NOTES ON HIGHER COLEMAN THEORY

GEORGE BOXER AND VINCENT PILLONI

Abstract. The aim of higher Coleman theory is to study the finite slope part of the cohomology of Shimura varieties using local cohomology methods. We survey this theory in the Siegel case.

Contents

These are the notes for a series of four lectures we gave at a Montreal conference in December 2020. The goal of these notes is to explain the main results of higher Coleman theory in the case of Siegel Shimura varieties and to give some idea of the proofs. The reader should consult [BP21] for a more complete treatment of the theory. We follow the original plan of the lectures. The first lecture is a general overview of the theory. The second lecture is dedicated to the construction of the local cohomologies. The third lecture proves essential results on these local cohomologies (vanishing and slope estimates). The last lecture is about the construction of eigenvarieties. We would like to thank the organizers Najmuddin Fakhruddin, Eknath Ghate, Arvind Nair, C. S. Rajan, and Sandeep Varma of the International

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1. Lecture 1: General overview of the theory

1.1. **Toy model: the cohomology of $B\backslash G$.** We would like to explain first a way to compute the cohomology of flag varieties using local cohomology. These are the techniques we will apply later to study the cohomology of Shimura varieties.

1.1.1. *Notations.* Let us fix some standard notations:

- G is a split reductive group over a field F.
- $T \subseteq B$ is a maximal torus contained in a Borel.
- $X^\star(T)$ is the character group of T, $X^\star(T)^+$ is the dominant cone.
- The flag variety is $FL = B\backslash G$.
- W is the Weyl group, $\ell : W \to [0, d = \mathrm{dim} FL]$ is the length and w_0 is the longest element.
- We have the Bruhat stratification: $FL = \coprod_{w\in W} B\backslash BwB$ and $\ell(w)$ is the dimension of the Bruhat cell $B\backslash BwB$.
- Let ρ be half the sum of the positive roots. The dotted action of W on $X^\star(T)$ is given by $w \cdot \kappa = w(\kappa + \rho) - \rho$.

We have a functor from representations of B to G-equivariant locally free sheaves on FL. To any character $\kappa \in X^\star(T)$ (which can be inflated to a one dimensional representation of B) we associate the line bundle $\mathcal{L}_\kappa$ defined as follows. Let $\pi : G \to FL$ be the projection. We let

$$\mathcal{L}_\kappa(U) = \{f \in \mathrm{H}^0(\pi^{-1}(U), \mathscr{O}_G),\ f(bu) = w_0\kappa(b)f(u)\ \forall (b, u) \in B \times \pi^{-1}(U)\}$$

The right translation action of G on FL induces a left action of G on $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)$.

1.1.2. *The Borel-Weil-Bott theorem.* The following theorem entirely describes the cohomology of the Flag variety in characteristic 0.

Theorem 1.1 (Borel-Weil-Bott)**.** *Assume that* $\mathrm{char}(F) = 0$. *Let* $\kappa \in X^\star(T)$ *then either:*

1. *There is no* $w \in W$ *such that* $w \cdot \kappa \in X^\star(T)^+$ *and* $\mathrm{H}^i(FL, \mathcal{L}_\kappa) = 0$ *for all* i,
2. *There is a unique* $w \in W$ *such that* $w \cdot \kappa \in X^\star(T)^+$ *is dominant, and* $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa) = \mathrm{H}^{\ell(w)}(FL, \mathcal{L}_\kappa)[-\ell(w)]$ *and* $\mathrm{H}^{\ell(w)}(FL, \mathcal{L}_\kappa)$ *is the highest weight* $w \cdot \kappa$ *representation of* G.

Remark 1.2. Situation of (1) happens if and only if $\kappa + \rho$ is not G-regular.

1.1.3. *The Cousin complex.* We will survey a proof of this theorem following [Kem78], section 12. We consider the stratification of FL induced from the Bruhat stratification:

$$FL = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_d \supseteq Z_{d+1} = \emptyset$$

where $Z_i = \cup_{w,\ell(w)=d-i}\overline{B\backslash BwB}$ is the closure of the union of the Bruhat cells of codimension i.

One can build the following complex, called the Grothendieck-Cousin complex:

$$\mathcal{C}ous(\kappa) : 0 \to \mathrm{H}^0_{Z_0/Z_1}(FL, \mathcal{L}_\kappa) \to \mathrm{H}^1_{Z_1/Z_2}(FL, \mathcal{L}_\kappa) \to \cdots \to \mathrm{H}^d_{Z_d/Z_1}(FL, \mathcal{L}_\kappa) \to 0$$

where $\mathrm{R}\Gamma_{Z_p/Z_{p+1}}(FL, \mathcal{L}_\kappa) := \mathrm{R}\Gamma_{Z_p\backslash Z_{p+1}}(FL\backslash Z_{p+1}, \mathcal{L}_\kappa)$, and by definition, we have triangles:

$$\mathrm{R}\Gamma_{Z_p\backslash Z_{p+1}}(FL \setminus Z_{p+1}, \mathcal{L}_\kappa) \to \mathrm{R}\Gamma(FL \setminus Z_{p+1}, \mathcal{L}_\kappa) \to \mathrm{R}\Gamma(FL \setminus Z_p, \mathcal{L}_\kappa) \stackrel{+1}{\to}$$

The differentials are given by the boundary maps in the long exact sequences associated to the exact triangles:

$$\mathrm{R}\Gamma_{Z_{p+1}\backslash Z_{p+2}}(FL\backslash Z_{p+2}, \mathcal{L}_\kappa) \to \mathrm{R}\Gamma_{Z_p\backslash Z_{p+2}}(FL\backslash Z_{p+2}, \mathcal{L}_\kappa) \to \mathrm{R}\Gamma_{Z_p\backslash Z_{p+1}}(FL\backslash Z_{p+1}, \mathcal{L}_\kappa) \stackrel{+1}{\to}$$

It also follows easily from the definition that

$$\mathrm{H}^\star_{Z_p/Z_{p+1}}(FL, \mathcal{L}_\kappa) = \oplus_{w, \ell(w)=d-p} \mathrm{H}^\star_w(FL, \mathcal{L}_\kappa)$$

where $\mathrm{H}^\star_w(FL, \mathcal{L}_\kappa)$ consists of the classes supported on $B\backslash BwB$. The relevance of the Cousin complex is given by the following proposition.

Proposition 1.3. *The complex $\mathcal{C}ous(\kappa)$ is quasi-isomorphic to* $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)$.

Sketch of Proof. There is a spectral sequence $E_1^{p,q} = \mathrm{H}^{p+q}_{Z_p/Z_{p+1}}(FL, \mathcal{L}_\kappa) \Rightarrow \mathrm{H}^{p+q}(FL, \mathcal{L}_\kappa)$. The Cousin complex is simply $E_1^{\bullet,0}$. One needs to prove that $E_1^{p,q} = 0$ if $q \neq 0$. Each of the Bruhat cell is a product of affine spaces indexed by certain root spaces. The vanishing therefore boils down to a computation on the affine space. Let $n \geq m$ and let $\mathbb{A}^m \hookrightarrow \mathbb{A}^n$ be the inclusion given by the vanishing of the last $n-m-1$ coordinates. A classical computation shows that $\mathrm{H}^\star_{\mathbb{A}^m}(\mathbb{A}^n, \mathscr{O}_{\mathbb{A}^n})$ is concentrated in degree $n-m$ and this is all we need. □

Here are some further properties of $\mathcal{C}ous(\kappa)$:

(1) This is a complex of infinite dimensional F vector spaces.
(2) The action of T is explicit on the complex.
(3) There is no G-action, but there is an infinisitesimal G-action. Thus, $\mathcal{C}ous(\kappa)$ is a complex of $\mathcal{U}(\mathfrak{g})$-modules and one can show that it belongs to the BGG category $\mathcal{O}$.

1.1.4. *An example.* We illustrate this by working out an example. Let

$$FL = \mathbb{P}^1 = B\backslash \mathrm{SL}_2 = B\backslash Bw_0B \coprod B\backslash B = \mathbb{A}^1 \coprod \{\infty\},$$

where B is the upper triangular Borel in the group SL_2 of determinant 1, 2×2 matrices. Let $\pi : \mathrm{SL}_2 \to FL$ be the projection. Here $\mathbb{A}^1 = \pi(w_0U)$ (where U is the unipotent radical in B), and $\{\infty\} = \pi(1)$. Let $k \in \mathbb{Z} = X^\star(T)$. The Cousin complex takes the following shape:

$$\mathcal{C}ous(k) : 0 \to \mathrm{H}^0(\mathbb{A}^1, \mathcal{L}_k) \to \mathrm{H}^1_\infty(\mathbb{P}^1, \mathcal{L}_k) \to 0$$

We define an isomorphism

$$\begin{array}{rcl} \mathrm{H}^0(\mathbb{A}^1, \mathcal{L}_k) & \to & F[X] \\ f & \mapsto & f(\pi(w_0 \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix})) \end{array}$$

One easily computes the action of T. For $t \in T$ and $P(X) \in F[X]$, $t.P(X) = t^kP(t^{-2}X)$. Therefore the weights of T on $\mathrm{H}^0(\mathbb{A}^1, \mathcal{L}_k)$ are $k, k-2, k-4, \cdots$.

We can also compute $\mathrm{H}^1_\infty(\mathbb{P}^1, \mathcal{L}_k)$. Let $\bar{U}$ be the opposite unipotent radical. We find that $\bar{U}$ maps isomorphically via π to a neighborhood of $\{\infty\} \in \mathbb{P}^1$ so that

$$\begin{aligned} \mathrm{H}^1_\infty(\mathbb{P}^1, \mathcal{L}_k) &= \mathrm{H}^1_\infty(\pi(\bar{U}), \mathcal{L}_k) \\ &= \mathrm{H}^1_1(\bar{U}, \pi^\star \mathcal{L}_k) \end{aligned}$$

We have $\bar{U} = \mathrm{Spec}\ k[Y]$, and $1 \in \bar{U}$ is given by $Y = 0$. Moreover, there is an isomorphism:

$$\begin{aligned} \mathrm{H}^0(\bar{U}, \pi^\star \mathcal{L}_k) &\to F[Y] \\ f &\mapsto f(\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}) \end{aligned}$$

We also compute the T-action. For $t \in T$ and $Q(Y) \in F[Y]$, $t.Q(Y) = t^{-k}Q(t^2 Y)$. There is a short exact sequence

$$0 \to \mathrm{H}^0(\bar{U}, \pi^\star \mathcal{L}_k) \to \mathrm{H}^0(\bar{U} \setminus \{1\}, \pi^\star \mathcal{L}_k) \to \mathrm{H}^1_1(\bar{U}, \pi^\star \mathcal{L}_k) \to 0$$

and therefore we deduce that $\mathrm{H}^1_\infty(\mathbb{P}^1, \mathcal{L}_k) \simeq k[Y, Y^{-1}]/k[Y]$ and the weights of T on $\mathrm{H}^1_\infty(\mathbb{P}^1, \mathcal{L}_k)$ are $-k-2, -k-4, \cdots$. We deduce that $\mathcal{C}ous(k)$ is given by the following complex:

$$0 \to F[X]e \to F(Y)/F[Y]f \to 0$$

where X has weight -2, Y has weight 2, e has weight k and f has weight $-k$. It remains to compute the differential which is given by the change of coordinate between X and Y. The following identity

$$\begin{pmatrix} 1 & X^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X^{-1} & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X^{-1} & 1 \end{pmatrix}$$

implies that on $\mathbb{A}^1 \setminus \{0\} = \mathrm{Spec}\ F[X, X^{-1}] = \mathrm{Spec}\ F[Y, Y^{-1}]$, the global sections of $\mathcal{L}_k$ are isomorphic to $F[X, X^{-1}] = F[Y, Y^{-1}]$ via the map $P(X) \mapsto Q(Y) = Y^k P(Y^{-1})$. We finally conclude that $\mathcal{C}ous(k)$ is given by the following complex:

$$\begin{aligned} 0 \to F[X]e &\to F(X)/F[X^{-1}]f \to 0 \\ P(X)e &\mapsto X^{-k}P(X)f \end{aligned}$$

There is a B action on the complex, where $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts on the X-coordinate via $X \mapsto X+1$. There is also an action of $\bar{\mathfrak{u}}$ where $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is given by ∂_Y in the Y-coordinate.

1.1.5. *Big weight part.* We now want to illustrate how we can use the knowledge of the action of T on the Cousin complex. We introduce a partial order on $X^\star(T)_\mathbb{R}$ by declaring that $\lambda \leq \mu$ if $\mu - \lambda$ is a non-negative linear combination of positive roots. The following lemma follows from explicit computations, and as already been observed in the example of $\mathbb{P}^1$:

Lemma 1.4. *The weights of T acting on $\mathrm{H}^\star_w(FL, \mathcal{L}_\kappa)$ are $\leq w^{-1}w_0.\kappa$.*

We see that the action of T can be used to separate the contribution from the various local cohomologies in the Cousin complex. We formalize this. Let $C(\kappa) = \{w \in W,\ w^{-1}w_0(\kappa + \rho) \in X^\star(T)^+\}$. Let $\nu = w^{-1}w_0 \cdot \kappa$ for any $w \in C(\kappa)$. Let $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)^{bw(\nu)}$ be the big weight part of $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)$, where the weights of T are $\not\leq w \cdot \nu$ for all $w \in W$ with $w \cdot \nu \neq \nu$. Then we find the following theorem:

Proposition 1.5. *The cohomology* $\mathrm{R}\Gamma(FL,\mathcal{L}_\kappa)^{bw(\nu)}$ *is a prefect complex of amplitude* $[\min_{w\in C(\kappa)} d-\ell(w), \max_{w\in C(\kappa)} d-\ell(w)]$.

Proof. It follows from lemma 1.4 that all the terms of the Cousin complex outside the range are not of big weight. □

This result is elementary and valid in all characteristics. One can think of this theorem as a classicity theorem as well, in the sense that it identifies certain local cohomology classes with global cohomology classes. Indeed, assume that $\kappa+\rho$ is G-regular so that $C(\kappa)=\{w\}$. The theorem says in this case that

$$\mathrm{R}\Gamma(FL,\mathcal{L}_\kappa)^{bw(\nu)} = \mathrm{H}^{d-\ell(w)}_w(FL,\mathcal{L}_\kappa)^{bw(\nu)}[-d+\ell(w)].$$

Example 1.6. Let us look again at the case of $\mathbb{P}^1$. If $k\geq 0$, we have $\nu=k$, we find that $\mathrm{R}\Gamma(\mathbb{P}_1,\mathcal{L}_k)^{bw(\nu)}=\mathrm{H}^0(\mathbb{A}_1,\mathcal{L}_k)^{bw(\nu)}$ is the space where the weights of T are $>-k-2$. If $k\leq -2$, we find that $\nu=-2-k$. $\mathrm{R}\Gamma(\mathbb{P}_1,\mathcal{L}_k)^{bw(\nu)}=\mathrm{H}^1_\infty(\mathbb{P}_1,\mathcal{L}_k)^{bw(\nu)}[-1]$ is the space where the weights of T are $>k$. If $k=-1$, the proposition is vacuous.

Using properties of $\mathcal{O}$ in characteristic 0 we can derive easily a full proof of theorem 1.1.

1.1.6. *Interpolation.* Another aspect of the story is interpolation. The sheaves $\mathcal{L}_\kappa$ cannot be interpolated in the variable $\kappa\in X^\star(T)$. For example, we know that $\mathrm{Pic}(\mathbb{P}^1)=\mathbb{Z}$. A representative of each isomorphism class is given by $\mathcal{L}_k$ for $k\in\mathbb{Z}$. We also cannot interpolate the cohomologies since they have different dimensions or sit in different degrees.

On the other hand, in a neighborhood of each Bruhat cell ($\mathbb{A}^1$ or ∞ in the case of $\mathbb{P}^1$), the sheaves $\mathcal{L}_\kappa$ are trivial and therefore all the local cohomologies $\mathrm{H}^\star_w(FL,\mathcal{L}_\kappa)$ are isomorphic as F-vector spaces. We actually claim that it is possible to interpolate the action of Lie algebra $\mathfrak{g}$.

We assume that $\mathrm{char}(F)=0$. Let $\widehat{B},\widehat{T}$ and $\widehat{G}$ be the formal completions of B,T and G at the identity. We have an isomorphism $\log:\widehat{T}\to\mathfrak{t}\otimes\widehat{\mathbb{G}_a}$. We let $\mathfrak{t}^\vee=\mathrm{Spec}\left(\mathrm{Sym}(X^\star(T)\otimes_{\mathbb{Z}}F)\right)$. This is the space of characters $\widehat{T}\to\widehat{\mathbb{G}_m}$. We let $\kappa^{un}:\widehat{T}\times\mathfrak{t}^\vee\to\widehat{\mathbb{G}_m}$ be the universal character.

Proposition 1.7. *Assume that* $\mathrm{char}(F)=0$. *For any* $w\in W$, *and any* $\kappa\in\mathfrak{t}^\vee$ *there are invertible,* $\hat{G}$*-equivariant sheaves* $\mathcal{L}_\kappa$ *defined in a neighborhood of the Bruhat cell* $B\backslash BwB$ *in* FL. *These sheaves interpolate the* G*-equivariant sheaves* $\mathcal{L}_\kappa$ *for* $\kappa\in X^\star(T)$. *Consequently, the local cohomology*

$$\mathrm{H}^{d-\ell(w)}_w(FL,\mathcal{L}_\kappa)$$

defines an family of $\mathfrak{g}$*-representations parametrized by* $\mathfrak{t}^\vee$.

Proof. Let us briefly explain this for the cohomology of the big cell $B\backslash Bw_0B$. We have the following uniformization of the big cell: $\hat{\pi}:\widehat{B}w_0U\to\widehat{B}\backslash\widehat{B}w_0U$. Moreover, $\hat{B}$ acts by left translation and $\hat{G}$ by right translation on $\widehat{B}w_0U$, and these actions commute with each other. Any character κ of $\widehat{T}$ can also be extended to $\widehat{B}$ by inflation. For any $\kappa\in\mathfrak{t}^\vee$, we define a sheaf $\mathcal{L}_\kappa$ over $\widehat{B}\backslash\widehat{B}w_0U$ as follows:

$$\mathcal{L}_\kappa(U)=\{f\in\mathrm{H}^0(\hat{\pi}^{-1}(U),\mathscr{O}_{\hat{\pi}^{-1}(U)}),\ f(\hat{b}u)=w_0\kappa(\hat{b})f(u),\ \forall(\hat{b},u)\in\widehat{B}\times\hat{\pi}^{-1}(U)\}.$$

This sheaf is trivial, because $\hat{\pi}$ has a section, but it carries a non-trivial $\widehat{G}$-equivariant action. We deduce that there is an action morphism

$$\mathrm{H}^0(B\backslash Bw_0B, \mathcal{L}_\kappa) \to \mathrm{H}^0(B\backslash Bw_0B, \mathcal{L}_\kappa) \otimes_F \mathscr{O}_{\widehat{G}}.$$

In other words, $\mathrm{H}^0(B\backslash Bw_0B, \mathcal{L}_\kappa)$ is a $\mathcal{U}(\mathfrak{g})$-module. □

Example 1.8. In the case of $\mathbb{P}^1$, one can compute that the weights of $\widehat{T}$ on $\mathrm{H}^0(\mathbb{A}^1, \mathcal{L}_\kappa)$ for $\kappa \in F$ are κ, $\kappa - 2$, $\kappa - 4$...

1.2. **Local cohomology on Shimura varieties.** Let p be a fixed prime number. We now want to introduce local cohomology on p-adic Shimura varieties. The theory has been developed for any abelian type Shimura datum (G, X), under the assumption that $G_{\mathbb{Q}_p}$ is quasi-split. But for simplicity, we will restrict ourselves to the Siegel Shimura datum in this note.

1.2.1. *The Shimura variety.* Let $g \geq 1$ be an integer and let $(G = \mathrm{GSp}_{2g}, X = \mathcal{H}_g)$ be the Siegel Shimura datum (for $\mathcal{H}_g$ the Siegel space of matrices $M \in \mathrm{M}_g(\mathbb{C})$, with $M^t = M$ and $\mathrm{Im}(M)$ is definite positive or negative). Let p be a prime and $K = K_pK^p \subseteq G(\mathbb{A}_f)$ is a neat compact open subgroup. The complex manifold

$$S_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

admits a model S_K which is an algebraic variety defined over $\mathbb{Q}$ and is a moduli space of polarized abelian varieties of dimension g, with level structure (depending on K). For a suitable combinatorial choice of polyhedral decomposition Σ, we have toroidal compactifications $S_K \hookrightarrow S^{tor}_{K,\Sigma}$ defined over $\mathbb{Q}$, as well as minimal compactification $S_K \hookrightarrow S^\star_K$ (see [FC90]). Let $D = S^{tor}_{K,\Sigma} \setminus S_K$.

1.2.2. *Vector bundles, local systems, vector bundles with connection.* Attached to (G, X) we have a conjugacy class of cocharacters $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$. We fix a representative of μ. We have two opposite (Siegel type) parabolics $P_\mu, P^{std}_\mu \hookrightarrow G$, with Levi M_μ (isomorphic to $\mathrm{GL}_g \times \mathbb{G}_m$). The Borel embedding is the holomorphic embedding $\beta : X \hookrightarrow G(\mathbb{C})/P^{std}_\mu(\mathbb{C})$ which sends a point $h \in X$ corresponding to a Hodge structure to the parabolic stabilizing the Hodge filtration. The map β is $G(\mathbb{R})$-equivariant. Let us fix a maximal torus and Borel: $T \subseteq B \subseteq P_\mu$.

Remark 1.9. Our choice of taking $B \subseteq P_\mu$ and not $B \subseteq P^{std}_\mu$ is guided by the fact that X and $G(\mathbb{C})/P^{std}_\mu(\mathbb{C})$ have opposite curvature.

We let $\mathrm{Rep}(M_\mu)$ and $\mathrm{Rep}(G)$ be the category of finite dimensional representations of M_μ and G respectively. We let $X^\star(T)$ be the characters of T, $X^\star(T)^+$ be the dominant characters for G, and $X^\star(T)^{M_\mu,+}$ be the dominant characters for M_μ. They label the irreducible representations of G and M_μ respectively. We let $VB(Sh^{tor}_{K,\Sigma})$ be the category of vector bundles on the toroidal compactification of the Shimura variety, $LS(Sh_K(\mathbb{C}))$ be the category of $\mathbb{Q}$-local systems, and $MIC(Sh^{tor}_{K,\Sigma})$ be the category of filtered vector bundles with integrable log-connections. We have functors (see for example [Har90]):

$$\begin{array}{rcl} \mathrm{Rep}(M_\mu) & \to & VB(S^{tor}_{K,\Sigma}) \\ \kappa \in X^\star(T)^{M_\mu,+} & \mapsto & \mathcal{V}_\kappa \\ \mathrm{Rep}(G) & \to & LS(S_K(\mathbb{C})) \\ \nu \in X^\star(T)^+ & \mapsto & \mathcal{W}_\nu \\ \mathrm{Rep}(G) & \to & MIC(S^{tor}_{K,\Sigma}) \\ \nu \in X^\star(T)^+ & \mapsto & \mathcal{W}_{\nu,dR} \end{array}$$

One can quickly define these functors over $\mathbb{C}$, by constructing G-equivariant vector bundles, local systems or integrable connections on $G(\mathbb{C})/P^{std}_\mu(\mathbb{C})$, pull back via β to $X \times G(\mathbb{A}_f)/K$ and descend them using the $G(\mathbb{Q})$-action. Using the modular interpretation on S_K, on can prove that the vector bundles or vector bundles with connexion are defined over $\mathbb{Q}$, and using the theory of toroidal compactifications, one can see that there are canonical extensions.

The vector bundles, local systems or integrable connections obtained via these functor only depend on the restriction of the representations to the derived groups. But they also carry an equivariant action of the Hecke algebra which depends on the representation itself.

1.2.3. *Some more notations.* To fix ideas, we assume that $G \hookrightarrow \mathrm{GL}_{2g}$ preserves up to a similitude factor the symplecting form given by the matrix $\begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}$ with s the antidiagonal $g \times g$ matrix with only 1 on the antidiagonal. We also assume that $P^{std}_\mu = \{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in G\}$ is the upper triangular by block Siegel parabolic. Note that $C = \nu s (A^{-1})^t s$ for $\nu \in \mathbb{G}_m$ the similitude factor. We let T be the diagonal torus. A typical element $t \in T$ writes $\mathrm{diag}(ct_1, \cdots, ct_g, ct_g^{-1}, \cdots, ct_1^{-1})$. We identify $X^\star(T) = \{(k_1, \cdots k_g; k) \in \mathbb{Z}^g \times \mathbb{Z}, \sum k_i = k \mod 2\}$, where $(k_1, \cdots k_g; k)t = c^k \prod_{i=1}^g t_i^{k_i}$.

We fix $B = \{\begin{pmatrix} A & 0 \\ D & C \end{pmatrix} \in G,\ A \text{ upper triangular}\}$[1] We deduce that $X^\star(T)^+ = \{(k_1, \cdots k_g; k) \in X^\star(T),\ 0 \geq k_1 \geq \cdots \geq k_g\}$ and $X^\star(T)^{M_\mu,+} = \{(k_1, \cdots k_g; k) \in X^\star(T), k_1 \geq \cdots \geq k_g\}$.

Having fixed all this, we may describe a little bit more explicitely the various automorphic sheaves. Let $A \to S^{tor}_{K,\Sigma}$ be the semi-abelian scheme. We let ω_A be its co-normal sheaf, which is locally free of rank g. We let $\mathrm{H}_1(A, \mathbb{Q})$ be the first relative Betti homology of A over $S_K(\mathbb{C})$. This is a rank $2g$, $\mathbb{Q}$-local system. We let $\mathcal{H}_{1,dR}(A)$ be the first relative log-de Rham homology of the Kuga Satake compactification of A over $S^{tor}_{K,\Sigma}$. This is a filtered rank $2g$ locally free sheaf equipped with an integrable log-connection.

(1) $g = 1$, $k \in \mathbb{Z}$, $\mathcal{V}_{(k,-k)} = \omega_A^k = \omega^k$ is the sheaf of weight k modular forms.
(2) $g = 2$, $k_1 \geq k_2 \in \mathbb{Z}$, $\mathcal{V}_{(k_1,k_2;-k_1-k_2)} = \mathrm{Sym}^{k_1-k_2}\omega_A \otimes \det^{k_2}\omega_A = \omega^{(k_1,k_2)}$ is the sheaf of weight (k_1, k_2) Siegel modular forms of genus 2.
(3) $g = 1$, $\nu \in \mathbb{Z}_{\geq 0}$, $\mathcal{W}_{(-\nu;\nu)} = \mathrm{Sym}^\nu \mathrm{H}_1(A)$, $\mathcal{W}_{(-\nu;\nu),dR} = \mathrm{Sym}^\nu \mathcal{H}_{1,dR}(A)$

[1] We appologize for this strange choice of Borel subgroup. It has the pleasant effect that the labelling of weights of automorphic vector bundles coincides with what we believe is the usual labelling.

(4) For any g, $k_1 \geq k_2 \geq \cdots \geq k_g \in \mathbb{Z}$, $\mathcal{V}_{(k_1,\cdots,k_g,-\sum k_i)}$ is the sheaf of weight $(k_1,\cdots,k_g)$ Siegel modular forms.
(5) For any g, $\mathcal{V}_{(g+1,\cdots,g+1,0)}$ is the canonical sheaf $\det(\Omega^1_{S^{tor}_{K,\Sigma}/\mathbb{Q}})$.
(6) For any g, $\mathcal{W}_{(0,\cdots,0,-1;1)} = \mathrm{H}_1(A)$, $\mathcal{W}_{(0,\cdots,0,-1;1),dR} = \mathcal{H}_{1,dR}(A)$.

1.2.4. *Hodge-Tate period map.* In the case of the Flag variety, the local cohomologies are defined using the Bruhat stratification. In the case of Shimura varieties, we are going to use the Hodge-Tate period map and the Bruhat stratification again. We use p-adic geometry, and we denote by $\mathcal{S}^{an}_K$ the adic Siegel variety over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$, and by $\mathcal{S}^{tor}_{K_pK^p,\Sigma}$ and $\mathcal{S}^{\star}_{K_pK^p}$ the adic toroidal and minimal compactifications.

We let $\mathcal{S}^{tor}_{K^p,\Sigma} = \lim_{K'_p\subseteq K_p} \mathcal{S}^{tor}_{K'_pK^p,\Sigma}$, $\mathcal{S}^{\star}_{K^p} = \lim_{K'_p\subseteq K_p} \mathcal{S}^{\star}_{K'_pK^p}$ and $\mathcal{S}^{an}_{K^p} = \lim_{K'_p\subseteq K_p} \mathcal{S}^{an}_{K_pK^p}$. These are perfectoid spaces (for the minimal compactification this is one of the main results of [Sch15], the case of toroidal compactifications is deduced in [PS16]). We have a commutative diagram:

$$\begin{array}{ccc} \mathcal{S}^{tor}_{K^p,\Sigma} & & \\ \downarrow & \overset{\pi^{tor}_{HT}}{\searrow} & \\ \mathcal{S}^{\star}_{K^p} & \xrightarrow{\pi_{HT}} & \mathcal{FL}_{G,\mu} \end{array}$$

where π_{HT} and π^{tor}_{HT} are the Hodge-Tate period maps. These maps are easy to describe on points outside the boundary. Let $x = \mathrm{Spa}(K,\mathcal{O}_K) \to \mathcal{S}^{an}_{K^p,\Sigma}$. We have the Hodge-Tate exact sequence:

$$0 \to Lie(A) \to V_p(A) \otimes_{\mathbb{Q}_p} K \to \omega_{A^t} \to 0$$

and a trivialization $\mathbb{Q}_p^{2g} \simeq V_p(A)$. Therefore, we get a Lagrangian subspace $Lie(A) \subseteq K^{2g}$ and this defines an element of $\pi_{HT}(x) \in \mathcal{FL}_{G,\mu}(K)$.

The important properties of these maps are:

(1) The map π_{HT} is $G(\mathbb{Q}_p)$-equivariant,
(2) The map π^{tor}_{HT} is K_p-equivariant,
(3) The map π_{HT} is affinoid: the pre-image of the standard affinoid rational subsets of $\mathcal{FL}_{G,\mu}$ is affinoid perfectoid.

By passing to the quotient by K_p, we deduce a continuous map of topological spaces:

$$\pi^{tor}_{HT,K_p} : |\mathcal{S}^{tor}_{K_pK^p,\Sigma}| \to |\mathcal{FL}_{G,\mu}/K_p|$$

1.2.5. *Bruhat stratification on the Flag variety and the Shimura variety.* Let W_M be the Weyl group of M_μ, W the Weyl group of G, and let ${}^MW \hookrightarrow W$ the minimal length set of representatives of $W_M\backslash W$. As W permutes the coordinates of the diagonal torus, we can realize $W \subseteq \mathcal{S}_{2g}$ as the set of permutations for which $w(i)+w(2g-i+1) = 2g+1$. We have that W_M is the subset of permutations which preserve the sets $\{1,\cdots,g\}$ and $\{g+1,\cdots,2g\}$. Finally, ${}^MW$ identifies with permutations w for which $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(g)$.

We have the Bruhat stratification of the special fiber of the Flag variety:

$$FL_{G,\mu,\mathbb{F}_p} = \coprod_{w\in {}^MW} P_{\mu,\mathbb{F}_p}\backslash P_{\mu,\mathbb{F}_p} w B_{\mathbb{F}_p}$$

We also see easily that for the action of $T(\mathbb{Q}_p)$ on $\mathcal{FL}_{G,\mu}$, there is a unique fixed point in each tube $]P_{\mu,\mathbb{F}_p}\backslash P_{\mu,\mathbb{F}_p} w B_{\mathbb{F}_p}[$ and therefore, ${}^MW = \mathrm{Fix}(T(\mathbb{Q}_p)|\mathcal{FL}_{G,\mu})$.

Let us consider the stratification by open subsets

$$\{FL^{\geq r}_{G,\mu,\mathbb{F}_p} = \cup_{\ell(w)\geq r} P_{\mu,\mathbb{F}_p}\backslash P_{\mu,\mathbb{F}_p} w B_{\mathbb{F}_p}\}_{0\leq r\leq d}$$

We then define $\overline{]FL^{\geq r}_k[} := Z_r$ (the closure of the tube). This gives a filtration

$$\mathcal{FL}_{G,\mu} = Z_0 \supset Z_1 \supset \cdots \supset Z_d \supset Z_{d+1} = \emptyset$$

by closed subspaces invariant under the Iwahori subgroup of $G(\mathbb{Z}_p)$. From now on we let K_p be the Iwahori subgroup.

We can consider the pullback of this stratification by π^{tor}_{HT,K_p} to get a filtration $\mathcal{S}^{tor}_{K,\Sigma} = \mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \cdots \supset \mathcal{Z}_d \supset \mathcal{Z}_{d+1} = \emptyset$ by closed subspaces.

Let us describe a little bit more this filtration. It follows from the definition that $\mathcal{Z}_p \setminus \mathcal{Z}_{p+1}$ contains all the $(\pi^{tor}_{HT,K_p})^{-1}(wK_p)$ for $\ell(w) = p$ and we can describe very explicitly this set. Over $\mathcal{S}^{an}_{K_pK^p}$, let $\mathrm{Fil}^i A[p]$ be the universal filtration on $A[p]$ corresponding to the K_p-level structure.

Lemma 1.10. *Let $w \in {}^MW$. Then $(\pi^{tor}_{HT,K_p})^{-1}(wK_p)$ is a union of connected components in (the closure of) the ordinary locus characterized as follows: for any $x \in (\pi^{tor}_{HT,K_p})^{-1}(wK_p) \cap \mathcal{S}^{an}_{K_pK^p}$, $\mathrm{Gr}^{w(i)}A[p]$ is a group of multiplicative reduction.*

Proof. One first checks that $(\pi^{tor}_{HT})^{-1}(w) \cap \mathcal{S}^{an}_{K^p}$ is the set of points $x \in \mathcal{S}^{an}_{K^p}$ with the property that for the Hodge-Tate exact sequence

$$0 \to Lie(A) \to V_p(A) \otimes_{\mathbb{Q}_p} k(x) \to \omega_{A^t} \to 0$$

and trivialization $V_p(A) = \mathbb{Q}_p^{2g}$, $Lie(A)$ is the $k(x)$-vector space generated by the vectors $e_{w(1)}, \cdots, e_{w(g)}$ of the canonical basis of $\mathbb{Q}_p^{2g}$. Since the Hodge-Tate periods are rational we deduce from [Far11], sect. 2.5 that A is ordinary. We have a multiplicative-étale exact sequence: $0 \to A[p^\infty]^m \to A[p^\infty] \to A[p^\infty]^{et} \to 0$ and $A[p^\infty]^m = \mathrm{Ker}(A[p^\infty] \to \omega_{A^t})$. The lemma follows. □

Remark 1.11. The set $(\pi^{tor}_{HT,K_p})^{-1}(K_p)$ is called the canonical locus, this is the locus where $\mathrm{Fil}^g A[p]$ is $A[p]^m$.

Remark 1.12. By [SW13], for any complete algebraically closed field extension $\mathbb{C}$ of $\mathbb{Q}_p$, the Hodge-Tate period map identifies the set of isomorphism classes of p-divisible groups G over $\mathcal{O}_{\mathbb{C}}$, of height $2g$, equipped with a trivialization $T_p(G) \simeq \mathbb{Z}_p^{2g}$ and such that $G \simeq G^D$ with $\mathcal{FL}_{G,\mu}(\mathbb{C})$. The fibers of the Hodge-Tate map are pefectoid Igusa varieties: they parametrize abelian schemes with a fixed p-divisible group and a fixed trivialization of the Tate module, determined by the point in the flag variety. The sets $\{(\pi^{tor}_{HT,K_p})^{-1}(wK_p)\}_{w\in {}^MW}$ are the images at level K_p of the various "ordinary" Igusa varieties

1.2.6. *Local cohomology and the spectral sequence.* For any weight $\kappa \in X^\star(T)^{M_\mu,+}$, we have the cohomology

$$\mathrm{R}\Gamma_{\mathcal{Z}_p/\mathcal{Z}_{p+1}}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa) := \mathrm{R}\Gamma_{\mathcal{Z}_p\setminus\mathcal{Z}_{p+1}}(\mathcal{S}^{tor}_{K,\Sigma} \setminus \mathcal{Z}_{p+1}, \mathcal{V}_\kappa),$$

and one can further see that there is a natural decomposition:

$$\mathrm{R}\Gamma_{\mathcal{Z}_p/\mathcal{Z}_{p+1}}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa) = \oplus_{w\in {}^MW,\ell(w)=p} \mathrm{R}\Gamma_w(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$$

and we have a spectral sequence:

$$\mathrm{H}^{p+q}_{\mathcal{Z}_p/\mathcal{Z}_{p+1}}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa) \Rightarrow \mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa).$$

So far the situation is really parallel to the case of the Flag variety. We would like to point out a number of analogies:

(1) The classical cohomology $\mathrm{H}^\star(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$ is a representation of the full Hecke algebra $\mathcal{H}_p = \mathcal{C}^\infty_c(K_p\backslash G(\mathbb{Q}_p)/K_p, \mathbb{Q}_p)$. The local cohomogies $\mathrm{H}^\star_w(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$ are only representations of the dilating Hecke algebra $\mathcal{H}_p^+$, generated by the double classes $[K_ptK_p]$, for $t \in T(\mathbb{Q}_p)^+ = \{t \in T(\mathbb{Q}_p), v(\alpha(t)) \geq 0 \ \forall \alpha \in \Phi^+\}$. This algebra is isomorphic to $\mathbb{Q}_p[T(\mathbb{Q}_p)^+/T(\mathbb{Z}_p)]$.
(2) The local cohomologies are infinite dimensional. In fact, $\mathrm{R}\Gamma_w(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$ can be represented by complexes of compact projective limits of Banach spaces.

Two very important inputs in the case of the Flag variety, were the properties that:

(1) The local cohomologies are supported in a single degree. This was (among other facts) a consequence of the affiness of the Bruhat cells.
(2) The action of the torus is explicit on the local cohomology.

In the case of Shimura varieties, the action of $\mathcal{H}_p^+$ is going to play the role of the torus action T on the Flag variety. The starting point is the following property:

Proposition 1.13. *Let $T(\mathbb{Q}_p)^{++} = \{t \in T(\mathbb{Q}_p), v(\alpha(t)) > 0 \ \forall \alpha \in \Phi^+\}$. The ideal $\mathcal{H}_p^{++} = ([K_ptK_p], \ t \in T(\mathbb{Q}_p)^{++}) \subseteq \mathcal{H}_p^+$ acts via compact operators on $\mathrm{R}\Gamma_w(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$.*

This proposition is proved by computing the dynamic of the action of $T(\mathbb{Q}_p)^{++}$ on the Flag variety. It makes sense to consider the finite slope part of the cohomology (this is the factor where the operators $[K_ptK_p], \ t \in T(\mathbb{Q}_p)^+$ act invertibly):

$$\mathrm{R}\Gamma_w(K^p, \kappa)^{+,fs} := \mathrm{R}\Gamma_w(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)^{fs}$$

$$\mathrm{R}\Gamma_w(K^p, \kappa, cusp)^{+,fs} := \mathrm{R}\Gamma_w(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa(-D))^{fs}$$

We remark that on classical cohomology we have an action of the full Hecke algbera $\mathcal{H}_p$, and all the operators $[K_ptK_p]$ are invertible in $\mathcal{H}_p$. Therefore, all the classical cohomology is of finite slope. We obtain the following theorem:

Theorem 1.14. *Let $\kappa \in X^\star(T)^{M_\mu,+}$ be a weight. There is a $\mathcal{H}_p^+$-equivariant spectral sequence $\mathbf{E}^{p,q}(K^p, \kappa)^+$ converging to classical cohomology $\mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$, such that*

$$\mathbf{E}_1^{p,q}(K^p, \kappa)^+ = \oplus_{w \in {}^MW, \ell(w) = p} \mathrm{H}_w^{p+q}(K^p, \kappa)^{+,fs}.$$

There are also spectral sequences $\mathbf{E}^{p,q}(K^p, \kappa, cusp)^+$ converging to $\mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa(-D))$ such that

$$\mathbf{E}_1^{p,q}(K^p, \kappa, cusp)^+ = \oplus_{w \in {}^MW, \ell(w) = p} \mathrm{H}_w^{p+q}(K^p, \kappa, cusp)^{+,fs}.$$

Example 1.15. We can describe the spectral sequence for modular curves. Let us write $W = {}^MW = \{1, w\}$

We have that $\mathrm{R}\Gamma_1(K^p, \kappa)^{+,fs}$ is concentrated in degree 0, and $\mathrm{H}^0_1(K^p, \kappa)^{+,fs}$ is the finite slope part (for the action of U_p) of the space of weight κ overconvergent modular forms with tame level K^p. We have that $\mathrm{R}\Gamma_w(K^p, \kappa)^{+,fs}$ is concentrated in degree 1, and $\mathrm{H}^1_w(K^p, \kappa)^{+,fs}$ is the finite slope part of the compactly supported cohomology of the dagger space ordinary locus with tame level K^p. The spectral sequence boils down to the following complex, which computes classical cohomology:

$$0 \to \mathrm{H}^0_1(K^p, \kappa)^{+,fs} \to \mathrm{H}^1_w(K^p, \kappa)^{+,fs} \to 0$$

Here is roughly how the differential is constructed. We have two components of the ordinary locus in $\mathcal{S}^{tor}_{K,\Sigma}$. The multiplicative locus corresponding to $1 \in W$ and the étale locus corresponding to $w \in W$. Any overconvergent form $f \in \mathrm{H}^0_1(K^p, \kappa)^{+,fs}$ can be analytically continued (by the finite slope property) to a form over the complement of the ordinary étale locus, and defines a class in $\mathrm{H}^1_w(K^p, \kappa)^{+,fs}$ measuring precisely the obstruction to extend this form to the étale components.

1.2.7. *Vanishing theorems on the local cohomology.* In order to study the spectral sequence, we need to see that some terms vanish. The main vanishing result is:

Theorem 1.16. *The cohomology complex* $\mathrm{R}\Gamma_w(K^p, \kappa, cusp)^{+,fs}$ *is concentrated in degrees* $[0, \ell(w)]$*, while* $\mathrm{R}\Gamma_w(K^p, \kappa)^{+,fs}$ *is concentrated in degrees* $[\ell(w), d]$.

Remark 1.17. Recall that $\mathrm{H}^\star_{\mathbb{A}^m}(\mathbb{A}^n, \mathscr{O}_{\mathbb{A}^n})$ is concentrated in degree $n-m$. It is rather easy to see that this cohomology is supported in degree $[0, n-m]$ by using the triangle:

$$\mathrm{R}\Gamma_{\mathbb{A}^m}(\mathbb{A}^n, \mathscr{O}_{\mathbb{A}^n}) \to \mathrm{R}\Gamma(\mathbb{A}^n, \mathscr{O}_{\mathbb{A}^n}) \to \mathrm{R}\Gamma_{\mathbb{A}^m}(\mathbb{A}^n \setminus \mathbb{A}^m, \mathscr{O}_{\mathbb{A}^n}) \stackrel{+1}{\to}$$

and the fact that $\mathbb{A}^n \setminus \mathbb{A}^m$ is covered by $n-m$-affines. The fact that the cohomology is in degree $n-m$ is more subtle and uses that $\mathbb{A}^m$ is defined by a regular sequence.

Corollary 1.18. *For the spectral sequence* $\mathbf{E}^{p,q}(K^p, \kappa, cusp)^+$*, we have* $\mathbf{E}^{p,q}_1(K^p, \kappa, cusp)^+ = 0$ *is* $q > 0$*, while for the spectral sequence* $\mathbf{E}^{p,q}(K^p, \kappa)^+$*, we have* $\mathbf{E}^{p,q}_1(K^p, \kappa)^+ = 0$ *for* $q < 0$.

The proof of theorem 1.16 in the cuspidal case uses the vanishing of the higher direct images of cuspidal sheaves from toroidal to minimal compactifications, and also the affineness of the Hodge-Tate period map, which follows (non-trivially!) from the affineness of the ordinary locus, while the non cuspidal case is deduced from this using duality (see theorem 1.19 below.)

1.2.8. *Duality.* In order to study duality, we develop a dual theory of local cohomology which is adapted to the algebra $\mathcal{H}^-_p \subseteq \mathcal{H}_p$ generated by the operators $[K_p t K_p]$ with $t \in T(\mathbb{Q}_p)^-$.

We are therefore able to define cohomology groups $\mathrm{R}\Gamma_w(K^p, \kappa)^{-,fs}$, $\mathrm{R}\Gamma_w(K^p, \kappa, cusp)^{-,fs}$ together with spectral sequences $\mathbf{E}^{p,q}(K^p, \kappa)^-$, $\mathbf{E}^{p,q}(K^p, \kappa, cusp)^-$, all carrying actions of $\mathcal{H}^-_p$.

Theorem 1.19. *We have pairings*

$$\langle, \rangle : \mathrm{H}^i_w(K^p, -w_{0,M}\kappa - 2\rho_{nc})^{\pm,fs} \times \mathrm{H}^{d-i}_w(K^p, \kappa, cusp)^{\mp,fs} \to \mathbb{Q}_p$$

compatible with the $\mathcal{H}^\pm_p$*-actions in the sense that we have* $\langle t\cdot, \cdot\rangle = \langle \cdot, t^{-1}\cdot\rangle$*. These pairings are non degenerate (equivalently they restrict to perfect pairings on the finite dimensional slope* $< C$ *part for any* $C \in \mathbb{Q}$*.) They are moreover compatible with the classical Serre duality pairings via the spectral sequences.*

1.2.9. *Cousin complex.* We first define interior cohomology:

$$\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_\kappa) = \mathrm{Im}(\mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_\kappa(-D)) \to \mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)).$$

We can also define interior overconvergent cohomologies:

$$\overline{\mathrm{H}}^{\ell(w)}_w(K^p, \kappa)^{+,fs} = \mathrm{Im}(\mathrm{H}^{\ell(w)}_w(K^p, \kappa, cusp)^{+,fs} \to \mathrm{H}^{\ell(w)}_w(K^p, \kappa)^{+,fs}).$$

Note that by theorem 1.16 this would vanish outside of degree $\ell(w)$.

Let us now define Cousin complex $\mathcal{C}ous(K^p,\kappa)^+ = \mathbf{E}_1^{\bullet,0}(K^p,\kappa,cusp)^{\pm}$ and similarly for cuspidal cohomology. We can define the interior Cousin complex $\overline{\mathcal{C}ous}(K^p,\kappa)^+$ as the image of the map from the cuspidal to the ordinary Cousin complex. In other words, the interior Cousin complex has the shape:

$$0 \to \overline{\mathrm{H}}^0_1(K^p,\kappa)^{+,fs} \to \oplus_{w\in {}^MW, \ell(w)=1}\overline{\mathrm{H}}^1_w(K^p,\kappa)^{+,fs} \to \cdots \to \overline{\mathrm{H}}^d_{w_0^M}(K^p,\kappa)^{+,fs} \to 0$$

Proposition 1.20. *$\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma},\mathcal{V}_\kappa)$ is a sub-quotient of $\mathrm{H}^i(\overline{\mathcal{C}ous}(K^p,\kappa)^{\pm})$.*

Remark 1.21. There is an analogous story for all abelian type Shimura varieties. When S_K is compact, the Cousin complex is exactly the first page of the spectral sequence, and computes exactly the classical cohomology.

1.2.10. *Slope estimates.* We have found an efficient way to compute the cohomology by using the local cohomologies. The next step is to obtain information on the $\mathcal{H}_p^{\pm}$-action on $\mathrm{R}\Gamma_w(K^p,\kappa)^{\pm,fs}$. In the case of the Flag varieties, it is possible to completely describe the action of the torus. In the case of Shimura varieties, it is not reasonable to hope for a complete description but we can find slope bounds. There is a valuation map $v : T(\mathbb{Q}_p) \to X_\star(T)$ extending the p-adic valuation $v : \mathbb{Q}_p^\times \to \mathbb{Z}$. For any character $\lambda : T(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_p^\times$, we can define its slope character $v(\lambda)$: applying the valuation on $\overline{\mathbb{Q}}_p^\times$ defines a map $v(\lambda) : X_\star(T) \to \mathbb{Q}$, or equivalently an element $v(\lambda) \in X^\star(T)_\mathbb{Q}$. Recall also that we have a partial order on $X^\star(T)_\mathbb{Q}$. We make the following conjecture:

Conjecture 1.22. *Fix $w \in {}^MW$, $\kappa \in X^\star(T)^{M_\mu,+}$. For any character λ of $T^+(\mathbb{Q}_p)$ on $\mathrm{R}\Gamma_w(K^p,\kappa)^{+,fs}$ or $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{+,fs}$ we have $v(\lambda) \geq w^{-1}w_{0,M}(\kappa+\rho)+\rho$.*

In full generality, we can prove:

Theorem 1.23. *Fix $w \in {}^MW$, $\kappa \in X^\star(T)^{M_\mu,+}$. For any character λ of $T^+(\mathbb{Q}_p)$ on $\mathrm{R}\Gamma_w(K^p,\kappa)^{+,fs}$ or $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{+,fs}$ we have $v(\lambda) \geq w^{-1}w_{0,M}(\kappa)$.*

Example 1.24. Let us spell out the meaning of this in the modular curve case. The spectral sequence is just the following complex, which computes classical cohomology:

$$0 \to \mathrm{H}^0_1(K^p,\kappa)^{+,fs} \to \mathrm{H}^1_w(K^p,\kappa)^{+,fs} \to 0$$

The conjecture 3.4 actually holds in this case, and it says that:

(1) If $k \geq 1$, the slopes of U_p are ≥ 0 on $\mathrm{H}^0_1(K^p,(k+2;-k))^{+,fs}$ and $\geq k-1$ on $\mathrm{H}^1_w(K^p,(k+2;-k)^{+,fs}$.
(2) If $k \leq 1$, The slopes of U_p are $\geq 1-k$ on $\mathrm{H}^0_1(K^p,(k;k))^{+,fs}$ and ≥ 0 on $\mathrm{H}^1_w(K^p,(k;k))^{+,fs}$.

In this example, we have normalized the central character appropriately so that U_p has non-negative slope and that ordinary classes have slope 0.

1.3. **Application 1: Vanishing theorems.** The first application is to the vanishing of small slope cohomologies.

1.3.1. *The cohomological range.* For any $\kappa \in X^\star(T)^{M_\mu,+}$, there is a range

$$[\ell_{\min}(\kappa),\ell_{\max}(\kappa)] \subseteq [0,d]$$

where $d = \dim S_K$ where we expect to see interesting (for example tempered) cohomology classes in the cohomology of the Shimura variety in that given weight. Here is a combinatorial description of this range. Let $C(\kappa)^+ = \{w \in W, w^{-1}w_{0,M}(\kappa + \rho) \in X^\star(T)^-_{\mathbb{Q}}\}$. Put $\ell_{\min}(\kappa) = \inf_{w\in C(\kappa)^+}\ell(w)$, $\ell_{\max}(\kappa) = \sup_{w\in C(\kappa)^+}\ell(w)$. It is worth remarking that if $\kappa+\rho$ is G-regular, $C(\kappa)^+$ has only one element and $\ell_{\min}(\kappa) = \ell_{\max}(\kappa)$.

1.3.2. *Small slope.* We fix once and for all an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. For any cohomology carrying an action of $\mathcal{H}_p^+$, we introduce a condition $+, ss$ which means small slope. This is a condition on the p-adic valuation of the eigensystem. This condition depends on the the cohomology and its weight (and there are even different small slope conditions for the same cohomology, depending on the situation...). We shall not try to spell out this condition and refer to the paper [BP21] instead. We will nevertheless try to give the condition in explicit examples. Let us also add that any ordinary class will always staisfy the small slope. We have another condition $+, sss$ which means strongly small slope and is slightly stronger than small slope. Sometimes we need to use this condition, but we always conjecture that $+, ss$ should be sufficient.

1.3.3. *Vanishing on coherent cohomology.*

Theorem 1.25. *The small slope interior cohomology* $\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)^{+,ss}$ *is concentrated in the range* $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$.

Remark 1.26. We also prove that the strongly small slope cuspidal cohomology is concentrated in degree $[0, \ell_{\max}(\kappa)]$ and the strongly small slope usual cohomology is in degree $[\ell_{\min}(\kappa), d]$.

Remark 1.27. Classical Archimedean result (Blasius-Harris-Ramakrishnan, Mirkovich, Schmid and Williams, see [Har90] theorems 3.4 and 3.5) asserts that the tempered at infinity interior cohomology is concentrated in the range $[\ell_{\min}(\kappa), \ell_{\max}(\kappa)]$. Small slope is some kind of "p-adic temperedness".

1.3.4. *Vanishing on Betti Cohomology.* For any $\nu \in X^\star(T)^+$, we let $\overline{\mathrm{H}}^\star(S_K(\mathbb{C}), \mathcal{W}^\vee_\nu) = \mathrm{Im}\big(\mathrm{H}^\star_c(S_K(\mathbb{C}), \mathcal{W}^\vee_\nu) \to \mathrm{H}^\star(S_K(\mathbb{C}), \mathcal{W}^\vee_\nu)\big)$. This is the interior Betti cohomology.

Theorem 1.28. *For any* $\nu \in X^\star(T)^+$, *the small slope interior Betti cohomology* $\overline{\mathrm{H}}^\star(S_K(\mathbb{C}), \mathcal{W}^\vee_\nu)^{+,ss}$ *is concentrated in the middle degree* $d = \dim S_K(\mathbb{C})$.

Let $\nu \in X^\star(T)^+$. For all $w \in {}^MW$, we let $\kappa_w = -w_{0,M}w(\nu+\rho) - \rho$. Using the de Rham comparaison theorem and Falting's dual BGG we deduce that for all $\nu \in X^\star(T)^+$, we have:

$$\overline{\mathrm{H}}^d(S_K(\mathbb{C}), \mathcal{W}^\vee_\nu)^{+,ss} \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{w\in {}^MW}\overline{\mathrm{H}}^{\ell(w)}(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa_w})^{+,ss} \otimes_{\mathbb{Q}} \mathbb{C}$$

Remark 1.29. We also prove that the strongly small slope compactly supported cohomology is concentrated in degree $[0, d]$ and the strongly small slope usual cohomology is in degree $[d, 2d]$.

Remark 1.30. In [CS17] Caraiani and Scholze proved a similar concentration result for Betti cohomology of unitary Shimura varieties under a genericity condition (and it works with torsion coefficients).

Remark 1.31. Temperdness at infinity, genericity at p and small slope at p are related. Take S_K = compact Shimura curve associated to a $D/\mathbb{Q}$ split at ∞ and p. Let $1 \in \mathrm{H}^0(S_K, \mathscr{O}_{S_K})$. We observe that 1 is:

(1) Non tempered at ∞, since $\mathrm{Vol}(\mathrm{PGL}_2(\mathbb{R})) = \infty$,
(2) Not small slope at p, because $U_p.1 = p$.
(3) Not generic, because $LL(1) = \mathrm{diag}(p^{\frac{1}{2}}, p^{-\frac{1}{2}})$.

1.4. **Application 2: classicality theorems.** The knowledge of slopes on cohomology can be used to prove classicality theorems (special cases have been obtained in [Col96], [Kas06], [Pil11], [BPS16]).

Theorem 1.32. *Let $\kappa \in X^\star(T)^{M_\mu,+}$ be a weight such that $\kappa+\rho$ is G-regular. Then we have a quasi-isomorphism:*

$$\mathrm{R}\Gamma_w(K^p,\kappa)^{+,sss} = \mathrm{R}\Gamma(S^{tor}_{K,\Sigma},\mathcal{V}_\kappa)^{+,sss}$$

where $\{w\} = C(\kappa)^+$, and similarly for cuspidal cohomology. If $\kappa \in X^\star(T)^+$, then we have $\mathrm{H}^0_{Id}(K^p,\kappa,cusp)^{+,ss} = \mathrm{H}^0(S^{tor}_{K,\Sigma},\mathcal{V}_\kappa(-D))^{+,ss}$.

Example 1.33. For the group $\mathrm{GL}_2/\mathbb{Q}$, the theorem is actually proved with the $+,ss$ condition and we have that:

$$\begin{aligned}\mathrm{H}^0_1(K^p,(k+2;-k))^{+,ss} &= \mathrm{H}^0(S^{tor}_{K,\Sigma},\mathcal{V}_{(k+2;-k)}(-D))^{+,ss},\ k\geq 0,\\ \mathrm{H}^1_w(K^p,(k;k))^{+,ss} &= \mathrm{H}^0(S^{tor}_{K,\Sigma},\mathcal{V}_{(k;k)}(-D))^{+,ss},\ k\leq 0.\end{aligned}$$

In the first equality, $+,ss$ is the condition that the slope of U_p is $< k-1$; in the second equality, $+,ss$ is the condition that the slope of U_p is $> 1-k$.

Example 1.34. For any $g \geq 1$, we have the operator $U_p = [K_ptK_p] \in \mathcal{H}_p^+$ where $t = \mathrm{diag}(p^{-1},\cdots,p^{-1},1,\cdots,1)$ (the $^{-1}$ sup-script is because we are using adelic operators which act on the left...). For a weight $\kappa = (k_1,\cdots,k_g;-\sum k_i - g - 1)$ with $k_1 \geq \cdots \geq k_g \geq g+1$ (so κ is in the holomorphic chamber) the theorem says that:

$$\mathrm{H}^0_1(K^p,\kappa,cusp)^{+,ss} = \mathrm{H}^0(S^{tor}_{K,\Sigma},\mathcal{V}_\kappa(-D))^{+,ss}$$

where the $+,ss$ condition is the condition that the U_p has slope is $< k_g - g$ (by our normalization of the central character, the U_p-slope is ≥ 0).

1.5. **Application 3: decomposition of de Rham cohomology.** One can also combine local cohomology and the BGG complex to analyze de Rham cohomology.

1.5.1. *Modular curves.* For $g=1$, and any $k\geq 0$, let $\nu = (-k;k) \in X^\star(T)^+$ be the highest weight of $\mathrm{Sym}^k\mathrm{Std}$. The BGG theory gives a (filtered) quasi-isomorphism between the de Rham complex and the dual BGG complex:

$$DR(\mathcal{W}^\vee_{\nu,dR}) = [\mathcal{V}_{(-k;-k)} \xrightarrow{\Theta^k} \mathcal{V}_{(k+2;-k)}]$$

We find that $\mathcal{W}^\vee_{\nu,dR} = \mathrm{Sym}^k\mathcal{H}^1_{dR}(A)$, and that $\mathcal{V}_{(-k;-k)} \simeq \omega^{-k}$, $\mathcal{V}_{(k+2;-k)} \simeq \omega^{k+2}$. These last two isomorphisms are up to some Hecke twist.

It is interesting to combine the BGG complex and local cohomology. We find the following double complex:

$$\begin{array}{ccccc}\mathcal{C}ous(K^p,(k+2;-k))^+ : & & \mathrm{H}^0_1(K^p,(k+2;-k))^{+,fs} & \longrightarrow & \mathrm{H}^1_w(K^p,(k+2;-k))^{+,fs}\\ \uparrow{\scriptstyle\Theta^k} & & \uparrow{\scriptstyle\Theta^k} & & \uparrow{\scriptstyle\Theta^k}\\ \mathcal{C}ous(K^p,(-k;-k))^+ : & & \mathrm{H}^0_1(K^p,(-k;-k))^{+,fs} & \longrightarrow & \mathrm{H}^1_w(K^p,(-k;-k))^{+,fs}\end{array}$$

whose total complex computes $\mathrm{R}\Gamma_{dR}(S^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu)$.

The slopes of U_p are $\geq k$ on $\mathrm{H}^0_1(K^p,(-k;-k))^{+,fs}$ and $\mathrm{H}^1_w(K^p,(k+2;-k))^{+,fs}$, we find that the small slope part $\mathrm{R}\Gamma_{dR}(\mathcal{S}^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu)^{+,ss}$ is concentrated in degree 1 and we have the decomposition:

$$\mathrm{H}^1_{dR}(\mathcal{S}^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu)^{+,ss} = \mathrm{H}^0_1(K^p,(k+2;-k))^{+,ss} \oplus \mathrm{H}^1_w(K^p,(-k;-k))^{+,ss}.$$

which is reminiscent of the Hodge decomposition over $\mathbb{C}$, where harmonic forms are used to split the Hodge filtration.

We also observe that the above diagram can be used to give a short proof of the following classical result:

Theorem 1.35 ([Col96])**.** *Let λ be an eigensystem for the Hecke algebra, corresponding to an eigenclass $c \in \mathrm{H}^0_1(K^p,(k+2;-k))^{+,fs}$ which is not in $\Theta^k(\mathrm{H}^0_1(K^p,(-k;-k))^{+,fs})$. Then λ is an eigensystem for an eigenclass in $\mathrm{H}^0(S^{tor}_K,\mathcal{V}_{(k+2;-k)})$.*

Proof. First consider the image c' of c in $\mathrm{H}^1_w(K^p,(k+2;-k))^{+,fs}$. If $c'=0$, then c comes from a class in $\mathrm{H}^0(S^{tor}_K,\mathcal{V}_{(k+2;-k)})$. Assume that $c'\neq 0$. We claim that the map $\Theta^k : \mathrm{H}^1_w(K^p,(-k;-k))^{+,fs} \to \mathrm{H}^1_w(K^p,(k+2;-k))^{+,fs}$ is surjective. This map is dual to $\Theta^k : \mathrm{H}^0_1(K^p,(-k;k),cusp)^{+,fs} \to \mathrm{H}^0_1(K^p,(k+2;k),cusp)^{+,fs}$, which is injective (on q-expansion it is given by $\sum a_nq^n \mapsto \sum n^k a_n q^n$). Therefore, if c' is the image via Θ^k of an eigenclass $c'' \in \mathrm{H}^1_w(K^p,(-k;-k))^{+,fs}$ with the same eigensystem as c and c'. Either c'' has non-trivial image in $\mathrm{H}^1(S^{tor}_K,\mathcal{V}_{(k+2;-k)})$ and we conclude: the cohomology $\mathrm{H}^1(S^{tor}_K,\mathcal{V}_{(k+2;-k)})$ is entirely described by cuspidal automorphic forms π on $\mathrm{GL}_2/\mathbb{Q}$ which are discrete series at ∞, and therefore π_∞ contains both an holomorphic (lowest weight) and antiholomorphic (highest weight) vector, so π contributes to $\mathrm{H}^0(S^{tor}_K,\mathcal{V}_{(k+2;-k)})$ as well. Otherwise c'' comes from an eigenclass $c''' \in \mathrm{H}^0_1(K^p,(-k;-k))^{+,fs}$ with the same eigensystem as c. Then we consider $c-\Theta^k(c''')$. Since c is not in the image of Θ^k by assumption, this class is non-trivial. It has trivial image in $\mathrm{H}^1_w(K^p,(k+2;-k))^{+,fs}$ by construction. It follows that $c-\Theta^k(c''')$ comes from an eigenclass in $\mathrm{H}^0(S^{tor}_K,\mathcal{V}_{(k+2;-k)})$. □

1.5.2. *The group* GSp_4. Let $\nu_1 \geq \nu_2 \geq 0$. Take $\nu = (-\nu_2,-\nu_1,\nu_1+\nu_2)$. We consider the vector bundle with connection $\mathcal{W}^\vee_{\nu,dR}$. For example, when $\nu_1=1$ and $\nu_2=0$, this is $\mathcal{H}^1_{dR}(A)$.

The Weyl group W is generated by the reflections s_1 and s_2 given by their action of $X^\star(T)$: $s_1(k_1,k_2;k)=(k_2,k_1;k)$ and $s_2(k_1,k_2;k)=(-k_1,k_2;k)$. The elements of ${}^MW$ are Id, s_2, s_2s_1, $s_2s_1s_2$. Let us denote by $w_i \in {}^MW$ the element of length i. We define $\kappa_i = \kappa_{w_i} = -w_{0,M}w_i(\nu+\rho)-\rho$. Noting that $\rho=(-1,-2;0)$, one finds quickly:

$$\begin{aligned}
\kappa_3 &= (-\nu_2,-\nu_1;-\nu_2-\nu_1)\\
\kappa_2 &= (\nu_2+2,-\nu_1;-\nu_2-\nu_1)\\
\kappa_1 &= (\nu_1+3,1-\nu_2;-\nu_2-\nu_1)\\
\kappa_0 &= (\nu_1+3,\nu_2+3;-\nu_2-\nu_1).
\end{aligned}$$

The BGG resolution is a quasi-isomorphism:

$$DR(\mathcal{W}^\vee_{\nu,dR}) = [\mathcal{V}_{\kappa_3} \to \mathcal{V}_{\kappa_2} \to \mathcal{V}_{\kappa_1} \to \mathcal{V}_{\kappa_0}]$$

Let us define

$$\begin{aligned}
\mathrm{R}\Gamma_{dR}(S^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu) &:= \mathrm{R}\Gamma(S^{tor}_{K,\Sigma},\mathrm{DR}(\mathcal{W}^\vee_\nu))\\
\mathrm{R}\Gamma_{dR,c}(S^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu) &:= \mathrm{R}\Gamma(S^{tor}_{K,\Sigma},\mathrm{DR}(\mathcal{W}^\vee_\nu(-D)))
\end{aligned}$$

Let

$$\overline{\mathrm{H}}^i_{dR}(S^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu)=\mathrm{Im}(\mathrm{H}^i_{dR,c}(S^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu)\to \mathrm{H}^i_{dR}(S^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu)).$$

Consider the following double complex:

$$\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu)^{+,fs}:=$$

$$\overline{\mathcal{C}ous}(K^p,\kappa_3)^{+,fs}\to\overline{\mathcal{C}ous}(K^p,\kappa_2)^{+,fs}\to\overline{\mathcal{C}ous}(K^p,\kappa_1)^{+,fs}\to\overline{\mathcal{C}ous}(K^p,\kappa_0)^{+,fs}$$

$$\begin{array}{ccccccc}
\overline{\mathrm{H}}^0_1(K^p,\kappa_0)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^1_{w_1}(K^p,\kappa_0)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^2_{w_2}(K^p,\kappa_0)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^3_{w_3}(K^p,\kappa_0)^{+,fs}\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\overline{\mathrm{H}}^0_1(K^p,\kappa_1)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^1_{w_1}(K^p,\kappa_1)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^2_{w_2}(K^p,\kappa_1)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^3_{w_3}(K^p,\kappa_1)^{+,fs}\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\overline{\mathrm{H}}^0_1(K^p,\kappa_2)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^1_{w_1}(K^p,\kappa_2)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^2_{w_2}(K^p,\kappa_2)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^3_{w_3}(K^p,\kappa_2)^{+,fs}\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
\overline{\mathrm{H}}^0_1(K^p,\kappa_3)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^1_{w_1}(K^p,\kappa_3)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^2_{w_2}(K^p,\kappa_3)^{+,fs} & \longrightarrow & \overline{\mathrm{H}}^3_{w_3}(K^p,\kappa_3)^{+,fs}
\end{array}$$

Proposition 1.36. *The cohomology groups* $\overline{\mathrm{H}}^i_{dR}(S^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu)$ *are subquotients of* $\mathrm{H}^i(\mathrm{Tot}(\overline{\mathcal{C}ous}(K^p,\mathcal{W}^\vee_\nu)^{+,fs}))$. *We have that* $(\overline{\mathrm{H}}^i_{dR}(S^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu))^{+,ss}=0$ *is* $i\neq 3$ *and we have the decomposition:*

$$\overline{\mathrm{H}}^3_{dR}(\mathcal{S}^{tor}_{K,\Sigma},\mathcal{W}^\vee_\nu)^{+,sss}=$$

$$\overline{\mathrm{H}}^0_1(K^p,\kappa_0)^{+,sss}\oplus\overline{\mathrm{H}}^1_{w_1}(K^p,\kappa_1)^{+,sss}\oplus\overline{\mathrm{H}}^2_{w_2}(K^p,\kappa_2)^{+,sss}\oplus\overline{\mathrm{H}}^3_{w_3}(K^p,\kappa_3)^{+,sss}$$

1.5.3. *The general case.* The decomposition of strongly small slope de Rham cohomology we observed in the case of modular curves and Siegel threefolds is general: the spectral sequence coming from the Bruhat stratification and the Hodge-to-de Rham spectral sequence are opposite to each other, giving the splitting.

1.6. **Eigenvarieties.** We now turn to eigenvarieties and present a construction generalizing [CM98] and [AIP15]

For all $\nu\in X^\star(T)^+$, we have that

$$\overline{\mathrm{H}}^d(S_K(\mathbb{C}),\mathcal{W}^\vee_\nu)^{+,ss}\otimes\mathbb{C}=\oplus_{w\in{}^MW}\overline{\mathrm{H}}^{\ell(w)}(S^{tor}_{K,\Sigma},\mathcal{V}_{\kappa_w})^{+,ss}\otimes\mathbb{C}.$$

We also have that $\overline{\mathrm{H}}^{\ell(w)}(S^{tor}_{K,\Sigma},\mathcal{V}_{\kappa_w})^{+,sss}=\overline{\mathrm{H}}^{\ell(w)}_w(K^p,\kappa_w)^{+,sss}$. We claim that it is possible to interpolate all the local cohomologies $\mathrm{R}\Gamma_w(K^p,\kappa_w)^{+,fs}$. The theory is probably the most interesting for the interior cohomology because it gives equidimensional eigenvarieties. Let $\mathcal{W}=\mathrm{Spa}(\mathbb{Z}_p[[T(\mathbb{Z}_p)]])\times_{\mathrm{Spa}\,\mathbb{Z}_p}\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ be the weight space. Let $\mathcal{H}^S$ be the spherical Hecke algebra away from p and all primes ℓ such that K_ℓ is not hyperspecial.

Theorem 1.37. *There is an eigenvariety* $\pi:\mathcal{E}^!\to\mathcal{W}$ *where the map* π *is quasi-finite.*

(1) $\mathcal{E}^!$ *is equidimensional of dimension* $g+1$

(2) *Any point* $x \in \mathcal{E}^!$ *corresponds to a pair* (λ_p, λ^S) *where* $\lambda_p : T(\mathbb{Q}_p) \to k(x)^\times$, *and* $\lambda^S : \mathcal{H}^S \to k(x)$,
(3) *The eigenvariety* $\mathcal{E}^!$ *carries coherent sheaves* $\mathcal{H}_w$ *for* $w \in {}^M W$, *which are torsion free as* $\pi^{-1}\mathscr{O}_{\mathcal{W}}$*-modules.*
(4) *For any* $\nu \in X^\star(T)^+$, *we have:*

$$\oplus_w \mathcal{H}_w|_\nu^{+,sss} = \oplus_w \overline{\mathrm{H}}^{\ell(w)}(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa_w})^{+,sss}$$

(5) *For any* $\kappa \in X^\star(T)^{+,M_\mu}$, *any eigenclass* $c \in \overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$ *gives a point* x_c *on the eigenvariety, and* $\pi(x_c) + \rho = -ww_{0,M}(\kappa + \rho)$ *for some* $w \in {}^M W$, $\ell(w) = i$.

Remark 1.38. The last point of the theorem is an advantage of our method to construct eigenvarieties. The property that interior cohomology eigenclasses deform analytically was only known under some stronger hypothesis (classes appearing in the cohomology of Shimura sets, non-critical classes, classes appearing in cuspidal H^0). See for example [Urb11].

Remark 1.39. We can also prove that Serre duality interpolates over the eigenvariety.

Let us conclude with an application to irregular Galois representations attached to automorphic forms on the group GSp_4.

Theorem 1.40. *Let* π *be a cuspidal automorphic form on the group* $\mathrm{GSp}_4/\mathbb{Q}$. *We assume that* π_∞ *is a limit of discrete series with infinitesimal character* $(\lambda, \lambda; -2\lambda)$ *for* $\lambda \in \mathbb{Z}_{>0}$. *Let* p *be a prime and let* $\iota : \mathbb{C} \hookrightarrow \overline{\mathbb{Q}}_p$ *be an embedding. There is a continuous Galois representation* $\rho_\pi : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ *such that :*

(1) ρ_π *is unramified at all primes* $\ell \neq p$ *such that* π_ℓ *is spherical and* $\iota^{-1}\rho_\pi(\mathrm{Frob}_\ell)^{ss}$ *corresponds to* $\pi_\ell \otimes |\nu|^{\frac{3}{2}}$ *via the unramified local Langlands correspondence.*
(2) ρ_π *has generalized Hodge-Tate weights* $(-2\lambda, -\lambda, -\lambda, 0)$. *If* π_p *his an unramified regular principal series then* π_p *is crystalline at* p *and* $\iota^{-1}WD(\rho_\pi|_{G_{\mathbb{Q}_p}})^{F-ss}$ *corresponds to* $\pi_p \otimes |\nu|^{\frac{3}{2}}$ *via the unramified local Langlands correspondence.*

Proof. One can realize π in the coherent cohomology $\overline{\mathrm{H}}^1(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$ and $\overline{\mathrm{H}}^2(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)$ where $\kappa = (\lambda+1, 2-\lambda; 2\lambda)$ and K is such that $\pi^K \neq 0$. If π_p has fixed vector under the Iwahori, π defines a point on the eigenvariety, and we can construct its Galois representation via analytic interpolation from the regular case (see [Mok14], sect. 3.2, for the existence of Galois representations in the regular case). We can then use the results of Kisin [Kis03] on the interpolation of crystalline periods. If π_p has no fixed vectors under the Iwahori, we need to use a base change to a totally real field. □

Remark 1.41. The novelty in this theorem is point 2, which is a consequence of the fact that we can give a construction via analytic interpolation using the eigenvariety, rather than congruences as in [PS16], [Box15] or [GK19].

2. Lecture 2: Overconvergent cohomologies and the spectral sequence

In this lecture we give more details on how the overconvergent cohomologies $\mathrm{R}\Gamma_w(K^p, \kappa)^{+,fs}$ are constructed.

2.1. **Cohomology with support.**

2.1.1. *Definition and basic properties.* Let X be a topological space, $U \subseteq X$ open, and $Z \subseteq X$ closed. Given an abelian sheaf $\mathscr{F}$ on X, the sections of X on U with support in $U \cap Z$ is defined to be

$$\Gamma_{U\cap Z}(U, \mathscr{F}) = \{s \in \Gamma(U, \mathscr{F}) \mid \mathrm{supp}(s) \subseteq U \cap Z\}.$$

This is a left exact functor of $\mathscr{F}$, and we may form its right derived functor $\mathrm{R}\Gamma_{U\cap Z}(U, \mathscr{F})$. We recall some of its basic properties:

(1) If $V = U \setminus Z$ there is an exact triangle

$$\mathrm{R}\Gamma_{U\cap Z}(U, \mathscr{F}) \to \mathrm{R}\Gamma(U, \mathscr{F}) \to \mathrm{R}\Gamma(V, \mathscr{F}) \stackrel{+1}{\to} .$$

(2) We have the following functorialities in U and Z: if $U \subset U'$ are open then there is a restriction map

$$\mathrm{res} : \mathrm{R}\Gamma_{U'\cap Z}(U', \mathscr{F}) \to \mathrm{R}\Gamma_{U\cap Z}(U, \mathscr{F})$$

and if $Z \subset Z'$ are closed, there is a "corestriction" map

$$\mathrm{cores} : \mathrm{R}\Gamma_{U\cap Z}(U, \mathscr{F}) \to \mathrm{R}\Gamma_{U\cap Z'}(U, \mathscr{F}).$$

(3) If Z_1, Z_2 are closed with $Z_1 \cap U$ and $Z_2 \cap U$ disjoint, then

$$\mathrm{R}\Gamma_{U\cap(Z_1\cup Z_2)}(U, \mathcal{F}) \simeq \mathrm{R}\Gamma_{U\cap Z_1}(U, \mathscr{F}) \oplus \mathrm{R}\Gamma_{U\cap Z_2}(U, \mathscr{F}).$$

(4) If U, U' and Z, Z' are closed with $U \cap Z = U' \cap Z'$, then we have the following "excision" isomorphism

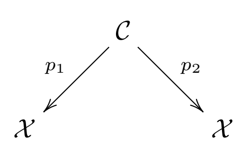
$$\mathrm{R}\Gamma_{U\cap Z}(U, \mathscr{F}) \simeq \mathrm{R}\Gamma_{U'\cap Z'}(U', \mathscr{F}).$$

In particular the local cohomology really only depends on the locally closed $U \cap Z$.

(5) If $X = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_r = \emptyset$ is a filtration by closed subsets then there is a spectral sequence

$$E_1^{p,q} = \mathrm{H}^{p+q}_{Z_p\setminus Z_{p+1}}(X \setminus Z_{p+1}, \mathscr{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F}).$$

2.1.2. *Action of a correspondence on cohomology with support.* Suppose we have a correspondence



where $\mathcal{X}$ and $\mathcal{C}$ are finite type adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and p_1 and p_2 are finite flat morphisms, so that in particular they are open and closed maps on underlying topological spaces (in our application, $\mathcal{X}$ and $\mathcal{C}$ will be toroidal compactifications of Siegel modular varieties, and so the maps p_i will not be finite flat at the boundary, but we suppress this for simplicity).

We call this correspondence T. For $x \in \mathcal{X}$ we have $T(x) = p_2(p_1^{-1}(x))$, and we also define the transpose correspondence T^t by exchanging the role of p_1 and p_2: $T^t(x) = p_1(p_2^{-1}(x))$. By our hypotheses T and T^t also preserve open and closed subsets of $\mathcal{X}$.

If $\mathscr{F}/\mathcal{X}$ is a locally free sheaf of finite rank, then we would like to define an action of T on the cohomology $\mathrm{R}\Gamma(\mathcal{X}, \mathscr{F})$, by pulling back by p_2 and taking a trace under p_1. In order to do this we also need a map of sheaves on $\mathcal{C}$, $T : p_2^\star \mathscr{F} \to p_1^\star \mathscr{F}$. Then the action of T on cohomology is given by the composition

$$\mathrm{R}\Gamma(X, \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{C}, p_2^\star \mathscr{F}) \to \mathrm{R}\Gamma(\mathcal{C}, p_1^\star \mathscr{F}) \to \mathrm{R}\Gamma(X, \mathscr{F})$$

where from left to right the maps are $p_2^\star$, T, and trace under p_1. We also call this composition T.

Now we turn to cohomology with support. In general we cannot expect that for any open U and closed Z, we have an action of T on $\mathrm{R}\Gamma_{U \cap Z}(U, \mathscr{F})$. We need to assume some compatibility between the correspondence T and the support conditions. The basic observation is that we do always have a map

$$T : \mathrm{R}\Gamma_{T(U) \cap Z}(T(U), \mathscr{F}) \to \mathrm{R}\Gamma_{U \cap T^t(Z)}(U, \mathscr{F}).$$

Before constructing this map in general, it may be helpful to consider two special cases:

- When $Z = \mathcal{X}$, there is a map
$$T : \mathrm{R}\Gamma(T(U), \mathscr{F}) \to \mathrm{R}\Gamma(U, \mathscr{F}).$$
- When $U = \mathcal{X}$, there is a map
$$T : \mathrm{R}\Gamma_Z(\mathcal{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{T^t(Z)}(\mathcal{X}, \mathscr{F}).$$

We can understand why these maps should exist by thinking about applying T to a local section s of $\mathcal{F}$. If we want Ts to be defined on the open U, then we need s to be defined on the open $T(U)$, and similarly if s is supported in the closed Z, then Ts will be supported in $T^t(Z) = \{x \in \mathcal{X} \mid T(x) \cap Z \neq \emptyset\}$.

In general the map T is constructed as the composition of five maps:

$$\mathrm{R}\Gamma_{T(U) \cap Z}(T(U), \mathscr{F}) \to \mathrm{R}\Gamma_{p_2^{-1} p_2 p_1^{-1}(U) \cap p_2^{-1}(Z)}(p_2^{-1} p_2 p_1^{-1}(U), p_2^\star \mathscr{F}) \to$$

$$\mathrm{R}\Gamma_{p_1^{-1}(U) \cap p_2^{-1}(Z)}(p_1^{-1}(U), p_2^\star \mathscr{F}) \to \mathrm{R}\Gamma_{p_1^{-1}(U) \cap p_2^{-1}(Z)}(p_1^{-1}(U), p_2^\star \mathscr{F}) \to \mathrm{R}\Gamma_{p_1^{-1}(U) \cap p_2^{-1}(Z)}(p_1^{-1}(U), p_1^\star \mathscr{F}) \to$$

$$\mathrm{R}\Gamma_{p_1^{-1}(U) \cap p_1^{-1} p_1 p_2^{-1}(Z)}(p_1^{-1}(U), p_1^\star \mathscr{F}) \to \mathrm{R}\Gamma_{U \cap T^t(Z)}(U, \mathscr{F})$$

Here in order the maps are:

- Pullback by p_2.
- Restriction for the inclusion $p_1^{-1}(U) \subseteq p_2^{-1} p_2 p_1^{-1}(U) = p_2^{-1} T(U)$.
- The map of sheaves $T : p_2^\star \mathscr{F} \to p_1^\star \mathscr{F}$.
- Corestriction for the inclusion $p_2^{-1}(Z) \subseteq p_1^{-1} p_1 p_2^{-1}(Z) = p_1^{-1} T^t(Z)$.
- Trace under p_1.

Now suppose that $T(U) \subseteq U$ and $T^t(Z) \subseteq Z$. Then we may build a diagram

$$\begin{array}{ccc} & \mathrm{R}\Gamma_{U \cap Z}(U, \mathscr{F}) & \\ \swarrow^{\text{res}} & & \nwarrow^{\text{cores}} \\ \mathrm{R}\Gamma_{T(U) \cap Z}(T(U), \mathscr{F}) & \xrightarrow{\quad T \quad} & \mathrm{R}\Gamma_{U \cap T^t(Z)}(U, \mathscr{F}) \end{array}$$

and hence we can define an action of the correspondence T on the local cohomology $\mathrm{R}\Gamma_{U \cap Z}(U, \mathscr{F})$.

2.1.3. *Compactness and finite slope parts.* In the classical theory of overconvergent modular forms, a crucial property is that the space of overconvergent modular forms (say of fixed radius of overconvergence) is a p-adic topological vector space, and the "U_p-type" operator acting on this space is compact.

We would like to have a corresponding property for cohomology with support. Let us first consider a special case, which corresponds to the classical situation. Assume that $Z = \mathcal{X}$, and that U and $T(U)$ are actually affinoid. Then $H^0(U, \mathscr{F})$ and $H^0(T(U), \mathscr{F})$ are in fact $\mathbb{Q}_p$-Banach spaces. Moreover, as a basic source of compactness, if we assume that $\mathcal{X}$ is proper, and that $\overline{T(U)} \subseteq U$ (that is, that U is a strict neighborhood of $T(U)$) then the restriction map

$$\mathrm{res} : H^0(U, \mathscr{F}) \to H^0(T(U), \mathscr{F})$$

is compact. Consequently, T acting on $H^0(U, \mathscr{F})$ is compact.

Now we consider the general situation. The first issue issue is that we are now working with complexes rather than spaces, and so the issue of topologies becomes more subtle. If we assume that U is quasi-compact (equivalently a finite union of affinoids) then we can compute $\mathrm{R}\Gamma(U, \mathscr{F})$ via a Čech complex for a finite cover of U by affinoid opens, and hence $\mathrm{R}\Gamma(U, \mathscr{F})$ may be computed by a finite complex of $\mathbb{Q}_p$-Banach spaces. While this complex will depend on the choice of the Čech cover, any two such choices will give isomorphic objects in the homotopy category $\mathcal{K}(\mathbf{Ban}(\mathbb{Q}_p))$ of bounded complexes of $\mathbb{Q}_p$-Banach spaces with homotopy classes of maps. More generally if U and $\mathcal{X} \setminus Z$ are quasi-compact, then the cohomology with support $\mathrm{R}\Gamma_{U \cap Z}(U, \mathscr{F})$ is also an object of $\mathcal{K}(\mathbf{Ban}(\mathbb{Q}_p))$, as we have the exact triangle

$$\mathrm{R}\Gamma_{U \cap Z}(U, \mathscr{F}) \to \mathrm{R}\Gamma(U, \mathscr{F}) \to \mathrm{R}\Gamma(U \setminus Z, \mathscr{F}) \stackrel{+1}{\to} .$$

Remark 2.1. One can also put topologies on the cohomology groups $H^i_{U \cap Z}(U, \mathscr{F})$. However they will in general be non Hausdorff and we will not use them.

Next we have the following generalization of the basic source of compactness recalled above:

Proposition 2.2. *Suppose that $\mathcal{X}$ is proper, $U \subseteq U' \subseteq \mathcal{X}$ are quasi-compact opens, and $Z \subseteq Z' \subseteq \mathcal{X}$ are closed and such that $\mathcal{X} \setminus Z'$ and $\mathcal{X} \setminus Z$ are quasi-compact. Further assume that $\overline{U} \subseteq U'$ and $Z \subseteq \mathrm{Int}(Z')$. Then the composition*

$$\mathrm{res} \circ \mathrm{cores} : \mathrm{R}\Gamma_{U' \cap Z}(U', \mathscr{F}) \to \mathrm{R}\Gamma_{U \cap Z'}(U, \mathscr{F})$$

in $\mathcal{K}(\mathbf{Ban}(\mathbb{Q}_p))$ is compact, in the sense that it can be represented by a map of complexes of $\mathbb{Q}_p$-Banach spaces which is term by term compact.

Remark 2.3. Note that one can take $U = U' = Z = Z' = \mathcal{X}$, in which case one recovers the finiteness of coherent cohomology on proper rigid analytic spaces.

Now we turn to the spectral theory. If V is a $\mathbb{Q}_p$-Banach space and T is a compact endomorphism of V, then the spectral theory of compact operators gives us for all $h \in \mathbb{Q}$ a T-stable slope decomposition $V = V^{\leq h} \oplus V^{>h}$, where $V^{\leq h}$ is finite dimensional and all the generalized eigenvalues of T on $V^{\leq h}$ have p-adic valuation $\leq h$, while for all α with $v(\alpha) \leq h$, $T - \alpha \mathrm{Id}$ is invertible on $V^{>h}$. The finite slope part is $V^{fs} = \bigcup_{h \in \mathbb{Q}} V^{\leq h}$. We note that T is invertible on V^{fs}, and in fact V^{fs} is the (algebraic) span of the generalized eigenvectors of T on V with nonzero eigenvalues. In the case of a complex $V^\bullet \in \mathcal{K}(\mathbf{Ban}(\mathbb{Q}_p))$ with a compact endomorphism T, we may form the finite slope part $V^{\bullet,fs}$, by picking a representative of T which is compact termwise, and forming the finite slope part term by term. The resulting complex doesn't depend on the choices (up to quasi-isomorphism.)

We now apply this discussion to the action of the correspondence T on the local cohomology $\mathrm{R}\Gamma_{U\cap Z}(U,\mathscr{F})$. Recall that in order to have such an action we had assumed that $U \subseteq \mathcal{X}$ is open and satisfies $T(U) \subseteq U$, and $Z \subseteq \mathcal{X}$ is closed and satisfies $T^t(Z) \subseteq Z$. We now impose the following further hypotheses: $\mathcal{X}$ is proper, U and $\mathcal{X} \setminus Z$ are quasi-compact, and we have "strict" inclusions $\overline{T(U)} \subseteq U$ and $T^t(Z) \subseteq \mathrm{Int}(Z)$. Then under these hypotheses, $\mathrm{R}\Gamma_{U\cap Z}(U,\mathscr{F})$ is an object of $\mathcal{K}(\mathbf{Ban}(\mathbb{Q}_p))$, and T^2 factors through res$\circ$cores and hence is compact (we note that all positive powers of T have the same finite slope part, and so it makes no difference whether T or just one of its powers is compact.) Thus under these hypotheses we may form the finite slope part of our cohomology with support: $\mathrm{R}\Gamma_{U\cap Z}(U,\mathscr{F})^{fs}$.

Remark 2.4. We remark that for technical reasons it is convenient to allow a bit more flexibility than we did in this section. For example one may want to allow U and $\mathcal{X} \setminus Z$ to be certain non quasi-compact opens, like Stein spaces (e.g. an open Tate disc) or more generally a finite union of (quasi-)Stein spaces. In this case, the cohomologies are not represented by complexes of Banach spaces, but inverse limits of complexes of Banach spaces. Nonetheless, variants of all the results of this section hold in this setting and we will use this implicitly in the rest of these lectures.

2.1.4. *Analytic continuation.* In the classical theory of overconvergent modular forms, a basic fact is that a finite slope overconvergent modular form may be analytically continued across the supersingular locus. This may be reformulated as follows: the restriction maps between spaces of overconvergent modular forms with different radii of overconvergence induce isomorphisms on the finite slope parts. We would like to have an analogous result for cohomology with support.

First we observe that the triangle diagram used to define T on $\mathrm{R}\Gamma_{U\cap Z}(U,\mathscr{F})$ may be extended to the following diamond diagram:

$$\begin{array}{ccccc} & & \mathrm{R}\Gamma_{U\cap Z}(U,\mathscr{F}) & & \\ & \swarrow^{\mathrm{res}} & & \nwarrow^{\mathrm{cores}} & \\ \mathrm{R}\Gamma_{T(U)\cap Z}(T(U),\mathscr{F}) & & \xrightarrow{\quad T \quad} & & \mathrm{R}\Gamma_{U\cap T^t(Z)}(U,\mathscr{F}) \\ & \nwarrow_{\mathrm{cores}} & & \swarrow_{\mathrm{res}} & \\ & & \mathrm{R}\Gamma_{T(U)\cap T^t(Z)}(T(U),\mathscr{F}) & & \end{array}$$

We may use this diagram to define an endomorphism T of $\mathrm{R}\Gamma_{T(U)\cap T^t(Z)}(T(U),\mathscr{F})$, and again we have that T^2 is compact (cores$\circ$res = res$\circ$cores in this diagram) so that it makes sense to form the finite slope part $\mathrm{R}\Gamma_{T(U)\cap T^t(Z)}(T(U),\mathscr{F})^{fs}$. Now our basic analytic continuation result will be that $\mathrm{R}\Gamma_{U\cap Z}(U,\mathscr{F})^{fs} \simeq \mathrm{R}\Gamma_{T(U)\cap T^t(Z)}(T(U),\mathscr{F})^{fs}$.

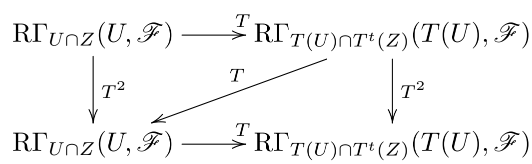
To see this, note that while there is in general no map between these two spaces defined just using restriction and corestriction, we do have maps

$$T := \mathrm{res} \circ T \circ \mathrm{res} : \mathrm{R}\Gamma_{U\cap Z}(U,\mathscr{F}) \to \mathrm{R}\Gamma_{T(U)\cap T^t(Z)}(T(U),\mathscr{F})$$

and

$$T := \mathrm{cores} \circ T \circ \mathrm{cores} : \mathrm{R}\Gamma_{T(U)\cap T^t(Z)}(T(U),\mathscr{F}) \to \mathrm{R}\Gamma_{U\cap Z}(U,\mathscr{F}).$$

These fit in to a commutative diagram

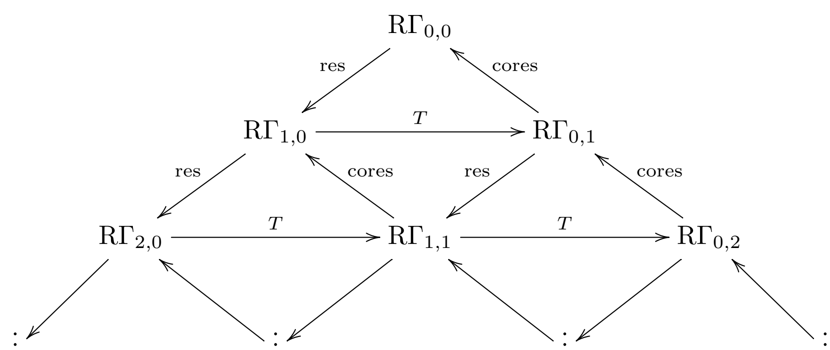


from which we see that the maps T induce isomorphisms on the finite slope parts, as they factor the vertical maps T^2, which become isomorphisms by definition.

Now we have managed to relate the cohomology of the "bigger" region $U \cap Z$, with the "smaller" region $T(U) \cap T^t(Z)$, but we would like to keep going and consider even smaller regions. Towards that end, we note that $U \supseteq T(U) \supseteq T^2(U) \supseteq \cdots$ and $Z \supseteq T^t(Z) \supseteq (T^t)^2(Z) \supseteq \cdots$ are nested, and we can consider the locally closeds $T^m(U) \cap (T^t)^n(Z)$ for $m, n \geq 0$. Let us abbreviate $\mathrm{R}\Gamma_{m,n} := \mathrm{R}\Gamma_{T^m(U)\cap(T^t)^n(Z)}(T^m(U), \mathscr{F})$. For $m, n \geq 0$ we have maps

(1) $\mathrm{res} : \mathrm{R}\Gamma_{m,n} \to \mathrm{R}\Gamma_{m+1,n}$,
(2) $\mathrm{cores} : \mathrm{R}\Gamma_{m,n+1} \to \mathrm{R}\Gamma_{m,n}$, and
(3) $T : \mathrm{R}\Gamma_{m+1,n} \to \mathrm{R}\Gamma_{m,n+1}$

which fit together into the following "infinite diamond" diagram, where we note that the diamond at the top is the diagram we considered above:



This diagram has the following commutativity property: any morphism $\mathrm{R}\Gamma_{m,n} \to \mathrm{R}\Gamma_{m',n'}$ obtained by following any path through this diagram depends only on the number k of maps labeled T used. We denote this map by T^k, and note that such a map T^k exists for $k >> 0$ (depending on n, m, n', m'.) Moreover these maps will even be compact for $k >> 0$, as they factor through the $\mathrm{res} \circ \mathrm{cores}$ at the top of the diagram. In particular the maps $T^k : \mathrm{R}\Gamma_{m,n} \to \mathrm{R}\Gamma_{m,n}$ are compact for $k >> 0$, and so it makes sense to form the finite slope part $\mathrm{R}\Gamma_{m,n}^{fs}$ (in practice all compositions $\mathrm{res} \circ \mathrm{cores}$ in this diagram will be compact, so we will not have to take k large, but the hypotheses that we have made in this general setting do not imply that.)

As a consequence, any map $T^k : \mathrm{R}\Gamma_{m,n} \to \mathrm{R}\Gamma_{m',n'}$ factors through a power of T on both the source and the target, and hence induces an isomorphism on finite slope parts. We conclude that all the finite slope cohomologies $\mathrm{R}\Gamma_{m,n}^{fs}$ are isomorphic

2.2. **The spectral sequence associated to the Bruhat stratification.** We begin by recalling some notation:

- We let $G = \mathrm{GSp}_{2g}(\mathbb{Z}_p^{2g}, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is the standard anti-diagonal symplectic form with $\langle e_i, e_{2g+1-i} \rangle = 1$.

- We let $T \subset B \subseteq P$ be the standard diagonal torus, upper triangular Borel, and upper triangular Siegel parabolic[2]. We also write $M \subset P$ for the standard Levi.
- We let $K_p \subset G(\mathbb{Z}_p)$ be the standard Iwahori of elements reducing mod p to $B(\mathbb{F}_p) \subset G(\mathbb{F}_p)$. We consider the Iwahori Hecke algebra $\mathcal{H}_p = C_c^\infty(K_p \backslash G(\mathbb{Q}_p)/K_p)$ with its convolution product for the Haar measure where K_p is given volume 1. We write $[K_p g K_p] \in \mathcal{H}_p$ for the characteristic function of the double coset $K_p g K_p$.

  An important role will be played by the commutative subalgebra $\mathcal{H}_p^+ \subset \mathcal{H}_p$, spanned by $[K_p t K_p]$ for $t \in T^+$, as well as its ideal $\mathcal{H}_p^{++}$ spanned by $[K_p t K_p]$ for $t \in T^{++}$, where
$$T^+ = \{t \in T(\mathbb{Q}_p) \mid v(\alpha(t)) \geq 0, \forall \alpha \in \Phi^+\}$$
$$T^{++} = \{t \in T(\mathbb{Q}_p) \mid v(\alpha(t)) > 0, \forall \alpha \in \Phi^+\}.$$
  Here Φ^+ denotes the positive roots of G. T^{++} may be viewed as those elements of $T(\mathbb{Q}_p)$ which "contract" the unipotent radical of B under left conjugation. We note that $[K_p t K_p][K_p t' K_p] = [K_p t t' K_p]$ for $t, t' \in T^+$.
- We let $\mathrm{FL} = P \backslash G$. We let $d = g(g+1)/2$ be its dimension. For a $\mathbb{Z}_p$-scheme S, an S point of FL corresponds to a surjection $\mathbb{Z}_p^{2g} \otimes \mathscr{O}_S \to \mathcal{E}$ where $\mathcal{E}$ is locally free of rank g and the kernel is isotropic. In the case that $\mathcal{E}$ is free (for example if $S = \mathrm{Spec}(k)$ for k a field) then by picking a basis for $\mathcal{E}$ we can describe the point by a g by $2g$ matrix. For example when $g = 1$ we recover the description of points of $\mathrm{FL} = \mathbb{P}^1$ in terms of homogenous coordinates $[x, y]$.
- We let W be the Weyl group of G. Via its action on basis vectors $e_1, \ldots, e_{2g}$ it may be viewed as the subgroup
$$W = \{w \in S_{2g} \mid w(2g+1-i) = 2g+1-w(i)\} \subseteq S_{2g}.$$
  We let $W_M \subset W$ be the Weyl group of the Levi M. It is isomorphic to S_g, and may be described as those elements of W preserving $\{1, \ldots, g\}$.

  Inside W we have simple reflections s_i for $i = 1, \ldots, g$. For $i = 1, \ldots, g-1$, $s_i = (i\, i+1)(2g+1-i\; 2g-i) \in W_M$, and $s_g = (g\, g+1)$.

  On W we have the length function $\ell : W \to \mathbb{N}$ as well as the Bruhat order $\leq$.
- We have the set ${}^M W \subseteq W$ of minimal length coset representatives for $W_M \backslash W$. Since this set plays an important role in this theory, we give various conditions on $w \in W$ which are equivalent to $w \in {}^M W$:
  - We have $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(g)$.
  - We have $w^{-1}\Phi_M^+ \subseteq \Phi^+$, where Φ^+ and Φ_M^+ denote the positive roots of G and M.
  - We have $wX^\star(T)^+ \subseteq X^\star(T)^{M,+}$.

  In view of this last characterization, we may view ${}^M W$ as labelling the G-Weyl chambers contained in the dominant M-Weyl chamber.

  When $g = 1$, we have ${}^M W = W = \{1, w_0\}$. When $g = 2$ we have ${}^M W = \{1, s_2, s_2 s_1, s_2 s_1 s_2\}$.

[2]This choice of Borel is different from the choice made in the first lecture, it is more adapted to computations with the flag variety.

- We have the longest elements of W, W_M, and ${}^MW$:
$$w_0 = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}, \quad w_{0,M} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \quad w_0^M = w_{0,M}w_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$
Here s is the g by g antidiagonal matrix with 1's along the antidiagonal.
There is an involution of ${}^MW$ defined by $w \mapsto w_{0,M}ww_0$. It satisfies $\ell(w) + \ell(w_{0,M}ww_0) = d$. (We remark that w_0 is in the center of W, and so $w_{0,M}ww_0 = w_0^M w$. However this fails for other Cartan types, so it is better to not use it.)

We have the Bruhat stratification
$$\mathrm{FL} = \coprod_{w \in {}^MW} C_w$$
where $C_w := P\backslash PwB$ are the Bruhat cells. We have $C_w \simeq \mathbb{A}^{\ell(w)}$, and $C_w \subseteq \overline{C_{w'}}$ if and only if $w \leq w'$.

We also consider:

- $C^w := P\backslash Pw\overline{B} = C_{w_{0,M}ww_0} \cdot w_0$, the opposite Bruhat cells.
- $X_w := \overline{C_w} = \cup_{w' \in {}^MW, w' \leq w} C_{w'}$.
- $Y_w := \cup_{w' \in {}^MW, w' \geq w} C_{w'}$.

We note that X_w is closed in FL, Y_w is open in FL, and $X_w \cap Y_w = C_w$.

In this section we will use the Bruhat stratification of $FL_{\mathbb{F}_p}$ to define a filtration of $\mathcal{FL}$ which we will use to construct our spectral sequence. Before giving the general case, we begin with an example.

Example 2.5. We consider the case $g = 1$. The Bruhat stratification of the special fiber is
$$\mathbb{P}^1_{\mathbb{F}_p} = C_{1,\mathbb{F}_p} \coprod C_{w_0,\mathbb{F}_p}.$$
Here $C_{1,\mathbb{F}_p} = \{\infty\}$ and $C_{w_0,\mathbb{F}_p} = \mathbb{A}^1_{\mathbb{F}_p}$. We may associate to each Bruhat cell its tube in the generic fiber $\mathbb{P}^{1,ad}$. They are described as follows:

- $]C_{1,\mathbb{F}_p}[$ is an open Tate disc about the point ∞. In coordinates its classical points are $\{[x, 1] \mid v(x) > 0\}$.
- $]C_{w_0,\mathbb{F}_p}[$ is a closed (affinoid) Tate disc about the point $0 = [1, 0]$. In coordinates its classical points are $\{[1, y] \mid v(y) \geq 0\}$.

We emphasize that both $]C_{1,\mathbb{F}_p}[$ and $]C_{w_0,\mathbb{F}_p}[$ are open in $\mathbb{P}^{1,ad}$, and although they contain all the classical points of $\mathbb{P}^{1,ad}$ they are not a cover (from the point of view of classical rigid analytic geometry they do not form an admissible cover.) In fact there is exactly one higher rank point, generalizing to the Gauss point of $]C_{w_0,\mathbb{F}_p}[$, which is not in either open. We can add this point to either tube, and thus obtain two different decompositions of $\mathbb{P}^{1,ad}$ into an open and a closed: either
$$\mathbb{P}^{1,ad} =]C_{1,\mathbb{F}_p}[\coprod \overline{]C_{w_0,\mathbb{F}_p}[}, \quad \text{or} \quad \mathbb{P}^{1,ad} = \overline{]C_{1,\mathbb{F}_p}[} \coprod]C_{w_0,\mathbb{F}_p}[.$$
In what follows, we will use the first decomposition. This is in a certain sense more natural from the point of view of the classical theory of overconvergent modular forms. We recall that $\pi^{-1}_{HT,K_p}(B \cdot 1 \cdot K_p)$ corresponds to the (closure of the) canonical ordinary locus, and so $\pi^{-1}_{HT,K_p}(]C_{1,\mathbb{F}_p}[)$ will be a neighborhood of the canonical ordinary locus.

If we instead used the second decomposition, we would get a theory of overconvergent modular forms living on neighborhoods of the anticanonical ordinary locus

$\pi^{-1}_{HT,K_p}(Bw_0K_p)$. In this case the Hecke operator $U_p^t = [K_p \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K_p]$ has the right dynamical properties. In fact, the theory that one obtains is related to the usual theory in two ways: the two theories are both "Serre dual" to each other, and isomorphic under the Atkin-Lehner involution. See the discussion of duality below.

We finally remark that the second decomposition is in a certain sense much more natural from the point of analytic geometry: there is a continuous specialization map $\mathrm{sp} : \mathbb{P}^{1,ad} \to \mathbb{P}^1_{\mathbb{F}_p}$, and the second decomposition is exactly the preimage of the Bruhat decomposition of the special fiber under sp. On the other hand in the first decomposition, we have exchanged which cell is open and which is closed: the cell corresponding to $w = 1$ is closed in the special fiber and open in the generic fiber.

Now we explain the general construction. We define a filtration by closed subsets $\mathcal{FL} = Z_0 \supset Z_1 \supset \cdots \supset Z_d \supset Z_{d+1} = \emptyset$ by taking

$$Z_r = \overline{]\bigcup_{w \in {}^MW, l(w) \geq r} Y_{w,\mathbb{F}_p}[}.$$

Here we note that the subsets $\cup_{w \in {}^MW, l(w) \geq r} Y_{w,\mathbb{F}_p}$ are open in the special fiber, but we are obtaining a closed subset of the generic fiber, as in the example of $\mathbb{P}^1$ above. Again this choice is motivated by the fact that the open $Z_0 \setminus Z_1$ for our filtration is $]C_{1,\mathbb{F}_p}[$, which is an open neighborhood of the point $P \cdot 1 \in \mathcal{FL}$. Then when we pull back by the Hodge-Tate period map, we will get an open neighborhood

$$(\pi^{tor}_{HT,K_p})^{-1}(P \cdot 1 \cdot K_p) \subset (\pi^{tor}_{HT,K_p})^{-1}(]C_{1,\mathbb{F}_p}[)$$

of the canonical ordinary locus.

More generally we have

$$\left(\bigcup_{w \in {}^MW, l(w) \geq r} Y_{w,\mathbb{F}_p} \right) \setminus \left(\bigcup_{w \in {}^MW, l(w) \geq r+1} Y_{w,\mathbb{F}_p} \right) = \coprod_{w \in {}^MW, l(w) = r} C_{w,\mathbb{F}_p}$$

and so

$$\coprod_{w \in {}^MW, l(w) = r}]C_{w,\mathbb{F}_p}[\subseteq Z_r \setminus Z_{r+1}$$

and this is an equality on classical points but a strict inclusion on adic space points when $r > 0$. In fact one can see that

$$Z_r \setminus Z_{r+1} = \coprod_{w \in {}^MW, l(w) = r}]X_{w,\mathbb{F}_p}[\cap \overline{]Y_{w,\mathbb{F}_p}[}.$$

We note that

$$]C_{w,\mathbb{F}_p}[=]X_{w,\mathbb{F}_p}[\cap]Y_{w,\mathbb{F}_p}[\subseteq]X_{w,\mathbb{F}_p}[\cap \overline{]Y_{w,\mathbb{F}_p}[} =]X_{w,\mathbb{F}_p}[\cap \overline{]C_{w,\mathbb{F}_p}[}$$

and so the right hand side can be viewed as a sort of "partial closure" of the tube $]C_{w,\mathbb{F}_p}[$. In the next section we will give a more concrete description of this partial closure using affine coordinates.

The filtration Z_r is invariant under the Iwahori, and so it descends to a filtration of $\mathcal{FL}/K_p$, which we can then pull back by π^{tor}_{HT,K_p} to obtain a filtration $\mathcal{S}^{tor}_{K,\Sigma} = \mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \cdots \supset \mathcal{Z}_d \supset \mathcal{Z}_{d+1} = \emptyset$.

We may now consider the spectral sequence associated to this filtration:

$$E_1^{p,q} = \mathrm{H}^{p+q}_{\mathcal{Z}_p \setminus \mathcal{Z}_{p+1}}(\mathcal{S}^{tor}_{K,\Sigma} \setminus \mathcal{Z}_{p+1}, \mathcal{V}_\kappa) \Rightarrow \mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa).$$

By the decomposition of $Z_p \setminus Z_{p+1}$ recorded above, the local cohomologies on the first page of this spectral sequence decompose into terms indexed by $w \in {}^MW$:

$$\mathrm{H}^{p+q}_{\mathcal{Z}_p\setminus\mathcal{Z}_{p+1}}(\mathcal{S}^{tor}_{K,\Sigma}\setminus\mathcal{Z}_{p+1},\mathcal{V}_\kappa) = \bigoplus_{w\in {}^MW,\ell(w)=p} \mathrm{H}^{p+q}_{(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,\mathbb{F}_p}[\cap\overline{]Y_{w,\mathbb{F}_p}[})}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,\mathbb{F}_p}[),\mathcal{V}_\kappa).$$

In order for this spectral sequence to be interesting, we need to consider the Hecke action on it. The abutment of this spectral sequence carries an action of the Iwahori Hecke algebra $\mathcal{H}_p$. However, as explained above, we can only expect that certain Hecke operators which have the right dynamical properites act on these local cohomology groups. In fact, it is exactly the subalgebra $\mathcal{H}_p^+$ which acts:

Theorem 2.6. *For each $w \in {}^MW$, there is an action of $\mathcal{H}_p^+$ on the local cohomologies $\mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,\mathbb{F}_p}[\cap\overline{]Y_{w,\mathbb{F}_p}[})}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,\mathbb{F}_p}[),\mathcal{V}_\kappa)$, and moreover the $[K_ptK_p]$ for $t \in T^{++}$ act compactly. Moreover the above spectral sequence is $\mathcal{H}_p^+$-equivariant.*

As a consequence of the theorem we can define

$$\mathrm{R}\Gamma_w(K^p,\kappa)^{+,fs} := \mathrm{R}\Gamma_{(\pi^{tor}_{HT,K_p})^{-1}(]X_{w,\mathbb{F}_p}[\cap\overline{]Y_{w,\mathbb{F}_p}[})}((\pi^{tor}_{HT,K_p})^{-1}(]X_{w,\mathbb{F}_p}[),\mathcal{V}_\kappa)^{+,fs},$$

where the finite slope part is taken with respect to any $[K_ptK_p]$ for $t \in T^{++}$. Then taking finite slope parts in the spectral sequence above, we obtain

$$E_1^{p,q} = \bigoplus_{w\in {}^MW,l(w)=p} \mathrm{H}^{p+q}_w(K^p,\kappa)^{+,fs} \Rightarrow \mathrm{H}^{p+q}(\mathcal{S}^{tor}_{K,\Sigma},\mathcal{V}_\kappa).$$

We note that the $[K_ptK_p]$ for $t \in T^+$ are units in the Iwahori Hecke algebra $\mathcal{H}_p$, and hence the abutment of the spectral sequence is already finite slope. In order to prove the theorem, using the results of the previous section, we need to understand something about the dynamics of $\mathcal{H}_p^+$ on $\mathcal{FL}/K_p$, and its interaction with the Schubert stratification.

2.3. **Dynamics on the flag variety.**

2.3.1. *The dynamics of T^+ on $\mathcal{FL}$.*

Example 2.7. We consider $g = 1$. The fixed points for the action of T on $\mathcal{FL} = \mathbb{P}^{1,ad}$ are $\infty = B1 = [0,1]$ and $0 = Bw_0 = [1,0]$. We compute

$$[x,1]\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix} = [px,1]$$

and so we see that ∞ is an attracting fixed point, while 0 is a repelling fixed point.

Example 2.8. We consider an example for $g = 2$, before trying to do the general case more systematically. Now there are four fixed points, corresponding to the elements $w = 1, s_2, s_2s_1, s_2s_1s_2$. In coordinates they are

$$\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}0&1&0&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix},\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}.$$

Let's try to understand the dynamics of T^+ near the fixed point corresponding to $w = s_2$. We can write down an affine neighborhood of this point, and compute the action of an element of T^{++} as follows:

$$\begin{bmatrix} x & 1 & z & 0 \\ y & 0 & -x & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^{-1} & 0 & 0 \\ 0 & 0 & p^{-2} & 0 \\ 0 & 0 & 0 & p^{-3} \end{pmatrix} = \begin{bmatrix} px & 1 & p^{-1}z & 0 \\ p^3 y & 0 & -px & 1 \end{bmatrix}$$

What we see is that Ps_2 is an attracting fixed point in two directions and a repelling fixed point in one. We can understand the situation a bit better by observing that these directions can be interpreted in terms of the Schubert cell C_{s_2} and its opposite C^{s_2}. Namely we have

$$C_{s_2} = \left\{ \begin{bmatrix} 0 & 1 & z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \quad C^{s_2} = \left\{ \begin{bmatrix} x & 1 & 0 & 0 \\ y & 0 & -x & 1 \end{bmatrix} \right\}.$$

Thus our affine neighborhood of Ps_2 may be described as a product $C_{s_2} \times C^{s_2}$ of the Schubert cell and its opposite, and T^{++} is expanding on C_{s_2} and contracting on C^{s_2}.

We can also consider the tube $]C_{s_2,\mathbb{F}_p}[$. In terms of our coordinates, its classical points are described by the conditions $v(x), v(y) > 0$, $v(z) \geq 0$. It is not stable by T^{++}. Indeed for the element in our calculation above, we see that if $v(z) < 1$, then

$$\begin{bmatrix} px & 1 & p^{-1}z & 0 \\ p^3 y & 0 & -px & 1 \end{bmatrix} \sim \begin{bmatrix} p^2xz^{-1} & pz^{-1} & 1 & 0 \\ p^3 y & 0 & -px & 1 \end{bmatrix}$$

reduces to the point $P1$, and hence lies in the tube $]C_{1,\mathbb{F}_p}[$. However we see that the tube over the closed Schubert cell $]X_{s_2,\mathbb{F}_p}[$ is stable by T^{++}.

Let us now return to the general case. We can write down an affine chart around the fixed point $P1$ by considering the opposite cell $C^1 = P\backslash P1\overline{U}$, where $\overline{U}$ is the unipotent radical of the parabolic opposite to P. To get a neighborhood of Pw for $w \in {}^MW$ we can translate by w: $C^1 \cdot w = P\backslash P1\overline{U} = P\backslash Pw(w^{-1}\overline{U}w)$. We have

$$w^{-1}\overline{U}w = (w^{-1}\overline{U}w \cap B) \times (w^{-1}\overline{U}w \cap \overline{B})$$

and we have that

$$C_w = P\backslash Pw(w^{-1}\overline{U}w \cap B), \quad C^w = P\backslash Pw(w^{-1}\overline{U}w \cap \overline{B})$$

and so our neighborhood $C^1 \cdot w$ is (T-equivariantly) isomorphic to $C_w \times C^w$, and we see that T^{++} is expanding on $C_w \simeq \mathbb{A}^{l(w)}$ and contracting on $C^w \simeq \mathbb{A}^{d-l(w)}$.

We can use these affine charts to write down certain rigid analytic neighborhoods of the point Pw. We would like to introduce coordinates corresponding to the x, y, z in the example above. For this we consider the root subgroups $U_\alpha \subset G$ for $\alpha \in \Phi$. They are isomorphic to $\mathbb{G}_a$, via a natural coordinate x_α, and for $n \in \mathbb{Q}$ we can introduce the rigid analytic subgroups $\mathcal{U}_{\alpha,n}$ (resp. $\mathcal{U}^\circ_{\alpha,n}$) of U_α^{ad}, which are the closed (resp. open) Tate discs defined by $v(x_\alpha) \leq n$ (resp. $v(x_\alpha) < n$). We have

$$w^{-1}\overline{U}w = \prod_{\alpha \in w^{-1}\Phi^{M,-} \cap \Phi^+} U_\alpha \times \prod_{\alpha \in w^{-1}\Phi^{M,-} \cap \Phi^-} U_\alpha$$

where $\Phi^{M,-} = \Phi^- \setminus \Phi^-_M$ is the set of roots occuring in $\overline{U}$. Then for $m, n \in \mathbb{Q}$ we can consider polydiscs

$$]C_w[_{m,n} := P\backslash Pw \cdot \prod_{\alpha \in w^{-1}\Phi^{M,-} \cap \Phi^+} \mathcal{U}_{\alpha,n} \times \prod_{\alpha \in w^{-1}\Phi^{M,-} \cap \Phi^-} \mathcal{U}^\circ_{\alpha,m} \subset \mathcal{FL}.$$

We note that $]C_{w,\mathbb{F}_p}[_{0,0}=]C_{w,\mathbb{F}_p}[$, which explains why we have used open and closed Tate discs in this way (see the example above.)

We can also introduce certain "partial closures". We let $]C_{w,\mathbb{F}_p}[_{m,\overline{n}}$ be the closure of $]C_{w,\mathbb{F}_p}[_{m,n}$ in $]C_{w,\mathbb{F}_p}[_{m,n-1}$. These are the kinds of locally closed sets that might arise as possible support conditions when we run the "analytic continuation" procedure on the flag variety, starting with the Bruhat stratification. In particular one can see that $]C_w[_{\overline{0},\overline{0}}=]X_{w,\mathbb{F}_p}[\cap\overline{]Y_{w,\mathbb{F}_p}[}$. Then some of the basic facts about the dynamics of T^+ on $\mathcal{FL}$ are summarized in the following easy proposition:

Proposition 2.9. (1) *For $t\in T^+$ we have*

$$]X_{w,\mathbb{F}_p}[\cdot t\subseteq]X_{w,\mathbb{F}_p}[,\quad\text{and}\quad\overline{]Y_{w,\mathbb{F}_p}[}\cdot t^{-1}\subseteq\overline{]Y_{w,\mathbb{F}_p}[}$$

and moreover these inclusions are strict when $t\in T^{++}$.

(2) *For $t\in T^{++}$ we have*

$$]X_{w,\mathbb{F}_p}[\cdot t^m\cap\overline{]Y_{w,\mathbb{F}_p}[}\cdot t^{-n}\subseteq]C_{w,\mathbb{F}_p}[_{m,\overline{n}}$$

2.3.2. *The dynamics of $\mathcal{H}_p^+$ on $\mathcal{FL}/K_p$.* The basic facts about the dynamics of $\mathcal{H}_p^+$ on $\mathcal{FL}/K_p$ are summarized in the following proposition:

Proposition 2.10. (1) *For $t\in T^+$ we have*

$$]X_{w,\mathbb{F}_p}[\cdot(K_ptK_p)\subseteq]X_{w,\mathbb{F}_p}[,\quad\text{and}\quad\overline{]Y_{w,\mathbb{F}_p}[}\cdot(K_pt^{-1}K_p)\subseteq\overline{]Y_{w,\mathbb{F}_p}[}$$

and moreover these inclusions are strict when $t\in T^{++}$.

(2) *For $t\in T^{++}$ we have*

$$]X_{w,\mathbb{F}_p}[\cdot(K_ptK_p)^m\cap\overline{]Y_{w,\mathbb{F}_p}[}\cdot(K_pt^{-1}K_p)^n\subseteq]C_{w,\mathbb{F}_p}[_{m,\overline{0}}\cdot K_p\cap]C_{w,\mathbb{F}_p}[_{0,\overline{n}}\cdot K_p$$

This proposition explains why the operators $t\in T^+$ act on the local cohomology.

We conclude this lecture by explaining that the finite slope part of the local cohomology depends only on neighborhoods of the ordinary locus. In order to see this, and following the results of section 2.1.4, it remains to describe $]C_{w,\mathbb{F}_p}[_{m,\overline{0}}\cdot K_p\cap]C_{w,\mathbb{F}_p}[_{0,\overline{n}}\cdot K_p$. When $w=\mathrm{Id}$, we find that $\{]C_{\mathrm{Id},\mathbb{F}_p}[_{m,\overline{0}}\cdot K_p\}_m$ form a system of neighborhoods of $\mathrm{Id}\cdot K_p$ in $\mathcal{FL}/K_p$. When $w=w_0^M$, we find that $\{]C_{w_0^M,\mathbb{F}_p}[_{0,\overline{n}}\cdot K_p\}_n$ form a system of neighborhoods of $w_0^M\cdot K_p$ in $\mathcal{FL}/K_p$. In general it is useful to allow K_p to be a deeper Iwahori subgroup (namely the elements of $G(\mathbb{Z}_p)$ which are congruent to elements of $B(\mathbb{Z}_p)$ modulo p^l for an integer $l\geq m$). The finite slope cohomologies for the various lwahori subgroups K_p are canonically identified using some elementary relations in the Iwahori-Hecke algebra. Under this assumption on K_p, we find that $]C_{w,\mathbb{F}_p}[_{m,\overline{0}}\cdot K_p\cap]C_{w,\mathbb{F}_p}[_{0,\overline{n}}\cdot K_p\subseteq]C_{w,\mathbb{F}_p}[_{m,\overline{n}}\cdot K_p$. The sets $\{]C_{w,\mathbb{F}_p}[_{m,\overline{n}}\}_{m,n}$ form a system of neighborhoods of $w\in\mathcal{FL}$.

3. Lecture 3: Vanishing, duality, slope bounds, and classicality

3.1. Vanishing.

We recall from the introduction the following vanishing theorem:

Theorem 3.1. *The cohomology complex $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{+,fs}$ is concentrated in the interval $[0,\ell(w)]$ and the cohomology complex $\mathrm{R}\Gamma_w(K^p,\kappa)^{+,fs}$ is concentrated in the interval $[\ell(w),d]$.*

Our goal is to explain how the theorem is proved in the cuspidal case. The non cuspidal case is deduced using duality. We begin with a simple example which illustrates a key point of the proof.

Example 3.2. For $r \in \mathbb{Q}_{>0}$ we write $D(p^{-r}) \subset \mathbb{A}^1$ for the closed Tate disc of radius p^{-r} (if X is the coordinate on $\mathbb{A}^1$, it is defined by $|X| \leq |p^r|$). Then consider

$$\begin{aligned} Z &:= \overline{D(1) \times D(1)} \quad \subseteq \quad \mathbb{A}^2 \\ U &:= D(p) \times D(1) \quad \subseteq \quad \mathbb{A}^2 \end{aligned}$$

We let $V = U \setminus Z$. Then V is the product $A \times D(1)$, where A is the "half" open annulus described by $\bigcup_{\epsilon<0} |p^\epsilon| \leq |X| \leq |p^{-1}|$.

Then we claim that for $\mathscr{F}$ a coherent sheaf on U, the local cohomology $\mathrm{R}\Gamma_{Z\cap U}(U, \mathscr{F})$ is concentrated in the interval $[0, 1]$. Indeed, this follows from the exact triangle

$$\mathrm{R}\Gamma_{Z\cap U}(U, \mathscr{F}) \to \mathrm{R}\Gamma(U, \mathscr{F}) \to \mathrm{R}\Gamma(V, \mathscr{F}) \to$$

as well as the fact that U and V have no higher coherent cohomology (they are examples of "quasi-stein" rigid analytic spaces.)

We may generalize this example slightly, we now consider $a, b \geq 0$ and:

$$\begin{aligned} Z &:= \overline{D(1)^a \times D(1)^b} \quad \subseteq \quad \mathbb{A}^{a+b} \\ U &:= D(p)^a \times D(1)^b \quad \subseteq \quad \mathbb{A}^{a+b} \end{aligned}$$

Then we claim that $\mathrm{R}\Gamma_{Z\cap U}(U, \mathscr{F})$ is concentrated in degree $[0, a]$. Indeed, again this follows from the exact triangle

$$\mathrm{R}\Gamma_{Z\cap U}(U, \mathscr{F}) \to \mathrm{R}\Gamma(U, \mathscr{F}) \to \mathrm{R}\Gamma(V, \mathscr{F}) \to$$

but now while U is still quasi-stein, $V := U \setminus Z$ is no longer quasi-stein when $a > 1$, but rather covered by a quasi-stein spaces (of the form an annulus in one coordinate times discs in the rest) and hence $\mathrm{R}\Gamma(V, \mathscr{F})$ is computed by a Čech complexes with amplitude $[0, a-1]$.

Now the vanishing theorem is proved by combining the following ingredients:

(1) If $\pi : S^{tor}_{K,\Sigma} \to S^\star_K$ denotes the map from the toroidal compactification to the minimal compactification, then we have $R^i\pi_\star\mathcal{V}_\kappa(-D) = 0$ for $i > 0$.
(2) The Hodge-Tate period map $\pi_{HT,K_p} : S^\star_K \to \mathcal{FL}/K_p$ is affine, in the sense that each classical point of $\mathcal{FL}/K_p$ has a basis of open neighborhoods whose pullbacks to $S^\star_K$ are affinoid.
(3) The locally closed $]C_{w,\mathbb{F}_p}[_{m,\overline{n}}$, which arise as support conditions which can be used to compute $\mathrm{R}\Gamma_w(K^p, \kappa, cusp)^{fs}$, are exactly like in the example above. In particular $\ell(w) = \#(w^{-1}\Phi^{M,-} \cap \Phi^+)$ is the number of "closed directions" in $]C_{w,\mathbb{F}_p}[_{m,\overline{n}}$.

3.2. **Duality.** We consider Serre duality on the Shimura variety $\mathcal{S}^{tor}_{K,\Sigma}$. We have $\mathcal{V}^\vee_\kappa \simeq \mathcal{V}_{-w_{0,M}\kappa}$ and moreover the canonical bundle of $\mathcal{S}^{tor}_{K,\Sigma}$ is $\mathcal{V}_{-2\rho_{nc}}(-D)$ where $2\rho_{nc} = \sum_{\alpha\in\Phi^{M,+}} \alpha$ is the sum of the positive roots of G that are not roots of M. Hence there is a perfect Serre duality pairing:

$$\mathrm{H}^i(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa) \times \mathrm{H}^{d-i}(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_{-w_{0,M}\kappa-2\rho_{nc}}(-D)) \to \mathbb{Q}_p.$$

Moreover this pairing is Hecke equivariant in the sense that the transpose of the Hecke operator $[KgK]$ is $[Kg^{-1}K]$.

For various reasons we would like to have a duality theory for local cohomologies. For example, since duality exchanges cuspidal and non cuspidal cohomologies, the non cuspidal part of the vanishing theorem 3.1 can be seen as the dual of the cuspidal part. The local cohomologies constructed in the previous section carry an action of the Hecke algebra $\mathcal{H}_p^+$, and so in view of the Hecke equivariance of Serre duality recalled above, it is natural to try to pair them with some other local cohomology groups carrying an action of the algebra $\mathcal{H}_p^-$, spanned by $[K_ptK_p]$ for $t \in T^-$. In fact, we can construct cohomology groups $\mathrm{R}\Gamma_w(K^p, \kappa)^{-,fs}$, $\mathrm{R}\Gamma_w(K^p, \kappa, cusp)^{-,fs}$ together with spectral sequences $\mathbf{E}^{p,q}(K^p, \kappa)^-$ and $\mathbf{E}^{p,q}(K^p, \kappa, cusp)^-$, all carrying actions of $\mathcal{H}_p^-$. These cohomology groups are obtained by "reversing" the support conditions used in the +-theory (more precisely we consider partial closures in the opposite directions, for example we may consider locally closeds like $]C_{w,\mathbb{F}_p}[_{\overline{n},m}$, the closure of $]C_{w,\mathbb{F}_p}[_{n,m}$ in $]C_{w,\mathbb{F}_p}[_{n+1,m}$.)

Then we construct pairings:

$$\langle , \rangle : \mathrm{R}\Gamma_w(K^p, -w_{0,M}\kappa - 2\rho_{nc})^{\pm,fs} \times \mathrm{R}\Gamma_w(K^p, \kappa, cusp)^{\mp,fs} \to \mathbb{Q}_p[-d]$$

This pairing is compatible with the $\mathcal{H}_p^{\pm}$-actions in the sense that we have $\langle [K_ptK_p]\cdot, \cdot\rangle = \langle \cdot, [K_ptK_p]\cdot\rangle$ for $t \in T^{\pm}$.

Let us illustrate how these pairings are constructed by continuing the example 3.2.

Example 3.3. We consider $a, b \geq 0$ and:

$$\begin{aligned} Z &:= \overline{D(1)^a \times D(1)^b} &\subseteq& \quad \mathbb{A}^{a+b} \\ U &:= D(p)^a \times D(1)^b &\subseteq& \quad \mathbb{A}^{a+b} \\ Z' &:= \overline{D(p)^a \times D(p^{-1})^b} &\subseteq& \quad \mathbb{A}^{a+b} \end{aligned}$$

The first cohomology we considered was $\mathrm{R}\Gamma_{Z\cap U}(U, \mathscr{F})$. The "closed" directions for $Z \cap U$ inside $\mathbb{A}^{a+b}$ are with respect the first a coordinates. We may also consider $\mathrm{R}\Gamma_{Z'\cap U}(U, \mathscr{F})$ where the "closed" directions are with respect to the last b coordinates. Assume that $\mathscr{F}$ is a locally free sheaf, let $\mathscr{F}^\vee$ be its dual, and $\omega_{\mathbb{A}^{a+b}}$ be the canonical sheaf of $\mathbb{A}^{a+b}$. We can define a cup-product:

$$\mathrm{R}\Gamma_{Z\cap U}(U, \mathscr{F}) \otimes^L_{\mathbb{Q}_p} \mathrm{R}\Gamma_{Z'\cap U}(U, \mathscr{F}^\vee \otimes \omega_{\mathbb{A}^{a+b}}) \to \mathrm{R}\Gamma_{Z\cap Z'}(U, \omega_{\mathbb{A}^{a+b}}).$$

by taking the intersection of supports. We observe that U is a neighborhood of $Z \cap Z'$. There is therefore a natural trace map $\mathrm{R}\Gamma_{Z\cap Z'}(U, \omega_{\mathbb{A}^{a+b}}) \to \mathbb{Q}_p[-d]$. This trace map can be obtained as the composite:

$$\mathrm{R}\Gamma_{Z\cap Z'}(U, \omega_{\mathbb{A}^{a+b}}) \to \mathrm{R}\Gamma(\mathbb{P}^{a+b}, \omega_{\mathbb{P}^{a+b}}) \to \mathbb{Q}_p[-d]$$

where the second map is the usual trace for projective duality, and where the first map decomposes further into:

$$\mathrm{R}\Gamma_{Z\cap Z'}(U, \omega_{\mathbb{A}^{a+b}}) \to \mathrm{R}\Gamma_{Z\cap Z'}(\mathbb{P}^{a+b}, \omega_{\mathbb{P}^{a+b}}) \to \mathrm{R}\Gamma(\mathbb{P}^{a+b}, \omega_{\mathbb{P}^{a+b}})$$

where the first map is excision and the second forgets the support.

Finally we explain another formulation of duality using w_0, which is what is explained in [BP20] in the case $g = 1$. We note that the involution $t \mapsto w_0t^{-1}w_0^{-1}$ of $T(\mathbb{Q}_p)$ preserve the monoid T^+ (actually it is even the identity on the maximal torus of the derived group, but this fails for other Cartan types.) Then there is also a pairing

$$\mathrm{R}\Gamma_w(K^p, -w_{0,M}\kappa - 2\rho_{nc})^{+,fs} \times \mathrm{R}\Gamma_{w_{0,M}ww_0}(K^p, \kappa, cusp)^{+,fs} \to \mathbb{Q}_p[-d]$$

for which the transpose of $[K_ptK_p]$ is $[K_p(w_0t^{-1}w_0^{-1})K_p]$ for $t \in T^+$. Note that here both sides of the pairing have the + theory, but now there are two different elements of ${}^MW$.

In fact this pairing is constructed from the one above, by producing isomorphisms

$$\mathrm{R}\Gamma_w(K^p,\kappa)^{-,fs} \simeq \mathrm{R}\Gamma_{w_{0,M}ww_0}(K^p,\kappa)^{+,fs},$$

and similarly for cuspidal cohomology. The idea for constructing such a map is to use right translation by w_0. Indeed, at least at the level of the flag variety, right multiplication by w_0 exchanges the fixed points $P\backslash Pw$ and $P\backslash Pw_{0,M}ww_0$, and it also exchanges the "closed" directions for the $+$ and $-$ theories.

When passing to Shimura varieties we need to be slightly careful about the level at p. When $g=1$, the element $\begin{pmatrix} 0 & -p^{-1} \\ 1 & 0 \end{pmatrix}$ normalizes K_p and induces the Atkin-Lehner involution of the modular curve at Iwahori level, which realizes the above isomorphisms. When $g>1$ however, there is no element normalizing the Iwahori K_p corresponding to w_0 (there is still the Atkin-Lehner involution given by $\begin{pmatrix} 0 & -p^{-1}I \\ I & 0 \end{pmatrix}$, but this now corresponds to $w_0^M \neq w_0$ when $g>1$.) Nonetheless this is not a serious problem: to construct these isomorphisms in general we need to consider different levels at p.

3.3. **Slope bounds.** We are able to bound below the slopes of the possible eigensystems for the action of $\mathcal{H}_p^+$ on the local cohomology $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{+,fs}$. If we have a Hecke eigenclass in one of our local cohomologies, we can associate a character $\lambda : T^+ \to \overline{\mathbb{Q}}_p^\times$, where for $t\in T^+$, $\lambda(t)$ is the eigenvalue of the Hecke operator $[K_ptK_p]$. For now we are interested only in the slopes (i.e. p-adic valuations) of these eigenvalues. It is helpful to observe that these slopes can be packaged together into a rational character $v(\lambda)\in X^\star(T)\otimes\mathbb{Q}$, which is characterized by $\langle v(\lambda),\mu\rangle = v(\lambda(\mu(p)))$ for $\mu\in X_\star(T)^+$ (here $\langle\cdot,\cdot\rangle$ is the pairing between $X^\star(T)$ and $X_\star(T)$.) We will also use the partial order on $X^\star(T)\otimes\mathbb{Q}$, defined by $\nu\geq 0$ if and only if $\nu = \sum_{\alpha\in\Phi^+} c_\alpha\alpha$ with $c_\alpha\in\mathbb{Q}_{\geq 0}$.

We make the following optimal conjectural lower bound on the slopes:

Conjecture 3.4. *Fix $w\in {}^MW$ and $\kappa\in X^\star(T)^{M_\mu,+}$. For any character λ of T^+ ocuring in $\mathrm{R}\Gamma_w(K^p,\kappa)^{+,fs}$ or $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{+,fs}$ we have $v(\lambda)\geq w^{-1}w_{0,M}(\kappa+\rho)+\rho$.*

In general, we are only able to prove a slightly weaker lower bound:

Theorem 3.5. *Fix $w\in {}^MW$ and $\kappa\in X^\star(T)^{M_\mu,+}$. For any character λ of T^+ occuring in $\mathrm{R}\Gamma_w(K^p,\kappa)^{+,fs}$ or $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{+,fs}$ we have $v(\lambda)\geq w^{-1}w_{0,M}(\kappa)$.*

Remark 3.6. Cohomology classes for which the inequality in the conjecture is an equality are ordinary cohomology classes.

Remark 3.7. To explain what the conjecture and theorem say from a more concrete perspective, we note that if $\mu_1,\dots,\mu_g\in X_\star(T)$ are the fundamental dominant coweights, then for $\nu\in X^\star(T)\otimes\mathbb{Q}$, $\nu\geq 0$ is equivalent to ν having trivial restriction to the center Z, and satisfying $\langle\nu,\mu_i\rangle\geq 0$ for $i=1,\dots,g$.

Consider the case $g=2$, and let

$$U_{p,1} = [K_p\mathrm{diag}(1,p^{-1},p^{-1},p^{-2})K_p], \quad U_{p,2} = [K_p\mathrm{diag}(1,1,p^{-1},p^{-1})K_p]$$

be the Hecke operators corresponding to the two fundamental dominant coweights. Then untangling the statement of the theorem and conjecture, they first give the valuation of eigenvalues of the diamond operator $[K_p\mathrm{diag}(p^{-1},p^{-1},p^{-1},p^{-1})K_p]$

(which is not so interesting) and then are equivalent to certain lower bounds on the valuations of eigenvalues of $U_{p,1}$ and $U_{p,2}$, which are given by some formula in terms of w and κ. See the tables below for these formulas.

Remark 3.8. In a number of cases, we can actually prove the bounds of the conjecture, for example for degree 0 cohomology, for most classical cohomology classes, and for classes which have large families of deformations on the eigenvariety.

Now we will sketch how these slope bounds are proved. The basic strategy is as follows:

(1) If we have an endomorphism T of a $\mathbb{Q}_p$-vector space V, a natural way to prove that the valuations of eigenvalues of T are $\geq C$ is to find a lattice $L \subset V$ for which $T(L) \subseteq p^C L$.

(2) In order to find a lattice in one of our local cohomologies $\mathrm{R}\Gamma_{U\cap Z}(U, \mathcal{V}_\kappa)$, we will use a lattice in the sheaf: an $\mathscr{O}^+_{\mathcal{S}^{tor}_{K,\Sigma}}$-subsheaf $\mathcal{V}^+_\kappa \subseteq \mathcal{V}_\kappa$. In order to find such lattices, because all the automorphic vector bundles $\mathcal{V}_\kappa$ are produced from ω_A by Schur functors, it suffices to treat the case of ω_A. Then there are two natural choices, either of which will work for our present purposes: we can use the subsheaf of integral differentials (at least on the good reduction locus) as in [BP20], or we can use the integral structure coming from the Hodge-Tate map as in [BP21]. We remark that it is highly non trivial that the cohomology of the sheaf $\mathcal{V}^+_\kappa$ actually defines a lattice in the finite slope part of the cohomology of $\mathcal{V}_\kappa$. This uses crucially a theorem of Bartenwerfer [Bar78].

(3) Now for $t \in T^+$ we consider the corresponding Hecke correspondence $\mathcal{S}^{tor}_{K,\Sigma} \xleftarrow{p_1} \mathcal{C} \xrightarrow{p_2} \mathcal{S}^{tor}_{K,\Sigma}$. We write $\mathcal{S}^{ord,w} \subset \mathcal{S}^{tor}_{K,\Sigma}$ for the component of the ordinary locus corresponding to $w \in {}^M W$ (so that the closure of $\mathcal{S}^{ord,w}$ is $(\pi^{tor}_{HT,K_p})^{-1}(P\backslash PwK_p)$.)

Then we show that over the "fixed part" of the correpsondence $p_1^{-1}(\mathcal{S}^{ord,w}) \cap p_2^{-1}(\mathcal{S}^{ord,w})$, the map of sheaves $p_2^\star \mathcal{V}_\kappa \to p_1^\star \mathcal{V}_\kappa$ used to define the action of the correspondence on cohomology, gives on integral sheaves a map $p_2^\star \mathcal{V}^+_\kappa \to p^{\langle wt, w_{0,M}\kappa\rangle} p_1^\star \mathcal{V}^+_\kappa$.

(4) Now our local cohomologies are computed by $\mathrm{R}\Gamma_{U\cap Z}(U, \mathcal{V})$ where $U \cap Z$ is a locally closed neighborhood of $\mathcal{S}^{ord,w}$, which can be made arbitrarily small by the analytic continuation argument. Moreover the action of $[K_p t K_p]$ on this local cohomology only depends on the map $p_2^\star \mathcal{V}_\kappa \to p_1^\star \mathcal{V}^+_\kappa$ on (neighborhoods of) $p_1^{-1}(U) \cap p_2^{-1}(Z)$. We deduce from the previous point that over such neighborhoods we have a map on integral sheaves $p_2^\star \mathcal{V}^+_\kappa \to p^{\langle wt, w_{0,M}\kappa\rangle - \varepsilon} p_1^\star \mathcal{V}^+_\kappa$, where $\varepsilon > 0$ can be made as small as desired by shrinking $U \cap Z$. This is enough to prove the slope bound of the theorem.

Remark 3.9. The difference between the theorem and the conjecture should come from the integral properties of the trace map $tr : (p_1)_\star \mathscr{O}_{\mathcal{C}} \to \mathscr{O}_{\mathcal{S}^{tor}_{K,\Sigma}}$. One may show using Fourier expansions or Serre-Tate coordinates that there is a factorization

$$tr_{p_1} : \mathrm{H}^0(p_1^{-1}(\mathcal{S}^{ord,w}) \cap p_2^{-1}(\mathcal{S}^{ord,w}), p_1^\star \mathcal{V}^+_\kappa) \to p^{\langle t, w^{-1} w_{0,M}\rho + \rho\rangle} \mathrm{H}^0(\mathcal{S}^{ord,w}, \mathcal{V}^+_\kappa).$$

However, because the map p_1 is étale, one cannot hope to have such a factorization locally on $\mathcal{S}^{ord,w}$. This makes studying the integral properties of the trace on higher cohomology a rather delicate point.

For the rest of the section we will try to explain step 3 in a bit more detail and give some examples for $g = 1$ and 2. For simplicity we will consider only the Hecke operator $U_{p,g} = [K_p \mathrm{diag}(1, \dots, 1, p^{-1}, \dots, p^{-1}) K_p]$ (this is often called just U_p in the theory of Siegel modular forms.) We recall the moduli interpretation of the corresponding Hecke correspondence $\mathcal{S}^{tor}_{K,\Sigma} \xleftarrow{p_1} \mathcal{C} \xrightarrow{p_2} \mathcal{S}^{tor}_{K,\Sigma}$, at least away from the boundary.

- A point of the Iwahori level Shimura variety $\mathcal{S}_K$ is a tuple $(A, \lambda, \{H_i\})$ where (A, λ) is an abelian variety of dimension g with a prime-to-p polarization and $\{H_i\}$ is an Iwahori level structure: a complete flag $0 = H_0 \subset H_1 \subset \cdots \subset H_{2g} = A[p]$ which is self dual for the Weil pairing. There is also a K^p-level structure that we ignore.
- A point of (the interior of) $\mathcal{C}$ is a tuple $(A, \lambda, \{H_i\}, L)$ where $(A, \lambda, \{H_i\})$ is as above, and $L \subset A[p]$ is a maximal isotropic subspace with $L \cap H_g = \{1\}$.
- The map p_1 is just "forget L".
- The map p_2 sends $(A, \lambda, \{H_i\}, L)$ to $(A/L, \lambda', \{H'_i\})$ where λ' is the induced principal polarization on A/L and $H'_i = (H_i + L)/L$ for $i = 0, \dots, g$, and then $H'_i = (H'_{2g-i})^{\perp}$ for $i = g+1, \dots, 2g$.

In order to carry out step 3 of our sketch above, we will need to consider points of $\mathcal{C}$ with good ordinary reduction, and study the induced map $\omega_{A/L} \to \omega_A$ on integral differentials. We recall that $A[p]$ contains a maximal isotropic canonical subgroup $H_{can} \subset A[p]$ which has multiplicative reudction. We have a factorization $A \to A/(H_{can} \cap L) \to A/L$ where the kernel of the first isogeny $H_{can} \cap L$ has multiplicative reduction, while the second isogeny is étale integrally. It follows that with respect to a suitable choice of basis for the source and target, the map $\omega_{A/L} \to \omega_A$ has the form $\mathrm{diag}(p, \dots, p, 1, \dots, 1)$ where the number of p's is $\dim(H_{can} \cap L)$.

To determine this, we can use the following basic dynamical lemma:

Lemma 3.10. *Let $(A, \lambda, \{H_i\}, L)$ be a point of $\mathcal{C}$ with good ordinary reduction, so that we have canonical subgroups $H_{can} \subset A[p]$ and $H'_{can} \subset A'[p]$. Then we have inequalities*

$$\dim H_{can} \cap H_g \leq g - \dim(H_{can} \cap L) \leq \dim H'_{can} \cap H'_g.$$

We conclude from this two key points: first the quantity $\dim(H_{can} \cap H_g)$ is non decreasing as we apply the correspondence $U_{p,g}$, and second if it stays constant, in particular if we are considering a point of one of the fixed parts $p_1^{-1}(\mathcal{S}^{ord,w}) \cap p_2^{-1}(\mathcal{S}^{ord,w})$, then $\dim(H_{can} \cap L) = g - \dim(H_{can} \cap H_g)$.

Now we describe the case of $g = 1, 2$ in more detail.

Example 3.11. We consider the case $g = 1$, so that ${}^{M}W = \{1, w_0\}$. The ordinary locus for the modular curve has two components: the canonical ordinary locus $\mathcal{S}^{ord,1}$ where $H_1 = H_{can}$, and the anti-canonical ordinary locus $\mathcal{S}^{ord,w_0}$ where $H_1 \neq H_{can}$.

The ordinary locus of the correspondence $\mathcal{C}$ has three components:

- There is the fixed part for $w = 1$, where H_1 is canonical and L is (necessarily) non canonical.
- There is the fixed part for $w = w_0$, where H_1 is non canonical but L is canonical, so that $(E/L, H'_1)$ is also non canonical.
- There is the "unstable locus" where H_1 is not canonical but L is also not canonical, so that $(E/L, H'_1)$ is canonical.

The third component does not figure in to our calculation.

When $w = 1$ we find that the isogeny $E \to E/L$ is integrally étale, and we deduce that U_p has slopes ≥ 0 on $\mathrm{H}^0_1(K^p, k)$. The conjecture, which is easily proven in this case, says that the slopes are actually ≥ 1, with the difference coming from the trace map (p_1 is Frobenius mod p.)

When $w = w_0$ we find that the isogeny $E \to E/L$ is integrally multiplicative, and we deduce that U_p has slopes $\geq p^k$ on $\mathrm{H}^1_{w_0}(K^p, k)$. Here the conjecture and the theorem agree, because the map p_1 is actually an isomorphism, so we cannot obtain and further divisibility from the trace.

Example 3.12. We consider the case $g = 2$, so that ${}^M W = \{1, s_2, s_2s_1, s_2s_1s_2\}$. The corresponding four components of the ordinary locus $\mathcal{S}^{ord,w}$ are determined by the relative position of H_{can} and $\{H_i\}$ according to the following table:

	1	s_2	s_2s_1	$s_2s_1s_2$
$\dim(H_{can} \cap H_1)$	1	1	0	0
$\dim(H_{can} \cap H_2)$	2	1	1	0

Given a point of the fixed part $p_1^{-1}(\mathcal{S}^{ord,w}) \cap p_2^{-1}(\mathcal{S}^{ord,w})$ of $\mathcal{C}$ for some $w \in {}^M W$, we deduce from the lemma that $\dim(H_{can} \cap L)$ and the map $\omega_{A/L} \to \omega_A$ (with respect suitable bases) are given by the following table:

	1	s_2	s_2s_1	$s_2s_1s_2$
$\dim(H_{can} \cap L)$	0	1	1	2
$\omega_{A/L} \to \omega_A$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$

Now for $k_1 \geq k_2$ we consider the automorphic vector bundle $\mathrm{sym}^{k_1-k_2}\omega_A \otimes \det^{k_2}\omega_A$. Applying this Schur functor to the matrices in the above table and recording the smallest power of p ocuring, we obtain the first row of the following table, giving the slope bounds of the theorem:

	1	s_2	s_2s_1	$s_2s_1s_2$
Slopes of $U_{p,2}$ are $\geq$	0	k_2	k_2	$k_1 + k_2$
Slopes of $U_{p,1}$ are $\geq$	k_2	k_2	$2k_2 + k_1$	$2k_2 + k_1$

Finally we record the slightly stronger bounds of the conjecture:

	1	s_2	s_2s_1	$s_2s_1s_2$
Slopes of $U_{p,2}$ are $\geq$	3	$k_2 + 1$	$k_2 + 1$	$k_1 + k_2$
Slopes of $U_{p,1}$ are $\geq$	$k_2 + 3$	$k_2 + 3$	$2k_2 + k_1$	$2k_2 + k_1$

3.4. **Classicality.** As an application of the slope bounds of the previous section we can prove some classicality theorems. The idea is quite simple: we impose some condition on the slopes so that the cohomologies $\mathrm{R}\Gamma_w(K^p, \kappa)$ vanish for all but one $w \in {}^M W$.

We first consider some examples for $g = 2$.

Example 3.13. We consider $w = 1$. From the above tables, we see that when $k_2 > 0$, we can kill the cohomologies for $w \neq 1$ by imposing the condition that the slopes of $U_{p,2}$ are $< k_2$. We deduce the following classicality theorem:

$$\mathrm{R}\Gamma_1(K^p, \kappa)^{fs, U_{p,2} < k_2} \simeq \mathrm{R}\Gamma(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)^{U_{p,2} < k_2}.$$

We note that this will actually be vacuous unless $k_2 > 3$ because the slopes of $U_{p,2}$ on $\mathrm{R}\Gamma_1(K^p, \kappa)^{fs}$ will be ≥ 3. If we assume the conjecture, then we can improve the small slope condition to $U_{p,2} < k_2 + 1$, and this will be non vacuous when $k_2 > 2$.

Example 3.14. Now we consider $w = s_2$. Then from the table we see that if $k_2 < 0$ and $k_1 + k_2 > 3$ then we have the classicality theorem:

$$\mathrm{R}\Gamma_{s_2}(K^p, \kappa)^{fs, U_{p,2}<0, U_{p,1}<2k_2+k_1} \simeq \mathrm{R}\Gamma(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)^{U_{p,2}<0, U_{p,1}<2k_2+k_1}.$$

If we assume the conjecture, then we can replace $k_2 < 0$ with $k_2 < 2$, and $U_{p,2} < 0$ with $U_{p,2} < 3$. We remark that we need the condition on $U_{p,2}$ to kill $\mathrm{R}\Gamma_1$, and the condition on $U_{p,1}$ to kill $\mathrm{R}\Gamma_w$ for $w = s_2 s_1, s_2 s_1 s_2$. This is in contrast to the previous example where we only needed the single Hecke operator $U_{p,2}$.

Now we try to explain something about the case of general g. We make the following somewhat combinatorial definitions:

Definition 3.15. *Let $\kappa \in X^\star(T)^{M_\mu,+}$ be a weight.*

- *We let $C(\kappa)^+ = \{w \in {}^M W \mid -w^{-1}w_{0,M}(\kappa+\rho) \in X^\star(T)^+_{\mathbb{Q}}\}$. We also let $\ell_{min}(\kappa) = \min_{w \in C(\kappa)^+} \ell(w)$ and $\ell_{max}(\kappa) = \max_{w\in C(\kappa)^+} \ell(w)$.*
- *We say that a character λ of T^+ satisfies $+, ss^M(\kappa)$*[3] *if $v(\lambda) \ngeq w^{-1}w_{0,M}(\kappa+\rho)+\rho$ for all $w \in {}^M W \setminus C(\kappa)^+$.*
- *We say that a character λ of T^+ satisfies $+, sss^M(\kappa)$ if $v(\lambda) \ngeq w^{-1}w_{0,M}(\kappa)$ for all $w \in {}^M W \setminus C(\kappa)^+$.*

The small slope and strongly small slope conditions are clearly motivated by the bounds of conjecture 3.4 and theorem 3.5, but we try to explain the meaning of the set $C(\kappa)^+$. In fact it has the following interpretation, which is a combinatorial exercise: the conjectural lower bound $w^{-1}w_{0,M}(\kappa+\rho)+\rho$ is independent of $w \in C(\kappa)^+$, and is moreover the minimum as w varies over ${}^M W$, in the sense that for $w \in C(\kappa)^+$ and $w' \in {}^M W \setminus C(\kappa)^+$, we have

$$w^{-1}w_{0,M}(\kappa+\rho)+\rho < w'^{-1}w_{0,M}(\kappa+\rho)+\rho.$$

In terms of these definitions we immediately have our general classicality theorems:

Theorem 3.16. *Let $\kappa \in X^\star(T)^{M_\mu,+}$ be a weight, and suppose that $C(\kappa)^+ = \{w\}$ has only a single element. Then we have an isomorphism*

$$\mathrm{R}\Gamma_w(K^p, \kappa)^{+, sss^M(\kappa)} \simeq \mathrm{R}\Gamma(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa)^{+, sss^M(\kappa)}.$$

Assuming conjecture 3.4, we can replace $+, sss^M(\kappa)$ with $+, ss^M(\kappa)$. The same results hold for cuspidal cohomology.

As another application of the slope bounds and these definitions we can prove a first vanishing theorem:

Theorem 3.17. *For $\kappa \in X^\star(T)^{M_\mu,+}$, $\mathrm{R}\Gamma(\mathcal{S}^{tor}_{K,\Sigma}, \mathcal{V}_\kappa(-D))^{+, sss^M(\kappa)}$ is concentrated in degrees $[0, \ell_{max}(\kappa)]$. Assuming conjecture 3.4, we can replace $+, sss^M(\kappa)$ with $+, ss^M(\kappa)$.*

Indeed, by the slope bounds, imposing the condition $+, sss^M(\kappa)$ kills the cohomologies $\mathrm{R}\Gamma_w(K^p, \kappa, cusp)^{+, fs}$ for $w \notin C(\kappa)^+$, and these remaining terms are concentrated in degrees $[0, \ell(w)]$ by theorem 3.1. The theorem follows from this and the spectral sequence.

[3] This $+, ss^M(\kappa)$-condition is the precise $+, ss$ condition we need to use for the vanishing theorems of interior classical cohomology

Remark 3.18. Using Serre duality for classical cohomology we can also deduce vanishing theorems for non-cuspidal and interior classical cohomology. However as stated in the first lecture, we can unconditionally prove a vanishing theorem for interior cohomology using only the condition $+, ss^M(\kappa)$. This uses the theory of the eigenvariety.

4. Lecture 4: Eigenvarieties

4.1. **Sheaf interpolation.** In order to construct the eigenvariety, one needs to construct p-adic variation of the local cohomologies. We will construct a p-adic variation of the automorphic vector bundles and take the local cohomology of these sheaves.

4.1.1. *The modular curve case.* As a warm up, we would like to briefly explain the definition of overconvergent modular forms of any p-adic weight for the group $\mathrm{GL}_2/\mathbb{Q}$ (following [AIS14] and [Pil13]). Let $k \in \mathbb{Z}$. Let N be a integer. A modular form f of weight k and level N over $\mathbb{Z}[1/N]$ is a function f on triples (E, ψ_N, ω) where E is an elliptic curve, ψ_N is a full level N structure, $\omega \in \omega_E \setminus \{0\}$ is a nowhere vanishing differential and such that:

1. f is functorial,
2. For all $\lambda \in \mathbb{G}_m$, $f((E, \mathrm{level}, \lambda\omega) = \lambda^{-k} f(E, \mathrm{level}, \omega)$,
3. $f(\mathrm{Tate}(q^{1/N}), \Psi_N, \omega_{can}) \in \mathbb{Z}[1/N, \zeta_N][[q^{1/N}]]$, for all possible level N structure Ψ_N on $\mathrm{Tate}(q^{1/N})$.

We now turn to overconvergent modular forms. We now work in the p-adic setting. An elliptic curve E such that $|\mathrm{Hasse}(E)| \geq p^{-\frac{1}{p^{n-1}(p+1)}}$ has a canonical subgroup H_n. When the elliptic curve has ordinary reduction the canonical subgroup $H_n \subset E[p^n]$ is locally étale isomorphic to μ_{p^n} and has multiplicative reduction.

Let us assume that $E \to \mathrm{Spa}(R, R^+)$. Moreover, after passing to a rational cover, we may assume that E and H_n have integral models over Spec R_0 an open and bounded subring of R^+, and that there is a map $H_n \hookrightarrow E[p^n]$ extending the inclusion on the generic fiber.

We have a commutative diagram:

$$\begin{array}{ccc} E[p^n] & \xrightarrow{\mathrm{HT}} & \omega_E/p^n \\ \downarrow & & \downarrow \\ H_n^D & \longrightarrow & \omega_{H_n} \end{array}$$

Moreover, there exists $\epsilon_n > 0$ (with $\epsilon_n \to 0$ when $n \to \infty$) such that the map $\omega_E/p^{n-\epsilon_n} \to \omega_{H_n}/p^{n-\epsilon_n}$ is an isomorphism. An n-overconvergent modular form f of weight $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ and level N is a function f on triples $(E, \psi_N, \omega \in \omega_E)$ where E is an elliptic curve with a canonical subgroup H_n, ψ_N is a full level N structure, $\omega \in \omega_E$ is an integral differential with $\omega|_{\omega_{H_n}} \in \mathrm{HT}(H_n^D \setminus H_n^D[p^{n-1}])$, and

1. f is functorial,
2. For all $\lambda \in \mathbb{Z}_p^\times$, $f((E, \psi_N, \lambda\omega) = \kappa(\lambda)^{-1} f(E, \psi_N, \omega)$,
3. $f(\mathrm{Tate}(q^{1/N}) \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p), \Psi_N, \omega_{can}) \in \mathbb{C}_p[[q^{1/N}]]$, for all possible level N structure Ψ_N on $\mathrm{Tate}(q^{1/N}) \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

There is a map from n-overconvergent to $n+1$-overconvergent modular forms and an overconvergent modular form is an object of the inductive limit. We claim that this is a good notion (and in particular this space is non-zero if $N \geq 3$!).

In the context of the first classical definition, we have the $\mathbb{G}_m$-torsor of non-zero invariant differentials over the modular curve, and an modular form is just a k-homogeneous function on this torsor (together with a growth condition at cusps). In the context of the second definition, we first replace the $\mathbb{G}_m$-torsor of non-zero invariant differentials over the modular curve by the torsor of differential form which are "congruent" to the image of a generator of the dual canonical subgroup via the Hodge-Tate map. In other words, we construct a reduction of group structure of the original $\mathbb{G}_m$-torsor to a torsor under a group $\mathbb{Z}_p^\times(1+p^{n-\epsilon'_n}\mathbb{D})$ for a suitable $\epsilon'_n > 0$ (with $\epsilon'_n \to 0$ as $n \to +\infty$) and $\mathbb{D}$ the unit ball. It is reasonable to look for κ-homogeneous functions on this torsor because κ extends uniquely to an analytic function on $\mathbb{Z}_p^\times(1+p^{n-\epsilon'_n}\mathbb{D})$ for n large enough.

4.1.2. *General case.* We would like to extend this to general Siegel varieties. This has been done in [AIP15], but we take the opportunity to phrase the construction in a more systematic way which works for any Shimura variety. We also remark that many results in the direction of general Shimura varieties have already been obtained ([Bra16], [Bra20], [Her19], [CHJ17], [BHW20], with the last references putting emphasis on the Hodge-Tate period map). Consider the following diagram:

$$\begin{array}{ccc} \mathcal{S}^{tor}_{K^p,\Sigma} & & \\ \downarrow \pi & \searrow^{\pi^{tor}_{HT}} & \\ \mathcal{S}^{tor}_{K^p,\Sigma} & & \mathcal{FL}_{G,\mu} \end{array}$$

Let $\mathcal{M}^{an}_{dR}$ be the torsor of basis of $\omega_{A^t} \oplus \mathrm{Lie}(A)$ compatible with the polarization up to a scalar. This is a $\mathcal{M}^{an}_\mu$-torsor.

Let $\mathcal{M}^{an}_{HT}$ be the $\mathcal{G}^{an}$-equivariant torsor $\mathcal{U}_{\mathcal{P}^{an}_\mu}\backslash\mathcal{G}^{an} \to \mathcal{P}^{an}_\mu\backslash\mathcal{G}^{an}$.

Proposition 4.1. *We have a canonical isomorphism of K_p-equivariant torsors $\pi^\star\mathcal{M}^{an}_{dR} = (\pi^{tor}_{HT})^\star\mathcal{M}^{an}_{HT}$.*

Proof. Straightforward from the definition. □

Let $\mathcal{G}_n \hookrightarrow \mathcal{G}^{an}$ be the subgroup of elements congruent to 1 modulo p^n. Let $\mathcal{M}_{\mu,n} \hookrightarrow \mathcal{M}^{an}_\mu$ be the subgroup of elements congruent to 1 modulo p^n. Let $K_{p,M}$ be the Iwahori subrgoup of $M_\mu(\mathbb{Z}_p)$.

Theorem 4.2. *For any $w \in {}^MW$, over $(\pi^{tor}_{HT,K_p})^{-1}(w\mathcal{G}_nK_p) \subseteq \mathcal{S}^{tor}_{K_pK^p,\Sigma}$, $\mathcal{M}_{dR}$ has a reduction to a $\mathcal{M}_nK_{p,M}$-torsor $\mathcal{M}_{dR,n}$.*

Proof. We first construct the reduction over the flag variety. We have a diagram:

$$\begin{array}{ccc} \mathcal{M}^{an}_{HT} & \longleftarrow & (w\mathcal{G}_nK_pw^{-1} \cap \mathcal{U}_{\mathcal{P}^{an}_\mu})\backslash(w\mathcal{G}_nK_pw^{-1}).w \\ \downarrow & & \downarrow \\ \mathcal{FL}_{G,\mu} & \longleftarrow & (w\mathcal{G}_nK_pw^{-1} \cap \mathcal{P}^{an}_\mu)\backslash(w\mathcal{G}_nK_pw^{-1}).w \end{array}$$

Let us call $\mathcal{M}_{HT,n} := (w\mathcal{G}_nK_pw^{-1} \cap \mathcal{U}_{\mathcal{P}_\mu^{an}})\backslash(w\mathcal{G}_nK_pw^{-1}).w$. Via left translation, it carries a structure of torsor under

$$\mathcal{M}_nK_{p,M} = w\mathcal{G}_nK_pw^{-1} \cap \mathcal{U}_{\mathcal{P}_\mu^{an}}\backslash w\mathcal{G}_nK_pw^{-1} \cap \mathcal{P}_\mu^{an}.$$

Via right translation, it has an action of K_p.

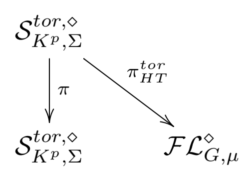
We now consider $\pi_{HT}^\star\mathcal{M}_{HT,n}$ and we claim that this torsor descends via π to $\mathcal{M}_{dR,n}$. On topological spaces $|\pi_{HT}^\star\mathcal{M}_{HT,n}| \hookrightarrow |\pi^\star\mathcal{M}_{dR}^{an}|$. Since $|\mathcal{M}_{dR}^{an}| = |\pi^\star\mathcal{M}_{dR}^{an}|/K_p$, and $\pi_{HT}^\star\mathcal{M}_{HT,n}$ is K_p-equivariant, we find that the topological space $|\pi_{HT}^\star\mathcal{M}_{HT,n}|$ descends to $|\mathcal{M}_{dR,n}| \subseteq |\mathcal{M}_{dR}^{an}|$. There is therefore a corresponding open adic subspace of $\mathcal{M}_{dR}^{an}$, and it is easy to check that this is an étale torsor under $\mathcal{M}_nK_{p,M}$.

□

Remark 4.3. We insist that pro-étale descent of étale torsors is usually not effective. Most G-equivariant étale sheaves on $\mathcal{FL}_{G,\mu}$ can be pulled back to $\mathcal{S}_{K^p}^{tor}$ but do not descend to étale sheaves on $\mathcal{S}_{K^p}^{tor}$.

Remark 4.4. This construction compares with the one of section 4.1.1: the subsets $(\pi_{HT,K_p}^{tor})^{-1}(\mathcal{G}_nK_p)$ for a system of strict neighborhoods of the multiplicative ordinary locus and the torsor $\mathcal{M}_{dR,n}$ can be described as a torsor of differential forms congruent to generators of the dual canonical subgroup via the Hodge-Tate period map.

Remark 4.5. We have not used seriously that $\mathcal{S}_{K^p,\Sigma}^{tor}$ was a perfectoid space. The argument was mostly at a topological level and we could have used the underlying diamonds. For a general Shimura variety, the results of [DLLZ19] allows to construct a diagram:



Together with a canonical identification $\pi^\star\mathcal{M}_{dR}^{an,\diamond} \simeq (\pi_{HT}^{tor})^\star\mathcal{M}_{HT}^{an,\diamond}$. This is all we need for the argument. We note that this was previously available (up to minor boundary issue) in the Hodge case by [CS17].

4.1.3. *Analytic induction.* For any $\kappa \in X^\star(T)^{M,+}$, we have the usual algebraic induction:

$$V_\kappa = \{f : M_\mu \to \mathbb{A}^1,\ f(mb) = -w_{0,M}(b)f(m), \forall(m,b) \in M_\mu \times B_{M_\mu}\}$$

where B_{M_μ} is the Borel subgroup of M_μ and $m \in M_\mu$ acting via $m.f = f(m^{-1}.\ )$. This is the highest weight κ representation. We can consider the analytic induction:

$$V_{\kappa_A}^{n-an} = \{f : \mathcal{M}_{\mu,n}K_{p,M} \to \mathbb{A}^1_{\mathrm{Spa}(A,A^+)},$$

$$f(mb) = -w_{0,M}(b)f(m), \forall(m,b) \in \mathcal{M}_{\mu,n}K_{p,M} \times (\mathcal{B}_{M_\mu} \cap \mathcal{M}_{\mu,n}K_{p,M})\}$$

for any character $\kappa_A : T(\mathbb{Z}_p) \to A^\times$ where A is a complete Tate algebra, and κ_A is n-analytic. The groups $\mathcal{M}_{\mu,n}K_{p,M}$ have Iwahori decompositions, and the A-Banach space $V_{\kappa_A}^{n-an}$ is orthonormalizable. We let $D_{\kappa_A}^{n-an}$ be the dual A-Banach space to $V_{\kappa_A}^{n-an}$.

4.1.4. *Sheaves.* Let $\nu_A : T(\mathbb{Z}_p) \to A^\times$ be a character. Let $w \in {}^MW$. Let $\kappa_A = -w_{0,M}w(\nu_A+\rho)-\rho$.

We define a sheaf $\mathcal{V}^{n-an}_{\nu_A}$ over $(\pi^{tor}_{HT,K_p})^{-1}(w\mathcal{G}^1_n K_p)$ modeled on $V^{n-an}_{\kappa_A}$ via:

$$\mathcal{V}^{n-an}_{\nu_A} = (\mathscr{O}_{\mathcal{M}_{dR,n}} \hat{\otimes}_{\mathbb{Q}_p} A(w_{0,M}\kappa))^{(\mathcal{B}_{M_\mu}\cap \mathcal{M}_{\mu,n}K_{p,M})}$$

We also let $\mathcal{D}^{n-an}_{\nu_A} = \underline{\mathrm{Hom}}_{cont}(\mathcal{V}^{n-an}_{\nu_A}, \Omega^d_{\mathcal{S}^{tor}_{K_pK^p,\Sigma}}\hat{\otimes}_{\mathbb{Q}_p} A)$, this is the continuous "Serre dual" of $\mathcal{V}^{n-an}_{\nu_A}$, and is modeled on $D^{n-an}_{\kappa_A}$.

Remark 4.6. The sheaves $\mathcal{V}^{n-an}_{\nu_A}$ and $\mathcal{D}^{n-an}_{\nu_A}$ are locally in the analytic topology of the form $M\hat{\otimes}_{\mathbb{Q}_p}\mathscr{O}_{\mathcal{S}^{tor}_{K_pK^p,\Sigma}}$ for M a projective Banach A-module (i.e. M is a direct factor of an orthonormalizable Banach A-module).

Remark 4.7. We have switched to label weights by ν rather than κ. This normalization will allow us to consider all $w \in {}^MW$ simultaneously later.

When $\kappa \in X^\star(T)^{M_\mu,+}$, $\nu \in X^\star(T)^+ - \rho_{nc}$, and $\kappa = -w_{0,M}w(\nu+\rho)-\rho$ we have natural maps:

$$\begin{array}{rcl} \mathcal{V}_\kappa & \to & \mathcal{V}^{n-an}_\nu \\ \mathcal{D}^{n-an}_\nu & \to & \mathcal{V}_{-w_{0,M}\kappa-2\rho_{nc}} \end{array}$$

4.2. **locally analytic overconvergent cohomologies.**

4.2.1. *Definition of the cohomologies.* For all $w \in {}^MW$, we can define locally analytic overconvergent cohomologies

$$\mathrm{R}\Gamma_{w,an}(K^p,\nu_A)^{\pm,fs}$$

These are obtained by taking local cohomologies of $\mathcal{V}^{n-an}_{\nu_A}$ (in the $+$-case) and $\mathcal{D}^{n-an}_{\nu_A}$ (in the $-$-case) an applying a finite slope projector in the same way as we did for classical sheaves. We also have cuspidal versions $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A,cusp)^{\pm,fs}$ By construction

$$\mathrm{R}\Gamma_{w,an}(K^p,\nu_A)^{\pm,fs},\ \mathrm{R}\Gamma_{w,an}(K^p,\nu_A,cusp)^{\pm,fs}$$

are objects of the derived category of A-modules. They admit some (non-canonical) better presentation. Namely, one can construct a complex of coherent sheaves $\mathrm{R}\Gamma_{w,an}\widetilde{(K^p,\nu_A)}^{\pm,fs}$ over $\mathrm{Spa}(A,A^+)\times\mathbb{G}_m^{an}$ supported on a closed subspace $\tilde{\mathcal{Z}} \hookrightarrow \mathrm{Spa}(A,A^+)\times\mathbb{G}_m^{an}$. The projection $\pi : \tilde{\mathcal{Z}} \to \mathrm{Spa}(A,A^+)$ is quasi-finite, partially proper and the complex $\mathrm{R}\Gamma_{w,an}\widetilde{(K^p,\nu_A)}^{\pm,fs}$ is a perfect complex of $\pi^{-1}\mathscr{O}_{\mathrm{Spa}(A,A^+)}$-modules. Moreover, $\pi_\star\mathrm{R}\Gamma_{w,an}\widetilde{(K^p,\nu_A)}^{\pm,fs} = \mathrm{R}\Gamma_{w,an}(K^p,\nu_A)^{\pm,fs}$. The same applies to cuspidal cohomology.

4.2.2. *The BGG spectral sequence.* When $\kappa \in X^\star(T)^{+,M}$, $\nu \in X^\star(T)^+ - \rho_{nc}$, and $\kappa = -w_{0,M}w(\nu+\rho)-\rho$, we have natural maps ,

$$\begin{array}{rcl} \mathrm{R}\Gamma_w(K^p,\kappa)^{+,fs} & \to & \mathrm{R}\Gamma_{w,an}(K^p,\nu)^{+,fs} \\ \mathrm{R}\Gamma_{w,an}(K^p,\nu)^{-,fs} & \to & \mathrm{R}\Gamma_w(K^p,-w_{0,M}\kappa-2\rho_{nc})^{-,fs} \end{array}$$

There is a spectral sequence from locally analytic overconvergent cohomology to overconvergent cohomology that will allow us to give sufficient conditions for the above map to induce quasi-isomorphisms.

Theorem 4.8. *There is a spectral sequence* $\mathbf{E}_w^{p,q}(K^p,\kappa)^+$ *converging to finite slope overconvergent cohomology* $\mathrm{H}_w^{p+q}(K^p,\kappa)^{+,fs}$, *such that*

$$\mathbf{E}_{w,1}^{p,q}(K^p,\kappa)^+ =$$

$$\oplus_{v\in W_M,\ell(v)=p}\mathrm{H}^q_{w,an}\big(K^p,(((w_{0,M}w)^{-1}vw_{0,M}w)\cdot\nu)\big)^{+,fs}.$$

There is a $\mathcal{H}_p^-$*-equivariant spectral sequence* $\mathbf{E}_w^{p,q}(K^p,\kappa)^-$ *converging to finite slope overconvergent cohomology* $\mathrm{H}_w^{p+q}(K^p,-w_{0,M}\kappa-2\rho_{nc})^{-,fs}$, *such that*

$$\mathbf{E}_{w,1}^{p,q}(K^p,\kappa)^- =$$

$$\oplus_{v\in W_M,\ell(v)=-p}\mathrm{H}^q_{w,an}\big(K^p,(((w_{0,M}w)^{-1}vw_{0,M}w)\cdot\nu)\big)^{-,fs}$$

We also have spectral sequences for cuspidal cohomology.

Example 4.9. In the $\mathrm{GL}_2/\mathbb{Q}$-case, the spectral sequence is trivial since $W_M=\{1\}$. This simply reflects the fact that $\mathcal{V}_\nu^{n-an}=\mathcal{V}_\kappa$.

Example 4.10. In the $\mathrm{GSp}_4/\mathbb{Q}$-case, the spectral sequence is induced by the exact triangles:

$$\mathrm{R}\Gamma_w(K^p,\kappa)^{+,fs}\to\mathrm{R}\Gamma_{w,an}(K^p,\nu)^{+,fs}\to\mathrm{R}\Gamma_{w,an}(K^p,w^{-1}w_{0,M}w\cdot\nu)^{+,fs}\stackrel{+1}{\to}$$

$$\mathrm{R}\Gamma_{w,an}(K^p,w^{-1}w_{0,M}w\cdot\nu)^{-,fs}\to\mathrm{R}\Gamma_{w,an}(K^p,\nu)^{-,fs}\stackrel{+1}{\to}\mathrm{R}\Gamma_w(K^p,-w_{0,M}\kappa-2\rho_{nc})^{-,fs}\stackrel{+1}{\to}$$

The following theorem follows by examining the possible slopes on cohomology in the BGG spectral sequence:

Theorem 4.11. *The maps*

$$\begin{array}{rcl}\mathrm{R}\Gamma_w(K^p,\kappa)^{+,sss} & \to & \mathrm{R}\Gamma_{w,an}(K^p,\nu)^{+,sss}\\ \mathrm{R}\Gamma_{w,an}(K^p,\nu)^{-,sss} & \to & \mathrm{R}\Gamma_w(K^p,-w_{0,M}\kappa-2\rho_{nc})^{-,sss}\end{array}$$

are quasi-isomorphisms. The same holds for cuspidal cohomology.

4.2.3. *Vanishing theorem.* The following result is probably the hardest technical result in our work.

Theorem 4.12. *The complex* $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A,cusp)^{+,fs}$ *is concentrated in degree* $[0,\ell(w)]$. *The complex* $\mathrm{R}\Gamma_{w,an}(K^p,\nu_A)^{+,fs}$ *is concentrated in degree* $[\ell(w),d]$.

The idea of the proof in the cuspidal case is the same as for the cohomologies $\mathrm{R}\Gamma_w(K^p,\kappa,cusp)^{+,fs}$: one has to prove the vanishing of the higher direct images for the projection to the minimal compactification and then use the affineness of π_{HT} on the minimal compactification. Unfortunately, the cohomology of Banach sheaves can be very pathological (the analogues of theorem A and B of Cartan fail in this context). The proof uses the theory of formal models to reduce the question to statements about quasi-coherent sheaves. We prove that the pre-image $(\pi^{tor}_{HT,K'_p})^{-1}(U)$ of any quasi-compact subset $U\subseteq\mathcal{FL}_{G,\mu}$ (necessarily invariant under a sufficiently small compact open subgroup K'_p) admits a "nice" formal model with a good description of the boundary. This is possible because the Hodge-Tate period morphism behaves well at the boundary of Shimura varieties.

4.3. **Construction of the eigenvariety.** Let us summarize all the complexes at hand. Let $\mathcal{W} = \mathrm{Spa}(\mathbb{Z}_p[[T(\mathbb{Z}_p)]]) \times_{\mathrm{Spa}(\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let $\mathrm{Spa}(A, A^+)$ be an affinoid open subset of $\mathcal{W}$ and let $\nu_A : T(\mathbb{Z}_p) \to A^\times$ be the universal character. For all $w \in {}^MW$, we have the complexes:

(1) $\mathrm{R}\Gamma_w(K^p, \nu_A)^{\pm, fs}$,
(2) $\mathrm{R}\Gamma_w(K^p, \nu_A, cusp)^{\pm, fs}$.

Those are related by pairings:

$$\langle , \rangle : \mathrm{R}\Gamma_{w,an}(K^p, \nu_A, cusp)^{\pm, fs} \times \mathrm{R}\Gamma_{w,an}(K^p, \nu_A)^{\pm, fs} \to A[-d]$$

The connection to classical cohomology is given via a series of spectral sequences. Let us summarize them.

For all points $x \in \mathrm{Spa}(A, A^+)$ with corresponding character ν, we have a Tor spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Tor}^A_{-p}(\mathrm{H}^p_{w,an}(K^p, \nu_A)^{+,fs}, k(\nu)) \Rightarrow \mathrm{H}^{p+q}_{w,an}(K^p, \nu)^{+,fs}.$$

For all $\kappa \in X^\star(T)^{+,M}$ and all $w \in {}^MW$, we let $\nu = -w^{-1}w_{0,M}(\kappa + \rho) - \rho$. We have the BGG spectral sequence:

$$\mathbf{E}^{p,q}_{w,1}(K_p, \kappa)^+ = \oplus_{v \in W_M, \ell(v) = p} \mathrm{H}^q_{w,an}\big(K^p, (((w_{0,M}w)^{-1} v w_{0,M} w) \cdot \nu)\big)^{+,fs} \Rightarrow \mathrm{H}^{p+q}_w(K^p, \kappa)^{+,fs}$$

For all $\kappa \in X^\star(T)^{+,M}$, we have the Bruhat stratification spectral sequence:

$$\mathbf{E}^{p,q}_1(K_p, \kappa)^+ = \oplus_{w \in {}^MW, \ell(w) = p} \mathrm{H}^{p+q}_w(K^p, \kappa)^{+,fs} \Rightarrow \mathrm{H}^{p+q}(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_\kappa)$$

There are also analogous spectral sequences for the cuspidal and the "-" theory. In particular if a Hecke eigensystem appears in a classical cohomology $\mathrm{H}^i(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_\kappa)$, we deduce that there exists:

- $j \in \mathbb{Z}$,
- $w_M \in W_M$,
- $w^M \in {}^MW$,

such that this eigensystem is in the support of $\mathrm{H}^j_{w,an}(K^p, \nu_A)^{+,fs}$ over the weight ν where $\nu = -w^{-1}w_{0,M}(\kappa + \rho) - \rho$ for $w = w_M w^M$.

Conversely, we know that for any $\kappa \in X^\star(T)^{+,M}$ with $\kappa + \rho$ regular, there is a unique $w \in {}^MW$ such that $\nu = -w^{-1}w_{0,M}(\kappa + \rho) - \rho \in X^\star(T)^+$ and

$$\mathrm{R}\Gamma_{w,an}(K^p, \nu)^{+,sss} = \mathrm{R}\Gamma(S^{tor}_{K^pK_p}, \mathcal{V}_\kappa)^{+,sss}.$$

There are analogous statements for cuspidal cohomology and the $-$ theory. Everything can be organized in the eigenvariety.

Theorem 4.13. *The eigenvariety $\pi : \mathcal{E} \to \mathcal{W}$ is locally quasi-finite and partially proper. It carries graded coherent sheaves*

$$\bigoplus_{w \in {}^MW, k \in \mathbb{Z}} \left(\mathcal{H}^{k,+,fs}_w \oplus \mathcal{H}^{k,-,fs}_w \oplus \mathcal{H}^{k,+,fs}_{w,cusp} \oplus \mathcal{H}^{k,-,fs}_{w,cusp}\right)$$

and we have maps $\mathcal{H}^S \otimes \mathbb{Q}_p[T(\mathbb{Q}_p] \to \mathscr{O}_\mathcal{E}$ where $\mathcal{H}^S$ is the spherical Hecke algebra away from the set of primes S containing p and the tame ramification.

They satisfy the following properties:

(1) *(Any classical, finite slope eigenclass gives a point of the eigenvariety) For any $\kappa \in X^\star(T)^{M,+}$, and any system of Hecke eigenvalues (λ_p, λ^S) occurring in $\mathrm{H}^i(S^{tor}_{K_pK^p}, \kappa)$ there is a $w = w_M w^M \in W$ and $k \in \mathbb{Z}$ so that if $\nu = -w^{-1}w_{0,M}(\kappa_{alg} + \rho) - \rho$, then (λ_p, λ^S) is a point of the eigenvariety $\mathcal{E}$ which*

lies in the support of $\mathcal{H}^{k,+,fs}_{w^M}$ *over* ν. *There is also a* $w' = w'_M w'^M \in W$ *and* $k' \in \mathbb{Z}$ *so that if* $\nu' = (w')^{-1}(\kappa+\rho)-\rho$, *then* (λ_p, λ^S) *is a point of the eigenvariety* $\mathcal{E}$ *which lies in the support of* $\mathcal{H}^{k,-,fs}_{w'^M}$ *over* ν'. *We have similar statements for cuspidal cohomology.*

(2) *(Small slope points of the eigenvariety in regular, locally algebraic weights are classical) Let* $\nu \in X^\star(T)^+$. *For any* $w \in {}^MW$ *let* $\kappa_w = -w_{0,M}w(\nu+\rho)-\rho$. *For any point* (λ_p, λ^S) *of* $\mathcal{E}$ *in the support of* $\oplus_{k\in\mathbb{Z}}\mathcal{H}^{k,+,sss}_w|_\nu$, *the system of eigenvalue is in the support of* $\oplus_k \mathrm{H}^k(S_{K_pK^p}, \mathcal{V}_{\kappa_w})^{+,sss}$. *Similarly, for any point* (λ_p, λ^S) *of* $\mathcal{E}$ *in the support of* $\oplus_{k\in\mathbb{Z}}\mathcal{H}^{k,-,sss}_w|_\nu$, *the system of eigenvalue is in the support of* $\oplus_k \mathrm{H}^k(S_{K_pK^p}, \mathcal{V}_{-w_{0,M}\kappa_w - 2\rho_{nc}})^{-,sss}$. *We have the same statements for cuspidal cohomology.*

(3) *(Serre duality interpolates over the eigenvariety) We have pairings:*

$$\mathcal{H}^{k,\pm,fs}_w \times \mathcal{H}^{d-k,\mp,fs}_{w,cusp} \to \pi^{-1}\mathscr{O}_{\mathcal{W}}.$$

and these pairings are compatible with Serre duality under the classicality theorems.

We finish by exhibiting some maximal dimensional components of the eigenvariety. We consider the map

$$\mathrm{H}^{\ell(w)}_{w-an}(K^p, \nu_A, cusp)^{+,fs} \to \mathrm{H}^{\ell(w)}_{w-an}(K^p, \nu_A)^{+,fs}$$

and denote its image by $\overline{\mathrm{H}}^{\ell(w)}(K^p, \nu_A)^{+,fs}$.

By the vanishing theorem, the right hand side is the first non-zero cohomology group of a complex of torsion free A-modules. Therefore it is a torsion free A-module. It follows that $\overline{\mathrm{H}}^{\ell(w)}(K^p, \nu_A)^{+,fs}$ is also a torsion free A-module. We can also define a subsheaf $\mathcal{H}_w$ of $\mathcal{H}^{\ell(w),+,fs}_w$ by taking the image of the above map (over each A). One sees that $\mathcal{H}_w$ is torsion free as a $\pi^{-1}\mathscr{O}_{\mathcal{W}}$-module.

By the vanishing theorem again, $\mathrm{H}^{\ell(w)}_{w-an}(K^p, \nu_A, cusp)^{+,fs}$ is the last nonzero cohomology group and hence the specialization map

$$\mathrm{H}^{\ell(w)}_{w-an}(K^p, \nu_A, cusp)^{+,fs} \otimes_A k(\nu) \to \mathrm{H}^{\ell(w)}_{w-an}(K^p, \nu, cusp)$$

is an isomorphism and it follows that the corresponding map on interior cohomology

$$\overline{\mathrm{H}}^{\ell(w)}_{w-an}(K^p, \nu_A, cusp)^{+,fs} \otimes_A k(\nu) \to \overline{\mathrm{H}}^{\ell(w)}_{w-an}(K^p, \nu, cusp)$$

is surjective.

By a final application of the vanishing theorem, the map

$$\mathrm{H}^{\ell(w)}_w(K^p, \kappa)^{+,fs} \to \mathrm{H}^{\ell(w)}_{w,an}(K^p, \nu)$$

is injective, and hence so is the corresponding map on interior cohomology.

Finally we recall that by proposition 1.20, any classical interior coherent cohomology class in $\overline{\mathrm{H}}^i(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_\kappa)$ occurs in some $\overline{\mathrm{H}}^{\ell(w)}_w(K^p, \kappa)$ for $\ell(w) = i$, and hence by the above arguments, occurs in the support of $\mathcal{H}_w$. To summarize we have the following theorem:

Theorem 4.14. *The eigenvariety carries torsion free* $\pi^{-1}\mathscr{O}_{\mathcal{W}}$*-sheaves* $\oplus_{w\in {}^MW}\mathcal{H}_w$ *whose support is a union of equidimensional components of the eigenvariety* $\mathcal{E}^! \subseteq \mathcal{E}$. *Any eigenclass in* $\overline{\mathrm{H}}^i(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_\kappa)$ *gives a point in the support of* $\mathcal{H}_w$ *for some* $w \in {}^MW$ *with* $\ell(w) = i$.

It seems natural to try to determine for which $w \in {}^M W$ the class c is in the support of $\mathcal{H}_w$. For example, is it the case that this set is exactly the set $C(\kappa)^+$ (at least when c corresponds to a generic eigensystem)?

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