

Proof. Let $m \geq 0$. We consider the following commutative diagram:

$$\begin{array}{ccc} Y_n & \xrightarrow{j_n} & \pi_{HT}^{-1}(X_w) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ Y_{n,m} & \xrightarrow{j_{n,m}} & \mathrm{Sh}_{K^p}^{\mathrm{tor}} \\ \uparrow{\scriptstyle j} & \nearrow{\scriptstyle i_{n,m}} & \\ Y_{n,m} \setminus Y_n & & \end{array}$$

We have to prove that we have a triangle

$$\mathrm{R}\Gamma(\mathrm{Sh}_{K^p}^{\mathrm{tor}}, (i_{n,m})_! \mathscr{F}') \to \mathrm{R}\Gamma(\mathrm{Sh}_{K^p}^{\mathrm{tor}}, (j_{n,m})! \mathscr{F}') \to \mathrm{R}\Gamma(\pi_{HT}^{-1}(X_w), (j_n)_! \mathscr{F}) \overset{+1}{\to}$$

We have the triangle $j_! j^{-1}\mathscr{F}' \to \mathscr{F}' \to i_*\mathscr{F} \overset{+1}{\to}$. We apply $(j_{n,m})_!$, and we get a triangle: $(i_{n,m})_! j^{-1}\mathscr{F}' \to (j_{n,m})_!\mathscr{F}' \to (j_{n,m})_! i_*\mathscr{F} \overset{+1}{\to}$. Observe that $(j_{n,m})_! i_*\mathscr{F} = i'_*(j_n)_!\mathscr{F}$ (as $i_* = i_!$ and $i'_* = i'_!$). We then apply $\mathrm{R}\Gamma(\mathrm{Sh}_{K^p}^{\mathrm{tor}}, -)$. □

4.6.34. *Higher Coleman functors.*

Definition 4.6.35. Using Lemma 4.6.12, we define the contravariant higher Coleman functor:

$$\begin{array}{rcl} HC_{w,\lambda} : \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda-\mathrm{alg}} & \to & D(\mathrm{Mod}_{B(\mathbf{Q}_p)}^{\lambda-\mathrm{sm}}(E)) \\ M & \mapsto & \mathrm{R}\Gamma_w(\mathrm{Sh}_{K^p}^{\mathrm{tor}}, VB^0(HCS_{w,\lambda}(M))) \end{array}$$

and the contravariant cuspidal higher Coleman functor:

$$\begin{array}{rcl} HC_{\mathrm{cusp},w,\lambda} : \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda-\mathrm{alg}} & \to & D(\mathrm{Mod}_{B(\mathbf{Q}_p)}^{\lambda-\mathrm{sm}}(E)) \\ M & \mapsto & \mathrm{R}\Gamma_w(\mathrm{Sh}_{K^p}^{\mathrm{tor}}, VB^0(HCS_{w,\lambda}(M)(-D))). \end{array}$$

We extend these functors to the derived category

$$HC_{w,\lambda}, HC_{\mathrm{cusp},w,\lambda} : D^b(\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda-\mathrm{alg}}) \to D(\mathrm{Mod}_{B(\mathbf{Q}_p)}^{\lambda-\mathrm{sm}}(E))$$

by putting $HC_{w,\lambda}(M) = \mathrm{R}\Gamma_w(\mathrm{Sh}_{K^p}^{\mathrm{tor}}, VB^{\mathrm{red}}(HCS_{w,\lambda}(M)))$ and similarly in the cuspidal case. We note that $HCS_{w,\lambda}$ is an exact functor $\mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda-\mathrm{alg}} \to LB_{(\mathfrak{g},B)}(C_w^\dagger)^{\mathfrak{u}_\mathfrak{p}^0}$, and VB^0 is also exact on $LB_{(\mathfrak{g},B)}(C_w^\dagger)^{\mathfrak{u}_\mathfrak{p}^0}$. Therefore, if M is a complex, $VB^{\mathrm{red}}(HCS_{w,\lambda}(M))$ is computed by applying $VB^0(HCS_{w,\lambda}(-))$ to each term of a complex representing M. Moreover, its i-th cohomology sheaf $\underline{H}^i(VB^{\mathrm{red}}(HCS_{w,\lambda}(M))$ is $VB^0(HCS_{w,\lambda}(H^i(M)))$.

4.6.36. *Formal models and the Hodge–Tate period map.* Our goal is to compute the cohomological amplitude of the higher Coleman functors. The main source of vanishing is the affineness of the Hodge–Tate period map. To perform the argument, we need to consider formal models for some of the spaces and sheaves introduced so far. We can consider the following diagram where $\mathrm{Sh}_{K^p}^*$ is the minimal compactification.

$$\begin{array}{ccc} \mathrm{Sh}_{K^p}^{\mathrm{tor}} & & \\ \downarrow & \searrow{\scriptstyle \pi_{HT}} & \\ \mathrm{Sh}_{K^p}^* & \xrightarrow{\pi_{HT}^*} & \mathcal{FL} \end{array}$$

In [BP21], sect. 4.4.31, we constructed a formal model of this diagram:

$$\begin{array}{ccc} \mathfrak{Sh}^{\mathrm{tor},mod}_{K^p} & & \\ \downarrow & \searrow & \\ \mathfrak{Sh}^{*,mod}_{K^p} & \longrightarrow & \mathfrak{FL} \end{array}$$

Moreover $\mathfrak{Sh}^{\mathrm{tor},mod}_{K^p} = \lim_{K_p} \mathfrak{Sh}^{\mathrm{tor},mod}_{K_pK^p}$ and $\mathfrak{Sh}^{*,mod}_{K^p} = \lim_{K_p} \mathfrak{Sh}^{*,mod}_{K_pK^p}$ for all K_p small enough, where $\mathfrak{Sh}^{\mathrm{tor},mod}_{K_pK^p}$ is a formal model of $\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}$ and $\mathfrak{Sh}^{*,mod}_{K_pK^p}$ is a formal model of $\mathrm{Sh}^{*}_{K_pK^p}$.

Let $\mathcal{U}$ be a quasi-compact open of $\mathcal{FL}$. Let $\mathfrak{U} \to \mathfrak{FL}$ be a formal model for the map $\mathcal{U} \to \mathcal{FL}$. Using the notation of [BP21], sect. 4.4.31, we have a formal model for $\pi_{HT}^{-1}(\mathcal{U})$, denoted by $\mathfrak{Sh}^{\mathrm{tor},mod}_{K^p,\mathfrak{U}}$ as well as a formal model for $(\pi^*_{HT})^{-1}(\mathcal{U})$ denoted by $\mathfrak{Sh}^{*,mod}_{K^p,\mathfrak{U}}$. There is a diagram:

$$\begin{array}{ccc} \mathfrak{Sh}^{\mathrm{tor},mod}_{K^p,\mathfrak{U}} & & \\ \downarrow & \searrow & \\ \mathfrak{Sh}^{*,mod}_{K^p,\mathfrak{U}} & \longrightarrow & \mathfrak{U} \end{array}$$

Moreover, we have $\pi_{HT}^{-1}(\mathcal{U}) = \lim_{K_p} \pi_{HT}^{-1}(\mathcal{U})_{K_p}$ for K_p small enough and accordingly $\mathfrak{Sh}^{\mathrm{tor},mod}_{K^p,\mathfrak{U}} = \lim_{K_p} \mathfrak{Sh}^{\mathrm{tor},mod}_{K_pK^p,\mathfrak{U}}$. Similarly, for K_p small enough $(\pi^*_{HT})^{-1}(\mathcal{U}) = \lim_{K_p}(\pi^*_{HT})^{-1}(\mathcal{U})_{K_p}$ and $\mathfrak{Sh}^{*,mod}_{K^p,\mathfrak{U}} = \lim_{K_p} \mathfrak{Sh}^{*,mod}_{K_pK^p,\mathfrak{U}}$.

4.6.37. *Formal Banach sheaves and formal Smith sheaves.* In this section we consider a flat p-adic formal scheme $\mathfrak{X}$ which locally of topologically finite type over Spf $\mathcal{O}_E$ (in other words, Zariski locally, $\mathfrak{X}$ is Spf A where A is a quotient of $\mathcal{O}_E\langle X_1, \cdots, X_n\rangle$). We write $\mathcal{X}$for the generic fiber of $\mathfrak{X}$. The sheaf $\mathcal{O}_{\mathfrak{X}}$ is a sheaf of solid $\mathcal{O}_E$-modules, as are all the sheaves we will consider. We define the scheme $X_n = \mathfrak{X} \times_{\mathrm{Spf}\ \mathcal{O}_E} \mathrm{Spec}\ \mathcal{O}_E/p^n\mathcal{O}_E$.

Definition 4.6.38.

(1) A locally trivial formal Banach sheaf over $\mathfrak{X}$ is a sheaf $\mathfrak{F}$ of $\mathcal{O}_{\mathfrak{X}}$-modules which is flat as an $\mathcal{O}_E$-module, and such that $\mathfrak{F} = \lim_n \mathscr{F}_n$ for $\mathscr{F}_n = \mathfrak{F}/p^n\mathfrak{F}$, and there exists a covering $\mathfrak{X} = \cup_i \mathfrak{V}_i$ and sets I_i and such that $\mathscr{F}_n|_{V_{i,n}} = \mathcal{O}_{V_{i,n}} \otimes_{\mathcal{O}_E/p^n\mathcal{O}_E} (\oplus_{s\in I_i} \mathcal{O}_E/p^n\mathcal{O}_E)$ and the transition maps $\mathscr{F}_n|_{V_{i,n}} \to \mathscr{F}_{n-1}|_{V_{i,n-1}}$ are the obvious ones.

(2) A very small formal Banach sheaf over $\mathfrak{X}$ is a sheaf $\mathfrak{F}$ of $\mathcal{O}_{\mathfrak{X}}$-modules which is flat as an $\mathcal{O}_E$-module, and such that $\mathfrak{F} = \lim_n \mathscr{F}_n$ for $\mathscr{F}_n = \mathfrak{F}/p^n\mathfrak{F}$, and $\mathscr{F}_1 = \mathscr{G}_1 \otimes_{\mathcal{O}_E} (\oplus_{s\in I} \mathcal{O}_E/p)$ for some coherent sheaf $\mathscr{G}_1$ and some set I.

Definition 4.6.39.

(1) A locally trivial formal Smith sheaf over $\mathfrak{X}$ a sheaf $\mathfrak{F}$ of $\mathcal{O}_{\mathfrak{X}}$-modules which is flat as an $\mathcal{O}_E$-module, such that $\mathfrak{F} = \lim_n \mathscr{F}_n$ for $\mathscr{F}_n = \mathfrak{F}/p^n\mathfrak{F}$, and there exists a covering $\mathfrak{X} = \cup_i \mathfrak{V}_i$ and sets I_i such that $\mathscr{F}_n = \mathcal{O}_{V_{i,n}} \otimes_{\mathcal{O}_E/p^n\mathcal{O}_E} (\mathcal{O}_E/p^n\mathcal{O}_E)^{I_i}$ with the obvious transition maps $\mathscr{F}_n|_{V_{i,n}} \to \mathscr{F}_{n-1}|_{V_{i,n-1}}$.

(2) A very small formal Smith sheaf over $\mathfrak{X}$ is a sheaf $\mathscr{F}$ of $\mathcal{O}_{\mathfrak{X}}$-modules which is flat as an $\mathcal{O}_E$-module, such that $\mathfrak{F} = \lim_n \mathscr{F}_n$ for $\mathscr{F}_n = \mathfrak{F}/p^n\mathfrak{F}$, and $\mathscr{F}_1 = \mathscr{G}_1 \otimes_E (\mathcal{O}_E/p\mathcal{O}_E)^I$ for a coherent sheaf $\mathscr{G}_1$ and some set I.

If $\mathfrak{F}$ is a locally trivial formal Smith sheaf, then it has a generic fiber $\mathscr{F}$ over $\mathcal{X}$ defined as follows: we take a covering $\mathcal{X} = \cup_i \mathfrak{V}_i$, such that $\mathfrak{F}|_{\mathfrak{V}_i} = \mathcal{O}_{\mathfrak{V}_i} \otimes \mathcal{O}_E^{I_i}$. Over the generic fibre $\mathcal{V}_i$ of $\mathfrak{V}_i$, we define $\mathscr{F}|_{\mathcal{V}_i} = \mathcal{O}_{\mathcal{V}_i} \otimes_E (\mathcal{O}_E^{I_i}[1/p])$ and we use the gluing data of $\mathfrak{F}$ to glue the $\mathscr{F}|_{\mathcal{V}_i}$.

A similar construction applies to a locally trivial Banach sheaf. See also [BP21, Thm. 2.5.9] for a more general statement.

Proposition 4.6.40. *If $\mathfrak{F}$ is a very small formal Banach sheaf or a very small formal Smith sheaf, $\mathfrak{X}$ admits an ample invertible sheaf and $\mathcal{X}$ is affinoid, then $H^i(\mathfrak{X}, \mathscr{F}) \otimes_{\mathcal{O}_E} E = 0$ for all $i > 0$.*

Proof. The formal Banach case is [BP21, Thm. 2.5.8]. The same proof with minor modification applies to the Smith case. □

4.6.41. *Cohomological amplitude of the higher Coleman functors.* Let $M \in \mathcal{O}_{\lambda-\mathrm{alg}}$. Let us consider the sheaf $\mathcal{F} := VB^0(HCS_{w,\lambda}(M))$. Theorem 4.5.20 addresses the local structure of this sheaf. We also need to produce some integral structure, as in the following lemma.

Lemma 4.6.42. *There exists a quasi-compact open subset $U_0 \subseteq C_w$ containing w such that if we set $V_0 = \pi_{HT}^{-1}(U_0)$, then we can write $\mathscr{F}|_{V_0} = \operatorname{colim} i_m^{-1}\mathscr{F}_m$ where:*

(1) $U_0 = \cap_m U_{0,m}$ where $\{U_{0,m}\}_{m\in\mathbf{Z}_{\geq 0}}$ is a decreasing sequence of quasi-compact open affinoid subsets of $\mathcal{FL}$;
(2) $V_{0,m} = \pi_{HT}^{-1}(U_{0,m})$ is a quasi-compact open subset of $\mathrm{Sh}^{\mathrm{tor}}_{K^p}$, stable under a compact open subgroup $K_{p,m} \subseteq G(\mathbf{Q}_p)$;
(3) $i_m : V_0 \to V_{0,m}$ is the natural inclusion;
(4) $\mathscr{F}_m$ is a sheaf over $V_{0,m}$;
(5) $V_{0,m,K_{p,m}} \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}_{K^pK_{p,m}}$ is a quasi-compact open which descends $V_{0,m}$ to finite level $K_{p,m}$, and $\pi_{K_{p,m}} : V_{0,m} \to V_{0,m,K_{p,m}}$ is the induced projection;
(6) We have a Banach sheaf $\mathscr{G}_{m,K_{p,m}}$ on $V_{0,m,K_{p,m}}$ and $\mathscr{F}_m = \pi_{K_{p,m}}^{-1}\mathscr{G}_{m,K_{p,m}} \otimes_{\pi_{K_{p,m}}^{-1}\mathcal{O}_{V_{0,m,K_{p,m}}}} \mathcal{O}^{\mathrm{sm}}_{V_{0,m}}$;
(7) The sheaves $\mathscr{G}_{m,K_{p,m}}$ arise as the generic fibers of locally trivial, very small, formal Banach sheaves $\mathfrak{G}_{m,K_{p,m}}$ over $\mathfrak{Sh}^{\mathrm{tor},mod}_{K^pK_{p,m},\mathfrak{U}_{0,m}}$.

Proof. The first 6 points follow from Theorem 4.5.20. We need to give a more explicit construction of $\mathscr{G}_{m,K_{p,m}}$ in order to be able to produce an integral structure. We will follow closely the proof of [BP21, Lem. 6.6.2]. As $M \in \mathcal{O}_{\lambda-\mathrm{alg}}$, we deduce that $\hat{M}^\vee(\lambda) = \operatorname{colim}_r M_r$ where each M_r is a Banach space representation of the group denoted $\mathrm{Stab}(w)_{r,s}$ in Definition 3.3.10 (with $Q = B$). Unraveling the definition, we deduce in particular that M_r is a Banach space representation of $M_{w,r}U_{M_w,s} =: M_{w,r,s} \hookrightarrow M_w$. Moreover, after possibly changing r and s, we can also assume that there is a lattice $M_r^+ \subseteq M_r$ with the property that the co-action map $M_r \to M_r \otimes \mathcal{O}_{M_{w,r,s}}$ induces a map $M_r^+ \to M_r^+ \otimes \mathcal{O}^+_{M_{w,r,s}}$, trivial on M_r^+/pM_r^+. We consider the $M_{w,r,s}$-torsor $w(U_{P_w} \cap G_rU_s)\backslash G_rU_s \to w(P_w \cap G_rU_s)\backslash G_rU_s$. This is a reduction of structure of the standard M-torsor $U_P\backslash G \to P\backslash G$ over $w(P_w \cap G_rU_s)\backslash G_rU_s$. Let us write $\mathcal{U}_r = w(P_w \cap G_rU_s)\backslash G_rU_s$, which is a quasi-compact open subset of $\mathcal{FL}$. Over $\pi_{HT}^{-1}(\mathcal{U}_r)$, we pull back to get a $M_{w,r,s}$-torsor that we denote by $\mathcal{M}_{dR,r,s,\mathcal{U}_r}$. For K_p small enough, by [BP21, prop. 4.6.12], it descends to a $M_{w,r,s}$-torsor $\mathcal{M}_{dR,r,s,\mathcal{U}_r,K_p}$ over $\pi_{HT}^{-1}(\mathcal{U}_r)_{K_p}$. Moreover, by [BP21, prop. 4.6.15] for any affinoid open subset $V_{K_p} \subseteq \pi_{HT}^{-1}(\mathcal{U})_{K_p}$, we can find $K_p' \subseteq K_p$ such that the torsor $\mathcal{M}_{dR,r,s,\mathcal{U}_r,K_p'}|_{V_{K_p'}}$ is trivial. After rescaling, we can assume that $U_{0,r} \subseteq \mathcal{U}_r$. We now attach to M_r a locally projective small formal Banach sheaf on the formal model $\mathfrak{Sh}^{\mathrm{tor},mod}_{K^pK_{p,r},\mathfrak{U}_{0,r}}$ of $V_{0,r,K_{p,r}} = \pi_{HT}^{-1}(U_{0,r})_{K_{p,r}}$ for a small enough $K_{p,r}$. Indeed, we pick a finite affine covering $\cup_i \mathfrak{V}_i = \mathrm{Spf}(A_i)$ of $\mathfrak{Sh}^{\mathrm{tor},mod}_{K^pK_{p,r},\mathfrak{U}_{0,r}}$. After replacing $K_{p,r}$ by a smaller compact open, we can assume that $\mathcal{M}_{dR,r,s,U_{0,r},K_{p,r}}$ is trivial over the generic fiber of every $\mathfrak{V}_i$. It follows that the torsor is described by a 1-cocycle $\{m_{i,j} \in M_{w,r,s}((A_{i,j}[1/p], A_{i,j}))\}_{i,j}$ (where $\mathfrak{V}_{i,j} = \mathfrak{V}_i \cap \mathfrak{V}_j = \mathrm{Spf}(A_{i,j})$). We can use the elements $m_{i,j}$ to glue the trivial formal Banach sheaf $\mathcal{O}_{\mathfrak{V}_i} \otimes_{\mathcal{O}_L} M_r^+$ to get the very small, locally trivial, formal Banach sheaf $\mathfrak{G}_{r,K_{p,r}}$. The very smallness property comes from the fact that the $m_{i,j}$ act trivially on M_r^+/p. □

We give a technical variant of this description, using formal Smith sheaves instead.

Lemma 4.6.43. *There exists a quasi-compact open subset $U_0 \subseteq C_w$ containing w such that if we set $V_0 = \pi_{HT}^{-1}(U_0)$, then we can write $\mathscr{F}|_{V_0} = \mathrm{colim}\, i_m^{-1}\mathscr{F}_m$ where:*

1. *$U_0 = \cap_m U_{0,m}$ where $\{U_{0,m}\}_{m \in \mathbf{Z}_{\geq 0}}$ is a decreasing sequence of quasi-compact open affinoid subsets of $\mathcal{FL}$,*
2. *$V_{0,m} = \pi_{HT}^{-1}(U_{0,m})$ is a quasi-compact open subset of $\mathrm{Sh}^{\mathrm{tor}}_{K^p}$, stable under a compact open subgroup $K_{p,m} \subseteq G(\mathbf{Q}_p)$,*
3. *$i_m : V_0 \to V_{0,m}$ is the natural inclusion,*
4. *$\mathscr{F}_m$ is a sheaf over $V_{0,m}$,*
5. *$V_{0,m,K_{p,m}} \hookrightarrow \mathrm{Sh}^{\mathrm{tor}}_{K^pK_{p,m}}$ is a quasi-compact open which descends $V_{0,m}$ to finite level $K_{p,m}$, and $\pi_{K_{p,m}} : V_{0,m} \to V_{0,m,K_{p,m}}$ is the induced projection,*
6. *We have a sheaf $\mathscr{G}'_{m,K_{p,m}}$ on $V_{0,m,K_{p,m}}$ and $\mathscr{F}_m = \pi_{K_{p,m}}^{-1}\mathscr{G}_{m,K_{p,m}} \otimes_{\pi_{K_{p,m}}^{-1}\mathcal{O}_{V_{0,m,K_{p,m}}}} \mathcal{O}^{\mathrm{sm}}_{V_{0,m}}$.*
7. *The sheaves $\mathscr{G}'_{m,K_{p,m}}$ arise as the generic fibers of locally trivial, very small, formal Smith sheaves $\mathfrak{G}_{m,K_{p,m}}$ over $\mathfrak{Sh}^{\mathrm{tor},mod}_{K^pK_{p,m},\mathfrak{U}_{0,m}}$.*

Proof. The argument is almost identical to the proof of Lemma 4.6.42. We have that $\hat{M}^\vee(\lambda) = \mathrm{colim}_r M_r$ where each M_r is a Banach space representation of the group denoted $\mathrm{Stab}(w)_{r,s}$ in Definition 3.3.10. Since this is a LB-space of compact type, it also admits a presentation as a LS-space of compact type, $\hat{M}^\vee(\lambda) = \mathrm{colim}_r M_r'$ where each M_r' is a Smith space representation of $\mathrm{Stab}(w)_{r,s}$ (see [RJRC22, Cor. 3.38]). We then pick lattices $(M_r')^+$ in M_r' and glue the sheaves $\mathcal{O}_{\mathfrak{V}_i} \otimes_{\mathcal{O}_L} (M_r')^+$ to get the very small formal Smith sheaf $\mathfrak{G}'_{r,K_{p,r}}$. □

For any $K_p \subseteq K_{p,m}$, we have maps $\pi_{K_p,K_{p,m}} : \mathfrak{Sh}^{\mathrm{tor},mod}_{K^pK_p,\mathfrak{U}_{0,m}} \to \mathfrak{Sh}^{\mathrm{tor},mod}_{K^pK_{p,m},\mathfrak{U}_{0,m}}$ and we write $\mathfrak{G}_{m,K_p} := \pi^*_{K_p,K_{p,m}}\mathfrak{G}_{m,K_{p,m}}$ (a locally trivial, very small formal Banach sheaf) and $\mathfrak{G}'_{m,K_p} := \pi^*_{K_p,K_{p,m}}\mathfrak{G}'_{m,K_{p,m}}$ (a locally trivial, very small formal Smith sheaf). We denote their generic fibers by $\mathcal{G}_{m,K_p}$ and $\mathcal{G}'_{m,K_p}$.

Lemma 4.6.44.

(1) For any affinoid $U' \subseteq U_{0,m}$, any compact open subgroup $K_p \subseteq K_{p,m'}$ fixing U', and letting $V' = \pi_{HT}^{-1}(U')$, we have $H^i(V'_{K_p}, \mathscr{G}_{m,K_p}(-D)) = 0$ for all $i > 0$.

(2) For any Stein space $S' \subseteq U_{0,m}$ stable under a compact open subgroup $K_p \subseteq K_{p,m'}$, let $V'' = \pi_{HT}^{-1}(S')$. We have $H^i_c(V''_{K_p}, \mathscr{G}'_{m,K_p}) = 0$ for all $i \neq d$.

Proof. The first part follows as in lemma 6.6.2 of [BP21]. We briefly recall the argument. We take a formal model $\mathfrak{U}' \to \mathfrak{FL}$ of $\mathcal{U}' \to \mathcal{FL}$. By Lemma 4.6.42, we have a locally trivial, very small, formal Banach sheaf $\mathfrak{G}_{m,K_p}$ over $\mathfrak{Sh}^{\mathrm{tor}}_{K_pK^p,\mathfrak{U}'}$. Let $\pi : \mathfrak{Sh}^{\mathrm{tor}}_{K_pK^p,\mathfrak{U}'} \to \mathfrak{Sh}^{*}_{K_pK^p,\mathfrak{U}'}$. We have that $\mathrm{R}^i\pi_*\mathfrak{G}_{m,K_p}(-D) = 0$ for all $i > 0$. Indeed, using the very smallness, this reduces to the vanishing ([BP21, Thm. 4.4.37]) of $\mathrm{R}^i\pi_*\mathcal{O}_{\mathfrak{Sh}^{\mathrm{tor}}_{K_pK^p,\mathfrak{U}'}}(-D)$. We deduce that $\pi_*\mathfrak{G}_{m,K_p}(-D)$ is a very small formal Banach sheaf and thus Proposition 4.6.40 implies that $H^i(\mathfrak{Sh}^{\mathrm{tor}}_{K_pK^p,\mathfrak{U}'}, \mathfrak{G}_{m,K_p}(-D)) \otimes E = 0$ for all $i > 0$.

We claim that $H^i(\mathfrak{Sh}^{\mathrm{tor}}_{K_pK^p,\mathfrak{U}'}, \mathfrak{G}_{m,K_p}(-D)) \otimes E = H^i(V'_{K_p}, \mathscr{G}_{m,K_p}(-D))$. To see this, take a Zariski open affine cover $\{\mathfrak{W}_s\}_s$ of $\mathfrak{Sh}^{\mathrm{tor}}_{K_pK^p,\mathfrak{U}'}$ with the property that $\mathfrak{G}_{m,K_p}(-D)|_{\mathfrak{W}_s}$ is a trivial formal Banach sheaf. By Corollary 4.6.19 $H^i(V'_{K_p}, \mathscr{G}_{m,K_p}(-D))$ is computed by the Čech complex associated to the generic fiber of this cover, which computes $H^i(\mathfrak{Sh}^{\mathrm{tor}}_{K_pK^p,\mathfrak{U}'}, \mathfrak{G}_{m,K_p}(-D)) \otimes E$. We deduce that $H^i(V'_{K_p}, \mathscr{G}_{m,K_p}(-D)) = 0$ for all $i > 0$, as required.

The second part follows from a certain form of duality. Let us define the "Serre dual" of $\mathscr{G}'_{m,K_p}$,

$$D(\mathscr{G}'_{m,K_p}) = \underline{\mathrm{Hom}}_{\mathcal{O}_{V_{0,m,K_p}}}(\mathscr{G}'_{m,K_p}, \Omega^d_{V_{0,m,K_p}}),$$

by which we mean the following. We take a finite Stein covering $V''_{K_p} = \cup_i V''_{K_p,i}$ with the property that $\Omega^d_{V_{m,K_p}}|_{V''_{K_p,i}}$ is trivial and that $\mathscr{G}'_{m,K_p} = \mathcal{O}_{V''_{K_p,i}} \otimes M_i$ for a Smith space M_i. Then $D(\mathscr{G}'_{m,K_p})|_{V''_{K_p,i}} = \mathcal{O}_{V''_{K_p,i}} \otimes M_i^\vee$ where $M_i^\vee$ is a Banach space. We consider the Čech complex of Fréchet spaces:

$$C : \prod_i H^0(V''_{K_p,i}, D(\mathscr{G}'_{m,K_p})) \to \prod_{i,j} H^0(V''_{K_p,i} \cap V''_{K_p,i}, D(\mathscr{G}'_{m,K_p})) \to \cdots$$

as well as the Čech complex of LS-spaces:

$$D : H^d_c(\cap_i V''_{K_p,i}, \mathscr{G}'_{m,K_p}) \to \prod_j H^d_c(\cap_{i\neq j} V''_{K_p,i}, \mathscr{G}'_{m,K_p}) \to \cdots$$

The two complexes are termwise dual of each other by the duality theory (see Corollary 4.6.26). The complex D computes $\mathrm{R}\Gamma_c(V''_{K_p}, \mathscr{G}'_{m,K_p})$ and the complex C computes $\mathrm{R}\Gamma(V''_{K_p}, D(\mathscr{G}'_{m,K_p}))$ by Corollary 4.6.20 and 4.6.26. Thus, it suffices to prove that the complex C has cohomology concentrated in degree 0. Write $S' = \cup S'_r$ as a countable increasing union of affinoids, and let $V''_r = \pi_{HT}^{-1}(S'_r)$ and $V''_r = \cup_i V''_{r,i}$. It follows from Lemma 4.6.43 that $D(\mathscr{G}'_{m,K_p})$ admits a formal model which is a very small formal Banach sheaf over a formal model of V''_{r,K_p}. We deduce that $\mathrm{R}\Gamma(V''_{r,K_p}, D(\mathscr{G}'_{m,K_p}))$ is concentrated in degree 0 by the same argument as for the first point of the lemma. Thus, the Čech complex C_r of $D(\mathscr{G}'_{m,K_p})$ with respect to $\{V''_{r,i,K_p}\}$ has only cohomology in degree 0. We then use topological Mittag-Leffler to deduce that $C = \lim_r C_r$ is concentrated in degree 0. ☐

Theorem 4.6.45. *For any $M \in \mathcal{O}_{\lambda-\mathrm{alg}}$, $HC_{\mathrm{cusp},w,\lambda}(M)$ has amplitude $[0,\ell(w)]$ and $HC_{w,\lambda}(M)$ has amplitude $[\ell(w),d]$.*

Proof. We first prove the claim regarding $HC_{\mathrm{cusp},w,\lambda}(M)$. Let us denote $\mathscr{F} = VB^0(HCS_{w,\lambda}(M))(-D)$. Let us fix a closed neighborhood Z_0 of w in C_w and an open neighborhood U_0 of Z_0 in C_w as in definition 4.6.28. Let $t \in T^{++}(\mathbf{Q}_p)$. It is easy to see that $\cup_{n\in\mathbf{Z}_{\geq 0}} Z_0 t^n = C_w$. As above, we put $V_0 = \pi_{HT}^{-1}(U_0)$ and $F_0 = \pi_{HT}^{-1} Z_0$. By Lemma 4.6.29 (noting that $Z_0 t^n$ and Z_n are cofinal), we have that $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F}) = \mathrm{colim}_n \mathrm{R}\Gamma_{F_0 t^n}(V_0 t^n, \mathscr{F})$. It follows that each $H^i_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F})$ is generated as a $B(\mathbf{Q}_p)$-module by the image of $H^i_{F_0}(V_0, \mathscr{F})$ in $H^i_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathscr{F})$. Thus it suffices to see that $\mathrm{R}\Gamma_{F_0}(V_0, \mathscr{F})$ has amplitude $[0,\ell(w)]$.

We are free to chose n_0 in definition 4.6.28, and to make F_0 and V_0 arbitrarily small. By Lemma 4.6.42, we can assume that $\mathscr{F}|_{V_0} = \mathrm{colim}\, \mathscr{F}_m$ where $\mathscr{F}_m = \mathscr{G}_{m,K_{p,m}} \otimes_{\mathcal{O}_{V_{0,m,K_{p,m}}}} \mathcal{O}_{V_{0,m}}(-D)$ where $\mathscr{G}_{m,K_{p,m}}$ is a Banach sheaf admitting a locally trivial, very small, formal model over $V_{0,m,K_{p,m}}$. By Lemma 4.6.30, we have that $\mathrm{R}\Gamma_{F_0}(V_0, \mathscr{F}) = \mathrm{colim}_m \mathrm{R}\Gamma_{F_0}(V_0, \mathscr{F}_m)$ and $\mathrm{R}\Gamma_{F_0}(V_0, \mathscr{F}_m) = \mathrm{colim}_{s,n} \mathrm{R}\Gamma_{F_{0,s,K_{p,n}}}(V_{0,s,K_{p,n}}, \mathscr{G}_{m,K_{p,n}}(-D))$ where $K_{p,n}$ tends to $\{e\}$. Thus it suffices to prove that $\mathrm{R}\Gamma_{F_{0,s,K_{p,n}}}(V_{0,s,K_{p,n}}, \mathscr{G}_{m,K_{p,n}}(-D))$ has cohomological amplitude in $[0,\ell(w)]$. Recall from definition 4.6.28 that $F_{0,s} = \pi_{HT}^{-1}(Z_{0,s})$ and $V_{0,s} = \pi_{HT}^{-1}(U_{0,s})$ for s large enough (and a fixed choice of n_0 big enough). We observe that $U_{0,s}$ is affinoid while $U_{0,s} \setminus Z_{0,s}$ is covered by $\ell(w)$ affinoids. Using the triangle:

$$\mathrm{R}\Gamma_{F_{0,s,K_{p,n}}}(V_{0,s,K_{p,n}}, \mathscr{G}_{m,K_{p,n}}(-D)) \to \mathrm{R}\Gamma(V_{0,s,K_{p,n}}, \mathscr{G}_{m,K_{p,n}}(-D))$$
$$\to \mathrm{R}\Gamma(V_{0,s,K_{p,n}} \setminus F_{0,s,K_{p,n}}, \mathscr{G}_{m,K_{p,n}}(-D)) \stackrel{+1}{\to}$$

together with Lemma 4.6.44, we arrive at the desired conclusion.

We now turn to the case of usual cohomology, which follows along similar lines to the above, using the presentation via cohomology with compact support. Let us denote now $\mathscr{F} = VB^0(HCS_{w,\lambda}(M))$. We first see by lemma 4.6.32 and a similar argument using the $B(\mathbf{Q}_p)$-action, that it is enough to check that $\mathrm{R}\Gamma_c(Y_0, \mathscr{F})$ is concentrated in degrees $[\ell(w),d]$ where $Y_0 = \pi_{HT}^{-1} X_0$ and X_0 is a Stein open neighborhood of w in C_w (see Definition 4.6.31). Next, we use Lemma 4.6.43 to see that $\mathscr{F}|_{Y_0} = \mathrm{colim}\, \mathscr{F}'_m$ where $\mathscr{F}'_m = \mathscr{G}'_{m,K_{p,m}} \otimes_{\mathcal{O}_{Y_{0,m,K_{p,m}}}} \mathcal{O}_{Y_{0,m}}$ and $Y_{0,m} = \pi_{HT}^{-1}(X_{0,m})$ for $X_{0,m}$ a Stein neighborhood of X_0 in $\mathcal{FL}$. We deduce that $\mathrm{R}\Gamma_c(Y_0, \mathscr{F}) = \mathrm{colim}\, \mathrm{R}\Gamma_c(Y_0, \mathscr{F}'_m)$. Let $\mathscr{G}'_m = \mathrm{colim}_{K_p} \mathscr{G}'_{m,K_p}$ where $\mathscr{G}'_{m,K_p}$ is the pull back of $\mathscr{G}'_{m,K_{p,m}}$ to Y_{0,m,K_p} for $K_p \subseteq K_{p,m}$ and $\mathscr{G}'_m$ is viewed as a sheaf on $Y_{0,m}$. We deduce from lemma 4.6.33 that we have a triangle

$$\mathrm{R}\Gamma_c(Y_{0,m} \setminus Y_0, \mathscr{G}'_m) \to \mathrm{R}\Gamma_c(Y_{0,m}, \mathscr{G}'_m) \to \mathrm{R}\Gamma_c(Y_0, \mathscr{F}'_m) \stackrel{+1}{\to}.$$

Recall that $W_{0,m,r} = \pi_{HT}^{-1}(T_{0,m,r})$ (see definition 4.6.31). We have that $\mathrm{R}\Gamma_c(Y_{0,m}, \mathscr{G}'_m) = \mathrm{colim}_{K_p} \mathrm{R}\Gamma_c(Y_{0,m,K_p}, \mathscr{G}'_{m,K_p})$ is concentrated in degree d by Lemma 4.6.44. Similarly, $\mathrm{R}\Gamma_c(Y_{0,m} \setminus Y_0, \mathscr{G}'_m) = \mathrm{colim}_r\, \mathrm{colim}_{K_p} \mathrm{R}\Gamma_c(W_{0,m,r,K_p}, \mathscr{G}'_{m,K_p})$, and $\mathrm{R}\Gamma_c(W_{0,m,r,K_p}, \mathscr{G}'_{m,K_p})$ has cohomology concentrated in degrees $[\ell(w)+1,d]$ by Lemma 4.6.44 (as $T_{0,m,r}$ is the union of $\ell(w)$ Stein spaces). □

4.6.46. *Finite slope projector.* We follow the notation introduced in Section 1.8.5. Let $\mathcal{Z}$ be the character space of $T(\mathbf{Q}_p)$. Fixing an isomorphism $T(\mathbf{Q}_p) = \mathbf{Z}^r \times T(\mathbf{Z}_p)$ (with r the rank of T^d, the maximal split torus in T), we see that $\mathcal{Z} = \mathcal{W} \times (\mathbf{G}_m^{an})^r$ with $\mathcal{W} = \mathrm{Spa}(\mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!], \mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!]) \times_{\mathrm{Spa}(\mathbf{Z}_p)} \mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$.

We fix an affinoid increasing covering of the Stein space $\mathcal{Z} = \cup_n \mathcal{Z}_n$. As in [And21], we let $D(\mathcal{Z}_n)$ denote the category $D((\mathcal{O}_{\mathcal{Z}_n}, \mathcal{O}^+_{\mathcal{Z}_n})_\blacksquare)$, and we let $D(\mathcal{Z})$ be the derived category of quasi-coherent sheaves over $\mathcal{Z}$ (see [And21, Thm. 1.6]). We have defined in Section 1.8.5 the monoids $T^{++}(\mathbf{Q}_p) \subseteq T^+(\mathbf{Q}_p) \subseteq T(\mathbf{Q}_p)$. Note that $T^+(\mathbf{Q}_p) = T(\mathbf{Z}_p) \times \mathbf{Z}^s \times \mathbf{Z}^{r-s}_{\geq 0}$, where s is the rank of the maximal split torus in $Z(G)$. We let $(\mathbf{Q}_p)_\blacksquare[T^+(\mathbf{Q}_p)]$ be the solidification of $\mathbf{Q}_p[T^+(\mathbf{Q}_p)]$, and we let $(\mathbf{Q}_p)_\blacksquare[T(\mathbf{Q}_p)]$ the solidification of $\mathbf{Q}_p[T(\mathbf{Q}_p)]$. The categories of solid modules over these rings are denoted by $\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p)$ and $\mathrm{Mod}_{T(\mathbf{Q}_p)}(\mathbf{Q}_p)$ (this is consistent with definition 2.2.21). Their derived categories are denoted $D(\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p))$ and $D(\mathrm{Mod}_{T(\mathbf{Q}_p)}(\mathbf{Q}_p))$.

We define a functor $f_n^* : D(\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p)) \to D(\mathcal{Z}_n)$, $f_n^* M = M \otimes_{(\mathbf{Q}_p)_\blacksquare[T^+(\mathbf{Q}_p)]} (\mathcal{O}_{\mathcal{Z}_n}, \mathcal{O}^+_{\mathcal{Z}_n})_\blacksquare$. This functor has a right adjoint given by the forgetful functor $(f_n)_* : D(\mathcal{Z}_n) \to D(\mathrm{Mod}_{T(\mathbf{Q}_p)}(\mathbf{Q}_p))$. These functors induce adjoint functors $f^* : D(\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p)) \to D(\mathcal{Z})$ and $f_* : D(\mathcal{Z}) \to D(\mathrm{Mod}_{T(\mathbf{Q}_p)}(\mathbf{Q}_p))$.

We finally define the finite slope functor:

$$(-)^{fs} = f_* f^* : D(\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p)) \to D(\mathrm{Mod}_{T(\mathbf{Q}_p)}(\mathbf{Q}_p)).$$

The unit of adjunction gives a natural map $M \to M^{fs}$ in $D(\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p))$. Note that $M^{fs} = \lim_n (f_n)_* f_n^* M$.

Remark 4.6.47. The finite slope functor is a localization functor, which factors over the functor $- \otimes_{(\mathbf{Q}_p)_\blacksquare[T^+(\mathbf{Q}_p)]} (\mathbf{Q}_p)_\blacksquare[T(\mathbf{Q}_p)]$. However, it is a stronger form of localization. For example, let us (abusively!) consider the case that $T^+(\mathbf{Q}_p) = \mathbf{Z}_{\geq 0}$ and $T(\mathbf{Q}_p) = \mathbf{Z}$. In this case, $(\mathbf{Q}_p)_\blacksquare[T^+(\mathbf{Q}_p)] = \mathbf{Q}_p[X]$ and $(\mathbf{Q}_p)_\blacksquare[T(\mathbf{Q}_p)] = \mathbf{Q}_p[X, X^{-1}]$. We can also suppose that $\mathcal{Z}_n = \mathrm{Spa}(\mathbf{Q}_p\langle p^n X, p^n X^{-1}\rangle, \mathbf{Z}_p\langle p^n X, p^n X^{-1}\rangle)$. We claim that the module $\mathbf{Q}_p((X))$ is a solid $\mathbf{Q}_p[X, X^{-1}]$-module, whose finite slope part is trivial. By definition we need to show that $\mathbf{Q}_p((X)) \otimes^L_{\mathbf{Z}_p[X]} \mathbf{Z}_p\langle p^n X, p^n X^{-1}\rangle = 0$ for every n. To see this, we first show note that $\mathbf{Q}_p((X))$ is a solid $\mathbf{Q}_p\langle p^{-s} X\rangle$ for every $s \geq 0$. Indeed $\mathbf{Q}_p[X]/(X^\ell)$ is a solid $\mathbf{Q}_p\langle p^{-s}X\rangle$-module for any $\ell, s \in \mathbf{Z}_{\geq 0}$ and so $\mathbf{Q}_p[\![X]\!] = \lim_\ell \mathbf{Q}_p[X]/(X^\ell)$ is also a $\mathbf{Q}_p\langle p^{-s}X\rangle$-module, and thus $\mathbf{Q}_p((X))$ is a solid $\mathbf{Q}_p\langle p^{-s}X\rangle$-module, as claimed. It remains to observe that if $s \geq n$, then $\mathbf{Q}_p\langle p^{-s}X\rangle \otimes^L_{\mathbf{Q}_p[X]} \mathbf{Q}_p\langle p^n X, p^n X^{-1}\rangle = 0$. To see this, note that this is represented by the following complex:

$$[\mathbf{Q}_p\langle U, p^n X, p^n X^{-1}\rangle \overset{p^{s-n}Up^nX^{-1}-1}{\longrightarrow} \mathbf{Q}_p\langle U, p^n X, p^n X^{-1}\rangle].$$

But $1 - p^{s-n}U(p^n X^{-1})$ is invertible, with inverse $\sum_{\ell \geq 0} (p^{s-n}U(p^nX^{-1}))^\ell$.

Sometimes, one wants to consider not only the finite slope part, but to specify the slope. For any rank one point $\mathrm{Spa}(C, \mathcal{O}_C) \to \mathcal{Z}$ corresponding to a character $\chi : T(\mathbf{Q}_p) \to C^\times$, we define the slope of χ as follows. Let $v : C \to \mathbf{R} \cup \{+\infty\}$ be the valuation, normalized by $v(p) = 1$. Composing the valuation and χ we obtain a map $v(\chi) : T(\mathbf{Q}_p) \to \mathbf{R}$. On the other hand, via the exact sequence (1.8.6) we can think of $v(\chi)$ as an element of $X^*(T^d)_{\mathbf{R}}$ (see also [BP21, Sect. 5.9]). This defines a continuous "slope" map $s : \mathcal{Z} \to X^*(T^d)_{\mathbf{R}}$ (which factors through the Berkovich space of $\mathcal{Z}$).

Remark 4.6.48. If (again abusively) we consider the case $T(\mathbf{Q}_p) = \mathbf{Z}$, then $\mathcal{Z} = \mathbf{G}_m^{an}$, and $s : \mathbf{G}_m^{an} \to \mathbf{R}$ extends the p-adic valuation on classical rigid analytic points.

We have a partial order relation on $X^*(T^d)_{\mathbf{R}}$ where $\lambda \geq \lambda'$ if $\lambda(v(t)) - \lambda'(v(t)) \geq 0$ for any $t \in T^+(\mathbf{Q}_p)$. For any $\lambda \in X^*(T^d)_{\mathbf{Q}}$, we can consider the cone $\{\lambda' \in X^*(T^d)_{\mathbf{R}}, \lambda' \leq \lambda\}$. The subset $s^{-1}(\{\lambda' \in X^*(T^d)_{\mathbf{R}}, \lambda' \leq \lambda\})$ is the closure of a unique rational open that we denote by $\mathcal{Z}_{\leq \lambda} \stackrel{i_\lambda}{\to} \mathcal{Z}$. We have functors $f^*_{\leq \lambda} : D(\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p)) \to D(\mathcal{Z}_{\leq \lambda})$, $f^*_{\leq \lambda} M = i^*_\lambda f^* M$. We also have a forgetful functor $(f_{\leq \lambda})_* : D(\mathcal{Z}_{\leq \lambda}) \to D(\mathrm{Mod}_{T(\mathbf{Q}_p)}(\mathbf{Q}_p))$. We finally define the slope $\leq \lambda$ functor:

$$(-)^{\leq \lambda} = (f_{\leq \lambda})_* f^*_{\leq \lambda} : D(\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p)) \to D(\mathrm{Mod}_{T(\mathbf{Q}_p)}(\mathbf{Q}_p)).$$

If $M \in D(\mathrm{Mod}_{T^+(\mathbf{Q}_p)}(\mathbf{Q}_p))$, then we have $\lim_\lambda M^{\leq \lambda} = M^{fs}$. We leave it to the reader to define the slope λ functor $M \mapsto M^{=\lambda}$ and the slope $\geq \lambda$ functor, $M \mapsto M^{\geq \lambda}$.

Remark 4.6.49. Let us assume that M is a Banach space equipped with a compact endomorphism X. In other words, we are (again, abusively) in the situation $T^+(\mathbf{Q}_p) = \mathbf{Z}_{\geq 0}$ and $T(\mathbf{Q}_p) = \mathbf{Z}$, and $1 \in \mathbf{Z}_{\geq 0}$ acts like X. It follows from [Ser62, Prop. 12] that for any $\lambda \in \mathbf{Q}$, $M^{\leq \lambda}$ is a direct summand of M and is finite dimensional. In particular, $f^* M$ defines a coherent sheaf on $\mathcal{Z}$ and M^{fs} is a pro-finite vector space.

4.6.50. *Hecke algebra action.* Let K_U be a compact open subgroup of $U(\mathbf{Q}_p) \subseteq B(\mathbf{Q}_p)$ which admits an Iwahori factorization. Let $\lambda \in X^*(T)_{\mathbf{Q}_p}$. For any $M \in \mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)$, we consider the submodule M^{K_U} of K_U-invariants. It is canonically a direct summand, since one can define a trace $\mathrm{Tr}_{K_U} : M \to M^{K_U}$. Indeed, we have $M = \mathrm{colim}_{K'} M^{K'}$ where K' runs through the compact open subgroups of $U(\mathbf{Q}_p)$. Then for any $K'_U \subseteq K_U$ we can define a normalized trace

$$\mathrm{Tr}_{K'_U/K_U} = \frac{1}{[K_U : K'_U]} \left(\sum_{k \in K_U/K'_U} k \right) : M^{K'_U} \to M^{K_U},$$

and passing to the inductive limit over $\underline{K}_U$ yields the map Tr_{K_U}.

Let $t \in T^+(\mathbf{Q}_p)$. We define an action of t on M^{K_U} as follows:

$$M^{K_U} \stackrel{t}{\to} M^{tK_Ut^{-1}} \stackrel{\mathrm{Tr}_{tK_Ut^{-1}/K_U}}{\to} M^{K_U}$$

Lemma 4.6.51. *The above rule defines an action of the commutative monoid $T^+(\mathbf{Q}_p)$ on M^{K_U}.*

Proof. This follows from [Cas, Lem. 4.1.5]. □

We deduce that we have an exact functor

$$(-)^{K_U} : D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)) \to D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{T^+(\mathbf{Q}_p)}(E)).$$

Lemma 4.6.52. *Let $K'_U \subseteq K_U$. We have a natural transformation $(-)^{K'_U} \Rightarrow (-)^{K_U}$, induced by the trace.*

Proof. We consider the map $\mathrm{Tr}_{K'_U/K_U} : M^{K'_U} \to M^{K_U}$. It is elementary to check that this map commutes with the action of t. □

We can now consider the composite functor:

$$(-)^{K_U, fs} : D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)) \stackrel{(-)^{K_U}}{\to} D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{T^+(\mathbf{Q}_p)}(E)) \stackrel{(-)^{fs}}{\to} D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{T(\mathbf{Q}_p)}(E))$$

Lemma 4.6.53. *Let $K'_U \subseteq K_U$. The natural transformation $(-)^{K'_U,fs} \Rightarrow (-)^{K_U,fs}$ is an isomorphism.*

Proof. We pick $t \in T^{++}(\mathbf{Q}_p)$ such that $tK_Ut^{-1} \subseteq K'_U$. We have a commutative diagram (where the vertical maps are trace maps, and the diagonal map is the composite $M^{K_U} \xrightarrow{t} M^{tK_Ut^{-1}} \xrightarrow{\mathrm{Tr}_{tK_Ut^{-1}/K'_U}} M^{K_U}$):

$$\begin{array}{ccc} M^{K_U} & \xrightarrow{t} & M^{K_U} \\ \uparrow & \searrow & \uparrow \\ M^{K'_U} & \xrightarrow{t} & M^{K'_U} \end{array}$$

On the finite slope quotient, the horizontal maps t are isomorphisms, showing that the maps $M^{K'_U,fs} \to M^{K_U,fs}$ are isomorphisms. ☐

In view of this lemma, we often simply write M^{fs} instead of $M^{K_U,fs}$, as the choice of K_U is irrelevant.

4.6.54. *The finite slope part of the higher Coleman functor.* We define the finite slope part of the higher Coleman functor:

$$\begin{array}{rcl} HC^{fs}_{w,\lambda} : \mathcal{O}_{\lambda-\mathrm{alg}} & \to & D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{T(\mathbf{Q}_p)}(E)) \\ M & \mapsto & \mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, VB^0(HCS_{w,\lambda}(M)))^{fs} \end{array}$$

$$\begin{array}{rcl} HC^{fs}_{\mathrm{cusp},w,\lambda} : \mathcal{O}_{\lambda-\mathrm{alg}} & \to & D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{T(\mathbf{Q}_p)}(E)) \\ M & \mapsto & \mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, VB^0(HCS_{w,\lambda}(M)(-D)))^{fs} \end{array}$$

4.6.55. *Comparison with higher Coleman theory* [BP21]. Let $\kappa \in X^*(T)^{M,+}$ and $\chi : T(\mathbf{Z}_p) \to \overline{\mathbf{Q}}_p^\times$ be a finite order character. We have defined in [BP21, Sect. 5] cohomology theories $\mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{+,fs}$ and $\mathrm{R}\Gamma_w(K^p,\kappa,\chi,\mathrm{cusp})^{+,fs}$.

Theorem 4.6.56. *We have canonical isomorphisms of smooth $T(\mathbf{Q}_p)$-modules (where the decomposition on the right hand side corresponds to the decomposition into isotypic parts for the action of $T(\mathbf{Z}_p)$):*

$$\begin{array}{rcl} HC^{fs}_{\mathrm{cusp},0,w}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}) \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p & = & \oplus_{\chi:T(\mathbf{Z}_p)\to\overline{\mathbf{Q}}_p^\times} \mathrm{R}\Gamma_w(K^p,\kappa,\chi,\mathrm{cusp})^{+,fs} \\ HC^{fs}_{0,w}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}) \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p & = & \oplus_{\chi:T(\mathbf{Z}_p)\to\overline{\mathbf{Q}}_p^\times} \mathrm{R}\Gamma_w(K^p,\kappa,\chi)^{+,fs} \end{array}$$

Proof. By Lemma 4.6.5, we have $VB^0 \circ HCS_{w,0}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}) = \omega^{\kappa,sm}|_{\pi^{-1}_{HT}(C^\dagger_w)}$, so that by definition we have $HC_{w,0}(L(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}) = \mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\kappa,\mathrm{sm}})$. Let Z_0 be a closed neighborhood of w in C_w stable under a compact open subgroup $K_U \subseteq U(\mathbf{Q}_p)$ (admitting an Iwahori factorization) and let U_0 be an open neighborhood of Z_0 in C_w. We let $F_0 = \pi^{-1}_{HT}(Z_0)$ and $V_0 = \pi^{-1}_{HT}(U_0)$. We can define an action of $t \in T^+(\mathbf{Q}_p)$ on $\mathrm{R}\Gamma_{F_0}(V_0,\omega^{\kappa,\mathrm{sm}})$ as follows (where the second and third maps are respectively given by restriction and the trace):

$$\mathrm{R}\Gamma_{F_0}(V_0,\mathscr{F})^{K_U} \xrightarrow{t} \mathrm{R}\Gamma_{F_0t^{-1}}(V_0t^{-1},\omega^{\kappa,\mathrm{sm}})^{tK_Ut^{-1}} \to$$
$$\mathrm{R}\Gamma_{F_0}(V_0,\mathscr{F})^{tK_Ut^{-1}} \to \mathrm{R}\Gamma_{F_0}(V_0,\omega^{\kappa,\mathrm{sm}})^{K_U}.$$

The map $\mathrm{R}\Gamma_{F_0}(V_0, \omega^{\kappa,\mathrm{sm}}) \to \mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}})$ induces a $T^+(\mathbf{Q}_p)$-equivariant map $\mathrm{R}\Gamma_{F_0}(V_0, \omega^{\kappa,\mathrm{sm}})^{K_U} \to \mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}})^{K_U}$ which is a quasi-isomorphism on the finite slope part. Indeed, it is easy to see that we have factorizations:

$$\begin{array}{ccc} \mathrm{R}\Gamma_{F_0}(V_0, \omega^{\kappa,\mathrm{sm}})^{K_U} & \xrightarrow{t} & \mathrm{R}\Gamma_{F_0}(V_0, \omega^{\kappa,\mathrm{sm}})^{K_U} \\ \uparrow & \searrow & \uparrow \\ \mathrm{R}\Gamma_{F_0 t^{-1}}(V_0 t^{-1}, \omega^{\kappa,\mathrm{sm}})^{K_U} & \xrightarrow{t} & \mathrm{R}\Gamma_{F_0 t^{-1}}(V_0 t^{-1}, \omega^{\kappa,\mathrm{sm}})^{K_U} \end{array}$$

On the finite slope part, the maps t are quasi-isomorphisms, therefore the maps $\mathrm{R}\Gamma_{F_0 t^{-1}}(V_0 t^{-1}, \omega^{\kappa,\mathrm{sm}})^{K_U,fs} \to \mathrm{R}\Gamma_{F_0}(V_0, \omega^{\kappa,\mathrm{sm}})^{K_U,fs}$ are also quasi-isomorphisms. Now we can take $t \in T^{++}(\mathbf{Q}_p)$. Since $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}})^{K_U} = \mathrm{colim}_n \, \mathrm{R}\Gamma_{F_0 t^n}(V_0 t^{-n}, \mathscr{F})^{K_U,fs}$, the colimit is constant and we conclude.

We now consider compact open subgroups $K_{p,n}$ of $G(\mathbf{Q}_p)$ which admits an Iwahori decomposition $K_{p,n} = K_U \times K_{\bar{B},n}$ where $K_{\bar{B},n}$ is the principal level p^n congruence subgroup in $\bar{B}(\mathbf{Q}_p)$. We let $K'_{p,n}$ be the compact open subgroup of $G(\mathbf{Q}_p)$ which admit an Iwahori decomposition $K'_{p,n} = K_U \times T(\mathbf{Z}_p) \times K_{\bar{U},n}$ where $K_{\bar{U},n}$ is the principal level p^n congruence subgroup in $\bar{U}(\mathbf{Q}_p)$. Note that $K_{p,n} \subseteq K'_{p,n}$ is a normal subgroup and that $K'_{p,n}/K_{p,n} = T(\mathbf{Z}_p/p^n\mathbf{Z}_p)$.

We have that $\mathrm{R}\Gamma_{F_0}(V_0, \omega^{\kappa,\mathrm{sm}})^{K_U} = \mathrm{colim}_{m,n} \, \mathrm{R}\Gamma_{F_{0,m,K_{p,n}}}(V_{0,m,K_{p,n}}, \omega^{\kappa}_{K_{p,n}})$. Moreover,

$$\mathrm{R}\Gamma_{F_{0,m,K_{p,n}}}(V_{0,m,K_{p,n}}, \omega^{\kappa}_{K_{p,n}}) \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p = \oplus_{\chi : T(\mathbf{Z}/p^n\mathbf{Z}_p) \to \overline{\mathbf{Q}}_p^{\times}} \mathrm{R}\Gamma_{F_{0,m,K'_{p,n}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi)).$$

We claim that each $\mathrm{R}\Gamma_{F_{0,m,K'_{p,n}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi))$ can be equipped with an action of $T^+(\mathbf{Q}_p)$ and on the finite slope quotient the maps:

$$\mathrm{R}\Gamma_{F_{0,m,K'_{p,n}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi)) \to \mathrm{R}\Gamma_{F_{0,m,K'_{p,n'}}}(V_{0,m,K'_{p,n'}}, \omega^{\kappa}_{K'_{p,n}}(\chi))$$

for $n' \geq n$ are quasi-isomorphisms, and the maps:

$$\mathrm{R}\Gamma_{F_{0,m,K'_{p,n}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi)) \to \mathrm{R}\Gamma_{F_{0,m',K'_{p,n}}}(V_{0,m',K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi))$$

for $m' \geq m$ are quasi-isomorphism. This follows from the property that for $t \in T^{++}(\mathbf{Q}_p)$ sufficiently regular, we have factorizations:

$$\begin{array}{ccc} \mathrm{R}\Gamma_{F_{0,m,K'_{p,n'}}}(V_{0,m,K'_{p,n'}}, \omega^{\kappa}_{K'_{p,n'}}(\chi)) & \xrightarrow{t} & \mathrm{R}\Gamma_{F_{0,m,K'_{p,n'}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n'}}(\chi)) \\ \uparrow & \searrow & \uparrow \\ \mathrm{R}\Gamma_{F_{0,m,K'_{p,n}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi)) & \xrightarrow{t} & \mathrm{R}\Gamma_{F_{0,m,K'_{p,n}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi)) \end{array}$$

as well as factorizations:

$$\begin{array}{ccc} \mathrm{R}\Gamma_{F_{0,m',K'_{p,n}}}(V_{0,m',K'_{p,n'}}, \omega^{\kappa}_{K'_{p,n}}(\chi)) & \xrightarrow{t} & \mathrm{R}\Gamma_{F_{0,m',K'_{p,n}}}(V_{0,m',K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi)) \\ \uparrow & \searrow & \uparrow \\ \mathrm{R}\Gamma_{F_{0,m,K'_{p,n}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi)) & \xrightarrow{t} & \mathrm{R}\Gamma_{F_{0,m,K'_{p,n}}}(V_{0,m,K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi)) \end{array}$$

We conclude, since by definition (see just after Theorem 5.4.14 in [BP21]) we have $\mathrm{R}\Gamma_{F_{0,m',K'_{p,n}}}(V_{0,m',K'_{p,n}}, \omega^{\kappa}_{K'_{p,n}}(\chi))^{fs} = \mathrm{R}\Gamma_w(K^p, \kappa, \chi)^{+,fs}$. □

Let $\nu : T(\mathbf{Z}_p) \to \mathbf{C}_p^\times$ be an analytic character. [BP21, Sect. 5] We have cohomology theories $\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{+,fs}$ and $\mathrm{R}\Gamma_{w,an}(K^p, \nu, \mathrm{cusp})^{+,fs}$, defined in [BP21, Sect. 6.4] (see just after Theorem 6.4.10 there). Let $d\nu \in X^*(T)_{\mathbf{C}_p}$ be the differential of ν. Define $\kappa \in X^*(T)_{\mathbf{C}_p}$ by the formula $d\nu = -w^{-1}w_{0,M}(\kappa + \rho) - \rho$.

Theorem 4.6.57. *We have canonical isomorphisms of $T(\mathbf{Q}_p)$-modules (where the decomposition on the right hand side corresponds to the decomposition into isotypic parts for the action of $T(\mathbf{Z}_p)$):*

$$\begin{aligned} HC^{fs}_{\mathrm{cusp},d\nu,w}(M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}) &= \oplus_{\nu' : T(\mathbf{Z}_p)\to\mathbf{C}_p^\times,\ d\nu'=d\nu} \mathrm{R}\Gamma_w(K^p, \nu', \mathrm{cusp})^{+,fs} \\ HC^{fs}_{d\nu,w}(M(\mathfrak{m}_w)_{-w^{-1}w_{0,M}\kappa}) &= \oplus_{\nu' : T(\mathbf{Z}_p)\to\mathbf{C}_p^\times,\ d\nu'=d\nu} \mathrm{R}\Gamma_w(K^p, \nu')^{+,fs} \end{aligned}$$

Proof. This is similar to the proof of Theorem 4.6.56 and left to the reader. □

We have the following theorem which slightly generalizes [BP21, Thm. 1.2.1, Thm. 6.7.3].

Theorem 4.6.58. *The functor $HC^{fs}_{w,\lambda}$ has cohomological amplitude $[\ell(w), d]$, and $HC^{fs}_{\mathrm{cusp},w,\lambda}$ has cohomological amplitude $[0, \ell(w)]$.*

Proof. Given Theorem 4.6.45, we simply need to see that the functor $(-)^{fs}$ is exact on higher Coleman theories. But one sees (see Theorem 4.6.56 and its proof) that the finite slope part is obtained by taking the finite slope part of an inductive system of complexes of Banach spaces acted on by a compact operator, and $(-)^{fs}$ is exact in this case, see [BP21, Proposition 5.1.4]. We remark that we could also deduce this theorem directly from [BP21, Thm. 1.2.1, Thm. 6.7.3] (which is the current theorem in the case of $HC^{fs}_{w,\lambda}$ and $HC^{fs}_{\mathrm{cusp},w,\lambda}$ applied to Vermas) by using a diagram chase similar to the proof of Theorem 2.3.32. □

4.6.59. *Bounds on slopes for higher Coleman theory.* Using [BP21] and [BP23], we can obtain bounds for the slopes.

Theorem 4.6.60. *Assume that either the Shimura variety is proper or that we are in the Siegel case. Let $w \in {}^MW$. Let $M \in \mathcal{O}(\mathfrak{m}_w, \mathfrak{b}_{M_w})_{\lambda-alg}$ be a module generated by a highest weight vector of weight ν. Then the slopes of $HC^{fs}_{\mathrm{cusp},w,\lambda}(M)$ and $HC^{fs}_{w,\lambda}(M)$ are $\geq \lambda - \nu + w^{-1}w_{0,M}\rho + \rho$.*

Remark 4.6.61. The finite slope projector and the notion of slope $\geq \lambda - \nu + w^{-1}w_{0,M}\rho + \rho$ were explained in section 4.6.46.

Remark 4.6.62. We conjecture ([BP21, Conj. 6.8.1]) that the theorem should hold for any Shimura variety. The slightly weaker bound $\geq \lambda - \nu$ is currently available in full generality by [BP21, Thm. 6.8.3].

Proof of Theorem 4.6.60. For $M = M(\mathfrak{m}_w)_\nu$ a Verma module with highest weight ν, this follows from [BP21, Thm. 6.10.1] for the proper case, and [BP23, Cor. 6.2.16] in the Siegel case, together with Theorem 4.6.57 (the paper [BP23] shows that the strongly small slope condition which is sometimes needed in [BP21] can be weakened to the small slope condition in the symplectic case). Since any module M as in the statement of the theorem admits a resolution by Verma modules $M(\mathfrak{m}_w)_{\nu'}$ with $\nu \geq \nu'$, we are done. □

4.7. **p-adic Eichler Shimura theory.** Let $\lambda \in X^*(T)_E$. We define functors:

$$\begin{array}{rcl} CC_\lambda : \mathcal{O}(\mathfrak{g},\mathfrak{b})_{\lambda-alg} & \to & D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{g}}(M, \mathrm{R}\Gamma(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}) \end{array}$$

$$\begin{array}{rcl} CC_{\mathrm{cusp},\lambda} : \mathcal{O}(\mathfrak{g},\mathfrak{b})_{\lambda-alg} & \to & D(\mathrm{Mod}^{\lambda-\mathrm{sm}}_{B(\mathbf{Q}_p)}(E)) \\ M & \mapsto & \mathrm{RHom}_{\mathfrak{g}}(M, \mathrm{R}\Gamma_c(\mathrm{Sh}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}). \end{array}$$

Theorem 4.7.1. *For any $M \in \mathcal{O}(\mathfrak{g},\mathfrak{b})_{\lambda-alg}$, we have*

$$CC_\lambda(M) \otimes \mathbf{C}_p = \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, VB^{\mathrm{red}}(\mathrm{Loc}(M)))$$

and

$$CC_{\mathrm{cusp},\lambda}(M) \otimes \mathbf{C}_p = \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, VB^{\mathrm{red}}(\mathrm{Loc}(M)) \otimes_{\mathscr{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}} \mathscr{I}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$$

Moreover, the action of $\mu \in Z(\mathfrak{m})$ on $\mathrm{Loc}(M)$ via the horizontal action induces an arithmetic Sen operator on the left hand side.

Proof. By Theorem 4.4.1, we have $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}} \otimes_{\mathbf{Q}_p} \mathbf{C}_p = \mathrm{R}\Gamma_{\mathrm{an}}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$. We consider the category $\mathrm{Mod}'_{\mathfrak{g}}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$ of sheaves of solid $\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}$-modules equipped with an action of $\mathfrak{g}$, and the similarly-defined category $\mathrm{Mod}'_{\mathfrak{g}}(\mathbf{C}_p)$.. These are abelian categories, with enough injectives. We consider the diagram of functors:

$$\begin{array}{ccc} \mathrm{Mod}'_{\mathfrak{g}}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}) & \longrightarrow & \mathrm{Mod}(\mathcal{O}^{\mathrm{sm}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}'_{\mathfrak{g}}(\mathbf{C}_p) & \longrightarrow & \mathrm{Mod}(\mathbf{C}_p) \end{array}$$

where the horizontal arrows are given by taking $\mathfrak{g}$-invariants and the vertical arrows are given by taking global sections. This diagram is 2-commutative, and it induces a 2-commutative diagram at the level of bounded derived categories. We deduce that $CC_\lambda(M) \otimes \mathbf{C}_p = \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathrm{RHom}_{\mathfrak{g}}(M, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}))$. We therefore need to show that $VB^{\mathrm{red}}(\mathrm{Loc}(M)) = \mathrm{RHom}_{\mathfrak{g}}(M, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$. To this end, the Chevalley–Eilenberg resolution yields

$$\mathrm{RHom}_{\mathfrak{g}}(M, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}) = [M^\vee \otimes \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \to M^\vee \otimes \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \otimes \mathfrak{g}^\vee \to \cdots \to M^\vee \otimes \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \otimes \Lambda^r \mathfrak{g}^\vee]$$

in degrees 0 up to $r = \dim \mathfrak{g}$. By Theorem 4.5.20(4) (and the flatness of $M^\vee \otimes \Lambda^i \mathfrak{g}^\vee$), we have

$$VB^0(M^\vee \otimes \mathcal{C}^{\mathrm{la}} \otimes \Lambda^i \mathfrak{g}^\vee) = M^\vee \otimes \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \otimes \Lambda^i \mathfrak{g}^\vee,$$

and we deduce (see Remark 4.5.18) that $VB^{\mathrm{red}}(\mathrm{Loc}(M)) = \mathrm{RHom}_{\mathfrak{g}}(M, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$. The cuspidal case is identical. The final claim regarding the Sen operator follows from Theorem 4.5.23. □

Theorem 4.7.2. *Assume that λ is non-Liouville. For any $M \in \mathcal{O}(\mathfrak{g},\mathfrak{b})_{\lambda-\mathrm{alg}}$, we have that $H^i(VB^{\mathrm{red}}(\mathrm{Loc}(M))) = VB^0(H^i(\mathrm{Loc}(M))$. We have a spectral sequence:*

$$E_1^{p,q} = \oplus_{w \in {}^M W, \ell(w) = p} H^{p+q}(HC_{w,\lambda}(E \otimes^L_{\mathfrak{u}_{P_w}} M))$$

converging to $H^{p+q}(CC_\lambda(M)) \otimes \mathbf{C}_p$. Similarly, we have a spectral sequence:

$$E_1^{p,q} = \oplus_{w \in {}^M W, \ell(w) = p} H^{p+q}(HC_{\mathrm{cusp},w,\lambda}(E \otimes^L_{\mathfrak{u}_{P_w}} M))$$

converging to $H^{p+q}(CC_{\mathrm{cusp},\lambda}(M)) \otimes \mathbf{C}_p$*. In all cases* $w\mu \in Z(\mathfrak{m}_w)$ *acting on* $H_*(\mathfrak{u}_{P_w}, M)$ *induces an arithmetic Sen operator.*

Proof. To prove the first part, it suffices to show that the cohomology sheaves $H^i(\mathrm{Loc}(M))$ is acyclic for the functor VB^{red}. We can check this acyclicity locally, and in particular after restricting to Bruhat strata, where it follows from Theorems 3.5.11 and 4.5.20(5). Finally, the spectral sequence is then a simple consequence of Proposition 4.6.13 together with Theorems 4.7.1 and 3.5.11. □

Let $\lambda \in X^*(T)_E$ be a non-Liouville weight. We apply the above theorem to the Verma module $M(\mathfrak{g})_\lambda$, giving the following results.

Corollary 4.7.3. *Assume that* λ *is antidominant in the sense of Remark 3.5.21, and that the Shimura variety is proper. Then* $CC_\lambda(M(\mathfrak{g})_\lambda)$ *is concentrated in the middle degree* d *and moreover, it has a decreasing filtration* $\mathrm{Fil}^i\mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda))$ *with*

- $\mathrm{Fil}^{d+1}\mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda)) = 0$,
- $\mathrm{Fil}^0\mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda)) = \mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda))$,
- $\mathrm{Gr}^p\mathrm{H}^d(CC_\lambda(M(\mathfrak{g})_\lambda)) = \oplus_{w\in {}^MW,\ell(w)=p} H^p(HC_{w,\lambda}(M(\mathfrak{m}_w)_{\lambda+w^{-1}w_{0,M}\rho+\rho}))$.

Proof. This follows from Theorem 4.7.2, because the spectral sequence degenerates by a combination of Corollary 3.5.20 (noting Remark 3.5.21) and Theorem 4.6.45 (noting that since the Shimura variety is proper, $HC_{\mathrm{cusp},w,\lambda} = HC_{w,\lambda}$). □

Remark 4.7.4. We see that in the antidominant case, the highest weights appearing in the p-adic Eichler–Shimura decomposition follow the exact same pattern as the highest weights appearing in the classical Eichler–Shimura theory.

We now consider the general case where λ need not be antidominant, and we take the "ordinary" part.

Theorem 4.7.5. *Assume that we are in the Siegel case or that the Shimura variety is proper. Let* $\lambda \in X^*(T)_E$ *be a non-Liouville weight. We have that* $CC^{fs}_\lambda(M(\mathfrak{g})_\lambda)$ *and* $CC^{fs}_{\mathrm{cusp},\lambda}(M(\mathfrak{g})_\lambda)$ *have slope* ≥ 0*. Moreover, we have a spectral sequence:*

$$E_1^{p,q} = \oplus_{w\in {}^MW,\ell(w)=p} H^{2p+q-d}(HC^{=0}_{w,\lambda}(M(\mathfrak{m}_w)_{\lambda+w^{-1}w_{0,M}\rho+\rho}))$$

converging to $H^{p+q}(CC^{=0}_\lambda(M(\mathfrak{g})_\lambda)) \otimes \mathbf{C}_p$*, and similarly for cuspidal cohomology.*

Proof. This is a combination of Theorem 4.7.2, Corollary 3.5.20 and Theorem 4.6.60. □

Finally we have the following corollary.

Corollary 4.7.6. *Assume that the Shimura variety is proper. Let* $\lambda \in X^*(T)_E$ *be a non-Liouville weight. Then* $CC^{=0}_\lambda(M(\mathfrak{g})_\lambda))$ *is concentrated in the middle degree* d *and moreover, it has a decreasing filtration* $\mathrm{Fil}^i\mathrm{H}^d(CC^{=0}_\lambda(M(\mathfrak{g})_\lambda))$ *with*

- $\mathrm{Fil}^{d+1}\mathrm{H}^d(CC^{=0}_\lambda(M(\mathfrak{g})_\lambda)) = 0$,
- $\mathrm{Fil}^0\mathrm{H}^d(CC^{=0}_\lambda(M(\mathfrak{g})_\lambda)) = \mathrm{H}^d(CC^{=0}_\lambda(M(\mathfrak{g})_\lambda))$,
- $\mathrm{Gr}^p\mathrm{H}^d(CC^{=0}_\lambda(M(\mathfrak{g})_\lambda)) = \oplus_{w\in {}^MW,\ell(w)=p} H^p(HC^{=0}_{w,\lambda}(M(\mathfrak{m}_w)_{\lambda+w^{-1}w_{0,M}\rho+\rho}))$.

Proof. This is a consequence of Theorem 4.7.5 and Theorem 4.6.57. □

Remark 4.7.7. Thus, we see that on the ordinary part the highest weights appearing in the p-adic Eichler–Shimura decomposition follow the same pattern as the highest weights appearing in the classical Eichler–Shimura theory.

4.8. **The classical Hodge–Tate decomposition for** GSp_4**.** We now specialize the theory to the group GSp_4 and first review the classical Hodge–Tate decomposition. From now on we use the notation for GSp_4 introduced in Section 1.8.8. Let $\kappa = (k_1, k_2; w) \in X^*(T)^+$ be a dominant weight for GSp_4. The dominance condition is $0 \geq k_1 \geq k_2$. We let V_κ be the corresponding highest weight representation. We let $\mathcal{V}^\vee_{\kappa,K_p}$ be the pro-Kummer étale local system on $\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}$, attached to $V^\vee_\kappa$.

The coherent weights appearing in the Hodge–Tate decomposition of the local system $\mathcal{V}^\vee_{\kappa,K_p}$ are the $\{-w_{0,M}(w.\kappa + 2\rho^M)\}$ where $w \in {}^MW$ (see [FC90], Thm. 6.2). We make this explicit: the set $\{-w_{0,M}(w.\kappa + 2\rho^M)\}$ consists exactly of $(3-k_2, 3-k_1; -w), (3-k_2, k_1+1; -w), (2-k_1, k_2; -w), (k_1, k_2; -w)$. We recall that our conventions are that the cyclotomic character $\mathbf{Q}_p(1)$ has Hodge–Tate weight -1 and that the Sen operator acts via 1 on the Sen module of $\mathbf{Q}_p(1)$, so that the (generalized) Hodge–Tate weights are the negatives of the eigenvalues of the Sen operator. Given our choice of parabolic P_μ, we have $\mu = (-1/2, -1/2; 1/2) \in X_*(T)_E$. By Theorem 4.5.20, μ is an arithmetic Sen operator.

Remark 4.8.1 (reality check)**.** This is consistent with the fact that the tautological exact sequence over $\mathcal{FL}$ is

$$0 \to \mathcal{L}_{(0,-1;1)} \to St \otimes \mathcal{O}_{\mathcal{FL}} \to \mathcal{L}_{(1,0;1)} \to 0$$

which pulls back to

$$0 \to Lie(A)_{K_p}(1) \otimes_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}}} \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \to T_pA \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \to (\omega_{A^t})_{K_p} \otimes_{\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}}} \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \to 0.$$

We see that $Lie(A)$ has Sen weight

$$1 = \langle (0, -1; 1), (-1/2, -1/2; 1/2) \rangle$$

(and Hodge–Tate weight -1), while w_{A^t} has weight $0 = \langle (1, 0; 1), (-1/2, -1/2; 1/2) \rangle$.

The Hodge–Tate weight attached to the sheaf $\omega^{(l_1,l_2;w)}$ is $\frac{l_1+l_2-w}{2}$. Thus, in the Hodge–Tate decomposition of the cohomology of $V^\vee_\kappa$, the Hodge–Tate weights are given by the formula: $(k_1, k_2; -w) \mapsto \frac{k_1+k_2+w}{2}$, $(2 - k_1, k_2; -w) \mapsto \frac{2-k_1+k_2+w}{2}$, $(3 - k_2, k_1 + 1; -w) \mapsto \frac{4-k_2+k_1+w}{2}$, $(3 - k_2, 3 - k_1; -w) \mapsto \frac{6-k_1-k_2+w}{2}$.

Theorem 4.8.2 ([FC90], Thm. 6.2)**.** *We have the following $G_{\mathbf{Q}_p} \times \mathbf{T}_{K_pK^p}$-equivariant isomorphisms:*

$$\mathrm{H}^i(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \mathcal{V}^\vee_{\kappa,K_p}) \otimes_{\mathbf{Q}_p} \mathbf{C}_p =$$

$$H^i(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^{(k_1,k_2;-w)}_{K_p})(\frac{-k_1 - k_2 - w}{2}) \oplus H^{i-1}(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^{(2-k_1,k_2;-w)}_{K_p})(\frac{-2 + k_1 - k_2 - w}{2})$$

$$\oplus H^{i-2}(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^{(3-k_2,k_1+1;-w)}_{K_p})(\frac{-4 + k_2 - k_1 - w}{2}) \oplus H^{i-3}(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^{(3-k_2,3-k_1;-w)}_{K_p})(\frac{-6 + k_2 + k_1 - w}{2})$$

$$\mathrm{H}^i_c(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \mathcal{V}^\vee_{\kappa,K_p}) \otimes_{\mathbf{Q}_p} \mathbf{C}_p =$$

$$H^i(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^{(k_1,k_2;-w)}_{K_p}(-D_{K_p}))(\frac{-k_1 - k_2 - w}{2}) \oplus$$

$$H^{i-1}(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^{(2-k_1,k_2;-w)}_{K_p}(-D_{K_p}))(\frac{-2 + k_1 - k_2 - w}{2}) \oplus$$

$$H^{i-2}(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^{(3-k_2,k_1+1;-w)}_{K_p}(-D_{K_p}))(\frac{-4 + k_2 - k_1 - w}{2}) \oplus$$

$$H^{i-3}(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \omega^{(3-k_2,3-k_1;-w)}_{K_p}(-D_{K_p}))(\frac{-6 + k_2 + k_1 - w}{2})$$

We can also state a similar result using completed cohomology. We first recall the following theorem:

Theorem 4.8.3. *Let V_κ be a finite dimensional representation of G of highest weight κ. Then*

$$\begin{aligned} CC_0(V_\kappa) = \mathrm{RHom}_{\mathfrak{g}}(V_\kappa, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}) &= \mathrm{colim}_{K_p}\, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \mathcal{V}^\vee_{\kappa,K_p}) \\ CC_{\mathrm{cusp},0}(V_\kappa) = \mathrm{RHom}_{\mathfrak{g}}(V_\kappa, \mathrm{R}\Gamma_c(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}) &= \mathrm{colim}_{K_p}\, \mathrm{R}\Gamma_c(\mathrm{Sh}^{\mathrm{tor}}_{K_pK^p}, \mathcal{V}^\vee_{\kappa,K_p}) \end{aligned}$$

Proof. See e.g. [Eme06, Cor. 2.2.18]. □

We deduce the following:

Corollary 4.8.4. *We have that*

$$\mathrm{H}^i(\mathrm{RHom}_{\mathfrak{g}}(V_\kappa, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}}))\otimes \mathbf{C}_p =$$

$$H^i(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(k_1,k_2;-w),\mathrm{sm}})(\frac{-k_1-k_2-w}{2}) \oplus H^{i-1}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2-k_1,k_2;-w),\mathrm{sm}})(\frac{-2+k_1-k_2-w}{2})$$

$$\oplus H^{i-2}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(3-k_2,k_1+1;-w),\mathrm{sm}})(\frac{-4+k_2-k_1-w}{2}) \oplus H^{i-3}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(3-k_2,3-k_1;-w),\mathrm{sm}})(\frac{-6+k_2+k_1-w}{2})$$

There is a similar statement for compactly supported cohomology and the cuspidal coherent cohomology.

Proof. This is simply obtained by passing to the limit over K_p in Theorem 4.8.2. But alternatively, this is a combination of Theorem 4.7.1, Lemma 4.6.5 and Proposition 3.5.17. □

4.9. **The p-adic Eichler–Shimura theory for** GSp_4**.** We continue to assume that $G = \mathrm{GSp}_4$, and let $\lambda \in X^*(T)_E$. We are ultimately interested in the case that $\lambda = (1,1;w)$. We will now specialize the results of Section 4.7, in order to compute the Hodge–Tate structure of the ordinary part of $CC_\lambda(M(\mathfrak{g})_\lambda) = \mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}})$. We will use some more suggestive notation for the higher Coleman sheaves now that we specialize to GSp_4. Let us briefly summarize who are the main players.

- For all $\kappa \in X^*(T)^{M,+}$, we have the classical modular sheaves $\omega^{\kappa,\mathrm{sm}}$, computing classical cohomology.
- Following Definition 4.6.6, we have the "big" sheaves $\omega^{\dagger,\kappa}_w$ on $\pi^{-1}_{HT}(C^\dagger_w)$ for all $\kappa \in X^*(T)_{\mathbf{C}_p}$ and all $w \in {}^MW$. If $\kappa \in X^*(T)^{M,+}$ we have maps $\omega^{\kappa,\mathrm{sm}}|_{\pi^{-1}_{HT}(C^\dagger_w)} \to \omega^{\dagger,\kappa}_w \otimes E(-w^{-1}w_{0,M}\kappa)$ by Proposition 4.6.9 (where the twist is a twist of the $B(\mathbf{Q}_p)$-action).
- We have higher Coleman theories $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w)$ for the big sheaves and $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,sm})$ for the classical sheaves.

The superscript $(-)^{\mathrm{ord}}$ means the ordinary part, which is the minimal slope part; we caution the reader that precisely what the "minimal slope" part occasionally depends on the context, but will always be spelled out. On $CC_\lambda(M(\mathfrak{g})_\lambda)$, the ordinary part is the slope $= 0$ part by Theorem 4.7.5. Theorem 4.7.5 specializes as follows.

Theorem 4.9.1. *There is a spectral sequence:*

$$E_1^{p,q} : \left(H^{2p+q-d}_{{}^pw}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,-w_{0,M}({}^pw\cdot\lambda+2\rho^M)}_{{}^pw}) \otimes \mathbf{C}_p(-{}^pw^{-1}w_{0,M}\rho-\rho)\right)^{\mathrm{ord}} \Rightarrow$$
$$H^{p+q}(\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}})^{\mathrm{ord}} \otimes \mathbf{C}_p)$$

and similarly:

$$E_1^{p,q} : \left(H^{2p+q-d}_{{}^p w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger, -w_{0,M}({}^p w\cdot\lambda + 2\rho^M)}_{{}^p w}(-D)) \otimes \mathbf{C}_p(-{}^p w^{-1} w_{0,M}\rho - \rho)\right)^{\mathrm{ord}} \Rightarrow$$
$$H^{p+q}(\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma_c(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}})^{\mathrm{ord}} \otimes \mathbf{C}_p).$$

If the Shimura variety were proper, we could use Corollary 4.7.6 to simplify the spectral sequence. In our case we will arrive to a similar conclusion after making a non-Eisenstein localization. We can give a first analysis of the spectral sequence with the help of some vanishing theorems. The following lemma comes as a complement to Theorem 4.6.58.

Lemma 4.9.2.

(1) *For all* $\kappa \in X^*(T)_E$*, we have that*
$$H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w(-D))^{fs} = H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w)^{fs} = 0$$
for all $w \neq \mathrm{Id}$ *and*
$$H^3_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w(-D))^{fs} = H^3_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w)^{fs} = 0$$
for all $w \neq w_0^M$*.*

(2) *For all* $\kappa \in X^*(T)^{M,+}$*, we have that*
$$H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}}(-D))^{fs} = H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}})^{fs} = 0$$
for all $w \neq \mathrm{Id}$ *and*
$$H^3_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}}(-D))^{fs} = H^3_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}})^{fs} = 0$$
for all $w \neq w_0^M$*.*

Proof. The statements regarding H^0 and H^3 are equivalent under Serre duality ([BP21, Thm. 6.7.2]). The vanishing of $H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w)^{fs}$ for $w \neq \mathrm{Id}$ follows from Theorem 4.6.58. The injective map of sheaves $\omega^{\dagger,\kappa}_w(-D) \to \omega^{\dagger,\kappa}_w$ induces an injective map $H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w(-D)) \to H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w)$ which implies the vanishing of $H^0_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w(-D))^{fs}$ for $w \neq \mathrm{Id}$. One argues similarly for the sheaf $\omega^{\kappa,\mathrm{sm}}$. ☐

Proposition 4.9.3.

(1) $\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma_c(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}})^{\mathrm{ord}}$ *is supported in degrees in the range* $[1, 3]$. *Moreover (with obvious notation) the graded pieces for the Hodge–Tate decompositions are:*
   (a) $H^3 \otimes \mathbf{C}_p : \mathrm{coker}(H^1_{{}^2 w}(-D) \to H^3_{{}^3 w}(-D)), H^2_{{}^2 w}(-D), H^1_{{}^1 w}(-D), H^0_{{}^0 w}(-D)$.
   (b) $H^2 \otimes \mathbf{C}_p : H^2_{{}^3 w}(-D), \mathrm{Ker}(H^1_{{}^2 w}(-D) \to H^3_{{}^3 w}(-D))$.
   (c) $H^1 \otimes \mathbf{C}_p : H^1_{{}^3 w}(-D)$.

(2) $\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \mathbf{Q}_p)^{\mathrm{la}})^{\mathrm{ord}}$ *is supported in degrees in the range* $[3, 5]$. *Moreover the graded for the Hodge–Tate decompositions are:*
   (a) $H^3 \otimes \mathbf{C}_p : H^3_{{}^3 w}, H^2_{{}^2 w}, H^1_{{}^1 w}, \mathrm{Ker}(H^0_{{}^0 w} \to H^2_{{}^1 w})$.
   (b) $H^4 \otimes \mathbf{C}_p : \mathrm{Coker}(H^0_{{}^0 w} \to H^2_{{}^1 w}), H^1_{{}^0 w}$.
   (c) $H^5 \otimes \mathbf{C}_p : H^2_{{}^0 w}$.

Proof. From Theorem 4.6.58 and Lemma 4.9.2, we have that the cohomology $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w)^{fs}$ is supported in the range:

- $[0, 2]$ for $w = {}^0 w$,
- $[1, 2]$ for $w = {}^1 w$,
- $[2]$ for $w = {}^2 w$,

- [3] for $w = {}^3w$.

On the other hand, the cohomology $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w(-D))^{fs}$ is supported in the range:

- [0] for $w = {}^0w$,
- [1] for $w = {}^1w$,
- $[1,2]$ for $w = {}^2w$,
- $[1,3]$ for $w = {}^3w$.

It follows that the spectral sequences of Theorem 4.9.1 degenerate on the second page. □

We now fix an irreducible residual representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{F}}_p)$. We define a maximal ideal $\mathfrak{m}_{\bar{\rho}}$ of the abstract spherical Hecke algebra of level prime to $S \cup \{p\}$ (where S is the set of primes at which $\overline{\rho}$ is ramified or K^p is not hyperspecial) by the formula:

$$P_\ell(X) \mod \mathfrak{m}_{\bar{\rho}} = \det(X - \bar{\rho}(\mathrm{Frob}_\ell)),\ \ell \notin S \cup \{p\}$$

where $P_\ell(X)$ is the Hecke polynomial defined in (1.8.27).

Theorem 4.9.4. *The map* $\mathrm{R}\Gamma_c(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \overline{\mathbf{Q}}_p)_{\mathfrak{m}_{\bar{\rho}}} \to \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \overline{\mathbf{Q}}_p)_{\mathfrak{m}_{\bar{\rho}}}$ *is a quasi-isomorphism.*

Proof. Indeed the cohomology of the boundary is Eisenstein by identical arguments to those of [NT16, §4]. □

Corollary 4.9.5. *The maps* $\mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w(-D))^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}} \to \mathrm{R}\Gamma_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_w)^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}}$ *are quasi-isomorphisms.*

Proof. While this could be proved by analyzing the cohomology of the boundary, we argue as follows. By Proposition 4.9.3 and Theorem 4.9.4, it suffices to show that the maps

$$H^0_{{}^0w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,-w_{0,M}({}^0w\cdot\lambda+2\rho^M)}_{{}^0w})^{\mathrm{ord}} \to H^2_{{}^1w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,-w_{0,M}({}^1w\cdot\lambda+2\rho^M)}_{{}^1w})^{\mathrm{ord}},$$

$$H^1_{{}^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,-w_{0,M}({}^2w\cdot\lambda+2\rho^M)}_{{}^2w}(-D))^{\mathrm{ord}} \to H^3_{{}^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,-w_{0,M}({}^3w\cdot\lambda+2\rho^M)}_{{}^3w}(-D))^{\mathrm{ord}}$$

are 0 after localizing at $\mathfrak{m}_{\bar{\rho}}$. The second statement follows from the first by duality, and the first statement follows from the fact that the natural map

$$H^0_{{}^0w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_{{}^0w}(-D))^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}} \to H^0_{{}^0w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_{{}^0w})^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}}$$

is an isomorphism, which is Lemma 4.9.6 below. □

Lemma 4.9.6. *The map* $H^0_{{}^0w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_{{}^0w}(-D))^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}} \to H^0_{{}^0w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,\kappa}_{{}^0w})^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}}$ *is an isomorphism.*

Proof. This is similar to [Pil20, Cor. 15.2.3.1] and [BCGP21, Lem. 3.10.7], except that we work with ordinary p-adic modular forms rather than classical forms. We translate the statement to a result about Hida complexes which can then be proved as in these previous results but working mod p and with the structure sheaf.

The Hida complexes we consider are constructed in [BP23] and also recalled below in Section 7.3; they are perfect complexes $M^\bullet_{{}^0w,\mathrm{cusp}}$ and $M^\bullet_{{}^0w}$ of $\Lambda = \mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!]$-modules, and there is a natural morphism $M^\bullet_{{}^0w,\mathrm{cusp}} \to M^\bullet_{{}^0w}$. The complex $M^\bullet_{{}^0w,\mathrm{cusp}}$ is projective in degree 0 (and in fact it is the classical object constructed by Hida [Hid02]), while $M^\bullet_{{}^0w}$ has amplitude $[0,2]$. We can also consider the boundary Hida complex $M^{\bullet}_{{}^0w,\partial} = \mathrm{cone}(M^{\bullet}_{{}^0w,\mathrm{cusp}} \to M^{\bullet}_{{}^0w})$. A priori this has amplitude $[-1,2]$ but we will recall below the simple geometric reason that it has amplitude $[0,2]$.

Below we shall prove that after non Eisenstein localization $M^{\bullet}_{{}^0w,\partial,\mathfrak{m}_{\overline{\rho}}}$ has amplitude $[1,2]$ (in fact it actually vanishes, but since we do not need this, we do not prove it). This statement implies that for any continuous homomorphism $\nu : T(\mathbf{Q}_p) \to \overline{\mathbf{Q}}_p^{\times}$ the morphism

$$H^0(M^{\bullet}_{{}^0w,\mathrm{cusp},\mathfrak{m}_{\overline{\rho}}} \otimes^L_{\Lambda,\nu} \overline{\mathbf{Q}}_p) \to H^0(M^{\bullet}_{{}^0w,\mathfrak{m}_{\overline{\rho}}} \otimes^L_{\Lambda,\nu} \overline{\mathbf{Q}}_p) \tag{4.9.7}$$

is an isomorphism, and by the comparison between higher Hida and Coleman theory [BP23, Thm 6.2.15] and Theorem 4.6.56 this is exactly the statement of the lemma.

To prove the claim, by Nakayama's lemma it suffices to prove that

$$H^0(M^{\bullet}_{{}^0w,\mathrm{cusp},\mathfrak{m}_{\overline{\rho}}} \otimes^L_{\Lambda} \mathbf{F}_p[T(\mathbf{F}_p)]) \to H^0(M^{\bullet}_{{}^0w,\mathfrak{m}_{\overline{\rho}}} \otimes^L_{\Lambda} \mathbf{F}_p[T(\mathbf{F}_p)])$$

is an isomorphism. We now translate this back into a statement about mod p modular forms on the ordinary locus which we prove by analyzing the boundary.

We consider $IG/\mathbf{F}_p$, the special fiber of the (ordinary) Igusa variety corresponding to the pro-p Iwahori subgroup of $P'(\mathbf{Q}_p)$, in the notation of [BP23, §3.4.5]. We let $\pi : IG^{\mathrm{tor}} \to IG^*$ be its (partial) toroidal and minimal compactifications. By the very construction of the Hida complexes, the map (4.9.7) is nothing but the natural map

$$H^0(IG^{\mathrm{tor}}, \mathcal{O}(-D))^{\mathrm{ord}}_{\mathfrak{m}_{\overline{\rho}}} \to H^0(IG^{\mathrm{tor}}, \mathcal{O})^{\mathrm{ord}}_{\mathfrak{m}_{\overline{\rho}}} \tag{4.9.8}$$

(For this and the meaning of the ordinary part, see [BP23, §5.2], but note that the setup is substantially simplified because $w = {}^0w$.) We remark that even without the non-Eisenstein localization this map is always injective, which justifies the assertion made above that the boundary cohomology always has amplitude $[0,2]$.

We now follow the strategy of [Pil20, Cor. 15.2.3.1] and [BCGP21, Lem. 3.10.7] to prove that (4.9.8) is surjective after non-Eisenstein localization. If not then there is a non-Eisenstein Hecke eigenvector occurring in $H^0(IG^{\mathrm{tor}}, \mathcal{O}_D)$. We write $D^* \subseteq IG^*$ for the (reduced) boundary. We have $\pi_*\mathcal{O}_D = \mathcal{O}_{D^*}$ so that $H^0(D, \mathcal{O}_D) = H^0(D^*, \mathcal{O}_{D^*})$. The boundary D^* is a union of (ordinary) Igusa curves crossing at cusps. We write $\tilde{D}^*$ for the normalization, which is a disjoint union of ordinary Igusa curves. We have an injective pullback map

$$H^0(D^*, \mathcal{O}_{D^*}) \to H^0(\tilde{D}^*, \mathcal{O}_{\tilde{D}^*})$$

There is a compatibility between the GSp_4 Hecke action and the GL_2 Hecke action at primes away from p and the tame level (see [BCGP21, Lem. 3.10.7] for a precise statement). This implies that the systems of Hecke eigenvalues in $H^0(\tilde{D}^*, \mathcal{O}_{\tilde{D}^*})$ are Eisenstein, as required. □

Theorem 4.9.9. *For any irreducible representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{F}}_p)$, the localization*

$$V := \mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \overline{\mathbf{Q}}_p)^{\mathrm{la}})^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}} = CC_{\lambda}(M(\mathfrak{g})_{\lambda})^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}}$$

is concentrated in degree 3*. Moreover, there is a $G_{\mathbf{Q}_p} \times \mathbf{T}_{K^p} \times T(\mathbf{Q}_p)$-equivariant filtration $\{Fil^i V_{\mathbf{C}_p}\}_{i=0,1,2,3}$ on $V \otimes \mathbf{C}_p$ and*

$$\mathrm{Gr}^i V_{\mathbf{C}_p} = \left(H^i_{{}^iw}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,-w_{0,M}({}^iw\cdot\lambda+2\rho^M)}_{{}^iw}) \otimes \mathbf{C}_p(-{}^iw^{-1}w_{0,M}\rho - \rho)\right)^{\mathrm{ord}}_{\mathfrak{m}_{\bar{\rho}}}.$$

The Sen operator is scalar on $\mathrm{Gr}^i V_{\mathbf{C}_p}$ *and acts via* $\frac{-\lambda_1-\lambda_2-w}{2}$, $\frac{-2+\lambda_1-\lambda_2-w}{2}$, $\frac{-4+\lambda_2-\lambda_1-w}{2}$, $\frac{-6+\lambda_1+\lambda_2-w}{2}$ *for* $i = 3, 2, 1, 0$ *respectively.*

Proof. This is immediate from Proposition 4.9.3, Theorem 4.9.4 and Corollary 4.9.5. □

4.10. **Sen and Cousin.** We now specialize to $\lambda = (1, 1; w)$.

4.10.1. *The Cousin map for the classical sheaves.* All the action will be happening on the union of Bruhat strata (in fact a Q-orbit) $C_{^3w,Q} = C_{^3w} \cup C_{^2w}$. We have an extension over $\pi_{HT}^{-1}(C_{^3w,Q})$, corresponding to the stratification of $C_{^3w,Q}$ into B-orbits, with $j_{^3w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}} : \pi_{HT}^{-1}(C_{^3w}) \hookrightarrow \pi_{HT}^{-1}(C_{^3w,Q})$:

$$0 \to (j_{^3w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_! \omega^{(1,1;-w),\mathrm{sm}}|_{\pi_{HT}^{-1}(C_{^3w})} \to \omega^{(1,1;-w),\mathrm{sm}}|_{\pi_{HT}^{-1}(C_{^3w,Q})} \to \omega^{(1,1;-w),\mathrm{sm}}|_{\pi_{HT}^{-1}(C_{^2w})} \to 0 \tag{4.10.2}$$

Proposition 4.10.3.

(1) *The natural map:*

$$\mathrm{R}\Gamma_c(\pi_{HT}^{-1}(C_{^3w,Q}), \omega^{(1,1;-w),\mathrm{sm}}) \to \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})$$

induces a quasi-isomorphism on the ordinary part (the slope $= -(1, 1; w)$*-part).*

(2) *Moreover,* $\mathrm{R}\Gamma_c(\pi_{HT}^{-1}(C_{^3w,Q}), \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}$ *is computed by the following complex in degrees* $2, 3$ *where Cous is induced by the class of the extension* (4.10.2)

$$H^2_{^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}} \xrightarrow{Cous} H^3_{^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}.$$

Proof. By Proposition 4.6.13, we have a spectral sequence (the Cousin spectral sequence) from local cohomologies converging to classical cohomology. The first statement is equivalent to the vanishing of the ordinary part of the higher Coleman theories for the elements ^0w and ^1w. By Theorem 4.6.60, we find that the slopes on $H^i_{^0w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})$ are $\geq -(2, 2; w)$ and the slopes on $H^i_{^1w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})$ are $\geq (0, -2; -w)$. Since $-(1, 1; w)+\gamma = -(2, 2; w)$ and $-(1, 1; w)+\alpha = (0, -2; -w)$ we conclude that the ordinary part vanishes. The second statement is a consequence of Theorem 4.6.58 and Lemma 4.9.2. □

4.10.4. *The Cousin map for the big sheaves.* Applying the functor VB^0 to the sheaf $HCS_{Q,^3w,\eta}(M(\mathfrak{m}_{^3w})_\lambda)$ of (3.6.5) and twisting the $B(\mathbf{Q}_p)$-action by $\lambda - \eta$ yields an extension over $\pi_{HT}^{-1}(C_{^3w,Q})$:

$$0 \to j_{^3w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}} \omega^{\dagger,(1,1;-w)}_{^3w} \to VB^0(HCS_{Q,^3w,\eta}(M(\mathfrak{m}_{^3w})_\lambda)) \otimes \mathbf{C}_p(\lambda-\eta) \to \omega^{\dagger,(1,1;-w)}_{^2w} \otimes \mathbf{C}_p((2, 0; 0)) \to 0 \tag{4.10.5}$$

The natural map in $\mathcal{O}(\mathfrak{m}_{^3w}, \mathfrak{b}_{M_{^3w}})$ (mapping a Verma of dominant weight to its finite dimensional quotient) $M(\mathfrak{m}_{^3w})_\lambda \to L(\mathfrak{m}_{^3w})_\lambda$ yields a map

$$HCS_{Q,^3w,\eta}(L(\mathfrak{m}_{^3w})_\lambda) \to HCS_{Q,^3w,\eta}(M(\mathfrak{m}_{^3w})_\lambda).$$

As in Lemma 4.6.5, applying VB^0 gives a map

$$\omega^{(1,1;-w),\mathrm{sm}} \otimes \mathbf{C}_p(\eta)|_{\pi_{HT}^{-1}(C_{^3w,Q})} \to VB^0(HCS_{Q,^3w,\eta}(M(\mathfrak{m}_{^3w})_\lambda)).$$

We deduce that there is a map of extensions from (4.10.2) to (4.10.5) (for clarity we drop the twist of the $B(\mathbf{Q}_p)$-action in this diagram):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & j_{{}^3w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}}\omega^{\dagger,(1,1;-w)}_{{}^3w} & \longrightarrow & VB^0(HCS_{Q,{}^3w,\eta}(M(\mathfrak{m}_{{}^3w})_\lambda)) & \longrightarrow & \omega^{\dagger,(1,1;-w)}_{{}^2w} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (j_{{}^3w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!\omega^{(1,1;-w),\mathrm{sm}}|_{\pi^{-1}_{HT}(C_{{}^3w})} & \longrightarrow & \omega^{(1,1;-w),\mathrm{sm}}|_{\pi^{-1}_{HT}(C_{{}^3w,Q})} & \longrightarrow & \omega^{(1,1;-w),\mathrm{sm}}|_{\pi^{-1}_{HT}(C_{{}^2w})} & \longrightarrow & 0
\end{array}$$

Proposition 4.10.6. *The maps*

$$\begin{array}{lcl}
\mathrm{R}\Gamma_{{}^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{(1,1;-w),\mathrm{sm}})\otimes\mathbf{C}_p((1,1;w))^{\mathrm{ord}} & \to & \mathrm{R}\Gamma_{{}^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^2w})\otimes\mathbf{C}_p((2,0;0))^{\mathrm{ord}} \\
\mathrm{R}\Gamma_{{}^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{(1,1;-w),\mathrm{sm}})\otimes\mathbf{C}_p((1,1;w))^{\mathrm{ord}} & \to & \mathrm{R}\Gamma_{{}^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^3w})^{\mathrm{ord}}
\end{array}$$

are quasi-isomorphisms.

Consequently we have a quasi-isomorphism

$$\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{(1,1;-w),\mathrm{sm}})\otimes\mathbf{C}_p((1,1;w))^{\mathrm{ord}}=$$

$$H^2_{{}^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^2w})\otimes\mathbf{C}_p((2,0;0))^{\mathrm{ord}}\stackrel{Cous}{\to}H^3_{{}^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^3w})^{\mathrm{ord}}$$

where the complex is in degree $[2,3]$ *and the map* $Cous$ *is induced by the class of the extension* (4.10.5)*.*

Proof. We have the BGG short exact sequence

$$0\to M(\mathfrak{m}_{{}^3w})_{(0,2;w)}\to M(\mathfrak{m}_{{}^3w})_{(1,1;w)}\to L(\mathfrak{m}_{{}^3w})_{(1,1;w)}\to 0.$$

Applying $HC_{{}^3w,\lambda}$ gives a triangle

$$HC_{{}^3w,\lambda}(L(\mathfrak{m}_{{}^3w})_{(1,1;w)})\to HC_{{}^3w,\lambda}(M(\mathfrak{m}_{{}^3w})_{(1,1;w)})\to HC_{{}^3w,\lambda}(M(\mathfrak{m}_{{}^3w})_{(0,2;w)})\stackrel{+1}{\to}$$

By Theorem 4.6.60, the ordinary part of $HC_{{}^3w,\lambda}(M(\mathfrak{m}_{{}^3w})_{(0,2;w)})$ is trivial, so that we get a quasi-isomorphism $HC_{{}^3w,\lambda}(L(\mathfrak{m}_{{}^3w})_{(1,1;w)}))^{\mathrm{ord}}\to HC_{{}^3w,\lambda}(M(\mathfrak{m}_{{}^3w})_{(1,1;w)})^{\mathrm{ord}}$. This translates into the quasi-isomorphism $\mathrm{R}\Gamma_{{}^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{(1,1;-w),\mathrm{sm}})\otimes\mathbf{C}_p((1,1;w))^{\mathrm{ord}}\to\mathrm{R}\Gamma_{{}^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^3w})^{\mathrm{ord}}$. The quasi-isomorphism $\mathrm{R}\Gamma_{{}^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{(1,1;-w),\mathrm{sm}})\otimes\mathbf{C}_p((1,1;w))^{\mathrm{ord}}\to\mathrm{R}\Gamma_{{}^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^2w})\otimes\mathbf{C}_p((2,0;0))^{\mathrm{ord}}$ follows by similar considerations. The second part is then immediate from Proposition 4.10.3. □

Remark 4.10.7. We also have a quasi-isomorphism

$$\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{(2,2;-w),\mathrm{sm}}(-D))\otimes\mathbf{C}_p((2,2;w))^{\mathrm{ord}}=$$

$$H^0_{{}^0w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^0w}(-D))\otimes\mathbf{C}_p((3,3;0))^{\mathrm{ord}}\stackrel{Cous}{\to}H^1_{{}^1w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(2,2;-w)}_{{}^1w}(-D))\otimes\mathbf{C}_p((-1,3;0))^{\mathrm{ord}}$$

where the complex is in degrees $[0,1]$. This statement is Serre dual to Proposition 4.10.3.

4.10.8. *The Sen map.* Let $V=H^3(\mathrm{RHom}_{\mathfrak{b}}(\lambda,\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\overline{\mathbf{Q}}_p)^{\mathrm{la}})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}})$. By Theorem 4.9.9, $V_{\mathbf{C}_p}$ carries a filtration where

- $\mathrm{Gr}^3V_{\mathbf{C}_p}=H^3_{{}^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^3w})\otimes\mathbf{C}_p(0,0;0)^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}}$,
- $\mathrm{Gr}^2V_{\mathbf{C}_p}=H^2_{{}^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)}_{{}^2w})\otimes\mathbf{C}_p(2,0;0)^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}}$,
- $\mathrm{Gr}^1V_{\mathbf{C}_p}=H^1_{{}^1w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(2,2;-w)}_{{}^1w})\otimes\mathbf{C}_p(-1,3;0)^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}}$,
- $\mathrm{Gr}^0V_{\mathbf{C}_p}=H^0_{{}^0w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(2,2;-w)}_{{}^0w})\otimes\mathbf{C}_p(3,3;0)^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}}$.

Acting on $V_{\mathbf{C}_p}$ we have a Sen operator whose eigenvalues are $-1-\frac{w}{2}, -1-\frac{w}{2}, -2-\frac{w}{2}, -2-\frac{w}{2}$. The generalized Hodge–Tate weight $1+\frac{w}{2}$-part of $V_{\mathbf{C}_p}$ fits in the following short exact sequence (where Θ is the Sen operator and we drop the twist of the $B(\mathbf{Q}_p)$-action to lighten the notation):

$$0 \to H^3_{3_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,(1,1;-w)}_{3_w})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}} \to V_{\mathbf{C}_p}[(\Theta+1+\frac{w}{2})^2] \to H^2_{2_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,(1,1;-w)}_{2_w})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}} \to 0. \tag{4.10.9}$$

Since $Sen := \Theta + 1 + \frac{w}{2}$ is nilpotent, it induces a map:

$$H^2_{2_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,(1,1;-w)}_{2_w}) \otimes \mathbf{C}_p(2,0;0)^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}} \stackrel{Sen}{\to} H^3_{3_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,(1,1;-w)}_{3_w})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}}. \tag{4.10.10}$$

Similarly, by looking at the weight $2+\frac{w}{2}$-part of V we obtain the following map:

$$H^0_{0_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,(2,2;-w)}_{0_w}) \otimes \mathbf{C}_p(3,3;0)^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}} \stackrel{Sen}{\to} H^1_{1_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,(2,2;-w)}_{1_w}) \otimes \mathbf{C}_p(-1,3;0)^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}}$$

4.10.11. *Comparison between the Sen and Cousin map.* The following theorem is one of the main results in this section. It is a generalization of [Pan22a, Thm. 5.3.18], in the modular curve case. We will follow the method of proof of [Pil24, Thm. 6.1].

Theorem 4.10.12. *The two maps*

$$Cous, Sen : H^2_{2_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,(1,1;-w)}_{2_w}) \otimes \mathbf{C}_p(2,0;0)^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}} \to H^3_{3_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\dagger,(1,1;-w)}_{3_w})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho}}$$

(coming respectively from Proposition 4.10.6 and (4.10.10)*)agree up to a non-zero scalar.*

Proof. We consider $\mathrm{RHom}_{\mathfrak{b},*_2}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})|_{\pi^{-1}_{HT}C_{3_{w,Q}}} = VB^{\mathrm{red}}(\mathrm{Loc}(M(\mathfrak{g})_\lambda)|_{\pi^{-1}_{HT}C_{3_{w,Q}}})$, which fits in the triangle (obtained by applying VB^{red} to Proposition 3.6.6):

$$\mathrm{Ext}^0_{\mathfrak{b},*_2}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{\pi^{-1}_{HT}(C_{3_{w,Q}})}) \to \mathrm{RHom}_{\mathfrak{b},*_2}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{\pi^{-1}_{HT}(C_{3_{w,Q}})})$$
$$\to \mathrm{Ext}^1_{\mathfrak{b},*_2}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{\pi^{-1}_{HT}(C_{3_{w,Q}})})[-1] \stackrel{+1}{\to}$$

where $\mathrm{Ext}^0_{\mathfrak{b},*_2}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{\pi^{-1}_{HT}C_{3_{w,Q}}}) = (j_{3_w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!\omega^{\dagger,(1,1;-w)}_{3_w}$ and $\mathrm{Ext}^1_{\mathfrak{b},*_2}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{\pi^{-1}_{HT}C_{3_{w,Q}}}) = \omega^{\dagger,(1,1;-w)}_{2_w} \otimes \mathbf{C}_p((2,0,0))$.

On $\mathrm{RHom}_{\mathfrak{b},*_2}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{\pi^{-1}_{HT}(C_{3_{w,Q}})})$, we have $(\Theta+1+\frac{w}{2})^2 = 0$. Let us introduce some simplifying notations and denote by:

- $X = (j_{3_w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!\omega^{\dagger,(1,1;-w)}_{3_w}$,
- $Y = \mathrm{RHom}_{\mathfrak{b},*_2}(\lambda, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{\pi^{-1}_{HT}(C_{3_{w,Q}})})$,
- $Z = \omega^{\dagger,(1,1;-w)}_{2_w} \otimes \mathbf{C}_p((2,0,0))[-1]$,
- $W = \mathrm{RHom}_{\mathfrak{b},*_2,\Theta}(\lambda, -1-\frac{w}{2}, \mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}|_{\pi^{-1}_{HT}(C_{3_{w,Q}})})$.

Applying [Sta13, Tag 05R0] to the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle (\Theta+1+\frac{w}{2})} \\ X & \longrightarrow & Y \end{array}$$

we obtain the following commutative diagram, where all lines and columns are part of distinguished triangles:

$$\begin{array}{ccccccc} X\oplus X[-1] & \longrightarrow & W & \longrightarrow & Z\oplus Z[-1] & \longrightarrow & X[1]\oplus X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (\Theta+1+\frac{w}{2})} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

The top horizontal triangle can be written as

$$(j_{{}^3w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!\omega^{\dagger,(1,1;-w)}_{{}^3w}\oplus(j_{{}^3w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}})_!\omega^{\dagger,(1,1;-w)}_{{}^3w}[-1]\to \mathrm{RHom}_{\mathfrak{b},*_2,\Theta}(\lambda,-1-\frac{w}{2},\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})$$
$$\to\omega^{\dagger,(1,1;-w)}_{{}^2w}[-1]\oplus\omega^{\dagger,(1,1;-w)}_{{}^2w}[-2]\stackrel{+1}{\to}$$

(compare (3.6.7); we again drop the twist of the $B(\mathbf{Q}_p)$-action for simplicity and we note that taking the $\Theta+1+\frac{w}{2}$-cohomology localizes over $\pi_{HT}^{-1}(C_{{}^3w,Q})$ so we also drop it from the notation.)

Taking cohomology yields a long exact sequence:

$$H^3(\mathrm{RHom}_{\mathfrak{b},*_2,\Theta}(\lambda,-1-\frac{w}{2},\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}}))\to H^2_{{}^2w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)})\stackrel{\delta}{\to}H^3_{{}^3w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p},\omega^{\dagger,(1,1;-w)})$$

where the map δ is induced by the class of the extension (extracted from the top horizontal triangle):

$$0\to j_{{}^3w,\mathrm{Sh}^{\mathrm{tor}}_{K^p}}\omega^{\dagger,(1,1;-w)}_{{}^3w}\to\mathrm{Ext}^1_{\mathfrak{b},*_2,\Theta}(\lambda,-1-\frac{w}{2},\mathcal{O}^{\mathrm{la}}_{\mathrm{Sh}^{\mathrm{tor}}_{K^p}})\to\omega^{\dagger,(1,1;-w)}_{{}^2w}\otimes\mathbf{C}_p((2,0;0))\to 0. \tag{4.10.13}$$

By definition, the map Sen is equal to δ. Now, by Theorem 3.6.9 and Theorem 4.5.20, the extensions (4.10.5) and (4.10.13) agree up to a non-zero scalar. Thus the maps $Cous, Sen$ agree up to a non-zero scalar, as claimed. □

4.11. **The Eichler–Shimura relation and semi-simplicity.** The following is a special case of a result of Nekovář, [Nek18]. If r is a finite-dimensional representation of a group Γ, and $g\in\Gamma$, then we write $\mathrm{char}_{r(g)}$ for the characteristic polynomial of $r(g)$.

Proposition 4.11.1. *Let $\rho: G_{\mathbf{Q}}\to\mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ and $s: G_{\mathbf{Q}}\to\mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be continuous representations (for some $n\geq 1$) and assume that*

(1) the Zariski closure of $\rho(G_{\mathbf{Q}})$ contains Sp_4, and
(2) for a density one set of primes l, we have $\mathrm{char}_{\rho(\mathrm{Frob}_l)}\big(s(\mathrm{Frob}_l)\big)=0$.

Then we have $s\cong\rho^{\oplus m}$ for some integer $m\geq 1$.

Proof. We claim that the result is an immediate application of [Nek18, Prop. 3.10] (with ℓ replaced by p), taking $\Gamma=\Gamma'=G_{\mathbf{Q}}$, $a=r=1$, and the representations ρ, ρ_1 of [Nek18, Prop. 3.10] to be s and ρ respectively.

The hypothesis (C') of [Nek18, Prop. 3.10] is immediate from our hypotheses, taking Σ to be the set of the Frob_l for primes l satisfying Condition (2). For hypothesis (A'), since the Zariski closure of $\rho(G_{\mathbf{Q}})$ contains Sp_4 by hypothesis (and is contained in GSp_4), the Lie algebra $\overline{\mathbf{Q}}_p\cdot\mathrm{Lie}\,(\rho(G_{\mathbf{Q}}))$ is equal to $\mathfrak{sp}_4$ or $\mathfrak{gsp}_4$, and the representation of this Lie algebra induced by ρ is the standard 4-dimensional representation, which is minuscule. We are thus in the situation of part (3) of [Nek18, Prop. 3.10], and the proposition follows. □

We have the following variant of Proposition 4.11 in the induced case:

Proposition 4.11.2. *Let $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ and $s : G_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be continuous representations and assume that*

(1) the Zariski closure of $\rho(G_{\mathbf{Q}})$ contains $\mathrm{SL}_2 \times \mathrm{SL}_2$,
(2) ρ is absolutely irreducible but becomes reducible on some index two subgroup G_E, and
(3) for a density one set of primes l, we have $\mathrm{char}_{\rho(\mathrm{Frob}_l)}\big(s(\mathrm{Frob}_l)\big) = 0$.

Then we have $s \cong \rho^{\oplus m}$ for some integer $m \geq 1$.

Proof. We may write $\rho|_{G_E} = \varrho \oplus \varrho^c$, where $\mathrm{Gal}(E/\mathbf{Q})$ permutes the factors. The assumption that ρ is symplectic (together with the assumptions on the image of ρ) implies that $\det(\varrho)$ and $\det(\varrho^c)$ are both the restriction to G_E of the similitude character of ρ.

Let t be an irreducible subquotient of $s|_{G_E}$. Assumption (3) implies that we have $\mathrm{char}_{\rho(h)}\big(t(h)\big) = 0$ for a dense set of elements $h \in G_E$. We may assume that ϱ, ϱ^c, and t all have models over the ring of integers $\mathcal{O}$ of some finite extension of $\mathbf{Q}_p$. Let T and P denote the Zariski closures of t and $\varrho \oplus \varrho^c$ respectively. By assumption, T is reductive, and the Zariski closure of $t \oplus \varrho \oplus \varrho^c$ inside $T \oplus P$ is also reductive, and by Goursat's lemma is the graph of some projections $\pi_1 : T \to G$, $\pi_2 : P \to G$ onto a common quotient G. By the Chebotarev density theorem and continuity, Assumption (3) implies that the minimal polynomial of any element in $K = \ker(\pi_1)$ divides $(X-1)^4$. This is because elements in the image of s which lie in K are limits of $s(\mathrm{Frob}_l)$ for Frobenius elements Frob_l, and by assumption these will satisfy $(s(\mathrm{Frob}_l)-1)^4\mathcal{O}^n \subset \pi^m\mathcal{O}^n$ for larger and larger m. This implies that K is unipotent, which — since T is reductive — implies that K is trivial. Hence $G = T$ and thus T is a quotient of P. But now from the fact that P contains $\mathrm{SL}_2 \times \mathrm{SL}_2$, we see that the only possibilities for t up to twist are $\mathrm{Sym}^i \varrho \otimes \mathrm{Sym}^j \varrho^c$, from which one easily sees that t must either be ϱ or ϱ^c, and thus any irreducible subquotient of $s|_{G_E}$ is either ϱ or ϱ^c.

We claim that $s|_{G_E}$ cannot contain any non-trivial extensions of ϱ by ϱ (equally, of ϱ^c by ϱ^c). To see this, note that a generic element in the image of ϱ^c acts invertibly on ϱ. Hence the assumptions imply that for of a dense set of $h \in G_E$, the characteristic polynomial of $\varrho(h)$ annihilates this extension of ϱ by ϱ. But then the results follow from the GL_2-version of [Nek18, Prop. 3.10] (first proved in [BLR91]).

Now return to representations of $G_{\mathbf{Q}}$. By what we have shown for $s|_{G_E}$, we deduce that every irreducible subquotient of s is isomorphic to ρ. Hence it suffices to rule out the case that s is of the form

$$0 \to \rho \to W \to \rho \to 0$$

for some non-split extension W. Note that the restriction $W|_{G_E}$ is an extension:

$$0 \to \varrho \oplus \varrho^c \to W|_{G_E} \to \varrho \oplus \varrho^c \to 0.$$

In particular, $W|_{G_E}$ corresponds to a $\mathrm{Gal}(E/\mathbf{Q})$-invariant class in

$$\mathrm{Ext}^1_{G_E}(\varrho, \varrho^c) \oplus \mathrm{Ext}^1_{G_E}(\varrho, \varrho) \oplus \mathrm{Ext}^1_{G_E}(\varrho^c, \varrho^c) \oplus \mathrm{Ext}^1_{G_E}(\varrho^c, \varrho).$$

As already shown, the corresponding extensions of ϱ by ϱ and ϱ^c by ϱ^c both split, so the projection of this class to both $\operatorname{Ext}^1_{G_E}(\varrho^c,\varrho)$ and $\operatorname{Ext}^1_{G_E}(\varrho,\varrho^c)$ (they are permuted by the Galois action) must be non-trivial. In particular, we may assume that $W|_{G_E}$ has a subquotient which is a genuine extension of ϱ by ϱ^c. Our assumptions on the Zariski closure of the image of ρ imply that a generic element of ρ is regular semi-simple. Thus, by Chebotarev and assumption (3), it follows that there is a dense set of elements of $G_{\mathbf{Q}}$ which act semi-simply on W. Moreover, since the tensor product of two semi-simple matrices is semi-simple, the same holds for the tensor product $W \otimes \rho^\vee$ and so consequently also for any subquotient of this representation. We have an exact sequence:

$$0 \to \rho\otimes\rho^\vee \to W\otimes\rho^\vee \to \rho\otimes\rho^\vee \to 0. \tag{4.11.3}$$

Write χ for the quadratic character of $\operatorname{Gal}(E/\mathbf{Q})$, and let η be the similitude character of ρ. We have (compare [BCGP21, §7.5.16]) a decomposition

$$\rho\otimes\rho^\vee \simeq \overline{\mathbf{Q}}_p \oplus \overline{\mathbf{Q}}_p(\chi) \oplus \operatorname{Ind}_{G_E}^{G_{\mathbf{Q}}} \operatorname{ad}^0(\varrho) \oplus s(\varrho)\otimes\eta^{-1} \oplus s(\varrho)\otimes\eta^{-1}\otimes\chi,$$

where $s(\varrho)$ is the Asai representation (i.e. the tensor induction). Hence taking suitable subquotients of (4.11.3), we arrive at a pair of extensions

$$\begin{gathered} 0 \to s(\varrho)\otimes\eta^{-1} \to U \to \overline{\mathbf{Q}}_p \to 0, \\ 0 \to s(\varrho)\otimes\eta^{-1}\otimes\chi \to V \to \overline{\mathbf{Q}}_p \to 0. \end{gathered} \tag{4.11.4}$$

We claim that at least one of these sequences must be non-split. The point is that

$$\operatorname{Ext}^1_{G_E}(\varrho,\varrho^c) = H^1(E, \operatorname{Hom}(\varrho,\varrho^c)),$$

but $\operatorname{Hom}(\varrho,\varrho^c)$ is the restriction of $s(\varrho)\otimes\eta^{-1}$ to G_E, and thus, by Shapiro's lemma, we have

$$\begin{aligned} \operatorname{Ext}^1_{G_E}(\varrho,\varrho^c) &= H^1(E, \left(s(\varrho)\otimes\eta^{-1}\right)|_{G_E}) \\ &= H^1(\mathbf{Q}, \left(s(\varrho)\otimes\eta^{-1}\right)\otimes \operatorname{Ind}_{G_E}^{G_{\mathbf{Q}}}\overline{\mathbf{Q}}_p) \\ &= H^1(\mathbf{Q}, s(\varrho)\otimes\eta^{-1}\otimes\chi) \oplus H^1(\mathbf{Q}, s(\varrho)\otimes\eta^{-1}); \end{aligned}$$

and by construction, the elements of the right hand side corresponding to our non-split extension of ϱ by ϱ^c coming from $W|_{G_E}$ are the extensions U, V of (4.11.4). We consider the case that U is non-split, the case of V being entirely similar.

To complete the proof, it suffices to show that for any such non-split extension, there cannot be a dense set of $g \in G_{\mathbf{Q}}$ which act semi-simply. Write $A = s(\varrho)\otimes\eta^{-1}$, and let G and H be the Zariski closures of the images of $G_{\mathbf{Q}}$ and G_E on U respectively. By constriction, the Zariski closure of the image of $G_{\mathbf{Q}}$ on A is the orthogonal group O_4, and the Zariski closure of the image of G_E on A is the index two subgroup SO_4 which is isomorphic to the image of $\mathrm{SL}_2\times\mathrm{SL}_2$. Any generic $h \in G_E$ *will* act semi-simply on U because it will have distinct eigenvalues. However, any $g \in G_{\mathbf{Q}} \smallsetminus G_E$ has 1 as an eigenvalue on both $A = s(\varrho)\otimes\eta^{-1}$ and $A\otimes\chi$ (the eigenvalues in either case take the form $1, -1, \lambda, -\lambda$ for some λ), so that in particular the eigenvalues of g on U are contained within the eigenvalues of g on A. By our semi-simplicity hypothesis, there is therefore a dense set of such g with the property that the image of g in $\operatorname{End}(U)$ is annihilated by the characteristic polynomial of g on A. By continuity, this extends to *all* elements of $G_{\mathbf{Q}} \smallsetminus G_E$ and also to all elements of $G \smallsetminus H$.

Now, G is a subgroup of the semi-direct product $N \rtimes \mathrm{O}_4$, where N is the standard representation of O_4 of dimension 4; and the projection $G \to \mathrm{O}_4$ is surjective. Since N is abelian, the conjugation action of G on N factors through this surjection, and since N is an irreducible representation of O_4, we see that either $G = \mathrm{O}_4$ (in which case the extension (4.11.4) splits, and we are done), or $G = N \rtimes \mathrm{O}_4$, in which case there are elements $g \in G \smallsetminus H$ whose minimal polynomial has degree 5, a contradiction. $\square$

4.12. **A classicality theorem.** Let S be the finite set of primes at which K^p is not hyperspecial, together with the prime p. In this section we will consider an ordinary overconvergent modular form $f \in H^0_{0_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}})^{\mathrm{ord}}$, which is an eigenform for $T(\mathbf{Q}_p)$ as well as for the spherical Hecke operators at the places not contained in S. We write $\chi_f^{sm} : T(\mathbf{Q}_p) \to \overline{\mathbf{Q}}_p^\times$ for the smooth character corresponding to f. Via the identification of dual groups, χ_f^{sm} induces a cocharacter $\mathbf{Q}_p^\times \to T(\overline{\mathbf{Q}}_p)$ that we denote by $t \mapsto \mathrm{diag}(\chi_1(t), \chi_2(t), \chi_3(t), \chi_4(t))$. We write

$$\mathfrak{m}_f \subseteq \overline{\mathbf{Q}}_p[T(\mathbf{Q}_p)] \otimes {\bigotimes_{l \notin S}}' \overline{\mathbf{Q}}_p[\mathrm{GSp}_4(\mathbf{Q}_l)//\mathrm{GSp}_4(\mathbf{Z}_l)]$$

for the maximal ideal corresponding to f.

Lemma 4.12.1. *We have a continuous semi-simple Galois representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$, which satisfies the following properties (where $P_\ell(X)$ is as in* (1.8.27)*):*

(1) ρ_f is unramified at primes $\ell \notin S$, and

$$P_\ell(X) \mod \mathfrak{m}_f = \det(X - \rho_f(\mathrm{Frob}_\ell)).$$

(2) $\rho_f|_{G_{\mathbf{Q}_p}}$ can be conjugated to a representation $\rho_f|_{G_{\mathbf{Q}_p}} : G_{\mathbf{Q}_p} \to B(\overline{\mathbf{Q}}_p)$ where the diagonal is given, via class field theory, by the cocharacter

$$z \mapsto \mathrm{diag}(\chi_4^{-1}(z) z^{-1-\frac{w}{2}}, \chi_3^{-1}(z) z^{-1-\frac{w}{2}}, \chi_2^{-1}(z)^{-1} z^{-2-\frac{w}{2}}, \chi_1^{-1}(z)^{-1} z^{-2-\frac{w}{2}})$$

for $z \in \mathbf{Z}_p^\times$, and

$$p \mapsto \mathrm{diag}(\chi_4^{-1}(p) p^{-1-\frac{w}{2}}, \chi_3^{-1}(p) p^{-1-\frac{w}{2}}, \chi_2^{-1}(p)^{-1} p^{-2-\frac{w}{2}}, \chi_1^{-1}(p)^{-1} p^{-2-\frac{w}{2}}).$$

Proof. Note that any cohomological, C-algebraic automorphic representation π has an associated Galois representation $\rho_{\pi,p}$ (see Section 1.8.12). This Galois representation is furthermore ordinary if π_p is ordinary. By a standard argument using p-adic families, we can interpolate the Galois representations $\rho_{\pi,p}$ associated to π of regular weight (see Section 1.8.12), and we define ρ_f to be the representation corresponding to the interpolation of $\rho_{\pi,p}^\vee \otimes \varepsilon^{-3}$. $\square$

Remark 4.12.2. The reason for considering ρ_f rather than $\rho_f^\vee \otimes \varepsilon^{-3}$ is that ρ_f is the Galois representation we are likely to realize in the completed cohomology, in view of Theorem 1.8.29.

We will also assume that $\overline{\rho}_f$ is irreducible and write $\mathfrak{m}_{\overline{\rho}_f}$ for the corresponding maximal ideal of the spherical Hecke algebra with $\overline{\mathbf{F}}_p$ coefficients, as in the previous sections. We let $V = H^3(\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \overline{\mathbf{Q}}_p)^{\mathrm{la}})^{\mathrm{ord}}_{\mathfrak{m}_{\overline{\rho}_f}})$, but we think of this space as $CC_0(M(\mathfrak{g})_\lambda)^{\mathrm{ord}}_{\mathfrak{m}_{\overline{\rho}_f}}$ and not as $CC_\lambda(M(\mathfrak{g})_\lambda)^{\mathrm{ord}}_{\mathfrak{m}_{\overline{\rho}_f}}$. In other words we now twist the $B(\mathbf{Q}_p)$ action to make it smooth (and not λ-smooth as in section 4.10).

Lemma 4.12.3. *Let $f \in H^0_{0_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}})^{\mathrm{ord}}$ be an ordinary overconvergent modular eigenform with Galois representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$. Let $\mathfrak{m}_f$ be the corresponding maximal ideal. We assume that $\overline{\rho}_f$ is irreducible and either*

1. *the Zariski closure of $\rho_f(G_{\mathbf{Q}})$ contains Sp_4; or*
2. *the Zariski closure of $\rho_f(G_{\mathbf{Q}})$ contains $\mathrm{SL}_2 \times \mathrm{SL}_2$, and ρ is irreducible but becomes reducible on some index two subgroup G_E.*

Let $V = H^3(\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \overline{\mathbf{Q}}_p)^{\mathrm{la}})^{\mathrm{ord}}_{\mathfrak{m}_{\overline{\rho}_f}})$. Then $V[\mathfrak{m}_f] = \rho_f \otimes_{\overline{\mathbf{Q}}_p} W$ for some finite-dimensional vector space $W \neq 0$.

Proof. Note that V carries a global Galois action since by results recalled in 4.4, $\mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \overline{\mathbf{Q}}_p) = \mathrm{R}\Gamma(\mathrm{Sh}^{alg}_{K^p}, \overline{\mathbf{Q}}_p)$. By Theorem 4.9.9, $V_{\mathbf{C}_p} := V \otimes_{\overline{\mathbf{Q}}_p} \mathbf{C}_p$ has a decreasing filtration with graded pieces $\{\mathrm{Gr}^i V_{\mathbf{C}_p}\}_{i=0,1,2,3}$ which are certain explicit spaces of ordinary higher Coleman theories localized at $\mathfrak{m}_{\overline{\rho}_f}$. In particular (bearing in mind Proposition 4.10.6),

$$\mathrm{Gr}^0 V_{\mathbf{C}_p} = (H^0_{0_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}}))^{\mathrm{ord}}_{\mathfrak{m}_{\overline{\rho}_f}}.$$

Let χ be the Nebentypus of f (the finite order character giving the action of $T(\mathbf{Z}_p)$ on f). Then each $\mathrm{Gr}^i V_{\mathbf{C}_p}[\chi]$ is finite dimensional (since we are fixing the slope to be ordinary, and also the action of $T(\mathbf{Z}_p)$). Taking the χ-isotypic part is an exact operation for smooth $T(\mathbf{Z}_p)$-modules in characteristic 0. We deduce that $V_{\mathbf{C}_p}[\chi]$ is finite-dimensional and has a filtration with graded pieces the $\{\mathrm{Gr}^i V_{\mathbf{C}_p}[\chi]\}_{i=0,1,2,3}$. Since $H^0_{0_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}})[\mathfrak{m}_f] \neq 0$, this implies that $V[\mathfrak{m}_f] \neq 0$ is finite dimensional. The result follows from the Eichler–Shimura relation (Theorem 1.8.29) together with Proposition 4.11.1 and Corollary 4.11.2. □

By Lemma 4.12.1, the representation $\rho_f|_{G_{\mathbf{Q}_p}}$ is ordinary, i.e. $\rho_f|_{G_{\mathbf{Q}_p}}$ preserves a Borel. The representation $\rho_f|_{G_{\mathbf{Q}_p}}$ is de Rham if it fits in an extension:

$$0 \to \rho_f^{(1+\frac{w}{2})}|_{G_{\mathbf{Q}_p}} \to \rho_f|_{G_{\mathbf{Q}_p}} \to \rho_f^{(2+\frac{w}{2})}|_{G_{\mathbf{Q}_p}} \to 0$$

where $\rho_f^{(1+\frac{w}{2})}|_{G_{\mathbf{Q}_p}}(1+\frac{w}{2})$ and $\rho_f^{(2+\frac{w}{2})}|_{G_{\mathbf{Q}_p}}(2+\frac{w}{2})$ are potentially unramified 2-dimensional representations. Equivalently, this means that the Sen operator of $D_{Sen}(\rho_f|_{G_{\mathbf{Q}_p}})$ is semi-simple with eigenvalues $-1-\frac{w}{2}$ and $-2-\frac{w}{2}$.

Theorem 4.12.4. *Let $f \in H^0_{0_w}(\mathrm{Sh}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}})^{\mathrm{ord}}$ be an ordinary overconvergent modular eigenform with Galois representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$. Let $\mathfrak{m}_f$ be the corresponding maximal ideal. Let $V = H^3(\mathrm{RHom}_{\mathfrak{b}}(\lambda, \mathrm{R}\Gamma(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \overline{\mathbf{Q}}_p)^{\mathrm{la}})^{\mathrm{ord}}_{\mathfrak{m}_{\overline{\rho}_f}})$. We assume that:*

1. *Either*
   - (a) *the Zariski closure of $\rho_f(G_{\mathbf{Q}})$ contains Sp_4; or*
   - (b) *the Zariski closure of $\rho_f(G_{\mathbf{Q}})$ contains $\mathrm{SL}_2 \times \mathrm{SL}_2$, and ρ_f is irreducible but becomes reducible on some index two subgroup G_E.*
2. *The representation $\rho_f|_{G_{\mathbf{Q}_p}}$ is de Rham.*
3. *There exists an integer n such that $\dim_{\mathbf{C}_p} \mathrm{Gr}^i V_{\mathbf{C}_p}[\mathfrak{m}_f] = n$ for each $0 \leq i \leq 3$, and $\dim_{\overline{\mathbf{Q}}_p} V[\mathfrak{m}_f] = 4n$.*
4. *The representation $\overline{\rho}_f$ is irreducible.*

Then f is a classical modular form.

Remark 4.12.5. The simplest way to verify the assumption (3) is to prove the multiplicity one statement that $\dim_{\mathbf{C}_p} H^i_{i_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}})[\mathfrak{m}_f] = 1$ for each i (when this statement is true). This forces $\dim_{\overline{\mathbf{Q}}_p} V[\mathfrak{m}_f] = 4$. Indeed, $\dim_{\overline{\mathbf{Q}}_p} V[\mathfrak{m}_f]$ is a multiple of 4 by Lemma 4.12.3.

Proof. By Proposition 4.10.6 and (4.10.9), we have an exact sequence (where Θ is the arithmetic Sen operator):

$$0 \to H^3_{3_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho_f}} \to V_{\mathbf{C}_p}[(\Theta+1+\frac{w}{2})^2] \to H^2_{2_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho_f}} \to 0$$

Since $\Theta + 1 + \frac{w}{2}$ is nilpotent, it induces a map:

$$H^2_{2_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho_f}} \overset{\Theta+1+\frac{w}{2}}{\to} H^3_{3_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}_{\mathfrak{m}_{\bar\rho_f}}.$$

We pass to $\mathfrak{m}_f$-isotypic components. Assumption (3) implies that the sequence:

$$0 \to H^3_{3_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}[\mathfrak{m}_f] \to V_{\mathbf{C}_p}[(\Theta+1+\frac{w}{2})^2][\mathfrak{m}_f] \to H^2_{2_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}[\mathfrak{m}_f] \to 0$$

remains exact. Therefore, we get a map

$$H^2_{2_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}[\mathfrak{m}_f] \overset{\Theta+1+\frac{w}{2}}{\to} H^3_{3_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}[\mathfrak{m}_f].$$

The kernel of this map is the space of classical forms $H^2(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})[\mathfrak{m}_f]$ because of Theorem 4.10.12 and Proposition 4.10.6. The map is zero since $V_{\mathbf{C}_p}[\mathfrak{m}_f] = D_{Sen}(\rho_f)^{\oplus n}$ by Lemma 4.12.3, and Θ identifies with the Sen operator of ρ_f which is semi-simple by assumption. It follows that $H^2(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}})[\mathfrak{m}_f] = H^2_{2_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(1,1;-w),\mathrm{sm}})^{\mathrm{ord}}[\mathfrak{m}_f]$. This implies that $\dim H^0(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}}))[\mathfrak{m}_f] = n$ (by the stability of the L-packet of automorphic forms corresponding to f) and therefore $H^0(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}}))[\mathfrak{m}_f] = H^0_{0_w}(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{(2,2;-w),\mathrm{sm}})^{\mathrm{ord}}[\mathfrak{m}_f]$, and we are done. □

Remark 4.12.6. The way we have explained the argument, we naturally proved the classicality of the relevant space of degree 2 cohomology classes, and used Arthur's classification of automorphic forms to deduce the classicality of f. This may seem a bit strange. We should first explain why we focused on degree 2 and degree 3 cohomology classes in the argument, instead of degree 0 and degree 1 cohomology classes. The reason is that the localization $\mathrm{Loc}(M(\mathfrak{g})_\lambda)$ is simpler on $C_{3_w,Q}$, but has some richer structure on its complement. This rich structure is irrelevant for the ordinary case, but would cause minor technical problems in Section 4.10.

Still focusing on degree 2 and 3 cohomology as we did, we could also have proven the classicality of the relevant space of degree 3 cohomology, and then used Serre duality instead of Arthur's classification to deduce the classicality of f. In order to do this, one would consider $V/\mathfrak{m}_f V$ instead of $V[\mathfrak{m}_f]$, and the same argument would apply with minor modifications.

5. An ordinary modularity lifting theorem for unitary groups with $p = 2$

Our goal in this section is to prove Theorem 5.7.14, which combines an ordinary 2-adic automorphy lifting theorem with a finiteness theorem for a universal deformation ring. This result is a slight variant on the 2-adic automorphy lifting theorems for unitary groups proved by Thorne in [Tho17]; we work with ordinary representations and use Ihara avoidance, and we use a slighter weaker definition of adequacy (see Definition 5.3.3). We emphasize that there are no significant innovations here, and indeed the arguments of [Tho17] go through verbatim using this weaker notion of adequacy. The one minor novelty in our arguments is an argument using base change to allow us to use an auxiliary prime in order to work at neat level; the usual choices of such primes for $p > 2$ rely on choosing a prime at which all Galois deformations are unramified, which is impossible when $p = 2$.

We have endeavored to write out the arguments in enough detail to make it easy for the reader familiar with automorphy lifting theorems for unitary groups which assume that $p > 2$ (e.g. [Tho12, BLGGT14]) but unfamiliar with [Tho17] to check the details (although where these details are literally identical to those of [Tho17] we do not repeat the proofs). We follow the notation of [Tho17] closely, although we assume throughout that $p = 2$, as the analogue for $p > 2$ of our results is already known (see e.g. [Tho17, Cor. 7.3]), and in any case the only use of automorphy lifting theorems for unitary groups in this paper is in the case $p = 2$.

In §5.1, we recall the notion of polarized representations and clarify the relationship between essentially self-dual representations $G_{F^+} \to \mathrm{GSp}_{2n}(R)$ and their associated polarized representations $G_F \to \mathcal{G}_{2n}(R)$. In §5.2, we recall the notion of strong residual oddness defined in [Tho17] and establish some basic facts concerning what this entails for polarized representations associated to essentially self-dual representations over totally real fields with image in $\mathrm{GSp}_{2n}(k)$. (In practice, we only use the special case corresponding to $\mathrm{GSp}_4(\mathbf{F}_2)$.) In §5.3, we discuss variants of the notion of adequateness in characteristic 2 as introduced in [Tho17]. Finally, in sections §5.4, §5.5, §5.6, and §5.7, we adapt the arguments of [Tho17] to our precise setting.

5.1. **Polarized representations.** Let $\mathcal{G}_n$ denote the semi-direct product of $\mathcal{G}_n^0 = \mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, \jmath\}$ where

$$\jmath(g, a)\jmath^{-1} = (ag^{-t}, a).$$

We let $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ be the character which sends (g, a) to a and sends $\jmath$ to -1. (This group, in the context of modularity lifting, was first introduced in [CHT08, §2.1].)

Let F be an imaginary CM field with maximal totally real subfield F^+. For each infinite place v we let $c_v \in G_{F^+}$ denote complex conjugation. Let c denote a fixed arbitrary choice of element in $G_{F^+} \smallsetminus G_F$ with $c^2 = e$ (so for example one could take $c = c_v$ for any $v|\infty$).

Definition 5.1.1. Let k be a perfect field, $\rho : G_F \to \mathrm{GL}_n(k)$ an absolutely irreducible representation, and $\mu : G_{F^+} \to k^\times$ a character. We say that (ρ, μ) is *polarized* if there is a perfect pairing $\langle \cdot, \cdot \rangle : k^n \times k^n \to k$ such that

$$\langle x, y \rangle = -\mu(c)\langle y, x \rangle,$$

and for all $g \in G_F$ we have

$$\langle \rho(g)x, \rho(cgc^{-1})y \rangle = \mu(g)\langle x, y \rangle.$$

We have ([Tho17, Lem. 2.2]):

Lemma 5.1.2. *If (ρ, μ) is polarized, then we may extend ρ to $r : G_{F^+} \to \mathcal{G}_n(k)$ with $\nu \circ r = \mu$ and $r^{-1}(\mathcal{G}_n^0(k)) = G_F$; and this extension is unique up to $\mathcal{G}_n^0(k)$-conjugacy.*

If R is any ring, then $\mathcal{G}_n^0(R)$ acts via conjugation on the set of homomorphisms $r : G_{F^+} \to \mathcal{G}_n(R)$ with $r^{-1}(\mathcal{G}_n^0(k)) = G_F$.

Remark 5.1.3. Note that the $\mathrm{GL}_1(R)$ factor of $\mathcal{G}_n^0(R)$ is not in general acting trivially via conjugation (while it is in the centre of $\mathcal{G}_n^0(R)$, it is not in the centre of $\mathcal{G}_n(R)$). Lemmas 2.1–2.5 of [Tho17] are analogues for $\mathcal{G}_n^0$-conjugacy of lemmas in [CHT08, §2.1] which work instead with GL_n-conjugacy. As well as giving cleaner statements (for example, the extension of r to ρ above is unique up to $\mathcal{G}_n^0(k)$-conjugacy, but the $\mathrm{GL}_n(k)$-conjugacy classes of ρ are in bijection with $k^\times/(k^\times)^2$), the versions of [Tho17] hold in the case of residue characteristic 2, unlike their analogues in [CHT08].

In Section 7.5, we will need to relate representations $G_{F^+} \to \mathrm{GSp}_4(R)$ to representations $G_{F^+} \to \mathcal{G}_4(R)$. We now explain how to do this following the construction of [CHT08, Lem. 2.1]. Write GSp_{2n} for the generalized symplectic group defined by an antisymmetric matrix J_{2n} (in particular, we can take J_4 to be the matrix J that we use to define GSp_4).

Definition 5.1.4. If R is any ring, then we extend the multiplier $\nu : \mathrm{GSp}_{2n}(R) \to R^\times$ to a homomorphism

$$\nu : \mathrm{GSp}_{2n}(R) \times \{\pm 1\} \to R^\times$$

via projection to the GSp_{2n}-factor, i.e. we set $\nu(g, a) := \nu(g)$.

Lemma 5.1.5. *There is an injective homomorphism*

$$r : \mathrm{GSp}_{2n}(R) \times \{\pm 1\} \to \mathcal{G}_{2n}(R)$$

defined as follows:

1. *$r((g, 1)) = (g, \nu(g))$.*
2. *$r((g, -1)) = (g, \nu(g)) \cdot (J_{2n}^{-1}, -1)\jmath$.*

This homomorphism is compatible with ν (defined on the source in Definition 5.1.4).

Proof. The only non-trivial thing to check is that $r((g, -1))r((h, -1)) = r((gh, 1))$, which amounts to the claim that

$$(J_{2n}^{-1}, -1)\jmath(h, \nu(h))(J_{2n}^{-1}, -1)\jmath = (h, \nu(h)).$$

The left hand side is (noting $\jmath = \jmath^{-1}$):

$$\begin{aligned}(J_{2n}^{-1}, -1)\jmath(h, \nu(h))(J_{2n}^{-1}, -1)\jmath &= (J_{2n}^{-1}, -1)\jmath(hJ_{2n}^{-1}, -\nu(h))\jmath^{-1} \\ &= (J_{2n}^{-1}, -1)(-\nu(h)(hJ_{2n}^{-1})^{-t}, -\nu(h)) \\ &= (-\nu(h)J_{2n}^{-1}(hJ_{2n}^{-1})^{-t}, \nu(h)),\end{aligned}$$

so we need to show that $-\nu(h)J_{2n}^{-1}(hJ_{2n}^{-1})^{-t} = h$. This can be rearranged to

$$h^t J_{2n} h = -\nu(h) J_{2n}^t,$$

and since $J_{2n}^t = -J_{2n}$ and $h \in \mathrm{GSp}_{2n}(R)$, we are done. □

Corollary 5.1.6. *If $\psi : G_{F^+} \to \mathrm{GSp}_{2n}(R)$ is a homomorphism, then there is a homomorphism $r_\psi : G_{F^+} \to \mathcal{G}_{2n}(R)$ defined by*

$$r_\psi(g) = \begin{cases} (\psi(g), \nu \circ \psi(g)) & \text{if } g \in G_F \\ (\psi(g), \nu \circ \psi(g)) \cdot (J_{2n}^{-1}, -1)\jmath & \text{if } g \in G_{F^+} \smallsetminus G_F. \end{cases}$$

Furthermore we have $r_\psi^{-1}(\mathcal{G}_{2n}^\circ(R)) = G_F$, and $\nu \circ r_\psi = \nu \circ \psi$.

Proof. There is obviously a homomorphism

$$\widetilde{\psi} : G_{F^+} \to \mathrm{GSp}_{2n}(R) \times \{\pm 1\}$$

given by the product of ψ and the projection onto $G_{F^+}/G_F \cong \{\pm 1\}$, i.e.

$$\widetilde{\psi}(g) = \begin{cases} (\psi(g), 1) & \text{if } g \in G_F \\ (\psi(g), -1) & \text{if } g \in G_{F^+} \smallsetminus G_F. \end{cases} \tag{5.1.7}$$

Furthermore we have $\nu \circ \widetilde{\psi} = \nu \circ \psi$. By definition we have $r_\psi = r \circ \widetilde{\psi}$, where r is as in Lemma 5.1.5, and the result follows immediately. □

Remark 5.1.8. If R is a (perfect) field, then we may apply Lemma 5.1.2 to the representation $\psi|_{G_F}$ of Corollary 5.1.6, and we see that the extension of $\psi|_{G_F}$ to a homomorphism $G_{F^+} \to \mathcal{G}_{2n}(R)$ is well-defined up to $\mathcal{G}^\circ_{2n}(R)$-conjugacy. Corollary 5.1.6 provides a particular choice of extension (depending of course on our choice of symplectic form, i.e. on J_{2n}). The reader may find it helpful to compare to the discussion at the end of [BLGGT14, §1.1], which in the case that R is a field and $\psi|_{G_F}$ is absolutely irreducible shows that the choice of a specific element in the $\mathcal{G}^\circ_{2n}(R)$-conjugacy class amounts to choosing $b_c \in \mathrm{GL}_{2n}(R)$ with

$$\psi(cgc^{-1}) \cdot b_c \psi(g)^t = \nu(g) b_c$$

for all $g \in G_F$. The implicit choice of such an element in Corollary 5.1.6 is $b_c := \psi(c) J_{2n}^{-1}$.

5.2. **Oddness.** We recall the following definition [Tho17, Defn. 3.3].

Definition 5.2.1. Suppose that (ρ, μ) is polarized, that k has characteristic 2, and that n is even. If v is an infinite place of F^+, then we say that (ρ, μ) is *strongly residually odd at* v if $r(c_v)$ is $\mathrm{GL}_n(k)$-conjugate to $(1_n, 1)\jmath$.

Remark 5.2.2. The idea behind Definition 5.2.1 is as follows. A representation $r : G_{F^+} \to \mathcal{G}_n(\overline{\mathbf{Q}}_p)$ is defined to be (totally) odd if $\nu \circ r(c_v) = -1$ for all infinite places of F^+ (see e.g. [BLGGT14, §2.1]). If $p > 2$, then a representation $\overline{r} : G_{F^+} \to \mathcal{G}_n(\overline{\mathbf{F}}_p)$ is odd if $\nu \circ \overline{r}(c_v) = -1$, and any lifting of $\overline{r}$ will automatically be odd. Furthermore, in either case there is a single GL_n-conjugacy class of elements $(x, 1)\jmath$ of order 2 (because all symmetric matrices are equivalent), so the analogue of Definition 5.2.1 holds automatically. If $p = 2$, in contrast, then the condition that $\nu \circ \overline{r}(c_v) = -1$ is automatic. However, if n is even, then there are two conjugacy classes of elements of the form $(x, 1)\jmath$ of order 2 (see [Tho17, Lem. 2.16]). In the situation of Definition 5.2.1, any lift of (ρ, μ) is automatically odd at v, i.e. the lift of μ is odd; this is one motivation for the terminology "strongly residually odd".

In the remainder of this section, we examine when the representations r_ψ of Corollary 5.1.6 are strongly residually odd at some v. We will ultimately only need a single example for GSp_4, but as it is straightforward to give a general treatment, we do so. Assume for the rest of this subsection that k has characteristic 2, so that in particular $J_{2n} = -J_{2n} = J_{2n}^{-1}$. Let $G = \mathrm{GL}_{2n}(k) \times k^\times$, and let $G.2 = G \rtimes \mathbf{Z}/2\mathbf{Z} = \mathcal{G}_{2n}(k)$ where the action by the order 2 element $\jmath$ is by the outer automorphism $\jmath(g, a)\jmath^{-1} = (ag^{-t}, a)$. Let $\chi : G.2 \to \mathbf{Z}/2\mathbf{Z}$ be the canonical projection. By Lemma 5.1.5, there is a natural map $\mathrm{Sp}_{2n}(k) \times \mathbf{Z}/2\mathbf{Z} \to G.2$ given by sending -1 to $J_{2n}\jmath$. If $A \in \mathrm{Sp}_{2n}(k)$ satisfies $A^2 = I$, then $(AJ_{2n}, 1) \cdot \jmath \in G.2 \smallsetminus G$ has order 2. Recall (e.g. from the proof of [Tho17, Lem. 2.16]) that any invertible symmetric matrix M in $\mathrm{GL}_{2n}(k)$ is either equivalent (under $M \mapsto g^t M g$) to J_{2n} or to I; the former if and only if all diagonal entries of M are zero. In the former case we say that M is alternating (since the corresponding non-degenerate pairing satisfies $B(x,x) = 0$ for all x) and otherwise we say that M is non-alternating.

Lemma 5.2.3. *If $A \in \mathrm{Sp}_{2n}(k)$ satisfies $A^2 = I$, then AJ_{2n} is a symmetric matrix and $AJ_{2n} \cdot \jmath \in G.2$ has order 2. This induces a map*

$$\left\{ \begin{array}{c} \textit{Conjugacy classes of } \mathrm{Sp}_{2n}(k) \\ \textit{of order dividing } 2 \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Conjugacy classes of } G.2 \smallsetminus G \\ \textit{of order } 2 \end{array} \right\}$$

The target has order 2 and consists of the conjugacy classes of $\jmath$ and $J_{2n} \cdot \jmath$. The fibre over $J_{2n} \cdot \jmath$ consists of A for which AJ_{2n} is alternating, and the fibre over $\jmath$ consists of A for which AJ_{2n} is not alternating. The set on the right remains unchanged if we only consider order 2 elements in $G.2 \smallsetminus G$ up to conjugation by $\mathrm{GL}_{2n}(k) \subset G \subset G.2$.

Proof. For any element $\gamma \in G.2 \smallsetminus G$, the $G.2$-conjugacy class of γ coincides with the G-conjugacy class of γ, since any element in $G.2$ either has the form $g \in G$ or $g\gamma$ with $g \in G$, and $(g\gamma)\gamma(g\gamma)^{-1} = g\gamma g^{-1}$. If $\gamma = (B, b)\jmath$ and $g = (I_{2n}, \lambda) \in G$, then

$$g\gamma g^{-1} = (I_{2n}, \lambda)(B, b)\jmath(I_{2n}, \lambda^{-1}) = (\lambda^{-1} \cdot B, b)\jmath,$$

but writing $\lambda^{-1} = \mu^2$ (possible since k is finite of characteristic 2), and taking $h = (\mu I_{2n}, 1) \in \mathrm{GL}_{2n}(k) \subset G$, we also have

$$h\gamma h^{-1} = (\mu I_{2n}, 1)(B, b)\jmath(\mu^{-1} I_{2n}, 1) = (\mu^2 \cdot B, b)\jmath = (\lambda^{-1} \cdot B, b)\jmath,$$

and so the G and $\mathrm{GL}_{2n}(k)$-conjugacy classes of $\gamma \in G.2 \smallsetminus G$ also coincide.

If $A^2 = I$, then $(A, -1) \in \mathrm{Sp}_{2n}(k) \times \mathbf{Z}/2\mathbf{Z}$ has order 2 so the image $AJ_{2n} \cdot \jmath \in G.2$ certainly has order 2 and does not lie in G. Moreover, this map certainly induces a map on conjugacy classes because if A is conjugate to A' in $\mathrm{Sp}_{2n}(k)$, then $(A, -1)$ is conjugate to $(A', -1)$ in $\mathrm{Sp}_{2n}(k) \times \mathbf{Z}/2\mathbf{Z}$. The condition that $(B, b) \cdot \jmath \in G.2$ has order 2 is equivalent to the equation

$$(B, b)\jmath(B, b)\jmath = (b \cdot B(B^t)^{-1}, b^2) = (I, 1),$$

which implies that $b^2 = 1$ and so $b = 1$ and $B = B^t$ is symmetric. Let $\Sigma_{2n} \subset \mathrm{GL}_{2n}(k)$ denote the set of symmetric matrices. Conjugation by $(M, 1) \in \mathrm{GL}_{2n}(k) \subset G$ replaces $(B, 1)\jmath$ by $(MBM^t, 1)\jmath$. Hence the order 2 conjugacy classes in $G.2 \smallsetminus G$ are the orbits of $\mathrm{GL}_{2n}(k)$ acting via MBM^t on Σ_{2n}. But the orbits of $\mathrm{GL}_{2n}(k)$ on this space are none other than the equivalence class of perfect pairings on k^{2n}, and as recalled above, there are two such orbits, corresponding to $B = I$ and $B = J_{2n}$. □

5.2.4. *Involutions in* $\mathrm{Sp}_{2n}(k)$. (See the proof of [FGS17, Lem. 4.3]). An involution A in $\mathrm{Sp}_{2n}(k)$ acting on the natural representation V preserves a flag

$$0 \subset (A - 1)V \subset V^A \subset V, \tag{5.2.5}$$

where we write $r := \dim((A-1)V) \le n$ for the rank of $A - 1$. With respect to this flag, one can (by [FGS17, Lem. 4.4]) write A in the form

$$A = \begin{pmatrix} I_r & 0 & S_r \\ 0 & I_{2n-2r} & 0 \\ 0 & 0 & I_r \end{pmatrix},$$

where S_r has rank r and $S_r J_r$ is symmetric. The parabolic stabilizing the flag (5.2.5) acts on the matrices of this form, and the orbit corresponds to all symmetric matrices equivalent to $S_r J_r$. Accordingly, if r is odd, the conjugacy class of A is determined by r. If $r > 0$ is even, there are two conjugacy classes corresponding to $S_r = I_r$ and $S_r = J_r$. In total there are $n+1+\lfloor n/2 \rfloor$ conjugacy classes. We see that

$$AJ_{2n} = \begin{pmatrix} S_r J_r & 0 & J_r \\ 0 & J_{2n-2r} & 0 \\ J_r & 0 & 0 \end{pmatrix} \tag{5.2.6}$$

This is non-alternating if either r is odd or r is even and $S_r = J_r$. Any involution $A \in \mathrm{GSp}_{2n}(k)$ must have $\nu(A)^2 = 1$ and thus $\nu(A) = 1$, and hence must lie in $\mathrm{Sp}_{2n}(k)$. In particular, from the discussion above and Lemma 5.2.3, we have the following:

Lemma 5.2.7. *Let $\overline{\psi} : G_{F^+} \to \mathrm{GSp}_{2n}(k)$ with k of characteristic 2, and let $r_{\overline{\psi}} : G_{F^+} \to \mathcal{G}_{2n}(k)$ be as in Lemma 5.1.6. Let v be an infinite place of F^+. Then the polarized pair $(\overline{\psi}|_{G_F}, \nu \circ \overline{\psi})$ is strongly residually odd at v if and only if either*

1. *$(\overline{\psi}(c_v) - I)$ has odd rank, or*
2. *$(\overline{\psi}(c_v) - I)$ has even rank $r > 0$, and the matrix $S_r J_r$ obtained from (5.2.6) with $A := \overline{\psi}(c_v)$ is non-alternating.*

Equivalently $(\overline{\psi}|_{G_F}, \nu \circ \overline{\psi})$ is strongly residually odd at v if and only if the quadratic form associated $\overline{\psi}(c_v)J_{2n}$ is equivalent to the one associated with I_{2n}, which occurs if and only if $\overline{\psi}(c_v)J_{2n}$ has at least one non-zero diagonal entry.

Remark 5.2.8. If $n = 1$, then either A is trivial or $A - I$ has rank $r = 1$. Hence in this case strong residual oddness is equivalent to $A \neq I$ (cf. [Tho17, Lem. 3.5(ii)]). This is no longer true for $n > 1$; there are n conjugacy classes of involutions giving rise to strongly residually odd representations and $1 + \lfloor n/2 \rfloor$ conjugacy classes of involutions which do not. In particular, for $2n = 4$, there are two conjugacy classes of involutions giving rise to strongly residually odd representations and two (one of which is the identity) which do not. Explicit representatives for the odd classes can be given as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the latter is conjugate to J_4, and an explicit representative for the non-trivial non-odd involution is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When we later fix (in Lemma 8.1.3) an explicit isomorphism $S_6 \simeq \mathrm{Sp}_4(\mathbf{F}_2)$, the first two elements can be identified with the images of $(1,2)$, and $(1,2)(3,4)$ respectively, whereas the latter can be identified with $(1,2)(3,5)(4,6)$ (See also the proof of Lemma 8.2.4.)

5.3. **Adequacy.** Let $\mathrm{ad} = \mathrm{Hom}(V, V)$ denote the adjoint representation and $\mathrm{ad}^0 \subset \mathrm{Hom}(V, V)$ the submodule of trace zero endomorphisms. We begin with the following lemma just to clarify that the definition of weakly adequate used in [Tho17, Defn. 2.20] (which is case (1) of Lemma 5.3.1 below) agrees with other definitions in the literature.

Lemma 5.3.1. *Let V be a finite-dimensional vector space over a finite field k, and let $H \subseteq \mathrm{GL}(V)$. The following conditions are equivalent:*

(1) *For each simple $\overline{k}[H]$-submodule $W \subset \mathrm{ad} \otimes \overline{k}$, there exists a semi-simple element $\sigma \in H$ with an eigenvalue $\alpha \in \overline{k}$ such that $\mathrm{tr}(e_{\sigma,\alpha} W) \neq 0$. (Here $e_{g,\alpha}$ is the g-equivariant projection onto the generalized α-eigenspace of g.)*
(2) *For each simple $\overline{k}[H]$-submodule $W \subset \mathrm{ad}^0 \otimes \overline{k}$, there exists a semi-simple element $\sigma \in H$ with an eigenvalue $\alpha \in \overline{k}$ such that $\mathrm{tr}(e_{\sigma,\alpha} W) \neq 0$.*
(3) *$\mathrm{End}(V)$ is spanned by the set H^{ss} of semi-simple elements of H.*

Proof. This follows directly from the proof of [Tho12, Lemma A.1]. More precisely, it is shown there that we have an equality

$$\begin{aligned} U &:= \{w \in \mathrm{ad} \otimes \overline{k} : \mathrm{tr}(gw) = 0 \quad \forall g \in H^{\mathrm{ss}}\} \\ &= \{w \in \mathrm{ad} \otimes \overline{k} : \mathrm{tr}(e_{g,\alpha} w) = 0 \quad \forall g \in H^{\mathrm{ss}}, \alpha \in \overline{k}\}. \end{aligned}$$

Note that U is an H-submodule of $\mathrm{ad} \otimes \overline{k}$; suppose that $w \in U$, $g \in H^{\mathrm{ss}}$, and $h \in H$. The element h via the natural action sends w to hwh^{-1}, and then

$$\mathrm{tr}(ghwh^{-1}) = \mathrm{tr}(h^{-1}ghw) = 0,$$

since $h^{-1}gh \in H^{\mathrm{ss}}$ and $w \in U$. Condition (3) is equivalent to $U = 0$, whereas conditions (1) and (2) above are equivalent to the intersection of U with the socle of $\mathrm{ad} \otimes \overline{k}$ (respectively, the socle of $\mathrm{ad}^0 \otimes \overline{k}$) being trivial. Since $U \subset \mathrm{ad}^0 \otimes \overline{k}$ (take $g = 1$), the result follows. □

Definition 5.3.2. We say that $H \subseteq \mathrm{GL}(V)$ is *weakly adequate* if the equivalent conditions of Lemma 5.3.1 hold for H. A representation $\rho : G \to \mathrm{GL}(V)$ is *weakly adequate* if $\mathrm{im}(\rho)$ is weakly adequate.

As remarked in [Tho17] (after Definition 2.20), if H is weakly adequate, then H acts absolutely irreducibly on V as a consequence of condition 3 of Lemma 5.3.1.

Definition 5.3.3. Let k be a subfield of $\overline{\mathbf{F}}_2$. We say that a finite subgroup $H \subset \mathrm{GL}_n(k)$ is *nearly adequate* if:

(1) H is weakly adequate.
(2) $H^1(H, k) = 0$.
(3) $H^1(H, \mathrm{ad}) = 0$.

We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *nearly adequate* if $\mathrm{im}(\rho)$ is nearly adequate.

Remark 5.3.4. The definition of a nearly adequate subgroup is almost the same as the definition of an adequate subgroup [Tho17, Defn. 2.20]. Indeed the only difference is that we are assuming that $H^1(H, \mathrm{ad}) = 0$, rather than the stronger assumption that $H^1(H, \mathrm{ad}/k) = 0$ (which implies the vanishing of $H^1(H, \mathrm{ad})$ in conjunction with the assumption that $H^1(H, k) = 0$).

The point of our definition is that, as we explain below, the arguments of [Tho17] apply unchanged with "adequate" relaxed to "nearly adequate", and in our applications we will need to work with representations which are nearly adequate, but not adequate. (See Lemma 8.2.2 and Remark 8.2.3.)

5.4. **Galois deformation theory.** We now recall some facts about Galois deformation theory when $p = 2$. The results we need are essentially identical to those of [Tho17, §2.1], except that we need to work relative to a larger coefficient ring (the weight space Λ), which we do following [KT17b, §4].

We continue to assume that F is an imaginary CM field with maximal totally real subfield F^+, and we assume that F/F^+ is everywhere unramified, and that all places of F^+ dividing 2 split in F. We write S_2 for the set of places of F^+ dividing 2, S_∞ for the set of places of F^+ dividing ∞, and S for a finite set of places of F^+ containing $S_2 \cup S_\infty$. Let $F(S)$ be the maximal Galois extension of F unramified outside of S, and write $G_{F^+,S} := \mathrm{Gal}(F(S)/F^+)$, $G_{F,S} := \mathrm{Gal}(F(S)/F)$. Throughout this section we will use the notation established in Section 1.8, specialized to the case $p = 2$, so that for example we have our field of coefficients $E/\mathbf{Q}_2$ with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field k.

We fix a representation $\overline{r} : G_{F^+,S} \to \mathcal{G}_n(k)$ such that $\overline{r}^{-1}(\mathcal{G}_n^0(k)) = G_{F,S}$, together with a character $\chi : G_{F^+,S} \to \mathcal{O}^\times$ with $\overline{\chi} = \nu \circ \overline{r}$ and $\chi(c_v) = -1$ for all $v \in S_\infty$. We abusively write $\overline{r}|_{G_{F,S}}$ for the representation given by restriction to $G_{F,S}$ and projection to the $\mathrm{GL}_n(k)$ factor in $\mathcal{G}_n^0(k)$. We assume that $\overline{r}|_{G_{F,S}}$ is absolutely irreducible. We often write $\overline{\rho}$ for $\overline{r}|_{G_{F,S}}$.

For each $v \in S$, we fix $\Lambda_v \in \mathrm{CNL}_{\mathcal{O}}$, and set $\Lambda = \widehat{\otimes}_{v\in S}\Lambda_v$, where the completed tensor product is taken over $\mathcal{O}$. For each $v \in S$, the canonical map $\Lambda_v \to \Lambda$ induces the forgetful functor $\mathrm{CNL}_\Lambda \to \mathrm{CNL}_{\Lambda_v}$.

As in [Tho17, Defn. 2.6], a *lifting* of $\overline{r}|_{G_{F_v^+}}$ to a CNL_{Λ_v}-algebra A is a continuous homomorphism $r_v : G_{F_v^+} \to \mathcal{G}_n(A)$ such that $r_v \bmod \mathfrak{m}_A = \overline{r}|_{G_{F_v^+}}$ and $\nu \circ r_v = \chi|_{G_{F_v^+}}$. We let $\mathrm{Lift}_v^\square : \mathrm{CNL}_{\Lambda_v} \to \mathrm{Sets}$ be the functor sending A to the set of liftings of $\overline{r}|_{G_{F_v^+}}$. The functor $\mathrm{Lift}_v^\square$ is representable by an object $R_v^\square \in \mathrm{CNL}_{\Lambda_v}$. If v splits in F, and r_v is a lifting of $\overline{r}|_{G_{F_v^+}}$, then $r_v(G_{F_v^+}) \subseteq \mathcal{G}_n^0(A)$, and we sometimes write $\rho_v : G_{F_v^+} \to \mathrm{GL}_n(A)$ for the projection of r_v to the GL_n factor.

A *local deformation problem* for $\overline{r}|_{G_{F_v^+}}$ is a representable subfunctor $\mathcal{D}_v \subseteq \mathrm{Lift}_v^\square$ such that for all $A \in \mathrm{CNL}_{\Lambda_v}$, the set $\mathcal{D}_v(A)$ is invariant under the conjugation action of $\widehat{\mathcal{G}}_n(A)$ on $\mathrm{Lift}_v(A)$.

A *global deformation problem* is a tuple

$$\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S}),$$

where:

- F, $\overline{r}$, $\mathcal{O}$, χ, S, and $\{\Lambda_v\}_{v\in S}$ are as above.
- For each $v \in S$, $\mathcal{D}_v$ is a local deformation problem for $\overline{\rho}|_{G_{F_v^+}}$.

As in the local case, a *lift* (or *lifting*) of $\overline{r}$ is a continuous homomorphism $r : G_F \to \mathcal{G}_n(A)$ to a CNL_Λ-algebra A, such that $r \bmod \mathfrak{m}_A = \overline{r}$. We say that two lifts $r_1, r_2 : G_F \to \mathcal{G}_n(A)$ are *strictly equivalent* if there is an $a \in \widehat{\mathcal{G}}_n(A)$ such that $r_2 = ar_1a^{-1}$. A *deformation of* $\overline{r}$ is a strict equivalence class of lifts of $\overline{r}$.

For a global deformation problem

$$\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S}),$$

we say that a lift $r : G_F \to \mathcal{G}_n(A)$ is of *type* $\mathcal{S}$ if $r|_{G_{F_v}} \in \mathcal{D}_v(A)$ for each $v \in S$. Note that if r_1 and r_2 are strictly equivalent lifts of $\overline{r}$, and r_1 is of type $\mathcal{S}$, then so is r_2. A *deformation of type* $\mathcal{S}$ is a strict equivalence class of lifts of type $\mathcal{S}$, and we denote by $\mathrm{Def}_{\mathcal{S}}$ the set-valued functor that takes a CNL_Λ-algebra A to the set of deformations $r : G_F \to \mathcal{G}_n(A)$ of type $\mathcal{S}$.

Given a subset $T \subseteq S$, a *T-framed lift of type* $\mathcal{S}$ is a tuple $(r, \{\alpha_v\}_{v\in T})$, where $r : G_F \to \mathcal{G}_n(A)$ is a lift of $\overline{r}$ of type $\mathcal{S}$ and $\alpha_v \in \widehat{\mathcal{G}}_n(A)$ for each $v \in T$. We say that two T-framed lifts $(r_1, \{\alpha_v\}_{v\in T})$ and $(r_2, \{\beta_v\}_{v\in T})$ to a CNL_Λ-algebra A are strictly equivalent if there is an $a \in \widehat{\mathcal{G}}_n(A)$ such that $r_2 = ar_1a^{-1}$ and $\beta_v = a\alpha_v$ for each $v \in T$. A strict equivalence class of T-framed lifts of type $\mathcal{S}$ is called a *T-framed deformation of type* $\mathcal{S}$. We denote by $\mathrm{Def}^T_{\mathcal{S}}$ the functor $\mathrm{CNL}_\Lambda \to \mathrm{Sets}$ taking A to the set of T-framed deformations to A of type $\mathcal{S}$.

Let $\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem, and let T be a subset of S. The functors $\mathrm{Def}_{\mathcal{S}}$ and $\mathrm{Def}^T_{\mathcal{S}}$ are representable (see [Tho17, Lem. 2.8, Lem. 2.10]); we denote their representing objects by $R_{\mathcal{S}}$ and $R^T_{\mathcal{S}}$, respectively. If $T = \emptyset$, then tautologically $R_{\mathcal{S}} = R^T_{\mathcal{S}}$, while if T is nonempty, $R^T_{\mathcal{S}}$ is a formally smooth $R_{\mathcal{S}}$-algebra of relative dimension $(n^2+1)\#T - 1$.

Let T be a (possibly empty) subset of S such that $\Lambda_v = \mathcal{O}$ for all $v \in S \smallsetminus T$. Write R_v for the representing object of $\mathcal{D}_v$, and define $R^{\mathrm{loc}}_{\mathcal{S},T} = \widehat{\otimes}_{v\in T} R_v$, with the completed tensor product being taken over $\mathcal{O}$. It is canonically a Λ-algebra, via the canonical isomorphism $\widehat{\otimes}_{v\in T}\Lambda_v \cong \widehat{\otimes}_{v\in S}\Lambda_v$. For each $v \in T$, the natural transformation $\mathrm{Def}^T_{\mathcal{S}} \to \mathcal{D}_v$ given by $(\rho, \{\alpha_v\}_{v\in T}) \mapsto \alpha_v^{-1}\rho|_{G_{F_v}}\alpha_v$ induces a morphism $R_v \to R^T_{\mathcal{S}}$ in CNL_{Λ_v}. We thus have a morphism $R^{\mathrm{loc}}_{\mathcal{S},T} \to R^T_{\mathcal{S}}$ in CNL_Λ.

In [Tho17, §2.2], the relative tangent space to this morphism is computed via Galois cohomology. (Strictly speaking this reference has $\Lambda_v = \mathcal{O}$ for all v, but the Λ-algebra structure does not intervene in the calculation.) More precisely, there is an explicit chain complex of k-vector spaces $C^i_{\mathcal{S},T}$ with cohomology groups $H^i_{\mathcal{S},T}$ of k-dimensions $h^i_{\mathcal{S},T}$, and by [Tho17, Lem. 2.12], we have $\dim_k \mathfrak{m}_{R^T_{\mathcal{S}}}/(\mathfrak{m}_{R^{\mathrm{loc}}_{\mathcal{S},T}}, \mathfrak{m}^2_{R^T_{\mathcal{S}}}) = h^1_{\mathcal{S},T}$, so that there is a surjection of $R^{\mathrm{loc}}_{\mathcal{S},T}$-algebras $R^{\mathrm{loc}}_{\mathcal{S},T}[\![X_1,\ldots,X_{h^1_{\mathcal{S},T}}]\!] \twoheadrightarrow R^T_{\mathcal{S}}$. Since we do not need any properties of $C^i_{\mathcal{S},T}$ and its cohomology groups beyond those proved in [Tho17, §2.2], we do not recall their definitions here. (It may however be helpful to point out that there is a typo in the definition of $C^i_{\mathcal{S},T}$ for $i \geq 3$: the sum over places $v \in S$ should be of $C^{i-1}(F^+_v, \mathrm{ad}\,\overline{r})$, not $C^{i-1}(F(S)/F^+, \mathrm{ad}\,\overline{r})$ as written in [Tho17].) We do however need to consider a certain dual Selmer group $H^1_{\mathcal{S}^\perp,T}$ of k-dimension $h^1_{\mathcal{S}^\perp,T}$, and we now turn to its definition (see (5.4.3) below).

We identify $\mathrm{ad}\,\overline{r}$ with $\widehat{\mathrm{GL}}_n(k[\epsilon])$, and we write $\mathfrak{g}_n\overline{r}$ for the adjoint representation on $\widehat{\mathcal{G}}_n(k[\epsilon])$, so that we have an exact sequence of $G_{F^+,S}$-modules

$$0 \to \mathrm{ad}\,\overline{r} \to \mathfrak{g}_n\overline{r} \to k \to 0. \tag{5.4.1}$$

For each $v \in S$, we as usual identify $\mathrm{Lift}^\square_v(k[\epsilon])$ with the cocycles $Z^1(F^+_v, \mathrm{ad}\,\overline{r})$, so that two liftings to $k[\epsilon]$ are $\mathcal{G}_n(k[\epsilon])$-conjugate if and only if the images of the corresponding cocycles in $H^1(F^+_v, \mathfrak{g}_n\overline{r})$ coincide. (If v splits in F, this is equivalent to their images coinciding in $H^1(F^+_v, \mathrm{ad}\,\overline{r})$.) We write $\mathcal{L}^1_v \subseteq Z^1(F^+_v, \mathrm{ad}\,\overline{r})$

for the cocycles corresponding to liftings in $\mathcal{D}_v(k[\epsilon])$, and $\mathcal{L}_v$ for the image of $\mathcal{L}_v^1$ in $H^1(F_v^+, \operatorname{ad}\overline{r})$. We write l_v^1, l_v for the dimensions of the k-vector spaces $\mathcal{L}_v^1, \mathcal{L}_v$ respectively. We define

$$\mu_v := \ker\left(H^1(F_v^+, \operatorname{ad}\overline{r}) \to H^1(F_v^+, \mathfrak{g}_n\overline{r})\right).$$

From the long exact sequence in cohomology associated to (5.4.1), there is a natural identification

$$\mu_v \simeq \operatorname{im}\left(H^0(F_v^+, k) \to H^1(F_v^+, \operatorname{ad}\overline{r})\right), \tag{5.4.2}$$

and so $\dim(\mu_v) \leq 1$. Note that since (by definition) $\mathcal{D}_v$ is stable under conjugation by $\widehat{\mathcal{G}}_n$, we have $\mu_v \subseteq \mathcal{L}_v$, and μ_v is trivial if v splits in F (since in this case (5.4.1) splits as a sequence of $G_{F_v^+}$-modules). (Note however that the places in S_∞ do not split in F, and indeed if (ρ, μ) is strongly residually odd at $v \in S_\infty$ in the sense of Definition 5.2.1, then μ_v is 1-dimensional by [Tho17, Lem. 2.17(ii)].)

The trace pairing $(X, Y) \mapsto \operatorname{tr}(XY)$ on $\operatorname{ad}\overline{r}$ is perfect and $G_{F^+,S}$-equivariant, so $\operatorname{ad}\overline{r}(1)$ is isomorphic to the Tate dual of $\operatorname{ad}\overline{r}$. (Of course, since $p = 2$, the Tate twist is trivial, and $\operatorname{ad}\overline{r}$ is self-dual, but to avoid confusing the reader who is used to the case $p > 2$, we preserve the Tate twist in our notation below.) For each $v \in S$ we let $\mathcal{L}_v^\perp \subseteq H^1(F_v^+, \operatorname{ad}\overline{r})$ be the annihilator of $\mathcal{L}_v$ under this pairing, and write $\mu_v^\perp \supseteq \mathcal{L}_v^\perp$ for the annihilator of μ_v.

For any $T \subseteq S$ as above (i.e. for any T such that $\Lambda_v = \mathcal{O}$ for all $v \in S \smallsetminus T$) we define the dual Selmer group

$$H^1_{\mathcal{S}^\perp,T} := \ker\left(H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1)) \to \prod_{v\in T} H^1(F_v^+, \operatorname{ad}\overline{r}(1))/\mu_v^\perp \times \prod_{v \in S\smallsetminus T} H^1(F_v^+, \operatorname{ad}\overline{r}(1))/\mathcal{L}_v^\perp\right). \tag{5.4.3}$$

As usual, we write $h^1_{\mathcal{S}^\perp,T} := \dim_k H^1_{\mathcal{S}^\perp,T}$. Assume that T is nonempty; then by [Tho17, Lem. 2.15] and our assumption that $\overline{r}|_{G_{F,S}}$ is absolutely irreducible, the $h^i_{\mathcal{S},T}$ vanish for $i \neq 1, 2$, and we have $h^2_{\mathcal{S},T} = h^1_{\mathcal{S}^\perp,T}$. (The assumption that T is nonempty guarantees the vanishing of $h^0_{\mathcal{S},T}$, and the vanishing of $h^i_{\mathcal{S},T}$ for $i \geq 4$ is automatic, as in the proof of [Tho17, Lem. 2.13].)

Remark 5.4.4 (Remarks on μ_v and our deformation problem.)**.** We now try to explain where the terms $\mu_v^\perp$ in (5.4.3) come from. A possibly unilluminating answer is that they are necessary in order to prove that $h^2_{\mathcal{S},T} = h^1_{\mathcal{S}^\perp,T}$. Indeed, the proof [Tho17, Lem. 2.15] is as usual via the Poitou–Tate sequence, and the $\mu_v^\perp$ arise because of the appearance of the cohomology groups $H^1(F_v^+, \operatorname{ad}\overline{r})^\eta$ in the long exact sequence in [Tho17] computing the $H^i_{\mathcal{S},T}$. Here $H^1(F_v^+, \operatorname{ad}\overline{r})^\eta$ is by definition the image of the map

$$H^1(F_v^+, \operatorname{ad}\overline{r}) \to H^1(F_v^+, \mathfrak{g}_n\overline{r}),$$

and μ_v is its kernel, whence the appearance of $\mu_v^\perp$ as the dual Selmer condition.

A possibly more helpful explanation is as follows. In the usual Kisin modification of the Taylor–Wiles method, when one presents a global deformation ring over a (completed tensor product of) local deformation rings at primes $v \in T$, the corresponding Selmer condition $\mathcal{L}_v$ at v is trivial. In our setting (exactly as in [Tho17]) we are considering deformations $\rho : G_{F_v^+} \to \mathcal{G}_n(A)$ which are equivalent under $\widehat{\mathcal{G}}_n(A)$. As in Remark 5.1.3, conjugation by the $\widehat{\operatorname{GL}}_1(A)$ factor of $\widehat{\mathcal{G}}(A)$

does not in general act trivially on deformations. The group μ_v exactly measures deformations ρ which are not equivalent under conjugation by $\widehat{\mathrm{GL}}_n(A)$ but become equivalent under conjugation by $\widehat{\mathcal{G}}(A)$.

An alternative way to view these general deformation problems would be to work purely with conjugate self-dual G_F-representations. In this setting, the Selmer groups are naturally subgroups of $H^1(F(S)/F, \mathrm{ad}\,\overline{r})^{\mathrm{Gal}(F/F^+)}$. To compare these approaches, note that, (since $\overline{r}$ is irreducible so $(\mathrm{ad}\,\overline{r})^{G_F} = k$) there is a natural inflation–restriction sequence:

$$\begin{aligned}0 &\to H^1(F/F^+, k) \to H^1(F(S)/F^+, \mathrm{ad}\,\overline{r}) \xrightarrow{\mathrm{res}} H^1(F(S)/F, \mathrm{ad}\,\overline{r})^{\mathrm{Gal}(F/F^+)} \\ &\to H^2(F/F^+, k) \to H^2(F(S)/F^+, k).\end{aligned} \tag{5.4.5}$$

The first group is 1-dimensional, and (for example by an explicit cocycle computation as in the proof of Lemma 5.5.3 below) its localization at any prime $v \in S$ agrees with μ_v. On the other hand, since F/F^+ is CM, for any real place v, the composite map

$$H^2(F/F^+, k) \to H^2(F(S)/F^+, k) \to H^2(F_v^+, k)$$

is injective (and indeed an isomorphism), so the restriction map in (5.4.5) is surjective.

5.5. **Taylor–Wiles systems.** We briefly recall the deformation condition that we use at Taylor–Wiles primes, and the notion of a Taylor–Wiles system, following [Tho17, §§2.3.2, 2.4]. A Taylor–Wiles prime is a finite place v of F^+ which splits in F and is such that $\overline{r}$ is unramified at v with $\overline{r}(\mathrm{Frob}_v)$ semi-simple. At such a v we choose an eigenvalue $\alpha_v \in k$ of multiplicity n_1, and decompose

$$\overline{\rho}|_{G_{F_v^+}} = \overline{A}_v \oplus \overline{B}_v \tag{5.5.1}$$

with $A_v(\mathrm{Frob}_v) = \alpha_v \cdot 1_{n_1}$. The local deformation problem $\mathcal{D}_v^{\mathrm{TW}}$ is given for each $R \in \mathrm{CNL}_\Lambda$ by declaring that $r \in \mathcal{D}_v^{\mathrm{TW}}(R)$ if there is a decomposition

$$\rho = A_v \oplus B_v$$

lifting (5.5.1), with B_v unramified and $A_v|_{I_{F_v^+}} = \psi_v \cdot 1_{n_1}$ for some $\psi_v : I_{F_v^+} \to R^\times$. Note that while $\mathcal{D}_v^{\mathrm{TW}}$ depends on the choice of α_v, it is omitted from the notation. Note also that $\mathcal{D}_v^{\mathrm{TW}}$ is indeed a local deformation problem, by [Tho12, Lem. 4.2].

We write $\Delta_v = k(v)^\times(2)$ for the 2-part of $k(v)^\times$. For any $\rho \in \mathcal{D}_v^{\mathrm{TW}}(R)$, the character $\psi_v \circ \mathrm{Art}_{F_v^+}$ gives a canonical homomorphism $\Delta_v \to R^\times$.

Definition 5.5.2. Let $\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem, and set $T = S \smallsetminus S_\infty$. For each $N \geq 1$, a *Taylor–Wiles datum of level N* is a pair $(\mathcal{Q}, (\alpha_v)_{v\in\mathcal{Q}})$ such that

(i) $\mathcal{Q}$ is a finite set of places of F^+.
(ii) For each $v \in \mathcal{Q}$, we have $v \notin S$, and v splits completely in $F(\zeta_{2^N})$.
(iii) For each $v \in \mathcal{Q}$, $\overline{\rho}(\mathrm{Frob}_v)$ is semi-simple, and $\alpha_v \in k$ is an eigenvalue of $\overline{\rho}(\mathrm{Frob}_v)$.

For each Taylor–Wiles datum $(\mathcal{Q}, (\alpha_v)_{v\in\mathcal{Q}})$, we define the corresponding *augmented global deformation problem*

$$\mathcal{S}_\mathcal{Q} = (F, \overline{r}, \mathcal{O}, \chi, S \cup \mathcal{Q}, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S} \cup \{\mathcal{D}_v^{\mathrm{TW}}\}_{v\in\mathcal{Q}}),$$

where for each $v \in \mathcal{Q}$ the local deformation problem $\mathcal{D}_v^{\mathrm{TW}}$ is defined using the choice of eigenvalue $\alpha_v \in k$. Write $\Delta_{\mathcal{Q}} := \prod_{v\in\mathcal{Q}} k(v)^\times(2)$. Then the canonical homomorphisms $\Delta_v \to R^\times$ give a canonical homomorphism $\mathcal{O}[\Delta_{\mathcal{Q}}] \to R_{\mathcal{S}_{\mathcal{Q}}}$, and we have a canonical identification $R_{\mathcal{S}_{\mathcal{Q}}} \otimes_{\mathcal{O}[\Delta_{\mathcal{Q}}]} \mathcal{O} = R_{\mathcal{S}}$.

The following lemma shows that it is not possible to kill all the classes in the dual Selmer group by adding Taylor–Wiles primes. We will shortly see that it is however possible to kill all but one class (more precisely, all but a one-dimensional space of classes), and that this is enough to patch.

Lemma 5.5.3. *Let $\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem. Set $T = S \smallsetminus S_\infty$. Then for N sufficiently large with respect to F^+, and any Taylor–Wiles datum $\mathcal{Q}$ of level N, we have $h^1_{\mathcal{S}_{\mathcal{Q}}^\perp,T} \geq 1$. In particular, taking $\mathcal{Q} = \emptyset$, we have $h^1_{\mathcal{S}^\perp,T} \geq 1$.*

Proof. Let K_m^+ be the maximal totally real subfield of $\mathbf{Q}(\zeta_{2^m})$. Assume without loss of generality that $K_{m-1}^+ \subseteq F^+ \subsetneq K_m^+$. Let L^+/F^+ be the totally real quadratic extension of F^+ given by $L^+ = F^+.K_m^+$, so $L^+ \subset F^+(\zeta_{2^m}) \subseteq F(\zeta_{2^m})$, and assume that $N \geq m$. We claim that the class ψ in $H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1))$ which is inflated from $H^1(L^+/F^+, k)$ (via the inclusion of the scalar matrices into $\operatorname{ad}\overline{r} = \operatorname{ad}\overline{r}(1)$) is necessarily contained in $H^1_{\mathcal{S}_{\mathcal{Q}}^\perp,T}$ for all $\mathcal{Q}$ of level $N \geq m$.

Since L^+ is totally real, ψ is trivial at all of the infinite places of F^+. In addition, each prime in $\mathcal{Q}$ splits completely in $F(\zeta_{2^N})$ and hence also in $F(\zeta_{2^m})$ and thus in L^+. Hence ψ is also trivial at all of the places in $\mathcal{Q}$.

It remains to show that (the restriction of) ψ is contained in $\mu_v^\perp$ for each finite place $v \in S$. Let $\Delta_v \subset H^1(F_v^+, \operatorname{ad}\overline{r})$ denote the the image of the map $H^1(F_v^+, k) \to H^1(F_v^+, \operatorname{ad}\overline{r})$ induced by the map $k \to \operatorname{ad}\overline{r}$. Certainly $\psi_v \in \Delta_v$; we now show that $\mu_v \subset \Delta_v$ and then analyze the pairing $\Delta_v \times \Delta_v \to k$.

From (5.4.2) we have $\mu_v = \operatorname{im}(\delta_v)$, where

$$\delta_v : k \to H^1(F_v^+, \operatorname{ad}\overline{r})$$

is the boundary map in the long exact sequence in cohomology obtained from the action of $G_{F_v^+}$ on (5.4.1). The short exact sequence (5.4.1) has a G_F-equivariant splitting $\mathfrak{g}_n\overline{r} \simeq \operatorname{ad}\overline{r} \oplus k$. Choose a lift $(0, 1)$ of $1 \in k$ compatible with this splitting. The corresponding cocycle $\phi = \delta_v(1)$ vanishes on G_F and sends $c \in G_{F^+} \smallsetminus G_F$ to $c(0, 1) - (0, 1) = (I_n, 0) \in \operatorname{ad}\overline{r}$, so $\phi \in \Delta_v$, i.e. $\mu_v \subset \Delta_v$.

Since the images of cocycles in Δ_v are contained in $k \subset \operatorname{ad}\overline{r}$ and since the self-duality on $\operatorname{ad}\overline{r}$ is given by $(X, Y) \mapsto \operatorname{Tr}(XY)$, the Tate pairing $\Delta_v \times \Delta_v \to k$ may be computed by first evaluating the pairing on $k \subset \operatorname{ad}\overline{r}$ and then multiplying by $n = \operatorname{Tr}(I_n)$. If n is even, it follows that $\Delta_v \subset \Delta_v^\perp$, so that $\Delta_v \subset \mu_v^\perp$, as required. This completes the proof when n is even (which is the case which we ultimately use). When n is odd, we must investigate more closely the pairing on $k \subset \operatorname{ad}\overline{r}$.

The Tate pairing

$$H^1(F_v^+, k) \times H^1(F_v^+, k) \to H^2(F_v^+, k) = k \tag{5.5.4}$$

for any v is given by the (local) Hilbert symbol:

$$\langle *, * \rangle_v : F^{+,\times}/F^{+,\times 2} \times F^{+,\times}/F^{+,\times 2} \to \mu_2. \tag{5.5.5}$$

More precisely, the relation between (5.5.4) and (5.5.5) is given by first identifying k with $\mu_2 \otimes k$ and $H^1(F_v^+, \mu_2)$ with $F^{+,\times}/F^{+,\times 2}$, and then tensoring (5.5.5) with k.

The class associated to $\mu_v \subset \Delta_v$ is the class $c_{F,v}$ coming by localization from the extension F/F^+, and the class associated to $\psi \subset \Delta_v$ is the class $c_{L^+,v}$ coming from L/L^+. It therefore suffices to show that

$$\langle c_{F,v}, c_{L^+,v} \rangle_v = 1.$$

By assumption, F/F^+ is unramified at all finite primes and L^+/L is unramified outside $v|2$, so the pairing vanishes at all finite primes away from $v|2$. (For primes of odd residue characteristic, the Hilbert symbol vanishes when restricted to units). Since L^+/F^+ is totally real, the Hilbert symbol also vanishes at $v|\infty$. Finally, we are assuming that the primes above 2 in F^+ are totally split in F/F^+, so $c_{F,v}$ is trivial for $v|2$ and the pairing also vanishes for $v|2$. □

The following is identical to [Tho17, Prop. 2.21] except with "adequate" replaced by "nearly adequate." The proof is identical, but we go through it in detail in order to show exactly where each hypothesis in Definition 5.3.3 is used (or more precisely, to show that the hypotheses in Definition 5.3.3 are the only ones used in the proof of [Tho17, Prop. 2.21], and the stronger assumption made there that $\overline{\rho}(G_F)$ is adequate is in fact never used).

Proposition 5.5.6. *Let $\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem, and let $T = S \smallsetminus S_\infty$. Assume that:*

- *(i) For each $v \in S_\infty$, $\mu(c_v) = -1$ and $\mathcal{D}_v = \mathrm{Lift}_v^\square$.*
- *(ii) $F = F^+(\sqrt{-1})$.*
- *(iii) If n is even, there exists an infinite place v of F^+ such that $(\overline{\rho}, \mu)$ is strongly residually odd at v.*
- *(iv) The group $\overline{\rho}(G_F) \subseteq \mathrm{GL}_n(k)$ is nearly adequate.*

Write $q = h^1_{\mathcal{S}^\perp,T} - 1$ and $g = q + \#T - 1 - [F^+ : \mathbf{Q}]n(n-1)/2$. Then for each $N \geq 1$ there are infinitely many Taylor–Wiles data $(\mathcal{Q}, (\alpha_v)_{v\in\mathcal{Q}})$ of level N such that $\#Q = q$ and the map $R^{\mathrm{loc}}_{\mathcal{S},T} \to R^T_{\mathcal{S}_\mathcal{Q}}$ can be extended to a surjection $R^{\mathrm{loc}}_{\mathcal{S},T}[\![X_1, \dots, X_g]\!] \twoheadrightarrow R^T_{\mathcal{S}_\mathcal{Q}}$.

Proof. We follow the proof of [Tho17, Prop. 2.21] very closely, assuming throughout that $p = 2$. As usual in arguments constructing Taylor–Wiles data, the proof begins by using the material on Galois cohomology recalled above to reduce to showing that for each $N \geq 1$, there are infinitely many Taylor–Wiles data $(\mathcal{Q}, (\alpha_v)_{v\in\mathcal{Q}})$ of level N such that $h^1_{\mathcal{S}^\perp_\mathcal{Q},T} = 1 = h^1_{\mathcal{S}^\perp,T} - \#\mathcal{Q}$. (This second equation is of course equivalent to $\#Q = q$ by the definition of q.) This reduction uses assumption ((i)) in the statement of the proposition, but makes no use of adequacy, so goes over unchanged under our assumptions. Since a Taylor–Wiles datum of level N is also a Taylor–Wiles datum of level M for any $M \leq N$, we can and do assume that N is sufficiently large to ensure that $F_N := F(\zeta_{2^N})$ strictly contains F. Let $[\psi] \in H^1_{\mathcal{S}^\perp,T} \subseteq H^1(F(S)/F^+, \mathrm{ad}\,\overline{r}(1))$ be a cohomology class with nonzero image in $H^1(F(S)/F_N, \mathrm{ad}\,\overline{r}(1))$. We claim that:

Claim 5.5.7. *There are infinitely many Taylor–Wiles data $(\{w\}, \alpha_w)$ of level N with $[\psi] \notin H^1_{\mathcal{S}^\perp_{\{w\}},T}$.*

Admitting the claim for now, the proof of the proposition is as follows. Write s for the dimension of the image of $H^1_{\mathcal{S}^\perp,T}$ in $H^1(F(S)/F_N, \mathrm{ad}\,\overline{r}(1))$. Applying the claim repeatedly, we see that there are infinitely many Taylor–Wiles data $(\mathcal{Q}, (\alpha_v)_{v\in\mathcal{Q}})$

of level N such that $\#Q = s$, $h^1_{\mathcal{S}^\perp_Q,T} = h^1_{\mathcal{S}^\perp,T} - \#Q$, and the morphism $H^1_{\mathcal{S}^\perp_Q,T} \to H^1(F(S)/F_N, \operatorname{ad}\overline{r}(1))$ is zero. We therefore have

$$H^1_{\mathcal{S}^\perp_Q,T} \subseteq H^1(F_N/F^+, \operatorname{ad}\overline{r}(1)^{G_{F_N}}) \tag{5.5.8}$$

(because by inflation-restriction, $H^1(F_N/F^+, \operatorname{ad}\overline{r}(1)^{G_{F_N}})$ is the kernel of the restriction map $H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1)) \to H^1(F(S)/F_N, \operatorname{ad}\overline{r}(1))$).

It only remains to show that $h^1_{\mathcal{S}^\perp_Q,T} = 1$. It is now time to use that $\overline{\rho}(G_F)$ is nearly adequate. We begin by using points (1) and 2 of Definition 5.3.3. The latter implies that (indeed, is equivalent to) $\overline{\rho}(G_F)$ has no normal subgroups of index 2, so that $\overline{\rho}(G_{F_N}) = \overline{\rho}(G_F)$. Then the former implies that $\overline{\rho}(G_{F_N})$ acts absolutely irreducibly, so that $\operatorname{ad}\overline{r}(1)^{G_{F_N}} = \operatorname{ad}\overline{r}^{G_{F_N}} = k$, the scalar matrices. In particular we have $H^1(F_N/F^+, \operatorname{ad}\overline{r}(1)^{G_{F_N}}) = H^1(F_N/F^+, k)$. We are assuming that $F = F^+(\sqrt{-1})$ (Assumption ((ii)) in the statement of the proposition) and that $F_N = F(\zeta_{2^N})$ is non-trivial over F. Together, these imply (since $p = 2$) that $H^1(F_N/F^+, k)$ is two-dimensional. By assumption ((iii)), together with [Tho17, Lem. 2.17(ii)] (and [Tho17, Lem. 2.16] in the case that n is odd), there is a place $v \in S_\infty$ such that the morphism $H^1(F_v^+, k) \to H^1(F_v^+, \operatorname{ad}\overline{r})$ is injective (this morphism being the one induced by the inclusion of the scalar matrices in $\operatorname{ad}\overline{r}$). In particular, for such a place the composite $H^1(F_N/F^+, k) \to H^1(F_v^+, k) \to H^1(F_v^+, \operatorname{ad}\overline{r}(1))$ is nonzero (because the first map is nonzero, for example because F/F^+ is an imaginary CM extension contained in F_N/F^+ and F_v^+ is real). (Note that in [Tho17] there is a typo, asserting that this composite is injective, but being nonzero is all that is needed.) Now, by definition (i.e. by (5.4.3)) the restriction to $H^1(F_v^+, \operatorname{ad}\overline{r}(1))$ of any class in $H^1_{\mathcal{S}^\perp_Q,T}$ vanishes; indeed, our choice of T gives $S_\infty = S \smallsetminus T$, and by assumption ((i)) in the statement of the proposition, we have $\mathcal{L}_v^\perp = 0$ for all $v \in \infty$. Going back to (5.5.8) (and recalling Lemma 5.5.3) we see that $h^1_{\mathcal{S}^\perp_Q,T} = 1$, and we are done.

It remains to prove Claim 5.5.7. Accordingly, we let

$$[\psi] \in H^1_{\mathcal{S}^\perp,T} \subseteq H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1))$$

be a cohomology class with nonzero image in $H^1(F(S)/F_N, \operatorname{ad}\overline{r}(1))$. By [Tho17, Lem. 2.19], finding a Taylor–Wiles datum $(\{w\}, \alpha_w)$ of level N with $[\psi] \notin H^1_{\mathcal{S}^\perp_{\{w\}},T}$ amounts to choosing w, α_w such that

- w splits completely in F_N, and $\overline{\rho}(\mathrm{Frob}_w)$ is semi-simple; and
- $\alpha_w \in k$ is an eigenvalue of $\overline{\rho}(\mathrm{Frob}_w)$ such that $\operatorname{tr} e_{\mathrm{Frob}_w,\alpha_w}\psi(\mathrm{Frob}_w) \neq 0$, where $e_{\mathrm{Frob}_w,\alpha_w}$ is the unique idempotent in $k[\overline{\rho}(\mathrm{Frob}_w)]$ whose image is the α_w-eigenspace of $\overline{\rho}(\mathrm{Frob}_w)$.

By Chebotarev, it therefore suffices to find $\sigma \in G_{F_N}$ and $\alpha \in k$ such that $\overline{\rho}(\sigma)$ is semi-simple, and α is an eigenvalue of $\overline{\rho}(\sigma)$ with $\operatorname{tr} e_{\sigma,\alpha}\psi(\sigma) \neq 0$.

Let K/F be the extension cut out by $\operatorname{ad}\overline{\rho}$, and write $K_N = K \cdot F_N$. Let f denote the image of $[\psi]$ under the restriction map

$$H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1)) \to H^1(F(S)/K_N, \operatorname{ad}\overline{r}(1))^{G_{F^+}}. \tag{5.5.9}$$

By the definition of K, the action of G_{K_N} on $\operatorname{ad}\overline{r}(1)$ is trivial so this image is a homomorphism $f : \mathrm{Gal}(F(S)/K_N) \to \operatorname{ad}\overline{r}$. We claim that $f \neq 0$. To see this, note that the restriction map (5.5.9) factors through the restriction map

$$H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1)) \to H^1(F(S)/F_N, \operatorname{ad}\overline{r}(1)),$$

and by assumption, the image of $[\psi]$ in $H^1(F(S)/F_N, \operatorname{ad}\overline{r}(1))$ is nonzero. It therefore suffices to show that the kernel of the restriction map

$$H^1(F(S)/F_N, \operatorname{ad}\overline{r}(1)) \to H^1(F(S)/K_N, \operatorname{ad}\overline{r}(1))$$

vanishes. By inflation-restriction, this kernel is $H^1(K_N/F_N, \operatorname{ad}\overline{r}(1))$. As we saw above, by Definition 5.3.3 (2) we have $\overline{\rho}(G_{F_N}) = \overline{\rho}(G_F)$, so that $H^1(K_N/F_N, \operatorname{ad}\overline{r}(1)) = H^1(\overline{\rho}(G_F), \operatorname{ad}\overline{r})$. This vanishes by Definition 5.3.3 (3), as required.

Let $V \subseteq \operatorname{ad}\overline{\rho}$ be the k-vector space generated by the image of f. Since f is G_{F_N}-equivariant (as it is restricted from $[\psi]$), V is a $k[G_{F_N}]$-module, and we let W be a simple $k[G_{F_N}]$-submodule of V. By Definition 5.3.3 (1) (and again using that $\overline{\rho}(G_{F_N}) = \overline{\rho}(G_F)$), we may find $\sigma_0 \in G_{F_N}$ and $\alpha_0 \in k$ such that $\overline{\rho}(\sigma_0)$ is semi-simple, and α_0 is an eigenvalue of $\overline{\rho}(\sigma_0)$ with $\operatorname{tr} e_{\sigma_0,\alpha_0} W \neq 0$.

If $\operatorname{tr} e_{\sigma_0,\alpha}\psi(\sigma_0) \neq 0$, then we are done, taking $\sigma = \sigma_0$ and $\alpha = \alpha_0$. Suppose instead that $\operatorname{tr} e_{\sigma_0,\alpha_0}\psi(\sigma_0) = 0$, and choose any $\tau \in K_N$ such that $\operatorname{tr} e_{\sigma_0,\alpha_0} f(\tau) \neq 0$. (Such a τ exists, because $\operatorname{tr} e_{\sigma_0,\alpha_0} W \neq 0$, and by definition V is spanned as a k-vector space by the varying $f(\tau)$.) We set $\sigma = \tau\sigma_0$, so that $\overline{\rho}(\sigma)$ is a scalar multiple of $\overline{\rho}(\sigma_0)$, and we let α be the corresponding scalar multiple of α_0, so that $e_{\sigma,\alpha} = e_{\sigma_0,\alpha_0}$. We have $\psi(\sigma) = \psi(\sigma_0) + \psi(\tau)$, so that $\operatorname{tr} e_{\sigma,\alpha}\psi(\sigma) = \operatorname{tr} e_{\sigma_0,\alpha_0} f(\tau) \neq 0$, as required. □

5.6. **Local deformation problems.** We now assume that all finite places $v \in S$ of F^+ split in F, and choose a place $\widetilde{v}$ of F above each $v \in S$. We write $\widetilde{S}$ for the set of places $\widetilde{v}$ with $v \in S$ finite, and $\widetilde{S}_2 \subset \widetilde{S}$ for the places lying over 2. For each $v \in S$ we can and do identify liftings of $\overline{r}|_{G_{F_v^+}}$ with liftings of $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$.

5.6.1. *Local deformation problems for $v \nmid 2$.* The following two lemmas are presumably well known, but for lack of a reference we give a proof.

Lemma 5.6.2. *Suppose that $v \nmid 2$. Then there is a finite extension $F'_{\widetilde{v}}/F_{\widetilde{v}}$ such that any lifting of $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ becomes unipotently ramified after restriction to $G_{F'_{\widetilde{v}}}$.*

Proof. Since the universal lifting ring of $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ is $\mathcal{O}$-flat by [Sho18, Thm. 2.5], it suffices to prove this for closed points of its generic fibre. Since this generic fibre has finitely many connected components, it suffices to prove the result for the closed points of any single connected component. For each connected component, it suffices to prove the result for a single point on that component by [BLGGT14, Lem. 1.3.4(1)] (a theorem of Choi), and the result is immediate. □

Lemma 5.6.3. *Suppose that $v \nmid 2$, that $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ is unramified, and that $\overline{\rho}(\mathrm{Frob}_{\widetilde{v}})$ is regular semi-simple. Then any lifting of $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is strictly equivalent to a direct sum of characters. In particular, there is a finite extension $F'_{\widetilde{v}}/F_{\widetilde{v}}$ such that any lifting of $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ becomes unramified after restriction to $G_{F'_{\widetilde{v}}}$.*

Proof. The second statement follows from the first by Lemma 5.6.2. Since $q_v \equiv 1 \pmod 2$, the first statement is standard, and may for example be proved by an identical argument to the proof of Lemma 6.1.6. □

Now let $\chi_{v,1}, \ldots, \chi_{v,n} : \mathcal{O}_{F_{\widetilde{v}}}^{\times} \to \mathcal{O}^{\times}$ be finite order characters, which are trivial modulo ϖ. Suppose that $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ is trivial. We write $\mathcal{D}_v^{\chi_v}$ for the set of liftings ρ of

$\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ to objects of $\mathrm{CNL}_{\mathcal{O}}$ such that for all $\sigma \in I_{F_{\widetilde{v}}}$, we have

$$\mathrm{char}_{\rho(\sigma)}(X) = \prod_{i=1}^{n} (X - \chi_{v,j}(\mathrm{Art}_{F_{\widetilde{v}}}(\sigma))^{-1}).$$

Write $R_v^{\chi_v}$ for the corresponding local lifting ring. The following is [Tho12, Prop. 3.16].

Proposition 5.6.4.

(1) Suppose that $\chi_{v,j} = 1$ for each j. Then each irreducible component of R_v^1 has dimension n^2+1, and every prime of R_v^1 minimal over ϖ contains a unique minimal prime. Every generic point of R_v^1 is of characteristic zero.

(2) Suppose that the $\chi_{v,j}$ are pairwise distinct. Then $\mathrm{Spec}\, R_v^{\chi_v}$ is irreducible of dimension n^2+1, and its generic point is of characteristic zero.

5.6.5. *Local deformation rings for $v|2$: ordinary deformation rings for $v|2$.* We now recall the ordinary deformation problems introduced in [Ger19, §3], and studied there and in [Tho15, §3.3.2]. Suppose that $v|2$ and that $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ can be conjugated to an upper-triangular representation whose diagonal characters are $\overline{\chi}_1, \dots, \overline{\chi}_n : G_{F_{\widetilde{v}}} \to k^\times$ (in that order). Let $\Lambda_{\mathrm{GL}_n,v}$ be the completed group ring of the group $I_{F_{\widetilde{v}}}^{\mathrm{ab}}(2)^n$, where (2) denotes pro-2 completion. Let $(\chi_1, \dots, \chi_n)$ denote the universal n-tuple of characters $I_{F_{\widetilde{v}}} \to \Lambda_{\mathrm{GL}_n,v}^\times$ lifting $(\overline{\chi}_1|_{I_{F_{\widetilde{v}}}}, \dots, \overline{\chi}_n|_{I_{F_{\widetilde{v}}}})$.

Proposition 5.6.6. *There is a local deformation problem $\mathcal{D}_v^{\triangle}$, represented by a $\mathrm{CNL}_{\Lambda_{\mathrm{GL}_n,v}}$-algebra $R_v^{\triangle}$ with the following properties.*

(1) $R_v^{\triangle}$ is reduced and 2-torsion free.

(2) Let E'/E be a finite extension with ring of integers $\mathcal{O}_{E'}$, and fix a morphism of local $\mathcal{O}$-algebras $\Lambda_{\mathrm{GL}_n,v} \to \mathcal{O}_{E'}$. Then a morphism of local $\Lambda_{\mathrm{GL}_n,v}$-algebras $R_v^{\square} \to \mathcal{O}_{E'}$ factors through $R_v^{\triangle}$ if and only if the corresponding representation $\rho : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}_{E'})$ is $\mathrm{GL}_n(\mathcal{O}_{E'})$-conjugate to an upper-triangular representation whose ordered diagonal characters $(\psi_1, \dots, \psi_n)$ are such that for each i, $\psi_i|_{I_{F_{\widetilde{v}}}}$ is equal to the pushforward of $(\chi_1, \dots, \chi_n)$ along $\Lambda_{\mathrm{GL}_n,v} \to \mathcal{O}_{E'}$.

(3) Suppose that $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ is trivial, and that $[F_v^+ : \mathbf{Q}_2] > n(n-1)/2+1$. Let Q be a minimal prime of $\Lambda_{\mathrm{GL}_n,v}$. Then $\mathrm{Spec}\, R_v^{\triangle}/Q$ is geometrically irreducible of dimension $[F_v^+ : \mathbf{Q}_2]n(n+1)/2 + n^2 + 1$, and $R_v^{\triangle}/(Q, \varpi)$ is generically reduced.

Proof. The lifting ring $R_v^{\triangle}$ is defined in [Ger19, §3]; see [Tho15, §3.3.2] for a summary of its definition. It is reduced and flat over $\mathcal{O}$ by construction. The remaining points are [Ger19, Lem. 3.3] and [Tho15, Prop. 3.14(3)]. □

We will use the following remark in the proof of Proposition 7.5.10.

Remark 5.6.7. Note that [Ger19, §3] defines $\mathrm{Spec}\, R_v^{\triangle}$ as the flat closure of the scheme-theoretic image of a projective morphism $\pi : \mathcal{G}_v \to \mathrm{Spec}\, R^{\square}$, it is shown in [Tho15, Lem. 3.11] that if $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ is trivial and $[F_v^+ : \mathbf{Q}_2] > n(n-1)/2+1$, then $\mathcal{G}_v$ is already flat over $\mathcal{O}$, so that $\mathrm{Spec}\, R_v^{\triangle}$ is equal to the scheme-theoretic image $\pi : \mathcal{G}_v \to \mathrm{Spec}\, R^{\square}$.

Definition 5.6.8. Let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F_{\widetilde{v}}, K)}$. We say that a continuous representation $\rho : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O})$ is *ordinary of weight* λ if:

(1) There exists a increasing invariant filtration Fil^i of $\mathcal{O}^n$, with each $\mathrm{gr}^i\,\mathcal{O}^n$ an $\mathcal{O}$-module of rank one.
(2) Write χ_i for the character $G_{F_{\widetilde{v}}} \to \mathcal{O}^\times$ giving the action on $\mathrm{gr}^i\,\mathcal{O}^n$. Then for every $\alpha \in F_{\widetilde{v}}^\times$ sufficiently close to 1, we have

$$(\chi_i \circ \mathrm{Art}_{F_{\widetilde{v}}}(\alpha)) = \prod_\tau (\tau(\alpha))^{-(\lambda_{\tau,n-i+1}+i-1)}.$$

5.7. **Automorphic forms on definite unitary groups.** We now introduce the spaces of automorphic forms that we work with, following [Tho17, §4] and [Ger19, §2]. We suppose throughout Subsection 5.7 that the following hypothesis holds.

Hypothesis 5.7.1.
- F/F^+ is everywhere unramified, and each place $v|2$ of F^+ splits in F.
- $n[F^+ : \mathbf{Q}] \equiv 0 \pmod 4$.

Let c denote the non-trivial element of $\mathrm{Gal}(F/F^+)$. By Hypothesis 5.7.1, we can find a unitary group G/F^+ which splits over F, and is such that:
- $G(F_v^+)$ is quasi-split at all finite places v of F^+.
- $G(F^+ \otimes_{\mathbf{Q}} \mathbf{R})$ is compact.

We can and do choose an integral model of G over $\mathcal{O}_{F^+}$ (which we continue to denote by G) in such a way that if v is a finite place of F^+ which splits as $v = \widetilde{v}\widetilde{v}^c$ in F, then there is an isomorphism

$$\iota_{\widetilde{v}} : G(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}}).$$

For each place $v|2$ of F^+ we choose a place $\widetilde{v}|v$ of F, and let $\widetilde{S}_2$ be the set of $\widetilde{v}$ for $v|2$. Let $\widetilde{I}_2$ denote the set of embeddings $F \hookrightarrow E$ inducing a place in $\widetilde{S}_2$. To each $\lambda \in (\mathbf{Z}_+^n)^{\widetilde{I}_2}$ there is an associated finite free $\mathcal{O}$-module M_λ with a continuous action of $\prod_{v\in S_2} G(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \prod_{v\in S_2} \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$, constructed as the tensor product over $\tau \in \widetilde{S}_2$ of the the algebraic representations of $\mathrm{GL}_n/\mathcal{O}$ with highest weight λ_τ.

We now write $S = S_\infty \sqcup T$, and let $R \subset T$ be a (possibly empty) set of places disjoint from S_2. For each place $v \in R$ we fix a choice of $\widetilde{v}$ (a place of F dividing v). Suppose that $U = \prod_v U_v$ is an open compact subgroup of $G(\mathbf{A}_{F^+}^\infty)$ such that $U_v \subset \iota_{\widetilde{v}}^{-1}\mathrm{Iw}(\widetilde{v})$ for $v \in R$. (Here $\mathrm{Iw}(\widetilde{v})$ is the Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ consisting of matrices which are upper-triangular modulo $\widetilde{v}$, with pro- v Iwahori subgroup $\mathrm{Iw}_1(\widetilde{v}) \subset \mathrm{Iw}(\widetilde{v})$.)

For each $v \in R$, we choose a character

$$\chi_v = \chi_{v,1} \times \cdots \times \chi_{v,n} : \mathrm{Iw}(\widetilde{v})/\mathrm{Iw}_1(\widetilde{v}) \to \mathcal{O}^\times,$$

the decomposition being with respect to the natural isomorphism

$$\mathrm{Iw}(\widetilde{v})/\mathrm{Iw}_1(\widetilde{v}) \cong (k(\widetilde{v})^\times)^n.$$

We set

$$M_{\lambda,\{\chi_v\}} = M_\lambda \otimes_{\mathcal{O}} \left(\bigotimes_{v\in R} \mathcal{O}(\chi_v) \right),$$

a representation of $G(\mathcal{O}_{F^+,2}) \times \prod_{v\in R} \mathrm{Iw}(\widetilde{v})$.

If A is an $\mathcal{O}$-module, and $U_v \subset G(\mathcal{O}_{F_v^+})$ for $v|2$, then we write $S_{\lambda,\{\chi_v\}}(U, A)$ for the set of functions

$$f : G(F^+)\backslash G(\mathbf{A}_{F^+}^\infty) \to M_{\lambda,\{\chi_v\}} \otimes_{\mathcal{O}} A$$

such that for every $u \in U$, we have $f(gu) = u_{S_2 \cup R}^{-1} f(g)$, where $u_{S_2\cup R}$ denotes the projection to $\prod_{v\in S_2\cup R} U_v$. If R is empty then we write $S_{\lambda,\{\chi_v\}}(U, A) = S_\lambda(U, A)$.

We will sometimes assume that U is sufficiently small in the following sense.

Definition 5.7.2. We say that U is *sufficiently small* if for some finite place v of F^+, the projection of U to $G(F_v^+)$ contains no element of finite order other than the identity.

Let w be a place of F split over F^+ and not contained in S, and let ϖ_w be a uniformizer of F_w. Write

$$\alpha^j_{\varpi_w} := \mathrm{diag}(\underbrace{\varpi_w, \ldots, \varpi_w}_{j}, 1, \ldots, 1).$$

The spaces $S_{\lambda,\{\chi_v\}}(U, A)$ receive an action of the Hecke operators

$$T^j_w := \iota_w^{-1}\left(\left[\mathrm{GL}_n(\mathcal{O}_{F_w})\alpha^j_{\varpi_w}\,\mathrm{GL}_n(\mathcal{O}_{F_w})\right]\right).$$

For integers $0 \le b \le c$, and $v \in S_2$, we consider the subgroup $\mathrm{Iw}(\widetilde{v}^{b,c}) \subset \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ defined as those matrices which are congruent to an upper-triangular matrix modulo $\widetilde{v}^c$ and congruent to a unipotent upper-triangular matrix modulo $\widetilde{v}^b$. We set $U(\mathfrak{l}^{b,c}) = \prod_{v\notin S_2} U_v \times \prod_{v\in S_2} \mathrm{Iw}(\widetilde{v}^{b,c})$. (Our use of $\widetilde{l}$ is in order to follow the notation of [Ger19].)

We now recall from [Ger19, Defn. 2.8] some additional Hecke operators at the places dividing 2. For each $v \in S_2$ we let $\varpi_{\widetilde{v}}$ be a uniformizer of $F_{\widetilde{v}}$. As above we write

$$\alpha^j_{\varpi_{\widetilde{v}}} = \mathrm{diag}(\underbrace{\varpi_{\widetilde{v}}, \ldots, \varpi_{\widetilde{v}}}_{j}, 1, \ldots, 1),$$

and we set

$$U^j_{\lambda,\varpi_{\widetilde{v}}} = (w_0\lambda)(\alpha^j_{\varpi_{\widetilde{v}}})^{-1}\left[U(\mathfrak{l}^{b,c})\iota_{\widetilde{v}}^{-1}(\alpha^j_{\varpi_{\widetilde{v}}})U(\mathfrak{l}^{b,c})\right],$$

where as usual w_0 is the longest element of the Weyl group. If $u \in T(\mathcal{O}_{F_{\widetilde{v}}})$ then we write

$$\langle u\rangle = \left[U(\mathfrak{l}^{b,c})\iota_{\widetilde{v}}^{-1}(u)U(\mathfrak{l}^{b,c})\right].$$

By [Ger19, Lem. 2.10], these operators commute with each other and act on the spaces $S_{\lambda,\{\chi\}}(U(\mathfrak{l}^{b,c}), A)$, compatibly with the inclusions

$$S_{\lambda,\{\chi\}}(U(\mathfrak{l}^{b,c}), \mathcal{O}) \subset S_{\lambda,\{\chi\}}(U(\mathfrak{l}^{b',c'}), \mathcal{O}),$$

where $b \le b'$ and $c \le c'$.

We write $\mathbf{T}^T_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{b,c}), A)$ for the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_{\lambda,\{\chi_v\}}(U, A))$ generated by the operators T^j_w and $(T^n_w)^{-1}$ as above and all the operators $\langle u\rangle := \prod_{v\in S_2}\langle u_v\rangle$ for

$$u = (u_v)_{v\in S_2} \in T(\mathcal{O}_{F^+,2}) = \prod_{v\in S_2} T(\mathcal{O}_{F_v^+}),$$

where T denotes the usual diagonal torus in GL_n. With these identifications, the operators $\langle u\rangle$ endow each Hecke algebra $\mathbf{T}^T_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{b,c}), A)$ with the structure of an algebra for the completed group ring

$$\Lambda = \mathcal{O}[\![T(\mathfrak{l})]\!], \tag{5.7.3}$$

where $T(\mathfrak{l})$ is defined by the exact sequence

$$0 \longrightarrow T(\mathfrak{l}) \longrightarrow \prod_{v\in S_2} T(\mathcal{O}_{F_v^+}) \longrightarrow \prod_{v\in S_2} k(v)^\times \longrightarrow 0.$$

We have the ordinary idempotent $e = \lim_{r\to\infty} U(\mathfrak{l})^{r!}$, where we set

$$U(\mathfrak{l}) = \prod_{v\in S_2}\prod_{j=1}^{n} U^j_{\lambda,\varpi_{\widetilde{v}}}.$$

We define the ordinary Hecke algebra

$$\mathbf{T}^{T,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{b,c}),A) = e\mathbf{T}^{T}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{b,c}),A);$$

equivalently, $\mathbf{T}^{T,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{b,c}),A)$ is the image of the Hecke algebra $\mathbf{T}^{T}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{b,c}),A)$ in $\mathrm{End}_{\mathcal{O}}\left(S^{\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{b,c}),A)\right)$.

We set

$$S_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{\infty}),E/\mathcal{O}) = \varinjlim_{c} S_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{c,c}),E/\mathcal{O}),$$

which receives a faithful action of the algebra

$$\mathbf{T}^{T}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{\infty}),E/\mathcal{O}) = \varprojlim_{c} \mathbf{T}^{T}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{c,c}),E/\mathcal{O}).$$

By [Ger19, Lem. 2.17], this algebra is naturally isomorphic to

$$\mathbf{T}^{T}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{\infty}),\mathcal{O}) = \varprojlim_{c} \mathbf{T}^{T}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{c,c}),\mathcal{O}).$$

We can again apply the idempotent e to these spaces and rings, in which case we again decorate them with 'ord' superscripts.

Specializing to the case $\lambda = 0$, we define a homomorphism $T(\mathfrak{l}) \to \mathbf{T}^{T,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{l}^{\infty}),\mathcal{O})^{\times}$ by

$$u \mapsto \left(\prod_{\tau\in\widetilde{I}_2}\prod_{i=1}^{n}\tau(u_i)^{1-i}\right)\langle u\rangle$$

(where the u_i are the coordinate entries of u, recalling that T is the usual diagonal maximal torus in GL_n). This gives rise to an $\mathcal{O}$-algebra homomorphism $\Lambda \to \mathbf{T}^{T,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$, and we write

$$\mathbf{T}^{T,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$$

for $\mathbf{T}^{T,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{l}^{\infty}),\mathcal{O})$ endowed with this Λ-algebra structure. This is the *universal ordinary Hecke algebra of level U*. It is a finite Λ-algebra by [Ger19, Cor. 2.21]. Along with all of the other Hecke algebras considered above, it is reduced (by [Ger19, Lem. 2.14]).

We can pass back from the universal ordinary Hecke algebra to the finite level Hecke algebras in the following way. Corresponding to each λ is a prime ideal $\wp_\lambda$ of $\mathrm{Spec}\,\Lambda$ as defined in [Ger19, Defn. 2.24(1)]; the prime ideals $\wp_\lambda$ are dense in $\mathrm{Spec}\,\Lambda$ (see the proof of [Ger19, Cor. 3.4]). By [Ger19, Lem. 2.25], we have a natural identification

$$\mathrm{Hom}_{\mathcal{O}}(\mathbf{T}^{T,\mathrm{ord}}_{0,\{\chi_v\}}(U(\mathfrak{l}^{\infty}),\mathcal{O})/\wp_\lambda,\overline{\mathbf{Q}}_2) = \mathrm{Hom}_{\mathcal{O}}(\mathbf{T}^{T,\mathrm{ord}}_{\lambda,\{\chi_v\}}(U(\mathfrak{l}^{1,1}),\mathcal{O}),\overline{\mathbf{Q}}_2). \quad (5.7.4)$$

We say that a RACSDC automorphic representation π for GL_n/F has weight $\lambda \in (\mathbf{Z}^n_+)^{\widetilde{I}_2}$ if its infinitesimal character agrees (after composing with our fixed isomorphism $\imath : \overline{\mathbf{Q}}_2 \xrightarrow{\sim} \mathbf{C}$) with that of the algebraic representation of weight λ. We say that π is ordinary if $(\imath^{-1}(\otimes_{v|2}\pi_v))^{\mathrm{ord}} \neq 0$. (See [BLGGT14, §2].) The relationship between the spaces of automorphic forms considered above and ordinary RACSDC representations is as follows. For each λ, write

$$\mathcal{A}_\lambda := \varinjlim_U S_\lambda(U, \overline{\mathbf{Q}}_2).$$

This is a semi-simple admissible $\overline{\mathbf{Q}}_2[G(\mathbf{A}^\infty_{F^+})]$-module, and by [Lab11, Cor. 5.3, Thm. 5.9] the irreducible submodules of $\imath\mathcal{A}_\lambda$ are the finite parts of automorphic representations of G/F^+ which arise as the descents of automorphic representations π of GL_n/F of weight λ. These automorphic representations π are isobaric direct sums of self dual representations, and in particular, they include the RACSDC representations of weight λ; and after localizing at a non-Eisenstein maximal ideal of an appropriate Hecke algebra (as we will always do below), the RACSDC representations are the only ones that contribute. Furthermore the irreducible submodules of $\imath\mathcal{A}_\lambda$ which have nonzero intersection with some $S^{\mathrm{ord}}_\lambda(U(\mathfrak{l}^{b,b}), \overline{\mathbf{Q}}_2)$ are precisely those which correspond to those π which are ordinary.

Now let π be an ordinary RACSDC automorphic representation π of GL_n/F, and assume that $\overline{\rho} = \overline{r}_{\pi,2} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_2)$ is irreducible. Assuming as always that our coefficient field E is large enough, we fix an extension of $\overline{\rho}$ to $\overline{r} : G_{F^+} \to \mathcal{G}_n(k)$. As above, we let $T \supset S_2$ be a finite set of finite places of F^+ which split in F. Again we consider a subset $R \subset (T \smallsetminus S_2)$, and for each $v \in R$ we fix characters $\chi_v : \mathrm{Iw}(\widetilde{v})/\mathrm{Iw}_1(\widetilde{v}) \to \mathcal{O}^\times$. We assume furthermore that:

Hypothesis 5.7.5.

- T contains all finite places lying under a place w of F at which π_w is ramified, and
- if $v \in R$ then $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ is trivial and $\pi_v^{\mathrm{Iw}(\widetilde{v})} \neq 0$.

Set $S = T \cup S_\infty$. If $v \notin S_2$ then we set $\Lambda_v = \mathcal{O}$, while if $v \in S_2$ we take $\Lambda_v = \Lambda_{\mathrm{GL}_n,v}$ where $\Lambda_{\mathrm{GL}_n,v}$ is as in Section 5.6.5. We define the global deformation problem

$$\mathcal{S}_{\{\chi_v\}} = \Big(F, \overline{r}, \mathcal{O}, \varepsilon^{1-n}\delta^n_{F/F^+}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}^{\chi_v}_v\}_{v\in R} \cup \{\mathcal{D}^{\triangle}_v\}_{v\in S_2} \cup \{\mathrm{Lift}^{\square}_v\}_{v\in S\smallsetminus(R\cup S_2)}\Big). \quad (5.7.6)$$

Using the natural isomorphism $\widehat{\bigotimes}_{v\in S_2}\Lambda_v \cong \Lambda$ provided by local class field theory, we see that $R^{\mathrm{loc}}_{\{\chi_v\}}$ is naturally a Λ-algebra.

Lemma 5.7.7. *Every irreducible component of* $\operatorname{Spec} R_{\mathcal{S}_{\{\chi_v\}}}$ *has dimension at least* $\dim\Lambda = 1 + [F : \mathbf{Q}]n$.

Proof. By Propositions 5.6.4 and 5.6.6, together with [Sho18, Thm. 2.5], the ring $R^{\mathrm{loc}}_{\mathcal{S}_{\{\chi_v\}},T}$ is equidimensional of dimension

$$\dim R^{\mathrm{loc}}_{\mathcal{S}_{\{\chi_v\}},T} = 1 + n^2\#T + [F^+ : \mathbf{Q}]n(n+1)/2. \quad (5.7.8)$$

It therefore suffices to show that there is a presentation of the form

$$R^{\mathrm{loc}}_{\{\chi_v\}}[[x_1, \dots, x_r]]/(f_1, \dots, f_{r+s}) \xrightarrow{\sim} R_{\mathcal{S}_{\{\chi_v\}}}$$

for some $r, s \geq 0$ with $s \leq n^2\#T + [F^+ : \mathbf{Q}]n(n-1)/2$. This follows from a standard deformation-obstruction argument, and can for example be proved exactly as in [CHT08, Cor. 2.2.12], using the complex $C^i_{\mathcal{S},T}$. Alternatively, the existence of such a presentation is a consequence of [Bal12, Prop. 4.2.5]. (As noted in [BG19, §4.1], it is assumed in [Bal12, §4.2] that the reductive group G there is connected, but in the proof of [Bal12, Prop. 4.2.5] this assumption is only used in order to cite results of [Til96] which do not use this assumption.) □

Proposition 5.7.9. *Suppose that for each $v \notin T$, the compact open subgroup U_v is hyperspecial; and that for each $v \in R$, we have $U_v \subset \iota_{\widetilde{v}}^{-1}\mathrm{Iw}(\widetilde{v})$.*

Then there is a maximal ideal $\mathfrak{m}$ of $\mathbf{T}^{T,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{l}^\infty),\mathcal{O})$ such that there is a surjection of Λ-algebras

$$R_{\mathcal{S}_{\{\chi_v\}}} \to \mathbf{T}^{T,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{l}^\infty),\mathcal{O})_{\mathfrak{m}}. \tag{5.7.10}$$

The corresponding (unique up to strict equivalence) representation

$$r_{\mathfrak{m}} : G_{F^+} \to \mathcal{G}_n(\mathbf{T}^{T,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{l}^\infty),\mathcal{O})_{\mathfrak{m}})$$

is characterized by the following property: if $v \notin T$ is a finite place of F^+ which splits as ww^c in F, then $r_{\mathfrak{m}}$ is unramified at w and w^c, and $r_{\mathfrak{m}}(\mathrm{Frob}_w)$ has characteristic polynomial

$$X^n + \cdots + (-1)^j(q_w)^{j(j-1)/2}T_w^jX^{n-j} + \cdots + (-1)^n(q_w)^{n(n-1)/2}T_w^n.$$

Proof. This is proved in exactly the same way as [Ger19, Prop. 2.29] (using [Ger19, Cor. 3.4] for the compatibility at the places $v \in S_2$), using [Tho17, Lem. 2.4] in place of [CHT08, Lem. 2.1.12] (which is used in the proof of [CHT08, Lem. 3.4.4], to which the proof of [Ger19, Prop. 2.29] refers). (See also [Tho17, Thm. 4.1] for a detailed proof of a very similar result.). □

Definition 5.7.11. If R is empty then we write $\mathcal{S}^{T,\mathrm{ord}}$ for $\mathcal{S}_{\{\chi_v\}}$ and $R^{T,\mathrm{ord}}$ for $R_{\mathcal{S}_{\{\chi_v\}}}$, and we write $\mathbf{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty))_{\mathfrak{m}}$ for $\mathbf{T}^{T,\mathrm{ord}}_{\{\chi_v\}}(U(\mathfrak{l}^\infty),\mathcal{O})_{\mathfrak{m}}$.

Before stating and proving the main result of this section, we make a definition, using the following (presumably well known) lemma.

Lemma 5.7.12. *If $w \nmid 2$ is a finite place of F, then there is a compact open subgroup U_w of $\mathrm{GL}_n(\mathcal{O}_{F_w})$, depending only on $\overline{r}_{\pi,2}|_{G_{F_w}}$, such that if π' is a RACSDC automorphic representation of GL_n/F with $\overline{r}_{\pi,2} \cong r_{\pi',2}$ then $(\pi'_w)^{U_w} \neq 0$.*

Proof. This follows from Lemma 5.6.2 and local-global compatibility, together with the compatibility of the local Langlands correspondence with conductors. □

Definition 5.7.13. Suppose that $v \in T \smallsetminus S_2$. Then we say that a compact open subgroup of $\mathcal{G}_n(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_{\widetilde{v}})$ is *sufficiently deep* if it satisfies the conclusion of Lemma 5.7.12 (for $w = \widetilde{v}$).

Theorem 5.7.14. *Let F be a CM field, and let $n \geq 2$. Fix a continuous representation*

$$\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_2)$$

satisfying the following hypotheses.

1. *There is an ordinary RACSDC automorphic representation π of GL_n/F such that $\overline{r}_{\pi,2} \cong \overline{\rho}$.*
2. *F/F^+ is everywhere unramified. All of the places $v|2$ of F^+ split in F, as do all places lying under a place at which π is ramified.*
3. *$n[F^+ : \mathbf{Q}] \equiv 0 \pmod 4$.*

(4) $\overline{\rho}(G_F)$ *is nearly adequate.*
(5) $\overline{\rho}(G_F)$ *contains a regular semi-simple element.*
(6) If n *is even, then there exists an infinite place* v *of* F^+ *such that the polarized pair* $(\overline{\rho}, \overline{\varepsilon}^{1-n}\delta^n_{F/F^+})$ *is strongly residually odd at* v.

Let T *be any finite set of finite places of* F^+ *which split in* F*, which contains all finite places lying under a place* w *of* F *at which* π_w *is ramified, and all places dividing* 2.

Then $R^{T,\mathrm{ord}}$ *is a finite* Λ*-algebra. Furthermore, if for each* $v \in T \smallsetminus S_2$ *the group* U_v *is sufficiently deep in the sense of Definition 5.7.13, then the morphism* $R^{T,\mathrm{ord}} \to \mathbf{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty))_{\mathfrak{m}}$ *given by* (5.7.10) *has nilpotent kernel, i.e.*

$$(R^{T,\mathrm{ord}})^{\mathrm{red}} \xrightarrow{\sim} \mathbf{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty))_{\mathfrak{m}}.$$

Remark 5.7.15. The assumption in Theorem 5.7.14 (5) that $\overline{\rho}(G_F)$ contains a regular semi-simple element is used in order to ensure that the automorphic forms that we consider are of neat level.

Proof of Theorem 5.7.14. We will begin by making a succession of solvable extensions of CM fields to put ourselves into a situation where we can apply the Taylor–Wiles patching method. In order to keep the notation compatible with that above we will continue to denote our CM field by F until the end of the argument, where we will descend to our original F.

We can and do replace F with a solvable extension (and replace T with the set of places lying over places in T) and enlarge our coefficient field E so that in addition to the hypotheses of the theorem, we have:

- $F = F^+(\sqrt{-1})$.
- if $v \in T$ then $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ is trivial.
- if $v \in S_2$ then $[F^+_v : \mathbf{Q}_2] > n(n-1)/2 + 1$.
- if $v \in T \smallsetminus S_2$ then:
  - $\pi_{\widetilde{v}}^{\mathrm{Iw}(\widetilde{v})} \neq 0$.
  - if $2^N \| (q_v - 1)$ then $2^N > n$ and $\mathcal{O}$ contains a 2^Nth root of unity.

(Note that if $(F')^+/F^+$ is a solvable extension of totally real fields then $(F')^+F/F$ is a solvable extension of CM fields, so we can choose a solvable CM extension to realize any finite set of local extensions. All of these conditions are local except for the first condition that $F = F^+(\sqrt{-1})$. Since arranging this only involves a quadratic extension, and $\overline{\rho}(G_F)$ has no normal subgroups of index 2 by the assumption that it is nearly adequate (which requires in particular that $H^1(\overline{\rho}(G_F), k) = 0$), this quadratic extension leaves $\overline{\rho}(G_F)$ unchanged.) Choose a finite place $v_1 \notin T$ of F^+ which splits in F as $v = \widetilde{v}_1\widetilde{v}_1^c$, for which $\overline{\rho}(\mathrm{Frob}_{\widetilde{v}_1})$ is regular semi-simple. (There are infinitely many such v_1 by our assumption (5).)

We replace T by $T \cup \{v_1\}$, and write $T = S_2 \sqcup \{v_1\} \sqcup R$. Note in particular that Hypotheses 5.7.1 and 5.7.5 hold. For each $v \in R$ we choose pairwise distinct characters $\chi_{v,1}, \dots, \chi_{v,n} : \mathcal{O}^\times_{F_{\widetilde{v}}} \to \mathcal{O}^\times$ which become trivial after reduction modulo ϖ. (We can do this by the conditions arranged in our initial base change.) We have the global deformation problem $\mathcal{S}_{\{\chi_v\}}$ defined in (5.7.6), and we write $\mathcal{S}_{\{1\}}$ for the global deformation problem defined in the same way but with all of the characters $\chi_{v,i}$ replaced by the trivial character. By the definitions of the local deformation problems $\mathcal{D}_v^{\chi_v}$ for $v \in R$, we can fix compatible isomorphisms $R^{\mathrm{loc}}_{\mathcal{S}_{\{\chi_v\}},T}/\varpi \cong R^{\mathrm{loc}}_{\mathcal{S}_{\{1\}},T}/\varpi$ and $R_{\mathcal{S}_{\{\chi_v\}}}/\varpi \cong R_{\mathcal{S}_{\{1\}}}/\varpi$.

We now specify open compact subgroups $U_v \subset G(F_v^+)$ as follows:

(1) $U_v = G(\mathcal{O}_{F_v^+})$ if $v \notin T$ is split in F.
(2) U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$ if v is inert in F.
(3) $U_v = \mathrm{Iw}(\widetilde{v})$ for $v \in R$.
(4) U_{v_1} is any torsion-free compact open subgroup of $G(F_{v_1}^+)$.

(Note in particular that the choice of U_{v_1} means that for any b, c, the group $U(\mathfrak{l}^{b,c})$ is sufficiently small in the sense of Definition 5.7.2. While we will not explicitly use this below, it is implicitly used multiple times, ultimately in the form of [Ger19, Lem. 2.6].)

Since by assumption π is ordinary and unramified outside of $R \cup S_2$, and since $\pi_{\widetilde{v}}^{\mathrm{Iw}(\widetilde{v})} \neq 0$ for all $v \in R$, there is a maximal ideal $\mathfrak{m}_1$ of $\mathbf{T}_{\{1\}}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})$ with residue field k such that $\overline{\rho} \cong \overline{r}_{\mathfrak{m}_1}|_{G_F}$. (As ever, we feel free to enlarge $\mathcal{O}$ if necessary.) Since the χ_v are trivial modulo ϖ, we have

$$S_{0,\{\chi_v\}}^{\mathrm{ord}}(U(\mathfrak{l}^\infty), k) = S_{0,\{1\}}^{\mathrm{ord}}(U(\mathfrak{l}^\infty), k), \tag{5.7.16}$$

so $\mathfrak{m}_1$ induces a unique maximal ideal $\mathfrak{m}_\chi$ of $\mathbf{T}_{\{\chi_v\}}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})$. After conjugating we can and do assume that $\overline{r}_{\mathfrak{m}_\chi} = \overline{r}_{\mathfrak{m}_1} = \overline{\rho}$.

Write

$$H_\chi := S_{0,\{\chi_v\}}^{\mathrm{ord}}(U(\mathfrak{l}^\infty), E/\mathcal{O})_{\mathfrak{m}_\chi}^\vee, \quad H_1 := S_{0,\{1\}}^{\mathrm{ord}}(U(\mathfrak{l}^\infty), E/\mathcal{O})_{\mathfrak{m}_1}^\vee.$$

By [Ger19, Cor. 2.21], H_1 is a faithful $\mathbf{T}_{\{1\}}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty), \mathcal{O})_{\mathfrak{m}_1}$-module, and is in particular an $R_{\mathcal{S}_{\{1\}}}$-module via (5.7.10). Similarly H_χ is an $R_{\mathcal{S}_{\{\chi_v\}}}$-module. By (5.7.16) we have a natural isomorphism

$$H_\chi/\varpi \cong H_1/\varpi,$$

which is compatible with the isomorphism $R_{\mathcal{S}_{\{\chi_v\}}}/\varpi \cong R_{\mathcal{S}_{\{1\}}}/\varpi$.

Write $q = h^1_{\mathcal{S}^\perp, T} - 1$ and $g = q + \#T - 1 - [F^+ : \mathbf{Q}]n(n-1)/2$. Write

$$S_\infty := \Lambda[\![X_1, \cdots, X_{q+(n^2+1)\#T-1}]\!]$$

with augmentation ideal $\mathfrak{a}_\infty = (X_1, \ldots, X_{q+(n^2+1)\#T-1})$. (The number of formal variables here is given by the number of Taylor–Wiles primes plus the relative dimension of $R_{\mathcal{S}}^T$ over $R_{\mathcal{S}}$.) We set

$$R_{\chi,\infty} := R^{\mathrm{loc}}_{\mathcal{S}_{\{\chi_v\}},T}[\![Y_1, \ldots, Y_g]\!], \quad R_{1,\infty} := R^{\mathrm{loc}}_{\mathcal{S}_{\{1\}},T}[\![Y_1, \ldots, Y_g]\!].$$

By (5.7.8) we have

$$\begin{aligned} \dim R_{\chi,\infty} = \dim R_{1,\infty} &= 1 + n^2\#T + [F^+ : \mathbf{Q}]n(n+1)/2 + g \\ &= [F^+ : \mathbf{Q}]n + (n^2+1)\#T + q \\ &= \dim S_\infty. \end{aligned} \tag{5.7.17}$$

Using Proposition 5.5.6 in place of [Tho12, Prop. 4.4], a standard patching argument exactly as in the proof of [Tho12, Thm. 8.6] provides us with the following:

- CNL_Λ-homomorphisms $S_\infty \to R_{1,\infty}$, $S_\infty \to R_{\chi,\infty}$.
- An $R_{1,\infty}$-module $H_{1,\infty}$, and an $R_{\chi,\infty}$-module $H_{\chi,\infty}$, each of which is free of finite rank over S_∞.

- A surjection of $R^{\mathrm{loc}}_{\mathcal{S}_{\{1\}},T}$-algebras $R_{1,\infty} \twoheadrightarrow R_{\mathcal{S}_{\{1\}}}$, which factors through a Λ-algebra map $R_{1,\infty}/\mathfrak{a}_\infty \to R_{\mathcal{S}_{\{1\}}}$; and similarly, a surjection of $R^{\mathrm{loc}}_{\mathcal{S}_{\{\chi_v\}},T}$-algebras $R_{\chi,\infty} \twoheadrightarrow R_{\mathcal{S}_{\{\chi_v\}}}$, which factors through a Λ-algebra map $R_{\chi,\infty}/\mathfrak{a}_\infty \to R_{\mathcal{S}_{\{\chi_v\}}}$.
- Isomorphisms $H_{1,\infty}/\mathfrak{a}_\infty \cong H_1$, $H_{\chi,\infty}/\mathfrak{a}_\infty \cong H_\chi$ compatible with the surjections $R_{1,\infty} \twoheadrightarrow R_{\mathcal{S}_{\{1\}}}$, $R_{\chi,\infty} \twoheadrightarrow R_{\mathcal{S}_{\{\chi_v\}}}$.
- Compatible identifications of all the above data for 1 and for χ after reducing modulo ϖ.

In particular since $H_{1,\infty}$ is a finite free S_∞-module, we deduce from (5.7.17) that

$$\operatorname{depth}_{R_{1,\infty}} H_{1,\infty} \geq \operatorname{depth}_{S_\infty} H_{1,\infty} = \dim S_\infty = \dim R_{1,\infty},$$

whence $\operatorname{depth}_{R_{1,\infty}} H_{1,\infty} = \dim R_{1,\infty}$, and the support of $H_{1,\infty}$ in $\operatorname{Spec} R_{1,\infty}$ is a union of irreducible components (see [Tay08, Lem. 2.3]). Similarly, the support of $H_{\chi,\infty}$ in $\operatorname{Spec} R_{\chi,\infty}$ is a union of irreducible components.

We now examine the irreducible components of $\operatorname{Spec} R_{\chi,\infty}$ and $\operatorname{Spec} R_{1,\infty}$. Bearing in mind Propositions 5.6.4 and 5.6.6, an identical argument to the proof of [BCG+25, Lem. 3.2.4] shows that for each minimal prime Q of Λ, we have the following properties.

(1) The generic points of $R_{\chi,\infty}/Q$ and $R_{1,\infty}/Q$ all have characteristic 0.
(2) The irreducible components $\mathcal{C}$ of $\operatorname{Spec} R_{\chi,\infty}/Q$ and $\operatorname{Spec} R_{1,\infty}/Q$ biject with the products of the corresponding sets of irreducible components $\mathcal{C}_v$ of the local deformation rings for $v \in R \cup \{v_1\}$.
(3) The irreducible components $\overline{\mathcal{C}}$ of $\operatorname{Spec} R_{\chi,\infty}/(Q,\varpi) = \operatorname{Spec} R_{1,\infty}/(Q,\varpi)$ biject with the products over T of the corresponding sets of the irreducible components $\overline{\mathcal{C}}_v$ of the special fibres of the deformation rings for $v \in R \cup \{v_1\}$.

In view of these statements we will use the notation $\mathcal{C} = \otimes_{v \in R \cup \{v_1\}} \mathcal{C}_v$ and $\overline{\mathcal{C}} = \otimes_{v \in R \cup \{v_1\}} \overline{\mathcal{C}}_v$.

(4) The irreducible components of $\operatorname{Spec} R_{\chi,\infty}/Q$ biject with the irreducible components of $\operatorname{Spec} R^{\square}_{v_1}$.
(5) For each irreducible component $\overline{\mathcal{C}} = \otimes_{v \in R \cup \{v_1\}} \overline{\mathcal{C}}_v$ of $\operatorname{Spec} R_{1,\infty}/(Q,\varpi)$, there are irreducible components $\mathcal{C}_v$ of $\operatorname{Spec} R^1_v$ for $v \in R$, and irreducible components $\mathcal{C}^1_{v_1}, \ldots \mathcal{C}^s_{v_1}$ of $\operatorname{Spec} R^{\square}_{v_1}$ (for some $s \geq 1$), such that the irreducible components of $\operatorname{Spec} R_{1,\infty}/Q$ generalizing $\overline{\mathcal{C}}$ are precisely the s components $\mathcal{C}^i_{v_1} \otimes_{v \in R} \mathcal{C}_v$.

Fix for the moment a minimal prime Q of Λ. The existence of π implies that the support of $H_{1,\infty}$ in $\operatorname{Spec} R_{1,\infty}$ is nonempty. Using the comparison modulo ϖ, the same is true of the support of $H_{\chi,\infty}$ in $\operatorname{Spec} R_{\chi,\infty}$. By points (4) and (5), we conclude that for each set X of irreducible components $\mathcal{C}_v$ of $\operatorname{Spec} R^1_v$ for $v \in R$, we can choose an irreducible component $\mathcal{C}_{X,v_1}$ of $\operatorname{Spec} R^{\square}_{v_1}$ such that $\mathcal{C}_{X,v_1} \otimes_{v \in R, \mathcal{C}_v \in X} \mathcal{C}_v$ is in the support of $H_{1,\infty}$. This choice of irreducible components corresponds to a quotient R^{loc}_X of $R^{\mathrm{loc}}_{\mathcal{S}_{\{1\}},T}$, and if we set

$$R_X := R_{\mathcal{S}_{\{1\}}} \otimes_{R^{\mathrm{loc}}_{\mathcal{S}_{\{1\}},T}} R^{\mathrm{loc}}_X,$$

then $\operatorname{Spec} R_X \subset \operatorname{Spec} R_{\mathcal{S}_{\{1\}}}$ is contained in the support of $H'_{1,\infty}$; equivalently, $\operatorname{Spec} R_X$ is contained in the support of H_1 in $\operatorname{Spec} R_{\mathcal{S}_{\{1\}}}$.

In particular $(R_X)^{\mathrm{red}}$ is a quotient of a Hecke algebra $\mathbf{T}^{T,\mathrm{ord}}_{\{1\}}(U(\mathfrak{l}^\infty),\mathcal{O})_{\mathfrak{m}_1}$, so it is a finite Λ-algebra, so that R_X itself is a finite Λ-algebra. (To see this, note by the topological Nakayama lemma it suffices to observe $R_X/\mathfrak{m}_\Lambda$ is Noetherian and zero-dimensional, thus finite.) Since R_X has dimension at least $\dim\Lambda$ by Lemma 5.7.7, we see that the morphism $\operatorname{Spec} R_X \to \operatorname{Spec}\Lambda/Q$ is dominant. In particular, we can choose a weight λ such that $R_X/\wp_\lambda R_X$ is a nonzero finite $\mathcal{O}$-algebra of dimension at least 1, and thus has a $\overline{\mathbf{Q}}_2$-point. Since $(R_X)^{\mathrm{red}}$ is a quotient of $\mathbf{T}^{T,\mathrm{ord}}_{\{1\}}(U(\mathfrak{l}^\infty),\mathcal{O})_{\mathfrak{m}_1}$, it follows from (5.7.4) that the Galois representation corresponding to this $\overline{\mathbf{Q}}_2$-point comes from an ordinary RACSDC representation of weight λ.

Repeating this construction for all choices of Q, we conclude that for each choice set Y of irreducible components $\mathcal{C}_v$ for $v\in R\cup S_2$, there is an ordinary RACSDC automorphic representation π_Y of GL_n/F such that:

- $\overline{r}_{\pi_Y,2}\cong\overline{\rho}$,
- $r_{\pi_Y,2}|_{G_{F_w}}$ is unramified for all places w not lying over a place in T,
- and for each $v\in R\cup S_2$, the representation $r_{\pi_Y,2}|_{G_{F_{\widetilde{v}}}}$ lies on $\mathcal{C}_v$ and on no other irreducible component (by the genericity of $\pi_{Y,v}$, see [BLGGT14, Lem. 1.3.2(1)]).

By Lemma 5.6.3, we can and do choose a solvable CM extension L/F, linearly disjoint from $\overline{F}^{\ker\overline{\rho}}$ over F, with the following property: for any Y as above, and any place w_1 of L^+ lying over v_1, the representation $r_{\pi_Y,2}|_{G_{F_{\widetilde{w}_1}}}$ is unramified (where we write $w_1=\widetilde{w}_1\widetilde{w}_1^c$).

We now repeat the patching argument above with F replaced by L. More precisely, we:

- replace R by the set R' of places of L lying over places in R;
- choose a place $v_1'\notin R'\cup S_2$ of L^+ splitting in L as $\widetilde{v}_1'(\widetilde{v}_1')^c$, with $\overline{\rho}(\mathrm{Frob}_{\widetilde{v}_1'})$ being regular semi-simple;
- and replace T by $T'=R'\cup\{v_1'\}$.

Writing H_1', $R_{1,\infty}'$ for the corresponding objects over L, we find in particular that we have the patched module $H_{1,\infty}'$, whose support in $\operatorname{Spec} R_{1,\infty}'$ is a union of irreducible components. Again, we write these irreducible components as $\mathcal{C}'=\otimes_{v'\in T'}\mathcal{C}_{v'}'$, and for each set Y as above we let $\mathcal{C}_Y'$ denote the irreducible component determined by letting $\mathcal{C}_{v_1'}'$ be the (unique) unramified component of $\operatorname{Spec} R^\square_{v_1'}$, and letting $\mathcal{C}_{v'}'$ for $v'|v\in R\cup S_2$ be the image of the component $\mathcal{C}_v$ for Y (via the natural morphism $\operatorname{Spec} R_v^\square\to\operatorname{Spec} R_{v'}^\square$).

By considering the base changes to L of the π_Y, we see that each component $\mathcal{C}_Y'$ is in the support of $H_{1,\infty}'$. The union of the irreducible components $\mathcal{C}_Y'$ corresponds to a quotient $R^{\mathrm{loc}}_{L/F,T'}$ of $R^{\mathrm{loc}}_{\mathcal{S}_{\{1\}},T'}$, and as above, if we set

$$R_{L/F,T'}':=R_{\mathcal{S}_{\{1\}}}'\otimes_{R^{\mathrm{loc}}_{\mathcal{S}_{\{1\}}',T'}}R^{\mathrm{loc}}_{L/F,T'},$$

then $\operatorname{Spec} R_{L/F,T'}'\subset\operatorname{Spec} R_{\mathcal{S}_{\{1\}}'}'$ is contained in the support of $H_{1,\infty}'$; equivalently, $\operatorname{Spec} R_{L/F,T'}'$ is contained in the support of H_1' in $\operatorname{Spec} R_{\mathcal{S}_{\{1\}}'}'$. Thus $(R_{L/F,T'}')^{\mathrm{red}}$ is a quotient of a Hecke algebra $\mathbf{T}^{T',\mathrm{ord}}_{\{1\}}(U(\mathfrak{l}^\infty),\mathcal{O})_{\mathfrak{m}_1}$, and in particular every homomorphism $R_{L/F,T'}'\to\mathcal{O}$ factors through $\mathbf{T}^{T',\mathrm{ord}}_{\{1\}}(U(\mathfrak{l}^\infty),\mathcal{O})_{\mathfrak{m}_1}$. Furthermore, it follows as above that $R_{L/F,T'}'$ is a finite Λ'-algebra.

We now return to the original situation of the statement of the theorem (so F is now the CM field that we started with, before we made any base changes, and T is as in the statement of the theorem). By the choice of L, we have a commutative diagram

$$\begin{array}{ccccc} \Lambda' & \longrightarrow & R'_{L/F,T'} & & \\ \downarrow & & \downarrow & & \\ \Lambda & \longrightarrow & R^{T,\mathrm{ord}} & \longrightarrow & \mathbf{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty))_{\mathfrak{m}} \end{array}$$

The morphism $R'_{L/F,T'} \to R^{T,\mathrm{ord}}$ is finite, by the obvious generalization of [BLGGT14, Lem. 1.2.3(1)] to the case $p = 2$, which has an identical proof up to replacing the appeal to [CHT08, Lem. 2.1.12] with a citation of [Tho17, Lem. 2.4]. It follows that $R^{T,\mathrm{ord}}$ is a finite Λ-algebra, as claimed.

Suppose now that the U_v for $v \in T \smallsetminus S_2$ are sufficiently deep. Since every irreducible component of $\operatorname{Spec} R^{T,\mathrm{ord}}$ has dimension at least that of $\operatorname{Spec}\Lambda$ by Lemma 5.7.7, it follows that each irreducible component dominates an irreducible component of $\operatorname{Spec}\Lambda$. It follows that the set of points $\operatorname{Spec}\overline{\mathbf{Q}}_2 \to \operatorname{Spec} R^{T,\mathrm{ord}}$ which lie over points of Λ given by the primes $\wp_\lambda$ is dense in $\operatorname{Spec} R^{T,\mathrm{ord}}$. It remains to show that each such point is in $\operatorname{Spec}\mathbf{T}^{T,\mathrm{ord}}(U(\mathfrak{l}^\infty))_{\mathfrak{m}}$. By (5.7.4) and the choice of U, it is enough to check that the corresponding Galois representations are automorphic. By solvable base change, it is enough to check this after restriction to G_L, where it follows from another application of (5.7.4) (and the observation above that $(R'_{L/F,T'})^{\mathrm{red}}$ is a quotient of $\mathbf{T}^{T',\mathrm{ord}}_{\{1\}}(U(\mathfrak{l}^\infty),\mathcal{O})_{\mathfrak{m}_1}$). □

6. Ordinary modularity lifting theorems for GSp_4: preliminaries

The goal of this section is — in part — to prove an ordinary modularity lifting theorem for GSp_4 for $p \geq 3$ over totally real fields in which p splits completely. Under suitable Taylor–Wiles hypotheses, this can be used to show that p-adic Galois representations coming from ordinary abelian surfaces give rise to quotients of a certain p-adic Hecke algebra, but does not yet show that such classes are classical. The modularity lifting theorem we prove in this section is (in the language of [CG18]) of $l_0 = 0$ type rather than $l_0 > 0$ type, and so is precisely amenable to the usual Taylor–Wiles method. Under a stronger hypothesis (that $\overline{\rho}$ is residually p-distinguished) our results are actually directly contained in [BCGP21] (although that paper is generally concerned with the more subtle $l_0 = 1$ situation), and versions of this theorem go back as far as [Pil12]. The methods we use here follow along generally similar grounds, with some important technical improvements due in several cases to Whitmore [Whi22].

In §6.1, we recall some general constructions and notation for GSp_4-deformation problems. In §6.2, we introduce the corresponding ordinary local deformation rings and study their local properties. In §6.3, we carry out the Taylor–Wiles argument (in part following [BCGP21] and [Whi22]). Finally, in §6.4, we explicitly analyze the subgroups of $\mathrm{GSp}_4(\mathbf{F}_3)$ which satisfy our running collection of "big–image" conditions.

6.1. **Notation and definitions.** We now turn to modularity lifting theorems for GSp_4. Our arguments have relatively little *direct* overlap with those of our earlier paper [BCGP21], although we will occasionally make references to it. In particular, in order to avoid confusing clashes of notation with our results for unitary groups in Section 5, we continue to write F^+ for a totally real field (whereas totally real fields were denoted F in [BCGP21]).

Accordingly we let F^+ denote a totally real field, and write S_p for the set of places of F^+ above p. We fix a continuous absolutely irreducible homomorphism $\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(k)$ with similitude $\overline{\varepsilon}^{-1}$. When $\overline{\rho}$ is explicitly considered as a symplectic representation (as in this section), we denote by $\mathrm{ad}\,\overline{\rho}$ and $\mathrm{ad}^0\,\overline{\rho}$ the adjoint G_{F^+} action with respect to GSp_4 and Sp_4 respectively (so $\dim \mathrm{ad}\,\overline{\rho} = 11$ and $\dim \mathrm{ad}^0\,\overline{\rho} = 10$). We warn the reader that there is some tension in this definition with the notation of §5 where $\mathrm{ad}\,\overline{r}$ denotes the adjoint action with respect to GL_n; we hope the precise meaning will always be clear from context.

Let S be a finite set of finite places of F^+ containing S_p and all places at which $\overline{\rho}$ is ramified. We write F_S^+ for the maximal subextension of $\overline{F^+}/F^+$ which is unramified outside S, and write $G_{F^+,S}$ for $\mathrm{Gal}(F_S^+/F^+)$. For each $v \in S$, we fix $\Lambda_v \in \mathrm{CNL}_{\mathcal{O}}$, and set $\Lambda = \widehat{\otimes}_{v\in S}\Lambda_v$, where the completed tensor product is taken over $\mathcal{O}$. Then CNL_Λ is a subcategory of CNL_{Λ_v} for each $v \in S$, via the canonical map $\Lambda_v \to \Lambda$.

Definition 6.1.1. A *lift*, also called a *lifting*, of $\overline{\rho}|_{G_{F_v^+}}$ is a continuous homomorphism $\rho : G_{F_v^+} \to \mathrm{GSp}_4(A)$ to a CNL_{Λ_v}-algebra A such that $\rho \bmod \mathfrak{m}_A = \overline{\rho}|_{G_{F_v^+}}$ and $\nu \circ \rho = \varepsilon^{-1}$.

We let $\mathcal{D}_v^\square$ denote the set-valued functor on CNL_{Λ_v} that sends A to the set of lifts of $\overline{\rho}|_{G_{F_v^+}}$ to A. This functor is representable, and we denote the representing object by $R_v^\square$. We can identify $\mathcal{D}_v^\square(k[\epsilon])$ with the group of 1-cocycles $Z^1(F_v, \mathrm{ad}^0\,\overline{\rho})$ by associating a cocycle ϕ to the lifting given by

$$\rho(\sigma) = (1 + \epsilon\phi(\sigma))\overline{\rho}(\sigma).$$

Note that two such liftings are $\widehat{\mathrm{GSp}_4}(k[\epsilon])$-conjugate if and only if the images of the corresponding 1-cocycles in $H^1(F_v, \mathrm{ad}\,\overline{\rho})$ are equal.

Definition 6.1.2. A *local deformation problem* for $\overline{\rho}|_{G_{F_v^+}}$ is a subfunctor $\mathcal{D}_v$ of $\mathcal{D}_v^\square$ satisfying the following:

- $\mathcal{D}_v$ is represented by a quotient R_v of $R_v^\square$.
- For all $A \in \mathrm{CNL}_{\Lambda_v}$, $\rho \in \mathcal{D}_v(A)$, and $a \in \widehat{\mathrm{GSp}}_4(A)$, we have $a\rho a^{-1} \in \mathcal{D}_v(A)$.

Definition 6.1.3. A *global deformation problem* is a tuple

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$$

where:

- $\overline{\rho}$, S, $\{\Lambda_v\}_{v\in S}$ are as above.
- For each $v \in S$, $\mathcal{D}_v$ is a local deformation problem for $\overline{\rho}|_{G_{F_v^+}}$.

As in the local case, a *lift* (or *lifting*) of $\overline{\rho}$ is a continuous homomorphism $\rho : G_{F^+,S} \to \mathrm{GSp}_4(A)$ to a CNL_Λ-algebra A, such that $\rho \bmod \mathfrak{m}_A = \overline{\rho}$ and $\nu \circ \rho = \varepsilon^{-1}$. We say that two lifts $\rho_1, \rho_2 : G_{F^+,S} \to \mathrm{GSp}_4(A)$ are *strictly equivalent* if there is an $a \in \widehat{\mathrm{GSp}}_4(A)$ such that $\rho_2 = a\rho_1 a^{-1}$. A *deformation of* $\overline{\rho}$ is a strict equivalence class of lifts of $\overline{\rho}$.

For a global deformation problem

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$$

we say that a lift $\rho : G_{F^+,S} \to \mathrm{GSp}_4(A)$ is of *type* $\mathcal{S}$ if $\rho|_{G_{F_v^+}} \in \mathcal{D}_v(A)$ for each $v \in S$. If ρ_1 and ρ_2 are strictly equivalent lifts of $\overline{\rho}$, and ρ_1 is of type $\mathcal{S}$, then so is ρ_2. A *deformation of type* $\mathcal{S}$ is a strict equivalence class of lifts of type $\mathcal{S}$, and we denote by $\mathcal{D}_{\mathcal{S}}$ the set-valued functor that takes a CNL_Λ-algebra A to the set of lifts $\rho : G_{F^+} \to \mathrm{GSp}_4(A)$ of type $\mathcal{S}$.

Given a subset $T \subseteq S$, a T*-framed lift of type* $\mathcal{S}$ is a tuple $(\rho, \{\gamma_v\}_{v\in T})$, where ρ is a lift of type $\mathcal{S}$, and $\gamma_v \in \widehat{\mathrm{GSp}}_4(A)$ for each $v \in T$. We say that two T-framed lifts $(\rho_1, \{\gamma_v\}_{v\in T})$ and $(\rho_2, \{\gamma'_v\}_{v\in T})$ to a CNL_Λ-algebra A are strictly equivalent if there is an $a \in \widehat{\mathrm{GSp}}_4(A)$ such that $\rho_2 = a\rho_1 a^{-1}$, and $\gamma'_v = a\gamma_v$ for each $v \in T$. A strict equivalence class of T-framed lifts of type $\mathcal{S}$ is called a T*-framed deformation of type* $\mathcal{S}$. We denote by $\mathcal{D}_{\mathcal{S}}^T$ the set valued functor that sends a CNL_Λ-algebra A to the set of T-framed deformations to A of type $\mathcal{S}$.

The functors $\mathcal{D}_{\mathcal{S}}$, $\mathcal{D}_{\mathcal{S}}^T$ are representable (as we are assuming that $\overline{\rho}$ is absolutely irreducible), and we denote their representing objects by $R_{\mathcal{S}}$ and $R_{\mathcal{S}}^T$ respectively. Assume now that T is chosen so that $\Lambda_v = \mathcal{O}$ for all $v \in S \smallsetminus T$. Write R_v for the representing object of $\mathcal{D}_v$, and define $R_{\mathcal{S},T}^{\mathrm{loc}} = \widehat{\otimes}_{v\in T} R_v$, with the completed tensor product being taken over $\mathcal{O}$. It is canonically a Λ-algebra, via the canonical isomorphism $\widehat{\otimes}_{v\in T}\Lambda_v \cong \widehat{\otimes}_{v\in S}\Lambda_v$. For each $v \in T$, the natural transformation $\mathrm{Def}_{\mathcal{S}}^T \to \mathcal{D}_v$ given by $(\rho, \{\alpha_v\}_{v\in T}) \mapsto \alpha_v^{-1}\rho|_{G_{F_v}}\alpha_v$ induces a morphism $R_v \to R_{\mathcal{S}}^T$ in CNL_{Λ_v}. We thus have a morphism $R_{\mathcal{S},T}^{\mathrm{loc}} \to R_{\mathcal{S}}^T$ in CNL_Λ.

If T is empty, then $R_{\mathcal{S}} = R_{\mathcal{S}}^T$, and otherwise the natural map $R_{\mathcal{S}} \to R_{\mathcal{S}}^T$ is formally smooth of relative dimension $11\#T - 1$. Indeed $\mathcal{D}_{\mathcal{S}}^T \to \mathcal{D}_{\mathcal{S}}$ is a torsor under $(\prod_{v\in T} \widehat{\mathrm{GSp}}_4)/\widehat{\mathbf{G}}_m$.

Definition 6.1.4. Let

$$\mathcal{T} := \Lambda[\![x_1, \ldots, x_{11\#T-1}]\!]$$

be the coordinate ring of $(\prod_{v\in T} \widehat{\mathrm{GSp}}_4)/\widehat{\mathbf{G}}_m$ over Λ.

The choice of a representative $\rho_{\mathcal{S}} \colon G_F \to \mathrm{GSp}_4(R_{\mathcal{S}})$ for the universal type $\mathcal{S}$ deformation determines a splitting of the torsor $\mathcal{D}_{\mathcal{S}}^T \to \mathcal{D}_{\mathcal{S}}$ and a canonical isomorphism

$$R_{\mathcal{S}}^T \cong R_{\mathcal{S}} \widehat{\otimes}_\Lambda \mathcal{T}. \tag{6.1.5}$$

The following lemma and its proof are standard, but we include them in order to reassure the reader that they remain valid for $p = 2$.

Lemma 6.1.6. *Suppose that* $q_v \equiv 1 \bmod p$*, and that* $\overline{\rho}|_{G_{F_v^+}}$ *is unramified, with* $\overline{\rho}(\mathrm{Frob}_v)$ *being regular semi-simple with (ordered) eigenvalues* $\overline{\alpha}_{v,1}$, $\overline{\alpha}_{v,2}$, $\overline{\alpha}_{v,2}^{-1}$, $\overline{\alpha}_4 = \overline{\alpha}_{v,1}^{-1}$. *Let* $\rho : G_{F_v^+} \to \mathrm{GSp}_4(A)$ *be any lift of* $\overline{\rho}$.

Then there are unique continuous characters $\gamma_i : G_{F_v^+} \to A^\times$ *for* $i = 1, 2$*, such that* ρ *is* $\mathrm{GSp}_4(A)$*-conjugate to a lift of the form*

$$\gamma_1 \oplus \gamma_2 \oplus \gamma_2^{-1}\varepsilon^{-1} \oplus \gamma_1^{-1}\varepsilon^{-1},$$

where $(\gamma_i \bmod \mathfrak{m}_A)(\mathrm{Frob}_v) = \overline{\alpha}_{v,i}$ *for each* $i = 1, 2$.

Proof. Let ϕ be a lifting of Frob_v to $G_{F_v^+}$. Then $\rho(\phi)$ is regular semi-simple, so is contained in $T(A)$ for a unique torus T, and we need to show that for each $\sigma \in I_{F_v^+}$, we have $\rho(\sigma) \in T(A)$. To do this we will prove by induction that for each $n \geq 1$ we have

$$(\rho \bmod \mathfrak{m}_A^n)(\sigma) \in T(A/\mathfrak{m}_A^n),$$

the case $n = 1$ being true by our hypotheses.

For the inductive step, we assume the result holds for n and deduce it for $n+1$; replacing A by $A/\mathfrak{m}_A^{n+1}$, we may furthermore assume that $\mathfrak{m}_A^{n+1} = 0$. By the inductive hypothesis (and the smoothness of T) we can write $\rho(\sigma) = tu$ where $t \in T(A)$ and $u \equiv 1 \pmod{\mathfrak{m}^n}$. Since $\overline{\rho}(\sigma) = 1$, we see that u and t necessarily commute (as do $\rho(\phi)$ and t, as both are contained in $T(A)$).

Now, since ρ is tamely ramified, we have $\rho(\phi)\rho(\sigma)\rho(\phi)^{-1} = \rho(\sigma)^{q_v}$. Since $q_v \equiv 1 \pmod{p}$, and $u \equiv 1 \pmod{\mathfrak{m}^n}$, we see that $u^{q_v} = u$, and thus that $\rho(\phi)u\rho(\phi)^{-1} = t^{q_v-1}u$. Using again that $u \equiv 1 \pmod{\mathfrak{m}^n}$ and that $\overline{\rho}(\phi)$ is regular semi-simple, it follows that $u = 1$, as required. □

6.2. **Ordinary deformation rings.** In this section we study some ordinary deformation rings for GSp_4. We assume that v is a place of F^+ lying over p such that $F_v^+ = \mathbf{Q}_p$. Write $\Lambda_{\mathrm{GSp}_4,v} = \mathcal{O}[\![(\mathcal{O}_{F_v^+}^\times(p))^2]\!]$, where $\mathcal{O}_{F_v^+}^\times(p)$ denotes the pro-p completion of $\mathcal{O}_{F_v^+}^\times$. There is a canonical character $I_{F_v^+} \to \mathcal{O}_{F_v^+}^\times(p)$ given by $\mathrm{Art}_{F_v^+}^{-1}$, and we define a pair of characters $\theta_i : I_{F_v^+} \to \Lambda_{\mathrm{GSp}_4,v}$, $i = 1, 2$ by letting θ_i correspond to the embedding $\mathcal{O}_{F_v^+}^\times(p) \to (\mathcal{O}_{F_v^+}^\times(p))^2$ given by the ith copy. When $p > 2$, $\Lambda_{\mathrm{GSp}_4,v} = \mathcal{O}[\![x_1, x_2]\!]$ is formally smooth, while when $p = 2$, $\operatorname{Spec} \Lambda_{\mathrm{GSp}_4,v}$ has 4 irreducible components but the generic fiber is regular.

Assume that $\overline{\rho}|_{G_{F^+v}}$ is ordinary, and fix a p-stabilization $(\overline{\chi}_1, \overline{\chi}_2)$ of $\overline{\rho}|_{G_{F^+v}}$, so that $(\overline{\chi}_1, \overline{\chi}_2)$ is an ordered pair of characters $G_{F_v^+} \to k^\times$. Then for any $A \in \mathrm{CNL}_{\mathcal{O}}$ there is an obvious bijection between homomorphisms $\Lambda_{\mathrm{GSp}_4,v} \to A$ and ordered pairs of characters $(\chi_1, \chi_2) : I_{F_v^+} \to A^\times$ lifting $(\overline{\chi}_1, \overline{\chi}_2)$, given by multiplying the characters (θ_1, θ_2) by the Teichmüller lifts of $(\overline{\chi}_1, \overline{\chi}_2)$.

Similarly, write $\widetilde{\Lambda}_{\mathrm{GSp}_4,v} = \mathcal{O}[\![\mathrm{Gal}(F_v^{+,\mathrm{ab}}/F_v^+)(p)^2]\!]$, where $\mathrm{Gal}(F_v^{+,\mathrm{ab}}/F_v^+)(p)$ is the pro-p completion of $\mathrm{Gal}(F_v^{+,\mathrm{ab}}/F_v^+)$. Then we have a universal pair of characters $(\widetilde{\chi}_1, \widetilde{\chi}_2) : G_{F_v^+} \to \widetilde{\Lambda}_{\mathrm{GSp}_4,v}$ lifting $(\overline{\chi}_1, \overline{\chi}_2)$.

We now introduce the ordinary deformation ring we consider, following [Ger19, §3]. Let $\mathcal{F}$ denote the flag variety for GSp_4 over $\mathcal{O}$, i.e. the variety whose S-points, for any $S/\operatorname{Spec} \mathcal{O}$, parameterize full flags

$$0 = \mathrm{Fil}_0 \subset \mathrm{Fil}_1 \subset \cdots \subset \mathrm{Fil}_4 = \mathcal{O}_S^4$$

with Fil_i being locally free of rank i and locally a direct summand, with the further property that $(\mathrm{Fil}_i)^\perp = \mathrm{Fil}_{4-i}$ for each i (where $^\perp$ is with respect to our usual symplectic form on $\mathcal{O}_S^4$).

Write $R_v^\square$ for the ring denoted $R_v^\square$ in Section 6.1 when $\Lambda_v = \widetilde{\Lambda}_{\mathrm{GSp}_4,v}$. One shows as in the proof of [Ger19, Lem. 3.2] that there is a closed subscheme $\mathcal{G}_v$ of $\mathcal{F} \times_{\mathcal{O}} \operatorname{Spec} R_v^\square$, such that for any $\mathcal{O}$-algebra A, the A-points of $\mathcal{G}_v$ are exactly the pairs $(\mathrm{Fil}_\bullet, R_v^\square \to A)$ consisting of a symplectic flag $\mathrm{Fil}_\bullet$ on A^4 and an $\mathcal{O}$-algebra morphism $R_v^\square \to A$, such that the pushforward of the universal lifting over $R_v^\square \to A$ preserves $\mathrm{Fil}_\bullet$, and for $i = 1, 2, 3, 4$ the action of $G_{F_v^+}$ on $\mathrm{Fil}_i / \mathrm{Fil}_{i-1}$ is via respectively (the pushforwards to A of) the characters $\widetilde{\chi}_1$, $\widetilde{\chi}_2$, $\varepsilon^{-1}\widetilde{\chi}_2^{-1}$, $\varepsilon^{-1}\widetilde{\chi}_1^{-1}$.

We write

$$R_v^\triangle := \mathrm{im}\left(R_v^\square \to \mathcal{O}_{\mathcal{G}_v}(\mathcal{G}_v)\right),$$

so that $\operatorname{Spec} R_v^{\triangle}$ is the scheme-theoretic image of the morphism $\mathcal{G}_v \to \operatorname{Spec} R_v^{\square}$. (Note that here we differ from [Ger19] by not passing to the p-torsion-free quotient.) We denote by $\mathcal{D}_v^{\triangle}$ the corresponding deformation problem.

Exactly as in the proof of [Ger19, Lem. 3.3], it follows immediately from the properness of $\mathcal{F}$ that if E'/E is a finite extension with ring of integers $\mathcal{O}_{E'}$ and $\widetilde{\Lambda}_{\mathrm{GSp}_4,v} \to \mathcal{O}_{E'}$ is a morphism of $\mathcal{O}$-algebras, then the $\mathcal{O}_{E'}$-points of $\operatorname{Spf} R_v^{\triangle}$ are exactly those lifts ρ of $\overline{\rho}$ having the property that there is a symplectic flag

$$0 = \mathrm{Fil}_0 \subset \mathrm{Fil}_1 \subset \cdots \subset \mathrm{Fil}_4 = \mathcal{O}_{E'}^4$$

as above such that for $i = 1, 2, 3, 4$ the action of $G_{F_v^+}$ on $\mathrm{Fil}_i / \mathrm{Fil}_{i-1}$ is via respectively the characters $\widetilde{\chi}_1$, $\widetilde{\chi}_2$, $\varepsilon^{-1}\widetilde{\chi}_2^{-1}$, $\varepsilon^{-1}\widetilde{\chi}_1^{-1}$. Equivalently, these are the lifts ρ which are ordinary with p-stabilization $(\widetilde{\chi}_1, \widetilde{\chi}_2)$ in the sense of Definition 1.8.10.

Remark 6.2.1. As in Definition 1.8.10, we say that $\overline{\rho}$ is *residually p-distinguished* if the 4 characters $\overline{\chi}_1, \overline{\chi}_2, \varepsilon^{-1}\overline{\chi}_2^{-1}, \varepsilon^{-1}\overline{\chi}_1^{-1}$ are pairwise distinct (if $p \neq 2$ this amounts to $\overline{\chi}_1 \neq \overline{\chi}_2$). In this case the filtration Fil_i in the definition of $\mathcal{G}_v$ is uniquely determined by the Galois representation, and it follows that the map $\mathcal{G}_v \to \operatorname{Spec} R_v^{\square}$ is a closed immersion. In [BCGP21, §7.3], we made this assumption and assumed $p > 2$ and studied $\mathcal{G}_v$ under the name $R_v^{B,\overline{\tilde{r}}_v}$.

Let E'/E be a finite extension, and let $x : \operatorname{Spec} E' \to \mathcal{G}_v[1/p]$ be a closed point. Let $\rho_x : G_{F_v^+} \to \mathrm{GSp}_4(E')$ be the pushforward of the universal lift coming from the composite $\operatorname{Spec} E' \to \mathcal{G}_v[1/p] \to \operatorname{Spec} R_v^{\square}$. Let $\operatorname{ad}^0 \rho_x$ denote the adjoint representation with respect to Sp_4, and define a decreasing filtration $\mathrm{Fil}^i \operatorname{ad}^0 \rho_x$ on ρ_x by

$$\mathrm{Fil}^i \operatorname{ad}^0 \rho_x := \{A \in \operatorname{ad}^0 \rho_x | A \,\mathrm{Fil}_j \,\rho_x \subseteq \mathrm{Fil}_{j-i} \,\rho_x \forall j\};$$

in particular, $\mathrm{Fil}^0 \operatorname{ad}^0 \rho_x$ is the subspace of $\operatorname{ad}^0 \rho_x$ preserving the flag $\mathrm{Fil}_\bullet \rho_x$. We have $\dim \operatorname{ad}^0 \rho_x = 10$, and $\dim \mathrm{Fil}^i \operatorname{ad}^0 \rho_x = 6, 4, 2, 1, 0$, for $i = 0, \ldots, 4$ respectively.

Lemma 6.2.2.

(1) *If* $H^2(G_{F_v^+}, \mathrm{Fil}^0 \operatorname{ad}^0 \rho_x) = 0$ *then* x *is a regular point of* $\mathcal{G}_v[1/p]$*; and* x *is contained in a unique irreducible component of* $\mathcal{G}_v[1/p]$*, and this component has dimension* 16*. We have*

$$H^2(G_{F_v^+}, \mathrm{Fil}^0 \operatorname{ad}^0 \rho_x) = 0$$

if and only if

$$H^0(G_{F_v^+}, \big(\operatorname{ad}^0 \rho_x / \mathrm{Fil}^1 \operatorname{ad}^0 \rho_x\big)(1)) = 0.$$

(2) *The equivalent conditions of part* (1) *hold under any of the following circumstances:*
  (a) *none of the specializations at* x *of the characters* $\widetilde{\chi}_1^2\varepsilon$, $\widetilde{\chi}_2^2\varepsilon$, $\widetilde{\chi}_1\widetilde{\chi}_2\varepsilon$, $\widetilde{\chi}_1\widetilde{\chi}_2^{-1}$ *are equal to* ε.
  (b) ρ_x *is pure and p-distinguished.*
  (c) ρ_x *is pure and potentially crystalline.*

(3) *If* ρ_x *is p-distinguished and the equivalent conditions of part* (1) *hold (in particular, if* ρ_x *is p-distinguished and pure) then the image of* x *in* $\operatorname{Spec} R_v^{\triangle}$ *is a regular point which is contained in a unique irreducible component of* $\operatorname{Spec} R_v^{\triangle}$*, which has relative (over* $\mathcal{O}$*) dimension 16.*

Proof. By a standard tangent-obstruction calculation exactly as in the proof of [Ger19, Lem. 3.7] (see also [Til96, §5.1] for the case of general algebraic groups), the tangent space to $\mathcal{G}_{\Lambda_{\mathrm{GSp}_4},v}[1/p]$ at x has dimension

$$16 + \dim_{E'} H^2(G_{F_v^+}, \mathrm{Fil}^0 \operatorname{ad}^0 \rho_x), \tag{6.2.3}$$

and there is an obstruction class in $H^2(G_{F_v^+}, \mathrm{Fil}^0 \operatorname{ad}^0 \rho_x)$ whose vanishing implies that x is a regular point of $\mathcal{G}_{\Lambda_{\mathrm{GSp}_4},v}[1/p]$.

For the remaining claim in (1), note that the space $\operatorname{ad}^0 \rho_x$ is self-dual under the trace pairing $(A, B) \to \mathrm{Tr}(AB)$ (since we are in characteristic zero). If $A \in \mathrm{Fil}^0 \operatorname{ad}^0 \rho_x$ and $B \in \mathrm{Fil}^1 \operatorname{ad}^0 \rho_x$ then $AB \in \mathrm{Fil}^1 \operatorname{ad}^0 \rho_x$ and so $\mathrm{Tr}(AB) = 0$. It follows that $\mathrm{Fil}^1 \operatorname{ad}^0 \rho_x \subset (\mathrm{Fil}^0 \operatorname{ad}^0 \rho_x)^{\perp}$. By considering the dimensions of these spaces (6 and $10-4 = 6$ respectively), we deduce that this is an equality, and hence the claim follows by Tate local duality.

For part (2a), note that $\mathrm{Fil}^0 \operatorname{ad}^0 \rho_x$ has a filtration with graded pieces of rank 1, and the characters through which $G_{F_v^+}$ acts on these graded pieces are 1, 1, $\widetilde{\chi}_1^2\varepsilon$, $\widetilde{\chi}_2^2\varepsilon$, $\widetilde{\chi}_1\widetilde{\chi}_2\varepsilon$, $\widetilde{\chi}_1\widetilde{\chi}_2^{-1}$. Part 2c follows from part (2a) because a potentially crystalline pure representation cannot contain two Jordan–Hölder factors differing by a cyclotomic twist.

We now turn to part (2b), so that ρ_x is pure and p-distinguished, and (since we have just established part 2c) we can furthermore assume that we are not potentially crystalline. Assume for the sake of contradiction that $H^0(G_{F_v^+}, (\operatorname{ad}^0 \rho_x / \mathrm{Fil}^1 \operatorname{ad}^0 \rho_x)(1)) \neq 0$. We now argue as in the proof of [Ger19, Lem. 3.7(3)]. By our assumption, there is some $A \in \operatorname{ad}^0 \rho_x - \mathrm{Fil}^1 \operatorname{ad}^0 \rho_x$ such that for all $\sigma \in G_{F_v^+}$, we have $\rho_x(\sigma) A \rho_x(\sigma)^{-1} = \varepsilon(\sigma)^{-1} A \bmod \mathrm{Fil}^1 \operatorname{ad}^0 \rho_x$; equivalently,

$$\varepsilon(\sigma)\rho_x(\sigma)A - A\rho_x(\sigma) \in \mathrm{Fil}^1 \operatorname{ad}^0 \rho_x. \tag{6.2.4}$$

We can and do conjugate ρ_x so that it is contained in the usual upper triangular Borel subgroup, so that each $\mathrm{Fil}_{x,i}$ is generated by $e_1, \dots, e_i$. Write

$$\rho_x = \begin{pmatrix} \widetilde{\chi}_{x,1} & * & * & * \\ 0 & \widetilde{\chi}_{x,2} & * & * \\ 0 & 0 & \widetilde{\chi}_{x,3} & * \\ 0 & 0 & 0 & \widetilde{\chi}_{x,4} \end{pmatrix}$$

(so $\widetilde{\chi}_{x,3} = \varepsilon^{-1}\widetilde{\chi}_{x,2}^{-1}$ and $\widetilde{\chi}_{x,4} = \varepsilon^{-1}\widetilde{\chi}_{x,1}^{-1}$). Let $1 \leq s \leq 4$ be minimal with $Ae_s \notin \mathrm{Fil}_{x,s-1}$ (such an s exists by the hypothesis that $A \notin \mathrm{Fil}^1 \operatorname{ad}^0 \rho_x$), and let $r \geq s$ be the unique integer with $Ae_s \in \mathrm{Fil}_{x,r} - \mathrm{Fil}_{x,r-1}$. By (6.2.4) and the assumption on s, we see that for all $\sigma \in G_{F_v^+}$, we have

$$\rho_x(\sigma)(Ae_s) \equiv \varepsilon(\sigma)^{-1} A \rho_x(\sigma) e_s \equiv (\varepsilon^{-1}\widetilde{\chi}_{x,s})(\sigma)(Ae_s) \pmod{\mathrm{Fil}_{x,s-1}}.$$

Since $r \geq s$ this congruence in particular holds modulo $\mathrm{Fil}_{x,r-1}$, whence $\widetilde{\chi}_{x,r} = \varepsilon^{-1}\widetilde{\chi}_{x,s}$, and consequently $r > s$; and furthermore we see that $E \cdot (Ae_s) + \mathrm{Fil}_{x,s-1}$ is $G_{F_v^+}$-stable. More precisely, we see that the 2-dimensional subquotient $(E \cdot (Ae_s) + \mathrm{Fil}_{x,s}) / \mathrm{Fil}_{x,s-1}$ of ρ_x is isomorphic to $\varepsilon^{-1}\widetilde{\chi}_{x,s} \oplus \widetilde{\chi}_{x,s}$.

It is immediate from the definition of purity that no twist of $\varepsilon \oplus 1$ can be a subrepresentation of ρ_x, so we must have $s = 2$ or $s = 3$. Since $\mathrm{Fil}_{\bullet}$ is symplectic, the possibility $(s, r) = (3, 4)$ is also ruled out (because we already saw that $(s, r) = (1, 2)$ is impossible), so we must have $s = 2$ and $r = 3$ or 4. However, since we are pure and not potentially crystalline, we see in either case that we have $\widetilde{\chi}_{x,1} = \widetilde{\chi}_{x,2}$, which contradicts our assumption that ρ_x is p-distinguished. So $H^0(G_{F_v^+}, (\mathrm{ad}^0 \rho_x / \mathrm{Fil}^1 \mathrm{ad}^0 \rho_x)(1)) = 0$ after all, as claimed.

Part (3) follows immediately as when ρ_x is p-distinguished, the map $\mathcal{G}_v \to \operatorname{Spec} R_v^{\triangle}$ is an isomorphism in a neighborhood of x. ☐

In the rest of this section, we assume $p > 2$. In [BCGP21, Prop. 7.3.4], we showed (in a somewhat hands-on manner) that if $\overline{\rho}$ is residually p-distinguished, then $\operatorname{Spec} R_v^{\triangle}[1/p]$ is irreducible. This was used in the proof of our modularity lifting theorem. We expect that the same holds in general, but we don't prove this. Instead we explain a softer way to proceed. We prove a series of Lemmas which will be used in our modularity lifting theorems.

Lemma 6.2.5. *Assume that $(\overline{\rho} \otimes \overline{\varepsilon})|_{G_{F_v^+}}$ is finite flat. Then any ordinary pure weight 2 crystalline lift lies on a unique irreducible component of $\operatorname{Spec} R_v^{\triangle}$, which is moreover independent of the lift. This component has relative dimension 16 over $\mathcal{O}$.*

Proof. We first show that a point ρ of $\operatorname{Spec} R_v^{\triangle}$ corresponding to an ordinary pure weight 2 crystalline lift lies on a unique irreducible component (if ρ were p-distinguished this was already part of Lemma 6.2.2 (3)). Consider the fiber in $\mathcal{G}_v$ over ρ in $\operatorname{Spec} R_v^{\triangle}$, or in other words consider the space of $G_{F_v^+}$-stable symplectic filtrations $\{\mathrm{Fil}_i\}$ on ρ on which $G_{F_v^+}$ acts on $\mathrm{Fil}_i / \mathrm{Fil}_{i-1}$ by $\chi_1, \chi_2, \varepsilon^{-1}\chi_2^{-1}, \varepsilon^{-1}\chi_1^{-1}$ for $i = 1, 2, 3, 4$. By assumption χ_1, χ_2 are unramified and ρ has two dimensional inertia invariants, which hence must be Fil_2. Then either $G_{F_v^+}$ has scalar action on Fil_2, in which case the fiber is the $\mathbf{P}^1$ of possible Fil_1's, or there is a unique line on which $G_{F_v^+}$ acts through $\chi_{1,x}$, and the fiber is a point. In particular either way this fiber is connected.

By Lemma 6.2.2 (2c) each point of this fiber is contained in a unique irreducible component of $\mathcal{G}_v$, and as the fiber is connected, the entire fiber is contained in this component. It follows that the image of this component in $\operatorname{Spec} R_v^{\triangle}$ is the unique irreducible component containing ρ.

Now we prove that all such ρ lie on the same irreducible component. Consider the closed subscheme $\mathcal{G}_v^{\mathrm{flat}} \subseteq \mathcal{G}_v$ whose points for any $\mathcal{O}$-algebra A are pairs $(\mathrm{Fil}_\bullet, \rho)$ where $\rho \otimes \varepsilon$ is finite flat and $\mathrm{Fil}_\bullet$ is a filtration with Fil_2 unramified. We claim that the formal completions of $\mathcal{G}_v^{\mathrm{flat}}$ at k'-points for k'/k finite are formally smooth. By a standard tangent-obstruction calculation as in Lemma 6.2.2 this amounts to the vanishing of $H^2_{\mathrm{flat}}(\mathrm{Fil}^0 \mathrm{ad}^0 \overline{\rho})$ (*cf.* the proof of [Kis09, Prop. 2.4.4]).

We now consider $R_v^{\triangle,\mathrm{flat}}$, the scheme-theoretic image of $\mathcal{G}_v^{\mathrm{flat}}$ in $R_v^{\triangle}$. It is irreducible by the same argument as above, as the fiber over $\overline{\rho}$ is either a point or $\mathbf{P}^1$. As every pure weight 2 crystalline point lies on this irreducible locus, they all lie on the same unique irreducible component of $R_v^{\triangle}$.

Finally the dimension can be computed at any p-distinguished point using Lemma 6.2.2 ☐

We finally prove a lemma which will help with "Ihara avoidance". Let $Q \subset R_v^{\triangle}$ be a minimal prime, corresponding to an irreducible component of $\operatorname{Spec} R_v^{\triangle}$.

Lemma 6.2.6. *Suppose that $\operatorname{Spec} R_v^{\triangle}/Q \to \operatorname{Spec} \Lambda_{\mathrm{GSp}_4,v}$ is surjective. Then there exists a minimal prime of $R_v^{\triangle}/(p)$ which contains Q and no other minimal prime of $R_v^{\triangle}$. Moreover $R_v^{\triangle}/Q$ has relative dimension 16 over $\mathcal{O}$.*

Proof. By the hypothesis we can take an $\mathbf{F}_q((t))$ valued point x of $\operatorname{Spec} R_v^{\triangle}/Q$ so that no ratio of the characters $\chi_{1,x}, \chi_{2,x}, \chi_{2,x}^{-1}\varepsilon^{-1}, \chi_{1,x}^{-1}\varepsilon^{-1}$ is 1 or ε (even on inertia). By the same argument as in Lemma 6.2.2 (2a) and (3) we have that the local ring $R_{v,x}^{\triangle}$ is regular. We now take any irreducible component of $\operatorname{Spec} R_v^{\triangle}/(p)$ containing x. $\square$

We expect that the hypothesis in Lemma 6.2.6 is always satisfied. Rather than attempt a direct local proof of this fact we will check it by global means in the application in the next section.

6.3. **An ordinary modularity lifting theorem for** GSp_4**,** $p > 2$**.** We explain how to prove a modularity lifting theorem for a p-adic Hida family of Hilbert–Siegel modular forms over a totally real field F^+ in which the odd prime p splits completely; this is a slight adaptation of the arguments of our earlier paper [BCGP21] and their improvements by Whitmore [Whi22]. Indeed, under a residually p-distinguished hypothesis, our theorem is a very special case of the theorems proved in those papers. (The entire difficulty in [BCGP21] was about proving modularity lifting theorems in the case $l_0 > 0$, but the $l_0 = 0$ case that we needed here is completely routine.) It would of course be more natural not to include the assumption that p splits completely in F^+, but as we do not know a reference for the relevant Hida families beyond this case, we leave such results for a future paper.

We let F^+ be a totally real field in which $p > 2$ splits completely and let π be an ordinary cuspidal automorphic representation for GSp_4/F^+ of central character $|\cdot|^2$ and weight $((k_v, l_v; 2))_{v|\infty}$. Using our fixed isomorphism $\imath : \mathbf{C} \cong \overline{\mathbf{Q}}_p$ we identify the places $v \mid \infty$ and $v \mid p$ without further comment. Recall from Theorem 1.8.17 that there is a Galois representation $\rho_{\pi,p} : G_{F^+} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ associated to π. We let $\overline{\rho} = \overline{\rho}_{\pi,p}$. We fix R, a finite set of finite, prime to p places of F^+ containing all the prime to p places where π is ramified.

We make the following assumptions:

Hypothesis 6.3.1.

(1) $\overline{\rho}$ is GSp_4-reasonable, in the sense of [Whi22, Defn. 3.19]. In particular, $\overline{\rho}$ is absolutely irreducible.
(2) $\overline{\rho}$ is tidy, in the sense of [BCGP21, Defn. 7.5.11].
(3) For each $v|p$, $\overline{\rho}|_{G_{F_v^+}}$ is ordinary of weight 2, with a fixed p-stabilization $(\overline{\alpha}_v, \overline{\beta}_v)$ which is compatible with a fixed choice of ordinary p-stabilization of π_v.
(4) For each $v \in S_p$, the representation $\rho_{\pi,p}|_{G_{F_v^+}}$ lies on a unique irreducible component of $R_v^{\triangle}$ (where $R_v^{\triangle}$ is defined via the p-stabilization $(\overline{\alpha}_v, \overline{\beta}_v)$, and $\rho_{\pi,p}|_{G_{F_v^+}}$ is a point of $R_v^{\triangle}$ via the chosen p-stabilization on π_v).
(5) For each place $v \in R$ we have:
 - $\overline{\rho}|_{G_{F_v^+}}$ is trivial.
 - $q_v \equiv 1 \pmod p$, and if $p = 3$, then $q_v \equiv 1 \pmod 9$.
 - $\pi_v^{\mathrm{Iw}(v)} \neq 0$.

Note that we do not assume that π is ordinary of weight 2 (it will be in the main application but we also allow π to have regular weight in order to prove Lemma 10.4.1).

Remark 6.3.2. The main result of this section will be a "minimal at p" modularity lifting theorem for ordinary p-adic modular forms. In particular hypothesis (5) will be used for Taylor's Ihara avoidance argument, in order to have no minimality hypotheses away from p. In the application, $F^+/\mathbf{Q}$ will be a solvable extension chosen to ensure this, and the main modularity lifting theorem for $\mathbf{Q}$ will be deduced using base change.

We also note that the theorem is only "minimal at p" due to our failure to completely analyze the deformation rings $R_v^{\triangle}$ in the previous section. However we emphasize that when $\overline{\rho}|_{G_{F_v^+}}$ is residually p-distinguished then $\operatorname{Spec} R_v^{\triangle}[1/p]$ is irreducible and hypothesis (4) is automatic.

Remark 6.3.3. The most important applications of the result of this section are in the case $p = 3$ and $\overline{\rho}(G_{F^+}) \subseteq \mathrm{GSp}_4(\mathbf{F}_3)$. In this case the condition that $\overline{\rho}$ is GSp_4-reasonable can be made completely explicit, see §6.4.

By the assumption that $\overline{\rho}(G_{F^+})$ is tidy, we can choose an unramified place w_0 of F^+ of residue characteristic greater than 5, with $q_{w_0} \not\equiv 1 \pmod{p}$, and such that no two eigenvalues of $\overline{\rho}(\mathrm{Frob}_{w_0})$ have ratio q_{w_0}. We set $S = S_p \cup R \cup \{w_0\}$.

By the above hypotheses, we are in the situation of [BCGP21, Hyp. 7.8.1], except that we have not assumed that we are residually p-distinguished. In the notation of [BCGP21, §7.8] and [Whi22, §7], we take $I = \emptyset$, so that by definition the ring Λ_I is equal to $\Lambda_{\mathrm{GSp}_4,F^+} := \widehat{\otimes}_{v|p}\Lambda_{\mathrm{GSp}_4,v}$. The only change that we make to the setup of [BCGP21, Whi22] is that for $v|p$ we use the deformation problem $\mathcal{D}_v^{\triangle}$, taking $\overline{\chi}_1, \overline{\chi}_2$ to be unramified with $\overline{\chi}_1(\mathrm{Frob}_v) = \overline{\alpha}_p$, $\overline{\chi}_2(\mathrm{Frob}_v) = \overline{\beta}_p$. Note that in the residually p-distinguished case that $\overline{\alpha}_p \neq \overline{\beta}_p$, this agrees with the deformation problem denoted $\mathcal{D}_v^{B,\overline{\alpha}_p}$ in [BCGP21, Whi22].

We can then carry out all of the constructions made in [BCGP21, §7.8] and [Whi22, §7], which for the most part make no use of the hypothesis that $\overline{\alpha}_p \neq \overline{\beta}_p$: the proof of [Whi22, Thm. 7.8] generalizing [BCGP21, Thm. 7.9.4] only uses the values of $\mathcal{D}_v^{\triangle}$ on $\mathcal{O}_{E'}$ for E'/E a finite extension (recall that in the non residually p-distinguished case we don't necessarily understand the values of $\mathcal{D}_v^{\triangle}$ on general complete Noetherian local rings due to its definition as a scheme-theoretic image). We now recall the key points, allowing ourselves to simplify the notation slightly in comparison to that of [BCGP21], by dropping the symbols "I" and "$?$" appearing there.

In particular, we have the global deformation problem

$$\mathcal{S}_1 = (\overline{\rho}, S, \{\Lambda_{\mathrm{GSp}_4,v}\}_{v\in S_p} \cup \{\mathcal{O}\}_{v\in S\smallsetminus S_p}, \{\mathcal{D}_v^{\triangle}\}_{v\in S_p} \cup \{\mathcal{D}_v^1\}_{v\in R} \cup \{\mathcal{D}_{w_0}^{\square}\}),$$

where $\mathcal{D}_v^1$ is defined in [BCGP21, §7.4.5] (it corresponds to unipotently ramified liftings). There is a surjection of $\Lambda_{\mathrm{GSp}_4,F^+}$-algebras $R_{\mathcal{S}_1} \to \mathbf{T}_{\mathcal{S}_1}$, where $\mathbf{T}_{\mathcal{S}_1}$ is the Hida Hecke algebra considered in [BCGP21, §7.9]: it acts faithfully on a finite free $\Lambda_{\mathrm{GSp}_4,F^+}$-module M^1, which is obtained from the ordinary part of the coherent H^0 of Hilbert–Siegel Shimura varieties.

By Hypothesis 6.3.1 (4), for each $v|p$ the representation $\rho_{\pi,p}|_{G_{F_v^+}}$ lies on a unique irreducible component of $\operatorname{Spec} R_v^{\triangle}$, which we denote $\operatorname{Spec} R_v^{\triangle,\pi}$. We let $\mathcal{D}_v^{\triangle,\pi}$ be the deformation problem determined by this irreducible component, and write

$$\mathcal{S}_{1,\pi} = (\overline{\rho}, S, \{\Lambda_{\mathrm{GSp}_4,v}\}_{v\in S_p} \cup \{\mathcal{O}\}_{v\in S\smallsetminus S_p}, \{\mathcal{D}_v^{\triangle,\pi}\}_{v\in S_p} \cup \{\mathcal{D}_v^1\}_{v\in R} \cup \{\mathcal{D}_{w_0}^{\square}\}).$$

We let $\mathbf{T}_{\mathcal{S}_{1,\pi}}$ denote $R_{\mathcal{S}_{1,\pi}} \otimes_{R_{\mathcal{S}_1}} \mathbf{T}_{\mathcal{S}_1}$. These should be thought of as "p-minimal" deformation rings and Hecke algebras, see also Remark 6.3.2.

Theorem 6.3.4. *Assume that we are in the above situation, so that in particular Hypothesis 6.3.1 holds. Then $R_{\mathcal{S}_{1,\pi}}$ is a finite $\Lambda_{\mathrm{GSp}_4,F^+}$-algebra, and the morphism $R_{\mathcal{S}_{1,\pi}} \to \mathbf{T}_{\mathcal{S}_{1,\pi}}$ has nilpotent kernel, i.e. $(R_{\mathcal{S}_{1,\pi}})^{\mathrm{red}} \xrightarrow{\sim} \mathbf{T}_{\mathcal{S}_{1,\pi}}$.*

Proof. We first verify that for $v \mid p$, the irreducible components $R_v^{\triangle,\pi}$ satisfy the hypothesis of Lemma 6.2.6. For this consider a minimal prime $Q_\pi \subset \mathbf{T}_{\mathcal{S}_1}$ contained in the prime ideal corresponding to π and the chosen p-stabilizations, and consider the composition

$$R_v^{\triangle} \to R_{\mathcal{S}_1} \to \mathbf{T}_{\mathcal{S}_1} \to \mathbf{T}_{\mathcal{S}_1}/Q_\pi.$$

By Hypothesis 6.3.1 (4), the composite must factor through the component $R_v^{\triangle,\pi}$. We now claim that the composite

$$\operatorname{Spec} \mathbf{T}_{\mathcal{S}_1}/Q_\pi \to \operatorname{Spec} R_v^{\triangle,\pi} \to \operatorname{Spec} \Lambda_{\mathrm{GSp}_4,v}$$

is surjective, and hence the second map is surjective, which is what we are trying to prove. To see this note that the composite is also

$$\operatorname{Spec} \mathbf{T}_{\mathcal{S}_1}/Q_\pi \to \operatorname{Spec} \Lambda_{\mathrm{GSp}_4,F^+} \to \operatorname{Spec} \Lambda_{\mathrm{GSp}_4,v}.$$

Here the second map is clearly surjective, while the first map is surjective because $\operatorname{Spec} \mathbf{T}_{\mathcal{S}_1}/Q_\pi$ is finite and torsion free as a $\Lambda_{\mathrm{GSp}_4,F^+}$-module, since $\mathbf{T}_{\mathcal{S}_1}$ acts faithfully on a finite free $\Lambda_{\mathrm{GSp}_4,F^+}$-module.

Now we proceed with the proof of the theorem. It is enough to prove that the support of M^1 in $\operatorname{Spec} R_{\mathcal{S}_1}$ contains $\operatorname{Spec} R_{\mathcal{S}_{1,\pi}}$. As in [Whi22, §7.3], we have a power series ring S_∞ over $\Lambda_{\mathrm{GSp}_4,F^+}$, and we write $\mathfrak{a}_\infty$ for the augmentation ideal $\ker(S_\infty \to \Lambda_{\mathrm{GSp}_4,F^+})$. We also have a power series ring R^1_∞ over $R^{\mathrm{loc}}_{\mathcal{S}_1,S}$. Set

$$R_{\infty,\pi} := R^1_\infty \otimes_{R^{\mathrm{loc}}_{\mathcal{S}_1,S}} R^{\mathrm{loc}}_{\mathcal{S}_{1,\pi},S}.$$

The patching argument of [Whi22, Prop. 7.11] provides us in particular with:

- $\Lambda_{\mathrm{GSp}_4,F^+}$-algebra morphisms $S_\infty \to R^1_\infty \to R_{\mathcal{S}_1}$;
- a R^1_∞ module M^1_∞ which is free as an S_∞ module (and hence has depth as an R^1_∞ equal to the dimension of S_∞).
- an isomorphism $M^1_\infty/\mathfrak{a}_\infty \cong M^1$;
- a commutative diagram of S_∞-algebras

$$\begin{array}{ccc} R^1_\infty & \longrightarrow & \mathrm{End}_{S_\infty}(M^1_\infty) \\ \downarrow & & \downarrow {\scriptstyle -\otimes_{S_\infty}\Lambda_{\mathrm{GSp}_4,F^+}} \\ R_{\mathcal{S}_1} & \longrightarrow & \mathrm{End}_{\Lambda_{\mathrm{GSp}_4,F^+}}(M^1) \end{array}$$

It thus suffices to prove that the support of M^1_∞ contains every irreducible component of $R_{\infty,\pi}$. Exactly as in the proof of Theorem 5.7.14, we know that the support of M^1_∞/ϖ contains $\operatorname{Spec} R_{\infty,\pi}/\varpi$. (This is Taylor's "Ihara avoidance" argument, using the data M^χ_∞ etc. from [Whi22, Prop. 7.11] which we have not recalled here.) Every irreducible component of the support of M_∞ has dimension equal to that of S_∞, and as every irreducible component of $\operatorname{Spec} R_{\infty,\pi}$ has dimension equal to that of S_∞, M_∞ will be supported on an irreducible component of $\operatorname{Spec} R_{\infty,\pi}$ as soon as it is supported on some point which is only contained in that irreducible component.

It thus suffices to show that for each irreducible component of $\operatorname{Spec} R_{\infty,\pi}$ there is a point of $\operatorname{Spec} R_{\infty,\pi}/p$ contained in it and in no other component. Since $R_{\infty,\pi}$

is a power series ring over a completed tensor product of local deformation rings, it suffices to check the same property for each factor, i.e. to check that the same holds for $R_v^{\triangle,\pi}$ (for $v \in S_p$), $\operatorname{Spec} R_v^1$ (for $v \in R$), and $\operatorname{Spec} R_{w_0}^{\square}$. The first of these follows from Lemma 6.2.6 (noting that we have verified the hypothesis above), the second from [BCGP21, Prop. 7.4.7], and the last from our choice of w_0, which guarantees that $R_{w_0}^{\square}$ is formally smooth over $\mathcal{O}$. □

6.4. **Subgroups of** $\mathrm{GSp}_4(\mathbf{F}_3)$**.** In this section, we shall identify the precise subgroups of $\mathrm{GSp}_4(\mathbf{F}_3)$ we are allowing for our modularity lifting theorems. We first identify the regular semi-simple elements in $\mathrm{GSp}_4(\mathbf{F}_3)$. We have:

Lemma 6.4.1. *There are three conjugacy classes of elements $g \in \mathrm{GSp}_4(\mathbf{F}_3)$ such that $\nu(g) = 1$ and g is regular semi-simple, namely:*

(1) The unique conjugacy class of elements of order 5*,*
(2) The unique conjugacy class of elements of order 10*,*
(3) The unique conjugacy class of elements of order 8 *which lie in* $\mathrm{Sp}_4(\mathbf{F}_3)$*.*

There are five conjugacy classes of elements $g \in \mathrm{GSp}_4(\mathbf{F}_3)$ such that $\nu(g) = -1$ and g is regular semi-simple, namely:

(1) Both conjugacy classes of elements of order 20*,*
(2) Three of the five conjugacy classes of elements of order 8*, namely those whose images in* $\mathrm{PSp}_4(\mathbf{F}_3)$ *lie in the conjugacy classes* $4D$ *or* $8A$ *but not* $4C$ *in the notation of Lemma 9.1.3.*

We now turn to reasonableness [Whi22, Defn. 3.19]. Although this definition does not *a priori* depend only on the image of $\overline{\rho}$, it shall turn out that that under our running assumptions this will be true.

Lemma 6.4.2. *Suppose that $A/\mathbf{Q}$ is an abelian surface, and that A has good reduction at some $p > 2$. Then the image of $\overline{\rho}_{A,p}|_{G_{\mathbf{Q}(\zeta_p)}}$ coincides with the image of $\overline{\rho}_{A,p}|_{G_{\mathbf{Q}(\zeta_{p^n})}}$ for all $n \geq 1$.*

Proof. Let $K/\mathbf{Q}_p$ be the fixed field of the kernel of $\overline{\rho}_{A,p}|_{G_{\mathbf{Q}_p}}$, which certainly contains $\mathbf{Q}_p(\zeta_p)$. Since $\mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^n})/\mathbf{Q}_p(\zeta_p))$ is cyclic, the lemma holds unless there exists an inclusion $\mathbf{Q}_p(\zeta_{p^2}) \subseteq K$. Assume such an inclusion exists. It follows that the root discriminant δ_K is divisible by the root discriminant of $\mathbf{Q}(\zeta_{p^2})$, which is $p^{(2p^2-3p)/p(p-1)} = p^{(2p-3)/(p-1)}$. The assumption that A has good reduction implies that $A[p]/\mathbf{Z}_p$ is a finite flat group scheme, which by [Fon85, 2.1 Thm. 1] implies that the root discriminant of K satisfies

$$v_p(\delta_K) < 1 + \frac{1}{p-1}.$$

Since $p \geq 3$, this contradicts our lower bound:

$$v_p(\delta_K) \geq v_p(\delta_{\mathbf{Q}(\zeta_{p^2})}) = 1 + \frac{1}{p-1} + \frac{p-3}{p-1}.$$ □

We deduce:

Lemma 6.4.3. *Let $A/\mathbf{Q}$ be an abelian surface with a prime to* 3 *polarization and good reduction at* 3*, and let*

$$\overline{\rho} = \overline{\rho}_{A,3} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$$

denote the corresponding mod 3 *representation. Then the following hypotheses:*

$\Gamma' \subset \mathrm{GSp}_4(\mathbf{F}_3)$ and $\Gamma = \Gamma' \cap \mathrm{Sp}_4(\mathbf{F}_3)$			Conditions				
LMFDB label	small group labels for Γ', Γ		(1)	(2)	(3)	(4)	
3.1620.1	<64,258>	<32,44>					✓
3.1620.2	<64,258>	<32,50>			✗		
3.1620.5	<64,152>	<32,44>					✓
3.1620.10	<64,152>	<32,8>					✓
3.1296.1	<80,29>	<40,3>					✓
3.810.1	<128,137>	<64,137>		✗			
3.810.2	<128,2023>	<64,137>					✓
3.810.5	<128,137>	<64,37>		✗			
3.810.6	<128,142>	<64,37>					✓
3.540.1	<192,1485>	<96,190>					✓
3.540.2	<192,1483>	<96,191>	✗				
3.540.3	<192,1485>	<96,202>			✗		
3.540.5	<192,1018>	<96,202>			✗		
3.540.7	<192,965>	<96,191>	✗				
3.405.1	<256,6671>	<128,937>					✓
3.270.1	<384,18045>	<192,989>					✓
3.216.1	<480,948>	<240,90>					✓
3.216.2	<480,947>	<240,89>	✗				
3.162.1	<640,21454>	<320,1581>		✗			
3.135.1	<768,1086054>	<384,618>					✓
3.135.2	<768,1086054>	<384,18130>					✓
3.45.1	2304	1152					✓
3.36.1	2880	1440	✗				
3.27.1	3840	1920					✓
1.1.1	103680	51840					✓

TABLE 6.4.4. Conjugacy classes of subgroups $\Gamma' \subset \mathrm{GSp}_4(\mathbf{F}_3)$ with $\nu(\Gamma') \neq 1$ and $\Gamma = \Gamma' \cap \mathrm{Sp}_4(\mathbf{F}_3)$ absolutely irreducible. LMFDB labels determine the conjugacy class of Γ', the small group labels [BEO01] determine Γ, Γ' up to abstract isomorphism.

(1) $\overline{\rho}$ *is* GSp_4*-reasonable in the sense of* [Whi22, Defn. 3.19]*,*
(2) $\overline{\rho}$ *is tidy in the sense of* [BCGP21, Defn. 7.5.11]*,*
(3) $\overline{\rho}(G_{\mathbf{Q}(\zeta_3)})$ *contains a regular semi-simple element,*
(4) $\overline{\rho}(G_{\mathbf{Q}}) \smallsetminus \overline{\rho}(G_{\mathbf{Q}(\zeta_3)})$ *contains a regular semi-simple element,*

are satisfied precisely if $\Gamma' = \overline{\rho}(G_{\mathbf{Q}})$ *in Table 6.4.4 has a tick, where otherwise the cross indicates the corresponding obstruction to condition* (1)*,* (2)*,* (3)*, or* (4)*. In particular, these conditions are all satisfied if* $\overline{\rho}(G_{\mathbf{Q}}) = \mathrm{GSp}_4(\mathbf{F}_3)$*.*

Proof. Note that if $\Gamma' = \overline{\rho}(G_{\mathbf{Q}})$, and $\Gamma = \overline{\rho}(G_{\mathbf{Q}(\zeta_3)})$, then $\Gamma = \Gamma' \cap \mathrm{GSp}_4(\mathbf{F}_3)$. Furthermore, reasonableness (which a priori depends on the image of $\overline{\rho}|_{G(\mathbf{Q}(\zeta_{3^n}))}$ for all n) only depends on the image of $\overline{\rho}|_{G(\mathbf{Q}(\zeta_3))}$ by Lemma 6.4.2. As noted in [Whi22, §4.3], the spanning condition of reasonableness is satisfied for all of these subgroups. We have listed the abstract isomorphism types of Γ' and Γ according to the small groups database [BEO01] when they are of small order. The LMFDB subgroup labels [LMF24] (which for proper subgroups are of the form `3.i.n` where `i` = $[\mathrm{GSp}_4(\mathbf{F}_3) : \Gamma']$) determine Γ' up to conjugacy in $\mathrm{GSp}_4(\mathbf{F}_3)$. The groups of larger order are described more explicitly in [BCGP21, Lemma 7.5.21]. The abstract isomorphism type of Γ' is already enough to determine Γ' up to conjugation in $\mathrm{GSp}_4(\mathbf{F}_3)$ (under the assumption that $\Gamma = \Gamma' \cap \mathrm{Sp}_4(\mathbf{F}_3)$ acts absolutely irreducibly) except for two pairs with $|\Gamma'| = 64$, one pair with $|\Gamma'| = 128$, and the pair of groups with $|\Gamma'| = 768$, and in all such cases they can be distinguished by the abstract isomorphism type of $\Gamma = \Gamma' \cap \mathrm{Sp}_4(\mathbf{F}_3)$. □

Remark 6.4.5 ($\mathrm{Sp}_4(\mathbf{F}_3)$ is not GL_4-adequate)**.** There is a natural inclusion

$$\overline{\rho} : \mathrm{Sp}_4(\mathbf{F}_3) \hookrightarrow \mathrm{GL}_4(\mathbf{F}_3). \tag{6.4.6}$$

It turns out that the image G of $\overline{\rho}$ is *not* adequate (in the sense of [Tho17, Defn. 2.20]), which is the reason why, when $p = 3$, we need to use GSp_4 modularity lifting theorems rather than $U(4)$ automorphy lifting theorems (in contrast to our treatment of the case $p = 2$ in Section 5). The failure of adequacy can be seen directly as follows. The group $G \simeq \mathrm{Sp}_4(\mathbf{F}_3)$ has exactly two irreducible representations V, V^σ of dimension 4 over $\mathbf{C}$. The representations are defined over the ring $\mathbf{Z}[\zeta_3]$ and are conjugate under the action of $\mathrm{Gal}(\mathbf{Q}(\zeta_3)/\mathbf{Q})$ [CCN+85]. The mod $\pi = (1 - \zeta_3)$ reduction of this representation is $\overline{\rho}$. The corresponding mod $\pi^2 = (3)$ reduction:

$$\rho : G \to \mathrm{SL}_4(\mathbf{Z}[\zeta]/3) = \mathrm{SL}_4(\mathbf{F}_3[\epsilon]/\epsilon^2)$$

gives a non-trivial deformation of $\overline{\rho}$ and a non-zero class in $H^1(G, \mathfrak{sl}_4) \subset H^1(G, \mathfrak{gl}_4)$. The deformation ρ is not, however, valued in $\mathrm{Sp}_4(\mathbf{F}_3[\epsilon]/\epsilon^2)$; this reflects the fact that V is not self-dual in characteristic zero; we have $V^\vee \simeq V^\sigma$. In particular, identifying $\mathfrak{gl}_4$ with $\mathrm{Hom}(\overline{\rho}, \overline{\rho}) \simeq \overline{\rho} \otimes \overline{\rho} \simeq \mathrm{Sym}^2(\overline{\rho}) \oplus \wedge^2(\overline{\rho})$, this cohomology class lives in $H^1(G, \wedge^2\overline{\rho})$. In contrast, for $\overline{\rho}$ to be adequate as a symplectic representation, it suffices that $H^1(G, \mathfrak{sp}_4) = 0$ where (in this example) $\mathfrak{sp}_4$ may be identified with $\mathrm{Sym}^2(\overline{\rho})$.

This example is similar to the failure of the image of the map

$$\overline{\rho} : \mathrm{SL}_2(\mathbf{F}_5) \hookrightarrow \mathrm{GL}_2(\mathbf{F}_5) \tag{6.4.7}$$

to be adequate. This failure of adequacy for (6.4.7) does not cause an issue in [Wil95]; one exploits the fact that the fixed field of the kernel of the adjoint representation of $\overline{\rho}_{E,5}$ for an elliptic curve $E/\mathbf{Q}$ does not contain ζ_5 (see the proof of [Wil95, Prop 1.11]). On the other hand, for an abelian surface A, the fixed of the kernel of the adjoint representation of $\overline{\rho}_{A,3}$ always contains ζ_3, so there is no way to avoid this cohomological obstruction. Hence this situation is more analogous to the problem of proving modularity lifting for elliptic curves over $\mathbf{Q}(\sqrt{5})$ using 5-adic modularity lifting theorems; see the introduction to [KT17a] for an exposition of this case, and an explanation of why there are classes in the dual Selmer group (so-called "Lie classes") that cannot be killed by Taylor–Wiles primes.

7. Multiplicity one theorems

Our classicality theorems in low weight (in particular Theorem 4.12.4) require as input a multiplicity one theorem in characteristic zero. The main goal of this section is to prove such a theorem. Note that multiplicity one really consists of two separate statements — firstly that the multiplicity is at least one (a p-adic modularity statement), and secondly that the multiplicity is at most one.

One approach to proving multiplicity one (following Diamond [Dia97]) would be to prove an $R = \mathbf{T}$ theorem for the corresponding ordinary Hida family and then, assuming the local deformation ring at p is formally smooth, deduce that the corresponding module M of modular forms is free, and moreover free of rank one by specialization at classical points. Such an argument would work if we made the additional hypotheses that $p > 2$ and that $\overline{\rho}$ was p-distinguished (as is done in [BCGP21]). Since we are not making such assumptions, a further argument is required. The first point to note is that we are working in characteristic zero and hence we only need prove that M is free (and non-zero) after localizing at a height one prime $\mathfrak{q}$ corresponding to our characteristic zero representation. To show that M is non-zero when $p = 2$ we are able to appeal to the $R = \mathbf{T}$ theorem for unitary groups that we proved in Section 5, while for $p > 2$ we use the $R = \mathbf{T}$ theorems for GSp_4 proved in Section 6.3. In either case, Diamond's argument applies (at least in principle) providing that the formal completion of the local deformation ring at $\mathfrak{q}$ is regular, something that is ultimately true under our hypotheses.

More precisely, what is ultimately required for our arguments is the following. First, we need an $R^{\mathrm{red}}[1/p] = \mathbf{T}[1/p]$ theorem in our higher Hida theory (not yet classical) situation. (In truth, when $p > 2$, we get away with a weaker version of such a theorem in a neighbourhood of the prime $\mathfrak{q}$, at the cost of some further local complications already considered in §6.) This proves that $M_{\mathfrak{q}}$ is non-zero. Second, we want to control the relative tangent space of R at the prime $\mathfrak{q}$. This is closely related to establishing the vanishing of the adjoint Bloch-Selmer group (in characteristic zero) of our characteristic zero representation. Theorems of this kind were proved by Newton and Thorne [NT23] in some generality for Galois representations associated to automorphic representations of GL_n of unitary type, and we follow their arguments closely. In fact our situation is for the most part simpler than theirs, since as we are assuming that $\overline{\rho}$ is absolutely irreducible we do not have to use pseudorepresentations. Finally, we need enough local properties of the local deformation ring at p in characteristic zero at $\mathfrak{q}$, and this is what ultimately requires the p-distinguished hypothesis in characteristic zero.

A summary of this section is as follows. In §7.1, we adapt the Galois-theoretic arguments of Newton–Thorne [NT23] to our setting. In §7.2 we set up the basic patching formalism required for our argument, and in §7.4, we show how this can be applied in the setting of GSp_4 over $\mathbf{Q}$, by patching modules coming from higher Hida theory, as recalled in §7.3. Note that there is quite a lot of overlap between the arguments of §7.4 and of similar ones in [BCGP21, §7.8, 7.9] — the difference being that the latter worked under a more restrictive hypothesis on $\overline{\rho}$ but also proved strong integral statements. Finally, in §7.5, we prove the desired multiplicity one theorem. Note that our arguments certainly require understanding the multiplicities of certain automorphic representations in cohomology, which ultimately uses Arthur's classification of discrete automorphic representations of GSp_4.

7.1. **Taylor–Wiles primes.** Let H be a compact subgroup of $\mathrm{GSp}_4(\mathcal{O})$. After replacing E by a finite extension, we can assume that for each element $h \in H$, the characteristic polynomial of h is already split over E. We will assume this without comment from now on. We write $\overline{H}$ for the image of H in $\mathrm{GSp}_4(k)$. Throughout this section, we write ad, ad^0 for the Lie algebras associated to GSp_4 and Sp_4 over $\mathcal{O}$.

Definition 7.1.1. A compact subgroup H of $\mathrm{GSp}_4(\mathcal{O})$ is *integrally enormous* if it acts absolutely irreducibly on E^4, and if for all simple $E[H]$-submodules $W \subset E \otimes \mathrm{ad}^0$, there exists an element $h \in H$ such that

- 1 is an eigenvalue for the action of h on W, and
- the image $\overline{h}$ of h in $\mathrm{GSp}_4(k)$ is regular semi-simple (i.e. has 4 distinct eigenvalues).

Lemma 7.1.2 (Examples of integrally enormous representations, I)**.**

(1) *If $\overline{H}$ contains a regular semi-simple element, and the Zariski closure of H contains Sp_4, then H is integrally enormous.*

(2) *Suppose that the action of $H \subset \mathrm{Sp}_4(\mathcal{O})$ is absolutely irreducible but becomes reducible after restriction to an index 2 subgroup G. Suppose that $(\overline{H} \smallsetminus \overline{G})$ contains a regular semi-simple element. Suppose that the Zariski closure of G is $\mathrm{SL}_2 \times \mathrm{SL}_2$. Then H is integrally enormous.*

Proof. We first consider case (1). Since Sp_4 acts irreducibly on both its standard representation and on ad^0, H acts irreducibly on both E^4 and $E \otimes \mathrm{ad}^0$. (Indeed if H preserves a subspace, we obtain a partial flag which is stabilized by the Zariski closure of H.) Furthermore every element $h \in \mathrm{GSp}_4(\mathcal{O})$ has 1 as an eigenvalue on $E \otimes \mathrm{ad}^0$; so if $h \in H$ is any element whose reduction $\overline{h}$ is regular semi-simple, then h satisfies the conditions of Definition 7.1.1.

We now consider case (2). Our argument is essentially a characteristic zero version of the proof of [BCGP21, Lemma 7.5.17] (though note that the roles of G and H are reversed). If $h \in (H \smallsetminus G) \cap \mathrm{Sp}_4$, then the eigenvalues of h are of the form $(\alpha, \alpha^{-1}, -\alpha, -\alpha^{-1})$, and by assumption we may assume that the image of h in $\mathrm{Sp}_4(k)$ lands in $(\overline{H} \smallsetminus \overline{G})$ and is regular semi-simple — the latter condition being equivalent to the condition that $\overline{\alpha}^4 \neq 1$. Now, following the proof of [BCGP21, Lemma 7.5.17], the representation $E \otimes \mathrm{ad}^0$ decomposes over the algebraic closure of E into two irreducible representations of dimension 6 and 4 on which h has eigenvalues $(1, -1, \alpha^2, -\alpha^2, \alpha^{-2}, -\alpha^{-2})$ and $(-1, 1, \alpha^2, \alpha^{-2})$ respectively, both of which contain 1 as an eigenvalue. ☐

We now let F^+ be a totally real number field in which p splits completely, and we fix a continuous representation $\rho : G_{F^+} \to \mathrm{GSp}_4(\mathcal{O})$ satisfying the following hypothesis.

Hypothesis 7.1.3. Assume that:

(1) ρ is unramified at all but finitely many places.
(2) $\nu \circ \rho = \varepsilon^{-1}$.
(3) $\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(k)$ is absolutely irreducible.
(4) $\rho(G_{F^+(\zeta_{p^\infty})})$ is integrally enormous.
(5) ρ is pure.
(6) for all $v \in S_p$, $\rho|_{G_{F_v^+}}$ is ordinary, semistable of weight 2, pure, and p-distinguished. We choose a p-stabilization (α_p, β_p) of $\rho|_{G_{F_v^+}}$ (and thus of $\overline{\rho}|_{G_{F_v^+}}$, so that by definition $\rho|_{G_{F_v^+}}$ corresponds to a point of $\operatorname{Spec} R_p^{\triangle}$).
(7) If $p = 2$ then $\overline{\rho}(G_{F^+(i)}) = \overline{\rho}(G_{F^+})$.
(8) If $p > 2$ then $\overline{\rho}(G_{F^+}) \smallsetminus \mathrm{Sp}_4(\mathbf{F}_p)$ contains a regular semi-simple element.

We will use the following result to verify condition (4) in Hypothesis 7.1.3.

Corollary 7.1.4 (Examples of integrally enormous representations, II)**.** *Suppose that $\rho : G_{F^+} \to \mathrm{GSp}_4(\mathcal{O})$ satisfies either of the following two sets of conditions:*

(A1) the Zariski closure of $\rho(G_{F^+})$ contains Sp_4, and
(A2) $\overline{\rho}(G_{F^+(\zeta_p)})$ contains a regular semi-simple element.

Or alternatively:

(B1) ρ is induced from a quadratic extension K/F^+ disjoint from the compositum F_∞ of $F^+(\zeta_{p^\infty})$ with the fixed field of the similitude character. The Zariski closure of $\rho(G_{F^+})$ contains $\mathrm{SL}_2 \times \mathrm{SL}_2$, and
(B2) $\overline{\rho}(G_{F_\infty}) \smallsetminus \overline{\rho}(G_{K.F_\infty})$ contains a regular semi-simple element.

Then $\rho(G_{F^+(\zeta_{p^\infty})})$ is integrally enormous.

Proof. We assume the first set of conditions. Since the regular semi-simple elements have order prime to p, and since $F^+(\zeta_{p^\infty})/F^+(\zeta_p)$ is a pro-p extension, $\overline{\rho}(G_{F^+(\zeta_{p^\infty})})$ contains a regular semi-simple element. Since the Zariski closure of $\rho(G_{F^+})$ contains Sp_4 by assumption, so does the Zariski closure of $\rho(G_{F^+(\zeta_{p^\infty})})$ (note that taking the derived subgroup is compatible with taking the Zariski closure, by [Bor91, 2.1(e)]). The result follows from Lemma 7.1.2(1).

Now we assume the second set of conditions. The property of being integrally enormous is inherited from subgroups so it suffices to show that $H := \rho(G_{F_\infty})$ is integrally enormous. By construction $H \subset \mathrm{Sp}_4(\mathcal{O})$ is absolutely irreducible but becomes reducible after restriction to $G = \rho(G_{K.F_\infty})$. Moreover, as in the first case, we deduce that the Zariski closure of H contains $\mathrm{SL}_2 \times \mathrm{SL}_2$. Hence the result follows from Lemma 7.1.2(2). □

We now use the notation for deformation rings that was introduced in Section 6.1. Let S be a finite set of places of F^+ containing S_p and the places where ρ is ramified. We define a global deformation problem $\mathcal{S}$ by

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S_p} \cup \{\mathcal{O}\}_{v\in S\smallsetminus S_p}, \{\mathcal{D}_v^{\triangle}\}_{v\in S_p} \cup \{\mathcal{D}_v^{\square}\}_{v\in S\smallsetminus S_p}), \tag{7.1.5}$$

where $\Lambda_v = \Lambda_{\mathrm{GSp}_4,v}$ and $\mathcal{D}_v^{\triangle}$ is as in Section 6.2. As in Section 6.1, we write $\Lambda_{\mathrm{GSp}_4,F^+} = \widehat{\otimes}_{v\in S_p}\Lambda_v$. Given a nonempty subset $T \subseteq S$, which we assume contains S_p, we fix an extension of ρ to a T-framed lifting $(\rho, \{\gamma_v\}_{v\in T})$ of $\overline{\rho}$ (i.e. fix choices of $\gamma_v \in \widehat{\mathrm{GSp}}_4(\mathcal{O})$ for each $v \in T$). Write $\mathfrak{q}$ (resp. $\mathfrak{q}_{S,T}$) for the kernel of the homomorphism $R_{\mathcal{S}} \to \mathcal{O}$ (resp. $R_{\mathcal{S}}^T \to \mathcal{O}$) corresponding to ρ, and write $\mathfrak{q}_T^{\mathrm{loc}}$ for the kernel of the composite $R_{\mathcal{S},T}^{\mathrm{loc}} \to R_{\mathcal{S}}^T \to \mathcal{O}$.

Write $W = \mathrm{ad}\,\rho$, $W_E = W \otimes_{\mathcal{O}} E$, $W_{E/\mathcal{O}} = W_E/W$, and for each $m \geq 1$ write $W_m = W \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$; and we similarly write $W^0 = \mathrm{ad}^0\,\rho$, $W^0_{E/\mathcal{O}} = W^0 \otimes_{\mathcal{O}} E/\mathcal{O}$, $W^0_m = W^0 \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$. We write

$$H^i(F_S^+/F^+, W)' := \mathrm{im}\Big(H^i(F_S^+/F^+, W^0) \to H^i(F_S^+/F^+, W)\Big),$$

and similarly we write $H^i(F_S^+/F^+, W_E)'$ and so on. For any place v of F^+ we also have cohomology groups $H^i(F_v^+, W)'$ etc. defined in the analogous way. We write $h^i(F_S^+, W_m)'$ for the length of the finite $\mathcal{O}$-module $H^i(F_S^+, W_m)'$, and similarly for $h^i(F_v^+, W_m)'$.

Remark 7.1.6. Write $\mathfrak{z}$ for the centre of ad. Since ad^0 and $\mathrm{ad}/\mathfrak{z}$ are isogenous, the quantities $h^i(F_S^+, W_m)' - h^i(F_S^+, W_m^0)$ are bounded independently of m (and similarly for $h^i(F_v, W_m)' - h^i(F_v, W_m^0)$, and for the various Selmer groups introduced below).

We define Selmer groups $H^1_{\mathcal{S},T}(F^+, W_m)$ by

$$H^1_{\mathcal{S},T}(F^+, W_m) := \ker\left(H^i(F_S^+/F^+, W_m)' \to \prod_{v\in T} H^i(F_v^+, W_m)'\right).$$

We write $h^1_{\mathcal{S},T}(F^+, W_m)$ for the length of $H^1_{\mathcal{S},T}(F^+, W_m)$. In the same way we define

$$H^1_{\mathcal{S},T}(F^+, W) := \ker\left(H^i(F_S^+/F^+, W)' \to \prod_{v\in T} H^i(F_v^+, W)'\right),$$

$$H^1_{\mathcal{S},T}(F^+, W_E) := \ker\left(H^i(F_S^+/F^+, W_E)' \to \prod_{v\in T} H^i(F_v^+, W_E)'\right),$$

$$H^1_{\mathcal{S},T}(F^+, W_{E/\mathcal{O}}) := \ker\left(H^i(F_S^+/F^+, W_{E/\mathcal{O}})' \to \prod_{v\in T} H^i(F_v^+, W_{E/\mathcal{O}})'\right).$$

Note that if we take direct limits via the injections $W_m \cong \varpi W_{m+1} \subset W_{m+1}$ we obtain

$$H^1_{\mathcal{S},T}(F^+, W_{E/\mathcal{O}}) = \varinjlim_m H^1_{\mathcal{S},T}(F^+, W_m),$$

while the Mittag-Leffler property means that if we take inverse limits with respect to the projection maps $W_{m+1} \to W_m$ we have

$$H^1_{\mathcal{S},T}(F^+, W) = \varprojlim_m H^1_{\mathcal{S},T}(F^+, W_m)$$

and thus

$$H^1_{\mathcal{S},T}(F^+, W_E) = \left(\varprojlim_m H^1_{\mathcal{S},T}(F^+, W_m)\right) \otimes_{\mathcal{O}} E.$$

Lemma 7.1.7. *Assume that $S_p \subseteq T$. For each $m \geq 1$, the length of the $\mathcal{O}/\varpi^m$-module*

$$\mathfrak{q}_{S,T}/(\mathfrak{q}_{S,T}^2, \mathfrak{q}_T^{\mathrm{loc}} \cdot R_{\mathcal{S}}^T, \varpi^m) \tag{7.1.8}$$

is

$$h^1_{\mathcal{S},T}(F^+, W_m) + \sum_{v\in T} h^0(G_{F_v^+}, W_m) - h^0(F_S^+, W_m).$$

Proof. The length of (7.1.8) is equal to the length of its $\mathcal{O}/\varpi^m$-dual, which equals

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_{S,T}/(\mathfrak{q}_{S,T}^2, \mathfrak{q}_T^{\mathrm{loc}} \cdot R_{\mathcal{S}}^T), \mathcal{O}/\varpi^m),$$

and elements of this latter group correspond to strict equivalence classes of T-framed liftings $(\rho', \{\gamma_v'\}_{v\in T})$ of type $\mathcal{S}$ of $\overline{\rho}$ to the ring $\mathcal{O} \oplus \epsilon\mathcal{O}/\varpi^m$, which furthermore satisfy:

1. $\rho' \pmod{\epsilon}$ is strictly equivalent to $(\rho, \{\gamma_v\}_{v\in T})$, and
2. for each $v \in T$, $(\gamma_v')^{-1}(\rho'|_{G_{F_v^+}})\gamma_v' = \gamma_v^{-1}(\rho|_{G_{F_v^+}})\gamma_v$.

(Recall that since we have assumed that $S_p \subseteq T$, we are not imposing any conditions at the places $v \in S \smallsetminus T$.) Suppose that $[\phi] \in H^1_{\mathcal{S},T}(F^+, W_m)$, and let ϕ be a lift of $[\phi]$ to $Z^1(F_S^+, W_m)$. By definition, for each $v \in T$ we can write $\phi|_{G_{F_v^+}} = d\psi_v$ for some $\psi_v \in Z^0(F_v^+, W_m) = W_m$. Then $((1+\epsilon\phi)\rho, \{(1-\epsilon\psi_v)\gamma_v\}_{v\in T})$ defines a T-framed lifting of $\overline{\rho}$, and we claim that the strict equivalence classes satisfying the two conditions above are exactly those containing $((1+\epsilon\phi)\rho, \{(1+\epsilon(a_v - \psi_v))\gamma_v\}_{v\in T})$ for some $[\phi] \in H^1_{\mathcal{S},T}(F^+, W_m)$ and $a_v \in H^0(G_{F_v^+}, W_m)$. (Implicit in the claim is that this set of strict equivalence classes does not depend on the choices of liftings ϕ and elements ψ_v.) Given this claim, the lemma follows immediately (because the only strict equivalences between such T-framed liftings are given by replacing all the a_v with $a_v + a$ for some $a \in H^0(F_S^+, W_m)$).

To establish the claim, we firstly check that $((1+\epsilon\phi)\rho, \{(1-\epsilon\psi_v)\gamma_v\}_{v\in T})$ is a lift of the required form. Condition (1) is obvious, while (2) is equivalent to asking that $\phi|_{G_{F_v^+}}(g) = \psi_v - g\psi_v$ for all $v \in T$, $g \in G_{F_v^+}$, which is true by the choice of the ψ_v.

Conversely, if $(\rho', \{\gamma_v'\}_{v\in T})$ satisfies (1), then after replacing ρ' by a strictly equivalent representation, we can and do assume that $(\rho', \{\gamma_v'\}_{v\in T}) \pmod{\epsilon} = (\rho, \{\gamma_v\}_{v\in T})$. We may write $\rho' = (1+\epsilon\phi)\rho$ for some $\phi \in Z^1(F_S^+, W_m)'$, and write $\gamma_v' = (1+\epsilon a_v)\gamma_v$ with $a_v \in W_m$. Then (2) is equivalent to asking that for each $v \in T$ and we have $\phi|_{G_{F_v^+}} = -da_v$. Thus we require that $[\phi] \in H^1_{\mathcal{S},T}(F^+, W_m)$, and given this, the possible a_v differ by elements of $H^0(G_{F_v^+}, W_m)$, as required. □

We define dual Selmer groups as follows. We let W^{0*} be the $\mathcal{O}$-module dual of W^0, so that $W_m^{0*} := W^{0*}/\varpi^m$ is the $\mathcal{O}/\varpi^m$-module dual of W_m (and similarly $W^{0*}_{E/\mathcal{O}} := W^{0*} \otimes_{\mathcal{O}} E/\mathcal{O} \simeq \operatorname{Hom}(W^0, E/\mathcal{O})$). Then we set

$$H^1_{\mathcal{S}^\perp,T}(F^+, W_m^{0*}(1)) := \ker\left(H^i(F_S^+/F^+, W_m^{0*}(1)) \to \prod_{v\in S\smallsetminus T} H^i(F_v^+, W_m^{0*}(1))\right),$$

and similarly

$$H^1_{\mathcal{S}^\perp,T}(F^+, W^{0*}(1)) := \ker\left(H^i(F_S^+/F^+, W^{0*}(1)) \to \prod_{v\in S\smallsetminus T} H^i(F_v^+, W^{0*}(1))\right),$$

$$H^1_{\mathcal{S}^\perp,T}(F^+, W_E^{0*}(1)) := \ker\left(H^i(F_S^+/F^+, W_E^{0*}(1)) \to \prod_{v\in S\smallsetminus T} H^i(F_v^+, W_E^{0*}(1))\right),$$

$$H^1_{\mathcal{S}^\perp,T}(F^+, W_{E/\mathcal{O}}^{0*}(1)) := \ker\left(H^i(F_S^+/F^+, W_{E/\mathcal{O}}^{0*}(1)) \to \prod_{v\in S\smallsetminus T} H^i(F_v^+, W_{E/\mathcal{O}}^{0*}(1))\right).$$

Just as for the Selmer groups, these satisfy the obvious compatibilities with direct and inverse limits.

We now show the existence of appropriate sets of Taylor–Wiles primes. We closely follow the proofs of [NT23, Cor. 2.21, Lem. 2.26, Cor. 2.27], beginning with the following lemma.

Lemma 7.1.9. *Suppose that $\rho : G_{F^+} \to \mathrm{GSp}_4(\mathcal{O})$ satisfies Hypothesis 7.1.3. Let $q := \operatorname{corank}_{\mathcal{O}} H^1_{\mathcal{S}^\perp,S}(F^+, W^{0*}_{E/\mathcal{O}}(1))$.*

Then there exist $\sigma_1, \ldots, \sigma_q \in G_{F^+(\zeta_{p^\infty})}$ such that

(a) for each $i = 1, \ldots, q$, $\overline{\rho}(\sigma_i)$ is regular semi-simple, and

(b) the kernel of the map

$$H^1_{\mathcal{S}^\perp,S}(F^+,W^{0*}_{E/\mathcal{O}}(1)) \to \bigoplus_{i=1}^{q} H^1(\widehat{\mathbf{Z}},W^{0*}_{E/\mathcal{O}}(1)) \cong \bigoplus_{i=1}^{q} W^{0*}_{E/\mathcal{O}}(1)/(\sigma_i-1)W^{0*}_{E/\mathcal{O}}(1)$$

(the product of the restriction maps $\mathrm{Res}^{G_{F^+,S}}_{\langle\sigma_i\rangle}$ *associated to the homomorphisms* $\widehat{\mathbf{Z}} \to G_{F^+,S}$*, the* i^{th} *such homomorphism sending* 1 *to* σ_i*) is a finite length* $\mathcal{O}$*-module.*

Proof. Since $H^1_{\mathcal{S}^\perp,S}(F^+,W^{0*}_{E/\mathcal{O}}(1))$ is cofinitely generated, it suffices (by an obvious inductive construction) to show that for any non-zero homomorphism $f : E/\mathcal{O} \to H^1_{\mathcal{S}^\perp,S}(F^+,W^{0*}_{E/\mathcal{O}}(1))$, we can find $\sigma \in G_{F^+(\zeta_{p^\infty})}$ such that $\overline{\rho}(\sigma)$ is regular semi-simple, and the restriction $\mathrm{Res}^{G_{F^+,S}}_{\langle\sigma\rangle} \circ f : E/\mathcal{O} \to W^{0*}_{E/\mathcal{O}}(1)/(\sigma-1)W^{0*}_{E/\mathcal{O}}(1)$ is non-zero.

Let L'_∞/F^+ be the extension cut out by $W^{0*}_E(1)$, and let $L_\infty = L'_\infty(\zeta_{p^\infty})$. We claim that $H^1(L_\infty/F^+,W^{0*}_E(1)) = 0$. To see this, note that the extension cut out by $W^*_E(1) = W^{0*}_E(1) \oplus E(1)$ is L_∞, and since $W^*_E(1)$ is pure, it follows from [Kis04, Lemma 6.2] that $H^1(L_\infty/F^+,W^*_E(1)) = 0$, and thus $H^1(L_\infty/F^+,W^{0*}_E(1)) = 0$ as claimed. Thus $H^1(L_\infty/F^+,W^{0*}_{E/\mathcal{O}}(1))$ is killed by a power of p (since it injects into the finitely generated $\mathcal{O}$-module $H^2(L_\infty/F^+,W^{0*}_{\mathcal{O}}(1))$), and hence the homomorphism

$$\mathrm{Res}^{G_{F^+,S}}_{G_{L_\infty,S_{L_\infty}}} \circ f : E/\mathcal{O} \to H^1(F_S/L_\infty,W^{0*}_{E/\mathcal{O}}(1))^{G_{F^+,S}} \cong \mathrm{Hom}_{G_{F^+,S}}(G_{L_\infty,S_{L_\infty}},W^{0*}_{E/\mathcal{O}}(1))$$

is still non-zero (here S_{L_∞} denotes the set of places of L_∞ lying over places in S).

Let $M \subset W^{0*}_{E/\mathcal{O}}(1)$ be the $\mathcal{O}$-submodule generated by the elements $f(x)(\sigma)$, $x \in E/\mathcal{O}$, $\sigma \in G_{L_\infty}$; it is a non-zero divisible $\mathcal{O}[G_{F^+(\zeta_{p^\infty})}]$-submodule of $W^{0*}_{E/\mathcal{O}}(1)$. By the assumption that $\rho(G_{F^+(\zeta_{p^\infty})})$ is integrally enormous, we deduce that there exists $\sigma \in G_{F^+(\zeta_{p^\infty})}$ such that $\overline{\rho}(\sigma)$ is regular semi-simple and $M \not\subset (\sigma-1)W^{0*}_{E/\mathcal{O}}(1)$. Consequently, there exists $m \geq 0$ and $\tau \in G_{L_\infty}$ such that $f(1/\varpi^m)(\tau) \notin (\sigma - 1)W^{0*}_{E/\mathcal{O}}(1)$.

If $f(1/\varpi^m)(\sigma) \notin (\sigma-1)W^{0*}_{E/\mathcal{O}}(1)$, then $\mathrm{Res}^{G_{F^+,S}}_{\langle\sigma\rangle} \circ f$ is non-zero, as required. On the other hand if $f(1/\varpi^m)(\sigma) \in (\sigma-1)W^{0*}_{E/\mathcal{O}}(1)$ then $\mathrm{Res}^{G_{F^+,S}}_{\langle\tau\sigma\rangle} \circ f$ is non-zero (because $f(1/\varpi^m)(\tau\sigma) = f(1/\varpi^m)(\tau) + f(1/\varpi^m)(\sigma)$ and $(\tau\sigma-1)W^{0*}_{E/\mathcal{O}}(1) = (\sigma-1)W^{0*}_{E/\mathcal{O}}(1)$). By construction, $\tau \in G_{L_\infty} \subset G_{F^+(\zeta_{p^\infty})}$ so $\tau\sigma \in G_{F^+(\zeta_{p^\infty})}$. Finally, since τ lies in G_{L_∞}, $\overline{\rho}(\tau)$ is scalar and hence $\overline{\rho}(\tau\sigma)$ is regular semi-simple, so we are done. □

Definition 7.1.10. A *set of Taylor–Wiles primes of level* N is a finite set of finite places Q of F^+, disjoint from S, such that for each $v \in Q$, we have $q_v \equiv 1 \bmod p^N$, and $\overline{\rho}(\mathrm{Frob}_v)$ is regular semi-simple.

Given a set of Taylor–Wiles primes Q, we define the augmented global deformation problem

$$\mathcal{S}_Q = (\overline{\rho}, S \cup Q, \{\Lambda_v\}_{v \in S_p} \cup \{\mathcal{O}\}_{v \in (S \cup Q) \smallsetminus S_p}, \{\mathcal{D}_v^{\triangle}\}_{v \in S_p} \cup \{\mathcal{D}_v^{\square}\}_{v \in (S \cup Q) \smallsetminus S_p}).$$

Lemma 7.1.11. *Suppose that $\rho : G_{F^+} \to \mathrm{GSp}_4(\mathcal{O})$ satisfies Hypothesis 7.1.3. Let $q := \mathrm{corank}_{\mathcal{O}} H^1_{\mathcal{S}^\perp, S}(F^+, W^{0*}_{E/\mathcal{O}}(1))$.*

Then there exist constants $d_1, d_2 \geq 0$ such that for each $N \geq 1$ we can find a set of Taylor–Wiles primes Q_N of level N such that

(1) $\# Q_N = q$.
(2) $h^1_{\mathcal{S}^\perp_{Q_N}, S}(F^+, W^{0*}_N(1)) \leq d_1$.
(3) for each $m \leq N$, the length of

$$\mathfrak{q}_{S \cup Q_N, S} / (\mathfrak{q}^2_{S \cup Q_N, S}, \mathfrak{q}^{\mathrm{loc}}_S \cdot R^S_{\mathcal{S}_{Q_N}}, \varpi^m)$$

is at most $d_2 + m\big(2q - 4[F^+ : \mathbf{Q}] + \# S - 1\big)$.

Proof. Let Q be any set of Taylor–Wiles primes. We have by definition the exact sequence

$$0 \to H^1_{\mathcal{S}^\perp_Q, S}(F^+, W^{0*}_N(1)) \to H^1_{\mathcal{S}^\perp, S}(F^+, W^{0*}_N(1)) \to \oplus_{v \in Q} H^1(k(v), W^{0*}_N(1)).$$

More generally, for a set $\mathcal{Q}$ of elements $\sigma_i \in G_{F^+(\zeta_{p^N})}$ with $\overline{\rho}(\sigma_i)$ semi-simple, let us denote by $H^1_{\mathcal{S}^\perp_{\mathcal{Q}}, S}(F^+, W^{0*}_N(1))$ the kernel of the map

$$H^1_{\mathcal{S}^\perp, S}(F^+, W^{0*}_N(1)) \to \oplus_{\mathcal{Q}} H^1(\langle \sigma_i \rangle, W^{0*}_N(1)).$$

Accordingly, if $Q = \{v_i\}$ is a set of Taylor–Wiles primes, and $\mathcal{Q} = \{\sigma_i\}$ with $\sigma_i = \mathrm{Frob}_{v_i}$, then

$$H^1_{\mathcal{S}^\perp_{\mathcal{Q}}, S}(F^+, W^{0*}_N(1)) = H^1_{\mathcal{S}^\perp_Q, S}(F^+, W^{0*}_N(1)).$$

Moreover, by the Chebotarev density theorem, for any such set $\mathcal{Q} = \{\sigma_i\}$, there exists a set of places v_i with $q_{v_i} \equiv 1 \bmod p^N$ such that the action of Frob_{v_i} on the finite module $W^{0*}_N(1)$ coincides with the action of σ_i, and such that for any $m \leq N$ and any $f \in H^1_{\mathcal{S}^\perp, S}(F^+, W^{0*}_m(1))$, we have $f(\sigma_i) = f(\mathrm{Frob}_{v_i})$. (Here we use that the $\mathcal{O}$-modules $H^1_{\mathcal{S}^\perp, S}(F^+, W^{0*}_m(1))$ are finitely generated and the modules $W^{0*}_m(1)$ are finite.) This implies the equality

$$H^1_{\mathcal{S}^\perp_{\mathcal{Q}}, S}(F^+, W^{0*}_m(1)) = H^1_{\mathcal{S}^\perp_Q, S}(F^+, W^{0*}_m(1)). \tag{7.1.12}$$

for any $m \leq N$. Comparing to Lemma 7.1.9, we see that we can and do choose a set of elements $\mathcal{Q} = \{\sigma_i\}$ so that the groups $H^1_{\mathcal{S}^\perp_{\mathcal{Q}}, S}(F^+, W^{0*}_{E/\mathcal{O}}(1))$ are finite length $\mathcal{O}$-modules of uniformly bounded length (indeed they are all isomorphic, but we do not need this), and let Q_N denote a corresponding set of Taylor–Wiles primes of level N so that equality 7.1.12 holds. We will now show that these sets in fact satisfy properties (2) and (3).

Considering the long exact sequences in Galois cohomology associated to the short exact sequence

$$0 \to W^{0*}_N(1) \to W^{0*}_{E/\mathcal{O}}(1) \stackrel{\varpi^N}{\to} W^{0*}_{E/\mathcal{O}}(1) \to 0$$

we see that we have a morphism

$$H^1_{\mathcal{S}^\perp_{Q_N}, S}(F^+, W^{0*}_N(1)) = H^1_{\mathcal{S}^\perp_{\mathcal{Q}}, S}(F^+, W^{0*}_N(1)) \to H^1_{\mathcal{S}^\perp_{\mathcal{Q}}, S}(F^+, W^{0*}_{E/\mathcal{O}}(1))[\varpi^N]$$

whose kernel has order bounded by that of $H^0(F^+, W^{0*}_{E/\mathcal{O}}(1))/\varpi^N$. Since as explained above $h^1_{\mathcal{S}^{\perp}_{\mathcal{Q}},S}(F^+, W^{0*}_{E/\mathcal{O}}(1))$ is uniformly bounded, in order to prove (2), it remains to show that $H^0(F^+, W^{0*}_{E/\mathcal{O}}(1))$ has finite length. It in turn suffices to check that $H^0(F^+, W^{0*}_E(1)) = 0$ (because then $H^0(F^+, W^{0*}_{E/\mathcal{O}}(1))$ is cofinitely generated and injects into the finitely generated $\mathcal{O}$-module $H^1(F^+, W^{0*}(1))$). But $H^0(F^+, W^{0*}_E(1)) = H^0(F^+, W^0_E(1))$, and we even have $H^0(F^+, W_E(1)) = 0$, because ρ is absolutely irreducible and not isomorphic to $\rho(1)$ (e.g. because it has different Hodge–Tate weights).

We now turn to (3). Write l for the length of a finite $\mathcal{O}$-module. By the Greenberg–Wiles formula together with Remark 7.1.6, the quantity

$$h^1_{\mathcal{S}_{Q_N},S}(F^+, W_m) - h^1_{\mathcal{S}^{\perp}_{Q_N},S}(F^+, W^{0*}_m(1)) + h^0(F^+, W^{0*}_m(1)) - h^0(F^+, W^0_m)$$
$$+ \sum_{v\in S} h^0(F^+_v, W^0_m) + \sum_{v\in Q_N} \big(h^0(F^+_v, W^0_m) - h^1(F^+_v, W^0_m)\big) + \sum_{v|\infty} l((1+c_v)W^0_m)$$

is uniformly bounded independently of N and $m \le N$ (and of our choice of Q_N).

Comparing to Lemma 7.1.7, we see that in order to establish (3), it suffices to show that the quantity

$$- h^1_{\mathcal{S}^{\perp}_{Q_N},S}(F^+, W^{0*}_m(1)) + h^0(F^+, W^{0*}_m(1))$$
$$+ \Big(h^0(F^+, W_m) - h^0(F^+, W^0_m) - m\Big) + \sum_{v\in S}\Big(h^0(F^+_v, W^0_m) - h^0(F^+_v, W_m) + m\Big)$$
$$+ \sum_{v\in Q_N}\Big(h^0(F^+_v, W^0_m) - h^1(F^+_v, W^0_m) + 2m\Big) + \sum_{v|\infty}\Big(l\big((1+c_v)W^0_m\big) - 4m\Big)$$

is uniformly bounded independently of N and of $m \le N$. We will do this by showing that each of the terms is uniformly bounded.

We begin with $h^1_{\mathcal{S}^{\perp}_{Q_N},S}(F^+, W^{0*}_m(1))$. Considering the morphisms

$$W^{0*}_m(1) \to W^{0*}_N(1) \to W^{0*}_{E/\mathcal{O}}(1)$$

we obtain a morphism

$$H^1_{\mathcal{S}^{\perp}_{Q_N},S}(F^+, W^{0*}_m(1)) = H^1_{\mathcal{S}^{\perp}_{\mathcal{Q}},S}(F^+, W^{0*}_m(1)) \to H^1_{\mathcal{S}^{\perp}_{\mathcal{Q}},S}(F^+, W^{0*}_N(1))$$

whose kernel is contained in the kernel of the morphism

$$H^1(F^+_S/F^+, W^{0*}_m(1)) \to H^1(F^+_S/F^+, W^{0*}_{E/\mathcal{O}}(1)).$$

This latter kernel is isomorphic to a subquotient of $H^0(F^+, W^{0*}_{E/\mathcal{O}}(1))$, which we showed above is a finite $\mathcal{O}$-module. The uniform boundedness of $h^1_{\mathcal{S}^{\perp}_{Q_N},S}(F^+, W^{0*}_m(1))$ for $m \le N$ then follows from that of $h^1_{\mathcal{S}^{\perp}_{Q_N},S}(F^+, W^{0*}_N(1))$, i.e. from (2).

To show that the term $h^0(F^+, W^{0*}_m(1))$ is uniformly bounded, we recall from above that $H^0(F^+, W^{0*}_E(1)) = 0$. It follows that we have an injective map:

$$H^0(F^+, W^{0*}_m(1)) \hookrightarrow H^1(F^+, W^{0*})[\varpi^m],$$

and we are done because $H^1(F^+, W^{0*})[\varpi^m]$ is uniformly bounded (since $H^1(F^+, W^{0*})$ is a finitely generated $\mathcal{O}$-module). The term

$$h^0(F^+, W_m) - h^0(F^+, W^0_m) - m$$

and the terms for $v \in S$ can be handled similarly, using that $H^0(F^+, W_E) = H^0(F^+, W_E^0) \oplus E$ (respectively $H^0(F_v^+, W_E) = H^0(F_v^+, W_E^0) \oplus E$).

If $v \in Q_N$, then since v splits in $F^+(\zeta_{p^N})$ and $m \le N$, the local Euler characteristic formula and Tate local duality give

$$h^1(F_v^+, W_m^0) - h^0(F_v^+, W_m^0) = h^2(F_v^+, W_m^0) = h^0(F_v^+, W_m^{0*}(1)) = h^0(F_v^+, W_m^{0*}),$$

and since $\overline{\rho}(\mathrm{Frob}_v)$ is regular semi-simple, we have $h^0(F_v^+, W_m^{0*}) = 2m$, so these terms vanish identically.

Finally the claim for places $v|\infty$ follows easily from $\dim_E H^0(F_v^+, W_E^0) = 4$ (which in turn follows from the assumption that $\nu \circ \rho = \varepsilon^{-1}$). ∎

Proposition 7.1.13. *Suppose that $\rho : G_{F^+} \to \mathrm{GSp}_4(\mathcal{O})$ satisfies Hypothesis 7.1.3. Let $q := \mathrm{corank}_{\mathcal{O}} H^1_{\mathcal{S}^\perp, S}(F^+, W^{0*}_{E/\mathcal{O}}(1))$.*

Then there exists an integer $d \ge 0$ such that for each $N \ge 1$ we can find a set of Taylor–Wiles primes Q_N of level N, together with a morphism in $\mathrm{CNL}_{\Lambda_{\mathrm{GSp}_4, F^+}}$

$$R^{\mathrm{loc}}_{\mathcal{S},S}[\![X_1, \ldots, X_g]\!] \to R^S_{\mathcal{S}_{Q_N}}, \tag{7.1.14}$$

such that:

(1) *$\#Q_N = q$.*
(2) *$g = 2q - 4[F^+ : \mathbf{Q}] + \#S - 1$.*
(3) *Let $\mathfrak{q}^{\mathrm{loc}}_{S,g}$ be the kernel of the composite morphism*

$$R^{\mathrm{loc}}_{\mathcal{S},S}[\![X_1, \ldots, X_g]\!] \to R^S_{\mathcal{S}_{Q_N}} \to \mathcal{O}$$

determined by ρ and (7.1.14). Then the finite $\mathcal{O}$-module

$$\mathfrak{q}_{S \cup Q_N, S}/(\mathfrak{q}^2_{S \cup Q_N, S}, \mathfrak{q}^{\mathrm{loc}}_{S,g} \cdot R^S_{\mathcal{S}_{Q_N}})$$

is killed by ϖ^d.

Proof. Choose the Taylor–Wiles primes Q_N as in Lemma 7.1.11, and let d be the constant d_2 in the statement of that lemma. We can without loss of generality assume that $N > d$ (because a set of Taylor–Wiles primes of level $N+1$ is also a set of Taylor–Wiles primes of level N). If $M = \mathfrak{q}_{S \cup Q_N, S}/(\mathfrak{q}^2_{S \cup Q_N, S}, \mathfrak{q}^{\mathrm{loc}}_S \cdot R^S_{\mathcal{S}_{Q_N}})$, the length of M/ϖ^m is at most $gm + d$ for all $m \le N$ by Lemma 7.1.11 (3). Thus, by [NT23, Lem. 2.20], we can find a map

$$\mathcal{O}^g \to M/\varpi^N = \mathfrak{q}_{S \cup Q_N, S}/(\mathfrak{q}^2_{S \cup Q_N, S}, \mathfrak{q}^{\mathrm{loc}}_S \cdot R^S_{\mathcal{S}_{Q_N}}, \varpi^N)$$

whose cokernel has length at most d, and is in particular killed by ϖ^d. By the topological version of Nakayama's lemma, we can find a presentation (7.1.14) such that

$$\mathfrak{q}_{S \cup Q_N, S}/(\mathfrak{q}^2_{S \cup Q_N, S}, \mathfrak{q}^{\mathrm{loc}}_{S,g} \cdot R^S_{\mathcal{S}_{Q_N}}, \varpi^N)$$

is killed by ϖ^d. Since we are assuming that $N > d$, this implies that

$$\mathfrak{q}_{S \cup Q_N, S}/(\mathfrak{q}^2_{S \cup Q_N, S}, \mathfrak{q}^{\mathrm{loc}}_{S,g} \cdot R^S_{\mathcal{S}_{Q_N}})$$

is killed by ϖ^d, as required. ∎

7.2. **An abstract patching argument.** Set $\Delta_\infty := \mathbf{Z}_p^{2q}$. Suppose that we have the following data.

Hypothesis 7.2.1.

(1) An $R_{\mathcal{S}}$-module M, which is finite free as a $\Lambda_{\mathrm{GSp}_4,F^+}$-module.
(2) For each integer $N \geq 1$, a finite quotient Δ_N of Δ_∞, such that $\Delta_\infty \twoheadrightarrow \Delta_\infty/p^N$ factors through Δ_N.
(3) For each $N \geq 1$, a homomorphism $\Lambda_{\mathrm{GSp}_4,F^+}[\Delta_N] \to R_{\mathcal{S}_{Q_N}}$, and a finite $R_{\mathcal{S}_{Q_N}}$-module M_N which is finite free as a $\Lambda_{\mathrm{GSp}_4,F^+}[\Delta_N]$-module.
(4) Isomorphisms of $\Lambda_{\mathrm{GSp}_4,F^+}$-modules
$$M_N \otimes_{\Lambda_{\mathrm{GSp}_4,F^+}[\Delta_N]} \Lambda_{\mathrm{GSp}_4,F^+} \xrightarrow{\sim} M,$$
(where the homomorphism $\Lambda_{\mathrm{GSp}_4,F^+}[\Delta_N] \to \Lambda_{\mathrm{GSp}_4,F^+}$ is the augmentation map), compatible with the actions of $R_{\mathcal{S}_{Q_N}}$ and $R_{\mathcal{S}}$ and the natural homomorphism $R_{\mathcal{S}_{Q_N}} \to R_{\mathcal{S}}$.

As in Definition 6.1.4, we let $\mathcal{T} = \Lambda_{\mathrm{GSp}_4,F^+}[\![x_1, \dots, x_{11\#S-1}]\!]$ be the coordinate ring of $(\prod_{v\in T} \widehat{\mathrm{GSp}_4})/\widehat{\mathbf{G}_m}$ over $\Lambda_{\mathrm{GSp}_4,F^+}$. Let
$$S_\infty := \Lambda_{\mathrm{GSp}_4,F^+}[\![\Delta_\infty]\!]\widehat{\otimes}_{\Lambda_{\mathrm{GSp}_4,F^+}}\mathcal{T},$$
and let $\mathbf{a}_\infty$ be the kernel of the map $S_\infty \to \Lambda_{\mathrm{GSp}_4,F^+}$ given by sending each element of Δ_∞ to 1 and each x_i to 0. Write $S_N := \Lambda_{\mathrm{GSp}_4,F^+}[\Delta_N]\widehat{\otimes}_{\Lambda_{\mathrm{GSp}_4,F^+}}\mathcal{T} = \mathcal{T}[\Delta_N]$, a quotient of S_∞. Let
$$R_\infty := R^{\mathrm{loc}}_{\mathcal{S},S}[\![X_1, \dots, X_g]\!].$$
For each $N \geq 1$ we write $R_N := R^S_{\mathcal{S}_{Q_N}}$ and $\mathfrak{q}_N = \mathfrak{q}_{S\cup Q_N,S}$, so that R_N is an R_∞-algebra via (7.1.14) and an S_N-algebra via (6.1.5) and Hypothesis 7.2.1(3). We set
$$M_N^\square := M_N \otimes_{R_{\mathcal{S}_{Q_N}}} R_N = M_N \otimes_{\Lambda_{\mathrm{GSp}_4,F^+}} \mathcal{T},$$
where the equality follows from (6.1.5). Write $R = R_{\mathcal{S}}$, which is naturally an R_N-algebra for each N.

Then Hypothesis 7.2.1 implies that:

- The R-module M is finite free as a $\Lambda_{\mathrm{GSp}_4,F^+}$-module.
- For each $N \geq 1$, we have a homomorphism $S_N \to R_N$, and $M_N^\square$ is an R_N-module which is finite free as an S_N-module.
- We have isomorphisms of $\Lambda_{\mathrm{GSp}_4,F^+}$-modules
$$M_N^\square/\mathbf{a}_\infty \xrightarrow{\sim} M,$$
compatible with the actions of R_N and R and the homomorphism $R_N \to R$.

Fix a non-principal ultrafilter $\mathcal{F}$ on $\mathbf{N}$, and write $\mathbf{\Lambda} := \prod_{N\in\mathbf{N}} \Lambda_{\mathrm{GSp}_4,F^+}$, and write $\mathbf{\Lambda}_x$ for the localization of $\mathbf{\Lambda}$ at the prime ideal
$$x := \{(x_N)_{N\in\mathbf{N}} | \exists I \in \mathcal{F} \text{ s.t. } \forall N \in I, x_N \in \mathfrak{m}_{\Lambda_{\mathrm{GSp}_4,F^+}}\}.$$
Then we set
$$M_\infty := \varprojlim_n \left(\mathbf{\Lambda}_x \otimes_{\mathbf{\Lambda}} \prod_{N\gg 0} M_N^\square/\mathbf{a}_\infty^n\right),$$

where the product is over the cofinite (by Hypothesis 7.2.1(2)) set of N for which $\mathbf{a}_\infty^n \supseteq \ker(S_\infty \to S_N)$, and we set

$$R^{\mathrm{patch}} := \varprojlim_n \left(\mathbf{\Lambda}_x \otimes_{\mathbf{\Lambda}} \prod_{N \geq 1} R_N / \mathfrak{m}_{R_N}^n \right).$$

By for example [GN20, Prop. 3.4.16], we have in particular produced the following structures.

- $\Lambda_{\mathrm{GSp}_4, F^+}$-algebra homomorphisms $S_\infty \to R_\infty \to R^{\mathrm{patch}} \twoheadrightarrow R$.
- A finite free S_∞-module M_∞, together with an isomorphism of $\Lambda_{\mathrm{GSp}_4, F^+}$-modules
$$M_\infty \otimes_{S_\infty} \Lambda_{\mathrm{GSp}_4, F^+} \xrightarrow{\sim} M.$$
- A commutative diagram of S_∞-algebras

$$\begin{array}{ccc} R_\infty & \longrightarrow & \mathrm{End}_{S_\infty}(M_\infty) \\ \downarrow & & \downarrow {\scriptstyle - \otimes_{S_\infty} \Lambda_{\mathrm{GSp}_4, F^+}} \\ R & \longrightarrow & \mathrm{End}_{\Lambda_{\mathrm{GSp}_4, F^+}}(M) \end{array}$$

Write $\mathfrak{q}^{\mathrm{patch}} \subset R^{\mathrm{patch}}$ and $q_\infty \subset R_\infty$ for the inverse images of $\mathfrak{q} \subset R$ (i.e. the kernels of the composite morphisms $R_\infty \to R^{\mathrm{patch}} \twoheadrightarrow R \to \mathcal{O}$ corresponding to ρ).

Lemma 7.2.2. *The $\mathcal{O}$-module $\mathfrak{q}^{\mathrm{patch}}/((\mathfrak{q}^{\mathrm{patch}})^2, \mathfrak{q}_\infty)$ is killed by ϖ^d, where d is as in Proposition 7.1.13.*

Proof. This is proved in exactly the same way as [NT23, Prop. 4.18]. By Proposition 7.1.13, for each $N \geq 1$ the $\mathcal{O}$-module $\mathfrak{q}_N/((\mathfrak{q}_N)^2, \mathfrak{q}_\infty)$ is killed by ϖ^d, so the cokernel of

$$\prod_{N \geq 1} \mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2 \to \prod_{N \geq 1} \mathfrak{q}_N/(\mathfrak{q}_N)^2$$

is killed by ϖ^d (here the map $\mathfrak{q}_\infty \to \mathfrak{q}_N$ is the one induced by the morphism $R_\infty \to R_N$). By an identical argument to the proof of [NT23, Lem. 4.16, 4.17], the image of $\prod_{N \geq 1} \mathfrak{q}_N$ (resp. $\prod_{N \geq 1} \mathfrak{q}_N^2$) in R^{patch} is $\mathfrak{q}^{\mathrm{patch}}$ (resp. $(\mathfrak{q}^{\mathrm{patch}})^2$).

It remains to show that the image of $\prod_{N \geq 1} \mathfrak{q}_\infty/(\mathfrak{q}_\infty^2)$ in $\mathfrak{q}^{\mathrm{patch}}/(\mathfrak{q}^{\mathrm{patch}})^2$ agrees with the image of $\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2$. This follows from [NT23, Lem. 4.19], exactly as in the proof of [NT23, Prop. 4.18]. $\square$

Our abstract freeness result is the following, where for the convenience of the reader we have recalled the running hypotheses and notation in the statement.

Proposition 7.2.3. *Assume that ρ satisfies Hypothesis 7.1.3, and that M satisfies Hypothesis 7.2.1. Write $\mathfrak{q}$ for the kernel of the homomorphism $R_{\mathcal{S}} \to \mathcal{O}$ corresponding to ρ. Then $M_{\mathfrak{q}}$ is a finite free $(R_{\mathcal{S}})_{\mathfrak{q}}$-module.*

Proof. By Lemma 6.2.2(3) and our assumption that $\rho|_{G_{F_v^+}}$ is ordinary, pure and p-distinguished for all $v|p$, together with [BCGP21, Lem. 7.1.3] and the assumption that ρ is pure, the local ring $(R_\infty)_{\mathfrak{q}_\infty}$ is regular. Write $(R_\infty)^\wedge_{\mathfrak{q}_\infty}$ for its $\mathfrak{q}_\infty$-adic completion, and similarly $(R^{\mathrm{patch}})^\wedge_{\mathfrak{q}^{\mathrm{patch}}}$ for the $\mathfrak{q}^{\mathrm{patch}}$-adic completion of $(R^{\mathrm{patch}})_{\mathfrak{q}^{\mathrm{patch}}}$. Then the morphism

$$(R_\infty)^\wedge_{\mathfrak{q}_\infty} \to (R^{\mathrm{patch}})^\wedge_{\mathfrak{q}^{\mathrm{patch}}}$$

is surjective, because by Lemma 7.2.2 the relative cotangent space $\mathfrak{q}^{\text{patch}}/((\mathfrak{q}^{\text{patch}})^2, \mathfrak{q}_\infty)$ vanishes (note that localization at $\mathfrak{q}^{\text{patch}}$ in particular inverts p).

Accordingly we can and do lift the morphism $S_\infty \to (R^{\text{patch}})^\wedge_{\mathfrak{q}^{\text{patch}}}$ to a morphism $S_\infty \to (R_\infty)^\wedge_{\mathfrak{q}_\infty}$. Since M_∞ is a finite free S_∞-module, we see that the $(R_\infty)^\wedge_{\mathfrak{q}_\infty}$-module $(M_\infty)^\wedge_{\mathfrak{q}_\infty}$ has depth $\dim S_\infty - 1$. By the definition of g together with Lemma 6.2.2(3)and [BG19, Thm. 3.3.3], we have $\dim (R_\infty)^\wedge_{\mathfrak{q}_\infty} = \dim S_\infty - 1$. Since $(R_\infty)^\wedge_{\mathfrak{q}_\infty}$ is regular, it follows from Auslander–Buchsbaum that $(M_\infty)^\wedge_{\mathfrak{q}_\infty}$ is a finite free $(R_\infty)^\wedge_{\mathfrak{q}_\infty}$-module. Quotienting by $\mathbf{a}_\infty$, we conclude that $M^\wedge_{\mathfrak{q}}$ is a finite free $R^\wedge_{\mathfrak{q}}$-module, and equivalently that $M_{\mathfrak{q}}$ is a finite free $R_{\mathfrak{q}}$-module, as required. □

7.3. **Higher Hida theory.** We now specialize to the case $F^+ = \mathbf{Q}$ (but still allow $p \geq 2$ to be arbitrary), and continue to assume that $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathcal{O})$ satisfies Hypothesis 7.1.3. In particular since $\rho|_{\mathbf{Q}_p}$ is ordinary, semistable of weight 2, and p-distinguished, and we have chosen a p-stabilization, we have determined an ordered pair (α_p, β_p) of distinct elements of $\mathcal{O}$ such that

$$\rho \cong \begin{pmatrix} \lambda_{\alpha_p} & 0 & * & * \\ 0 & \lambda_{\beta_p} & * & * \\ 0 & 0 & \varepsilon^{-1}\lambda_{\beta_p}^{-1} & 0 \\ 0 & 0 & 0 & \varepsilon^{-1}\lambda_{\alpha_p}^{-1} \end{pmatrix}$$

where λ_x is the unramified character with $\lambda_x(\mathrm{Frob}_p) = x$.

Our next goal is to explain the construction of the data as in Hypothesis 7.2.1 which we will use to prove our multiplicity one theorems. This amounts to constructing Taylor–Wiles systems out of higher Hida theory modules, which we already did for usual (i.e. H^0) Hida modules in [BCGP21, §7.8, 7.9], and we will follow the account there where possible. We do make some changes to the setup however: we specialize to the case $F^+ = \mathbf{Q}$, allow $p = 2$, work with different level structures at primes where ρ ramifies, and use a different argument to ensure that we can work at neat level.

We begin by recalling some of the main results of [BP23], specialized to the case of GSp_4. For each $w \in {}^M W$, and neat tame level K^p, in [BP23, §1.4, 5.4], we have defined perfect complexes of $\mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!]$-modules, which we denote by $M^\bullet_{w,\mathrm{cusp}}$ and $M^\bullet_w$. By [BP23, Prop. 5.6.3] the complexes $M^\bullet_{w,\mathrm{cusp}}$ have amplitude in the range $[0, \ell(w)]$, and the complexes $M^\bullet_w$ have amplitude in the range $[\ell(w), 3]$. These complexes have an action of $T(\mathbf{Q}_p)$ (extending the $T(\mathbf{Z}_p)$-action) and an action of the Hecke algebra $\mathbf{T}^p$ of prime to p level. We now set

$$M_w := \begin{cases} H^{\ell(w)}(M^\bullet_{w,\mathrm{cusp}}) \otimes_{\mathbf{Z}_p} \mathcal{O}, & \text{if } \ell(w) = 0, 1, \\ H^{\ell(w)}(M^\bullet_w) \otimes_{\mathbf{Z}_p} \mathcal{O}, & \text{if } \ell(w) = 2, 3. \end{cases}$$

When we want to stress the dependence on the tame level K^p, we write $M^\bullet_{w,\mathrm{cusp},K^p}$, $M^\bullet_{w,K^p}$, and M_{w,K^p}-respectively. We now summarize the main properties of these modules, writing $\mathrm{Iw}(p) \subset \mathrm{GSp}_4(\mathbf{Z}_p)$ for the Iwahori subgroup.

Theorem 7.3.1. *The following properties hold:*

1. *The modules M_w are finite projective $\mathcal{O}[\![T(\mathbf{Z}_p)]\!]$-modules.*
2. *For any dominant algebraic character $\lambda \in X^*(T)^+$, let*

$$\kappa = -w_{0,M} w(\lambda + \rho) - \rho.$$

There are Hecke equivariant isomorphisms

$$M_w \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!],\lambda} E(-\lambda) = \begin{cases} H^{\ell(w)}(\mathrm{Sh}^{\mathrm{tor}}_{\mathrm{Iw}(p)K^p}, \omega^\kappa(-D))^{\mathrm{ord}}, & \text{if } \ell(w) = 0, 1, \\ H^{\ell(w)}(\mathrm{Sh}^{\mathrm{tor}}_{\mathrm{Iw}(p)K^p}, \omega^\kappa)^{\mathrm{ord}}, & \text{if } \ell(w) = 2, 3. \end{cases}$$

(3) *For any algebraic character* $\lambda \in X^*(T)$ *with*

$$\kappa = -w_{0,M} w(\lambda + \rho) - \rho \in X^*(T)^{M,+}$$

there are Hecke equivariant isomorphisms

$$M_w \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!],\lambda} E(-\lambda) = \begin{cases} H^{\ell(w)}_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}}(-D))^{\mathrm{ord},T(\mathbf{Z}_p)}, & \text{if } \ell(w) = 0, 1, \\ H^{\ell(w)}_w(\mathrm{Sh}^{\mathrm{tor}}_{K^p}, \omega^{\kappa,\mathrm{sm}})^{\mathrm{ord},T(\mathbf{Z}_p)}, & \text{if } \ell(w) = 2, 3. \end{cases}$$

(4) *We have a perfect duality pairing:*

$$M_w \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} M_{w_{0,M} w w_0} \to \mathcal{O}[\![T(\mathbf{Z}_p)]\!]$$

interpolating the classical Serre duality.

(5) *If we have a normal subgroup* $K_1^p \subseteq K_2^p$ *then* M_{w,K_1^p} *is a finite projective* $\mathcal{O}[\![T(\mathbf{Z}_p)]\!][K_2^p/K_1^p]$*-module and the pullback and trace induce isomorphisms*

$$M_{w,K_2^p} \xrightarrow{\sim} M_{w,K_1^p}^{K_2^p/K_1^p} \qquad (M_{w,K_1^p})_{K_2^p/K_1^p} \xrightarrow{\sim} M_{w,K_2^p}.$$

Proof. By duality ([BP23, Thm 5.5.2]), point (1) follows from the vanishing theorem [BP23, Prop. 5.6.3] combined with the fact that when $\ell(w) = 1$ the H^0 with support vanishes. Point (2) is the classicality theorem [BP23, Cor 4.5.5] combined with the control theorem [BP23, Thm 5.3.5]. Point (3) is the comparison between higher Hida and Coleman theories [BP23, Thm 6.2.9] together with Theorem 4.6.56. As already remarked, the duality in point (4) is [BP23, Thm 5.5.2].

We give some justification for point (5) as this is not explained in [BP23]. The projectivity is a consequence of the other statements and Lemma 7.3.2 below. By the vanishing results just recalled it suffices to show that the corresponding statements for the higher Hida complexes. In other words, pullback and trace induce quasi-isomorphisms

$$M^\bullet_{w,K_2^p} \to R\Gamma(K_2^p/K_1^p, M^\bullet_{w,K_1^p}), \quad M_{w,K_1^p} \otimes^L_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!][K_2^p/K_1^p]} \mathcal{O}[\![T(\mathbf{Z}_p)]\!] \to M^\bullet_{w,K_2^p},$$

and the same statement holds for the cuspidal complexes. For this, the key point is to show that for a suitable choice of cone decomposition Σ, for the integral models $\pi : \mathrm{Sh}^{\mathrm{tor}}_{K_pK_1^p,\Sigma} \to \mathrm{Sh}^{\mathrm{tor}}_{K_pK_2^p,\Sigma}$ considered in [BP23] there is an action of K_2^p/K_1^p on $\mathrm{Sh}^{\mathrm{tor}}_{K_pK_1^p,\Sigma}$, and the natural map

$$\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK_2,\Sigma}} \to R\Gamma(K_2^p/K_1^p, R\pi_*\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK_1^p,\Sigma}})$$

is an isomorphism, and the analogous statements for the sheaves $\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK_i,\Sigma}}(-D)$, and for traces. For this, the facts that $R\pi_*\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK_1^p,\Sigma}} = \pi_*\mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK_1^p,\Sigma}}$ and $\pi_*\mathcal{O}^{K_2^p/K_1^p}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK_1^p,\Sigma}} = \mathcal{O}_{\mathrm{Sh}^{\mathrm{tor}}_{K_pK_2^p,\Sigma}}$ are standard and follow from the local description of the toroidal boundary (see [Lan17, Prop. 7.5] for example). We just need to explain why there is no higher K_2^p/K_1^p cohomology (note that the action of K_2^p/K_1^p may not be free and p may divide the order of K_2^p/K_1^p); however it again follows from the local description of the toroidal boundary that the stabilizers have order prime to p. □

We used the following (presumably standard) lemma above.

Lemma 7.3.2. *Let R be a Noetherian local ring with residue field k of characteristic p. Let G be a finite group with Sylow p-subgroup H, and let M be a finitely generated $R[G]$-module. Then the following conditions are equivalent:*

(1) M is a projective $R[G]$-module.
(2) M is a projective (equivalently free) $R[H]$-module.
(3) $M \otimes^L_{R[H]} R$ is a free R-module concentrated in degree 0.
(4) $M \otimes^L_{R[H]} k$ is concentrated in degree 0.

Proof. The equivalence of the first two conditions follows from the usual averaging argument to promote a splitting as $R[H]$-modules to a splitting as $R[G]$-modules. Since $R[H]$ is a local ring, the equivalence of the second, third and fourth conditions is immediate from the existence of minimal free resolutions, and in particular from [GN20, Lem. 2.1.7, Prop. 2.1.9]. □

Remark 7.3.3. Let us spell out the Hecke action at p. On $M_w \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!],\lambda} E(-\lambda)$, $T(\mathbf{Q}_p)$ acts via a smooth character, trivial on $T(\mathbf{Z}_p)$ (this is the reason for the twist by λ). The isomorphism $M_w \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!],\lambda} E(-\lambda) = H^{\ell(w)}(\mathrm{Sh}^{\mathrm{tor}}_{\mathrm{Iw}(p)K^p}, \omega^\kappa)^{\mathrm{ord}}$ matches the action of $t \in T^+(\mathbf{Q}_p)$ with the action of the double class $[\mathrm{Iw}(p)t\mathrm{Iw}(p)]$ (note that on the left hand side, the unitary action of $T(\mathbf{Q}_p)$ is twisted by $-\lambda$).

Remark 7.3.4. Let κ be such that $\kappa = -w_{0,M}w(\lambda+\rho) - \rho$ with $\lambda+\rho$ being G-dominant. Let c be an eigenclass in $H^{\ell(w)}(\mathrm{Sh}^{\mathrm{tor}}_{\mathrm{Iw}(p)K^p}, \omega^\kappa)^{\mathrm{ord}}$, corresponding to an automorphic representation π. The torus $T(\mathbf{Q}_p)$ acts on the Jacquet module of π via a smooth character χ_c^{sm}. Assume furthermore that π_∞ is a discrete series representation (which is automatic if λ is sufficiently regular), so that we have a Galois representation $\rho_{\pi,p} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ associated to π (see Theorem 1.8.13).

Then $\rho_{\pi,p}|_{G_{\mathbf{Q}_p}}$ is conjugate to a $B(\overline{\mathbf{Q}}_p)$-valued representation, which we can describe explicitly as follows (see Remark 1.8.15). If $\lambda = (\lambda_1, \lambda_2; w)$, the Hodge–Tate weights of ρ_π are given in increasing order by

$$\frac{\lambda_1+\lambda_2-w}{2}, \frac{2-\lambda_1+\lambda_2-w}{2}, \frac{4+\lambda_1-\lambda_2-w}{2}, \frac{6-\lambda_1-\lambda_2-w}{2}.$$

If we use the upper triangular Borel in $B(\mathbf{Q}_p)$ in $\mathrm{GSp}_4(\mathbf{Q}_p)$, and use the local class field theory map $\mathbf{Q}_p^\times \to G^{\mathrm{ab}}_{\mathbf{Q}_p}$, the character on the diagonal is given by

$$t \mapsto \mathrm{diag}(t^{\frac{-\lambda_1-\lambda_2+w}{2}}, t^{\frac{-2+\lambda_1-\lambda_2+w}{2}}, t^{\frac{-4-\lambda_1+\lambda_2+w}{2}}, t^{\frac{\lambda_1+\lambda_2+w-6}{2}}),\ t \in \mathbf{Z}_p^\times$$

while $p \in \mathbf{Q}_p^\times$ goes to

$$\begin{aligned}\mathrm{diag}(\chi_c^{sm}((1,1,p,p))p^{\frac{-\lambda_1-\lambda_2+w}{2}}, \chi_c^{sm}((p,1,p,1))p^{\frac{\lambda_1-\lambda_2+w}{2}},\\ \chi_c^{sm}((1,p,1,p))p^{\frac{-\lambda_1+\lambda_2+w}{2}}, \chi_c^{sm}((p,p,1,1))p^{\frac{\lambda_1+\lambda_2+w}{2}}).\end{aligned}$$

In view of Remark 7.3.4, we now give a more Galois-theoretic parametrization of the Higher Hida theories. Recall that in Section 6.2 we defined a complete local Noetherian $\mathcal{O}$-algebra $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}} = \mathcal{O}[\![(\mathbf{Z}_p^\times(p))^2]\!]$. We have two maps $\mathbf{Z}_p^\times \to \mathbf{Z}_p^\times(p) \times \mathbf{Z}_p^\times(p)$ (given by the projections to each factor), corresponding to $\chi_1, \chi_2 : \mathbf{Z}_p^\times \to \Lambda^\times_{\mathrm{GSp}_4,\mathbf{Q}}$, and there is an associated homomorphism $\psi : \mathbf{Z}_p^\times \to T(\Lambda_{\mathrm{GSp}_4,\mathbf{Q}})$ which is given by

$$\psi(z) = \mathrm{diag}(\chi_1(z), \chi_2(z), z^{-1}\chi_2^{-1}(z), z^{-1}\chi_1(z)^{-1}).$$

On the other hand, consider the universal character

$$\chi^{un} : T(\mathbf{Z}_p) \to (\mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!])^\times.$$

Using Lemma 7.3.5 and the identification of T and $\widehat{T}$, we can view $\chi^{un}\rho\nu^{-\frac{3}{2}} : T(\mathbf{Z}_p) \to (\mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!])^\times$ as a homomorphism

$$\chi^{un}\rho\nu^{-\frac{3}{2}} : \mathbf{Z}_p^\times \to T(\mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!]).$$

Lemma 7.3.5. *There is a unique algebra map $f : \mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!] \to \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$ such that $f \circ \chi^{un}\rho\nu^{-\frac{3}{2}} = \psi$.*

Proof. Using Lemma 7.3.6 and the identification of T and $\widehat{T}$, ψ corresponds to a character $T(\mathbf{Z}_p) \to (\Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^\times$, and the lemma follows from the universal properties of χ^{un} and of $\mathbf{Z}_p[\![T(\mathbf{Z}_p)]\!]$. □

We used the following (presumably standard) lemma above.

Lemma 7.3.6. *If $T/\mathbf{Z}$ is a split torus with dual $\widehat{T}$, then for any two commutative rings R, S, there is a natural bijection between group homomorphisms*

$$T(R) \to S^\times$$

and group homomorphisms

$$R^\times \to \widehat{T}(S).$$

Proof. This follows from the case $T = \mathbf{G}_m$, which is obvious. More precisely, since $X_*(\widehat{T}) = X^*(T)$ is a free $\mathbf{Z}$-module, we have

$$\begin{aligned}
\mathrm{Hom}(T(R), S^\times) &= \mathrm{Hom}(X_*(T) \otimes R^\times, S^\times) \\
&= \mathrm{Hom}(X_*(T), \mathrm{Hom}(R^\times, S^\times)) \\
&= X^*(T) \otimes \mathrm{Hom}(R^\times, S^\times) \\
&= X_*(\widehat{T}) \otimes \mathrm{Hom}(R^\times, S^\times) \\
&= \mathrm{Hom}(R^\times, X_*(\widehat{T}) \otimes S^\times) \\
&= \mathrm{Hom}(R^\times, \widehat{T}(S)),
\end{aligned}$$

as required. □

We now consider the action of the centre of the group $\mathrm{GSp}_4(\mathbf{A}_f)$. The action of $Z(\mathrm{GSp}_4(\mathbf{A}_f))$ on the toroidal compactification of our Shimura varieties factors into an action of $Z(\mathbf{Q})\backslash Z(\mathrm{GSp}_4(\mathbf{A}_f)) = \{\pm 1\}\backslash \prod_\ell \mathbf{Z}_\ell^\times$ (as can be seen by considering complex uniformization). From a modular perspective, $\mathbf{Z}_\ell^\times$ acts on the ℓ-adic Tate module of an abelian surface by scalar multiplication. The fact that we quotient by $\{\pm 1\}$ witnesses the fact that -1 is an automorphism of any abelian surface. This action extends to an action on the M_{w,K^p} (as part of the Hecke action on these modules).

When we have a $\mathbf{Z}_p$-module M, equipped with an action of $Z(\mathbf{Q})\backslash Z(\mathrm{GSp}_4(\mathbf{A}_f))$, we let $M^{|\cdot|^2}$ be the submodule where the centre acts via the character $(z_\ell)_\ell \mapsto z_p^{-2}$. More generally, if we have a finite set of primes S, we let $M^{|\cdot|^2,S}$ be the submodule where the group $\prod_{\ell\notin S} \mathbf{Z}_\ell^\times$ acts via the character $(z_\ell)_{\ell\notin S} \mapsto z_p^{-2}$.

Remark 7.3.7. This definition is motivated by Theorem 7.3.1 (2). Under the classicality theorem, this condition corresponds to fixing the central character of automorphic forms contributing to coherent cohomology to be $|.|^2$.

By abuse of language, we say that the centre acts by $|.|^2$ on $M^{|\cdot|^2}$.

On the module $M_{w,K^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$, the subgroup $\mathbf{Z}_p^\times$ of $Z(\mathrm{GSp}_4(\mathbf{A}_f))$ acts via $z \mapsto z^{-2}$. Therefore, fixing the central action to be $|.|^2$ amounts to asking that the group $\prod_{\ell \neq p} \mathbf{Z}_\ell^\times$ acts trivially. We would therefore morally have that $(M_{w,K^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2} = M_{w,ZK^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$ where $ZK^p = \prod_{\ell \neq p} Z(\mathbf{Z}_\ell)K_\ell$. Note however that the group ZK^p is not neat (a condition we have imposed so far on our tame level). We now explain a construction which addresses this issue.

We choose a prime $r > 5$ such that $r \not\equiv 1 \pmod p$ and $r \not\equiv 1 \pmod 4$ if $p = 2$. We let $\mathrm{Iw}_1(r)$ denote the subgroup of matrices which are upper-triangular and unipotent modulo r. If K^p is a compact open subgroup with $K_r = \mathrm{Iw}_1(r)$, then K^p is neat by [BCGP21, Lem. 7.8.3] (applied to $v = r$).

For any finite set of primes S (possibly empty) not containing p, we let $Z^S K^p = \prod_{\ell \notin S \cup \{p\}} Z(\mathbf{Z}_\ell)K_\ell \prod_{\ell \in S} K_\ell$.

Lemma 7.3.8. *Let $K_1^p \subseteq K_2^p$ be compact open subgroups, K_1^p normal in K_2^p, and the r-components of K_1^p, K_2^p both being $\mathrm{Iw}_1(r)$. Let S be a finite set of primes, with $r \in S$, and $p \notin S$.*

(1) The module
$$(M_{w,K_1^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2,S}$$
is a finite projective $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[Z^S K_2^p / Z^S K_1^p]$ module.

(2) The module
$$(M_{w,K_1^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2}$$
is a finite projective $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[ZK_2^p / ZK_1^p]$ module.

Proof. Since $r \in S$, it follows that the groups $Z^S K_i^p$ are neat for $i = 1, 2$. On the other hand, we have an identification
$$(M_{w,K_1^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2,S} = M_{w,Z^S K_1^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}},$$
so (1) follows from Theorem 7.3.1. Next, we can take $S = \{r\}$. Then $(M_{w,K_1^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2}$ is obtained from $(M_{w,K_1^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2,S}$ by considering the invariants for $(\mathbf{Z}/r\mathbf{Z})^\times$. Observe that $Z^S K_2^p / Z^S K_1^p = ZK_2^p / ZK_1^p$, so
$$(M_{w,K_1^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2,S}$$
is a finite projective $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[Z^S K_1^p / Z^S K_2^p]$ module. Note that $(\mathbf{Z}/r\mathbf{Z})^\times$ acts via $(\mathbf{Z}/r\mathbf{Z})^\times / \{\pm 1\}$ and this group has order prime to p. Therefore the invariants are a direct factor, as required. □

7.4. **Taylor–Wiles systems.** We continue to fix a continuous representation $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathcal{O})$ satisfying Hypothesis 7.1.3.

Definition 7.4.1. A *neat prime for $\overline{\rho}$* is a prime $r > 5$ such that

- $r \neq p$,
- $r \not\equiv 1 \pmod p$ and $r \not\equiv 1 \pmod 4$ if $p = 2$.
- $\rho|_{G_{\mathbf{Q}_r}}$ is unramified,

- $\overline{\rho}(\mathrm{Frob}_r)$ is regular semi-simple,

together with a fixed ordering $(\overline{\alpha}_{r,1}, \overline{\alpha}_{r,2}, r\overline{\alpha}_{r,2}^{-1}, r\overline{\alpha}_{r,1}^{-1})$ of the eigenvalues of $\overline{\rho}(\mathrm{Frob}_r)$.

By Hypothesis 7.1.3 parts (4), (7), and (8) we can and do choose a neat prime r.

Definition 7.4.2. We let S be the union of $\{r\}$ and the set of primes at which ρ is ramified (so in particular $p \in S$, because ε is ramified at p), and choose sets of Taylor–Wiles primes Q_N as in Proposition 7.1.13.

For any prime l, we let $\mathrm{Iw}(l)$ denote the subgroup of $\mathrm{GSp}_4(\mathbf{Z}_l)$ consisting of matrices which are upper-triangular modulo l, and we let $\mathrm{Iw}_1(l)$ denote the subgroup of matrices which are upper-triangular and unipotent modulo l.

Definition 7.4.3. We define an open compact subgroup $K^p = \prod_l K_l$ of $\mathrm{GSp}_4(\mathbf{A}^{\infty,p})$ as follows:

- If $l \notin S$, then $K_l = \mathrm{GSp}_4(\mathbf{Z}_l)$.
- $K_r = \mathrm{Iw}_1(r)$.
- If $l \in S \smallsetminus \{r\}$, then we allow any choice of open compact $K_l \subseteq \mathrm{GSp}_4(\mathbf{Z}_l)$.

We have compact subgroups $K_0^p(Q_N)$, $K_1^p(Q_N)$ of K^p given by

- If $l \notin Q_N$, then $K_0^p(Q_N)_l = K_1^p(Q_N)_l = K_l^p$.
- If $l \in Q_N$, then $K_0^p(Q_N)_l = \mathrm{Iw}(l)$, $K_1^p(Q_N)_l = \mathrm{Iw}_1(l)$.

These groups are neat by [BCGP21, Lem. 7.8.3].

We let
$$\mathbf{T}^S = \bigotimes_{l \notin S} \mathcal{O}[\mathrm{GSp}_4(\mathbf{Q}_l) /\!/ \mathrm{GSp}_4(\mathbf{Z}_l)]$$
be the ring of spherical Hecke operators away from the bad places, and similarly we set
$$\mathbf{T}^{S \cup Q_N} = \bigotimes_{l \notin S \cup Q_N} \mathcal{O}[\mathrm{GSp}_4(\mathbf{Q}_l) /\!/ \mathrm{GSp}_4(\mathbf{Z}_l)].$$

We will also make use of some Hecke operators at Iwahori level, which we recall from [BCGP21, §2.4]. Let l be a prime. Assume that E is large enough to contain a square root of l; we fix such a choice $l^{1/2}$. The reader can easily check that nothing before [BCGP21, Lem. 2.4.3] makes any use of the running assumption made there that $p \neq 2$; indeed, these results are for the most part over E, and use only that it is a field of characteristic zero containing $l^{1/2}$. We define
$$\mathcal{H} = \mathcal{H}(l) = \mathcal{O}[G(\mathbf{Q}_l) /\!/ \mathrm{Iw}(l)],$$
$$\mathcal{H}_1 = \mathcal{H}_1(l) = \mathcal{O}[G(\mathbf{Q}_l) /\!/ \mathrm{Iw}_1(l)].$$
With T denoting our usual maximal torus in GSp_4, we set
$$T(\mathbf{Z}_l)_1 := (\ker T(\mathbf{Z}_l) \to T(\mathbf{F}_l)),$$
and exactly as in [BCGP21, Prop. 2.4.2] we have an injective homomorphism
$$T(\mathbf{Q}_l)/(\ker T(\mathbf{Z}_l) \to T(\mathbf{F}_l)) \to (\mathcal{H}_1)^\times. \tag{7.4.4}$$

The injection (7.4.4) induces an injective homomorphism $\mathcal{O}[T(\mathbf{Q}_l)/T(\mathbf{Z}_l)_1] \to \mathcal{H}_1$, and we identify $\mathcal{O}[T(\mathbf{Q}_l)/T(\mathbf{Z}_l)_1]$ with its image in $\mathcal{H}_1$.

Definition 7.4.5. Assume that $l \equiv 1 \pmod{p}$. Given elements $\overline{\alpha}_{l,1}, \overline{\alpha}_2 \in k^\times$, we let $\mathfrak{m}_{\overline{\alpha}_1,\overline{\alpha}_2}$ denote the kernel of the homomorphism $\mathcal{O}[T(\mathbf{Q}_l)/T(\mathbf{Z}_l)_1] \to k$ induced by the character $T(\mathbf{Q}_l)/T(\mathbf{Z}_l)_1 \to k^\times$ sending $T(\mathbf{Z}_l) \mapsto 1$, $\beta_{l,0} \mapsto 1$, $\beta_{l,1} \mapsto \overline{\alpha}_{l,1}$, $\beta_{l,2} \mapsto \overline{\alpha}_{l,1}\overline{\alpha}_{l,2}$.

We define the following elements of $T(\mathbf{Q}_p)$ which act on M_w:

$$\begin{aligned} U_{p,1} &= \operatorname{diag}(1,1,p,p) \\ U_{p,2} &= \operatorname{diag}(p,1,p^2,p) \\ U_{p,0} &= \operatorname{diag}(p,p,p,p) \end{aligned}$$

We let $\mathfrak{m}^{\mathrm{an}} \subset \mathbf{T}^S$ be the maximal ideal corresponding to $\overline{\rho}$; so by definition $\mathfrak{m}^{\mathrm{an}}$ contains ϖ, and the polynomials $\det(X - \overline{\rho}(\mathrm{Frob}_l))$ and $Q_l(X)$ are congruent modulo $\mathfrak{m}^{\mathrm{an}}$ for each $l \notin S$. We define $\mathfrak{m}^{\mathrm{an},Q_N} \subset \mathbf{T}^{S\cup Q_N}$ in the same way. We let

$$\mathbf{T} = \mathbf{T}^S[U_{p,0}, U_{p,1}, U_{p,2}]$$

and

$$\mathbf{T}^{Q_N} = \mathbf{T}^{S\cup Q_N}[U_{p,0}, U_{p,1}, U_{p,2}],$$

and additionally we let $\mathfrak{m} \subset \mathbf{T}$ be the maximal ideal

$$\mathfrak{m} = (\mathfrak{m}^{\mathrm{an}}, U_{p,0} - 1, U_{p,1} - \alpha_p, U_{p,2} - \alpha_p\beta_p)$$

and we let $\mathfrak{m}^{Q_N} \subset \mathbf{T}^{Q_N}$ be the maximal ideal

$$\mathfrak{m}^{Q_N} = (\mathfrak{m}^{\mathrm{an},Q_N}, U_{p,0} - 1, U_{p,1} - \alpha_p, U_{p,2} - \alpha_p\beta_p).$$

For each $q \in Q_N$, fix an ordering $\overline{\alpha}_{q,1}, \overline{\alpha}_{q,2}, \overline{\alpha}_{q,2}^{-1}, \overline{\alpha}_{q,1}^{-1}$ of the eigenvalues of $\overline{\rho}(\mathrm{Frob}_q)$. Set $\Delta_N = \Delta_{Q_N} = \prod_{q\in Q_N} \mathbf{F}_q^\times(p)^2$. We now fix a choice of $w \in {}^M W$, and consider the finite free $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$-module

$$M := \mathrm{Hom}_{\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}}((M_{K^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2}, \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})_{\mathfrak{m},\mathfrak{m}_r}, \tag{7.4.6}$$

and the finite $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[\Delta_{Q_N}]$-modules

$$M_N := \mathrm{Hom}_{\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}}((M_{K_1^p(Q_N)} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2}, \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})_{\mathfrak{m}^{Q_N},\mathfrak{m}_r,\mathfrak{m}_{Q_N}},$$

where:

- M_{K^p} and $M_{K_1^p(Q_N)}$ denote the modules M_w of Theorem 7.3.1, taking K^p there to be respectively our K^p and $K_1^p(Q_N)$.
- $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$ is an $\mathcal{O}[\![T(\mathbf{Z}_p)]\!]$-algebra via Lemma 7.3.5.
- The localizations $\mathfrak{m}, \mathfrak{m}^{Q_N}$ are defined above.
- The localization $\mathfrak{m}_r$ and the localization $\mathfrak{m}_{Q_N}$ are with respect to the maximal ideals $\mathfrak{m}_l$ of the subalgebras $\mathcal{O}[T(\mathbf{Q}_l)/T(\mathbf{Z}_l)_1]$ of the pro-l Iwahori Hecke algebras $\mathcal{H}_1(l)$ for $l \in Q_N \cup \{r\}$ as in Definition 7.4.5.
- The action of Δ_{Q_N} on $M_{K_1^p(Q_N)}$ is induced by the actions of $\mathcal{O}[T(\mathbf{Q}_q)/T(\mathbf{Z}_q)_1]$ for $q \in Q_N$, by regarding $\mathbf{F}_q^\times(p)^2$ as the maximal p-power quotient of $T(\mathbf{F}_q)/Z(\mathbf{F}_q)$, where Z denotes the centre of GSp_4.
- The superscript $|\cdot|^2$ denotes that we are fixing the central character.

Lemma 7.4.7. *M is a free $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$-module, and M_N is a free $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[\Delta_{Q_N}]$-module.*

Proof. Since $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[\Delta_{Q_N}]$ is a local ring, this follows from Lemma 7.3.8(2), since $\mathrm{Iw}(q)Z(\mathbf{Z}_q)/\mathrm{Iw}_1(q)Z(\mathbf{Z}_q) \simeq T(\mathbf{F}_q)/Z(\mathbf{F}_q)$. □

The following is essentially [GG12, Lem. 7.1.1], adapted slightly to allow $p = 2$; we will use it in the proof of Proposition 7.4.10.

Lemma 7.4.8. *Let Γ be a profinite group, and let $S \subset R$ be complete local Noetherian rings with $\mathfrak{m}_R \cap S = \mathfrak{m}_S$ and common residue field k. Let $\rho : \Gamma \to \mathrm{GSp}_4(R)$ be a continuous representation. Suppose that ρ mod $\mathfrak{m}_R$ is absolutely irreducible, that $\operatorname{tr} \rho(\Gamma) \subset S$, and that $\nu \circ \rho(\Gamma) \subset S^\times$. Then there is a $\widehat{\mathrm{GSp}}_4(R)$-conjugate of ρ whose image is contained in $\mathrm{GSp}_4(S)$.*

Proof. By [CHT08, Lem. 2.1.10], there is some $B \in \widehat{\mathrm{GL}}_4(R)$ such that $\rho' := B\rho B^{-1}$ is valued in $\mathrm{GL}_4(S)$. Since $J^{-1}\rho J = (\nu \circ \rho)\rho^{-t}$, we have

$$(BJB^t)^{-1}\rho'(BJB^t) = (\nu \circ \rho')(\rho')^{-t}.$$

By choosing a symplectic basis for the alternating form determined by BJB^t, it follows that ρ' is $\widehat{\mathrm{GSp}}_4(S)$-conjugate to a representation ρ'' valued in $\mathrm{GSp}_4(S)$. By Schur's lemma [CHT08, Lem. 2.1.8], we see the element in $\widehat{\mathrm{GL}}_4(R)$ conjugating ρ to ρ' is necessarily contained in $\widehat{\mathrm{GSp}}_4(R)$, as required. ☐

Definition 7.4.9. Let $\mathbf{T}_{\mathcal{S}}$ (resp. $\mathbf{T}_{\mathcal{S}_{Q_N}}$) denote the image of $\mathbf{T}$ (resp. of $\mathbf{T}^{Q_N}$) in $\mathrm{End}_{\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}}(M)$ (resp. in $\mathrm{End}_{\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}}(M_N)$). (These are objects of $\mathrm{CNL}_{\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}}$, but under the present hypotheses we do not know that these algebras are nonzero (because we do not know that $\overline{\rho}$ is modular).)

Recall that we have defined the global deformation problems

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_{\mathrm{GSp}_4,p}\} \cup \{\mathcal{O}\}_{l \in S \smallsetminus \{p\}}, \{\mathcal{D}_p^{\triangle}\} \cup \{\mathcal{D}_l^{\square}\}_{l \in S \smallsetminus \{p\}}),$$

$$\mathcal{S}_{Q_N} = (\overline{\rho}, S \cup Q_N, \{\Lambda_{\mathrm{GSp}_4,p}\} \cup \{\mathcal{O}\}_{l \in (S \cup Q_N) \smallsetminus \{p\}}, \{\mathcal{D}_p^{\triangle}\} \cup \{\mathcal{D}_l^{\square}\}_{l \in (S \cup Q_N) \smallsetminus \{p\}}).$$

The deformation ring $R_{\mathcal{S}_{Q_N}}$ is a $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[\Delta_{Q_N}]$-algebra via Lemma 6.1.6 (i.e. Δ_{Q_N} acts via the characters $\gamma_{q,i} \circ \mathrm{Art}_{\mathbf{Q}_q}$). For ease of notation, we sometimes (e.g. in the statement of the following proposition) adopt the convention that $Q_0 = \emptyset$, so that for example $\mathbf{T}_{\mathcal{S}_{Q_N}} = \mathbf{T}_{\mathcal{S}}$.

Proposition 7.4.10. *For each $N \geq 0$, the action of Δ_{Q_N} on M_N makes $\mathbf{T}_{\mathcal{S}_{Q_N}}$ a $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[\Delta_{Q_N}]$-algebra, and there is a $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[\Delta_{Q_N}]$-algebra homomorphism $R_{\mathcal{S}_{Q_N}} \to \mathbf{T}_{\mathcal{S}_{Q_N}}$ with corresponding representation $\rho_{\mathcal{S}_{Q_N}} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{T}_{\mathcal{S}_{Q_N}})$ determined by the property that $\det(X - \rho_{\mathcal{S}_{Q_N}}(\mathrm{Frob}_l)) = Q_l(X)$ for all $l \notin S \cup Q_N$.*

Proof. Since M_N is a finite free $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$-module, this follows from local-global compatibility for the Galois representations at a dense set of points of regular weight. More precisely, it can be proved in exactly the same way as the case $I = \emptyset$ of [BCGP21, Thm. 7.9.4], using Lemma 7.4.8 in place of [GG12, Lem. 7.1.1]. (The fact that $\mathbf{T}_{\mathcal{S}_{Q_N}}$ is automatically a $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[\Delta_{Q_N}]$-algebra was not recorded in [BCGP21]; it follows from local-global compatibility at the places in Q_N.) ☐

In particular, Proposition 7.4.10 makes M into an $R_{\mathcal{S}}$-module, and each M_N into an $R_{\mathcal{S}_{Q_N}}$-module. We can also regard M as an $R_{\mathcal{S}_{Q_N}}$-module via the natural map $R_{\mathcal{S}_{Q_N}} \to R_{\mathcal{S}}$. For each $N \geq 1$ we fix a surjection $\Delta_\infty \twoheadrightarrow \Delta_{Q_N}$, and write Δ_N for the corresponding quotient of Δ_∞. The kernel of this surjection is contained in $(p^N\mathbf{Z}_p)^{2q}$, since each $v \in Q_N$ satisfies $q_v \equiv 1 \bmod p^N$. At this point we have established points (1)–(3) of Hypothesis 7.2.1, so it only remains to check (4), which is the content of Lemma 7.4.15 below.

Before proving it, we recall some standard facts about Iwahori Hecke algebras that were explained in [BCGP21, §2.4] under the unnecessary assumption that $p \neq 2$. Indeed the only place in [BCGP21, §2.4] that relies on the assumption $p \neq 2$

is [BCGP21, Lem. 2.4.34], i.e. the statement that the spherical invariants of a module for the Iwahori Hecke algebra are a direct summand. This is no longer valid for $p = 2$, but we will avoid this problem by making use of Lemma 7.3.8. We do need to make use of the (proofs of) [BCGP21, Lem. 2.4.36, 2.4.37], but for the convenience of the reader we will recall the necessary arguments as we use them (making it clear as we do so that they remain valid for $p = 2$).

Suppose now that q is a prime with $q \equiv 1 \pmod{p}$, and let $\mathcal{H} = \mathcal{H}(q)$ be the corresponding Iwahori Hecke algebra. The Bernstein presentation of $\mathcal{H}$ is valid for all p, so we can write

$$\mathcal{H} = \mathcal{O}[X_*(T)]\widetilde{\otimes}_{\mathcal{O}}\mathcal{O}[\mathrm{Iw}(q)\backslash \mathrm{GSp}_4(\mathbf{Z}_q)/\mathrm{Iw}(q)],$$

where the twisted tensor product is determined by the relations [BCGP21, (2.4.32)]. The centre of $\mathcal{H}$ is $\mathcal{O}[X_*(T)]^W$ (where as usual $W \cong D_8$ is the Weyl group of GSp_4),

$$\mathcal{O}[X_*(T)]^W = \mathcal{O}[\mathrm{GSp}_4(\mathbf{Q}_q)/\!/ \mathrm{GSp}_4(\mathbf{Z}_q)]$$

given by $x \mapsto [\mathrm{GSp}_4(\mathbf{Z}_q)]x$ (where we are regarding x as an element of $\mathcal{H}$); this isomorphism agrees with the usual Satake isomorphism. (Indeed this presentation, and the compatibility with the Satake isomorphism, are valid over any ring containing an invertible square root of q; see for example [Vig05] or the very general results of [Bou21].)

Since we are assuming that $q = 1$ in k, we deduce exactly as in [BCGP21, Lem. 2.4.33] that reduction modulo ϖ induces a natural isomorphism

$$\mathcal{H} \otimes_{\mathcal{O}} k \cong k[X_*(T) \rtimes W].$$

Since we are assuming that $q \equiv 1 \pmod{p}$, and in any case in our applications of these results in the global setting there is a twist which makes all of the powers of q integral, we will ignore all powers of $q^{1/2}$ from now on.

Exactly as in [BCGP21], we let x_0, x_1, and x_2 denote the following three cocharacters:

$$\begin{aligned} x_0 &: t \to \mathrm{diag}(t, t, 1, 1), \\ x_1 &: t \to \mathrm{diag}(1/t, 1, 1, t), \\ x_2 &: t \to \mathrm{diag}(1, 1/t, t, 1). \end{aligned}$$

Then $x_0^2x_1x_2$ is the cocharacter $t \mapsto \mathrm{diag}(t, t, t, t)$ and

$$\mathcal{O}[X_*(T)] = \mathcal{O}[x_0, x_1, x_2, (x_0^2x_1x_2)^{-1}] = \mathcal{O}[x_0, x_1, x_2, (x_0x_1x_2)^{-1}].$$

The action of W preserves $(x_0, x_0x_1, x_0x_2, x_0x_1x_2)$ considered as an unordered quadruple. Recalling that we are ignoring powers of $q^{1/2}$, under the identification of $k[X_*(T)]^W$ with the spherical Hecke algebra we have

$$\begin{aligned} Q_q(X) &= X^4 - T_{q,1}X^3 + (T_{q,2} + 2T_{q,0})X^2 - T_{q,0}T_{q,1}X + T_{q,0}^2 \\ &= (X - x_0)(X - x_0x_1)(X - x_0x_2)(X - x_0x_1x_2). \end{aligned}$$

Then $k[X_*(T)]^W = k[e_1, e_2, (e_3/e_1)^{\pm 1}]$ where

$$\sum e_iX^i = (X - x_0)(X - x_0x_1)(X - x_0x_2)(X - x_0x_1x_2).$$

Now suppose that $\overline{\alpha}_{q,1}, \overline{\alpha}_{q,2} \in k^\times$ are such that $\overline{\alpha}_{q,1}, \overline{\alpha}_{q,2}, \overline{\alpha}_{q,2}^{-1}, \overline{\alpha}_{q,1}^{-1}$ are pairwise distinct, and set $\gamma_0 := \overline{\alpha}_{v,1}$, $\gamma_1 := \overline{\alpha}_{v,2}\overline{\alpha}_{v,1}^{-1}$, $\gamma_2 := (\overline{\alpha}_{v,1}\overline{\alpha}_{v,2})^{-1}$.

We let $\mathfrak{n} \subset \mathcal{O}[X_*(T)]^W$ be the maximal ideal generated by ϖ and the $e_i - e_i(\gamma_0, \gamma_1, \gamma_2)$, and for each $w \in W$ we let $\mathfrak{m}_w$ be the maximal ideal of $\mathcal{O}[X_*(T)]$

generated by ϖ and the $w\cdot x_i - \gamma_i$. By our assumption on $\overline{\alpha}_{q,1}, \overline{\alpha}_{q,2}$, the 8 ideals $\mathfrak{m}_w$ are pairwise distinct, and exactly as in the proof of [BCGP21, Lem. 2.4.36], we see that $\mathcal{O}[X_*(T)]_{\mathfrak{n}}$ is a semi-local ring whose maximal ideals are the $\mathfrak{m}_w$. In particular if M is an $\mathcal{O}[X_*(T)]$-module, we can write

$$M_{\mathfrak{n}} = \oplus_{w\in W} M_{\mathfrak{m}_w}.$$

Lemma 7.4.11. *Suppose that R is a commutative ring, and that G is a finite group. If H is a subgroup of G then we write $e_H := \sum_{h\in H} h \in R[G]$, and let $[G]_H = \sum_{g\in G/H} g \in R[G]$ (the coset representatives being chosen arbitrarily). Then if M is a finite projective left $R[G]$-module, we have*

(1) $M^H = e_H M$.
(2) $M^G = [G]_H M^H$.
(3) If S is any R-algebra, then $(M\otimes_R S)^H = M^H \otimes_R S$.
(4) The natural maps $M^H \to M$ and $M \to M_H$ induce isomorphisms of R-modules

$$\mathrm{Hom}_R(M_H, R) \xrightarrow{\sim} \mathrm{Hom}_R(M,R)^H,$$

$$\mathrm{Hom}_R(M,R)_H \xrightarrow{\sim} \mathrm{Hom}_R(M^H, R).$$

Proof. Writing M as a direct summand of a free $R[G]$-module, we reduce to the case $M = R[G]$. Then the first part is an easy calculation, while the second part follows from the first and the relation $[G]_H e_H = e_G$, which is immediate from the definitions. The third part is immediate from the first.

Turning to the final part, it is easy to see that the isomorphism $\mathrm{Hom}_R(M_H, R) \xrightarrow{\sim} \mathrm{Hom}_R(M,R)^H$ holds for any finite left $R[G]$-module, projective or otherwise. It remains to show that if M is projective then $\mathrm{Hom}_R(M,R)_H \xrightarrow{\sim} \mathrm{Hom}_R(M^H,R)$. We may again assume that $M = R[G]$, and since $R[G]$ is a free $R[H]$-module, we can furthermore assume that $H = G$. We can identify $R[G]$ with $\mathrm{Hom}_R(R[G],R)$ by sending $[1]$ to the map ϕ such that $\phi\big(\sum_g x_g g\big) = x_1$. Combining this with the usual identification of $R[G]_G$ with R via the trace map, we see that $\mathrm{Hom}_R(M,R)_G$ is a free R-module of rank one generated by the image of ϕ. By part (1), we have $R[G]^G = R\cdot e_G$. Since $\phi(e_G) = 1$, we are done. ∎

Lemma 7.4.12. *Suppose that $\overline{\alpha}_{q,1}, \overline{\alpha}_{q,2} \in k^\times$ are such that $\overline{\alpha}_{q,1}, \overline{\alpha}_{q,2}, \overline{\alpha}_{q,2}^{-1}, \overline{\alpha}_{q,1}^{-1}$ are pairwise distinct, and define ideals $\mathfrak{m}_w$, $\mathfrak{n}$ as above.*

Let R be an object of $\mathrm{CNL}_{\mathcal{O}}$, and let N be an R-module with a smooth action of $\mathrm{GSp}_4(\mathbf{Q}_q)$, with the property that if $K_2 \trianglelefteq K_1$ are compact open subgroups of $\mathrm{GSp}_4(\mathbf{Q}_q)$, then N^{K_2} is a finite projective $R[K_1/K_2]$-module.

Then for each $w \in W$ the projection

$$\mathrm{pr}_w : (N^{\mathrm{GSp}_4(\mathbf{Z}_q)})_{\mathfrak{n}} \to (N^{\mathrm{Iw}(q)})_{\mathfrak{m}_w} \tag{7.4.13}$$

is an isomorphism.

Proof. Take $K_1 = \mathrm{GSp}_4(\mathbf{Z}_q)$, $K_2 = \ker\big(\mathrm{GSp}_4(\mathbf{Z}_q) \to \mathrm{GSp}_4(\mathbf{F}_q)\big)$. Set $G = K_1/K_2 = \mathrm{GSp}_4(\mathbf{F}_q)$, and let $H = \mathrm{Iw}(q)/K_2 = B(\mathbf{F}_q) < G$. Write $M = N^{K_2}$, so that by assumption M is a finite projective $R[G]$-module, and we have $M^G = N^{\mathrm{GSp}_4(\mathbf{Z}_q)}$ and $M^H = N^{\mathrm{Iw}(q)}$.

By Nakayama's lemma and Lemma 7.4.11(3), we can and do assume from now on that $R = k$. Let $[\mathrm{GSp}_4(\mathbf{Z}_q)] \in \mathcal{H}$ be the Hecke operator which is the indicator function on $\mathrm{GSp}_4(\mathbf{Z}_q)$. By Lemma 7.4.11(2) we have

$$N^{\mathrm{GSp}_4(\mathbf{Z}_q)} = [\mathrm{GSp}_4(\mathbf{F}_q)]_{B(\mathbf{F}_q)}(N^{\mathrm{Iw}(q)}) = [\mathrm{GSp}_4(\mathbf{Z}_q)](N^{\mathrm{Iw}(q)}),$$

where the second equality follows from the very definition of the Hecke operators. We claim that (under our assumption that $R = k$) the Hecke operator $[\mathrm{GSp}_4(\mathbf{Z}_q)]$ is an inverse to pr_w.

To see this, note firstly that by the Bruhat decomposition we have

$$[\mathrm{GSp}_4(\mathbf{Z}_q)] = \sum_{w \in W} w$$

(where we are using the identification $\mathcal{H} \otimes_{\mathcal{O}} k = k[X_*(T) \rtimes W]$). We have the decomposition

$$(N^{\mathrm{Iw}(q)})_{\mathfrak{n}} = \oplus_{w \in W} (N^{\mathrm{Iw}(q)})_{\mathfrak{m}_w},$$

so we can write any $x \in (N^{\mathrm{Iw}(q)})_{\mathfrak{n}}$ as $x = \sum_{w \in W} x_w$ for unique elements $x_w \in (N^{\mathrm{Iw}(q)})_{\mathfrak{m}_w}$, and in particular if $x \in (N^{\mathrm{GSp}_4(\mathbf{Z}_q)})_{\mathfrak{n}}$ then $\mathrm{pr}_w(x) = x_w$. By the definition of the $\mathfrak{m}_w$, we see that the action of W on $(N^{\mathrm{Iw}(q)})_{\mathfrak{n}}$ is via $w_1 x_{w_2} = x_{w_1 w_2}$ for all $w_1, w_2 \in W$.

We can therefore compute that if $x \in (N^{\mathrm{GSp}_4(\mathbf{Z}_q)})_{\mathfrak{n}}$ then $[\mathrm{GSp}_4(\mathbf{Z}_q)] \circ \mathrm{pr}_w(x) = [\mathrm{GSp}_4(\mathbf{Z}_q)] x_w = \sum_{w' \in W} w' x_w = \sum_{w' \in W} x_{w'w} = x$, while if $y \in (N^{\mathrm{Iw}(q)})_{\mathfrak{m}_w}$ then $\mathrm{pr}_w \circ [\mathrm{GSp}_4(\mathbf{Z}_q)](y) = \mathrm{pr}_w\big(\sum_{w' \in W} w' y\big) = y$, as required. □

Remark 7.4.14. The second half of the proof of Lemma 7.4.12 is essentially identical to the proof of [Whi22, Prop. 5.10] in the special case that $G = \mathrm{GSp}_4$ and $P = B$. Indeed note that while it is assumed there that $p \nmid \#W$, all that is used in the proof is that $p \nmid \#W_L$, where L is the Levi factor of P; and for $P = B$ the group W_L is trivial.

We now establish Hypothesis 7.2.1 (4).

Lemma 7.4.15. *The natural pullback map $M_{K^p} \to M_{K_1^p(Q_N)}$ induces an isomorphism of $R_{\mathcal{S}_{Q_N}}$-modules $(M_N)_{\Delta_{Q_N}} \to M$.*

Proof. First, we have that

$$(M_N)_{\Delta_{Q_N}} = \mathrm{Hom}_{\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}}\big((M_{K_0^p(Q_N)} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2}, \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}\big)_{\mathfrak{m}^{Q_N},\mathfrak{m}_r,\mathfrak{m}_{Q_N}}$$

by part (4) of Lemma 7.4.11 and Lemma 7.4.7. We claim that the map

$$(M_{K^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2}_{\mathfrak{m},\mathfrak{m}_r} \to (M_{K_0^p(Q_N)} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2}_{\mathfrak{m}^{Q_N},\mathfrak{m}_r,\mathfrak{m}_{Q_N}}$$

is an isomorphism. It suffices to see (imposing that the central character acts by $|\cdot|^2$) that the map

$$(M_{K^p} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2, Q_N \cup \{r\}}_{\mathfrak{m},\mathfrak{m}_r} \to (M_{K_0^p(Q_N)} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2, Q_N \cup \{r\}}_{\mathfrak{m}^{Q_N},\mathfrak{m}_r,\mathfrak{m}_{Q_N}}$$

is an isomorphism. Here the subscript $(|\cdot|^2, Q_N \cup \{r\})$ indicates that the central character acts by $|\cdot|^2$ up to primes in $Q_N \cup \{r\}$ (as in Lemma 7.3.8).

Let $K(Q_N) = \prod_l K(Q_N)_l \subset \mathrm{GSp}_4(\mathbf{A}^{\infty,p})$ be defined by

$$\begin{cases} K^p(Q_N)_l = K_l^p & \text{if } l \notin Q_N, \\ K(Q_N)_l = \ker\big(\mathrm{GSp}_4(\mathbf{Z}_l) \to \mathrm{GSp}_4(\mathbf{F}_l)\big) & \text{if } l \in Q_N \end{cases}$$

By Lemma 7.3.8(1), $(M_{K(Q_N)} \otimes_{\mathcal{O}[\![T(\mathbf{Z}_p)]\!]} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}})^{|\cdot|^2, Q_N \cup \{r\}}_{\mathfrak{m}^{Q_N},\mathfrak{m}_r}$ is a finite projective $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[\prod_{q \in Q_N} \mathrm{GSp}_4(\mathbf{F}_q)]$-module. We conclude by Lemma 7.4.12. □

We summarize our results so far in the following proposition; we remind the reader that at this point we do not know that $M_{\mathfrak{q}}$ is nonzero. Indeed, even if we knew that ρ was modular, it could be that our choice of subgroups K_l for $l \in S$ forces $M_{\mathfrak{q}}$ to be zero. In Proposition 7.5.10 we will establish sufficient conditions under which $M_{\mathfrak{q}}$ is free of rank 1.

Proposition 7.4.16. *Assume that ρ satisfies Hypothesis 7.1.3, and let $\mathcal{S}$ be the deformation problem* (7.1.5) *with S as in Definition 7.4.2. Write $\mathfrak{q}$ for the kernel of the homomorphism $R_{\mathcal{S}} \to \mathcal{O}$ corresponding to ρ. Define M as in* (7.4.6)*. Then $M_{\mathfrak{q}}$ is a finite free $(R_{\mathcal{S}})_{\mathfrak{q}}$-module.*

Proof. By Lemmas 7.3.8 and 7.4.15, our construction of the modules M_N above gives the data of Hypothesis 7.2.1. The result is then immediate from Proposition 7.2.3. □

7.5. **Multiplicity one.** Before establishing our main multiplicity one results we begin with some background material and preliminary lemmas. We refer to Section 1.8.22 for the relationship between cuspidal automorphic representations π of $\mathrm{GSp}_4/\mathbf{Q}$ which are of general type, and cuspidal automorphic representations Π of $\mathrm{GL}_4/\mathbf{Q}$ of symplectic type.

Assume from now on that $F = \mathbf{Q}$, and that furthermore π has central character $\omega_\pi = |\cdot|^2$. We now recall some consequences of the theory of newforms due to Roberts and Schmidt [RS07]. (This theory assumes that we are working with representations of trivial central character, but this is harmless, as we can reduce to this case by twisting π by the everywhere unramified character $|\cdot|$.) Recall that for each prime l and each $n \geq 0$, the paramodular group of level n is

$$\mathrm{Par}(l^n) := \{g \in \mathrm{GSp}_4(\mathbf{Q}_l) \mid \nu(g) \in \mathbf{Z}_l^\times\} \cap \begin{pmatrix} \mathbf{Z}_l & \mathbf{Z}_l & \mathbf{Z}_l & l^{-n}\mathbf{Z}_l \\ l^n\mathbf{Z}_l & \mathbf{Z}_l & \mathbf{Z}_l & \mathbf{Z}_l \\ l^n\mathbf{Z}_l & \mathbf{Z}_l & \mathbf{Z}_l & \mathbf{Z}_l \\ l^n\mathbf{Z}_l & l^n\mathbf{Z}_l & l^n\mathbf{Z}_l & \mathbf{Z}_l \end{pmatrix}$$

We say that π_l is *paramodular* if $(\pi_l)^{\mathrm{Par}(l^n)} \neq 0$ for some n. The minimal such n is the *paramodular level* N_{π_l} of π_l.

The following result summarizes the facts that we need about paramodular vectors in cuspidal automorphic representations π of $\mathrm{GSp}_4/\mathbf{Q}$.

Proposition 7.5.1. *Suppose that π is of general type.*

1. *For each prime l, there is a unique paramodular representation in the L-packet containing π_l, namely the unique generic representation.*
2. *If π_l is generic, then $(\pi_l)^{\mathrm{Par}(l^{N_{\pi_l}})}$ is one-dimensional.*
3. *The paramodular level N_{π_l} coincides with the conductor of the corresponding L-parameter $\mathrm{rec}_{\mathrm{GT}}(\pi_l)$.*
4. *If π is regular algebraic and $\rho_{\pi,p}$ is irreducible, then N_{π_l} coincides with the conductor of $\rho_{\pi,p}|_{G_{\mathbf{Q}_l}}$.*

Proof. The first part is [Sch18, Thm. 1.1], the second is [RS07, Thm. 7.5.1], and the third is [JLRS23, Thm. 2.3.5]. The last claim is [BCGP21, Thm. 2.7.1(2)] (see e.g. [Ulm16] for the various equivalent definitions of the conductor of a Galois representation). □

For the following lemma we return to the setting of Section 6.1.

Lemma 7.5.2. *Suppose that $v \nmid p$ and that x_1, x_2 are two closed points of* $\operatorname{Spec} R_v^{\square}[1/p]$ *which lie on a common irreducible component, and are such that the corresponding lifts ρ_{x_1}, ρ_{x_2} of $\overline{\rho}|_{G_{F_v^+}}$ are both pure. Then we have the equality of conductors $a(\rho_{x_1}) = a(\rho_{x_2})$.*

Proof. Let x_i' be the image of x_i in the spectrum of the GL_4 lifting ring for $\overline{\rho}|_{G_{F_v^+}}$. These points lie on a common irreducible component by the assumption on x_1, x_2, and the purity of the ρ_{x_i} ensures that this is the unique irreducible component that either lies on (e.g. by [BG19, Cor. 3.3.4] and the definition of purity), The result follows immediately from [BLGGT14, Lem. 1.3.4(2)] (a lemma of Choi). □

We now establish some instances of solvable descent for GSp_4.

Proposition 7.5.3.

(1) *If Π is a $|\cdot|^2$-self dual regular algebraic cuspidal automorphic representation of $\mathrm{GL}_4/\mathbf{Q}$, then $(\Pi, |\cdot|^2)$ is of symplectic type.*

(2) *Let $F/\mathbf{Q}$ be a solvable Galois extension with F CM. Suppose that*

$$\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$$

has multiplier ε^{-1}, that $\rho|_{G_F}$ is irreducible, and that there is a RACSDC automorphic representation π_F of GL_4/F such that $\rho|_{G_F} \cong \rho_{\pi_F,p} \otimes \varepsilon$. Then there is a regular algebraic cuspidal automorphic representation π of $\mathrm{GSp}_4/\mathbf{Q}$ with central character $|\cdot|^2$, such that $\rho \cong \rho_{\pi,p}$.

(3) *Let $F^+/\mathbf{Q}$ be a solvable Galois extension with F^+ totally real. Suppose that $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ has multiplier ε^{-1}, that $\rho|_{G_{F^+}}$ is irreducible, and that there is a regular algebraic cuspidal automorphic representation π_{F^+} of GSp_4/F^+, with central character $|\cdot|^2$, and such that $\rho|_{G_{F^+}} \cong \rho_{\pi_{F^+},p}$.*

Then there is a regular algebraic cuspidal automorphic representation π of $\mathrm{GSp}_4/\mathbf{Q}$ with central character $|\cdot|^2$, such that $\rho \cong \rho_{\pi,p}$.

Proof. We begin with part (1). If the pair $(\Pi, |\cdot|^2)$ is not of symplectic type, then it is of orthogonal type, so it descends to an automorphic representation π^α of some $\mathrm{GSpin}_4^\alpha/\mathbf{Q}$, with central character $|\cdot|^2$. The central character of π^α_∞ can be read off from its L-parameter. Under our assumption that Π is regular algebraic (i.e. C-algebraic), it follows from [Pat15, Lem. 3.2(3), 3.4] that the central character must be odd, which means in particular that it cannot equal $|\cdot|^2$. This contradiction implies that $(\Pi, |\cdot|^2)$ is of symplectic type, as claimed.

By part (1), in each of parts (2) and (3) it suffices to show that there is a $|\cdot|^2$-self dual regular algebraic cuspidal automorphic representation Π of $\mathrm{GL}_4/\mathbf{Q}$ with $\rho \cong \rho_{\Pi,p}$. Indeed by (1) such a Π is of symplectic type, and we can take π to be a descent of Π. Then part (2) is a standard consequence of solvable descent for GL_4, and in particular is a special case of [BLGGT14, Lem. 2.2.2] (bearing in mind [BLGGT14, Lem. 2.2.1], which takes care of the twist by ε). Finally for part (3), since $\rho_{\pi_{F^+},p}$ is irreducible, we see that π_{F^+} is of general type. Its transfer Π_{F^+} is $|\cdot|^2$-self dual, and the result follows from another application of [BLGGT14, Lem. 2.2.2]. □

We now return to our running hypotheses, so that $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathcal{O})$ is a continuous representation satisfying Hypothesis 7.1.3, r is a fixed neat prime for $\overline{\rho}$, and $\mathcal{S}$ is the deformation problem (7.1.5) with S as in Definition 7.4.2. Write $\mathfrak{q}$ for the kernel of the homomorphism $R_{\mathcal{S}} \to \mathcal{O}$ corresponding to ρ.

Lemma 7.5.4. *Every irreducible component of* $\operatorname{Spec} R_{\mathcal{S}}$ *containing* $\mathfrak{q}$ *has dimension at least* $\dim \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$.

Proof. We claim there is a presentation

$$R_p^{\triangle}[\![x_1,\dots,x_r]\!]/(f_1,\dots,f_{r+14}) \xrightarrow{\sim} R_{\mathcal{S}}$$

for some $r \geq 0$. Indeed since $\{p\} \subsetneq S$, the map [Bal12, (4.2.1)] is injective, and the existence of such a presentation is a consequence of [Bal12, Prop. 4.2.5, Rem. 4.2.6]. Now by Hypothesis 7.1.3 (6) and Lemma 6.2.2 (3), $\rho|_{G_{\mathbf{Q}_p}}$ lies on a unique irreducible component of $\operatorname{Spec} R_p^{\triangle}$ which has dimension $\dim \Lambda_{\mathrm{GSp}_4,\mathbf{Q}} + 14$ and the result follows.

□

We now put ourselves in the setting of Section 7.4, and if $l \in S \smallsetminus \{r\}$, then we take

$$K_l = \mathrm{Par}(l^{a(\rho|_{G_{\mathbf{Q}_l}})}). \tag{7.5.5}$$

Define M as in (7.4.6). Let $\mathfrak{q}'$ be the kernel of a homomorphism $R_{\mathcal{S}} \to \mathcal{O}$ corresponding to a lift $\rho' : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathcal{O})$ of $\overline{\rho}$, and assume that $\mathfrak{q}'$ and $\mathfrak{q}$ lie on a common irreducible component of $\operatorname{Spec} R_{\mathcal{S}}[1/p]$ (in particular, we could take $\mathfrak{q}' = \mathfrak{q}$, but we will also consider other choices below). Then by the definition of M, we have

$$\dim_E (M/\mathfrak{q}'M)[1/p] = \dim_E \left((M_{K^p})_{\mathfrak{m}^p,\mathfrak{m}_r,|\cdot|^2} \otimes_{\mathcal{O}} E \right)[\mathfrak{q}']. \tag{7.5.6}$$

Definition 7.5.7. If $R \in \mathbf{Z}_{\geq 1}$, we say that a $\overline{\mathbf{Q}}_p$-point of $\operatorname{Spec} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$ is of R-*regular classical weight* if the corresponding characters $\theta_1, \theta_2 : I_{\mathbf{Q}_p} \to \overline{\mathbf{Q}}_p^{\times}$ are algebraic with respective Hodge–Tate weights h_1, h_2 satisfying $h_2 - h_1, 1 - 2h_2 \geq R$.

Lemma 7.5.8. *For all sufficiently large R, if the point of* $\operatorname{Spec} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$ *determined by* $\mathfrak{q}'$ *is of R-regular classical weight, and* $M_{\mathfrak{q}'}$ *is nonzero, then* $\dim_E (M/\mathfrak{q}'M)[1/p] = 1$.

Proof. We claim that $\dim_E (M/\mathfrak{q}'M)[1/p]$ is equal to

$$\sum_{\pi} \dim_E (\pi^{\infty})^{K^p \mathrm{Iw}(p), U_{p,1}=\alpha'_p, U_{p,2}=\alpha'_p\beta'_p, \beta_{r,1}=\alpha'_{r,1}, \beta_{r,2}=\alpha'_{r,1}\alpha'_{r,2}} \tag{7.5.9}$$

where the sum is over the cuspidal automorphic representations π of weight determined by $\mathfrak{q}'$ with central character $|\cdot|^2$, with π_∞ respectively holomorphic if $l(w) = 0$ or 3, and generic if $l(w) = 1$ or $l(w) = 2$, and which satisfy $\rho_{\pi,p} \cong \rho'$; and α'_p, β'_p are the lifts of $\overline{\alpha}_p, \overline{\beta}_p$ determined by $\mathfrak{q}'$, and similarly for $\alpha'_{r,1}, \alpha'_{r,2}$ (where the $\beta_{r,i}$ act as in (7.4.4)). Indeed (7.5.6) and Theorem 7.3.1 (2) reduce this to the corresponding assertion about the (cuspidal or otherwise) coherent cohomology of Shimura varieties, which holds by a standard argument using [Har90, BHR94, HZ01]. More precisely, the results of Harris–Zucker allow us to reduce to the case of interior cohomology, and the argument is then identical to that of the proof of [BCGP21, Thm. 3.10.1].

Since $\overline{\rho}$ is absolutely irreducible, so is ρ', so any such π is of general type. We need to show that there is a unique π with a nonzero contribution to (7.5.9), and that this contribution is 1. By strong multiplicity one, it suffices to show that for each prime l, there is a unique π_l in the L-packet corresponding to $\rho'|_{G_{\mathbf{Q}_l}}$ which contributes, and that it contributes with multiplicity one. For $l \neq p, r$, this follows from our choice of K^p, together with Proposition 7.5.1 and Lemma 7.5.2. For $l = p$ it follows from ordinarity and the assumption that $\mathfrak{q}'$ is in R-regular classical weight that π_p is an irreducible unramified principal series representation, and that the simultaneous eigenspaces for the $U_{p,i}$-operators are 1-dimensional (see [BCGP21, Prop. 2.4.24, 2.4.26]). Finally for $l = r$, since ρ' and ρ lie on a common irreducible component of $\operatorname{Spec} R_{\mathcal{S}}[1/p]$, we know that $\rho|_{G_{\mathbf{Q}_r}}$ has unipotent ramification, so π_r has $\mathrm{Iw}(r)$-fixed vectors. Since $\overline{\rho}(G_{\mathbf{Q}_r})$ is regular semi-simple, we again conclude that the simultaneous $\beta_{r,1}, \beta_{r,2}$-eigenspaces are 1-dimensional, as required. □

We now prove our multiplicity one criterion. For the convenience of the reader, we incorporate our running hypotheses into the statement of the result.

Proposition 7.5.10. *Suppose that $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathcal{O})$ satisfies the following conditions.*

(1) ρ is unramified at all but finitely many primes.
(2) $\nu \circ \rho = \varepsilon^{-1}$.
(3) $\rho(G_{\mathbf{Q}(\zeta_{p^\infty})})$ is integrally enormous.
(4) ρ is pure.
(5) $\rho|_{G_{\mathbf{Q}_p}}$ is ordinary, semistable of weight 2, and p-distinguished.

Suppose furthermore that either $p = 2$, and there is a solvable CM extension $F/\mathbf{Q}$ and an ordinary RACSDC automorphic representation π of GL_4/F such that:

(A1) $\overline{r}_{\pi,2} \cong \overline{\rho}|_{G_F}$.
(A2) $\overline{\rho}(G_F)$ is nearly adequate.
(A3) $\overline{\rho}(G_F)$ contains a regular semi-simple element.
(A4) There exists an infinite place v of F^+ such that the polarized pair $(\overline{\rho}|_{G_F}, 1)$ is strongly residually odd at v.
(A5) $\overline{\rho}(G_{\mathbf{Q}(i)}) = \overline{\rho}(G_{\mathbf{Q}})$.

Or alternatively, suppose that $p > 2$, and there exists an ordinary cuspidal automorphic representation π of $\mathrm{GSp}_4/\mathbf{Q}$ with central character $|\cdot|^2$ such that:

(B1) $\overline{\rho}_{\pi,p} \simeq \overline{\rho}$.
(B2) $\overline{\rho}$ is GSp_4-reasonable, in the sense of [Whi22, Defn. 3.19].
(B3) $\overline{\rho}$ is tidy, in the sense of [BCGP21, Defn. 7.5.11].
(B4) $\overline{\rho}(G_{\mathbf{Q}}) \smallsetminus \mathrm{Sp}_4(\mathbf{F}_p)$ contains a regular semi-simple element.
(B5) There is a compatible choice of p-stabilizations of π_p and $\rho|_{G_{\mathbf{Q}_p}}$ such that $\rho_{\pi,p}|_{G_{\mathbf{Q}_p}}$ lies on a unique component of $\operatorname{Spec} R_p^{\triangle}$ and $\rho|_{G_{\mathbf{Q}_p}}$ lies on the same component.

Let r be a neat prime for $\overline{\rho}$, and define $S, \mathcal{S}, M$ as in Definition 7.4.2 and (7.1.5), (7.4.6) *respectively, where we make the choice of level structure* (7.5.5). *Let $\mathfrak{q}$ be the prime of $R_{\mathcal{S}}$ determined by ρ. Then $M_{\mathfrak{q}}$ is a free $(R_{\mathcal{S}})_{\mathfrak{q}}$-module of rank* 1.

Proof. Note firstly that in either case ρ satisfies Hypothesis 7.1.3 (with F^+ there equal to $\mathbf{Q}$), because $\overline{\rho}$ is absolutely irreducible by whichever of (A2) and (B2) applies, and if $p = 2$ then $\overline{\rho}(G_{\mathbf{Q}(i)}) = \overline{\rho}(G_{\mathbf{Q}})$ by (A5). We showed in Section 7.4 that the $R_{\mathcal{S}}$-module M is part of a set of data satisfying Hypothesis 7.2.1, so $M_{\mathfrak{q}}$ is a free $(R_{\mathcal{S}})_{\mathfrak{q}}$-module by Proposition 7.2.3.

Recall that we have a surjection $R_{\mathcal{S}} \to \mathbf{T}_{\mathcal{S}}$, where the Hecke algebra $\mathbf{T}_{\mathcal{S}}$ is defined in Definition 7.4.9, and acts faithfully on M by definition. We next show in the case $p = 2$ that the induced map $R_{\mathcal{S}}^{\mathrm{red}}[1/p] \to \mathbf{T}_{\mathcal{S}}[1/p]$ is an isomorphism. In the case $p > 2$ we prove the weaker statement that the image of the corresponding map on spectra contains an irreducible component containing $\mathfrak{q}$. In either case it follows that $M_{\mathfrak{q}}$ is nonzero, and we will conclude by showing that its rank is one.

We begin with the case $p = 2$. After possibly replacing F^+ with a quadratic extension, we can and do assume that all places of F^+ lying over S split in F, as well as any place lying under a place at which π is ramified. After making a further solvable extension of totally real fields, we can furthermore assume that F/F^+ is everywhere unramified and for each place $v|2$ of F, $\overline{\rho}|_{G_{F_v}}$ is trivial and we have $[F_v : \mathbf{Q}_2] > 7$; so by Remark 5.6.7 restriction induces a map from the deformation problem $\mathcal{D}_2^{\triangle}$ of Section 6.2 to the deformation problems $\mathcal{D}_v^{\triangle}$ of Proposition 5.6.6. Let T be the set of places of F^+ which lie over places in S, and let $R^{T,\mathrm{ord}}$ be the global deformation problem for $\overline{\rho}|_{G_{F^+}}$ defined in Definition 5.7.11. This is by definition a Λ-algebra, where Λ is as in (5.7.3). Then we have a morphism of Λ-algebras $R^{T,\mathrm{ord}} \to R_{\mathcal{S}}$ (the Λ-algebra structure on $R_{\mathcal{S}}$ comes from the natural map $\Lambda_{\mathrm{GSp}_4,F^+} \to \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$), defined by applying the construction of Corollary 5.1.6 to $(\rho\otimes\varepsilon^{-1})|_{G_{F^+}}$. (Note that this construction indeed gives a morphism of deformation problems rather than just framed deformation problems, because conjugating a lift of $\overline{\rho}$ by a matrix $B \in \widehat{\mathrm{GSp}_4}(R)$ corresponds to conjugating the corresponding lift of $r_{\overline{\rho}|_{G_{F^+}}}$ by $(B, \nu\circ(B)) \in \widehat{\mathcal{G}}_4^{\circ}(R)$.)

By our assumptions, we can apply Theorem 5.7.14, and conclude in particular that $R^{T,\mathrm{ord}}$ is a finite Λ-algebra. We have the commutative diagram

$$\begin{array}{ccccc} \Lambda & \longrightarrow & R^{T,\mathrm{ord}} & & \\ \downarrow & & \downarrow & & \\ \Lambda_{\mathrm{GSp}_4,\mathbf{Q}} & \longrightarrow & R_{\mathcal{S}} & \longrightarrow & \mathbf{T}_{\mathcal{S}}, \end{array}$$

and since $R^{T,\mathrm{ord}} \to R_{\mathcal{S}}$ is a finite morphism (by a standard argument exactly as in the proof of [BLGGT14, Lem. 1.2.3]), we deduce that $R_{\mathcal{S}}$ is a finite Λ-algebra, and thus a finite $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$-algebra. Combining this finiteness with Lemma 7.5.4, it follows that every irreducible component of $\operatorname{Spec} R_{\mathcal{S}}[1/2]$ dominates an irreducible component of $\operatorname{Spec}\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}[1/2]$, and thus that there is a dense set of closed points of $\operatorname{Spec} R_{\mathcal{S}}[1/2]$ of R-regular classical weight. Let $\rho' : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_2)$ be the lift of $\overline{\rho}$ corresponding to such a point; then Theorem 5.7.14 shows that $\rho'|_{G_F}$ is automorphic. By solvable descent (see Proposition 7.5.3 (2)), we deduce that ρ' corresponds to a point of $\operatorname{Spec}\mathbf{T}_{\mathcal{S}}$. Since this applies to a dense set of points, we see that the surjection $R_{\mathcal{S}}^{\mathrm{red}}[1/2] \to \mathbf{T}_{\mathcal{S}}[1/2]$ is an isomorphism, as claimed.

We now turn to the case $p > 2$. We choose a solvable totally real extension $F^+/\mathbf{Q}$ disjoint from $\overline{\mathbf{Q}}^{\ker\overline{\rho}}$, in which p splits completely, and so that for every prime $l \neq p$ for which either $\rho|_{G_{\mathbf{Q}_l}}$ or $\rho_{\pi,p}|_{G_{\mathbf{Q}_l}}$ ramifies, if $v \mid l$ is a prime of F^+, then:

- $q_v \equiv 1 \pmod p$, and if $p = 3$, then $q_v \equiv 1 \pmod 9$
- $\overline{\rho}|_{G_{F_v^+}}$ is trivial.
- If ρ' is any lift of $\overline{\rho}|_{G_{\mathbf{Q}_q}}$ then $\rho'|_{G_{F_v^+}}$ is unipotently ramified. (In particular $\rho|_{G_{F_v^+}}$ and $\rho_{\pi,p}|_{G_{F_v^+}}$ are unipotently ramified.)

For the third point, see Lemma 5.6.2 which we stated for $p = 2$, but whose proof makes no use of this assumption. We put ourselves in the setting of Section 6.3 with the automorphic representation π there being the base change π_{F^+} of our π

to F^+, and R the set of places lying above the primes $l \neq p$ where either $\rho|_{G_{\mathbf{Q}_l}}$ or $\rho_{\pi,p}|_{G_{\mathbf{Q}_l}}$ ramifies. Then Hypothesis 6.3.1 is satisfied.

We have a diagram

$$\begin{array}{ccccc} \Lambda_{\mathrm{GSp}_4,F^+} & \longrightarrow & R_{\mathcal{S}_1} & & \\ \downarrow & & \downarrow & & \\ \Lambda_{\mathrm{GSp}_4,\mathbf{Q}} & \longrightarrow & R_{\mathcal{S}} & \longrightarrow & \mathbf{T}_{\mathcal{S}} \end{array}$$

where $R_{\mathcal{S}}$ is a finite $R_{\mathcal{S}_1}$-algebra (again, this follows exactly as in [BLGGT14, Lem. 1.2.3], using Lemma 7.4.8 in place of [CHT08, Lem. 2.1.12]).

Consider a minimal prime Q of $R_{\mathcal{S}}$ contained in the prime corresponding to ρ. By assumption (B5), the map $R_{\mathcal{S}_1} \to R_{\mathcal{S}}/Q$ factors through the quotient $R_{\mathcal{S}_1,\pi_{F^+}}$ defined in section 6.3, which is finite over $\Lambda_{\mathrm{GSp}_4,F^+}$ by Theorem 6.3.4. It follows that $R_{\mathcal{S}_1}/Q$ is finite over $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$. As it also has dimension at least that of $\Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$ by Lemma 7.5.4, it follows that $\operatorname{Spec} R_{\mathcal{S}_1}/Q \to \operatorname{Spec} \Lambda_{\mathrm{GSp}_4,\mathbf{Q}}$ is surjective. Arguing exactly in the previous case we use Theorem 6.3.4 and Proposition 7.5.3 (3) at a dense set of closed points of R-regular classical weight to show that Q contains the kernel of $R_{\mathcal{S}} \to \mathbf{T}_{\mathcal{S}}$, and consequently $M_{\mathfrak{q}}$ is nonzero, as claimed.

It remains to show that $M_{\mathfrak{q}}$ is of rank 1. Let η be the generic point of any irreducible component of $\operatorname{Spec} R_{\mathcal{S}}[1/p]$ containing ρ. Since $M_{\mathfrak{q}}$ is a nonzero free $(R_{\mathcal{S}})_{\mathfrak{q}}$-module, it suffices to show that M_η has rank 1 over $(R_{\mathcal{S}})_\eta$. Since the rank can only increase under specialization, it suffices to show that there is some other $\rho' : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ corresponding to a closed point of the component determined by η, given by an ideal $\mathfrak{q}'$ such that $M_{\mathfrak{q}'}$ is free of rank 1 over $(R_{\mathcal{S}})_{\mathfrak{q}'}$.

To see this, note that we have seen above that $\operatorname{Spec} R_{\mathcal{S}}[1/p]$ has a dense set of closed points of R-regular classical weight. Let $\mathfrak{q}'$ correspond to such a point (on the component η); then we are done by Lemma 7.5.8. □

Theorem 7.5.11. *Suppose that ρ satisfies the hypotheses of Proposition 7.5.10, and that in addition either:*

(1) the Zariski closure of $\rho(G_{\mathbf{Q}})$ contains Sp_4; or
(2) the Zariski closure of $\rho(G_{\mathbf{Q}})$ contains $\mathrm{SL}_2 \times \mathrm{SL}_2$, and ρ is irreducible but becomes reducible on some index two subgroup G_E.

Then ρ is modular.

Proof. This is immediate from Proposition 7.5.10 and (7.5.6) (with $\mathfrak{q}' = \mathfrak{q}$), Theorem 7.3.1 (3) and Theorem 4.12.4, bearing in mind Remark 4.12.5. □

8. A 2-adic modularity theorem for abelian surfaces

In this section, we prove a modularity theorem (Theorem 8.3.2) for abelian surfaces $A/\mathbf{Q}$ which are ordinary at 2 and whose mod-2 representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_2) \simeq S_6$ has a very particular form. Specifically, we demand that the image Γ of $G_{\mathbf{Q}}$ contains a copy of $A_5 \subset \mathrm{GSp}_4(\mathbf{F}_2)$ with index at most two which acts absolutely irreducibly on $\mathbf{F}_2^4$, and additionally require that the image of complex conjugation is non-trivial and lands in $A_5 \subseteq \Gamma$. After passing to the totally real field F^+ (at most quadratic) such that $\overline{\rho}|_{G_{F^+}}$ has image A_5, we may identify $\overline{\rho}$ with the symmetric cube of a 2-dimensional representation with image A_5, and this allows us to deduce that $\overline{\rho}$ is residually modular (in regular weight) using known cases of the Artin conjecture for totally real fields together with symmetric cube functoriality. In light of our previous modularity lifting theorems (in particular Theorem 7.5.11), the remaining work required to show that A is modular is to show that the representation $\overline{\rho}$ is nearly adequate in the sense of Definition 5.3.3 and strongly residually odd in the sense of Definition 5.2.1. Using our results from §5.2 and §5.3, this reduces to some facts concerning the modular representation theory of A_5 in characteristic 2.

In §8.1, we recall some standard facts about 2-torsion of abelian surfaces and fix once and for all a choice of isomorphism $S_6 \to \mathrm{GSp}_4(\mathbf{F}_2)$. In §8.2, we carry out the necessary group-theoretic arguments concerning the mod-2 representation theory of A_5. Finally, in §8.3, we prove that the representations $\overline{\rho}$ we are considering are residually modular (although not *a priori* in singular weight), and then use Theorem 7.5.11 to prove the desired modularity theorem.

8.1. **The 2-torsion of an abelian surface.** We begin by recalling some standard facts concerning the relationship between Weierstrass points on a genus two curve and the 2-torsion on its Jacobian. One source for the facts cited in this section is the introduction to [BFvdG08].

Let A be the Jacobian of a genus two curve X over a field of characteristic $\neq 2$. There is a Weil pairing on $A[2]$ which defines a symplectic pairing $\langle , \rangle$. If one denotes the Weierstrass points (over the algebraic closure) by r_i for $i = 1, \ldots, 6$, then for $i \neq j$ the element $r_i - r_j$ has order 2, and is thus a non-trivial element of $A[2]$. The 2-torsion points $r_i - r_j$ for $i < j$ are distinct, and they are precisely the nonzero elements of $A[2]$. Moreover, with respect to the Weil pairing, one has:

$$\langle r_i - r_j, r_k - r_l \rangle = \#\{i, j\} \cap \{k, l\} \pmod 2 \tag{8.1.1}$$

for $i \neq j$, $k \neq l$.

In [BPP+19, 5.1], the following identification $\iota : S_6 \xrightarrow{\sim} \mathrm{Sp}_4(\mathbf{F}_2) = \mathrm{GSp}_4(\mathbf{F}_2)$ is given: let $U := \mathbf{F}_2^6$ with the bilinear form $\langle x, y \rangle = \sum_{i=1}^6 x_i y_i$, let $U^0 \subset U$ denote the trace zero subspace, and let L be the span of $(1, 1, \ldots, 1) \in U^0$. Let S_6 act in the obvious way on $= \mathbf{F}_2^6$. Then $A[2] \simeq U^0/L$ where the Weil pairing is the pairing inherited from U. To see that this isomorphism is compatible with the action on the Weierstrass points, it suffices to identify U with the $\mathbf{F}_2$-space generated by r_i for $i = 1, \ldots, 6$. Certainly the $r_i - r_j$ land in U^0, and so it suffices to show that the divisor $\sum(r_i)$ is congruent modulo 2 to a principal divisor. If one writes an affine model for X as $y^2 = (x - r_1)(x - r_2) \cdots (x - r_6)$ and $\{1, 2, 3, 4, 5, 6\} = \{i, j, k\} \cup \{i', j', k'\}$ is any partition, then

$$r_i - r_{i'} + r_j - r_{j'} + r_k - r_{k'} = (x - r_i) + (x - r_j) + (x - r_k) - (y)$$

is principal. Finally, the compatibility of the Weil pairing is a consequence of equation (8.1.1).

Under this identification, there are two conjugacy classes of subgroup $S_5 \subset S_6$, which one can denote $S_5(a)$ and $S_5(b)$, where $S_5(b)$ is the subgroup which has a fixed point (we use here the same notation as [BPP+19, §5.1]), that is, $S_5(b)$ is the standard copy of S_5 in S_6 (and below $A_5(b)$ denotes the copy of A_5 in $S_5(b)$). It follows that X has a rational Weierstrass point, if and only if $\overline{\rho}_{A,2}$ factors through a conjugate of $S_5(b)$.

The S_6-representation U is the natural permutation representation. Hence U as an $A_5(b)$ representation is also the direct sum of the trivial representation and the standard representation. The Brauer character of U satisfies

$$\chi(1) = 5 + 1, \quad \chi((1,2,3,4,5)) = \chi((1,3,5,2,4)) = 0 + 1, \quad \chi((1,2,3)) = 2 + 1.$$

Consequently, if we let $V = A[2]$ as an A_5-representation, the Brauer character of V is

$$\chi(1) = 4, \quad \chi((1,2,3,4,5)) = \chi((1,3,5,2,4)) = -1, \quad \chi((1,2,3)) = 1.$$

Lemma 8.1.2. *The representation $V \otimes \overline{\mathbf{F}}_2$ is the unique irreducible modular representation of A_5 over $\overline{\mathbf{F}}_2$ of dimension 4.*

Proof. This follows directly from the Brauer character table of A_5; see Lemma 8.2.1. □

We can make the identification $\iota : S_6 \xrightarrow{\sim} \mathrm{Sp}_4(\mathbf{F}_2) = \mathrm{GSp}_4(\mathbf{F}_2)$ completely explicit:

Lemma 8.1.3. *An explicit isomorphism $S_6 \to \mathrm{GSp}_4(\mathbf{F}_2)$ is given by:*

$$(12)(34)(56) \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (12) \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(12345) \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix},$$

Proof. Let $e_1 = r_1 - r_2$, $e_2 = r_3 - r_4$, $e_3 = r_3 - r_5$, and $e_4 = r_1 - r_6$. Then the $e_i \in U^0$ span $U^0/L \simeq A[2]$, and the corresponding Weil pairing agrees with our usual choice of symplectic form J. □

8.2. **The modular representations of A_5.** We now establish some easy group-theoretic lemmas and also prove some facts concerning mod-2 representations of A_5. Everything here is elementary, but is included for completeness. We begin by describing the irreducible modular representations of A_5 in characteristic 2.

Lemma 8.2.1. *Let k be a subfield of $\overline{\mathbf{F}}_2$ which contains $\mathbf{F}_4$. Then the irreducible representations of A_5 over k are as follows; moreover, these representations are all absolutely irreducible, and in particular all absolutely irreducible representations of A_5 over $\overline{\mathbf{F}}_2$ are defined over $\mathbf{F}_4$.*

(1) *The trivial representation k.*
(2) *A two-dimensional representation U obtained by choosing an identification $A_5 \simeq \mathrm{SL}_2(\mathbf{F}_4)$ and then taking the tautological representation of $\mathrm{SL}_2(\mathbf{F}_4)$ over k.*
(3) *The conjugate U^σ of U by $\mathrm{Gal}(\mathbf{F}_4/\mathbf{F}_2)$ acting on k.*
(4) *A four-dimensional representation V which is defined over $\mathbf{F}_2$, which may be identified with $U \otimes U^\sigma$, and also with $\mathrm{Sym}^3(U)$. This lifts to the unique irreducible representation of A_5 in characteristic zero of dimension four. The representation V has a regular semi-simple element of order 5.*

Furthermore: there are exactly two blocks of A_5, consisting of the trivial block and a block of defect zero consisting only of V. In particular, V defines a projective module for $k[A_5]$. The Brauer character table of A_5 is given as follows (where ζ is a 5th root of unity):

The Brauer character table of A_5

A_5	*dim*	$(1,2,3,4,5)$	$(1,3,5,2,4)$	$(1,2,3)$
k	1	1	1	1
U	2	$\zeta+\zeta^{-1}$	$\zeta^2+\zeta^{-2}$	-1
U^σ	2	$\zeta^2+\zeta^{-2}$	$\zeta+\zeta^{-1}$	-1
V	4	-1	-1	1

Proof. The group A_5 has 4 conjugacy classes of order prime to 2, and thus has 4 distinct irreducible representations over k. The trace of $(1,2,3,4,5)$ on U is $\zeta + \zeta^{-1}$. For the Brauer character table, see [Kar95, Ch 4, Example 8.5] or [Ser77a, Example 18.6]. From this table, the identifications $V = U \otimes U^\sigma$ and $V = \mathrm{Sym}^3(U)$ follow. (One can also deduce these identifications from the Steinberg tensor product theorem.) The facts concerning the blocks can be read off from the decomposition matrix and Cartan matrix given in [Kar95, Ch 4, Example 8.5]. The projectivity of V is also immediate from [Ser77a, Prop. 46], since $4 = \dim(V)$ is the largest power of 2 dividing $|A_5| = 4 \cdot 15$. The fact that the order 5 elements act with distinct eigenvalues on V is also apparent from the character table. □

Lemma 8.2.2. *The representation V of $G = A_5$ is nearly adequate (in the sense of Definition 5.3.3).*

Proof. By definition, we need to show that

(1) V is weakly adequate.
(2) $H^1(G,k) = 0$.
(3) $H^1(G, \mathrm{ad}\, V) = 0$.

The first claim follows directly from [GHT17, Prop 9.1] since $A_5 = \mathrm{SL}_2(\mathbf{F}_4)$ and $4 = 2^2 > 3$. (We also give a simple direct proof in Lemma 8.2.9 below.) The second claim is immediate from the fact that A_5 is perfect. For the third claim, recall that V is projective, and thus $\mathrm{ad}\, V \simeq V \otimes V$ is also projective (e.g. using that a projective module is a direct summand of a free module, and tensor products commute with direct sums and preserve freeness.) Hence we deduce that $H^n(G, \mathrm{ad}\, V) = 0$ for $n > 0$. □

Remark 8.2.3. From the exact sequence $0 \to k \to \mathrm{ad} \to \mathrm{ad}/k \to 0$, we deduce that $H^1(G, \mathrm{ad}/k) \simeq H^2(G,k) \simeq k$ since the Schur multiplier of A_5 is $\mathbf{Z}/2\mathbf{Z}$. Thus V is not adequate in the sense of [Tho17, Defn. 2.20].

Lemma 8.2.4. *Suppose that F^+ is totally real and that $\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(\mathbf{F}_2)$ is an absolutely irreducible representation with image $G = A_5(b)$. If v is an infinite place such that $\overline{\rho}(c_v)$ is non-trivial, then $(\overline{\rho}|_{G_F}, 1)$ is strongly residually odd at v in the sense of Definition 5.2.1.*

Proof. non-trivial involutions A in $A_5(b) \subset \mathrm{GSp}_4(\mathbf{F}_2) \simeq S_6$ are characterized by being squares of order 4 elements. This is most obvious by thinking about conjugacy classes in S_6 and noting that non-trivial involutions in A_5 have the cycle shape $(**)(**)$; this conjugacy class is also preserved by the outer automorphism so this description does not depend on any choice of isomorphism from $\mathrm{GSp}_4(\mathbf{F}_2)$ to S_6. Judiciously choosing a suitable order 4 element σ of $\mathrm{GSp}_4(\mathbf{F}_2)$, we find that any such A is conjugate in $\mathrm{GSp}_4(\mathbf{F}_2)$ to

$$A \sim \sigma^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I_2 & J_2 \\ 0 & I_2 \end{pmatrix},$$

so that in the notation of Section 5.2.4, $S_2 J_2 = J_2 \cdot J_2 = I_2$ which is manifestly not alternating. (In the explicit isomorphism $S_6 \simeq \mathrm{GSp}_4(\mathbf{F}_2)$ of Lemma 8.1.3, we have $\sigma = (1423)(56)$ and $\sigma^2 = (12)(34)$.) Hence the result follows from Lemma 5.2.7. (See also Remark 5.2.8.) $\square$

Proposition 8.2.5. *Suppose that F^+ is a totally real field, and that $\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(\mathbf{F}_2)$ has image $A_5(b)$. Suppose that there is an infinite place v of F^+ such that $\overline{\rho}(c_v) \neq 1$. Then for any imaginary CM quadratic extension F/F^+, the polarized pair $(\overline{\rho}|_{G_F}, 1)$ determined by $\overline{\rho}$ is nearly adequate and strongly residually odd at v. Furthermore, $\overline{\rho}(G_F)$ contains a regular semi-simple element.*

Proof. The representation of $G = A_5(b)$ in $\mathrm{GSp}_4(\overline{\mathbf{F}}_2)$ is irreducible and so coincides with the representation V of dimension 4 in Lemma 8.2.1 part (4), and so in particular has a regular semi-simple element of order 5. By Lemma 8.2.2, V is nearly adequate. Since G is perfect, we have $\overline{\rho}(G_F) = G$, so the polarized pair $(\overline{\rho}|_{G_F}, 1)$ is indeed nearly adequate. Finally, it is strongly residually odd at v by Lemma 8.2.4. $\square$

We end this section with our promised direct proof that V is weakly adequate. Recall firstly that if W is a representation of a finite group G over a field k of characteristic 2, then in addition to the usual short exact sequence

$$0 \to \wedge^2 W \to W \otimes W \to \mathrm{Sym}^2 W \to 0, \tag{8.2.6}$$

there is a short exact sequence

$$0 \to W(1) \to \mathrm{Sym}^2 W \to \wedge^2 W \to 0, \tag{8.2.7}$$

where the first map is the inclusion of the subspace spanned by the $x \otimes x$ for $x \in W$, and the second map is the one induced by $x \otimes y \mapsto x \wedge y$. We can and do identify $W(1)$ with the Frobenius twist of W.

Lemma 8.2.8. *The socle of $V \otimes V$ is $k \oplus V$.*

Proof. Consider (8.2.6) and (8.2.7) with $G = A_5$ and $W = V$. Then $V = V(1)$, and since V is projective, we see that $V \otimes V$ splits as a direct sum of V and an extension of $\wedge^2 V$ by itself. Since U, U^σ and consequently V are all self-dual, and since $V \otimes V \cong \mathrm{Hom}(V, V)$ contains exactly one copy of k in its socle by Schur's lemma, it suffices to prove that the socle of $\wedge^2 V$ is k.

Now considering (8.2.6) and (8.2.7) with $W = U$ we see that $U \otimes U$ admits a filtration with successive graded pieces k, U^σ, k. Similarly, $U^\sigma \otimes U^\sigma$ has a filtration with graded pieces k, U, k. Tensoring these together, we see that the Jordan–Hölder factors (with multiplicity) of $\wedge^2 V$ are k, k, U, U^σ.

Since $\wedge^2 V$ is $\mathrm{Gal}(\mathbf{F}_4/\mathbf{F}_2)$-invariant, if either U or U^σ occurs in the socle of $\wedge^2 V$, then they both do. Since $\wedge^2 V$ is self dual, however, if U and U^σ appear in the socle of $\wedge^2 V$, then they also appear in the cosocle, and therefore occur as direct summands. If this occurs, then since $H^1(A_5, k) = 0$, the representation $\wedge^2 V$ would be semi-simple, contradicting the presence of exactly one copy of k in its socle. This contradiction completes the proof. □

The following lemma gives our second proof that V is weakly adequate (the first was in the proof of Lemma 8.2.2).

Lemma 8.2.9. *The representation V of A_5 is weakly adequate.*

Proof. Let M be the subspace of $\mathrm{End}(V) = \mathrm{Hom}(V, V) \cong V \otimes V$ generated by the semi-simple elements of A_5. Note that M is an A_5-module: if $[h]$ is semi-simple, then so is $g.[h] = [ghg^{-1}]$. To prove that V is weakly adequate, it suffices (by definition) to verify any of the equivalent conditions of Lemma 5.3.1; we shall verify condition (3), namely, that $M = \mathrm{End}(V)$. If $M \to \mathrm{End}(V)$ is not surjective, then $\mathrm{End}(V)$ has a simple quotient Q such that the composite map $M \to Q$ is zero. The map from $\mathrm{End}(V)$ to any simple quotient factors through the cosocle of $\mathrm{End}(V)$, hence it suffices to show that M surjects onto the cosocle of $\mathrm{End}(V)$. Since V is self-dual, this cosocle is isomorphic to $k \oplus V$ by Lemma 8.2.8. The corresponding map $\mathrm{End}(V) \to k$ is the trace map, and any non-trivial element of A_5 of odd order has nonzero trace on V, so M surjects onto k.

It remains to show that M meets the direct summand V, which we will do by showing that each element of A_5 of order 3 contributes to this summand, using the description of this summand coming from (8.2.7) with $W = V$. Let e_1, e_2 be the standard basis of U, and let e_1^σ, e_2^σ be the corresponding basis of U^σ, so that the $e_i \otimes e_j^\sigma$ for $i = 1, 2$ give a basis $v_{i,j}$ for V. The two elements of order 3 in A_5 correspond to the diagonal matrices $(\omega, \omega^{-1}) \in \mathrm{SL}_2(\mathbf{F}_4)$, where $\omega^3 = 1$, and each $v_{i,j}$ is an eigenvector for these matrices (with eigenvalues $1, 1, \omega, \omega^{-1}$). The same is true for the $v_{i,j} \otimes v_{i,j}$, and these give a basis modulo $\wedge^2 V$ for the direct summand $V \subset V \otimes V$, so we are done. □

8.3. **A 2-adic ordinary modularity theorem.** In this section we will establish our 2-adic modularity theorem (Theorem 8.3.2). We begin by proving the following lemma which establishes residual modularity in our situation; a closely related result was also obtained by Tsuzuki and Yamauchi, see [TY22, Thm. 4.7].

Lemma 8.3.1. *Let F^+ be a totally real field, and let*

$$\overline{\rho} : G_{F^+} \to \mathrm{GSp}_4(\mathbf{F}_2) \simeq S_6$$

be a continuous Galois representation with the following properties:

(1) The image of $\overline{\rho}$ is either $S_5(b)$ or $A_5(b)$.
(2) The image of each complex conjugation has order 2 and lands in $A_5(b)$.

Then there exists a solvable extension of totally real fields E^+/F^+, and an imaginary CM quadratic extension E/E^+, such that:

- *$\overline{\rho}(G_E) = \overline{\rho}(G_{E^+}) = A_5(b)$, and*
- *there is an ordinary RACSDC representation π of GL_4/E with $\overline{r}_{\pi,2} \cong \overline{\rho}|_{G_E}$.*

Proof. Let E^+/F^+ denote the extension of degree at most 2 corresponding to the kernel of the composite $G_{F^+} \to S_5(b) \to \mathbf{Z}/2\mathbf{Z}$. Then $\overline{\rho}(G_{E^+}) = A_5$, and E^+ is totally real by the assumption on complex conjugations. Making a further solvable base change, we can and do assume that for each place $v|2$, $\overline{\rho}|_{G_{E^+_v}}$ is trivial.

Let

$$\overline{\varrho} : G_{E^+} \to \mathrm{SL}_2(\mathbf{F}_4) \simeq A_5$$

denote the residual 2-dimensional Galois representation associated to this A_5-extension. (There are two such representations which are permuted by the outer automorphism; choose either.) Note that $\overline{\rho} = \mathrm{Sym}^3 \overline{\varrho}$ by Lemma 8.2.1 (4).

By a theorem of Tate [Ser77b, Thm 4], the composite $\overline{\varrho} : G_{E^+} \to A_5 \hookrightarrow \mathrm{PGL}_2(\mathbf{C})$ lifts to a representation

$$\varrho : G_{E^+} \to \mathrm{GL}_2(\mathbf{C})$$

with finite image (which will be some central extension of A_5). Since the image of complex conjugation under $\overline{\varrho}$ is non-trivial in $\mathrm{PGL}_2(\mathbf{C})$, the image in $\mathrm{GL}_2(\mathbf{C})$ is non-scalar and hence ϱ is odd. By the odd Artin conjecture for GL_2 (i.e. by the main results of [PS16b, Sas19]), ϱ is modular. More precisely, ϱ is the Galois representation associated to an ordinary Hilbert modular eigenform f of parallel weight 1 (the ordinarity being a consequence of local-global compatibility, and the assumption that $\overline{\rho}|_{G_{E^+_v}}$ is trivial for all $v|2$.). In particular f is contained in a Hida family. Specializing this Hida family to parallel weight 2, and making a further solvable base change if necessary, we obtain an ordinary cuspidal automorphic representation π_{E^+} of GL_2 / E^+ of weight 0 and trivial central character, with $\overline{r}_{\pi,2} \cong \overline{\varrho}|_{G_{E^+}}$.

Let E/E^+ be an imaginary quadratic CM extension, and let π be the base change of π_{E^+} to E. Then the symmetric cube $\mathrm{Sym}^3 \pi$ (which exists by [KS02]) is an ordinary RACSDC automorphic representation of GL_4 / E. Since $\overline{\rho} = \mathrm{Sym}^3 \overline{\varrho}$, we are done. ☐

We now prove the main result of this section.

Theorem 8.3.2. *Suppose that $A/\mathbf{Q}$ is an abelian surface such that*

(1) $A_5(b) \subseteq \overline{\rho}_{A,2}(G_{\mathbf{Q}}) \subseteq S_5(b)$.
(2) The image of complex conjugation has order 2 and lands in $A_5(b)$.
(3) A has good ordinary or semistable reduction at 2, and $\rho_{A,2}|_{G_{\mathbf{Q}_2}}$ is ordinary and 2-distinguished.

Then A is modular. More precisely, there is a weight 2 cuspidal automorphic representation π for $\mathrm{GSp}_4 / \mathbf{Q}$ which is ordinary at 2, and satisfies $\rho_{\pi,p} \cong \rho_{A,p}$ for all p.

Proof. As recalled in Section 1.8.23, the representation $\rho_{A,2}$ unramified at all but finitely many primes, is pure, and $\nu \circ \rho_{A,2} = \varepsilon^{-1}$. By Lemma 8.3.1, there is an imaginary CM field E and an ordinary RACSDC automorphic representation π of GL_4 / E such that $E/\mathbf{Q}$ is solvable, $\overline{\rho}_{A,2}(G_{E^+}) = A_5(b)$, and $\overline{r}_{\pi,2} \cong \overline{\rho}_{A,2}|_{G_E}$. Making a further solvable extension, we can and do assume that furthermore E/E^+ is everywhere unramified, and all of the places $v|2$ of E^+ split in E, as do all places lying under a place at which π is ramified, and all places lying over a place at which A does not have good reduction.

By Theorem 7.5.11 it therefore suffices to check that:

(a) the Zariski closure of $\rho_{A,2}(G_{\mathbf{Q}})$ contains Sp_4.
(b) $\overline{\rho}_{A,2}(G_{\mathbf{Q}(i)}) = \overline{\rho}_{A,2}(G_{\mathbf{Q}})$.
(c) $\rho_{A,2}(G_{\mathbf{Q}(\zeta_{2^\infty})})$ is integrally enormous.
(d) $\overline{\rho}_{A,2}(G_E)$ is nearly adequate.
(e) $\overline{\rho}_{A,2}(G_E)$ contains a regular semi-simple element.

(f) There exists an infinite place v of E^+ such that the polarized pair $(\overline{\rho}_{A,2}|_{G_E}, 1)$ is strongly residually odd at v.

Since $A_5(b) \subseteq \overline{\rho}_{A,2}(G_{\mathbf{Q}})$, it follows from [Zar00, Thm. 2.1] that $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$. By [Ser00, Thm 3] (see also [Pin98, Thm 5.14]), this implies that the Zariski closure of $\rho_{A,2}(G_{\mathbf{Q}})$ contains Sp_4, which verifies condition (a). To see that $\overline{\rho}_{A,2}(G_{\mathbf{Q}(i)}) = \overline{\rho}_{A,2}(G_{\mathbf{Q}})$ (condition (b)), we note that the assumptions (1) and (2) imply that $\overline{\rho}_{A,2}(G_{\mathbf{Q}})$ contains at most one normal subgroup of index 2, and that such a subgroup corresponds to a real quadratic field, and in particular not to $\mathbf{Q}(i)$. For condition (c), note that $A_5(b) \subseteq \overline{\rho}_{A,2}(G_{\mathbf{Q}})$, so we see that $\overline{\rho}_{A,2}(G_{\mathbf{Q}})$ contains a regular semi-simple element (of order 5), and then by Corollary 7.1.4 we deduce that $\rho_{A,2}(G_{\mathbf{Q}(\zeta_{2^\infty})})$ is integrally enormous. Conditions (d), (e), and (f) are immediate from Proposition 8.2.5. □

9. Local Geometry of curves with a Weierstrass point

The goal of this section is to complete the proofs of our main modularity theorems (Theorem A and B) using a 2-3 switch. By Theorems 7.5.11 and 8.3.2, we have established the following (we omit the full list of hypotheses):

(1) A 2-adic ordinary modularity theorem in weight 2 under a hypothesis on the residual image: the image Γ of $G_{\mathbf{Q}}$ in $\mathrm{GSp}_4(\mathbf{F}_2)$ contains a copy of A_5 with index at most two acting absolutely irreducibly, and moreover complex conjugation is non-trivial and lies in $A_5 \subseteq \Gamma$.
(2) A 3-adic ordinary modularity lifting theorem in weight 2.

We combine these two results as follows. Given an abelian surface $A/\mathbf{Q}$ with good ordinary reduction at 3 (satisfying our supplementary hypotheses), we construct a second abelian surface $B/\mathbf{Q}$ with $A[3] = B[3]$, such that B also has good ordinary reduction at 3, and so that the result described in point (1) can be applied to establish the modularity of B. This implies that the 3-adic representation associated to A is residually modular and hence that A is modular using point (2).

The construction of B uses the rationality of a certain twisted moduli space $P(\overline{\rho})$ of principally polarized abelian surfaces (introduced in [BCGP21, §10.2], see Definition 9.2.1 below). The space $P(\overline{\rho})$ is closely related to the moduli space $\mathcal{M}_2^w(\overline{\rho})$ of genus two curves X with a fixed Weierstrass point and fixed level 3 structure $\overline{\rho} = \overline{\rho}_{\mathrm{Jac}(X),3}$. Concretely, the Torelli map $\mathcal{M}_2^w(\overline{\rho}) \to P(\overline{\rho})$ is an isomorphism onto its (open) image. This relationship suggests a natural approach to understanding points on $P(\overline{\rho})$ with suitable local properties at $p = 2$ and $p = 3$, including having good ordinary reduction when $p = 3$ and good reduction when $p = 2$. Namely, we can consider genus 2 curves over $\mathbf{F}_p$ and $\mathbf{Q}_p$ with a rational Weierstrass point with the corresponding local properties. We carry out this analysis in §9.1 for $p = 2$ and §9.3 for $p = 3$.

This would suffice to prove *some* version of our main theorem (with a more restrictive hypothesis at 2 but still applying to a positive proportion of genus 2 curves). However, we can push these arguments further by exploiting the fact that the Jacobian of a genus 2 curve can have good reduction even when the original curve does not. Moreover, a principally polarized abelian surface need not even be a Jacobian. We carry out such auxiliary constructions (for $p = 2$) in §9.2. It also follows from the results of that section that weakening the hypothesis at 2 any further would require some new ideas. Note that many of the arguments in this section could potentially become much simpler (and stronger) once our modularity lifting theorems are generalized to totally real fields (since passing to finite extensions makes finding local points much easier). It seemed potentially useful, however, to push our current methods as far as possible until such results are available.

Finally, in §9.4, we carry out the details of the 2-3 switch using the results in the previous three sections and then complete the proof of our main theorems in §9.5.

9.1. **Genus 2 curves locally at 2.** In this section, we discuss some explicit computations with genus 2 curves (with or without rational Weierstrass points) over $\mathbf{F}_2$ and also over local fields. Note that if $B/\mathbf{Q}_2$ is an abelian surface with good reduction, then $\overline{\rho}_{B,3}(\mathrm{Frob}_2) \in \mathrm{GSp}_4(\mathbf{F}_3)$ has similitude character $\varepsilon^{-1}(\mathrm{Frob}_2) = -1$, and thus the image of $\overline{\rho}_{B,3}(\mathrm{Frob}_2)$ in the group $\mathrm{PGSp}_4(\mathbf{F}_3)$ does not land in $\mathrm{PSp}_4(\mathbf{F}_3)$.

Remark 9.1.1 (Reminder concerning conventions)**.** If B is an abelian surface over a field of characteristic prime to p, our convention (see Section 1.8.23) is that T_pB and $B[p]$ correspond to $\rho^\vee_{B,p}$ and $\overline{\rho}^\vee_{B,p}$. Note, however, that since these representations are self-dual up to twist, the associated projective representations are independent of this choice.

Definition 9.1.2. Let $B/\mathbf{Q}_p$ be an abelian surface for which the associated Galois representation on T_pB is ordinary. We say that B is *p-distinguished* if the corresponding representation $\rho_{B,p} : G_{\mathbf{Q}_p} \to \mathrm{GSp}_4(\mathbf{Q}_p)$ is p-distinguished in the sense of Definition 1.8.10. For example, if B has good ordinary reduction, then B is p-distinguished if and only if the characteristic polynomial $Q(x)$ of Frob_p on $T_\ell B$, $\ell \neq p$, has pairwise distinct roots, or equivalently if $Q(x)$ is not a square. If $B_0/\mathbf{F}_p$ is ordinary, we say that B_0 is p-distinguished if the characteristic polynomial $Q(x)$ of Frobenius has pairwise distinct roots. If $X/\mathbf{Q}_p$ is a genus 2 curve, we say that X is p-distinguished if $\mathrm{Jac}(X)$ is p-distinguished.

We begin with some basic group–theoretic facts, which can easily be extracted from [CCN+85, pp. 26–27]:

Lemma 9.1.3. *There are* 10 *conjugacy classes of elements in* $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$. *Their orders and the characteristic polynomials of any lift to* $\mathrm{GSp}_4(\mathbf{F}_3)$ *are given by the following table. Here the name of the conjugacy class* (*with the first number indicating the order of the element*) *follows the same convention as the Atlas* [CCN+85]*:*

$\langle g \rangle$	$P(x) \in \mathbf{F}_3[x]$	*Size*
$2C$	$x^4 + 2x^2 + 1$	36
$2D$	$x^4 + x^2 + 1$	540
$4C$	$x^4 \pm x^3 + 2x^2 \mp x + 1$	540
$4D$	$x^4 + 1$	1620
$6G$	$x^4 + 2x^2 + 1$	1440
$6H$	$x^4 + 2x^2 + 1$	1440
$6I$	$x^4 + x^2 + 1$	4320
$8A$	$x^4 \pm x^3 + x^2 \mp x + 1$	6480
$10A$	$x^4 \pm x^3 \mp x + 1$	5184
$12C$	$x^4 \pm x^3 + 2x^2 \mp x + 1$	4320

There are, in particular, 9 *different possible characteristic polynomials of elements of* $\mathrm{GSp}_4(\mathbf{F}_3) \setminus \mathrm{Sp}_4(\mathbf{F}_3)$. *There is a natural permutation representation*

$$\mathrm{PGSp}_4(\mathbf{F}_3) \to S_{40}$$

coming from the action on the 40 *points* $(\mathbf{F}_3^4 - 0)/\{\pm 1\}$*, and there is also a unique transitive action* $\mathrm{PGSp}_4(\mathbf{F}_3) \to S_{27}$ *whose stabilizer is the maximal subgroup* $2^4 : S_5 \subset \mathrm{PGSp}_4(\mathbf{F}_3)$*; see* [CCN+85, p. 26]. *The conjugacy class of* $g \in \mathrm{PGSp}_4(\mathbf{F}_3) \setminus \mathrm{PSp}_4(\mathbf{F}_3)$ *is determined by the conjugacy classes of its images in* S_{40} *and* S_{27}*, and even the image in* S_{40} *suffices except for the following classes:*

g	S_{40}	S_{27}
$4C$	$(4,4,4,4,4,4,4,4,4,4)$	$(1,2,2,2,4,4,4,4,4)$
$4D$	$(4,4,4,4,4,4,4,4,4,4)$	$(1,1,1,1,1,2,4,4,4,4,4)$
$6G$	$(2,2,6,6,6,6,6,6)$	$(1,1,1,2,2,2,3,3,3,3,6)$
$6H$	$(2,2,6,6,6,6,6,6)$	$(3,3,3,3,3,6,6)$

Lemma 9.1.4. *Consider the genus* 2 *curves* C_i *over* $\mathbf{Q}$*:*

$$C_1 : y^2 + (x^3+1)y = x^2 + x$$
$$C_2 : y^2 + (x^3+1)y = -x^5 + x^3 + 2x^2 + x - 1$$
$$C_3 : y^2 + (x^2+x)y = x^5 + 2x^4 + x^3 - 16x^2 - 8x - 1$$

Then the C_i *have good ordinary reduction at* 2 *and a rational Weierstrass point. Moreover, the* C_i *are* 2*-distinguished. The corresponding characteristic polynomials of* Frob_2 *are as follows:*

$$P_1 : x^4 + 2x^3 + 3x^2 + 4x + 4,$$
$$P_2 : x^4 + x^2 + 4,$$
$$P_3 : x^4 - x^2 + 4,$$

and the conjugacy classes of $\overline{\rho}_{C_i,3}(\mathrm{Frob}_2)$ *in* $\mathrm{PGSp}_4(\mathbf{F}_3)$ *have type* $10A$*,* $6I$*, and* $6H$ *respectively.*

Proof. These three curves have conductors 249, 975, and 1947 respectively (they are taken from the LMFDB [LMF24]). The characteristic polynomials of Frobenius at 2 can be obtained by an explicit point count; they are irreducible over $\mathbf{Q}$, which proves they are 2-distinguished. One can determine directly from the characteristic polynomial that the conjugacy class of $\overline{\rho}_{C_i,3}(\mathrm{Frob}_2)$ must be $10A$ for $i = 1$; $2D$ or $6I$ for $i = 2$; and $2C$, $6G$, or $6H$ for $i = 3$. Given a genus two curve with a rational Weierstrass point, one can write down the general degree 40 polynomial whose splitting field is $\mathrm{PGSp}_4(\mathbf{F}_3)$ (see [CCR20, §3]), and then compute a degree 27 resolvent. From this one can determine the correct conjugacy class using Lemma 9.1.3. (For C_2, the image in S_{40} is already enough to determine that the element has order 6 and so must be $6I$.) □

Remark 9.1.5. The computations in this section and the next are all done using `magma` [BCP97], and the explicit code with documentation can be found at [BCGP25]).

We now consider what happens as we loop over all ordinary genus two curves over $\mathbf{F}_2$.

Lemma 9.1.6. *Let* $X_0/\mathbf{F}_2$ *be an ordinary smooth genus two curve.*

(1) *The action of* Frob_2 *on* $\mathrm{Jac}(X_0)[3]/\{\pm 1\}$ *has conjugacy class of type* $6G$, $6H$, $6I$, $8A$, *or* $10A$.
(2) *If* X_0 *has a smooth lift* $X/\mathbf{Z}_2$ *with a* $\mathbf{Q}_2$*-rational Weierstrass point, then the action of* Frob_2 *on* $\mathrm{Jac}(X_0)[3]/\{\pm 1\}$ *has conjugacy class of type* $6H$, $6I$, *or* $10A$.
(3) *If* $\overline{\rho} : G_{\mathbf{Q}_2} \to \mathrm{GSp}_4(\mathbf{F}_3)$ *is an unramified representation with similitude* $\overline{\varepsilon}^{-1}$, *and the image of* $\overline{\rho}(\mathrm{Frob}_2)$ *in* $\mathrm{PGSp}_4(\mathbf{F}_3)$ *has conjugacy class of type* $6H$, $6I$, *or* $10A$, *then there is an ordinary smooth genus 2 curve* $X/\mathbf{Z}_2$ *with* $\overline{\rho} \cong \overline{\rho}_{X,3}$, *such that* X *has a* $\mathbf{Q}_2$*-rational Weierstrass point, and is* 2*-distinguished.*

Proof. We may write any smooth genus two curve X_0 over $\mathbf{F}_2$ in the form

$$y^2 + h(x)y = f(x), \tag{9.1.6}$$

where $\deg(f(x)) \leq 6$ and $\deg(h(x)) \leq 3$. We may enumerate all such equations. We do not concern ourselves with identifying either isomorphism classes of curves or of their Jacobians, and so in particular when we talk of "curves" below we really mean curves with a given Weierstrass equation as in (9.1.6). Let $A = \mathrm{Jac}(X_0)$. We find that:

(1) There are 2^{11} possible pairs of $h(x)$ and $f(x)$.
(2) There are 768 curves which are smooth of genus 2.
(3) There are 384 ordinary curves, of which:
  (a) 32 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $6G$,
  (b) 16 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $6H$,
  (c) 48 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $6I$,
  (d) 96 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $8A$,
  (e) 192 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $10A$.
(4) The are are 384 non-ordinary curves, of which:
  (a) 48 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $6G$,
  (b) 144 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $12C$,
  (c) 48 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $4D$,
  (d) 48 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $8A$,
  (e) 96 have $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ of type $10A$.
(5) If X_0 is ordinary and is additionally the reduction of a smooth curve over $\mathbf{Z}_2$ with a $\mathbf{Q}_2$-rational Weierstrass point, then $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ has type $6H$, $6I$, or $10A$.

We first explain how to distinguish between the various conjugacy classes in parts (3) and (4), and then we explain part (5).

(1) By point counting, we can compute the characteristic polynomial $Q(x)$ of Frobenius. The characteristic polynomials of the conjugacy classes $8A$ and $10A$ are not congruent modulo 3 to the characteristic polynomial of any other class in $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$, so in these cases we are done. This is also enough to determine the counts in part (4).
(2) The conjugacy classes $\{2C, 6G, 6H\}$, $\{2D, 6I\}$, and $\{4C, 12C\}$ are complete sets of conjugacy classes in $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$ with the same characteristic polynomial. We now show how to distinguish the elements of order 2 (respectively, 4) from the elements of order 6 (respectively, 12) (none of the elements of order 2 or 4 of this type actually occur). If $g \in \mathrm{PGSp}_4(\mathbf{F}_3)$ is any element, and we choose a lift in $\mathrm{GSp}_4(\mathbf{F}_3)$, then g^2 is independent

of the choice of lift. If g is of type $2C$ or $2D$, the square of any lift is scalar and given by I and $-I$ respectively, and so g^4 will be trivial. If we start with $g \in \mathrm{PGSp}_4(\mathbf{F}_3)$ of order 6, however, then for any lift, the element $g^4 \in \mathrm{GSp}_4(\mathbf{F}_3)$ will not be trivial, since it will have order divisible by 3. This allows us to distinguish the classes of types $2C$ and $2D$ from the classes of order divisible by 3 by computing $\mathrm{Jac}(X_0)(\mathbf{F}_{16})[3]$. Similarly, if g is of type $4C$ or $12C$, then, for any lift, the element $g^8 \in \mathrm{GSp}_4(\mathbf{F}_3)$ will be trivial if and only if g has type $4C$. We have the following table (where as above $Q(x)$ denotes the characteristic polynomial of Frobenius):

$Q(x) \mod 3$	$\langle g \rangle$	$\dim \ker(g^8 - 1)$
$x^4 + 2x^2 + 1$	$2C$	4
$x^4 + 2x^2 + 1$	$6G$	2
$x^4 + 2x^2 + 1$	$6H$	2
$x^4 + x^2 + 1$	$2D$	4
$x^4 + x^2 + 1$	$6I$	2
$x^4 \pm x^3 + 2x^2 \mp x + 1$	$4C$	4
$x^4 \pm x^3 + 2x^2 \mp x + 1$	$12C$	2

We find in all the 96 ordinary cases and 192 non-ordinary cases when $Q(x)$ mod 3 is a polynomial corresponding to one of the conjugacy classes in this table, there is an isomorphism

$$\mathrm{Jac}(X_0)(\mathbf{F}_{256})[3] \simeq (\mathbf{Z}/3\mathbf{Z})^2.$$

This rules out the case that $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$ has order either 2 or 4. When $Q(x)$ mod 3 is either $x^4 + x^2 + 1$ or $x^4 \pm x^3 + 2x^2 \mp x + 1$, this suffices to determine the conjugacy class exactly for the 48 ordinary curves lying in $\{2D, 6I\}$, and the 144 non-ordinary curves lying in $\{4C, 12C\}$.

(3) For the remaining 48 ordinary curves and 48 non-ordinary curves where the conjugacy class is either of type $6G$ or $6H$, we first compute the degree 40 polynomial corresponding to the $\mathrm{PGSp}_4(\mathbf{F}_3)$ representation, and then compute the degree 27 resolvent, and then use the table in Lemma 9.1.3.

(4) To establish point (5), we need to show that all of the ordinary curves where $\overline{\rho}_{B,3}(\mathrm{Frob}_2)$ has conjugacy class $6G$ or $8A$ do not lift to a smooth curve $X/\mathbf{Z}_2$ with a rational Weierstrass point. By Lemma 9.1.8 below, the Jacobian of such a curve X has a rational 2-torsion point, so that in particular, the polynomial $Q(x)$ (mod 2) would need to have 1 as a root. However, the 32 curves of type $6G$ and the 96 curves of type $8A$ have the property that $Q(x) \equiv x^2(x^2 + x + 1)$ (mod 2), so no such lift can exist.

It remains to note that if $\overline{\rho}(\mathrm{Frob}_2)$ is of type $6H$, $6I$, or $10A$, then an appropriate X exists by Lemma 9.1.4. □

Definition 9.1.7. Say that an abelian variety $A/\mathbf{Q}_p$ has *semistable ordinary reduction* if it has semistable reduction and the abelian part of the special fibre of the Néron model is ordinary. (In particular, good ordinary reduction is a special case of semistable ordinary reduction.)

Recall from Section 8.1 that we have fixed an identification $S_6 = \mathrm{GSp}_4(\mathbf{F}_2)$.

Lemma 9.1.8. *Let $B/\mathbf{Q}_2$ be an abelian surface with semistable ordinary reduction. Suppose that the image of $\overline{\rho}_{B,2}$ lands inside $S_5(b) \subset S_6 = \mathrm{GSp}_4(\mathbf{F}_2)$. Then:*

1. *The image of $\overline{\rho}_{B,2}$ is a 2-group.*
2. *There exists a rational 2-torsion point $P \in B[2](\mathbf{Q}_2)$.*

In particular, this holds if $B = \mathrm{Jac}(X)$, and $X/\mathbf{Q}_2$ has good ordinary reduction and a rational Weierstrass point.

Proof. The ordinary assumption implies that the image of $G_{\mathbf{Q}_2}$ in $\mathrm{GSp}_4(\mathbf{F}_2) = \mathrm{Aut}(B[2])$ lands (up to conjugation) in the Siegel parabolic:

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \cap \mathrm{GSp}_4(\mathbf{F}_2). \tag{9.1.9}$$

To see this, it suffices to show that $G_{\mathbf{Q}_2}$ preserves an (isotropic) subspace $(\mathbf{Z}_2)^2$ inside $T_2(B)$. If B has good ordinary reduction, the subspace is the kernel of the mod-2 reduction. More generally, if B has semistable reduction, we can use the description of the Tate module given in [GRR72, Exp.9, IX]. There is a $G_{\mathbf{Q}_2}$-equivariant filtration $T_2(B)_t \subset T_2(B)_f \subset T_2(B)$ of (saturated) $\mathbf{Z}_2$-modules of ranks $t > 0$ and $t + 2a$ where $2(t + a) = 4$. Moreover, by the orthogonality theorem [GRR72, Thm 2.4, Exp.9, IX], $T_2(B)_t^{\perp} = T_2(B)_f$. If B is purely toric, then $t = 2$ and $T_2(B)$ gives the desired space. If $t = 1$, then the abelian part of B is an abelian variety, and the kernel of reduction gives a rank one $G_{\mathbf{Q}_2}$-stable submodule of $T_2(B)_f/T_2(B)_t$, and the inverse image of this in $T_2(B)_f$ is the desired submodule.

This group (9.1.9) is a subgroup of $\mathrm{GSp}_4(\mathbf{F}_2)$ order 48 which is isomorphic to $S_4 \times S_2$, and is the normalizer of the element:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.1.10}$$

There are two (non-conjugate) subgroups of order 48 in S_6; one given by the centralizer of $(**)(**)(**)$ in S_6 and the other by the centralizer of $(**)$; they are permuted by the outer automorphism. Under our fixed isomorphism, the element (9.1.10) is conjugate to $(**)(**)(**)$ by Lemma 8.1.3. (In fact the other conjugacy class of subgroups of order 48 is given by the Klingen parabolic.) But now the assumption that the image of $\overline{\rho}_{B,2}$ lands inside $S_5(b)$ implies that the image of $G_{\mathbf{Q}_2}$ lands inside the intersection of $S_5(b)$ with the centralizer of an element of the form $(**)(**)(**)$. If that intersection is not a 2-group, then it contains an element of order 3. But the conjugacy class of elements of order 3 inside the normalizer of $(**)(**)(**)$ consists of elements with cycle shape $(***)(***)$, whereas the conjugacy class of elements of order 3 in $S_5(b) \subset S_6$ consists of elements with cycle shape $(***)$, and thus the intersection is a 2-group, proving (1). Hence the image of $\overline{\rho}_{B,2}$ is certainly contained within the 2-Sylow of $\mathrm{GSp}_4(\mathbf{F}_2)$, so the action of $G_{\mathbf{Q}_2}$ fixes a 2-torsion point, proving part (2). □

9.2. **Abelian surfaces with semistable ordinary reduction at 2.** In this section, we study abelian surfaces $A/\mathbf{Q}_2$ with either good ordinary or semistable ordinary reduction at 2.

Let F be a number field or a local field of characteristic zero, and suppose that $\overline{\rho} : G_F \to \mathrm{GSp}_4(\mathbf{F}_3)$ has similitude character $\overline{\varepsilon}^{-1}$. We now recall some rational varieties associated to $\overline{\rho}$ constructed in [BCGP21, §10.2].

Definition 9.2.1. Let $P = P(\overline{\rho})$ be the fine moduli space over F parametrizing principally polarized abelian surfaces A with a given symplectic isomorphism $A[3] \simeq \overline{\rho}^\vee$ and a fixed odd theta characteristic.

Let $\mathcal{M}_2^w(\overline{\rho})$ be the fine moduli space over F parametrizing genus two curves X/F with a fixed Weierstrass point and a fixed symplectic isomorphism $\mathrm{Jac}(X)[3] \simeq \overline{\rho}^\vee$.

More explicitly (see [BCGP21, Defn. 10.2.2]) the space $P(\overline{\rho})$ can be defined as follows: we let B be the moduli space of principally polarized abelian surfaces A with a given symplectic isomorphism $A[3] \simeq \overline{\rho}^\vee$, and let $B(2) \to B$ be the $S_6 \cong \mathrm{PSp}_4(\mathbf{F}_2)$-cover corresponding to a full level 2 structure. Then P is the intermediate cover corresponding to the subgroup $S_5(b) \subset S_6$. In particular we note that a principally polarized abelian surface A/F gives rise to a point in $P(\overline{\rho})(F)$ if, in additional to having a symplectic isomorphism $A[3] \simeq \overline{\rho}^\vee$, the image of $\overline{\rho}_{A,2}$ is conjugate to a subgroup of $S_5(b)$.

The space $P(\overline{\rho})$ is smooth and rational [BCGP21, Thm 10.2.3]. The Torelli map $\mathcal{M}_2^w(\overline{\rho}) \to P(\overline{\rho})$ is an open immersion, and hence $\mathcal{M}_2^w(\overline{\rho})$ is also smooth and rational, and dense in $P(\overline{\rho})$.

An unramified representation

$$\overline{\rho} : G_{\mathbf{Q}_2} \to \mathrm{GSp}_4(\mathbf{F}_3)$$

with similitude character $\overline{\varepsilon}^{-1}$ is given up to conjugation and unramified twist by a conjugacy class in $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$. Given such a class, the goal of this section is (when possible) to find a point $A \in P(\overline{\rho})(\mathbf{Q}_2)$ which either has good ordinary or semistable ordinary reduction and is in addition 2-distinguished. Naturally, one such source of representations comes from a point $X \in \mathcal{M}_2^w(\overline{\rho})(\mathbf{Q}_2)$ where X has good ordinary reduction at 2, however, this turns out not to exhaust the list of possibilities. There are three reasons for this. The first is that $A = \mathrm{Jac}(X)$ can have good reduction even when X does not. The second is that A can have bad reduction and yet $\overline{\rho}_{A,3}$ can still be unramified (although such A will necessarily be semistable). The third is that some of the most accessible points of P lie on the complement of the image of $\mathcal{M}_2^w(\overline{\rho})$, namely, direct products of elliptic curves. We exploit a number of these phenomena to find points for various different representations $\overline{\rho}$.

Since we shall only consider $\overline{\rho} : G_{\mathbf{Q}_2} \to \mathrm{GSp}_4(\mathbf{F}_3)$ which are unramified, we begin with following, which is a (specialization of a) standard result:

Lemma 9.2.2. *Let $A/\mathbf{Q}_2$ be an abelian variety. Suppose that $\overline{\rho}_{A,3}$ is unramified. Then A has semistable reduction.*

Proof. The assumption that $\overline{\rho}_{A,3}$ is unramified implies that the action of inertia on $T_3(A)$ is unipotent, so the claim follows from Grothendieck's semi-stability Theorem [GRR72, Exp.9, IX]. □

The ultimate goal of this section is to prove the following theorem:

Theorem 9.2.3. *Let $\overline{\rho} : G_{\mathbf{Q}_2} \to \mathrm{GSp}_4(\mathbf{F}_3)$ be unramified with similitude character $\overline{\varepsilon}^{-1}$. There exists a point $A \in P(\overline{\rho})(\mathbf{Q}_2)$ which has semistable ordinary reduction and is 2-distinguished if and only if the conjugacy class of the image of $\overline{\rho}(\mathrm{Frob}_2)$*

in $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$ *is not of type* $4C$ *or* $12C$*. Moreover, one can additionally take* A *to have good reduction if and only if the conjugacy class of the image of* $\overline{\rho}(\mathrm{Frob}_2)$ *in* $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$ *is of type* $4D$*,* $6H$*,* $6I$*, or* $10A$*. This is summarized by the table below.*

$\langle g \rangle \subset \mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$	*good ordinary*	*semistable ordinary*
$2C$	✗	✓
$2D$	✗	✓
$4C$	✗	✗
$4D$	✓	✓
$6G$	✗	✓
$6H$	✓	✓
$6I$	✓	✓
$8A$	✗	✓
$10A$	✓	✓
$12C$	✗	✗

Proof. (Most of) the proof is carried out in the remainder of this section, and we give the proof by the order in which the argument occurs, namely:

(1) When $\langle g \rangle$ is one of the conjugacy classes $6H$, $6I$, and $10A$, the result follows directly from the fact that there exist genus two curves $X/\mathbf{Q}_2$ with a rational Weierstrass point, good ordinary reduction, and with 2-distinguished Jacobians, by Lemma 9.1.6(3).
(2) When $\langle g \rangle$ has the form $4C$ or $12C$, the result follows by Lemma 9.2.7.
(3) When $\langle g \rangle$ has the form $8A$, the good reduction case is covered by Lemma 9.2.6, and the semistable reduction case by Lemma 9.2.8.
(4) When $\langle g \rangle$ has the form $2D$ or $4D$, the semistable reduction case follows from Lemma 9.2.8, which also covers the good reduction case for the conjugacy class $4D$.
(5) When $\langle g \rangle$ has the form $2C$ or $6G$, the good reduction case follows from Lemma 9.2.11 and Lemma 9.2.12 (together with an examination of Table 9.2.5).
(6) The semistable reduction case for the conjugacy class $2C$ is Lemma 9.2.14.
(7) The semistable reduction case for the conjugacy class $6G$ is Lemma 9.2.17.
(8) The good reduction case for the conjugacy class $2D$ is Lemma 9.2.19. □

Definition 9.2.4. An ordinary Weil polynomial of weight one for p is a degree 4 polynomial $X^4 + aX^3 + bX^2 + paX + p^2 \in \mathbf{Z}[X]$ all of whose roots have absolute value $p^{1/2}$ and for which $(b, p) = 1$.

If $A/\mathbf{Q}_2$ has good ordinary reduction, then certainly the characteristic polynomial $Q(x)$ of Frobenius at 2 will (by the Weil conjectures) be an ordinary Weil polynomial of weight one for $p = 2$.

There are 16 possible ordinary Weil polynomials of weight one for $p = 2$, listed in factored form in Table 9.2.5, together with the list of corresponding conjugacy classes in $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$ (as described in Lemma 9.1.3) whose conjugacy class admits a lift to $\mathrm{GSp}_4(\mathbf{F}_3)$ with the given characteristic polynomial over $\mathbf{F}_3[x]$.

Lemma 9.2.6. *There does not exist a principally polarized abelian surface* $A/\mathbf{Q}_2$ *with good ordinary reduction at* 2 *such that* $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ *has conjugacy class* $8A$*,*

$Q(x)$	$Q(x) \bmod 3$	$\langle g\rangle$
$x^4 - x^2 + 4$	$x^4 - x^2 + 1$	$2C$, $6G$, $6H$
$x^4 - 3x^3 + 5x^2 - 6x + 4$	$x^4 - x^2 + 1$	
$x^4 + 3x^3 + 5x^2 + 6x + 4$	$x^4 - x^2 + 1$	
$x^4 + x^2 + 4$	$x^4 + x^2 + 1$	$2D$, $6I$
$(x^2 - x + 2)^2$	$x^4 + x^3 + 2x^2 + 2x + 1$	$4C$, $12C$
$x^4 + x^3 - x^2 + 2x + 4$	$x^4 + x^3 + 2x^2 + 2x + 1$	
$(x^2 + x + 2)^2$	$x^4 + 2x^3 + 2x^2 + x + 1$	$4C$, $12C$
$x^4 - x^3 - x^2 - 2x + 4$	$x^4 + 2x^3 + 2x^2 + x + 1$	
$x^4 - 3x^2 + 4$	$x^4 + 1$	$4D$
$(x^2 + x + 2)(x^2 - x + 2)$	$x^4 + 1$	
$x^4 + x^3 + x^2 + 2x + 4$	$x^4 + x^3 + x^2 + 2x + 1$	$8A$
$x^4 - x^3 + x^2 - 2x + 4$	$x^4 + 2x^3 + x^2 + x + 1$	$8A$
$x^4 - 2x^3 + 3x^2 - 4x + 4$	$x^4 + x^3 + 2x + 1$	$10A$
$x^4 + x^3 + 3x^2 + 2x + 4$	$x^4 + x^3 + 2x + 1$	
$x^4 + 2x^3 + 3x^2 + 4x + 4$	$x^4 + 2x^3 + x + 1$	$10A$
$x^4 - x^3 + 3x^2 - 2x + 4$	$x^4 + 2x^3 + x + 1$	

Table 9.2.5. Mod 3 reduction of ordinary Weil polynomials of weight one for $p = 2$

and $\overline{\rho}_{A,2}$ has image inside $S_5(b)$. If one further insists that A is 2-distinguished, then $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ can not have conjugacy class $4C$ and $12C$.

Proof. Consider first the case of $4C$ and $12C$. Up to unramified twist, the characteristic polynomial of $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ is then

$$(x^2 - x + 2)^2 \bmod 3.$$

If $Q(x) \equiv (x^2 - x + 2)^2 \bmod 3$ is an ordinary Weil polynomial, this forces (by Table 9.2.5) either the equality $Q(x) = (x^2 - x + 2)^2$ or

$$Q(x) = x^4 + x^3 - x^2 + 2x + 4 \equiv x^2(x^2 + x + 1) \bmod 2.$$

The first case is ruled out by the 2-distinguished condition. For the second, it implies that the action of Frob_2 on $A[2](\overline{\mathbf{F}}_2)$ has order 3. But this contradicts Lemma 9.1.8.

Now consider $8A$. The characteristic polynomial up to twist is

$$(x + 1)(x - 1)(x^2 + x + 2) \bmod 3,$$

in which case (see Table 9.2.5) there is a unique possibility

$$Q(x) = x^4 + x^3 + x^2 + 2x + 4 \equiv x^2(x^2 + x + 1) \bmod 2,$$

and we are again done by Lemma 9.1.8. □

We can upgrade this lemma as follows:

Lemma 9.2.7. *Suppose that $A/\mathbf{Q}_2$ is a principally polarized abelian surface with (potentially) semistable ordinary reduction and such that:*

(1) $\overline{\rho}_{A,3}$ is unramified.
(2) $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ has projective conjugacy class $4C$ or $12C$.

(3) $\overline{\rho}_{A,2}$ has image inside some conjugate of $S_5(b)$.

Then A has good reduction at 2 and is not 2-distinguished.

Proof. The conditions imply that $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ has, up to unramified twist, characteristic polynomial

$$(x^2+x+2)^2 \bmod 3.$$

If α is a root of this polynomial, then $-\alpha$ is clearly not a root. But that implies that no two roots have ratio $2 \equiv -1 \bmod 3$. In particular we see that $H^2(G_{\mathbf{Q}_2}, \overline{\rho}_{A,3}) = 0$, so all lifts of $\overline{\rho}_{A,3}$ are unramified. In particular $\rho_{A,3}|_{G_{\mathbf{Q}_2}}$ is unramified, and by Néron–Ogg–Shafarevich, we deduce that A must have good ordinary reduction at 2. The result now follows from Lemma 9.2.6. □

We now move on to the classes $2D$, $4D$, and $8A$, which we can construct directly using products of elliptic curves.

Lemma 9.2.8. *Let $X/\mathbf{Q}_2$ be an elliptic curve with split multiplicative reduction and such that the Tate parameter q is a perfect 6th power. Let $Y/\mathbf{Q}_2$ be an elliptic curve with good ordinary reduction with characteristic polynomial $Q(x) = x^2+x+2$ and with $\overline{\rho}_{Y,2}$ trivial. Let X' and Y' denote the unramified quadratic twists of X and Y respectively. Let $A = X \times Y$, and let $B = Y \times Y'$, and $C = X \times X'$. Then A, B, and C are principally polarized abelian surfaces with the following properties:*

(1) A and C have semistable ordinary reduction, and B has good ordinary reduction.

(2) $\mathbf{Q}(A[3])$, $\mathbf{Q}(B[3])$, and $\mathbf{Q}(C[3])$ are unramified at 2.

(3) $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ has projective conjugacy class $8A$, $\overline{\rho}_{B,3}(\mathrm{Frob}_2)$ has projective conjugacy class $4D$, and $\overline{\rho}_{C,3}(\mathrm{Frob}_2)$ has projective conjugacy class $2D$.

(4) $\overline{\rho}_{A,2}$, $\overline{\rho}_{B,2}$, and $\overline{\rho}_{C,2}$ have image inside $S_5(b) \subset \mathrm{GSp}_4(\mathbf{F}_2)$ up to conjugacy.

(5) A, B, and C are 2-distinguished.

Proof. First we note that both X and Y exist; there exists a Tate curve for any $q \in \mathbf{Q}_2$ with $v(q) > 0$, and one can take Y to be $y^2+xy+y = x^3-x^2-6x-4$, which is actually the base change to $\mathbf{Q}_2$ of an elliptic curve E over $\mathbf{Q}$ of conductor 17 with $E[2] \simeq (\mathbf{Z}/2\mathbf{Z})^2$ as a $G_{\mathbf{Q}}$-module.

The surfaces A, B, and C have a principal polarization coming from the principal polarization on each elliptic curve. They clearly all have semistable ordinary reduction, and B in addition has good ordinary reduction. The assumption that the Tate parameter q is a cube implies that the action of $G_{\mathbf{Q}_2}$ on $X[3]$ is isomorphic to $\mu_3 \oplus \mathbf{Z}/3\mathbf{Z}$ as a Galois representation, and hence is unramified. Moreover, we deduce that there is also an isomorphism $X'[3] \simeq \mathbf{Z}/3\mathbf{Z} \oplus \mu_3$, since the unique unramified quadratic character is the cyclotomic character. The second claim then follows since Y and Y' have good reduction.

The element $\overline{\rho}_{Y,3}(\mathrm{Frob}_2) \in \mathrm{GL}_2(\mathbf{F}_3)$ has characteristic polynomial $x^2+x+2 \bmod 3$, and $\overline{\rho}_{X,3}(\mathrm{Frob}_2)$ has characteristic polynomial (x^2-1) from the description above. Thus $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ has characteristic polynomial

$$(x^2-1)(x^2+x+2) = x^4+x^3+x^2-x+1 \bmod 3.$$

The only elements with this characteristic polynomial modulo 3 have conjugacy class $8A$ in $\mathrm{PGSp}_4(\mathbf{F}_3)$. For B, we see that $Q(x) = (x^2+x+2)(x^2-x+2)$, and thus (from Table 9.2.5) the only possibility is that $\overline{\rho}_{B,3}(\mathrm{Frob}_2)$ has projective conjugacy class $4D$. For C, we see that the characteristic polynomial of Frobenius on $C[3]$ is $(x^2-1)^2$ mod 3 and that Frobenius clearly has order 2, so the projective conjugacy class is $2D$.

We now show that the mod 2 reductions are conjugate to a subgroup of $S_5(b)$. The assumption that q is a square implies that $\overline{\rho}_{X,2}$ (and its quadratic twist) are trivial. On the other hand, by construction, $\overline{\rho}_{Y,2}$ and thus its quadratic twist are also trivial. So $\overline{\rho}_{A,2}$, $\overline{\rho}_{B,2}$, and $\overline{\rho}_{C,2}$ are also trivial and the claim follows.

It remains to show that A, B, and C are 2-distinguished. In each case, we can compute directly the unit Frobenius eigenvalues on the semi-simplification of the Tate module. If α denotes the unit root of $x^2+x+2=0$, then for A, B, and C they are given by $\{1,\alpha\}$, $\{\alpha,-\alpha\}$, and $\{1,-1\}$ respectively. Since $\alpha \neq 1$ and $\alpha \neq -\alpha$, these pairs all consist of distinct numbers and we are done. ☐

9.2.9. *The cases* $2C$*,* $6G$*, and* $6H$. We now turn to the cases of $2C$, $6G$, and $6H$, where (see Table 9.2.5) the characteristic polynomial of $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ is

$$x^4 - x^2 + 1 = (x^2+1)^2 \bmod 3 \tag{9.2.10}$$

We have the following:

Lemma 9.2.11. *If $A/\mathbf{Q}_2$ has good ordinary reduction and*

$$Q(x) = x^4 \pm 3x^3 + 5x^2 \pm 6x + 4 \equiv x^2(x^2+x+1) \bmod 2,$$

then the image of $\overline{\rho}_{A,2}$ is not conjugate to a subgroup of $S_5(b)$.

Proof. This follows from Lemma 9.1.8, exactly as in the proof of Lemma 9.2.6. ☐

Lemma 9.2.11 implies that, in the good reduction case with conjugacy classes $2C$, $6G$, and $6H$, the only possibility for $Q(x)$ is $x^4 - x^2 + 4$.

Lemma 9.2.12. *Let $A/\mathbf{F}_2$ be an abelian surface with $Q(x) = (x^4 - x^2 + 4)$. Then the action of* Frob_2 *on* $A[3](\overline{\mathbf{F}}_2)$ *is not semi-simple and the projective image of* $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ *has conjugacy class* $6H$.

Proof. Note firstly that the abelian surface $A = \mathrm{Jac}(C_3)$ where C_3 is as in Lemma 9.1.4 satisfies the hypothesis and conclusions of the lemma. Let B be another abelian surface with $Q(x) = x^4 - x^2 + 4$; it suffices to show that $A[3] \cong B[3]$. There is an inclusion $\mathbf{Z}[\phi] \subset \mathrm{End}(A)$ where ϕ is the Frobenius endomorphism which satisfies $\phi^4 - \phi^2 + 4 = 0$. Let $\psi = \phi^2 + 1$, so $\psi^\vee = 3 - \psi$ and $\psi^\vee \circ \psi = [6]$. Note that Frob_2 acts non-semi-simply on $A[3]$. Since Frob_2 on $A[3]$ has characteristic polynomial $(x^2+1)^2$ mod 3, and x^2+1 mod 3 is irreducible, it follows that the only proper $\mathrm{Gal}(\overline{\mathbf{F}}_2/\mathbf{F}_2)$-equivariant submodule of $A[3]$ is $\ker(\psi) \cap A[3]$.

Since $Q(x)$ determines A up to isogeny, there is an isogeny $\chi : A \to B$. Either χ has order prime to 3 or $\ker(\psi) \cap A[3] \subset \ker(\chi)$, in which case there is a factorization:

$$\begin{array}{ccc} A & \xrightarrow{\ \chi\ } & B \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle [2]} \\ A & \xrightarrow{\ \chi'\ } & B \end{array}$$

where the 3-part of the degree of χ' is less than that of χ. By induction, there exists an isogeny of A to B of order prime to 3, which implies that $A[3]$ and $B[3]$ are isomorphic, as required. ☐

Lemma 9.2.13. *Suppose that $A/\mathbf{Q}_2$ is an abelian surface such that:*

(1) $\overline{\rho}_{A,3}|_{G_{\mathbf{Q}_2}}$ *is unramified.*
(2) $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ *has projective conjugacy class* $2C$ *or* $6G$*.*
(3) The image of $\overline{\rho}_{A,2}$ *is conjugate to a subgroup of* $S_5(b)$*.*

Then $A/\mathbf{Q}_2$ *has semistable ordinary reduction with purely toric reduction.*

Proof. Suppose that A had good reduction. Condition (2) implies that $Q(x) \equiv x^4 - x^2 + 1 \bmod 3$. By Lemmas 9.2.11 and 9.2.12, assumptions (1), and (3) (taking into account Table 9.2.5) imply that $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ has projective conjugacy class $6H$, contradicting condition (2). Thus A cannot have good reduction. By Lemma 9.2.2, A has semistable reduction. As in the proof of Lemma 9.1.8 there is a $G_{\mathbf{Q}_2}$-equivariant filtration $T_3(A)_t \subset T_3(A)_f \subset T_3(A)$ of (saturated) $\mathbf{Z}_3$-modules of ranks t and $t + 2a$ where $2(t + a) = 4$. Since A does not have good reduction, we have $t > 0$. If $t = 1$, then $T_3(A)_t/3$ is a Galois invariant line inside $A[3]$, but this is not compatible with the fact that $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ has characteristic polynomial $(x^2 + 1)^2$, and $(x^2 + 1)$ has no roots over $\mathbf{F}_3$. So $t = 2$ and A has purely toric (and hence semistable ordinary) reduction. □

On the other hand, we have the following variation on Lemma 9.2.8.

Lemma 9.2.14. *There exists a principally polarized abelian surface* $A/\mathbf{Q}_2$ *satisfying the following:*

(1) A *has semistable ordinary reduction.*
(2) $\overline{\rho}_{A,3}$ *is unramified.*
(3) $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ *has projective conjugacy class* $2C$*,*
(4) $\overline{\rho}_{A,2}$ *has image inside* $S_5(b) \subset \mathrm{GSp}_4(\mathbf{F}_2)$ *up to conjugacy.*
(5) A *is* 2*-distinguished.*

Proof. Let $X/\mathbf{Q}_2$ be an elliptic curve with split multiplicative reduction and such that the Tate parameter $q \in \mathbf{Q}_2^\times$ is a perfect cube, and that $q \in 5\cdot(\mathbf{Q}_2^\times)^2$. With these choices $\overline{\rho}_{X,3}$ is the trivial representation while $\overline{\rho}_{X,2}$ is unramified and $\overline{\rho}_{X,2}(\mathrm{Frob}_2)$ is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We now take $A = X \otimes \mathbf{Z}^2$ where $G_{\mathbf{Q}_2}$ acts on $\mathbf{Z}^2$ via an unramified quotient with Frob_2 acting by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{9.2.15}$$

or in other words A is descended from $(X \times X)_{\mathbf{Q}_{16}}$ where Frobenius is twisted by the automorphism $(a, b) \mapsto (b, -a)$.

Certainly A is semistable ordinary. Since X is principally polarized and since the map $G_{\mathbf{Q}_2} \to \mathrm{GL}_2(\mathbf{Z})$ associated to (9.2.15) is self-dual, it follows that A is isomorphic to its dual and hence is also principally polarized. Moreover we have, up to conjugation,

$$\overline{\rho}_{A,3}(\mathrm{Frob}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with order 4 and projective image $2C$. On the other hand,

$$\overline{\rho}_{A,2}(\mathrm{Frob}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The centralizer of this element has order 16, which means (with respect to the isomorphism in Lemma 8.1.3, although this conjugacy class is preserved by the outer automorphism) that it is conjugate to $(**)(**)$ in S_6 and so is conjugate to an element of $S_5(b)$. □

This leads us to $6G$ as the last remaining semistable case of Theorem 9.2.3. There is a construction in this case along the lines of Lemma 9.2.14, although the details are more cumbersome. Instead, we use a different idea motivated by Lemma 9.2.13. Consider the genus two curve

$$Y : y^2 + y = x^5 - x^4 + x^3$$

with good reduction at 2 (it has conductor 797). We have $Q(x) \equiv x^4 - x^2 + 1 \bmod 3$, and so (the projective image of) $\overline{\rho}_{\mathrm{Jac}(Y),3}(\mathrm{Frob}_2)$ has conjugacy class $2C$, $6G$, or $6H$. By computing the corresponding degree 40 and degree 27 polynomials, we find that it has conjugacy class $6G$. This does not contradict Lemma 9.2.13 because $Q(x) = x^4 + 2x^2 + 1$ and Y is not ordinary. We may write $Y/\mathbf{Q}$ as

$$Y : y^2 = x^5 + ax^3 + bx^2 + cx + d, \tag{9.2.16}$$

with

$$(a, b, c, d) = \left(\frac{48}{5}, \frac{704}{25}, \frac{3072}{125}, \frac{821504}{3125} \right).$$

We can think of $\mathcal{M}_2^w(\overline{\rho})$ explicitly as the moduli space of genus two curves X given by $y^2 = x^5 + Ax^3 + Bx^2 + Cx + D$ with a (symplectic) isomorphism $\overline{\rho}_{\mathrm{Jac}(X),3} \simeq \overline{\rho}_{\mathrm{Jac}(Y),3}$. In [CCR20, Thm 2], an explicit parametrization $\mathbf{P}^3_{\mathbf{Q}}(s,t,u,v) \to \mathcal{M}_2^w(\overline{\rho})$ is given; that is, A, B, C, and D are explicit polynomials in (s,t,u,v) whose specialization to $(1,0,0,0)$ gives the parameters (a,b,c,d) of equation (9.2.16). By Lemma 9.2.13, any specialization of this family which does *not* have good reduction is necessarily semistable ordinary with purely toric reduction. Moreover, it will also necessarily be 2-distinguished; the pair of eigenvalues of Frobenius on the unramified quotient will be Galois invariant and yet be roots of $(x^2+1) \bmod 3$. Thus in practice we can choose random points on this family to find one which does not have good reduction, and then we are done. The specialization of this family to the point $(0,0,4,1)$ is the curve $y^2 = x^5 + Ax^3 + Bx^2 + Cx + D$, with (after scaling down by $(4^{36}, 8^{36}, 16^{36}, 32^{36})$ from the formulas in [CCR20])

$$\begin{aligned} A &= 672315215064342/5, \\ B &= -1977458186203677223733 32/25, \\ C &= -303874847142831213230465179932 3/125, \\ D &= 40513003622207649845365025720900145337 2/3125. \end{aligned}$$

Thus we have produced a curve of the required form. As a sanity check, the conductor has the form $2^2 \cdot N$ where $(2, N) = 1$, and the Euler factor at 2 is $x^2 + 1$, and one can indeed compute the 3-torsion division polynomial of degree 40 and its resolvent of degree 27 and find that they give an extension unramified at 2 with $\overline{\rho}(\mathrm{Frob}_2)$ of conjugacy class $6G$ (as they should). On the other hand, after reducing this modulo a large enough power of 2 (2^{12} in this case), we get a more manageable example (except now a different *global* representation):

Lemma 9.2.17. *Let X be the curve*

$$y^2 + (-x^2 - x - 1)y = x^5 - 4x^4 + 47x^3 - 43x^2 - 2x + 8.$$

Then $A = \mathrm{Jac}(X)/\mathbf{Q}_2$ has purely toric reduction, and is 2-distinguished. Furthermore $A[3]$ is unramified, $\overline{\rho}_{A,2}$ has image conjugate to a subgroup of $S_5(b)$, and $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ has conjugacy class $6G$.

Proof. We may compute directly using division polynomials that $A[3]$ is unramified at 2 and $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ has conjugacy class $6G$. Since X has a $\mathbf{Q}_2$ (even a $\mathbf{Q}$) Weierstrass point, the image of $\overline{\rho}_{A,2}$ is conjugate to a subgroup of $S_5(b)$. The conductor at 2 of A is 2^2, so by Lemma 9.2.13, A has purely toric reduction, and the eigenvalues of Frobenius on the unramified quotient are $\pm i$, so A is 2-distinguished. □

9.2.18. *The case* $2D$. We finish the proof of Theorem 9.2.3 by ruling out $2D$ in the case of good ordinary reduction.

Lemma 9.2.19. *There does not exist an abelian surface $A/\mathbf{Q}_2$ with good ordinary reduction and $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ projectively conjugate to $2D$.*

Proof. From Table 9.2.5, we see that such an A must satisfy $Q(x) = x^4 + x^2 + 4 \equiv (x-1)^2(x+1)^2 \bmod 3$. From Lemma 9.1.4 (in particular the curve C_2), we see that there exists a smooth ordinary $X/\mathbf{F}_2$ with the same $Q(x)$, and thus $\mathrm{Jac}(X)$ is isogenous to $A/\mathbf{F}_2$. Let $\chi : \mathrm{Jac}(X) \to A$ be an isogeny, which we may assume is not divisible by [3]. Since $\overline{\rho}_{\mathrm{Jac}(X),3}(\mathrm{Frob}_2)$ is conjugate to $6I$, the *minimal* polynomial of Frob_2 on $A[3]$ is $(x-1)^2(x+1)^2$. Thus the only Galois invariant subspaces of $A[3]$ contain either the intersection of $A[3]$ with the kernel of $\psi = \phi - 1$ or $\psi = \phi + 1$ respectively. Thus (as in Lemma 9.2.12) we may reduce to the case when χ has degree prime to 3, which implies that $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ is conjugate to $\overline{\rho}_{\mathrm{Jac}(X),3}(\mathrm{Frob}_2)$ of class $6I$. □

9.3. **Genus 2 curves locally at 3.** In this section, we carry out some computations similar to §9.1 except now over $\mathbf{F}_3$. (The `magma` files for these computations can also be found in [BCGP25], as noted in Remark 9.1.5.)

Recall that an irreducible monic polynomial $Q(x) \in \mathbf{Z}[x]$ with roots of absolute value $q^{1/2}$ (for q some power of p) corresponds (by Honda–Tate theory) to an isogeny class of simple abelian varieties A of dimension $\frac{\deg(Q)}{2} \cdot [E : F]^{1/2}$ over $\mathbf{F}_q$, where $F = \mathbf{Q}(\alpha) = \mathbf{Q}[x]/Q(x)$ and E is a certain division algebra whose centre is F and whose invariants (also determined by $Q(x)$) are trivial away from primes dividing p and ∞. However, if one also assumes that $Q(x)$ is ordinary in the sense that it has degree $2g$ and for g of the embeddings $F \hookrightarrow \overline{\mathbf{Q}}_p$, the valuation of α is zero, this forces F to be totally complex and the invariants of E to be trivial at $v|p$, which implies that $E = F$ and $\dim(A) = g$.

Specializing to the case $g = 2$, recall (Definition 9.2.4) that by an ordinary Weil polynomial of weight one for p, we mean a degree 4 polynomial $X^4 + aX^3 + bX^2 + paX + p^2 \in \mathbf{Z}[X]$ all of whose roots have absolute value $p^{1/2}$ and for which $(b, p) = 1$.

Lemma 9.3.1. *Table 9.3.3 contains the following data concerning all pairs consisting of a smooth genus two curve together with an explicit Weierstrass equation $X : y^2 = f(x)$ with $f(x) \in \mathbf{F}_3[x]$. The columns indicate:*

- *All* 40 *ordinary Weil polynomials* $Q(x)$ *of weight one for* $p = 3$. *By Honda–Tate theory, these correspond to isogeny classes of ordinary abelian surfaces* $A/\mathbf{F}_3$.
- *The reduction of* $Q(x) \bmod 3$.
- *Whether the isogeny class of* $A/\mathbf{F}_3$ *contains the Jacobian of an ordinary curve* $X/\mathbf{F}_3$ *with a rational Weierstrass point.*
- *Whether the isogeny class of* $A/\mathbf{F}_3$ *contains the Jacobian of an ordinary curve* $X/\mathbf{F}_3$.
- *How many such* X *have a Jacobian with the corresponding* $Q(x)$.

Of the 3^7 *possible* $f(x)$ *of degree* ≤ 6, *we find that:*

(1) *There are* 1296 *curves which are smooth of genus* 2.
(2) *There are* 864 *ordinary curves.*
(3) *Exactly* 10 *of these ordinary curves are not* 3*-distinguished; equivalently, the polynomial* $Q(x)$ *is a square. Moreover, these are precisely the curves for which:*
$$\mathrm{Jac}(X)(\overline{\mathbf{F}}_3)[3] \simeq \chi \otimes (\mathbf{Z}/3\mathbf{Z})^2$$
as a $G_{\mathbf{F}_3}$*-representation, where* $\chi^2 = 1$. *None of these curves have a rational Weierstrass point.*

If one enumerates curves together with a generalized Weierstrass equation
$$y^2 + h(x)y = f(x),$$
where $\deg(f(x)) \leq 6$ *and* $\deg(h(x)) \leq 3$, *all the relative ratios remain unchanged.*

Proof. This is a straightforward computation, although we explain point (3). If there is an isomorphism $\mathrm{Jac}(X)(\overline{\mathbf{F}}_3)[3] \simeq \chi \otimes (\mathbf{Z}/3\mathbf{Z})^2$ as a $G_{\mathbf{F}_3}$-representation, then $Q(x) \equiv x^2(x \pm 1)^2 \pmod 3$. Similarly, if $Q(x)$ is a square then it is a square modulo 3, and thus $Q(x) \equiv x^2(x \pm 1)^2 \pmod 3$.

We may make a quadratic twist to reduce to the case $\chi = 1$ and $Q(x) \equiv x^2(x-1)^2$. We are reduced to checking that if $\mathrm{Jac}(X)$ is in the isogeny class corresponding to $Q(x)$ with $Q(x) \equiv x^2(x-1)^2 \pmod 3$, then $Q(x)$ is a square if and only if $\mathrm{Jac}(X)(\mathbf{F}_3)[3] = (\mathbf{Z}/3\mathbf{Z})^2$. One checks this directly for each of the $4+24+1+8+48+8+48 = 141$ curves X corresponding to such a $Q(x)$. One further checks that the 5 such X where $Q(x)$ is a square do not have rational Weierstrass points. □

Remark 9.3.2. The fact that some $Q(x)$ in Table 9.3.3 do not arise from any X means that there exist ordinary abelian surfaces over $\mathbf{F}_3$ which are not isogenous to Jacobians of genus two curves. The simple (although not absolutely simple) examples in our table (with $Q(x) = x^4 - 5x^2 + 9$ and $Q(x) = x^4 - 4x^2 + 9$) actually generalize to similar examples over $\mathbf{F}_q$ for any odd q, see [How04].

Given a finite flat ordinary mod 3 representation $\overline{\rho}^\vee : G_{\mathbf{Q}_3} \to \mathrm{GSp}_4(\mathbf{F}_3)$, we would like to realize it as the 3-torsion in the Jacobian of a genus 2 curve with good ordinary reduction and a rational Weierstrass point. We shall do this (under some restrictions) in Lemma 9.3.7, using the following lemma.

Lemma 9.3.4. *Let* $p > 2$ *be prime, and let* $\mathcal{O}$ *be the ring of integers in a finite extension of* $\mathbf{Q}_p$ *with residue field* $\mathbf{F}$. *Let* $G_1/\mathcal{O}$ *be a principally quasi-polarized finite flat group scheme of rank* 4. *Suppose that* $A_0/\mathbf{F}$ *is a principally polarized abelian surface with* $A_0[p] \simeq G_{1,\mathbf{F}}$ *compatibly with the quasi-polarization. Then there exists a lift of* A_0 *to a principally polarized abelian surface* $A/\mathcal{O}$ *with* $A[p] \simeq G_1$.

$Q(x)$	$Q(x)$ mod 3	$\mathrm{Jac}(X_{\mathrm{WP}})$	$\mathrm{Jac}(X)$	$\#X$
$9-5x^2+x^4$	x^2+x^4	✗	✗	0
$9-2x^2+x^4$	x^2+x^4	✓	✓	30
$9+x^2+x^4$	x^2+x^4	✗	✓	24
$9+4x^2+x^4$	x^2+x^4	✓	✓	24
$9-9x+7x^2-3x^3+x^4$	x^2+x^4	✓	✓	24
$9+9x+7x^2+3x^3+x^4$	x^2+x^4	✓	✓	24
$9-4x^2+x^4$	$2x^2+x^4$	✗	✗	0
$(3-x+x^2)(3+x+x^2)$	$2x^2+x^4$	✗	✓	24
$(3-2x+x^2)(3+2x+x^2)$	$2x^2+x^4$	✓	✓	36
$(3-2x+x^2)(3-x+x^2)$	$2x^2+x^4$	✗	✗	0
$(3+2x+x^2)(3+x+x^2)$	$2x^2+x^4$	✗	✗	0
$9-x^2+x^4$	$2x^2+x^4$	✗	✓	24
$9-9x+5x^2-3x^3+x^4$	$2x^2+x^4$	✓	✓	24
$9+9x+5x^2+3x^3+x^4$	$2x^2+x^4$	✓	✓	24
$(3-x+x^2)^2$	$x^2+x^3+x^4$	✗	✓	4
$(3-x+x^2)(3+2x+x^2)$	$x^2+x^3+x^4$	✓	✓	24
$(3+2x+x^2)^2$	$x^2+x^3+x^4$	✗	✓	1
$9-6x+x^2-2x^3+x^4$	$x^2+x^3+x^4$	✗	✓	8
$9-6x+4x^2-2x^3+x^4$	$x^2+x^3+x^4$	✓	✓	48
$9+3x-2x^2+x^3+x^4$	$x^2+x^3+x^4$	✓	✓	8
$9+3x+x^2+x^3+x^4$	$x^2+x^3+x^4$	✓	✓	48
$(3-2x+x^2)^2$	$x^2+2x^3+x^4$	✗	✓	1
$(3-2x+x^2)(3+x+x^2)$	$x^2+2x^3+x^4$	✓	✓	24
$(3+x+x^2)^2$	$x^2+2x^3+x^4$	✗	✓	4
$9-3x-2x^2-x^3+x^4$	$x^2+2x^3+x^4$	✓	✓	8
$9-3x+x^2-x^3+x^4$	$x^2+2x^3+x^4$	✓	✓	48
$9+6x+x^2+2x^3+x^4$	$x^2+2x^3+x^4$	✗	✓	8
$9+6x+4x^2+2x^3+x^4$	$x^2+2x^3+x^4$	✓	✓	48
$9-12x+8x^2-4x^3+x^4$	$2x^2+2x^3+x^4$	✗	✓	6
$9-3x-x^2-x^3+x^4$	$2x^2+2x^3+x^4$	✓	✓	24
$9-3x+2x^2-x^3+x^4$	$2x^2+2x^3+x^4$	✓	✓	48
$9-3x+5x^2-x^3+x^4$	$2x^2+2x^3+x^4$	✓	✓	24
$9+6x+2x^2+2x^3+x^4$	$2x^2+2x^3+x^4$	✓	✓	36
$9+6x+5x^2+2x^3+x^4$	$2x^2+2x^3+x^4$	✗	✓	24
$9-6x+2x^2-2x^3+x^4$	$2x^2+x^3+x^4$	✓	✓	36
$9-6x+5x^2-2x^3+x^4$	$2x^2+x^3+x^4$	✗	✓	24
$9+3x-x^2+x^3+x^4$	$2x^2+x^3+x^4$	✓	✓	24
$9+3x+2x^2+x^3+x^4$	$2x^2+x^3+x^4$	✓	✓	48
$9+3x+5x^2+x^3+x^4$	$2x^2+x^3+x^4$	✓	✓	24
$9+12x+8x^2+4x^3+x^4$	$2x^2+x^3+x^4$	✗	✓	6

TABLE 9.3.3. Data from Lemma 9.3.1

Proof. By [Wed01, (2.17)], there is a lift of $A_0[p^\infty]$ to a principally quasi-polarized p-divisible group $G/\mathcal{O}$ with $G[p] \simeq G_1$. By Serre–Tate theory [Kat81, Thm 1.2.1], there exists a principally polarized formal abelian surface $A/\mathcal{O}$ with $A_{\mathbf{F}} = A_0$ and $A[p^\infty] \simeq G$. Since deformations of polarized abelian surfaces are effective, we are done. $\square$

Remark 9.3.5. While we do not use this fact, we note that if A_0 in Lemma 9.3.4 is of the form $\mathrm{Jac}(C_0)$ for a smooth genus 2 curve $C_0/\mathbf{F}$, then necessarily $A = \mathrm{Jac}(C)$ for a lift C of C_0 to $\mathcal{O}$. To see this, note that since deformations of curves are effective, it suffices to show that taking the functor taking a formal lift of C_0 to its Jacobian is an isomorphism to the deformation problem of lifting $J_0 = \mathrm{Jac}(C_0)$ to a principally polarized abelian variety. Since both deformation problems are formally smooth of dimension 3, it is enough to show that the morphism on tangent spaces is injective. This is classical; see [Lan21, §2.1] for an exposition.

Corollary 9.3.6. *Let $\overline{\rho}^\vee : G_{\mathbf{Q}_3} \to \mathrm{GSp}_4(\mathbf{F}_3)$ be a finite flat representation with similitude character $\overline{\varepsilon}$. Assume that $\overline{\rho}^\vee$ is ordinary, so it is an extension of an unramified 2-dimensional representation $\overline{V}$ by its Cartier dual.*

Suppose that there exists an ordinary principally polarized abelian surface $A_0/\mathbf{F}_3$ with the following properties:

1. *There is an isomorphism of $G_{\mathbf{F}_3}$-representations $\overline{V} \simeq A_0[3]^{\text{ét}}$, and*
2. *The image of $\overline{\rho}_{A_0,2}$ is conjugate to a subgroup of $S_5(b)$.*

Then there exists a genus 2 curve $X/\mathbf{Q}_3$ with a $\mathbf{Q}_3$-rational Weierstrass point such that $J = \mathrm{Jac}(X)$ has good ordinary reduction, and $\overline{\rho}_{J,3} \simeq \overline{\rho}$. Moreover, if A_0 is 3-distinguished, then so is J.

Proof. The p-divisible group $A_0[3^\infty]$ is the direct product of an étale part V and its Cartier dual. By abuse of notation, we may also consider V as an unramified representation of $G_{\mathbf{Q}_3}$ (equivalently, the generic fibre of an étale 3-divisible group). We are assuming that $\overline{V}$ is isomorphic to the unramified quotient of $\overline{\rho}$.

We can therefore apply Lemma 9.3.4 to A_0 where G_0 taken to be the (unique) finite flat group scheme with generic fibre $\overline{\rho}^\vee$. Let A be the resulting lift of A_0. Since $\overline{\rho}_{A_0,2}$ has image inside a conjugate of $S_5(b)$, so does $\overline{\rho}_{A,2}$ (since A has good reduction, these two representations are the same). Hence A gives a $\mathbf{Q}_3$-rational point of $P(\overline{\rho})$ (see Definition 9.2.1). By a version of Krasner's Lemma due to Kisin [Kis99, Thm. 5.1], all the properties listed hold in any open ball around $A \in P(\overline{\rho})$, and hence there exists a $\mathbf{Q}_3$-point $J = \mathrm{Jac}(X)$ in the corresponding (dense) open subscheme $\mathcal{M}_2^w(\overline{\rho})$, and we are done. $\square$

We use this to deduce the following:

Lemma 9.3.7. *Let $\overline{\rho} : G_{\mathbf{Q}_3} \to \mathrm{GSp}_4(\mathbf{F}_3)$ be an ordinary representation with similitude factor $\overline{\varepsilon}^{-1}$, and suppose that $\overline{\rho}^\vee$ is finite flat. Then there exists a genus two curve $X/\mathbf{Q}_3$ with a rational Weierstrass point such that $\overline{\rho}_{\mathrm{Jac}(X),3} \cong \overline{\rho}$, and $\mathrm{Jac}(X)$ has good ordinary reduction and is 3-distinguished.*

Proof. Write $\overline{q}(x) \in \mathbf{F}_3[x]$ for the characteristic polynomial of the Frobenius on the unramified 2-dimensional quotient of $\overline{\rho}^\vee$. Note that $\overline{q}(x)$ determines the unramified quotient $\overline{V}$ of $\overline{\rho}$ unless it has repeated roots, in which case there are two possible $\overline{V}$; one semi-simple and one non-semi-simple. By Corollary 9.3.6, it suffices to find for each such $\overline{V}$ an $A_0/\mathbf{F}_3$ satisfying the hypotheses of Corollary 9.3.6. We first consider Jacobians $A_0 = \mathrm{Jac}(X_0)$ of smooth ordinary genus 2 curves $X_0/\mathbf{F}_3$ with an $\mathbf{F}_3$-rational Weierstrass point such that the characteristic polynomial $Q(x)$ of Frobenius at 3 on T_3X_0 lifts $x^2\overline{q}(x)$.

There are six possibilities $x^2 \pm 1$ and $x^2 \pm x \pm 1$ for $\overline{q}(x)$, and the existence of such an X_0 follows immediately from Lemma 9.3.1, in particular from Table 9.3.3. We can also give explicit examples of such curves $X_0 : y^2 = f(x)$ as follows, noting that (after taking into account unramified quadratic twists) we only need to consider four of the six cases.

$Q(x)$	$\overline{q}(x)$	$f(x)$
$x^4 + 3x^3 + 7x^2 + 9x + 9$	$x^2 + 1$	$x^5 + 2x + 1$
$x^4 + 3x^3 + 5x^2 + 9x + 9$	$x^2 - 1$	$x^5 + x^3 + x + 1$
$x^4 - x^3 + 2x^2 - 3x + 9$	$x^2 - x - 1$	$x^5 + x^2 + x$
$x^4 + x^3 + x^2 + 3x + 9$	$x^2 + x + 1$	$x^5 + x^4 + x^2 + 1$

In the ambiguous case where $\overline{q}(x)$ has repeated roots, it follows from Lemma 9.3.1(3) that in all examples which arise (including the final example above) the representation $\overline{V}$ is not semi-simple. Hence it remains to consider the case when Frob_3 acts on $\overline{V}$ by a scalar. In this case, we shall construct A_0 directly. After an unramified quadratic twist (if necessary), we may assume that $\overline{V}$ is trivial. Let $E_{-1}/\mathbf{F}_3$ and $E_2/\mathbf{F}_3$ denote the elliptic curves with $a_3 = -1$ and $a_3 = 2$ respectively. Note that they are both ordinary and they each have a rational point over $\mathbf{F}_3$. Let $A_0 = E_{-1} \times E_2$. Then $A_0/\mathbf{F}_3$ is principally polarized and 3-distinguished, since $Q(x) = (x^2 + x + 3)(x^2 - 2x + 3)$. Moreover, $\overline{\rho}_{E_{-1},2}(\mathrm{Frob}_3)$ has order 2 and $\overline{\rho}_{E_2,2}(\mathrm{Frob}_3)$ has order 3. It follows that $\overline{\rho}_{A,2}(\mathrm{Frob}_3)$ has order 6 and characteristic polynomial $(x-1)^2(x^2+x+1)$ mod 2. This uniquely identifies the conjugacy class as the element $(***)(**) \in S_5(b) \subset \mathrm{GSp}_4(\mathbf{F}_2)$, since the characteristic polynomial of the other conjugacy class of order six elements $(******) \in S_6 \simeq \mathrm{GSp}_4(\mathbf{F}_2)$ is equal to $(x^2 + x + 1)^2$.

One can verify these claims directly using Lemma 8.1.3, where, for example,

$$(12)(345) \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (125346) \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

Alternatively, the claim about conjugacy classes is equivalent to the claim that the eigenvalues of the semi-simple element $(***) \in A_5(b)$ are $1, 1, \omega, \omega^{-1}$ for a primitive third root of unity ω and not ω and ω^{-1} with multiplicity two; equivalently that the Brauer character of V on $(***)$ evaluates to $1 + 1 + \omega + \omega^{-1} = 1$ rather than $2\omega + 2\omega^{-1} = -2$, and this follows from Lemma 8.2.1. (The 4-dimensional representation of $A_5(a)$ coming from $\mathrm{GSp}_4(\mathbf{F}_2)$, in contrast, is isomorphic over $\mathbf{F}_4$ to $U \oplus U^\sigma$.) See also [BPP$^+$19, Lemma 5.1.7]. □

Remark 9.3.8. Although $A_0/\mathbf{F}_3 = E_{-1} \times E_2$ is principally polarized, it is not a Jacobian of an ordinary curve $X/\mathbf{F}_3$ (by Lemma 9.3.1 (3)), and so the X whose existence is proven in Lemma 9.3.7 must have bad reduction at 3, even though the Jacobian of X has good reduction at 3. From Table 9.3.3, we see there are exactly five isogeny classes of principally polarized ordinary abelian surfaces $A_0/\mathbf{F}_3$ which are 3-distinguished and with $Q(x) \equiv x^2(x^2 + x + 1) = x^2 + x^3 + x^4$ mod 3. It turns out that in four out of these five examples, it is *not* possible to find an $A_0/\mathbf{F}_3$ in the corresponding isogeny class with $A_0[3](\mathbf{F}_3) = (\mathbf{Z}/3\mathbf{Z})^2$. This can be proved by an argument similar to Lemma 9.2.12; for each of the five isogeny classes there exists a Jacobian $B_0/\mathbf{F}_3$ with $B_0[3](\mathbf{F}_3) = \mathbf{Z}/3\mathbf{Z}$. Suppose there exists an isogeny $\chi : B_0 \to A_0$ with $A_0[3](\mathbf{F}_3) = (\mathbf{Z}/3\mathbf{Z})^2$. The kernel of χ must contain $B_0[3](\mathbf{F}_3)$. Now suppose that the characteristic polynomial $Q(x)$ of Frobenius satisfies $Q(1) \equiv \pm 3 \bmod 9$, which occurs in precisely four of these cases. It follows that (up to isogenies of degree prime to 3) the map χ will factor through $1 - \phi$ where ϕ is the Frobenius morphism, and since this reduces the power of three dividing the degree, we reduce to the case when χ has degree prime to three and we obtain a contradiction. In the remaining case (which we exploited above), we have $Q(x) = (3 - x + x^2)(3 + 2x + x^2)$ and so $Q(1) \equiv 0 \bmod 9$, and now such an isogeny is possible.

9.4. **A 2-3 switch.** We begin with the following approximation lemma.

Lemma 9.4.1. *Let Z be a rational variety over $\mathbf{Q}$. Let S be a finite set of places of $\mathbf{Q}$, and for each $v \in S$, let Ω_v be a non-empty open subset of $Z(\mathbf{Q}_v)$ (for the v-adic topology). Then there exists a rational point $P \in Z(\mathbf{Q})$ with $P_v \in \Omega_v$ for all $v \in S$, and such that P avoids any fixed thin subset of $Z(\mathbf{Q})$.*

Proof. Apart from the statement that we may avoid any fixed thin subset of $Z(\mathbf{Q})$, this is a special case of [Ser08, Lem. 3.5.5], and our proof is an obvious variation on the arguments of [Ser08, §3.4, §3.5]. We may assume that S is nonempty. After shrinking Z if necessary, we may assume that $Z \hookrightarrow \mathbf{P}^n_{\mathbf{Q}}$ is an open immersion. Here we use that Z is smooth; this guarantees that, for any open $U \subset Z$, $\Omega_v \cap U(\mathbf{Q}_v) \subset U(\mathbf{Q}_v)$ is non-empty. Since $Y = \mathbf{P}^n_{\mathbf{Q}} \smallsetminus Z$ is closed, $Y(\mathbf{Q}_v) \subset \mathbf{P}^n_{\mathbf{Q}}(\mathbf{Q}_v)$ is closed, and so $\Omega_v \subset Z(\mathbf{Q}_v) \subset \mathbf{P}^n_{\mathbf{Q}}(\mathbf{Q}_v)$ is open. Since

$$\mathbf{P}^n_{\mathbf{Q}}(\mathbf{Q}) \cap \Omega_v \subset \mathbf{P}^n_{\mathbf{Q}}(\mathbf{Q}) \cap Z(\mathbf{Q}_v) = Z(\mathbf{Q}),$$

we can and do assume that $Z = \mathbf{P}^n_{\mathbf{Q}}$.

The number of points in $\mathbf{P}^n_{\mathbf{Q}}(\mathbf{Q})$ which are of height at most H and are contained in Ω_v for all $v \in S$ grows at the rate of a positive constant times H^n (the precise constant depending on the open sets $\{\Omega_v\}$), whereas the number of points in any fixed thin set is bounded by $O(H^{n-1/2} \log H)$ by [Ser08, Thm. 3.4.4], and the result follows. □

We now construct a suitable abelian surface through which to do our 2-3-switch.

Lemma 9.4.2. *Suppose that*

$$\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$$

has similitude $\overline{\varepsilon}^{-1}$, that $\overline{\rho}^\vee|_{G_{\mathbf{Q}_3}}$ is ordinary and finite flat, and that $\overline{\rho}|_{G_{\mathbf{Q}_2}}$ is unramified.

1. *The following conditions are equivalent:*
   1. *(a) The image of $\overline{\rho}(\mathrm{Frob}_2)$ in $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$ is not conjugate to $4C$ or $12C$ (see Lemma 9.1.3).*
   2. *(b) $\overline{\rho}|_{G_{\mathbf{Q}_2}} \simeq \overline{\rho}_{A,3}$, where A is the Jacobian of a genus 2 curve $Y/\mathbf{Q}_2$ with a rational Weierstrass point, and where A has either good ordinary or semistable ordinary reduction at 2 and is 2-distinguished.*

(2) *Assume that the equivalent conditions in* (1) *hold. Then there exists a genus two curve* $X/\mathbf{Q}$ *with a rational Weierstrass point, with* $B = \mathrm{Jac}(X)$ *having the following properties:*
   (a) $\overline{\rho}_{B,3} \cong \overline{\rho}$.
   (b) *B has good ordinary or semistable ordinary reduction at* 2*, and is* 2*-distinguished.*
   (c) *B has good ordinary reduction at* 3*.*
   (d) *The representation*
$$\overline{\rho}_{B,2} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_2)$$
   has image $S_5(b)$*, and the image of complex conjugation has conjugacy class* $(**)(**)$*.*

   Moreover, $\mathrm{End}(B_{\overline{\mathbf{Q}}}) = \mathbf{Z}$*.*

Proof. We recall from Definition 9.2.1; that $\mathcal{M}_2^w(\overline{\rho})$ and $P = P(\overline{\rho})$ are the fine moduli spaces over $\mathbf{Q}$ parametrizing respectively genus 2 curves X with a rational Weierstrass point together with a symplectic isomorphism $\overline{\rho}_{\mathrm{Jac}(X),3} \cong \overline{\rho}$, and principally polarized abelian surfaces with a fixed odd theta characteristic and a symplectic isomorphism $\overline{\rho}_{A,3} \cong \overline{\rho}$.

We claim that condition (1a) is equivalent to condition (1b) by Theorem 9.2.3. More precisely, that theorem shows that (1a) implies there exists a point $A \in P(\overline{\rho})(\mathbf{Q}_2)$ with either good ordinary or semistable ordinary reduction (and which is 2-distinguished), whereas condition (1b) shows that there is a point $A = \mathrm{Jac}(Y) \in P(\overline{\rho})(\mathbf{Q}_2)$ which lies in the image of $\mathcal{M}_2^w(\overline{\rho})$. The variety $P(\overline{\rho})$ is smooth and the map $\mathcal{M}_2^w(\overline{\rho}) \to P(\overline{\rho})$ is an open immersion. Moreover, all the properties listed hold in any open ball around any such point A by [Kis99, Thm. 5.1]. Hence given $A \in P(\overline{\rho})(\mathbf{Q}_2)$ there exists a point $B \in P(\overline{\rho})(\mathbf{Q}_2)$ with the same properties but lying in the image of $\mathcal{M}_2^w(\overline{\rho})$.

Having established this equivalence, we now turn to the proof of part (2), so we in particular assume that condition (1b) holds. We now use Lemma 9.4.1 (applied to $Z = \mathcal{M}_2^w(\overline{\rho})$) to produce a suitable point $X/\mathbf{Q}$. Our set S will consist of the primes 2, 3, ∞. The corresponding thin set inside $Z(\mathbf{Q}) \subset P(\mathbf{Q})$ is the union of the rational points in the images of $P_G(\mathbf{Q})$, where $P_G \to P$ is the cover corresponding to imposing that the image of $\overline{\rho}_{B,2}$ lands inside a strict subgroup $G \subset S_5(b)$. There are finitely many such G and the degree of P_G over P is $[S_5(b) : G] > 1$, so this is indeed a thin set.

(1) Suppose that $p = 2$. Condition (1b) implies that there exists a point X in $Z(\mathbf{Q}_2)$ with the required properties ($\mathrm{Jac}(X)/\mathbf{Q}_2$ with good ordinary reduction or semistable ordinary reduction and 2-distinguished in characteristic zero). By [Kis99, Thm. 5.1], there exists an open ball $\Omega_2 \subset P(\mathbf{Q}_2)$ around X consisting of points which also have good ordinary reduction and are 2-distinguished.
(2) Suppose that $p = 3$. Then there exists a suitable point $X \in Z(\mathbf{Q}_3)$ by Lemma 9.3.7. As above, we take Ω_3 to be a suitable open ball around X.
(3) For $p = \infty$, we choose $\Omega_\infty \subset Z(\mathbf{R})$ to be a sufficiently small open ball around any point with the correct local properties, namely $y^2 = f(x)$ for any separable $f(x) \in \mathbf{R}[x]$ of degree 5 with exactly one real root.

The existence of X and B then follows from Lemma 9.4.1. Since the image of $\overline{\rho}_{B,2}$ is $S_5(b)$, it follows from [Zar00] that $\mathrm{End}(B_{\overline{\mathbf{Q}}}) = \mathbf{Z}$. □

Remark 9.4.3. The application of Lemma 9.3.7 in the proof of Lemma 9.4.2 can obviously additionally be used to show that B can be chosen to be 3-distinguished, but we shall not use this fact, so we have not explicitly recorded it.

Remark 9.4.4. Suppose that $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$ has multiplier ε^{-1}, and $\overline{\rho}|_{G_{\mathbf{Q}_3}}$ is ordinary and (dual to) finite flat. Then, exactly as in the proof of Lemma 9.4.2 (now ignoring the conditions at 2) obtains infinitely many genus two curves $X/\mathbf{Q}$ with a rational Weierstrass point and such that $A = \mathrm{Jac}(X)$ has good ordinary reduction at 3, such that $\overline{\rho}_{X,3} \cong \overline{\rho}$. This was implicitly assumed in the proof of [BCGP21, Theorem 10.2.1].

9.5. **Proof of Theorems A and B.** In this section, we prove Theorem A, which we restate as Theorem 9.5.2 below, except that the hypothesis on the image of $\overline{\rho}_{A,3}$ has been relaxed. (Note that if $\overline{\rho}_{A,3}$ is surjective, then $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$ is automatic, so Theorem 9.5.2 really does imply Theorem A.) We begin, however, with the following modularity lifting theorem.

Theorem 9.5.1. *Suppose that $p > 2$, and that $A/\mathbf{Q}$ and $B/\mathbf{Q}$ are abelian surfaces such that:*

1. $\overline{\rho}_{A,p} \cong \overline{\rho}_{B,p}$.
2. *A and B both have good ordinary reduction at p, and $\rho_{A,p}|_{G_{\mathbf{Q}_p}}$ is p-distinguished.*
3. *B is modular; more precisely, there is a weight 2 cuspidal automorphic representation π for $\mathrm{GSp}_4/\mathbf{Q}$ of level prime to p, which is ordinary at p and satisfies $\rho_{\pi,p} \cong \rho_{B,p}$.*
4. *The Zariski closure of $\rho_{A,p}(G_{\mathbf{Q}})$ contains Sp_4.*
5. *$\overline{\rho}_{A,p}$ is GSp_4-reasonable, in the sense of* [Whi22, Defn. 3.19].
6. *$\overline{\rho}_{A,p}$ is tidy, in the sense of* [BCGP21, Defn. 7.5.11].
7. *$\overline{\rho}_{A,p}(G_{\mathbf{Q}(\zeta_p)})$ contains a regular semi-simple element.*
8. *$\overline{\rho}_{A,p}(G_{\mathbf{Q}}) \smallsetminus \mathrm{Sp}_4(\mathbf{F}_p)$ contains a regular semi-simple element.*

Then A is modular. More precisely, there exists a cuspidal automorphic representation π for $\mathrm{GL}_4/\mathbf{Q}$ (the transfer of a cuspidal automorphic representation of $\mathrm{GSp}_4/\mathbf{Q}$ of weight 2) such that $L(s, H^1(A), s) = L(s, \pi)$.

Proof. We deduce the theorem from Theorem 7.5.11 (taking ρ there to be $\rho_{A,p}$). By our assumption (4), it suffices to check that hypotheses (1)–(5) and (B1)–(B5) of Proposition 7.5.10 hold.

Most of these conditions hold either explicitly by our assumptions, or by the purity of Galois representations associated to abelian surfaces. The only remaining conditions are:

(a) $\rho(G_{\mathbf{Q}(\zeta_{p^\infty})})$ is integrally enormous.
(b) We can choose p-stabilizations of $\rho_{A,p}|_{G_{\mathbf{Q}_p}}$ and $\rho_{B,p}|_{G_{\mathbf{Q}_p}}$ such that the representations $\rho_{B,p}|_{G_{\mathbf{Q}_p}}$ lies on a unique irreducible component of $\mathrm{Spec}\, R_p^{\triangle}$ and $\rho_{A,p}|_{G_{\mathbf{Q}_p}}$ lies on the same component.

Part (a) follows from Corollary 7.1.4. For Part (b), we firstly choose a p-stabilization of $\rho_{B,p}|_{G_{\mathbf{Q}_p}}$, and thus of $\overline{\rho}_{B,p}|_{G_{\mathbf{Q}_p}} = \overline{\rho}_{A,p}|_{G_{\mathbf{Q}_p}}$. Then at least one of the p-stabilizations of $\rho_{A,p}|_{G_{\mathbf{Q}_p}}$ is compatible with this fixed choice, and we conclude by Lemma 6.2.5 and the assumption that A and B both have good ordinary reduction at p. $\square$

We are now ready to prove our main theorem.

Theorem 9.5.2. *Let $A/\mathbf{Q}$ be an abelian surface with a polarization of degree prime to 3. Suppose that the following conditions hold:*

(1) The image of the mod 3 representation:

$$\overline{\rho}_{A,3} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{F}_3)$$

is one of the 15 subgroups listed in Lemma 6.4.3, and $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$.

(2) $\overline{\rho}_{A,3}|_{G_{\mathbf{Q}_2}}$ is unramified, and the image of $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ inside $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$ does not have conjugacy class $4C$ or $12C$ (see Lemma 9.1.3). Equivalently, the characteristic polynomial of $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ is not $(x^2 \pm x + 2)^2$.

(3) A has good ordinary reduction at 3 and is 3-distinguished.

Then A is modular. More precisely, there exists a cuspidal automorphic representation π for $\mathrm{GL}_4/\mathbf{Q}$ (the transfer of a cuspidal automorphic representation of $\mathrm{GSp}_4/\mathbf{Q}$ of weight 2) such that $L(s, H^1(A)) = L(s, \pi)$.

Proof. By Lemma 9.4.2 (2) (which applies to $\overline{\rho}_{A,3}$, since condition (1a) of Lemma 9.4.2 holds by our assumption (2)), there exists a genus two curve $X/\mathbf{Q}$ with a rational Weierstrass point, with $B = \mathrm{Jac}(X)$ having the following properties:

- $\overline{\rho}_{A,3} \cong \overline{\rho}_{B,3}$.
- B has semistable ordinary or good ordinary reduction at 2, and is 2-distinguished.
- B has good ordinary reduction at 3.
- The representation

$$\overline{\rho}_{B,2} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_2)$$

  has image $S_5(b)$, and the image of complex conjugation has conjugacy class $(**)(**)$.
- $\mathrm{End}(B_{\overline{\mathbf{Q}}}) = \mathbf{Z}$.

We shall first apply Theorem 8.3.2 (a 2-adic modularity theorem; we take A there to be our B) to deduce that B is modular. To recall, the hypotheses of Theorem 8.3.2 are as follows:

(i) $A_5(b) \subseteq \overline{\rho}_{B,2}(G_{\mathbf{Q}}) \subseteq S_5(b)$.
(ii) The image of each complex conjugation has order 2 and lands in $A_5(b)$.
(iii) $\rho_{B,2}|_{G_{\mathbf{Q}_2}}$ is ordinary and 2-distinguished.

All of these conditions are guaranteed by the properties of B listed above, noting that $(**)(**)$ is a non-trivial conjugacy class contained in $A_5(b)$. Thus B is modular. More precisely, there is a weight 2 cuspidal automorphic representation π for $\mathrm{GSp}_4/\mathbf{Q}$ which in particular satisfies $\rho_{\pi,3} \cong \rho_{B,3}$; furthermore π is necessarily of level prime to 3 and is ordinary at 3 by local-global compatibility.

We now wish to use Theorem 9.5.1 at $p = 3$ to deduce that A is modular, so we need to check the conditions of that theorem. We established that B is modular above, and thus condition (3) holds. The isomorphism $\overline{\rho}_{A,3} \cong \overline{\rho}_{B,3}$ holds by the construction of B, hence we have condition (1). Both A and B have good ordinary reduction at 3 and A is furthermore 3-distinguished (by assumption for A and by construction for B), and thus we have condition (2). We are assuming that $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$, so condition (4) holds by [Ser00, Thm 3]. Finally conditions (5), (6), (7) and (8) hold by Lemma 6.4.3 and our assumptions on A. ∎

Theorem 9.5.3. *Let X be a smooth genus two curve over $\mathbf{Q}$. Suppose that:*

1. *The image of $\overline{\rho}_{\mathrm{Jac}(X),3} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$ is one of the 15 subgroups listed in Lemma 6.4.3, and $\mathrm{End}(\mathrm{Jac}(X)_{\overline{\mathbf{Q}}}) = \mathbf{Z}$.*
2. *X has good ordinary reduction at 2.*
3. *X has good ordinary reduction at 3.*
4. *$\mathrm{Jac}(X)$ is 3-distinguished.*

Then X is modular.

Proof. Let $A = \mathrm{Jac}(X)$ (so that A is in particular principally polarized). It suffices to verify the conditions (1)–(3) of Theorem 9.5.2. Condition (1) is identical to our first condition. Since we are assuming that X has good ordinary reduction at 2 and 3, so does A. Condition (2) follows from Lemma 9.1.6(1). Finally condition (3) is immediate from our assumptions. Hence A (and thus X) is modular. □

We now deduce Theorem B, which we restate here, again with a weakening of the assumption that $\overline{\rho}_{X,3}$ is surjective.

Theorem 9.5.4. *Let $X : y^2 = f(x)$ with $\deg(f(x)) = 5$ be a smooth genus two curve over $\mathbf{Q}$. Suppose that:*

1. *The image of $\overline{\rho}_{X,3} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$ is one of the 15 subgroups listed in Lemma 6.4.3, and $\mathrm{End}(\mathrm{Jac}(X)_{\overline{\mathbf{Q}}}) = \mathbf{Z}$.*
2. *X has good ordinary reduction at 2.*
3. *X has good ordinary reduction at 3.*

Then X is modular.

Proof. By Theorem 9.5.3, it suffices to show that $\mathrm{Jac}(X)$ is 3-distinguished. Since X has a rational Weierstrass point (by our assumption that $f(x)$ has degree 5), this follows immediately from Lemma 9.3.1(3). □

Note that Theorem A and Theorem 9.5.2 do not require that A has good reduction at 2, only that $\overline{\rho}_{A,3}$ is unramified at 2. Here we answer a question of Drew Sutherland, who asks if the conditions of our main theorem are easy to verify computationally if $\mathbf{Q}(A[3])$ is unramified at 2 but A has *bad* reduction at 2. It turns out that the answer is surprisingly simple.

Theorem 9.5.5. *Let $A/\mathbf{Q}$ be an abelian surface with a polarization of degree prime to 3. Suppose the following holds:*

1. *The image of the mod 3 representation:*
$$\overline{\rho}_{A,3} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{F}_3)$$
is one of the 15 subgroups listed in Lemma 6.4.3, and $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$.
2. *$\overline{\rho}_{A,3}|_{G_{\mathbf{Q}_2}}$ is unramified.*
3. *A has good ordinary reduction at 3 and the characteristic polynomial of Frobenius at 3 does not have repeated roots.*
4. *A has bad reduction at 2.*

Then A is modular.

Proof. We shall apply Theorem 9.5.2. It suffices to show that, under the assumption that A has *bad* reduction at 2, that the action of Frob_2 on $A[3]$ does not have characteristic polynomial $(x^2 \pm x + 2)^2$. By Lemma 9.2.2, we deduce that A has semistable reduction. Hence, as in the proof of Lemma 9.1.8, we deduce the existence of a $G_{\mathbf{Q}_2}$-equivariant filtration $T_2(B)_t \subset T_2(B)_f \subset T_2(B)$ of (saturated) $\mathbf{Z}_3$-modules of ranks $t > 0$ and $t + 2a$ where $2(t + a) = 4$. If $t = 1$, then $A[3]$ has a $G_{\mathbf{Q}_2}$-stable line. But $x^2 \pm x + 2$ has no eigenvalues in $\mathbf{F}_3$, which concludes the proof in this case. Assume that $t = 2$, so A has purely multiplicative reduction. It follows that A has split multiplicative reduction over some minimal unramified extension $K/\mathbf{Q}_2$. There is a corresponding action of $\mathrm{Gal}(K/\mathbf{Q}_2)$ on $\mathbf{G}_m \times \mathbf{G}_m$ which gives the descent data to $\mathbf{Q}_2$; this determines a finite order element of $\mathrm{GL}_2(\mathbf{Z})$, and such elements can only have orders 1, 2, 3, 4, or 6. (The characteristic polynomial of this element will be, up to normalization, the L-factor of A at $p = 2$.) On the other hand, the action of $G_{\mathbf{Q}_2}$ on $T_3(B) \otimes \mathbf{Q}_3$ factors through $\mathrm{Gal}(K/\mathbf{Q}_2)$. By considering the action on the unramified quotient $A[3]$, we deduce that $\mathrm{Gal}(K/\mathbf{Q}_2)$ has order divisible by 8, since $x^4 + 1 = (x^2 + x + 2)(x^2 - x + 2) \bmod 3$. This is a contradiction. □

Remark 9.5.6. An alternative argument is to note that if the characteristic polynomial of Frob_2 on $\overline{\rho}_{A,3}$ is $(x^2 \pm x + 2)^2$, then the ratio of any two eigenvalues is never equal to $2 = -1 \bmod 3$, and so $\overline{\rho}_{A,3}|_{G_{\mathbf{Q}_2}}$ has no ramified lifts.

10. Complements

This final section includes a number of results which are complementary to the main theorems of our paper (and in particular are not used elsewhere).

In §10.1, we explain how our main theorems apply (relative to a certain natural way of enumerating genus two curves) to slightly over 10% of all such curves, and we compare this to the data in the LMDFB [LMF24]. In §10.2, we prove the automorphy of any abelian surface $A/\mathbf{Q}$ which falls into 32 of the 34 possible Galois types. In §10.3, we prove some residual modularity theorems (Serre's conjecture) for mod-2 representations $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_2)$ with image A_6 or S_6. Finally, in §10.4, we point out that a sufficiently strong version of Serre's conjecture for GSp_4 in *regular* weight would be enough to prove the modularity of all abelian surfaces $A/\mathbf{Q}$.

10.1. **Examples.** Suppose one samples genus two curves

$$X : y^2 + h(x)y = f(x)$$

with $h(x), f(x) \in \mathbf{Z}[x]$ of degrees ≤ 3 and ≤ 6 in any way in which the distributions modulo 2 and 3 are equidistributed, and considers curves X with the following properties:

(1) X has good reduction at 2,
(2) X has good ordinary reduction at 3,
(3) $\overline{\rho}_{\mathrm{Jac}(X),3}(\mathrm{Frob}_2)$ does not have characteristic polynomial $x^4 \pm x^3 + 2x^2 \pm x + 1$, equivalently, is not projectively conjugate to $4C$ or $12C$,
(4) The characteristic polynomial $Q_3(x)$ of Frob_3 has distinct eigenvalues.

Then from Lemmas 9.1.6 and 9.3.1 these X form a subset of density

$$\frac{768 - 144}{2^{11}} \cdot \frac{864 - 10}{3^7} = \frac{13}{16} \cdot \frac{3}{8} \cdot \frac{854}{2187} = \frac{5551}{46656} = 0.1189\ldots$$

(Note that 13/16 is the density of allowable elements for $\overline{\rho}_{X,3}(\mathrm{Frob}_2)$, and 3/8 is the density of curves with good reduction at 2.) Since $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$ holds for a set of density one, we see in particular that Theorem 9.5.2 applies to a positive (although not so large, slightly over 10%) proportion of all genus 2 curves. (The main theorem of [Wil95] also applies to a positive but strictly less than one proportion of all genus 1 curves by any natural counting.)

Another point of comparison is with the curves in the database [LMF24]: There are 66158 genus two curves X in [LMF24, BSS$^+$16]:

(1) Of those, 63107 have $\operatorname{End} A_{\overline{\mathbf{Q}}} = \mathbf{Z}$, where $A = \operatorname{Jac}(X)$.
(2) Of those, 22158 have good reduction at 2 and 3. (In the range of the data, a genus 2 curve X has good reduction at p (for any p) if and only if $A = \operatorname{Jac}(X)$ has good reduction at p.)
(3) Of those, 21552 have surjective mod 3 representations.
(4) Of those, 14856 have ordinary reduction at $p = 3$.
(5) Of those, our theorem applies to 11384 curves, where the distribution of various conjugacy classes and 3-distinguishedness conditions is indicated in Table 10.1.

Table 10.1.1. $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ distributions of $A = \operatorname{Jac}(X)$ where X has good reduction at 2, good ordinary reduction at 3, $\overline{\rho}_{A,3}$ is surjective, and X is taken from [BSS$^+$16], together with a count of those to whom Theorem 9.5.2 applies.

$\overline{\rho}_{A,3}(\mathrm{Frob}_2)$	ordinary at 2		non-ordinary at 2	
	3-dist	not 3-dist	3-dist	not 3-dist
$2C/6G/6H$	1048	8	840	7
$4D$	0	0	890	2
$2D/6I$	825	1	0	0
$8A$	1233	9	854	6
$10A$	3407	48	2287	4
$4C/12C$	0	0	3369	18
All	6513	66	8240	37
Theorem 9.5.2 applies	6513	0	4871	0

If one allows $\overline{\rho}_{A,3}$ to be any of the 15 subgroups listed in Lemma 6.4.3, there are three additional curves, precisely one of which we can deduce is modular by Theorem 9.5.2. This is the curve `7889.b.55223.1` of conductor $7^3 \cdot 23$. The representation $\overline{\rho}_{A,3}$ in this case (with image of order 2304) is induced from a representation $\overline{\rho}_{E,3} : G_F \to \mathrm{GL}_2(\mathbf{F}_3)$, where E is a modular elliptic curve over $F = \mathbf{Q}(\sqrt{-7})$ with [LMF24] label `2.0.7.1-322.1-a1` and conductor of norm $322 = 2 \cdot 7 \cdot 23$.

There are 41324 curves in the [LMF24] database with $\overline{\rho}_{A,3}$ surjective, $\operatorname{End}_{\overline{\mathbf{Q}}}(A) = \mathbf{Z}$, and such that A has good reduction at 3. Of those, 19772 of these curves have *bad* reduction at 2. We find that for precisely 360 of these curves, A is ordinary at 3 and $\overline{\rho}_{A,3}$ is unramified at 2. Of those, 359 are 3-distinguished and thus modular by Theorem 9.5.5. In particular, Theorem A applies to precisely $11384 + 359 = 11743$ of the 66158 curves in the [LMF24]. (One can verify modularity for more of the curves in [LMF24] by including quadratic twists.) The smallest conductor of such an A with bad reduction at 2 is 1982; the corresponding A is the Jacobian of the curve

$$y^2 + (x+1)y = -x^5 + x^4 - x^3 + x^2.$$

10.2. **Automorphy for abelian surfaces with small Sato–Tate group.** In this section, we prove the automorphy (in the sense of Definition 1.8.25) for 32 of the 34 Galois types (in the sense of [FKRS12]) of abelian surfaces A over $\mathbf{Q}$. We closely follow [BCGP21, §9.2] and use freely the notation of that section, as well as the results summarized there from [FKRS12, Joh17] (see also [Tay20]). We say that a Galois representation is "finite up to twist" if it is a twist by a character of a representation with finite image.

Recall that the Galois type of $A/\mathbf{Q}$ is $\mathbf{A}$ precisely when $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$, and $A/\mathbf{Q}$ has type $\mathbf{B}[C_2]$ if there exists a quadratic field $K/\mathbf{Q}$ so that $\mathrm{End}(A) = \mathbf{Z}$ but $\mathrm{End}(A_K) \otimes \mathbf{Q}$ is either $\mathbf{Q} \oplus \mathbf{Q}$ or a real quadratic field. (In [BCGP21, §9.2], we call an abelian surface $A/\mathbf{Q}$ "challenging" precisely when it is one of these two types.)

The main theorem of this section is as follows.

Theorem 10.2.1. *Let $A/\mathbf{Q}$ be an abelian surface. Suppose that the Galois type of A is neither $\mathbf{A}$ nor $\mathbf{B}[C_2]$. Then A is modular.*

Remark 10.2.2 (Abelian surfaces of Galois type $\mathbf{B}[C_2]$)**.** A natural source of abelian surfaces of type $\mathbf{B}[C_2]$ are those of the form $\mathrm{Res}_{K/\mathbf{Q}}(E)$ for a non-CM elliptic curve E which is not isogenous to its $\mathrm{Gal}(K/\mathbf{Q})$-conjugate. If $K/\mathbf{Q}$ is real quadratic, then E is automorphic for GL_2/K by [FLHS15] and then A is automorphic for $\mathrm{GL}_4/\mathbf{Q}$. The modularity of elliptic curves E over imaginary quadratic fields K is known in many cases (but not yet all) by [CN23]. On the other hand, for A of type $\mathbf{B}[C_2]$, the endomorphism algebra $\mathrm{End}(A_K) \otimes \mathbf{Q}$ could also be a real quadratic field E rather than $\mathbf{Q} \times \mathbf{Q}$, in which case A/K will be a simple abelian surface of GL_2-type. This happens, for example, when A is the Jacobian of the genus 2 curve

$$y^2 + (x^3+1)y = x^6 + 2x^3 - x$$

with $E = K = \mathbf{Q}(\sqrt{5})$ [LMF24, genus 2 curve `12500.a.12500.1`]. The modularity of such abelian surfaces remains open in general even for real quadratic fields K.

Proof of Theorem 10.2.1. Following the discussion in [BCGP21, §9.2] and [BCGP21, Prop 9.2.1], all abelian surfaces $A/\mathbf{Q}$ can be divided up into a number of possible Galois types, which, writing $\{\rho_{A,p}\}$ for the compatible system of Galois representations $\{H^1(A_{\overline{\mathbf{Q}}}, \overline{\mathbf{Q}}_p)\}$, fall into the following categories independently of p:

(1) strongly irreducible (type $\mathbf{A}$),
(2) reducible (type $\mathbf{B}[C_1]$, $\mathbf{C}$, $\mathbf{E}[C_n]$, some $\mathbf{D}$, some $\mathbf{F}$),
(3) potentially abelian but not reducible (of type the remaining $\mathbf{D}$ and $\mathbf{F}$ cases),
(4) induced from a quadratic extension $K/\mathbf{Q}$ but not potentially abelian, in which case either:
    (a) the two 2-dimensional representations over K are equivalent up to twist (type $\mathbf{E}[D_n]$), or
    (b) the two 2-dimensional representations over K are not equivalent up to twist (type $\mathbf{B}[C_2]$).

We will prove automorphy in all cases except those of type $\mathbf{A}$ and those of type $\mathbf{B}[C_2]$.

In the reducible cases, it follows from [FKRS12] that the compatible system associated to A can be written as a direct sum of two irreducible, odd, regular, weakly compatible systems of Galois representations over $\mathbf{Q}$. These are modular by [KW09]. In case $\mathbf{E}[D_n]$, we see (as in the proof of [BCGP21, Prop 9.2.1]) that there exists a quadratic extension $K/\mathbf{Q}$ and an odd irreducible regular weakly compatible system $\mathcal{S} = \{s_p\}$ of $G_{\mathbf{Q}}$ such that $\rho_p \simeq s_p \otimes \mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \psi^{-1} = \mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} s_p|_{G_K} \otimes \psi^{-1}$ for some fixed finite order character ψ. Once more $\mathcal{S}$ is automorphic for $\mathrm{GL}_2/\mathbf{Q}$ by [KW09], and then A is is automorphic for $\mathrm{GL}_4/\mathbf{Q}$, as required.

It remains to consider the cases where ρ_p is (absolutely) irreducible but potentially abelian. Since the representations ρ_p have similitude character ε^{-1} and ρ_p is not finite up to twist (since it has distinct Hodge–Tate weights), this last case follows from Lemma 10.2.5 below. □

In the remainder of this section we prove Lemma 10.2.5, which was used in the proof of Theorem 10.2.1. We begin with some preliminary lemmas, the first of which concerns representations which have potentially abelian image.

Lemma 10.2.3. *Let F be a number field, and let $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a continuous irreducible representation which is de Rham at all places dividing p and potentially abelian over a finite Galois extension L/F. Then there exist integers a, b, with $ab = n$, and b pairwise distinct characters $\chi_i : G_L \to \overline{\mathbf{Q}}_p^\times$ such that*

$$\rho|_{G_L} \simeq \bigoplus_{i=1}^{b} (\chi_i)^{\oplus a}.$$

The action of $\mathrm{Gal}(L/F)$ on the characters χ_i via $\chi_i^\sigma(g) = \chi_i(\sigma g \sigma^{-1})$ induces a map

$$\mathrm{Gal}(L/F) \to S_b$$

with transitive image. Let $\mathrm{Gal}(L/K_i)$ be the stabilizer of χ_i. Then there exists an irreducible representation V_i of G_{K_i} such that $V_i|_{G_L} \cong (\chi_i)^{\oplus a}$, and

$$\rho \simeq \mathrm{Ind}_{G_{K_i}}^{G_F} V_i.$$

If ρ is not finite up to twist, then:

(1) the characters χ_i are associated to algebraic Hecke characters of non-parallel weight.
(2) $b > 1$.
(3) If $a = 1$, then each K_i contains an imaginary CM field.

Proof. Since the χ_i eigenspace is mapped to the χ_i^σ eigenspace under the action of $\rho(\sigma)$ for any lift of $\sigma \in \mathrm{Gal}(L/F)$ to G_F, the group $\mathrm{Gal}(L/F)$ acts transitively on the characters (since otherwise the direct sum of the eigenspaces for χ_i^σ for any given i would be a non-trivial G_F-invariant subspace of ρ, and we are assuming that ρ is irreducible). Similarly, the multiplicity of each χ_i is independent of i. Let V be the vector space underlying the representation ρ, and let V_i denote the subspace on which G_L acts by χ_i. Since G_F preserves the decomposition $V = \bigoplus V_i$, it follows that V_i extends to a representation of G_{K_i} where $\mathrm{Gal}(L/K_i)$ stabilizes χ_i. By the orbit–stabilizer theorem, $[K_i : F] = b$. By Frobenius reciprocity, there is a non-trivial map $V \to \mathrm{Ind}_{G_{K_i}}^{G_F} V_i$, which (because V is irreducible) is an isomorphism, and hence V_i must also be irreducible.

Assume for the remainder of the proof that ρ is not finite up to twist. Each character χ_i is de Rham and thus is either a finite order character times an integer power of the cyclotomic character or has non-parallel weight. In the first case, after twisting ρ by a power of the cyclotomic character, we may assume that χ_i has finite image. But then V_i and hence ρ also have finite image, contrary to our assumption. If $b = 1$, then the projective image of ρ restricted to G_L is trivial, and hence the image of the projective representation $P\rho$ is finite. From the vanishing of $H^2(F, \overline{\mathbf{Q}}_p^\times)$ [Ser77b, Thm. 4], it follows $P\rho$ lifts to a genuine representation $\widehat{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ which has finite image. Since ρ is irreducible, $\widehat{\rho} \simeq \rho \otimes \chi$ for some χ, and thus ρ is finite up to twist, once more contrary to our assumption.

If $a = 1$, then the action of G_{K_i} on V_i is via a character ψ_i which restricts to χ_i over G_L. We have already shown that χ_i has non-parallel weight, and thus ψ_i also corresponds to an algebraic Hecke character of non-parallel weight, which implies that K_i contains an imaginary CM field. □

Lemma 10.2.4. *Let F be a number field, and let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be a continuous irreducible representation which is de Rham at p and potentially abelian over a finite extension. Then either:*

(1) ρ is finite up to twist, or
(2) ρ is automorphic for a cuspidal automorphic representation π for GL_2/F which is the automorphic induction of an algebraic Hecke character.

Proof. Suppose that $\rho|_{G_L}$ becomes reducible over a Galois extension L/F, and write $\rho|_{G_L} \simeq \chi_1 \oplus \chi_2$. If $\chi_1 = \chi_2$, then ρ is finite up to twist by Lemma 10.2.3. Thus we may assume that $\chi_1 \neq \chi_2$, and then by Lemma 10.2.3, we see that $K = K_i/F$ is cyclic of degree 2 and $\rho = \mathrm{Ind}_{G_K}^{G_F} \psi_p$ where ψ_p corresponds to an algebraic Hecke character ψ of G_K. □

Lemma 10.2.5. *Let $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ be a continuous irreducible representation which is de Rham at p. Suppose that:*

(1) There exists a Galois extension $L/\mathbf{Q}$ so that the image of $\rho|_{G_L}$ is abelian.
(2) If ν is the similitude character, then $\nu(c) = -1$, where c is complex conjugation.
(3) ρ is not finite up to twist.

Then ρ is modular.

Proof. By Lemma 10.2.3, there is a decomposition $\rho|_{G_L} \simeq \bigoplus_{i=1}^{b} (\chi_i)^{\oplus a}$ with $ab = 4$. Since ρ is not finite up to twist, it follows from Lemma 10.2.3 that $b > 1$. Suppose that $b = 2$. Then $\rho \simeq \mathrm{Ind}_{G_{K_i}}^{G_{\mathbf{Q}}} V_i$ where $[K_i : \mathbf{Q}] = 2$ and $\dim(V_i) = 2$. Note that since $\mathrm{Gal}(L/K_i)$ is the stabilizer of a point with respect to the map $\mathrm{Gal}(L/\mathbf{Q}) \to S_2$, the field $F = K_i$ does not depend on i. We know that V_1 and V_2 are irreducible and potentially abelian, so by Lemma 10.2.4 either both V_i are automorphic for GL_2/F, in which case ρ is modular, and we are done; or both V_i are finite up to twist, which we assume from now on.

If we write $V = \mathrm{Ind}_{G_F}^{G_{\mathbf{Q}}} V_i$, then

$$\wedge^2 V = \mathrm{Ind}_{G_F}^{G_{\mathbf{Q}}} \det(V_i) \oplus \mathrm{Asai}_{F/\mathbf{Q}}(V_i),$$

where $\mathrm{Asai}_{F/\mathbf{Q}}(V_i)|_{G_F} \simeq V_1 \otimes V_2$ (this only characterizes the representation over $G_{\mathbf{Q}}$ up to quadratic twist but it is all that we will use in this argument). Since V admits a symplectic form which is Galois invariant up to a similitude character, we know that $\wedge^2 V$ must contain a character. We deduce either that $\det(V_i)$ extends to $\mathbf{Q}$ or $\mathrm{Asai}(F/\mathbf{Q})(V_i)$ is reducible. Suppose firstly that $\det(V_i)$ is the restriction to G_G of a character χ of $G_{\mathbf{Q}}$. The image of χ lands in $\mathcal{O}_E^\times$ for some finite extension $E/\mathbf{Q}_p$, and so χ factors through a quotient of $G_{\mathbf{Q}}$ of the form $\mathbf{Z}_p^r \oplus T$ for some finite group T.

Define a new character $\psi : G_{\mathbf{Q}} \to \mathcal{O}_E^\times$ by sending topological generators σ of each of these $\mathbf{Z}_p$ factors to any square root of $\chi(\sigma)$, so that $\chi\psi^{-2}$ has finite order. Then $\det(V_i \otimes \psi|_{G_F}^{-1})$ has finite order, and $V_i \otimes \psi|_{G_F}^{-1}$ is finite up to twist, so $V_i \otimes \psi|_{G_F}^{-1}$ has finite image. Since

$$V = \operatorname{Ind}_{G_F}^{G_{\mathbf{Q}}} V_i = \psi \otimes \operatorname{Ind}_{G_F}^{G_{\mathbf{Q}}}(V_i \otimes \psi|_{G_F}^{-1}),$$

and $V_i \otimes \psi|_{G_F}^{-1}$ has finite image, we see that ρ is finite up to twist, contradicting our assumptions.

Hence we may assume that the Asai representation contains a character as a constituent, and in particular its restriction $V_1 \otimes V_2$ to G_F does as well, and thus (since the V_i are irreducible over F) we have $V_1 \simeq V_2 \otimes \phi$ for some ϕ. This implies that the projective representations associated to V_1 and V_2 are isomorphic. Since V_1 and V_2 are $G_{\mathbf{Q}}$-conjugate, it follows that the projective representation associated to V_i extends to $\mathbf{Q}$, and thus (by Tate's theorem) V_i itself lifts (up to twist) to a representation V of $G_{\mathbf{Q}}$. It then follows that

$$\rho \simeq V \otimes \operatorname{Ind}_{G_F}^{G_{\mathbf{Q}}} \chi$$

where now $F/\mathbf{Q}$ is an imaginary quadratic field, χ is an algebraic Hecke character, and V is a representation of $G_{\mathbf{Q}}$ of finite image. If V is induced, then V and ρ are automorphic, so we may assume that V is not induced. The $G_{\mathbf{Q}}$-module V admits a unique symplectic form invariant up to a similitude character which is given by $\det(V)$, but V does not admit any corresponding orthogonal form, since V is not induced. On the other hand, we also see that $W = \operatorname{Ind}_{G_F}^{G_{\mathbf{Q}}} \chi$ admits a symplectic form which is invariant up to similitude character $\det(W)$, and an orthogonal form which is invariant up to similitude character $\det(W)\eta_{F/\mathbf{Q}}$, where $\eta_{F/\mathbf{Q}}$ is the quadratic character associated to the imaginary quadratic field $F/\mathbf{Q}$. We deduce that the unique symplectic form on $\rho = V \otimes W$ has similitude character $\det(V)\det(W)\eta_{F/\mathbf{Q}}$, which is odd if and only if $\det(V)$ is odd, since $\det(W)\eta_{F/\mathbf{Q}}$ is even. Thus the oddness assumption implies that V is an *odd* Artin representation, and thus V is modular by known cases of the Artin conjecture [PS16b, Sas19], and the automorphy of ρ follows.

Finally, suppose that $b = 4$, so $a = 1$. By Lemma 10.2.3, there exists a degree 4 field $K/\mathbf{Q}$ such that $\rho \simeq \operatorname{Ind}_{G_K}^{G_{\mathbf{Q}}} \chi$, where χ corresponds to an algebraic Hecke character of non-parallel weight, so that $K/\mathbf{Q}$ contains an imaginary CM field. In particular, either K is itself an imaginary CM field, and thus contains a real quadratic subfield $E = K^+$, or K contains an imaginary quadratic subfield E. In either case, we see that $\varrho = \operatorname{Ind}_{G_K}^{G_E} \chi$ corresponds to a cuspidal automorphic representation of GL_2/E, and by another application of automorphic induction we deduce that $\rho = \operatorname{Ind}_{G_E}^{G_{\mathbf{Q}}} \varrho$ is automorphic. □

Remark 10.2.6. Various rationality considerations (see [FKRS12]) imply that, if ρ is a potentially abelian Galois representation associated to an abelian surface $A/\mathbf{Q}$, then ρ is actually potentially abelian over a *solvable* extension of $\mathbf{Q}$, which can be used to simplify the argument in this case, On the other hand. Lemma 10.2.5 is conjecturally still true without either the oddness assumption, or the finiteness up to twist condition, although presumably extremely difficult. In the first case, it would include the automorphy of representations of the form $V \otimes \operatorname{Ind}_{G_F}^{G_{\mathbf{Q}}} \chi$ where χ is an algebraic Hecke character of a CM field F and V is an even Galois representation with projective image A_5, which would imply the automorphy of such a V.

Similarly, assume oddness holds but drop the finiteness up to twist condition. The group S_6 is a subgroup of $\mathrm{PGSp}_4(\mathbf{C})$; this S_6 can also be seen inside $\mathrm{PGSp}_4(\mathbf{F}_3)$. Let $L/\mathbf{Q}$ be any S_6 extension such that complex conjugation is odd. Then there is a projective representation $G_{\mathbf{Q}} \to \mathrm{Gal}(L/\mathbf{Q}) \simeq S_6 \hookrightarrow \mathrm{PGSp}_4(\mathbf{C})$. Any lift to an Artin representation $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{C})$ will be odd, but the modularity of ρ is unknown for any representation with projective image S_6 regardless of the image of complex conjugation. Fortunately, neither case is relevant for the automorphy of abelian surfaces over $\mathbf{Q}$.

10.3. **Residual modularity theorems (modulo 2).** The goal in this section is to prove some residual modularity theorems for mod-2 representations with image A_6 or S_6. We will need the following variation of Lemma 9.3.4 for the prime $p = 2$.

Lemma 10.3.1. *Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbf{Q}_2$ with residue field $\mathbf{F}$. Let $G_1/\mathcal{O}$ be a 2-torsion finite flat group scheme of order $16 = 2^4$, together with an isomorphism $\lambda : G_1 \to G_1^\vee$ such that $\lambda^\vee = -\lambda$. Suppose that $A_0/\mathbf{F}$ is a principally polarized ordinary abelian surface with $A_0[2]/\mathbf{F} \simeq G_1/\mathbf{F}$. Then there exists a lift of A_0 to a principally polarized abelian surface $A/\mathcal{O}$ with $A[p] \simeq G_1$.*

Remark 10.3.2 (Remarks on the proof and the statement of Lemma 10.3.1)**.** Let $p = 2$. By Serre–Tate theory, we are reduced to finding an appropriate lifting $A_0[p^\infty]$ to a Barsotti–Tate group, together with a lifting of λ to make $A_0[p^\infty]$ a quasi-polarized BT. By a result of Grothendieck [Ill85], there is no issue in lifting $A_0[p^\infty]$ as a Barsotti–Tate group, so the subtlety is imposing the polarization. We proved an analogous statement in Lemma 9.3.4 (without any ordinary hypothesis) using results from [Wed01]. Wedhorn's argument in [Wed01, (2.17)] involves certain constructions in which one obtains a polarization by an averaging procedure involving dividing by 2 — this naturally causes issues when $p = 2$. One difficulty is that, when $p = 2$, one needs to decide what it means for a pairing on a finite flat group scheme G to be alternating rather than skew symmetric. If $G = \alpha_2/\mathbf{F}_2$, then G is Cartier self-dual via the map $\alpha_2 \times \alpha_2 \to \mathbf{G}_m$ given on points by $(x, y) \mapsto 1 + xy$. This is alternating on points (since $x^2 = 0$ for $x \in \alpha_2(A)$) but one does not want to regard it as an alternating pairing. Instead, following [LH13, §3.2] (where the idea is attributed in part to de Jong), one could define a pairing $G \times G \to \mathbf{G}_m$ of finite flat group schemes to be *strongly alternating* if, fpqc locally on the base, there is is a central extension

$$1 \to \mathbf{G}_m \to \widetilde{G} \to G \to 1 \tag{10.3.3}$$

such that the pairing arises from the commutator pairing on this extension.

One strategy would be to determine the precise conditions for any $G = G_1$ of exponent 2 admitting an isomorphism $\lambda : G_1 \xrightarrow{\sim} G_1^\vee$ with $\lambda^\vee = \lambda$ (since $p = 2$ there is no choice of sign) to give rise to a corresponding Heisenberg group extension $\widetilde{G}$ of the form (10.3.3), and then to prove a version of this lemma without any ordinary hypothesis on $A_0/\mathbf{F}$, but with a suitably modified definition of what it means for $G/\mathcal{O}$ to be quasi-polarized.

Alternatively, instead of addressing any of the more subtle issues which might arise in the general case, we exploit the assumption that $A_0/\mathbf{F}$ is ordinary. In this case, the assumption that a commutative finite flat group scheme G is an extension of an étale group scheme by a multiplicative (dual to étale) group scheme simplifies the situation considerably: There is a connected-étale sequence

$$1 \to G^0 \to G \to G^{\text{ét}} \to 1,$$

where G^0 is multiplicative and so its Cartier dual is étale. Now any isomorphism $\lambda : G \simeq G^\vee$ clearly has the property that the induced map

$$G^0 \to G \overset{\lambda}{\simeq} G^\vee \to (G^0)^\vee = ((G^0)^\vee)^{\text{ét}} \tag{10.3.4}$$

is trivial, and so any such λ will automatically be alternating on the generic fibre. In particular, we only work with the assumption that there exists an isomorphism $\lambda : G_1 \to G_1^\vee$ with $\lambda^\vee = -\lambda$ (the sign makes no difference for finite flat group schemes annihilated by 2), even though one expects this may will be the "wrong" definition in the non-ordinary case.

Proof of Lemma 10.3.1. The assumption that $A_0/\mathbf{F}$ is ordinary implies that the corresponding 2-divisible group splits into toroidal (ind-multiplicative) and (ind)-étale parts which are Cartier dual to each other. Over $\mathcal{O}/\pi_K^n$ or over $\mathcal{O}$, these factors have unique lifts, and the lifts of $A_0[2^\infty]$ are equivalent to the category of extensions of these factors ([Mes72, Prop 2.1]). But over $\mathcal{O}$, 2-divisible groups are determined by their generic fibres, and so the lifts are classified in terms of Galois cohomology. More precisely, if V denotes the free (rank 2) $\mathbf{Z}_2$ module corresponding to the Pontryagin dual of the (ind)-étale part of $A_0[2^\infty]$, and $W = V^\vee$ the $\mathbf{Z}_2$-dual of V, then the extensions of interest are computed by the group

$$H^1_f(K, W \otimes W(1)).$$

The result [Ill85] then implies the surjectivity of the reduction map:

$$H^1_f(K, W \otimes W(1)) \to H^1_f(K, \overline{W} \otimes \overline{W}(1)),$$

where $\overline{W} = W/2$. Now we wish to impose the condition that there exists a suitable polarization λ. There is an exact sequence of flat $\mathbf{Z}_2$-modules

$$0 \to S(W) \to W \otimes W \to \wedge^2(W) \to 0,$$

where $S(W)$ is the submodule generated by $x \otimes x$ for all $x \in W$. The vector space $S(W) \otimes K$ is isomorphic to $\mathrm{Sym}^2(W) \otimes K$, but this is not used below. By purity, the (generalized) eigenvalues of Frobenius on $W \otimes W$ cannot have absolute value 1 and so in particular are $\neq 1$. It follows that for M equal to any of $S(W)$, $W \otimes W$, or $\wedge^2 W$, we have $H^2(K, M(1)) = 0$, and $H^1_f(K, M(1)) = H^1(K, M(1))$. We say that a class η in $H^1_f(K, W \otimes W(1)) \subset H^1(K, W \otimes W(1))$ is alternating if it lies in the image of $H^1(K, S(W)(1))$, and we denote the alternating classes by $H^1_f(K, W \otimes W(1))^{\mathrm{Alt}}$. We similarly write $H^1_f(K, \overline{W} \otimes \overline{W}(1))^{\mathrm{Alt}}$ for the same condition modulo 2. (The reason this is the correct choice is ultimately explained by equation (10.3.4) below.) We have a commutative diagram as follows.

$$\begin{array}{ccccccc} H^1_f(K, S(W)(1)) & \longrightarrow & H^1_f(K, W \otimes W(1)) & \longrightarrow & H^1_f(K, \wedge^2(W)(1)) & \longrightarrow & 0 \\ \downarrow & & \twoheaddownarrow & & \downarrow & & \\ H^1_f(K, S(\overline{W})(1)) & \longrightarrow & H^1_f(K, \overline{W} \otimes \overline{W}(1)) & \longrightarrow & H^1_f(K, \wedge^2(\overline{W})(1)) & & \end{array}$$

Because the H^2 groups vanish, the kernel of each vertical map consists of classes divisible by 2. Now take an alternating class $\overline{\eta} \in H^1_f(K, \overline{W} \otimes \overline{W}(1))^{\mathrm{Alt}}$. It lifts to $\eta \in H^1_f(K, W \otimes W(1))$, which then maps to a class $\nu \in H^1_f(K, \wedge^2(W)(1))$ whose reduction $\overline{\nu}$ is trivial. But that implies that ν is divisible by 2, and thus, writing $\nu = 2\gamma$, and lifting γ to $\widetilde{\gamma} \in H^1_f(K, W \otimes W(1))$, we see that $\eta - 2\widetilde{\gamma} \in H^1_f(K, W \otimes W(1))^{\mathrm{Alt}}$, and hence there is a surjection

$$H^1_f(K, W \otimes W(1))^{\mathrm{Alt}} \to H^1_f(K, \overline{W} \otimes \overline{W}(1))^{\mathrm{Alt}}.$$

It now suffices to show that $H^1_f(K, W \otimes W(1))^{\mathrm{Alt}}$ classifies possible principally quasi-polarized Barsotti–Tate groups $G/\mathcal{O}$ lifting $A_0[2^\infty]$, equivalently, a BT $G/\mathcal{O}$ together with an isomorphism $\lambda : G \to G^\vee$ with $\lambda^\vee = -\lambda$, whereas $H^1_f(K, \overline{W} \otimes \overline{W}(1))^{\mathrm{Alt}}$ classifies finite flat group schemes over $\mathcal{O}$ lifting $A_0[2]$ together with an isomorphism $\lambda : G \to G^\vee$ with $\lambda^\vee = -\lambda$. In both settings, the corresponding lifts are determined by extensions of (fixed) étale by multiplicative group schemes, and these extensions are determined by their generic fibres. In either case, λ induces a skew-symmetric pairing J on the generic fibre which (as explained in the discussion surrounding (10.3.4) using the ordinary hypothesis) is alternating. Moreover, the generic fibre of the connected part (respectively, étale part) is isotropic with respect to this pairing. That implies that the action on the generic fibre factors through the generalized symplectic group, and in particular that the extension class of the étale by multiplicative part is alternating in the sense described above. (This is equivalent to the computation that

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^T = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{10.3.4}$$

if and only if A is symmetric.) Conversely, once the image of the Galois representation lies in the generalized symplectic group compatible with the connected part being isotropic, the Barsotti–Tate group (or finite flat group scheme) admits a suitable λ. We deduce that G_1 corresponds to a class in $H^1_f(K, \overline{W} \otimes \overline{W}(1))^{\mathrm{Alt}}$, and that there exists a lift of $A_0[2^\infty]$ to a principally quasi-polarized BT $G/\mathcal{O}$ with $G[2] \simeq G_1$, and we conclude as in the proof of Lemma 9.3.4. □

We also offer the following alternative proof using stacks, for those who are gripped to the pages of this manuscript and don't wish it to end:

Alternate proof of Lemma 10.3.1. Let us consider the p-divisible group $\mu^2_{p^\infty} \oplus \mathbf{Q}_p/\mathbf{Z}^2_p$ equipped with its standard polarization over $\overline{\mathbf{F}}_p$. By Serre–Tate theory, the moduli space of polarized extension of $0 \to \mu^2_{p^\infty} \to \mathcal{G} \to \mathbf{Q}_p/\mathbf{Z}^2_p \to 0$ on local Artinian $W(\overline{\mathbf{F}}_p)$-algebras is represented by

$$\widehat{\mathbf{G}}^3_m = \operatorname{Spf} W(\overline{\mathbf{F}}_p)[\![X_1, X_2, X_3]\!].$$

Identify $\mathbf{F} = \mathbf{F}_q$ and let ϕ be the q-th power Frobenius which topologically generates $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F})$. The p-divisible group $A_0[p^\infty]$ is isomorphic over $\overline{\mathbf{F}}_p$ to $\mu^2_{p^\infty} \oplus \mathbf{Q}_p/\mathbf{Z}^2_p$. This implies that the moduli stack of deformations (as polarized extensions) of $A_0[p^\infty]$ to local $W(\mathbf{F})$-Artinian algebras is represented by

$$\widehat{\mathbf{G}}^3_m/\phi^{\hat{\mathbf{Z}}}$$

where ϕ acts naturally on $W(\overline{\mathbf{F}}_p)$ and the action on the Serre–Tate parameters is induced by the action of Frobenius on $T_pA_0^{\text{ét}} \otimes T_p(A_0^m)^D$. Let $W(\overline{\mathbf{F}}_p)/p^n = W_n(\overline{\mathbf{F}}_p)$.

We can think of $\widehat{\mathbf{G}}_m^3/\phi^{\hat{\mathbf{Z}}}$ as the ind-stack $\operatorname{colim} \operatorname{Spec} W_n(\overline{\mathbf{F}}_p)[\![X_1, X_2, X_3]\!]/\phi^{\hat{\mathbf{Z}}}$ and each $\operatorname{Spec} W_n(\overline{\mathbf{F}}_p)[\![X_1, X_2, X_3]\!]/\phi^{\hat{\mathbf{Z}}}$ is the inverse limit on r of:

$$\operatorname{Spec} W_n(\overline{\mathbf{F}}_{q^r})[\![X_1, X_2, X_3]\!]/\phi^{\mathbf{Z}/q^r\mathbf{Z}}$$

where r is large enough so that the action on the Serre–Tate parameters of ϕ^{q^r} is trivial modulo p^n. The map

$$\operatorname{Spec} W_n(\overline{\mathbf{F}}_{q^r})[\![X_1, X_2, X_3]\!] \to \operatorname{Spec} W_n(\overline{\mathbf{F}}_{q^r})[\![X_1, X_2, X_3]\!]/\phi^{\mathbf{Z}/q^r\mathbf{Z}}$$

is formally étale as the group $\mathbf{Z}/q^r\mathbf{Z}$ is étale. The moduli of polarized extensions

$$0 \to \mu_p^2 \to G \to (\mathbf{Z}/p\mathbf{Z})^2 \to 0$$

on local Artinian $W(\overline{\mathbf{F}}_p)$-algebras is the quotient stack $\widehat{\mathbf{G}}_m^3/\widehat{\mathbf{G}}_m^3$ where each copy of $\widehat{\mathbf{G}}_m$ acts on itself through the map $(1+x, 1+y) \mapsto (1+x)^p(1+y)$. (The moduli of all extensions is given by $\widehat{\mathbf{G}}_m^4/\widehat{\mathbf{G}}_m^4$, and the inclusion of the polarized extensions into all extensions corresponds in the previous argument to the inclusion of $S(W)$ into $W \otimes W$.) The map $\widehat{\mathbf{G}}_m^3 \to \widehat{\mathbf{G}}_m^3/\widehat{\mathbf{G}}_m^3$ is formally smooth. We now consider the map

$$\widehat{\mathbf{G}}_m^3/\phi^{\hat{\mathbf{Z}}} \to \widehat{\mathbf{G}}_m^3/\widehat{\mathbf{G}}_m^3/\phi^{\hat{\mathbf{Z}}}. \tag{10.3.4}$$

The map (10.3.4) sends a deformation (as a polarized extension) of A_0 to a deformation (as a polarized extension) of $A_0[p]$. The map (10.3.4) is moreover the inductive limit of the maps

$$\widehat{\mathbf{G}}_m^3|_{W_n(\overline{\mathbf{F}}_p)}/\phi^{\hat{\mathbf{Z}}} \to \widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}/\widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}/\phi^{\hat{\mathbf{Z}}}. \tag{10.3.4}$$

In turn, these maps (10.3.4) are the inverse limit of the maps:

$$\widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}/\phi^{\mathbf{Z}/q^r\mathbf{Z}} \to \widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}/\widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}/\phi^{\mathbf{Z}/q^r\mathbf{Z}}$$

This last map is formally smooth because the groups $\widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}$ and $\mathbf{Z}/q^r\mathbf{Z}$ are both formally smooth.

By assumption, we begin with $G_1/\mathcal{O}$, which is a point $\operatorname{Spf} \mathcal{O} \to \widehat{\mathbf{G}}_m^3/\widehat{\mathbf{G}}_m^3/\phi^{\hat{\mathbf{Z}}}$, and our goal is to lift this to an Spf $\mathcal{O}$-point of $\widehat{\mathbf{G}}_m^3/\phi^{\hat{\mathbf{Z}}}$. Let ϖ be a uniformizer of $\mathcal{O}$. Assume that we have found a lift of $G_1|_{\operatorname{Spec}(\mathcal{O}/\varpi^k)}$ to $G \to \operatorname{Spec}(\mathcal{O}/\varpi^k)$; we shall upgrade it to a lift on $\operatorname{Spec}(\mathcal{O}/\varpi^{k+1})$ of $G_1|_{\operatorname{Spec}(\mathcal{O}/\varpi^{k+1})}$, and then we are done by induction.

There exists, n, r such that we have a commutative diagram (given by the solid arrows):

$$\begin{array}{ccc}
\operatorname{Spec}(\mathcal{O}/\varpi^k) & \longrightarrow & \widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}/\phi^{\mathbf{Z}/q^r\mathbf{Z}} \\
\downarrow & \nearrow & \downarrow \\
\operatorname{Spec}(\mathcal{O}/\varpi^{k+1}) & \longrightarrow & \widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}/\widehat{\mathbf{G}}_m^3|_{W_n(\mathbf{F}_{q^r})}/\phi^{\mathbf{Z}/q^r\mathbf{Z}}
\end{array}$$

By formal smoothness, we can produce the lift given by the dotted arrow, completing the proof. $\square$

10.3.5. *Moduli of X with fixed* $\mathrm{Jac}(X)[2]$. Let F be a global or local field of characteristic zero, and let

$$\overline{\rho}: G_F \to \mathrm{GSp}_4(\mathbf{F}_2)$$

be a continuous representation. Under the identification (Lemma 8.1.3) of S_6 with $\mathrm{GSp}_4(\mathbf{F}_2)$, there exists a corresponding degree 6 separable polynomial $f(x)$ such that $\overline{\rho}$ is isomorphic to the 2-torsion representation on the Jacobian of $y^2 = f(x)$. Let $K = F[x]/f(x) \simeq \prod F_i$, which is a degree 6 étale F-algebra. Given $\theta \in K$, the multiplication by θ map $K \to K$ naturally has a characteristic polynomial of degree 6 with roots we denote by $\sigma\theta$. If we fix a basis for K over F, for example given by the powers of x, this map is compatible with extensions of F. If we identify K with F^6, then by the primitive element theorem, the $\sigma\theta$ will be distinct for θ outside a finite number of hyperplanes (which are defined over the splitting field of $f(x)$ and compatible with field homomorphisms $F \to F'$). If the $\sigma\theta$ are distinct, then

$$X : y^2 = \prod (x - \sigma\theta),$$

will be a smooth genus two curve over F with $\mathrm{Jac}(X)[2] \simeq \overline{\rho}$. We have therefore constructed a smooth rational variety $Z(\overline{\rho}) \subset \mathbf{P}^6$ over F given by the complement of finitely many hyperplanes, whose F-rational points give smooth genus two curves X with 2-torsion given by $\overline{\rho}$. Moreover, the construction of $Z(\overline{\rho})$ (having fixed K) is compatible with both extensions of F and completions at primes of F. There is a map from $Z(\overline{\rho})$ to the corresponding moduli stack $\mathcal{M}_2(\overline{\rho})$, but to avoid any issues concerning fields of moduli versus fields of definition it is fine for our purposes to work directly with $Z(\overline{\rho})$.

Lemma 10.3.6. *Let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_2)$ be a continuous representation unramified at* 3*. Assume that there exists a finite flat model $\mathcal{W}/\mathbf{Z}_2$ for $\overline{\rho}$ over $\mathbf{Z}_2$ which is isomorphic to its Cartier dual and which is ordinary, that is, the extension of an étale group scheme by a multiplicative group scheme. Suppose that $\overline{\rho}(\mathrm{Frob}_3)$ is non-trivial. There exists an abelian surface $A/\mathbf{Q}$ such that:*

(1) A has good ordinary reduction at 3 *and is* 3*-distinguished.*
(2) A has good ordinary reduction at 2*. There is an isomorphism of finite flat group schemes $A[2]/\mathbf{Z}_2 \simeq \mathcal{W}$, and the characteristic polynomial $Q(x)$ of Frobenius at* 2 *satisfies*

$$Q(x) \not\equiv x^4 \pm x^3 + 2x^2 \mp x + 1 \bmod 3. \tag{10.3.7}$$

(3) $\overline{\rho}_{A,3}$ is surjective.
(4) $\overline{\rho}_{A,2} \cong \overline{\rho}$.

Proof. Let $Z = Z(\overline{\rho})$. The conditions we are imposing at 2 and 3 are open conditions in $Z(\mathbf{Q}_2)$ and $Z(\mathbf{Q}_3)$ respectively. The condition that $\overline{\rho}_{A,3}$ is surjective holds outside a thin set. Since Z is smooth and rational, by Lemma 9.4.1 once we find suitable points on $Z(\mathbf{Q}_2)$ and $Z(\mathbf{Q}_3)$, we obtain an A which has the required properties.

Let us consider $Z(\mathbf{Q}_3)$. With $F = \mathbf{Q}$, let K denote the degree 6 étale F-algebra corresponding to $\overline{\rho}$ as described above. Since we are assuming that $\overline{\rho}$ is unramified at 3, we certainly have that K is unramified at 3, so the possible completions $K \otimes \mathbf{Q}_3$ are determined by a partition of 6. There are exactly 11 such partitions, the partition $6 = 1 + 1 + 1 + 1 + 1 + 1$ corresponding to the case when $\overline{\rho}(\mathrm{Frob}_3)$ is trivial, which we are excluding. For the remaining 10 partitions, we now produce an explicit $X : y^2 = f(x)$ with good ordinary reduction at 3 which is 3-distinguished. We actually write down $X/\mathbf{Q}$ with these properties. Note that for $y^2 = f(x)$ where one of the Weierstrass points is at ∞, the corresponding partition corresponds to the factorization of $f(x)$ over $\mathbf{Q}_3$ plus another copy of $\mathbf{Q}_3$. In other words, the partition corresponds to the degrees of the (unramified) fields of definition of the 6 Weierstrass points of X over $\mathbf{Q}_3$.

$f(x)$	partition	N
$4x^5 + 32x^4 + 64x^3 + x^2 + 4x$	$[1, 1, 1, 1, 2]$	1051
$4x^5 - 7x^2 + 4x$	$[1, 1, 1, 3]$	709
$4x^5 - 11x^4 + 6x^3 + 3x^2 - 2x + 1$	$[1, 1, 2, 2]$	1415
$x^6 + 4x^5 - 6x^4 - 32x^3 + x^2 + 64x + 28$	$[1, 1, 4]$	389
$x^6 + 2x^5 + 5x^4 + 4x^3 - 4x - 8$	$[1, 2, 3]$	847
$x^6 + 2x^5 + 3x^4 - x^2 + 2x + 1$	$[1, 5]$	349
$x^6 + 4x^4 + 6x^3 - 8x^2 + 1$	$[2, 2, 2]$	7165
$x^6 + 2x^4 + 2x^3 + 5x^2 + 2x + 1$	$[2, 4]$	353
$x^6 + 2x^5 + 5x^4 - 10x^3 + 10x^2 - 4x + 1$	$[3, 3]$	4889
$x^6 + 4x^5 - 6x^4 + 2x^3 + x^2 - 2x + 1$	$[6]$	1343

Let us now turn to the prime 2. By Lemma 10.3.1, the required abelian surface $A/\mathbf{Z}_2$ will exist provided that there is an $A_0/\mathbf{F}_2$ with $A_0[2] \simeq \mathcal{W}/\mathbf{F}_2$ (also satisfying equation (10.3.7)). The finite flat group scheme $\mathcal{W}/\mathbf{Z}_2$ is an extension of an étale group scheme $\overline{V}$ by its Cartier dual $\overline{V}^\vee(1)$, and in particular $\mathcal{W}/\mathbf{F}_2$ is determined by $\overline{V}$. There are three possibilities for $\overline{V}$:

(1) $\overline{V}$ is trivial as a $G_{\mathbf{F}_2}$-module,
(2) $G_{\mathbf{F}_2}$ acts on $\overline{V}$ via a (non-semi-simple) element of order 2,
(3) $G_{\mathbf{F}_2}$ acts on $\overline{V}$ via a (semi-simple) element of order 3.

It now suffices to find an $A_0/\mathbf{F}_2$ of each form.

One subtlety is that, given $\overline{\rho}$, the finite flat group scheme $\mathcal{W}/\mathbf{Z}_2$ is not determined by $\overline{\rho}$. Consider the following two examples:

(1) The action of $G_{\mathbf{F}_2}$ on $\overline{V}$ is trivial, and the extension class of $\overline{V} = (\mathbf{Z}/2\mathbf{Z})^2$ by its Cartier dual $\overline{V}^\vee(1) = (\mu_2)^2$ is a direct sum of two extensions corresponding to the *unramified* class in $H^1_f(\mathbf{Q}, \mu_2) = \mathbf{Z}_2^\times/\mathbf{Z}_2^{\times 2}$.
(2) The action of $G_{\mathbf{F}_2}$ on $\overline{V}$ has order 2, and the group scheme is $\overline{V} \oplus \overline{V}^\vee(1)$.

In both cases, the representation $\overline{\rho}$ is unramified of order 2, and the non-trivial elements in the images are given by

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. Under the isomorphism of Lemma 8.1.3, these are equal to $(12)(34)(56) \in S_6$ and $(12)(35)(46) \in S_6$, and so are conjugate. (They are not, however, conjugate inside the Siegel parabolic $C((12)(34)(56)) = S_4 \times S_2$ described in Lemma 9.1.8.)

We now consider the following examples of genus 2 curves given by the equations $y^2 = f(x)$ where $f(x)$ is listed in the table below. One can check that the corresponding minimal models have good ordinary reduction at 2 and compute the corresponding polynomial $Q(x)$. Since these curves are defined over $\mathbf{Q}$, one can also compute the global conductor, which is indicated in the table by N.

$f(x)$	partition	$Q(x)$	N
$-4x^5 + x^4 + 6x^3 - 3x^2 - 4x$	$[1, 1, 1, 1, 1, 1]$	$x^4 - x^2 + 4$	3451
$x^6 - 12x^4 + 2x^3 + 16x^2 + 8x + 1$	$[2, 2, 2]$	$x^4 + 2x^3 + 3x^2 + 4x + 4$	2225
$x^6 - 4x^5 + 2x^4 + 2x^3 + x^2 - 2x + 1$	$[3, 3]$	$x^4 + 3x^3 + 5x^2 + 6x + 4$	713

Here the partition indicates the factorization of $f(x)$ over $\mathbf{Q}_2$; the corresponding Galois extension is cyclic and unramified of degree 1, 2, and 3 (in that order). It follows immediately in the first and last cases that $A[2]/\mathbf{Z}_2$ is the *split* extension of $\overline{V}$ by $\overline{V}^{\vee}(1)$, where $G_{\mathbf{F}_2}$ acts on $\overline{V}$ through a cyclic group of order 1 and 3 respectively. In the second case, we still need to check (in light of the example above) that $G_{\mathbf{F}_2}$ acts on $\overline{V}$ through a cyclic group of order 2. In this case, a smooth model $C/\mathbf{Z}_2$ is given by the equation

$$y^2 + (x^3 + 1)y = -3x^4 + 4x^2 + 2x,$$

from which we find that $\mathrm{Jac}(C)(\mathbf{F}_2) \simeq \mathbf{Z}/14\mathbf{Z}$. If $\overline{V}$ was trivial, then $\mathrm{Jac}(C)(\mathbf{F}_2)$ would contain $(\mathbf{Z}/2\mathbf{Z})^2$ as a subgroup, which it does not. Hence we deduce that the action of $G_{\mathbf{F}_2}$ on $\overline{V}$ is through a cyclic group of order 2 (which suffices for our purposes, but is not sufficient to determine $\mathcal{W}$ as an extension). Thus we obtain a suitable $A_0/\mathbf{F}_2$ in all possible cases. □

Remark 10.3.8. Note that the method of proof of Lemma 10.3.6 fails when $\overline{\rho}(\mathrm{Frob}_3)$ is trivial. This would imply that X has 6 Weierstrass points over $\mathbf{F}_3$, but if X has good reduction at 3 these points are distinct, and they are exactly the ramification points over the map to $\mathbf{P}^1$. But $\mathbf{P}^1(\mathbf{F}_3)$ has only $4 < 6$ points, so this is impossible. It seems unlikely one can avoid this using X for which $A = \mathrm{Jac}(X)$ has good reduction but X does not; at least the idea of using a product of elliptic curves does not work, since if $\#E[2](\mathbf{F}_3) = 4$, then by the Hasse bounds $\#E(\mathbf{F}_3) = 4$ and $a_3 = 0$, and E is supersingular.

By combining this with our main modularity theorem for abelian surfaces, we deduce the following:

Theorem 10.3.9. *Let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_2)$ be a continuous representation which is unramified at* 3 *and such that $\overline{\rho}(\mathrm{Frob}_3)$ is non-trivial. Suppose as in Lemma 10.3.6 that there exists a finite flat model $\mathcal{W}/\mathbf{Z}_2$ for $\overline{\rho}$ over $\mathbf{Z}_2$ which is isomorphic to its Cartier dual and which is ordinary. Then $\overline{\rho}$ is ordinarily modular of weight* 2 *and level prime to* 6.

Proof. Consider the abelian surface $A/\mathbf{Q}$ with $\overline{\rho}_{A,2} = \overline{\rho}$ whose existence follows from Lemma 10.3.6. It suffices to show that A is modular. Condition (10.3.7) guarantees (by Lemma 9.1.3) that $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ is not of conjugacy class $4C$ or $12C$ in $\mathrm{PGSp}_4(\mathbf{F}_3) \smallsetminus \mathrm{PSp}_4(\mathbf{F}_3)$. Moreover, A has good ordinary reduction and 3, is 3-distinguished, and $\overline{\rho}_{A,3}$ is surjective. Thus A satisfies the conditions of Theorem 9.5.2 and hence A is modular. □

Remark 10.3.10. If one proved a version of Lemma 10.3.1 in the non-ordinary case, one could improve the statement of Theorem 10.3.9. But note that in either case the required assumption on $\overline{\rho}$ is stronger than merely the assumption that $\overline{\rho}$ is finite flat, that is, arises as the generic fibre of some $\mathcal{W}/\mathbf{Z}_2$ without any duality assumption. For example, if $\overline{\rho}|_{\mathbf{Q}_2}$ is unramified with image of order 5, then $\overline{\rho}$ is both ordinary and finite flat, and yet there does not exist any abelian surface $A/\mathbf{Z}_2$ with $A[2] \simeq \overline{\rho}$. The issue is that the only finite flat $\mathcal{W}/\mathbf{Z}_2$ with generic fibre $\overline{\rho}$ are either étale or multiplicative, and so certainly not Cartier self-dual. This is analogous to the fact that an unramified representation $\overline{\rho} : G_{\mathbf{Q}_2} \to \mathrm{GL}_2(\mathbf{F}_2)$ with image of order 3 is ordinary and finite flat in the usual sense but does not come from the 2-torsion of an elliptic curve with good reduction, although after one extends the coefficients of $\overline{\rho}$ to $\mathbf{F}_4$ it does come from the 2-torsion of an abelian surface with endomorphisms by $\mathbf{Z}[(1+\sqrt{5})/2]$ with good reduction at 2 (for example, the modular abelian surface $J_0(23)/\mathbf{Z}_2$.) Note that this subtlety only arises (over $\mathbf{Q}$) for $p = 2$, since for $p > 2$ any finite flat $\mathcal{W}/\mathbf{Z}_p$ is determined by its generic fibre and so the Cartier self-duality of $\mathcal{W}/\mathbf{Z}_p$ follows from the corresponding property of $\overline{\rho}$.

10.4. **Consequences of Serre's Conjecture in regular weight.** The odd Artin conjecture for odd 2-dimensional *complex* representations of $G_{\mathbf{Q}}$ is a consequence ([KW09, Cor 10.2]) of Serre's Conjecture for odd 2-dimensional mod-p representations of $G_{\mathbf{Q}}$ (this implication was proved by Khare in [Kha97], using the weight lowering results [Gro90, CV92]). It seems worthwhile remarking here that as a consequence of our main theorems, an analogous deduction is valid for abelian surfaces over $\mathbf{Q}$.

Lemma 10.4.1 (Serre's Conjecture in regular weight implies modularity)**.** *Suppose that for every residual representation:*

$$\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_p)$$

satisfying the following conditions:

1. *$\overline{\rho}$ has multiplier $\overline{\varepsilon}^{-1}$,*
2. *$\overline{\rho}$ is absolutely irreducible,*
3. *the semi-simplification of $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ is a direct sum of characters,*

there exists an ordinary cuspidal automorphic representation π of $\mathrm{GSp}_4/\mathbf{Q}$ of regular weight, level prime to p, and central character $|\cdot|^2$, such that

$$\overline{\rho}_{\pi,p} \cong \overline{\rho}.$$

Then all abelian surfaces $A/\mathbf{Q}$ are modular.

Remark 10.4.2. There are several possible natural variations on the hypotheses of this Lemma; for example, one could only demand the statement for p sufficiently large. We have simply chosen one such version for illustrative purposes.

Proof of Lemma 10.4.1. By Theorem 10.2.1, we may assume that A is "challenging" in the terminology of [BCGP21, §9], i.e. that either $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$, or that A has Galois type $\mathbf{B}[C_2]$, so there exists a quadratic extension $K/\mathbf{Q}$ so that $E = \mathrm{End}(A_K)\otimes\mathbf{Q} = \mathrm{End}(A_{\overline{\mathbf{Q}}})\otimes\mathbf{Q}$ is either $\mathbf{Q}\oplus\mathbf{Q}$ or a real quadratic field. By [BCGP21, Lemma 9.2.5], there is a density one set of primes $p > 2$ such that A is ordinary at p and residually p-distinguished in the sense of [BCGP21, Def 7.3.1], and moreover that $\overline{\rho} = \overline{\rho}_{A,p}$ satisfies the hypotheses listed in the statement of this lemma, as well as being vast in the sense of [BCGP21, Defn. 7.5.6] (and in particular reasonable in the sense of [Whi22, Defn. 3.19]) and tidy in the sense of [BCGP21, Defn. 7.5.11]. Furthermore, we may assume that if A has Galois type $\mathbf{B}[C_2]$ then p splits in E.

We now deduce the modularity of A as a consequence of Theorem 7.5.11 for $\rho = \rho_{A,p}$. It suffices to check the conditions of that theorem; and in particular it suffices to check that hypotheses (1)–(5) and (B1)–(B5) of Proposition 7.5.10 hold (the other condition of Theorem 7.5.11 being immediate from the definition of "challenging"). Since A is an abelian surface with good ordinary reduction, and is additionally residually p-distinguished, the only conditions that need to be checked are that:

(a) $\rho_{A,p}(G_{\mathbf{Q}(\zeta_{p^\infty})})$ is integrally enormous,
(b) $\overline{\rho}_{A,p}(G_{\mathbf{Q}}) \smallsetminus \mathrm{Sp}_4(\mathbf{F}_p)$ contains a regular semi-simple element,
(c) There are choices of p-stabilizations such that $\rho_{\pi,p}|_{G_{\mathbf{Q}_p}}$ lies on a unique irreducible component of $\operatorname{Spec} R_p^{\triangle}$ and $\rho_{A,p}|_{G_{\mathbf{Q}_p}}$ lies on the same component.

Suppose firstly that A has Galois type $\mathbf{B}[C_2]$. Then for sufficiently large primes p (splitting in E) the mod p representations

$$\overline{\rho}_{A,p} : G_{K(\zeta_{p^\infty})} \to \mathrm{SL}_2(\mathcal{O}_E/p) = \mathrm{SL}_2(\mathbf{F}_p) \times \mathrm{SL}_2(\mathbf{F}_p)$$
$$\overline{\rho}_{A,p} : G_K \to \{(A,B) \in \mathrm{GL}_2(\mathbf{F}_p) \times \mathrm{GL}_2(\mathbf{F}_p),\ \det(A) = \det(B)\}$$

are surjective. (This follows from [BCGP21, Lem. 9.1.10(3)], and for $E = \mathbf{Q} \oplus \mathbf{Q}$ goes back to [Ser72].) Thus we may additionally assume that p is chosen so that $K \not\subseteq \mathbf{Q}(\zeta_{p^\infty})$ and $\rho_{A,p}(G_{\mathbf{Q}(\zeta_{p^\infty})})$ is precisely $\mathrm{SL}_2(\mathbf{F}_p) \wr \mathbf{Z}/2\mathbf{Z}$. As explained in the proof of [BCGP21, Lemma 7.5.18], for $p \geq 3$, the set $\overline{\rho}_{A,p}(G_{\mathbf{Q}(\zeta_{p^\infty})}) \smallsetminus \overline{\rho}(G_{G(\zeta_{p^\infty})}) = \mathrm{SL}_2(\mathbf{F}_p) \wr \mathbf{Z}/2\mathbf{Z} \smallsetminus \mathrm{SL}_2(\mathbf{F}_p)^2$ always contains a regular semi-simple element with eigenvalues $(\zeta_8, \zeta_8^{-1}, -\zeta_8, -\zeta_8^{-1})$. Thus condition (a) follows from Corollary 7.1.4. For $p > 5$, $\overline{\rho}_{A,p}(G_{\mathbf{Q}})$ contains $(A,B) = (\mathrm{diag}(1,6), \mathrm{diag}(2,3))$, which is regular semi-simple and which does not lie in $\mathrm{Sp}_4(\mathbf{F}_p)$, which verifies condition (b) in this case as well.

Finally if $\mathrm{End}(A_{\overline{\mathbf{Q}}}) = \mathbf{Z}$, then for sufficiently large p we have $\overline{\rho}_{A,p}(G_{\mathbf{Q}}) = \mathrm{GSp}_4(\mathbf{F}_p)$ and $\overline{\rho}_{A,p}(G_{\mathbf{Q}(\zeta_{p^\infty})}) = \mathrm{Sp}_4(\mathbf{F}_p)$. Hence both $\overline{\rho}_{A,p}(G_{\mathbf{Q}}) \smallsetminus \mathrm{Sp}_4(\mathbf{F}_p)$ and $\overline{\rho}_{A,p}(G_{\mathbf{Q}(\zeta_{p^\infty})})$ contain regular semi-simple elements for large enough p (for example, the same elements that were used above). This verifies condition (b) in this case, and $\rho_{A,p}(G_{\mathbf{Q}(\zeta_{p^\infty})})$ is integrally enormous by Corollary 7.1.4, verifying condition (a).

For condition (c), we firstly choose a p-stabilization of π_p, and thus of $\overline{\rho}_{A,p}|_{G_{\mathbf{Q}_p}}$. Then at least one of the p-stabilizations of $\rho_{A,p}|_{G_{\mathbf{Q}_p}}$ is compatible with this fixed choice. Having made this choice, since $\overline{\rho}_{A,p}|_{G_{\mathbf{Q}_p}}$ is residually p-distinguished it follows from [BCGP21, Prop. 7.3.4] that $\operatorname{Spec} R_p^{\triangle}[1/p]$ is irreducible, and we are done. □

References

[Ado76] A. Adolphson, *An index theorem for p-adic differential operators*, Trans. Amer. Math. Soc. **216** (1976), 279–293. 29

[AGI+] Hiraku Atobe, Wee Teck Gan, Atsushi Ichino, Tasho Kaletha, Mínguez Alberto, and Sug Woo Shin, *Local intertwining relations and co-tempered a-packets of classical groups*, preprint. 8

[AIS15] Fabrizio Andreatta, Adrian Iovita, and Glenn Stevens, *Overconvergent Eichler–Shimura isomorphisms*, J. Inst. Math. Jussieu **14** (2015), no. 2, 221–274. 6

[AL03] H. H. Andersen and N. Lauritzen, *Twisted Verma modules*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 1–26. 30

[And21] Grigory Andreychev, *Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces*, arXiv 2105.12591, 2021. 40, 94

[Art04] James Arthur, *Automorphic representations of* GSp(4), Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 65–81. 7

[Bal12] Sundeep Balaji, *G-valued potentially semi-stable deformation rings*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–The University of Chicago. 134, 176

[BB83] Alexander Beilinson and Joseph Bernstein, *A generalization of Casselman's submodule theorem*, Representation theory of reductive groups (Park City, Utah, 1982), Progr. Math., vol. 40, Birkhäuser Boston, Boston, MA, 1983, pp. 35–52. 58

[BCG+25] George Boxer, Frank Calegari, Toby Gee, James Newton, and Jack Thorne, *The Ramanujan and Sato–Tate conjectures for Bianchi modular forms*, 2025, to appear in Forum Math. Pi. 137

[BCGP21] George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni, *Abelian surfaces over totally real fields are potentially modular*, Publ. Math. Inst. Hautes Études Sci. **134** (2021), 153–501. 3, 4, 7, 8, 11, 12, 15, 16, 104, 105, 111, 139, 140, 143, 145, 146, 147, 149, 150, 151, 152, 153, 162, 163, 167, 168, 170, 171, 172, 174, 176, 177, 186, 192, 206, 211, 222, 223

[BCGP25] ———, *Auxiliary magma files GitHub repository*, `https://github.com/fcale75/newabeliansurfacesmagmafiles/`, 2025. 188, 199

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). 188

[BEO01] Hans Ulrich Besche, Bettina Eick, and E. A. O'Brien, *The groups of order at most 2000*, Electron. Res. Announc. Amer. Math. Soc. **7** (2001), 1–4. 150, 151

[BFvdG08] Jonas Bergström, Carel Faber, and Gerard van der Geer, *Siegel modular forms of genus 2 and level 2: cohomological computations and conjectures*, Int. Math. Res. Not. IMRN (2008), Art. ID rnn 100, 20. 180

[BG19] Rebecca Bellovin and Toby Gee, *G-valued local deformation rings and global lifts*, Algebra Number Theory **13** (2019), no. 2, 333–378. 134, 163, 175

[BHR94] Don Blasius, Michael Harris, and Dinakar Ramakrishnan, *Coherent cohomology, limits of discrete series, and Galois conjugation*, Duke Math. J. **73** (1994), no. 3, 647–685. 176

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609. 14, 15, 115, 117, 128, 132, 138, 139, 175, 178, 179

[BLR91] Nigel Boston, Hendrik W. Lenstra, Jr., and Kenneth A. Ribet, *Quotients of group rings arising from two-dimensional representations*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 4, 323–328. 110

[Bor91] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. 154

[Bou21] Reda Boumasmoud, *A tale of parahoric–Hecke algebras, Bernstein and Satake homomorphisms*, arXiv 2106.12500, 2021. 171

[BP21] George Boxer and Vincent Pilloni, *Higher Coleman theory*, arXiv 2110.10251, 2021. 6, 14, 33, 41, 67, 68, 82, 83, 86, 89, 90, 91, 92, 94, 96, 97, 98, 103

[BP23] ———, *Higher Hida theory for Siegel modular forms*, 2023, preprint. 98, 104, 105, 163, 164

[BPP+19] Armand Brumer, Ariel Pacetti, Cris Poor, Gonzalo Tornaría, John Voight, and David S. Yuen, *On the paramodularity of typical abelian surfaces*, Algebra Number Theory **13** (2019), no. 5, 1145–1195. 180, 203

[Bra47] Richard Brauer, *On Artin's L-series with general group characters*, Ann. of Math. (2) **48** (1947), 502–514. 3

[BSS+16] Andrew R. Booker, Jeroen Sijsling, Andrew V. Sutherland, John Voight, and Dan Yasaki, *A database of genus-2 curves over the rational numbers*, LMS J. Comput. Math. **19** (2016), no. suppl. A, 235–254. 3, 210

[Car14] Ana Caraiani, *Monodromy and local-global compatibility for $l = p$*, Algebra Number Theory **8** (2014), no. 7, 1597–1646. 15

[Cas] William Casselman, *Introduction to admissible representations of p-adic groups*, Available at `https://www.math.ubc.ca/~cass/research/publications.html`. 95

[CC22] Frank Calegari and Shiva Chidambaram, *Rationality of twists of the Siegel modular variety of genus 2 and level 3*, Proc. Amer. Math. Soc. **150** (2022), no. 5, 1975–1984. 4

[CCG20] Frank Calegari, Shiva Chidambaram, and Alexandru Ghitza, *Some modular abelian surfaces*, Math. Comp. **89** (2020), no. 321, 387–394. 3

[CCN+85] John H. Conway, Robert T. Curtis, Simon P. Norton, Richard A. Parker, and Robert A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray. 151, 187, 188

[CCR20] Frank Calegari, Shiva Chidambaram, and David P. Roberts, *Abelian surfaces with fixed 3-torsion*, ANTS XIV—Proceedings of the Fourteenth Algorithmic Number Theory Symposium, Open Book Ser., vol. 4, Math. Sci. Publ., Berkeley, CA, 2020, pp. 91–108. 188, 198

[CE12] Frank Calegari and Matthew Emerton, *Completed cohomology—a survey*, Nonabelian fundamental groups and Iwasawa theory, London Math. Soc. Lecture Note Ser., vol. 393, Cambridge Univ. Press, Cambridge, 2012, pp. 239–257. 71

[CG18] Frank Calegari and David Geraghty, *Modularity lifting beyond the Taylor–Wiles method*, Inventiones mathematicae **211** (2018), no. 1, 297–433. 139

[Chi90] Bruno Chiarellotto, *Duality in rigid analysis*, p-adic analysis (Trento, 1989), Lecture Notes in Math., vol. 1454, Springer, Berlin, 1990, pp. 142–172. 85

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Pub. Math. IHES **108** (2008), 1–181. 115, 116, 134, 139, 170, 179

[CL10] Pierre-Henri Chaudouard and Gérard Laumon, *Le lemme fondamental pondéré. I. Constructions géométriques*, Compos. Math. **146** (2010), no. 6, 1416–1506. 8

[Cla66] D. N. Clark, *A note on the p-adic convergence of solutions of linear differential equations*, Proc. Amer. Math. Soc. **17** (1966), 262–269. 29, 30

[CN23] Ana Caraiani and James Newton, *On the modularity of elliptic curves over imaginary quadratic fields*, arXiv 2301.10509, 2023. 211

[CO75] William Casselman and M. Scott Osborne, *The $\mathfrak{n}$-cohomology of representations with an infinitesimal character*, Compositio Math. **31** (1975), no. 2, 219–227. 27

[CS] Dustin Clausen and Peter Scholze, *Lectures on condensed mathematics*, Course notes. 18, 19, 30

[CV92] Robert F. Coleman and José Felipe Voloch, *Companion forms and Kodaira–Spencer theory*, Invent. Math. **110** (1992), no. 2, 263–281. 4, 222

[Dia97] Fred Diamond, *The Taylor–Wiles construction and multiplicity one*, Invent. Math. **128** (1997), no. 2, 379–391. 152

[DLLZ23] Hansheng Diao, Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu, *Logarithmic adic spaces: some foundational results*, p-adic Hodge theory, singular varieties, and non-abelian aspects, Simons Symp., Springer, Cham, [2023] ©2023, pp. 65–182. 71, 72, 75

[Eme06] Matthew Emerton, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math. **164** (2006), no. 1, 1–84. 102

[Fal87] Gerd Faltings, *Hodge–Tate structures and modular forms*, Math. Ann. **278** (1987), no. 1-4, 133–149. 6

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford. 18, 70, 101

[FGS17] Jason Fulman, Robert Guralnick, and Dennis Stanton, *Asymptotics of the number of involutions in finite classical groups*, J. Group Theory **20** (2017), no. 5, 871–902. 118

[FKRS12] Francesc Fité, Kiran S. Kedlaya, Víctor Rotger, and Andrew V. Sutherland, *Sato–Tate distributions and Galois endomorphism modules in genus 2*, Compos. Math. **148** (2012), no. 5, 1390–1442. 211, 214

[FLHS15] Nuno Freitas, Bao V. Le Hung, and Samir Siksek, *Elliptic curves over real quadratic fields are modular*, Invent. Math. **201** (2015), no. 1, 159–206. 211

[Fon85] Jean-Marc Fontaine, *Il n'y a pas de variété abélienne sur* $\mathbf{Z}$, Invent. Math. **81** (1985), no. 3, 515–538. 149

[Ger19] David Geraghty, *Modularity lifting theorems for ordinary Galois representations*, Math. Ann. **373** (2019), no. 3-4, 1341–1427. 129, 130, 131, 132, 134, 136, 142, 143, 144

[GG12] Toby Gee and David Geraghty, *Companion forms for unitary and symplectic groups*, Duke Math. J. **161** (2012), no. 2, 247–303. 169, 170

[GHT17] Robert Guralnick, Florian Herzig, and Pham Huu Tiep, *Adequate subgroups and indecomposable modules*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 4, 1231–1291. 182

[GK00] Elmar Grosse-Klönne, *Rigid analytic spaces with overconvergent structure sheaf*, J. Reine Angew. Math. **519** (2000), 73–95. 44

[GN20] Toby Gee and James Newton, *Patching and the completed homology of locally symmetric spaces*, Journal of the Institute of Mathematics of Jussieu (2020), 1–64. 162, 165

[Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. (2) **9** (1957), 119–221. 84

[Gro90] Benedict H. Gross, *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Math. J. **61** (1990), no. 2, 445–517. 4, 222

[GRR72] Alexander Grothendieck, Michèle Raynaud, and Dock S. Rim, *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Mathematics, Vol. 288, Springer-Verlag, Berlin-New York, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de Michèle Raynaud et Dock S. Rim. 191, 192

[GT11] Wee Teck Gan and Shuichiro Takeda, *The local Langlands conjecture for* $\mathrm{GSp}(4)$, Ann. of Math. (2) **173** (2011), no. 3, 1841–1882. 12, 14

[GT19] Toby Gee and Olivier Taïbi, *Arthur's multiplicity formula for* $\mathbf{GSp}_4$ *and restriction to* $\mathbf{Sp}_4$, J. Éc. polytech. Math. **6** (2019), 469–535. 15

[Har90] Michael Harris, *Automorphic forms and the cohomology of vector bundles on Shimura varieties*, Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), Perspect. Math., vol. 11, Academic Press, Boston, MA, 1990, pp. 41–91. 14, 176

[Haz70] Mihil Hazevinkel, *A duality theorem for cohomology of Lie algebras*, Mat. Sb. (N.S.) **83(125)** (1970), 639–644. 26

[Hen09] Guy Henniart, *Sur la fonctorialité, pour* $\mathrm{GL}(4)$*, donnée par le carré extérieur*, Mosc. Math. J. **9** (2009), no. 1, 33–45, back matter. 16

[Hid02] Haruzo Hida, *Control theorems of coherent sheaves on Shimura varieties of PEL type*, J. Inst. Math. Jussieu **1** (2002), no. 1, 1–76. 104

[How04] Everett W. Howe, *On the non-existence of certain curves of genus two*, Compos. Math. **140** (2004), no. 3, 581–592. 200

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. 12

[Hub94] R. Huber, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551. 84

[Hub96] Roland Huber, *étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996. 71, 84

[Hum08] James E. Humphreys, *Representations of semisimple Lie algebras in the BGG category* $\mathscr{O}$, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008. 25, 34

[HZ01] Michael Harris and Steven Zucker, *Boundary cohomology of Shimura varieties. III. Coherent cohomology on higher-rank boundary strata and applications to Hodge theory*, Mém. Soc. Math. Fr. (N.S.) (2001), no. 85, vi+116. 176

[Ill85] Luc Illusie, *Déformations de groupes de Barsotti–Tate (d'après A. Grothendieck)*, Astérisque (1985), no. 127, 151–198, Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84). 215, 216

[JLRS23] Jennifer Johnson-Leung, Brooks Roberts, and Ralf Schmidt, *Stable Klingen Vectors and Paramodular Newforms*, Lecture Notes in Mathematics, vol. 2342, Springer, Cham, 2023. 174

[Joh17] Christian Johansson, *On the Sato–Tate conjecture for non-generic abelian surfaces*, Trans. Amer. Math. Soc. **369** (2017), no. 9, 6303–6325, With an appendix by Francesc Fité. 211

[Kar95] Gregory Karpilovsky, *Group representations. Vol. 4*, North-Holland Mathematics Studies, vol. 182, North-Holland Publishing Co., Amsterdam, 1995. 182

[Kat81] N. Katz, *Serre–Tate local moduli*, Algebraic surfaces (Orsay, 1976–78), Lecture Notes in Math., vol. 868, Springer, Berlin-New York, 1981, pp. 138–202. 202

[Ked10] Kiran S. Kedlaya, *p-adic differential equations*, Cambridge Studies in Advanced Mathematics, vol. 125, Cambridge University Press, Cambridge, 2010. 29

[Kha97] Chandrashekhar Khare, *Remarks on mod p forms of weight one*, Internat. Math. Res. Notices (1997), no. 3, 127–133. 222

[Kie67] Reinhardt Kiehl, *Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie*, Invent. Math. **2** (1967), 256–273. 84

[Kim03] Henry H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. 16

[Kis99] Mark Kisin, *Local constancy in p-adic families of Galois representations*, Math. Z. **230** (1999), no. 3, 569–593. 202, 205

[Kis04] ———, *Geometric deformations of modular Galois representations*, Invent. Math. **157** (2004), no. 2, 275–328. 157

[Kis09] ———, *Modularity of 2-adic Barsotti–Tate representations*, Invent. Math. **178** (2009), no. 3, 587–634. 145

[KS02] Henry H. Kim and Freydoon Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and the symmetric cube for* GL_2, Ann. of Math. (2) **155** (2002), no. 3, 837–893, With an appendix by Colin J. Bushnell and Guy Henniart. 185

[KT17a] Chandrashekhar Khare and Jack A. Thorne, *Automorphy of some residually* S_5 *Galois representations*, Math. Z. **286** (2017), no. 1-2, 399–429. 151

[KT17b] ———, *Potential automorphy and the Leopoldt conjecture*, Amer. J. Math. **139** (2017), no. 5, 1205–1273. 121

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger, *Serre's modularity conjecture. I*, Invent. Math. **178** (2009), no. 3, 485–504. 211, 212, 222

[L9̈0] Werner Lütkebohmert, *Formal-algebraic and rigid-analytic geometry*, Math. Ann. **286** (1990), no. 1-3, 341–371. 21, 85

[Lab11] J.-P. Labesse, *Changement de base CM et séries discrètes*, On the stabilization of the trace formula, Stab. Trace Formula Shimura Var. Arith. Appl., vol. 1, Int. Press, Somerville, MA, 2011, pp. 429–470. 133

[Lan13] Kai-Wen Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Mathematical Society Monographs Series, vol. 36, Princeton University Press, Princeton, NJ, 2013. 70

[Lan17] ———, *Integral models of toroidal compactifications with projective cone decompositions*, International Mathematics Research Notices **11** (2017), 3237–3280. 164

[Lan21] Aaron Landesman, *The Torelli map restricted to the hyperelliptic locus*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 354–378. 202

[Lan22] Kai-Wen Lan, *Closed immersions of toroidal compactifications of Shimura varieties*, Math. Res. Lett. **29** (2022), no. 2, 487–527. 70

[LH13] Jacob Lurie and Mike Hopkins, *Ambidexterity in* $K(n)$*-local stable homotopy theory*, unpublished note, 2013. 215

[LLZ23] Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu, *de Rham comparison and Poincaré duality for rigid varieties*, Peking Math. J. **6** (2023), no. 1, 143–216. 6

[LMF24] The LMFDB Collaboration, *The L-functions and modular forms database*, `https://www.lmfdb.org`, 2024, [Online; accessed 1 June 2024]. 3, 151, 188, 209, 210, 211

[LS18] Kai-Wen Lan and Benoît Stroh, *Nearby cycles of automorphic étale sheaves*, Compositio Mathematica **154** (2018), no. 1, 80–119. 17

[MA13] Hamza Menken and Abdulkadir Aşan, *On some properties of Liouville numbers in the non-Archimedean case*, Eur. J. Pure Appl. Math. **6** (2013), no. 2, 239–246. 29

[Mes72] William Messing, *The crystals associated to Barsotti–Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics, Vol. 264, Springer-Verlag, Berlin-New York, 1972. 216

[Mil14] Dragan Miličić, *Variations on a Casselman–Osborne theme*, Developments and retrospectives in Lie theory, Dev. Math., vol. 38, Springer, Cham, 2014, pp. 275–289. 27

[Nek18] Jan Nekovář, *Eichler–Shimura relations and semisimplicity of étale cohomology of quaternionic Shimura varieties*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 5, 1179–1252. 109, 110

[NT16] James Newton and Jack A. Thorne, *Torsion Galois representations over CM fields and Hecke algebras in the derived category*, Forum Math. Sigma **4** (2016), Paper No. e21, 88. 104

[NT23] ———, *Adjoint Selmer groups of automorphic Galois representations of unitary type*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 5, 1919–1967. 152, 156, 160, 162

[oGVAG09] University of Georgia VIGRE Algebra Group, *On Kostant's theorem for Lie algebra cohomology*, Representation theory, Contemp. Math., vol. 478, Amer. Math. Soc., Providence, RI, 2009, University of Georgia VIGRE Algebra Group: Irfan Bagci, Brian D. Boe, Leonard Chastkofsky, Benjamin Connell, Bobbe J. Cooper, Mee Seong Im, Tyler Kelly, Jonathan R. Kujawa, Wenjing Li, Daniel K. Nakano, Kenyon J. Platt, Emilie Wiesner, Caroline B. Wright and Benjamin Wyser, pp. 39–60. 60

[Pan22a] Lue Pan, *On locally analytic vectors of the completed cohomology of modular curves*, Forum Math. Pi **10** (2022), Paper No. e7, 82. 5, 6, 29, 30, 72, 73, 108

[Pan22b] ———, *On locally analytic vectors of the completed cohomology of modular curves II*, 2022. 5

[Pat15] Stefan Patrikis, *On the sign of regular algebraic polarizable automorphic representations*, Math. Ann. **362** (2015), no. 1-2, 147–171. 175

[Pil12] Vincent Pilloni, *Modularité, formes de Siegel et surfaces abéliennes*, J. Reine Angew. Math. **666** (2012), 35–82. 139

[Pil20] ———, *Higher coherent cohomology and p-adic modular forms of singular weights*, Duke Math. J. **169** (2020), no. 9, 1647–1807. 104, 105

[Pil24] Vincent Pilloni, *Faisceaux equivariants sur* $\mathbf{P}^1$ *et faisceaux automorphes*, Annales de la Faculté des sciences de Toulouse : Mathématiques **6e série, 33** (2024), no. 4, 1155–1213 (fr). 5, 64, 73, 108

[Pin98] Richard Pink, *l-adic algebraic monodromy groups, cocharacters, and the Mumford–Tate conjecture*, J. Reine Angew. Math. **495** (1998), 187–237. 186

[PS16a] Vincent Pilloni and Benoît Stroh, *Cohomologie cohérente et représentations Galoisiennes*, Ann. Math. Qué. **40** (2016), no. 1, 167–202. 70

[PS16b] ———, *Surconvergence, ramification et modularité*, Astérisque (2016), no. 382, 195–266. 4, 185, 214

[Qua23] Julian Quast, *Deformations of g-valued pseudocharacters*, 2023. 14

[RC22] Juan Esteban Rodríguez Camargo, *Locally analytic completed cohomology*, arXiv 2209.01057, 2022. 6, 71, 72, 73, 74, 75, 78, 81

[RC23] ———, *Geometric Sen theory over rigid analytic spaces*, arXiv 2205.02016, 2023. 6, 71, 73, 74, 75, 77

[RC24] ———, *The analytic de Rham stack in rigid geometry*, 2024. 40

[RJRC22] Joaquín Rodrigues Jacinto and Juan Esteban Rodríguez Camargo, *Solid locally analytic representations of p-adic Lie groups*, Represent. Theory **26** (2022), 962–1024. 18, 19, 20, 21, 22, 23, 26, 42, 49, 72, 75, 80, 84, 86, 91

[RS07] Brooks Roberts and Ralf Schmidt, *Local newforms for GSp(4)*, Lecture Notes in Mathematics, vol. 1918, Springer, Berlin, 2007. 174

[Sas19] Shu Sasaki, *Integral models of Hilbert modular varieties in the ramified case, deformations of modular Galois representations, and weight one forms*, Invent. Math. **215** (2019), no. 1, 171–264. 4, 185, 214

[Sch13a] Tobias Schmidt, *Verma modules over p-adic Arens–Michael envelopes of reductive Lie algebras*, J. Algebra **390** (2013), 160–180. 18, 24, 25, 37

[Sch13b] Peter Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi **1** (2013), e1, 77. 71

[Sch15] ———, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) **182** (2015), no. 3, 945–1066. 70, 72

[Sch18] Ralf Schmidt, *Packet structure and paramodular forms*, Trans. Amer. Math. Soc. **370** (2018), no. 5, 3085–3112. 174

[Ser62] Jean-Pierre Serre, *Endomorphismes complètement continus des espaces de Banach p-adiques*, Inst. Hautes Études Sci. Publ. Math. (1962), no. 12, 69–85. 95

[Ser72] ———, *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), no. 4, 259–331. 223

[Ser77a] ———, *Linear representations of finite groups*, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. 182

[Ser77b] ———, *Modular forms of weight one and Galois representations*, Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London, 1977, pp. 193–268. 185, 213

[Ser00] ———, *Lettre à Marie-France Vignéras du 10/2/1986*, Œuvres. Collected papers. IV, Springer-Verlag, Berlin, 2000, 1985–1998, pp. 38–55. 186, 207

[Ser08] ———, *Topics in Galois theory*, second ed., Research Notes in Mathematics, vol. 1, A K Peters, Ltd., Wellesley, MA, 2008, With notes by Henri Darmon. 204

[Sho18] Jack Shotton, *The Breuil–Mézard conjecture when $l \neq p$*, Duke Math. J. **167** (2018), no. 4, 603–678. 128, 133

[ST02] Peter Schneider and Jeremy Teitelbaum, *Locally analytic distributions and p-adic representation theory, with applications to* GL_2, J. Amer. Math. Soc. **15** (2002), no. 2, 443–468. 24

[ST03] ———, *Algebras of p-adic distributions and admissible representations*, Invent. Math. **153** (2003), no. 1, 145–196. 24, 54, 72

[Sta13] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2013. 22, 108

[Tay08] Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations. II*, Pub. Math. IHES **108** (2008), 183–239. 137

[Tay20] Noah Taylor, *Sato–Tate distributions on Abelian surfaces*, Trans. Amer. Math. Soc. **373** (2020), no. 5, 3541–3559. 211

[Tho12] Jack A. Thorne, *On the automorphy of l-adic Galois representations with small residual image*, J. Inst. Math. Jussieu **11** (2012), no. 4, 855–920, With an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Thorne. 115, 120, 124, 129, 136

[Tho15] ———, *Automorphy lifting for residually reducible l-adic Galois representations*, J. Amer. Math. Soc. **28** (2015), no. 3, 785–870. 129

[Tho17] ———, *A 2-adic automorphy lifting theorem for unitary groups over CM fields*, Math. Z. **285** (2017), no. 1-2, 1–38. 7, 8, 114, 115, 116, 117, 119, 120, 121, 122, 123, 124, 126, 127, 130, 134, 139, 151, 182

[Til96] Jacques Tilouine, *Deformations of Galois representations and Hecke algebras*, Published for The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1996. 134, 144

[TY07] Richard Taylor and Teruyoshi Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493 (electronic). 14

[TY22] Nobuo Tsuzuki and Takuya Yamauchi, *Automorphy of mod 2 Galois representations associated to the quintic Dwork family and reciprocity of some quintic trinomials*, arXiv 2008.09852, 2022. 4, 184

[Ulm16] Douglas Ulmer, *Conductors of ℓ-adic representations*, Proc. Amer. Math. Soc. **144** (2016), no. 6, 2291–2299. 174

[Vig05] Marie-France Vignéras, *Pro-p-Iwahori Hecke ring and supersingular $\overline{\mathbf{F}}_p$-representations*, Math. Ann. **331** (2005), no. 3, 523–556. 171

[Wed01] Torsten Wedhorn, *The dimension of Oort strata of Shimura varieties of PEL-type*, Moduli of abelian varieties (Texel Island, 1999), Progr. Math., vol. 195, Birkhäuser, Basel, 2001, pp. 441–471. 202, 215

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. 25, 28

[Whi22] Dmitri Whitmore, *The Taylor–Wiles method for reductive groups*, arXiv 2205.05062, 2022. 139, 146, 147, 148, 149, 150, 173, 177, 206, 222

[Wil95] Andrew Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551. 4, 151, 210

[Zar00] Yuri G. Zarhin, *Hyperelliptic Jacobians without complex multiplication*, Math. Res. Lett. **7** (2000), no. 1, 123–132. 186, 205

Email address: g.boxer@imperial.ac.uk

Department of Mathematics, Imperial College London, London SW7 2AZ, UK

Email address: fcale@math.uchicago.edu

The University of Chicago, 5734 S University Ave, Chicago, IL 60637, USA

Email address: toby.gee@imperial.ac.uk

Department of Mathematics, Imperial College London, London SW7 2AZ, UK

Email address: vincent.pilloni@universite-paris-saclay.fr

Département de Mathématique, Université Paris-Sud, Bâtiment 307, 81405 Orsay cedex, France